Conditional approaches to sums of cubes

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Sec 0: Sums of three cubes (Intro)

Let $g := x^3 + y^3 + z^3$, so that $g(\mathbb{Z}^3)$ consists of *sums of three cubes*, i.e. integers a represented by g over $\mathbb{Z}$.

Question (Integral Hasse principle)

Is every *admissible*[a] integer a represented by g (over $\mathbb{Z}$)?

[a] i.e. locally represented; i.e. $\not\equiv \pm 4 \bmod 9$

Example

- Booker '19: YES for $a = 33$, since

$$(8866128975287528)^3 + (-8778405442862239)^3 + (-2736111468807040)^3 = 33.$$

- Wooley '95+: YES for $\gg A^{0.917}$ ints $a \leq A$ ($A \to \infty$).

Example (Cont'd)

- Hooley '86+: YES for $\gg_\epsilon A^{1-\epsilon}$ ints $a \le A$, under Hypo HW ($\approx$ modularity + GRH for Hasse–Weil L-functions).

Theorem (W.)

Assume standard NT conj's on L-functions (e.g. Hypo HW + "RMT") & "unlikely" divisors ("$p^2 \mid \Delta(\boldsymbol{c})$"). Then 100% *(resp.* $> 0\%$*) of admiss. ints lie in* $g(\mathbb{Z}^3)$ *(resp.* $g(\mathbb{Z}^3_{>0})$*).*

Remark (Re: 100% Hasse)

- For $5x^3 + 12y^3 + 9z^3$ (in place of $x^3 + y^3 + z^3$), $\exists$ Hasse failures (Cassels–Guy '66 + ϵ).
- For $x^2 + y^2 + z^2 - xyz$ (Markoff), $\exists$ uncond. proof of 100% Hasse (Ghosh–Sarnak '17).

Sec ℓ^1: Zero/Level sets (Counting basics)

For $P = x_1^3 + \cdots + x_s^3$ $(s = 3, 6)$, $K \subset \mathbb{R}^s$ nice (cpt, semi-alg), $X \to \infty$, let $N_{P-a,K}(X) := \#\{\boldsymbol{x} \in \mathbb{Z}^s \cap XK : P = a\}$ $(a \in \mathbb{Z})$.

Example

$K = [-1, 1]^s \implies XK = [-X, X]^s$,

$$\begin{aligned} \mathbb{Z}^s \cap XK &\xrightarrow{P} \mathbb{Z} \\ \boldsymbol{x} &\mapsto P \ll X^3. \end{aligned}$$

So $N_{P-a,K}(X)$ is $\asymp X^{s-3}$ *on avg* (in ℓ^1) over $a \ll X^3$.

HL ("randomness") prediction: $N_{P-a,K}(X) \approx\approx X^{s-3} \prod_{v \leq \infty} \sigma_v$.
(Here and elsewhere, $\approx\approx$ means I may be lying a bit.)

Sec ℓ^2: Doubling (Rags to riches)

Let $g := y_1^3 + y_2^3 + y_3^3$. From $\mathbb{Z}^3 \xrightarrow{g} \mathbb{Z}$, get (the 2nd moment map, or "fiber-wise square")

$$\mathbb{Z} \leftarrow \mathbb{Z}^3 \times_g \mathbb{Z}^3 = \{(\boldsymbol{y}, \boldsymbol{z}) \in (\mathbb{Z}^3)^2 : g(\boldsymbol{y}) = g(\boldsymbol{z})\}.$$

Here $g(\boldsymbol{y}) = g(\boldsymbol{z}) \iff F(\boldsymbol{y}, -\boldsymbol{z}) = 0$ ($F := x_1^3 + \cdots + x_6^3$).

Observation (Classical)

Let $K = [-1, 1]^6$. If $N_{F,K}(X) \ll X^3$ ($X \to \infty$), then $> 0\%$ of $\mathbb{Z}$ lies in $g(\mathbb{Z}^3_{>0})$.

Proof.

C–S ineq (2nd moment method). □

Hooley '86a: HL ("randomness") prediction misses triv. sol's (e.g. $x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 0$); maybe the truth is HLH?

Conjecture (HLH)

For any nice $K \subset \mathbb{R}^6$,

$$N_{F,K}(X) = c_{\mathrm{HL},F,K} \cdot X^3 + \#\{\text{triv. } \boldsymbol{x} \in \mathbb{Z}^6 \cap XK\} + o(X^3)$$

$(X \to \infty)$.

Theorem (S. Diaconu '19 + ϵ)

Say, $\forall$ nice $K \subset \mathbb{R}^6$, HLH holds. Then 100% *Hasse holds.*

Proof.

Something like a variance analysis (cf. Ghosh–Sarnak '17 for "borderline" problems like $g = a$). The details are subtle. $\square$

Sec 3: What's known?

Hua '38: $N_{F,K}(X) \ll X^{7/2+\epsilon}$ (by Cauchy b/w structure and randomness in $4, 8$ vars, resp.).
Vaughan '86+: " " $\ll X^{7/2}(\log X)^{\epsilon-5/2}$ (by new source of randomness).
Hooley '86+: " " $\ll X^{3+\epsilon}$, under Hypo HW ($\approx$ modularity + GRH for Hasse–Weil L-functions).

Remark

A large-sieve hypo[a] would suffice (W.).
(It's open! But)
$\exists$ uncond. apps to $x^2 + y^3 + z^3$ (W., via Brüdern '91 + Duke–Kowalski '00 + Wiles et al).

[a] a la Bombieri–Vinogradov

Hooley used an "upper-bound precursor" to the δ-method.

Proposition (δ-method: Kloosterman '26, Duke–Friedlander–Iwaniec '93, Heath-Brown '96)

$$N_{F,K}(X) \approx\approx \mathbb{E}_{\boldsymbol{c} \ll X^{1/2}} \mathbb{E}_{n \le X^{3/2}} [n^{-1} S_{\boldsymbol{c}}(n)] =: \star$$

($\boldsymbol{c} \in \mathbb{Z}^6$), where

$$S_{\boldsymbol{c}}(n) := \sum_{a \bmod n}{}^{'} \sum_{\boldsymbol{x} \in (\mathbb{Z}/n)^6} e_n(aF(\boldsymbol{x}) + \boldsymbol{c} \cdot \boldsymbol{x}).$$

($e_n(t) := e^{2\pi i t/n}$) (Don't worry about the "$'$"; it means $a \perp n$)

Remark

Here $\boldsymbol{c} = 0$ captures major arcs (roughly speaking), producing HL but not full HLH. And $\boldsymbol{c} \neq 0$ captures...

“Pf”.

Idea (“Kloosterman method”) is to treat classical major and minor arcs uniformly (using Poisson summation[a]), and average over a mod n.

$$N_{F,K}(X) \approx\approx \sum_{n\le X^{3/2}} \frac{1}{nX^{3/2}} \sum_{a \bmod n}{}' \sum_{\boldsymbol{x}\ll X} e_n(aF(\boldsymbol{x})) \quad (\circ\text{-method})$$

$$\approx\approx \sum_{n\le X^{3/2}} \frac{1}{nX^{3/2}} \mathbb{E}_{\boldsymbol{c}\ll n/X}[S_{\boldsymbol{c}}(n)] \quad (\text{“complexity”}\ \ n/X)$$

$$\approx\approx \mathbb{E}_{n\le X^{3/2}} \mathbb{E}_{\boldsymbol{c}\ll X^{1/2}}[n^{-1} S_{\boldsymbol{c}}(n)] = \star.$$

Idea’: In gen’l (for $n \gg X$ large), $\sum'_{a \bmod n} \sum_{\boldsymbol{x}\ll X} e_n(aF(\boldsymbol{x}))$ is incomplete mod n, but still a wt’d avg of the complete sums $S_{\boldsymbol{c}}(n)$, if we sample over enough $\boldsymbol{c}$’s (Nyquist–Shannon). □

[a] with $\boldsymbol{c}=0$ “purely probabilistic”, and $\boldsymbol{c}\neq 0$ subtler

The $S_{\boldsymbol{c}}(n)$'s relate to $\mathcal{V}_{\boldsymbol{c}} := \{[\boldsymbol{x}] \in \mathbb{P}^5 : F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0\}$.
Fact: $\exists$ disc poly $\Delta \in \mathbb{Z}[\boldsymbol{c}]$ measuring singularities of $\mathcal{V}_{\boldsymbol{c}}$.

Lemma (Hooley)

If $\Delta(\boldsymbol{c}) \neq 0$, then $\widetilde{S}_{\boldsymbol{c}}(n) := n^{-7/2} S_{\boldsymbol{c}}(n)$ look (to 1st order) like the coeffs $\mu_{\boldsymbol{c}}(n)$ of $1/L(s, V_{\boldsymbol{c}})$ ($V_{\boldsymbol{c}} := (\mathcal{V}_{\boldsymbol{c}})_{\mathbb{Q}}$).

Partial proof sketch.

Here F is homog (& a is summed), so $S_{\boldsymbol{c}}(n)$ is multiplicative.
Locally: If $p \nmid \boldsymbol{c}$, then $\widetilde{S}_{\boldsymbol{c}}(p) = \widetilde{E}_{\boldsymbol{c}}(p) + O(p^{-1/2})$, where $\widetilde{E}_{\boldsymbol{c}}(p) := p^{-3/2}[\#\mathcal{V}_{\boldsymbol{c}}(\mathbb{F}_p) - \#\mathbb{P}^3(\mathbb{F}_p)]$. Now use LTF. $\square$

Exercise (Cf. Hooley, "$\underline{\underline{2}}\times$-Kloosterman")

"Assume" $\forall \boldsymbol{c}, n, N$: $\Delta(\boldsymbol{c}) \neq 0$, $\widetilde{S}_{\boldsymbol{c}}(n) = \mu_{\boldsymbol{c}}(n)$, $\sum_{n \leq N} \mu_{\boldsymbol{c}}(n) \ll \|\boldsymbol{c}\|^{\epsilon} N^{1/2+\epsilon}$. Then $\star \ll X^{3+\epsilon}$.

Sec 4: What's new?

Theorem (W.)

Assume standard NT conj's on

- $L(s, V_{\boldsymbol{c}}), L(s, V_{\boldsymbol{c}}, \bigwedge^2), L(s, V(F))$ *(Hypo HW2 + Ratios Conj's + Krasner*[a]*), and*
- *"unlikely" divisors (Square-free Sieve Conjecture for* $\Delta(\boldsymbol{c})$*).*

Then for any nice $K \subset \mathbb{R}^6$ *w/* $K \cap \operatorname{hess} F = \emptyset$*,*[b] *we have* $N_{F,K}(X) \ll X^3$*, & in fact HLH Conj. holds. (Actual hypo's for former are cleaner than those for latter.)*

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[a] "effective version of Kisin's thesis (*Local constancy in p-adic families of Galois representations*)"

[b] This could probably be removed with enough work, but is mild enough for our main qualitative needs.

Glossary for hypo's

1. Hypo HW2: Similar in spirit to Hooley's Hypo HW.
2. Ratios Conj's: Give predictions of Random Matrix Theory (RMT) type for mean values of $1/L(s, V_{\boldsymbol{c}})$ and $1/L(s_1, V_{\boldsymbol{c}})L(s_2, V_{\boldsymbol{c}})$ over families of $\boldsymbol{c}$'s.[1]
3. Krasner: Need $L_p(s, V_{\boldsymbol{c}})$ to only depend on $\boldsymbol{c} \bmod p\Delta(\boldsymbol{c})^{1000}$ (cf. Kisin's thesis).
4. SFSC: Need (for $Z \geq 1$, $P \leq Z^3$)

$$\Pr\left[\boldsymbol{c} \in [-Z, Z]^6 : \exists\, p \in [P, 2P] \text{ with } p^2 \mid \Delta(\boldsymbol{c})\right] \ll P^{-\delta}.$$

[1] How does $\boldsymbol{c} \mapsto L(s, V_{\boldsymbol{c}})$ behave on average? RMT predictions originated for L-zeros "in the bulk" from Montgomery–Dyson, and "near 1/2" from Katz–Sarnak. CFKRS (2005) developed *full main term* predictions for L-powers, and CFZ (2008) for L-ratios.

Proof hint.

We want to bound/estimate (via δ-method)

$$N_{F,K}(X) \approx\approx \mathbb{E}_{\boldsymbol{c} \ll X^{1/2}} \mathbb{E}_{n \leq X^{3/2}} [n^{-1} S_{\boldsymbol{c}}(n)].$$

Exponent numerics over various loci (if $d = 3$, $s = 6$):

$$\underbrace{s - d}_{\boldsymbol{c}=0,\ n \text{ small}} = \underbrace{\frac{s}{2} + \cancel{O(\epsilon)}}_{\Delta(\boldsymbol{c})=0,\ n \text{ large}} = \underbrace{\frac{d}{4}(s - \underline{\underline{2}}) + \cancel{O(4\epsilon)}}_{\Delta(\boldsymbol{c}) \neq 0}$$
$$= 3 + \cancel{O(5\epsilon)}.$$

Main terms of HLH: $\Delta(\boldsymbol{c}) = 0$ (key: $S_{\boldsymbol{c}}(n)$ is biased for special $\boldsymbol{c}$'s). Conditional/hardest part: $\Delta(\boldsymbol{c}) \neq 0$ (which "factors" into certain mean-value and pointwise estimates over $\boldsymbol{c}$). $\square$

A sample mean-value ingredient

Over $\Delta(\boldsymbol{c}) \neq 0$, the reciprocal L-functions $1/L(s, V_{\boldsymbol{c}})$ are the main players. The Ratios Conjectures imply e.g. the following:

Conjecture (R2', roughly)

For certain holomorphic $f(s)$, e.g. e^{s^2}, we have

$$\mathbb{E}'_{\boldsymbol{c} \ll X^{1/2}} \left| \int_{(\sigma)} ds \, \frac{\zeta(2s)^{-1} L(s+1/2, V(F))^{-1}}{L(s, V_{\boldsymbol{c}})} \cdot f(s) N^s \right|^2 \ll_f N$$

($\sigma > 1/2$; $1 \ll N \ll X^{3/2}$).

- There are no $\log N$ or $\log X$ factors on the RHS! Such factors are determined by the "symmetry type" of the underlying family of L-functions.
- This is enough "RMT input" for $N_{F,K}(X) \ll X^3$.

More on mean values (Cancellation over $\boldsymbol{c}$)

Also, for some $\delta > 0$, one expects the following:

Conjecture (R1, roughly)

$$\mathbb{E}'_{\boldsymbol{c} \ll X^{1/2}} \left[\frac{1}{L(s, V_{\boldsymbol{c}})} - \underbrace{\zeta(2s) L(s + 1/2, V(F))}_{\text{polar factors}} A_F(s) \right] \ll_{\sigma,t} X^{-\delta}$$

(over $\Delta(\boldsymbol{c}) \neq 0$) (for $X \geq 1$; $s = \sigma + it$; $\sigma > 1/2$)
Here $A_F(s) \ll 1$ for $\Re(s) \geq 1/2 - \delta$.

Remark

For $N_{F,K}(X) \ll X^3$, we only use (R2'). But for HLH (which requires "cancellation over $\boldsymbol{c}$"), we use a "slight adelic perturbation" of (R1).

A sample pointwise ingredient

We also use partial results[2] toward a conjectural dichotomy$/\mathbb{F}_p$, amusingly parallel to HLH:

Conjecture (Randomness vs. structure over $\mathbb{F}_p$)

If $p \geq 100$ and $\boldsymbol{c} \in \mathbb{F}_p^6$ with $|\#\mathcal{V}_{\boldsymbol{c}}(\mathbb{F}_p) - \#\mathbb{P}^3(\mathbb{F}_p)| \geq 10^{10} p^{3/2}$, then $\mathcal{V}_{\boldsymbol{c}} \bmod p$ contains a plane (i.e. $c_i^3 = c_j^3$ in pairs).

Remark (R. Kloosterman)

A char. 0 analog of a stronger conj. (in the nodal case) holds (with a Hodge-theoretic proof).

(Lindner '20 proves partial results towards the "stronger conjecture".)

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[2] proven using "worst-case" results of Skorobogatov '92 (or Katz '91) and "average-case" results of Lindner '20

A cartoon of today's main players

1. Let $g(\boldsymbol{y}) := y_1^3 + y_2^3 + y_3^3$ first.
2. Let $F(\boldsymbol{x}) := x_1^3 + \cdots + x_6^3$ second.

$$\underbrace{\mathbb{A}^3 \xrightarrow{g} \mathbb{A}^1 \xleftarrow{g} \mathbb{A}^3 \times_g \mathbb{A}^3 \cong \{(\boldsymbol{y}, \boldsymbol{z}) \in (\mathbb{A}^3)^2 : g(\boldsymbol{y}) = g(\boldsymbol{z})\}}_{\text{Cf. Hardy–Littlewood (1925)}}$$

$$\{(\boldsymbol{y}, \boldsymbol{z}) \in (\mathbb{A}^3)^2 : g(\boldsymbol{y}) = g(\boldsymbol{z})\} \cong \{F(\boldsymbol{x}) = 0\} = C(\mathcal{V})$$

$$\underbrace{C(\mathcal{V}) \dashrightarrow \mathcal{V} \xleftarrow{[\boldsymbol{x}]} \{([\boldsymbol{x}], [\boldsymbol{c}]) \in \mathcal{V} \times (\mathbb{P}^5)^\vee : \boldsymbol{c} \cdot \boldsymbol{x} = 0\} \xrightarrow{[\boldsymbol{c}]} (\mathbb{P}^5)^\vee}_{\text{Cf. Kloosterman (1926), Heath-Brown (1983), Hooley (1986), \ldots}}$$

Analogs?

- $c^2 + b^4 + a^4 = t$ has some similarity to $c^3 + b^3 + a^3 = t$.
- Allowing *negative* integers, one might go significantly further with "exceptional sets" for *non-critical* problems, like $c^2 + b^3 + a^3 = t$ or $c^2 + b^2 + a^3 = t$, than for the critical $c^3 + b^3 + a^3 = t$. Even conjecturally, the limits of variance analysis are unclear, in view of Brauer–Manin obstructions.

Deformations?

- Let $N_{(q)}(X) := \#\{\boldsymbol{x} \in \mathbb{Z}^6 \cap [-X, X]^6 : q \mid x_1^3 + \cdots + x_6^3\}$. It is routine to estimate $N_{(q)}(X)$ if $q \leq X^{1-\delta}$. The delta method gives a way to estimate $N_{(q)}(X)$ for $q > 6X^3$. What can be proven in between these extremes?
- (Based on a comment from Wooley.) Let $N^{(\gamma)}(X)$ be the number of integral solutions to

$$x_1^3 + x_2^3 + x_3^3 = y_1^3 + y_2^3 + y_3^3$$

with $x_1, y_1 \in [10X^\gamma, 20X^\gamma]$ and $x_2, y_2, x_3, y_3 \in [X, 2X]$. Then $N^{(3/2)}(X) \asymp X^{7/2}$ unconditionally, while $N^{(1)}(X) \ll X^{7/2}$ unconditionally and $N^{(1)}(X) \asymp X^3$ conditionally. What about for $\gamma \in (1, 3/2)$?