MOP 2018: MANIPULATION AND BOUNDING (06/12, B)

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1. Algebraic manipulation

Problem 1.1 (Frobenius)**.** How many *odd* coefficients does $(x+y+z)^{15}$ have?

Problem 1.2 (Putnam?)**.** Given an odd prime p, prove that the function $F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}$ is injective on residues modulo p.

Problem 1.3. Find all pairs $p, q \in \mathbb{R}[x]$ such that $p(x)q(x+1) - p(x+1)q(x) = 1$.

Problem 1.4 (Rationalizing the denominator)**.** Let $\zeta_n := e^{2\pi i/n}$. Prove that $\mathbb{Q}[\zeta_n] = \mathbb{Q}(\zeta_n)$, where $\mathbb{Q}[\alpha] := \{P(\alpha) : P \in \mathbb{Q}[x]\}$ and $\mathbb{Q}(\alpha) := \{a/b : a \in \mathbb{Q}[\alpha],\ b \in \mathbb{Q}[\alpha] \setminus 0\}$.

Problem 1.5 (Hilbert's theorem 90, special case)**.** Prove that $\alpha \in \mathbb{Q}(\zeta_4)$ satisfies $\alpha\overline{\alpha} = 1$ if and only if there exists $z \in \mathbb{Q}(\zeta_4)$ such that $\alpha = \overline{z}/z$. Use this to solve $x^2 + y^2 = 1$ over $\mathbb{Q}$.

Problem 1.6 (ISL 2012 N8)**.** If $p > 100$ is prime and $r \in \mathbb{Z}$, then $(x^5 + r)^{(p-1)/2} + 1$ leaves a *nonzero polynomial* remainder modulo p (not all coefficients zero) when divided by $x^p - x$.

Problem 1.7 (USAMO 1988/5; AoPS)**.** For certain integers $b_k > 0$, the polynomial product

$$(1-z)^{b_1}(1-z^2)^{b_2}(1-z^3)^{b_3}(1-z^4)^{b_4}(1-z^5)^{b_5}\cdots(1-z^{32})^{b_{32}}$$

has the surprising property that if we multiply it out and discard all terms involving z to a power larger than 32, what is left is just $1 - 2z$. Determine, with proof, b_{32}.

What is the most natural formulation of the following problem?

Problem 1.8 (TST 2010)**.** Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and, for $n > 1$,

$$a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/3 \rfloor} + \ldots + a_{\lfloor n/n \rfloor} + 1.$$

Prove that there are infinitely many n such that $a_n \equiv n \pmod{2^{2010}}$.

2. Analysis: heuristics for estimates and bounding

Many "analysis-flavored" problems rely mainly on common-sense, bare-hands deductions based on inequalities, size considerations, infinitesimal, or asymptotic heuristics. Sometimes, you might want to use not just *local methods* (how individual points or pieces or a small portion works), but also *global reasoning* (how the pieces add up or interact with each other).

2.1. Using bounds effectively.

Problem 2.1 (China MO 2007)**.** Given a bounded sequence of real numbers such that $a_n < \frac{1}{2n+2007} + \sum_{k=n}^{2n+2006} \frac{a_k}{k+1}$ for all $n \geq 1$, prove that $a_n < \frac{1}{n}$ for all positive integers n.

Problem 2.2 (MOP 2007)**.** If $z \in \mathbb{C} \setminus \mathbb{R}$ but $(z^{2n+1} - 1)/(z^{2n} - z) \in \mathbb{R}$, show that $|z| = 1$.

Problem 2.3 (Putnam 2006, $k = 2$ case)**.** If $a_0 > 0$ and $a_{n+1} = a_n + a_n^{-1/2}$ for $n \geq 0$, estimate the growth of a_n. Is it exponential? Linear or quadratic? Some other power?

Problem 2.4. Determine the minimum constant $c \in \mathbb{R}$ such that for *every* finite sequence of reals $a_1, b_1, \ldots, a_n, b_n \in [1,2]$ with $\sum a_i^2 = \sum b_i^2$, we have $\sum a_i^3/b_i \le c \sum a_i^2$.

2.2. **Oscillation/cancellation.** In the d-dimensional lattice $\mathbb{Z}^d$, a random walk of N steps will, on average, take the random-walker to a squared distance of dN (substantially smaller than the maximum possible square distance, N^2). Many times in life, we don't have true *randomness*, but just *pseudo-randomness*, which can help us bound certain quantities.

Problem 2.5 (Abel?)**.** Prove that $\sum_{k\ge 1} \frac{z^k}{k}$ converges for all $z \in \mathbb{C} \setminus \{1\}$ with $|z| \le 1$.

Problem 2.6 (Gibbs?)**.** Prove that $\sum_{k=1}^{n} \frac{\sin kx}{k} \le 2\sqrt{\pi}$ for any real x and integer $n \ge 1$.

3. Test your instincts

3.1. **Polynomial roots.**

Problem 3.1 (Continuity of complex roots)**.** Let $f \in \mathbb{C}[x]$ be a *monic* degree n polynomial with roots $z_1, \ldots, z_n \in \mathbb{C}$. Let g be another monic degree n polynomial such that $g - f$ has all coefficients of absolute value at most ϵ. For each index $k \in \{1, \ldots, n\}$, prove that g has a root $z \in \mathbb{C}$ of distance at most $D = D_{f,\epsilon}$ from z_k, where D depends only on $n, z_1, \ldots, z_n, \epsilon$.

Problem 3.2. Let $f(x) = x^3 - ux^2 + vx - w$ be a cubic with u, v fixed positive reals and w a real variable. If f is maximal in w among f with all real roots, show that $\operatorname{Disc} f = 0$.

Here are two problems about polynomials with restricted coefficients. What is the best way to package and exploit the conditions on the coefficients?

Problem 3.3 (USA TST 2014/4)**.** Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying $\sum_{i=1}^{n-1} |c_i - 1| < 1$. Prove that

$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$

has no real roots.

Problem 3.4 (Putnam 1972)**.** The polynomial $p(x)$ has all coefficients 0 or 1, and $p(0) = 1$. Show that if the complex number z is a root, then $|z| \ge (\sqrt{5} - 1)/2$.

3.2. **Miscellaneous.**

Problem 3.5 (China 2009)**.** Let a, b, m, n be positive integers with $a \le m < n < b$. Prove that there exists a nonempty subset S of $\{ab, ab+1, \ldots, ab+a+b\}$ such that $(\prod_{x\in S} x)/mn$ is the square of a rational number.

Problem 3.6 (TST 2010)**.** Determine whether or not there exists a positive integer k such that $p = 6k+1$ is a prime and $\binom{3k}{k} \equiv 1 \pmod{p}$.

Problem 3.7 (Ostrowski's theorem)**.** Find all functions $f \colon \mathbb{Q} \to \mathbb{R}_{\ge 0}$ such that:

- f vanishes precisely at 0 (and is positive elsewhere);
- f is multiplicative, i.e. $f(ab) = f(a)f(b)$ for all a, b; and
- f satisfies the triangle inequality, i.e. $f(a+b) \le f(a) + f(b)$ for all a, b.[1]

Remark 3.8. Ostrowski illustrates concretely how "reduction mod p" and "size" are not completely separate concepts: there are *p-adic* notions of size that have to do with reduction modulo powers of p. People usually define the p-adic absolute value to be $|n|_p := p^{-v_p(n)}$, but something like $e^{-v_p(n)}$ would be fine too: the point is that p^{9001} is p-adically *small*.

[1] Such functions are called *absolute values*.