Wednesday, April 21, 2021 6:31 PM

These are very rough handwritten notes for an informal talk; use with caution.

L-series: arbitrary Dirichlet series

$$L(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$$

L-functions: axioms (Selberg)

(1) multiplicativity: $a_{mn} = a_m \cdot a_n$ if $(m,n)=1$

cf. Chinese remainder theorem

Euler product

$L(s) = \prod_p L_p(s)$

$L_p(s) = 1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \dots$

(2) functional equation, $L(s)$, $\overline{L}(1-s)$ related up to some factors

cf. Poisson summation formula

(3) analytic except at $s=1$ possibly (maybe poles)

(4) $a_n \ll_\varepsilon n^\varepsilon$

$(s) \cdots$

Note: Not all interesting L-series are L-functions.

Ex: w/o multiplicativity, "twisted multiplicativity" relations,

Kloosterman sums $K(a,b;n) := \sum_{xy \equiv 1 (n)} e_n(ax+by)$, $e_n := e\left(\frac{2\pi i}{n} \cdots\right)$

exp(2pi i/n ...), not e(2pi i/n ...)

$$K(1,1;mn) = \underbrace{K(\underline{n^{-1}},\underline{n^{-1}};m)\,K(\underline{m^{-1},m^{-1}};n)}_{\text{comes from}}$$

$$\frac{1}{mn} \equiv \underbrace{\frac{n^{-1}}{m} + \frac{m^{-1}}{n}}_{\text{Lagrange interpolation}} \pmod 1$$

or rather partial fractions?

Thursday, April 22, 2021 9:43 AM

$\underline{\text{Ex}}$ $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$

$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$\chi_d : \mathbb{Z} \quad n \mapsto \left(\frac{n}{d}\right)$

not quite correct, unless maybe if d is roughly a prime...

+1 if n is a square mod d

-1 if n is not

0 in some edge cases

$\chi_4(p) = 1$ if p is 1 mod 4

$\chi_4(p) = -1$ if p is 3 mod 4

$L(s, \chi_d)$ if $d \neq 1$: coefficients $\chi_d(n)$ oscillate,

In general, $L(s)$ a "irreducible" L-series, $L(s) \neq \zeta(s)$

then expect $a_1 + \dots + a_N \ll_\varepsilon \text{Cond}(L)^\varepsilon \cdot N^{\frac{1}{2}+\varepsilon}$

this bound on a_1+...+a_N is implied by GLH but probably not equivalent... (because unlike for 1/L, there's no direct zero/pole interpretation, so for sum a_n/n^(1/2 + it) partial sum n <= N, we lose too much in the t-aspect using partial summation).

also GLH probably needs deg L to be fixed for this uniformity to work out.

$$\underbrace{1 \quad \cdots \quad N}_{\text{General Lindelöf hypothesis}} \quad {}_{\varepsilon}\ \underbrace{\mathrm{Cond}(L)}_{\substack{\text{conductor} \\ \text{measures complexity of } L \\ \text{e.g. } \asymp |d| \text{ for } L(s,\chi_d)}} \cdot N$$

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) =: \sum_{n \geq 1} \underbrace{\mu(n)}_{\substack{\text{Möbius} \\ (-1)^{\#\text{ prime factors}}}} n^{-s}$$

$$\text{RH} \iff \mu(1) + \cdots + \mu(N) \ll_\varepsilon N^{\frac{1}{2}+\varepsilon}$$

In general, for $\displaystyle\frac{1}{L(s)} =: \sum_{n \geq 1} \mu_L(n) n^{-s}$

$$\text{GRH} \iff \mu_L(1) + \cdots + \mu_L(N) \ll_\varepsilon \mathrm{Cond}(L)^{\varepsilon} N^{\frac{1}{2}+\varepsilon}$$

<u>Note:</u> Actually GRH $\Rightarrow$ GLH.

This is b/c GRH is really a statement about zeros of $\log L$, or $\underline{L'}$

about zeros or $\underbrace{\log L}$, or $\frac{L'}{L}$

which control both L, $\frac{L'}{L}$

(GRH $\iff$ $L(s)$ has no $\underbrace{\text{zeros } \mathrm{Re}(s) > \frac{1}{2}}_{\text{"conspiracies"}}$)

Rmk 1: $\sum \frac{\chi_4(n)}{n^{\frac{1}{2}}}$ ~~[illegible]~~ converges,

more structured

less random

~~[illegible]~~

but $\sum \frac{\mu(n)\chi_4(n)}{n^{\frac{1}{2}}}$ diverges and

more random

is expected to have erratically oscillating partial sums (coin flip heads/tails)

Rmk 2: $\prod_p (1-\chi_4(p)p^{-\frac{1}{2}})$ should converge to $\frac{1}{L(\frac{1}{2},\chi_4)} \neq 0$

may be wrong up to constant factor... also is this conditional or unconditional?

In fact, "BS BSD" says

$$\prod_{p\le x}(1-\tilde{\alpha}_E(p)p^{-\frac{1}{2}})(1-\tilde{\beta}_E(p)p^{-\frac{1}{2}})$$

$p \le x$

should ~~look~~ look

like $(\log x)^{r_E}$

probably should be (log x)^(-r_E) (missing minus sign)

as $x \to \infty$

$E: y^2 = x^3 + \underline{a}x + \underline{b}$ / $\mathbb{Q}$

integers

For almost all primes p, $E \bmod p$ is an elliptic curve with

$(p+1) - p^{\frac{1}{2}}\underbrace{(\tilde{\alpha}_E(p) + \tilde{\beta}_E(p))}$

sqrt error

$\le 2\sqrt{p}$

$|\tilde{\alpha}|, |\tilde{\beta}| = 1$

Néron

$(\log x)^{r_E}$

$\propto \#\{(x,y) \text{ on } E(\mathbb{Q}) \text{ of height} \le X\}^2$

Notes: "BS BSD" $\Longrightarrow$ GRH for $L(s,E)$

$\Longrightarrow$ BSD

Thursday, April 22, 2021 10:10 AM

Average behavior of L-functions in families

Examples: - 50-50 ranks of elliptic curves in natural families

e.g. $dy^2 = x^3 + ax + b$

a, b fixed

- 50% should have rk r_0
- 50% should have rk $r_0 + 1$
- 0% rk $\geq r_0 + 2$

(BSD on average, + Goldfeld conjecture)

- Zeros, values, etc. of L-functions often behave more simply on average than individually (e.g. extreme or worst-case behavior) ...

yet average behavior is often enough

yet average ... is ... belongs

in "applications" (or interesting in own right)

Nowadays predictions in families are unified (hopefully!) conjecturally via random matrix theory (RMT) predictions

RMT predictions for L vs. $\frac{1}{L}$

L reflect mixture of randomness ($\ll \sqrt{\mathrm{Cond}(L)}$) and structure ($\gg \sqrt{\mathrm{Cond}(L)}$ due to duality, functional eqn, Poisson sum...)

$\frac{1}{L}$ reflect "randomness" of Möbius $\mu(n)$

$$\sum_{d \le X} L\left(\tfrac{1}{2}, \chi_d\right)^k = ??? \quad \text{(RMT prediction)}$$

$$\sum \frac{1}{\quad} = ??? \quad \text{(RMT prediction)}$$

$$\sum_{d \le X} \frac{1}{L(s,\chi_d)^k} = ??? \quad \text{(RMT predictions)}$$

$$\mathrm{Re}(s) > \frac{1}{2}$$

Unconditional applications of philosophy of families

1) Positivity & amplification ("embedding into families")

Kloosterman 1926: $|K(1,1;p)| \leq p$ (p a prime)

nontriv. estimate:

$$\underbrace{|K(1,1;p)|^4}_{\text{complicated}} \leq \underbrace{\sum_{a(p)} |K(a,1;p)|^4}_{\text{simpler you can understand (compute!)}}$$

$x_1+x_2+x_3+x_4=0$

$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=0$

$$\ll p^3$$

$$\Rightarrow |K(1,1;p)| \ll p^{\frac{3}{4}}$$

GRH over finite fields: e.g.

curve

e.g.

V a smooth projective ~~[illegible]~~ curve $/\mathbb{F}_p$

$$\#\{V(\mathbb{F}_p)\} = p+1+O(\sqrt{p})$$

Deligne (1973) proved the general version of the RH by positivity + amplification.

Thursday, April 22, 2021 10:30 AM

Conditional applications of RMT philosophy

1) Järviniemi – Teräväinen 2020: assume GRH + RMT philosophy.

Then 2^n+5 is almost always composite.

2) W. 2021: assume GRH, ——— + technical ingredients.

Then $x^3+y^3+z^3$ captures almost all integers $a \not\equiv 4, 5 \pmod 9$.

Both use variance analysis (Chebyshev's inequality, a la ...) to go from a relatively sparse individual settings

settings

to a "richer" average setting

susceptible to "rich statistical (namely, equidistribution) predictions"