Adic spaces

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Conventions :

- Every ring is commutative.
- $\mathbb{N}$ is the set of nonnegative integers.

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I.1 Valuations

I.1.1 Valuations and valuation rings

Notation I.1.1.1. If R is a local ring, we denote its maximal ideal by $\mathfrak{m}_R$.

Definition I.1.1.2. If $A \subset B$ are local rings, we say that B *dominates* A if $\mathfrak{m}_A \subset \mathfrak{m}_B$ (which is equivalent to $\mathfrak{m}_A = A \cap \mathfrak{m}_B$).

Remark I.1.1.3. If K is a field, domination is an order relation on local subrings of K.

Proposition I.1.1.4. *Let K be a field and $R \subset K$ be a subring. The following are equivalent :*

(a) R is local and it is maximal for the relation of domination (amnog local subrings of K);

(b) for every $x \in K^\times$, we have $x \in R$ or $x^{-1} \in R$;

(c) $\mathrm{Frac}(R) = K$, and the set of ideals of R is totally ordered for inclusion;

(c) $\mathrm{Frac}(R) = K$, and the set of principal ideals of R is totally ordered for inclusion.

If these conditions are satisfied, we say that R is a valuation subring of K.

Remark I.1.1.5. We use the convention that K is a valuation subring of itself.

Now we define valuations.

Definition I.1.1.6. An *ordered abelian group* is an abelian group $(\Gamma, +)$ with an order relation $\leq$ such that for all $a, b, c \in \Gamma$, $a \leq b \Rightarrow a + c \leq b + c$.

We will only be interested in totally ordered abelian groups. Here are some examples.

Example I.1.1.7.

- $(\mathbb{R}, +)$ with its usual order relation; more generally, any subgroup of $(\mathbb{R}, +)$, for example $(\mathbb{Z}, +)$.
- $(\mathbb{R}_{>0}, \times)$ with its usual order relation. Note that this is isomorphic (as an ordered group) to $(\mathbb{R}, +)$ by the map $\log$.
- $(\mathbb{R} \times \mathbb{R}, +)$ with the lexicographic order; more generally, $(\mathbb{R}^n, +)$ with the lexicographic order, for any positive integer n.

Remark I.1.1.8. Let $(\Gamma, \times)$ be a totally ordered abelian group .

(1) Γ is torsionfree : Indeed, if $\gamma \in \Gamma$ is a torsion element, then there is some positive integer n such that $\gamma^n = 1$. If $\gamma \leq 1$, then $1 \leq \gamma \leq \gamma^n = 1$, so $\gamma = 1$. If $\gamma \leq 1$, then $1 \geq \gamma \geq \gamma^2 = 1$, so $\gamma = 1$.

(2) Suppose that Γ is not the trivial group. Then, for every $\gamma \in \Gamma$, there exists $\gamma' \in \Gamma$ such that $\gamma' < \gamma$. Indeed, if $\gamma > 1$ then $\gamma^{-1} < \gamma$, if $\gamma < 1$ then $\gamma^2 < \gamma$, and if $\gamma = 1$ then there exists some $\delta \in \Gamma - \{1\}$ (because Γ is not trivial), and then either $\delta < 1$ or $\delta^{-1} < 1$.

Notation I.1.1.9. Let Γ be a totally ordered abelian group . We will want to add an element to Γ that is bigger or smaller than all the elements of Γ. More precisely :

(a) If the group law of Γ is written additively, we denote the unit element of Γ by 0, and we write $\Gamma \cup \{\infty\}$ for the union of Γ and of an element ∞, and we extend $+$ and $\leq$ to this set by the following rules : for every $a \in \Gamma$,

- $a + \infty = \infty + a = \infty$;
- $a \leq \infty$.

(b) If the group law of Γ is written multiplicatively, we denote the unit element of Γ by 1, and we write $\Gamma \cup \{0\}$ for the union of Γ and of an element 0, and we extend $\times$ and $\leq$ to this set by the following rules : for every $a \in \Gamma$,

- $a \times 0 = 0 \times a = 0$;
- $0 \leq a$.

Definition I.1.1.10. Let R be a ring.

(i) An *additive valuation* on R is a map $v : R \to \Gamma \cup \{\infty\}$, where $(\Gamma, +)$ is a totally ordered abelian group , satisfying the following conditions :

- $v(0) = \infty$, $v(1) = 0$;
- $\forall x, y \in R$, $v(xy) = v(x) + v(y)$;
- $\forall x, y \in R$, $v(x + y) \geq \min(v(x), v(y))$.

The *value group* of v is the subgroup of Γ generated by $\Gamma \cap v(R)$. The *kernel* (or *support*) of v is $\mathrm{Ker}(v) = \{x \in R \mid v(x) = \infty\}$; it is a prime ideal of R.

(ii) A *multiplicative valuation* (or *non-Archimedean absolute value*) on R is a map $|.| : R \to \Gamma \cup \{0\}$, where $(\Gamma, \times)$ is a totally ordered abelian group , satisfying the following conditions :

- $|0| = 0$, $|1| = 1$;
- $\forall x, y \in R$, $|xy| = |x||y|$;
- $\forall x, y \in R$, $|x + y| \leq \max(|x|, |y|)$.

The *value group* of $|.|$ is the subgroup of Γ generated by $\Gamma \cap |R|$. The *kernel* (or *support*) of $|.|$ is $\mathrm{Ker}(|.|) = \{x \in R \mid |x| = 0\}$; it is a prime ideal of R.

Remark I.1.1.11. Of course, whether we write the group law of a given totally ordered abelian group Γ additively or multiplicatively is an arbitrary choice. Fix Γ. Note that a map $v : R \to \Gamma \cup \{\infty\}$ is an additive valuation if and only if $-v$ is a multiplicative valuation (with the obvious convention that minus the biggest element is a new smallest element). This is a bit unfortunate, as both expressions "additive valuation" and "multiplicative valuation" are often shortened to "valuation", but hopefully the meaning is always clear from context. Modulo this sign issue, both definitions are equivalent, and the notions of value group and kernel are the same on both sides.

In these notes, we will eventually take all valuations to be multiplicative (as this is the usual convention for adic spaces), and "valuation" will mean "multiplicative valuation". Note however that some commutative algebra references, Matsumura's [21] or Bourbaki's [5] for example, use additive valuations.

Example I.1.1.12. (1) Let $\Gamma = (\{1\}, \times)$. Suppose that R is an integral domain. Then there is a valuation $|.|_{\mathrm{triv}} : R \to \Gamma \cup \{0\}$, called the *trivial valuation*, defined by : $|0| = 0$ and $|x| = 1$ for every $x \neq 0$. Its value group is $\{1\}$ and its kernel is $\{0\}$.

(2) More generally, if R is a ring and $\wp$ is a prime ideal of R, then composing the quotient map $R \to R/\wp$ with the trivial valuation on $R/\wp$ gives a valuation on R with value group $\{1\}$ and kernel $\wp$. We call this the *trivial valuation on R with kernel $\wp$*. We will often denote it by $|.|_{\wp,\mathrm{triv}}$.

(3) For every prime number ℓ, the usual ℓ-adic valuation $v_\ell : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ is an additive valuation on $\mathbb{Q}$ (and on all its subrings), and the ℓ-adic absolute value $|.|_\ell : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ is a multiplicative valuation on $\mathbb{Q}$. They are related by $|.|_\ell = \ell^{-v_\ell}$, and for our purposes they are interchangeable.

(4) By Ostrowski's theorem, every nontrivial valuation on $\mathbb{Q}$ is of the form $|.|_\ell^s$, for some prime number ℓ and some $s \in \mathbb{R}_{>0}$. It is easy to deduce from this that the valuations on $\mathbb{Z}$ are the trivial valuation, the valuation described in point (2) (one for each prime number ℓ), and the valuation $|.|_\ell^s$, for ℓ a prime number and $s \in \mathbb{R}_{>0}$.

Definition I.1.1.13. Let R be a ring and $|.|_1$, $|.|_2$ be two valuations on R. We denote by Γ_1 and Γ_2 their respective value groups. We say that $|.|_1$ and $|.|_2$ are *equivalent* if there exists an isomorphism of ordered groups $\varphi : \Gamma_1 \xrightarrow{\sim} \Gamma_2$ such that $\varphi \circ |.|_1 = |.|_2$ (with the convention that $\varphi(0) = 0$).

We compare the notions of valuations on a field and of valuation subrings of this field.

Proposition I.1.1.14. *Let K be a field.*

(i) If $v : K \to \Gamma \cup \{0\}$ is a valuation, then $R := \{x \in K \mid |x| \leq 1\}$ is a valuation subring of K, and its maximal ideal is given by the formula $\mathfrak{m}_R = \{x \in K \mid |x| < 1\}$.

(ii) Let R be a valuation subring of K. Let $\Gamma = K^\times/R^\times$; for $a, b \in K^\times$, we write $aR^\times \leq bR^\times$ if $ab^{-1} \in R$. Then this makes Γ into a totally ordered abelian group , and the map $|.| : K \to \Gamma \cup \{0\}$ defined by $|0| = 0$ and $|a| = aR^\times$ if $a \neq 0$ is a valuation on K.

(iii) The constructions of (i) and (ii) induce inverse bijections between the set of valuation subrings of K and the set of equivalence classes of valuations on K.

Notation I.1.1.15. If K is a field and $R \subset K$ is a valuation subring, we denote by $\Gamma_R = K^\times/R^\times$ the value group of the corresponding valuation on K, and we call it the value group of R. We also denote by $|.|_R : K \to \Gamma_R \cup \{0\}$ the valuation defined by R.

Example I.1.1.16. The valuation subring of $\mathbb{Q}$ corresponding to the ℓ-adic absolute value $|.|_\ell$ is $\mathbb{Z}_{(\ell)}$, with maximal ideal $\ell\mathbb{Z}_{(\ell)}$.

I.1.2 Some properties of valuations and valuation rings

Proposition I.1.2.1. *Let K be a field and $R \subset K$ be a valuation subring. Then :*

(i) R is integrally closed in K.

(ii) Every finitely generated ideal of R is principal (in other words, R is a Bézout domain).

(iii) If $I \subset R$ is a finitely generated ideal, then $\wp = \sqrt{I}$ is a prime ideal of R, and it is minimal among prime ideals of R containing I.

Theorem I.1.2.2. *Let K be a field and A be a subring of K.*

(i) ([21] Theorem 10.2) Let $\wp$ be a prime ideal of A. Then there exists a valuation subring $R \supset A$ of K such that $\wp = A \cap \mathfrak{m}_R$.

(ii) ([21] Theorem 10.4) Let B be the integral closure of A in K. Then B is the intersection of all the valuation subrings of K containing A.

Proposition I.1.2.3. *([5] §2 №3 prop. 1 and prop. 2, and №4 prop. 4.) Let K be a field, $|.| : K \to \Gamma \cup \{0\}$ be a multiplicative valuation and K' be an extension of K. Then there exists a multiplicative valuation $|.|' : K' \to \Gamma' \cup \{0\}$ such that $|.|'_{|K}$ is equivalent to $|.|$.*

Moreover, if $x_1, \dots, x_n \in K'$ are algebraically independent over K and $\gamma_1, \dots, \gamma_n \in \Gamma$, then we can find such a $|.|'$ such that $|x_i|' = \gamma_i$ for every $i \in \{1, \dots, n\}$.

Corollary I.1.2.4. *(Proposition 2.25 of [26].) Let K'/K be a field extension, let R' be a valuation subring of K', and set $R = K \cap R'$. Then R is a valuation subring of K, and the map $S' \longmapsto K \cap S'$ induces surjections*

$$\{\text{valuation subrings } R' \subset S' \subset K'\} \to \{\text{valuation subrings } R \subset S \subset K\}$$

and

$$\{\text{valuation subrings } S' \subset R' \subset K'\} \to \{\text{valuation subrings } S \subset R \subset K\}.$$

Moreover, if the extension K'/K is algebraic, then these surjections are actually bijections.

I.1.3 Rank of a valuation

We first define the height of a totally ordered abelian group Γ.

Definition I.1.3.1. ([5] §4 №2 Définition 1 p. 108) Let Γ be a totally ordered abelian group . A *convex subgroup* (or *isolated subgroup*) of Γ is a subgroup Δ of Γ such that, for all $a, b, c \in \Gamma$, if $a \leq b \leq c$ and $a, c \in \Delta$, then $b \in \Delta$.

Remark I.1.3.2. The condition in definition I.1.3.1 is also equivalent to the following condition (using additive conventions for Γ) : (*) for all $a, b \in \Gamma$ such that $0 \leq a \leq b$, if $b \in \Delta$, then $a \in \Delta$. Indeed, the condition of definition I.1.3.1 is obviously stronger than (*). Conversely, suppose that Δ satisfies (*), and let $a, b, c \in \Gamma$ such that $a \leq b \leq c$ and $a, c \in \Delta$; then $0 \leq b - a \leq c - a$ and $c - a \in \Delta$, so $b - a \in \Delta$ by (*), hence $b = (b - a) + a \in \Delta$.

Example I.1.3.3.
(1) $\{0\}$ and Γ are convex subgroups of Γ.
(2) If $\Gamma = \mathbb{R} \times \mathbb{R}$ with the lexicogrpahic order, then $\{0\} \times \mathbb{R}$ is a convex subgroup of Γ.

Proposition I.1.3.4. *([5] §4 №4 p. 110) The set of all convex subgroups of* Γ*, ordered by inclusion, is a well-ordered set. Its ordinal is called the* height *of* Γ *and denoted by* $\mathrm{ht}(\Gamma)$.

Example I.1.3.5.
(1) If $\mathrm{ht}(\Gamma) = 0$, then Γ has only one convex subgroup. As $\{0\}$ and Γ are always convex subgroups, this means that Γ is the trivial group.
(2) The condition $\mathrm{ht}(\Gamma) = 1$ means that Γ is nontrivial and that the only convex subgroups of Γ are the trivial subgroup and Γ itself. For example, if Γ is a nontrivial subgroup of $(\mathbb{R}, +)$, then Γ has height 1. (We will see in proposition I.1.3.6 that the converse is true.)
(3) $\Gamma = \mathbb{R}^n$ (with the lexicographic order) has height n. (See proposition I.1.4.1.)

In these notes, we mostly care about the distinction "height ≤ 1" versus "height > 1".

Proposition I.1.3.6. *([21] Theorem 10.6 or [5] §4 №5 proposition 8 p. 112) Let* $(\Gamma, +)$ *be a nontrivial totally ordered abelian group . The following are equivalent :*

(i) $\mathrm{ht}(\Gamma) = 1$*;*

(ii) for all $a, b \in \Gamma$ *such that* $a > 0$ *and* $b \geq 0$*, there exists* $n \in \mathbb{N}$ *such that* $b \leq na$*;*

(iii) there exists an injective morphism of ordered groups $\Gamma \to (\mathbb{R}, +)$.

Proof.

(i)⇒(ii) Consider the smallest convex subgroup Δ of Γ containing a. Condition (i) means that $\Delta = \{0\}$ or $\Delta = \Gamma$, so $\Delta = \Gamma$. This obviously implies (ii).

(ii)⇒(iii) We will just indicate the construction of the map $\varphi : \Gamma \to \mathbb{R}$. Fix some $a \in \Gamma$ such that $a > 0$. Let $b \in \Gamma$, and let $n_0 = \min\{n \in \mathbb{Z} \mid na \leq b\}$ (this makes sense by (ii)) and $b_1 = b - n_0 a$. We define sequences $(n_i)_{i\geq 1}$ of $\mathbb{N}$ and $(b_i)_{i\geq 1}$ of Γ by induction on i in the following way : Suppose that we have defined $b_1, \dots, b_i$ and $n_1, \dots, n_{i-1}$, for some $i \geq 1$. Then we set $n_i = \min\{n \in \mathbb{N} \mid n_i a \leq 10 b_i\}$ and $b_{i+1} = 10 b_i - n_i a$. Then we take $\varphi(b) = n_0 + \sum_{i\geq 1} n_i 10^{-i}$.

□

Definition I.1.3.7. The *rank* of a valuation (resp. of a valuation ring) if the height of its value group.

I.1.4 Comparing valuation subrings of a field

Proposition I.1.4.1. *([5] §4 №2 proposition 3 p. 108 and §4 №4 Exemples p. 111) Let Γ be a totally ordered abelian group .*

(i) If $\varphi : \Gamma \to \Gamma'$ is a morphism of ordered groups, then $\operatorname{Ker}\varphi$ is a convex subgroup of Γ.

(ii) Let Δ be a convex subgroup of Γ. We define a relation $\leq$ on Γ/Δ in the following way : if $c, c' \in \Gamma/\Delta$, then $c \leq c'$ if and only if there exists $x \in c$ and $x' \in c'$ such that $x \leq x'$ in Γ. Then this makes Γ/Δ into an ordered group (necessarily totally ordered) if and only if Δ is a convex subgroup.

(iii) In the situation of (ii), we have $\operatorname{ht}(\Gamma) = \operatorname{ht}(\Delta) + \operatorname{ht}(\Gamma/\Delta)$.

Theorem I.1.4.2. *([21] theorem 10.1 or [5] §4 №1 p. 106) Let K be a field.*

(i) Let $B \subset A \subset K$ be two subrings of K, and suppose that B is a valuation subring of K. Then :

(a) A is also a valuation subring of K.

(b) $\mathfrak{m}_A \subset \mathfrak{m}_B$, with equality if and only if $A = B$.

(c) $\mathfrak{m}_A$ is a prime ideal of B, and we have $A = B_{\mathfrak{m}_A}$.

(d) $B/\mathfrak{m}_A$ is a valuation subring of the field $A/\mathfrak{m}_A$.

(ii) Conversely, let $A \subset K$ be a valuation subring, let $\overline{B}$ be a valuation subring of $A/\mathfrak{m}_A$, and denote by B the inverse image of $\overline{B}$ in A. Then B is a valuation subring of K (in particular, $K = \operatorname{Frac}(B)$), its maximal ideal is the inverse image of the maximal ideal of $\overline{B}$, and we have an exact sequence of ordered groups :

$$1 \to \Gamma_{\overline{B}} \to \Gamma_B \to \Gamma_A \to 1,$$

where the maps $\Gamma_{\overline{B}} = (A/\mathfrak{m}_A)^\times/(B/\mathfrak{m}_A)^\times \simeq A^\times/B^\times \to \Gamma_B = K^\times/B^\times$ and $\Gamma_B = K^\times/B^\times \to \Gamma_A = K^\times/A^\times$ are the obvious ones.

Example I.1.4.3. Take $K = k((u))((t))$ and $A = k((u))[[t]]$ (the valuation subring corresponding to the t-adic valuation on K). Then $A/\mathfrak{m}_A = k((u))$ has the u-adic valuation, and the corresponding valuation subring is $\overline{B} = k[[u]]$. Its inverse image in A is

$$B = \{f = \sum_{n\geq 0} f_n t^n \in k((u))[[t]] \mid f_0 \in k[[u]]\}.$$

This is a valuation subring of rank 2 of K, and its value group is $\mathbb{Z}\times\mathbb{Z}$ (with the lexicographic order).

Corollary I.1.4.4. *([5] §4 №1 proposition 1 p. 106 and §4 №3 proposition 4 p. 109) Let K be a field and $B \subset K$ be a valuation subring.*

(i) The map $\wp \longmapsto B_\wp$ is an order-reversing bijection from the set of prime ideals of B to the set of subrings of K containing B; its inverse is $A \longmapsto \mathfrak{m}_A$.

(ii) The map $A \longmapsto \mathrm{Ker}(\Gamma_B \to \Gamma_A)$ is an order-preserving bijection from the set of subrings $A \supset B$ of K to the set of convex subgroups of Γ_B. Its inverse sends a convex subgroup H of Γ_B to the subring $A := B_\wp$, where $\wp = \{x \in B \mid \forall\delta \in H, |x|_B < \delta\}$.

Corollary I.1.4.5. *Let K be a field and $R \subset K$ be a valuation subring. Then the rank of R is equal to the Krull dimension of R.*

Corollary I.1.4.6. *([5] §4 №5 proposition 6 p. 111) Let K be a field and $R \subset K$ be a valuation subring. Then R has rank 1 if and only if it is maximal among all the subrings of K distinct from K.*

Note also the following result :

Proposition I.1.4.7. *([5] §3 №6 proposition 9 p. 105) Let K be a field and $R \subset K$ be a valuation subring. The following are equivalent :*

(i) R is a principal ideal domain.

(ii) R is Noetherian.

(iii) The ordered group Γ_R is isomorphic to $(\mathbb{Z}, +, \leq)$.

(iv) The ideal $\mathfrak{m}_R$ is principal and $\bigcap_{n\geq 0} \mathbb{R}^n = (0)$.

If these conditions are satisfied, we say that R is a discrete valuation ring *and that the corresponding valuation on K is* discrete.

I.1.5 Valuation topology and microbial valuations

Definition I.1.5.1. Let R be a ring and $|.| : R \to \Gamma \cup \{0\}$ be a valuation on R. The *valuation topology* on R associated to $|.|$ is the topology given by the base of open subsets $B(a,\gamma) = \{x \in R \mid |x - a| < \gamma\}$, for $a \in R$ and $\gamma \in \Gamma$.

This makes R into a topological ring, and the map $|.|$ is continuous if we put the discrete topology on $\Gamma \cup \{0\}$. (See [5] §5 Nº1 for the case of a field, the general case is similar.)

Remark I.1.5.2. Let R and $|.|$ be as in definition I.1.5.1.

(1) The valuation topology is Hausdorff if and only $\mathrm{Ker}\,|.| = \{0\}$ (which implies that R is a domain).

(2) If the value group of $|.|$ is trivial, then the valuation topology is the discrete topology.

(3) For every $a \in R$ and every $\gamma \in \Gamma$, let $\overline{B}(a,\gamma) = \{x \in R \mid |x-a| \leq \gamma\}$. This is an open subset in the valuation topology; indeed, for every $x \in \overline{B}(a,\gamma)$, we have $B(x,\gamma) \subset \overline{B}(a,\gamma)$.

Suppose that the value group of $|.|$ is not trivial. Then the sets $\overline{B}(a,\gamma)$ also form a base of the valuation topology. Indeed, for every $a \in R$ and every $\gamma \in \Gamma$, we have $B(a,\gamma) = \bigcup_{\delta<\gamma} \overline{B}(a,\delta)$. (This uses remark I.1.1.8(2), and it is not true in general if the value group is trivial, as then all the sets $\overline{B}(a,\gamma)$ are equal to R, so they cannot form a base of the discrete topology unless $R = \{0\}$.)

Definition I.1.5.3. Let R be a topological ring. An element $x \in R$ is called *topologically nilpotent* if 0 is a limit of the sequence $(x^n)_{n\geq 0}$

Theorem I.1.5.4. *Let K be a field, let $|.| : K \to \Gamma \cup \{0\}$ be a valuation, and let R be the corresponding valuation ring. We put the valuation topology on K.*

Then the following are equivalent :

(i) The topology on K coincides with the valuation topology defined by a rank 1 valuation on K.

(ii) There exists a nonzero topologically nilpotent element in K.

(iii) R has a prime ideal of height 1.[1]

If these conditions are satisfied, we say that the valuation $|.|$ is microbial. *Moreover, if the valuation is microbial, then :*

(a) If $\varpi \in K^\times$ is topologically nilpotent, we have $K = R[\varpi^{-1}]$, and, if $\varpi \in R$, then the subspace topology on R coincides with the ϖR-adic topology and also with the ϖ-adic topology.

(b) If $\wp$ is a prime ideal of height 1 of R, then the valuation subring $R_\wp$ has rank 1, and the corresponding valuation defines the same topology as $|.|$.

Note that, as the ideals of R are totally ordered by inclusion, R has at most one prime ideal of height 1.

[1] That is, a prime ideal that is minimal among the set of nonzero prime ideals of R.

Example I.1.5.5. We use the notation of example I.1.4.3. We claim that the valuation subrings A and B define the same topology on $K = k((u))((t))$. Indeed, the valuation corresponding to A is the t-adic valuation, which has rank 1, so we can try to apply theorem I.1.5.4 to B. We know that B has Krull dimension 2, so any prime ideal that is neither (0) not maximal has height 1; this is the case for $I := tB$, which is prime and not maximal because $B/I = k[[u]]$ by definition of B. Also, we see easily that $I = \mathfrak{m}_A$, so we are done by point (b) of the theorem.

Lemma I.1.5.6. *Let K be a field and $|.|$ be a valuation on K with valuation ring R. Then R and $\mathfrak{m}_R$ are open for the valuation topology.*

Proof. In the notation of definition I.1.5.1 and remark I.1.5.2, we have $R = \overline{B}(0,1)$ and $\mathfrak{m}_R = B(0,1)$.

□

Lemma I.1.5.7. *Let R be a ring and $|.|$ be a valuation on R. If $x \in R$ is topologically nilpotent for the valuation topology, then $|x| < 1$. If moreover $|.|$ has rank 1, then the converse is true.*

Proof. Suppose that $|x| \geq 1$. Then, for every integer $n \geq 1$, we have $|x^n| \geq 1$, so $x^n \notin B(0,1)$, which implies that 0 is not a limit of the sequence $(x^n)_{n\geq 0}$.

Conversely, suppose that the valuation has rank 1 and that $|x| < 1$. Let Γ be the value group of $|.|$, and let $\gamma = |x|^{-1} > 1$. Let $\delta \in \Gamma$. By proposition I.1.3.6, there exists $n \in \mathbb{N}$ such that $\delta^{-1} < \gamma^m$ for every $m \geq n$, and then we have : for every $m \geq n$, $|x^m| = \gamma^{-m} < \delta$, that is, $x^m \in B(0,\delta)$. This shows that 0 is a limit of the sequence $(x_n)_{n\geq 0}$.

□

Remark I.1.5.8. The converse in lemma I.1.5.7 is not true for a valuation of rank > 1.

For example, take $K = k((u))((t))$ with the rank 2 valuation defined by the valuation ring B of example I.1.4.3. Then $u \in \mathfrak{m}_B$ so it has valuation < 1, but it is not topologically nilpotent. Indeed, by example I.1.5.5, the valuation topology on K coincides with the the t-adic topology, and the sequence $(u^n)_{n\geq 0}$ does not tend to 0 in the t-adic topology.

Lemma I.1.5.9. *Let K be a field and $|.|$ be a valuation on K with valuation ring R. If $\varpi \in K^\times$ is topologically nilpotent for the valuation topology, then $K = R[\varpi^{-1}]$. Moreover, there exists a positive integer r such that $\varpi^r \in R$ (or even $\varpi^n \in \mathfrak{m}_R$), and then the subspace topology on R coincides with the $\varpi^r R$-adic topology.*

Proof. By lemma I.1.5.6, R and $\mathfrak{m}_R$ are open neighborhoods of 0 in K. As the sequence $(\varpi^n)_{n\geq 0}$ converges to 0, we have $\varpi^n \in \mathfrak{m}_R$ for n big enough. So we may assume that $\varpi \in R$.

We first show that $K = R[\varpi^{-1}]$. Let $x \in K$. We want to prove that $x\varpi^n \in R$ for n big enough. This is obvious if $x = 0$, so we may assume $x \neq 0$. By definition of R, we have $x\varpi^n \in R$ if and only if $|x\varpi^n| \leq 1$, i.e. if and only if $|\varpi^n| \leq |x|^{-1}$. As ϖ is topologically nilpotent, we have $\varpi^n \in B(0, |x|^{-1})$ for n big enough, which implies the desired result.

Now we prove that the subspace topology on R is the ϖR-adic topology. First, as $\varpi \in K^\times$, multiplication by ϖ is a homeomorphism of K. So the subsets $\varpi^n R$ are open for every $n \in \mathbb{Z}$. It remains to show that the family $(\varpi^n R)_{n\geq 0}$ is a basis of neighborhoods of 0. Let $\gamma \in \Gamma$, where γ is the value group of $|.|$. We want to find some $n \geq 0$ such that $\varpi^n R \subset B(0, \gamma)$. Let $n \in \mathbb{N}$. Then $\varpi^n R \subset B(0, \gamma)$ if and only if, for every $a \in R$, $|a\varpi^n| < \gamma$. As every $a \in R$ has valuation ≤ 1, this will hold if $|\varpi^n| < \gamma$. But $\varpi^n \to 0$ as $n \to +\infty$, so $|\varpi^n| < \gamma$ for n big enough.

□

Proof of theorem I.1.5.4. Note that point (a) follows from lemma I.1.5.9.

(i)⇒(ii) Let $|.|'$ be a rank 1 valuation on K defining the same topology as $|.|$. Then any $x \in K^\times$ with $|x|' < 1$ is topologically nilpotent by lemma I.1.5.7. (Such an element exists because the value group of $|.|'$ has rank 1, so it is not trivial, so it at least one element < 1, and this element must be the image of a nonzero element of K.)

(ii)⇒(iii) Let $\varpi \in K^\times$ be a topologically nilpotent element. By lemma I.1.5.9, we may assume that $\varpi \in \mathfrak{m}_R$, and the subspace topology on R is the I-adic topology, where $I = \varpi R$. Let $\wp = \sqrt{I}$. By proposition I.1.2.1, $\wp$ is the smallest prime ideal of R containing I. We show that $\wp$ has height 1. Let $\mathfrak{q} \subsetneq \wp$ be another prime ideal of R; we want to show that $\mathfrak{q} = (0)$. Note that $\mathfrak{q}$ does not contain I (otherwise it would contain $\wp$), and so $\varpi \notin \mathfrak{q}$. As the ideals of R are totally ordered by inclusion (see proposition I.1.1.4, we must have $\mathfrak{q} \subset I$. We show by induction on n that $\mathfrak{q} \subset I^n$ for every $n \geq 1$:

- We just did the case $n = 1$.
- Suppose that $\mathfrak{q} \subset I^n = \varpi^n R$. Let $x \in \mathfrak{q}$. As $\mathfrak{q} \subset \varpi R$, we can write $x = \varpi y$, with $y \in R$. As $\mathfrak{q}$ is prime and $\varpi \notin \mathfrak{q}$, this implies that $y \in \mathfrak{q}$, and so $x = \varpi y \in \varpi I^n = I^{n+1}$.

But we have seen that the topology on R is the I-adic topology, and it is Hausdorff because $\operatorname{Ker}|.| = \{0\}$ (remember that $\operatorname{Ker}|.|$ is an ideal of K). So $\bigcap_{n\geq 1} I^n = (0)$, and $\mathfrak{q} = (0)$.

(iii)⇒(i) Let $\wp$ be a height 1 prime ideal of R, and let $R' = R_\wp$. Then R' is also a valuation subring of R. Also, the Krull dimension of R' is the height of $\wp$, i.e. 1, so the rank of the valuation corresponding to R' is 1 by corollary I.1.4.5. So we need to show that the valuation topology corresponding to R' coincides with the valuation topology corresponding to R (this will also prove point (b)). Let $|.|$ (resp. $|.|'$) be the valuation corresponding to R (res. R'), and $\Gamma = K^\times/R^\times$ (resp. $\Gamma' = K^\times/R'^\times$) be its value. We denote the obvious projection $\Gamma \to \Gamma'$ by π; this is order-preserving by theorem I.1.4.2(ii). Note that Γ is not trivial (because R has a prime ideal of height 1, so it cannot be equal to K).

Let $a \in R$ and $\gamma \in \Gamma$. If $|x - a| \leq \gamma$, then $|x - a|' \leq \pi(\gamma)$ (because π is order-preserving). So

$$\overline{B}(a, \gamma) \subset \{x \in R \mid |x - a|' \leq \pi(\gamma)\}.$$

Also, if $|x-a| \geq \gamma$, then $|x-a|' \geq \pi(\gamma)$. So

$$B(a,\gamma) \supset \{x \in R \mid |x-a|' < \pi(\gamma)\}.$$

Thanks to remark I.1.5.2(3), this implies that the two valuation topologies on K are equal.

□

I.1.6 The Riemann-Zariski space of a field

This is a simple example of valuation spectrum, and it will also be useful to understand the general case.

Let K be a field and $A \subset K$ be a subring.

Definition I.1.6.1. We say that a valuation subring $R \subset K$ *has a center in* A if $A \subset R$; in that case, the *center of* R *in* A is the prime ideal $A \cap \mathfrak{m}_R$ of A.

Definition I.1.6.2. The *Riemann-Zariski space* of K over A is the set $RZ(K,A)$ of valuation subrings $R \supset A$ of K. We put the topology on it with the following base of open subsets (sometimes called *Zariski topology*) : the sets

$$U(x_1,\dots,x_n) = RZ(K,A[x_1,\dots,x_n]) = \{R \in RZ(K,A) \mid x_1,\dots,x_n \in R\},$$

for $x_1,\dots,x_n \in K$. (If $|.|$ is the valuation on K corresponding to R, the condition that $R \supset A$ becomes $|a| \leq 1$ for every $a \in A$, and the condition that $R \in RZ(K,A)$ becomes $|x_i| \leq 1$ for $1 \leq i \leq n$.)

If A is the image of $\mathbb{Z}$, we write $RZ(K,\mathbb{Z}) = RZ(K)$ and we call it the *Riemann-Zariski space of* K; this is the set of all valuation subrings of K.

Note that $U(x_1,\dots,x_n) \cap U(y_1,\dots,y_m) = U(x_1,\dots,x_n,y_1,\dots,y_m)$, so this does define a topology on $RZ(K,A)$.

Example I.1.6.3. Let k be a field, X/k be a smooth projective geometrically connected curve and K be the function field of X. Then $RZ(K,k)$ is canonically isomorphic to X as a topological space.

More generally, if K/k is a finitely generated field extension, then $RZ(K,k)$ is isomorphic (as a topological space) to the inverse limit of all the projective integral k-schemes with function field K.

See also example I.2.1.6 for a description of $RZ(\mathbb{Q})$.

Remark I.1.6.4. Let $R, R' \in RZ(K,A)$. Then R is a specialization of R' in $RZ(K,A)$ (i.e. R is in the closure of $\{R'\}$) if and only if $R \subset R'$.

Indeed, R is a specialization of R' if and only every open set of $RZ(K,A)$ that contains R also contains R'. This means that, for all $a_1,\dots,a_n \in K$ such that $a_1,\dots,a_n \in R$, we must also have $a_1,\dots,a_n \in R'$, so it is equivalent to the fact that $R \subset R'$.

Here is a particular case of the topological results to come.

Proposition I.1.6.5. *([21] theorem 10.5) $RZ(K, A)$ is quasi-compact.*

Proof. We write $X = RZ(K, A)$. Let $\mathscr{A}$ be a family of closed subsets of X having the finite intersection property. We must show that $\bigcap_{F \in \mathscr{A}} F \neq \varnothing$. Using Zorn's lemma, we can replace with a family of closed subsets of X having the finite intersection property and maximal among all families of closed subsets of X having the finite intersection property. Then the following hold (otherwise, we could enlarge $\mathscr{A}$ without losing the finite intersection property) :

(a) if $F_1, \dots, F_r \in \mathscr{A}$, then $F_1 \cap \dots \cap F_r \in \mathscr{A}$;

(b) if $Z_1, \dots, Z_r$ are closed subsets of X and $Z_1 \cup \dots \cup Z_r \in \mathscr{A}$, then at least one of the Z_i is in $\mathscr{A}$;

(c) if $F \in \mathscr{A}$ and $Z \supset F$ is a closed subset of X, then $Z \in \mathscr{A}$.

We claim that

$$\bigcap_{F \in \mathscr{A}} F = \bigcap_{a \in K | X - U(a) \in \mathscr{A}} (X - U(a)).$$

Indeed, it is obvious that the right hand side is contained in the left hand side. Conversely, let $x \in X$, and suppose that $x \notin \bigcap_{F \in \mathscr{A}} F$. Then there exists $F \in \mathscr{A}$ such that $x \notin F$. As $X - F$ is open, there exists $a_1, \dots, a_n \in K$ such that $x \in U(a_1, \dots, a_n)$ and $U(a_1, \dots, a_n) \cap F = \varnothing$. Then $F \subset X - U(a_1, \dots, a_n) = \bigcup_{i=1}^n (X - U(a_i))$, so, by (b) and (c), there exists $i \in \{1, \dots, n\}$ such that $X - U(a_i) \in \mathscr{A}$. As $x \in U(a_1, \dots, a_n) \subset U(a_i)$, we have $x \in X - U(a_i)$. This finishes the proof of the claim.

Now let $C = \{a \in K^\times \mid X - U(a^{-1}) \in \mathscr{A}\}$. If a point $x \in X$ corresponds to a valuation subring $A \subset R \subset K$, then, for every $a \in K^\times$:

$$x \in X - U(a^{-1}) \Leftrightarrow a^{-1} \notin R \Leftrightarrow a \in \mathfrak{m}_R.$$

So :

$$\bigcap_{F \in \mathscr{A}} F = \bigcap_{a \in C} (X - U(a^{-1})) = \{R \in RZ(K, A) \mid \mathfrak{m}_R \supset C\}.$$

Let I be the ideal of $A[C]$ generated by C. If $1 \in I$, then there exist $a_1, \dots, a_n \in C$ such that $1 \in \sum_{i=1}^n a_i A[C]$, and then $\bigcap_{i=1}^n (X - U(a_i^{-1})) = \varnothing$, which contradicts the fact that $\mathscr{A}$ has the finite intersection property. So $1 \notin I$, hence I is contained in a prime ideal of A, so, by theorem I.1.2.2(i), there exists a valuation subring $A \subset R \subset K$ such that $I \subset \mathfrak{m}_R$, and then $R \in \bigcap_{F \in \mathscr{A}} F$.

□

I.2 The valuation spectrum of a ring

I.2.1 Definition

Definition I.2.1.1. Let A be a commutative ring. The *valuation spectrum* $\mathrm{Spv}(A)$ of A is the set of equivalence classes of valuations on A, equipped for the topology generated by the subsets

$$U\left(\frac{f_1,\dots,f_n}{g}\right) = \{|.| \in \mathrm{Spv}(A) \mid |f_1|,\dots,|f_n| \le |g| \ne 0\},$$

for all $f_1,\dots,f_n,g \in A$. (Note that this subset is empty if $g = 0$.)

Note that

$$U\left(\frac{f_1,\dots,f_n}{g}\right) \cap U\left(\frac{f'_1,\dots,f'_m}{g'}\right) = U\left(\frac{f_1g',\dots,f_ng',f'_1g,\dots,f'_mg}{gg'}\right).$$

In particular, we can use the subsets $U\left(\frac{f}{g}\right)$, $f,g \in A$, to generate the topology of $\mathrm{Spv}(A)$.

Remark I.2.1.2. Let X be the set of pairs $(\wp, R)$, where $\wp$ is a prime ideal of A and R is a valuation subring of $\mathrm{Frac}(A/\wp)$. Then $\mathrm{Spv}(A)$ and X are naturally in bijection. Indeed, if $(\wp, R) \in X$, then we get a valuation on A by composing the quotient map $A \to A/\wp$ with the valuation on $\mathrm{Frac}(A/\wp)$ defined by R as in proposition I.1.1.14(ii). Conversely, if $|.|$ is a valuation on A, then $\wp := \mathrm{Ker}\,|.|$ is a prime ideal of A, and $|.|$ defines a valuation on the domain $A/\wp$, hence also on its fraction field; we take for R the valuation subring of that valuation, as in proposition I.1.1.14(i). It follows easily from proposition I.1.1.14(iii) that these maps are inverse bijections.

For $f_1,\dots,f_n,g \in A$, the image of the open subset $U\left(\frac{f_1,\dots,f_n}{g}\right)$ by this bijection is

$$\{(\wp, R) \in X \mid g \notin \wp \text{ and } \forall i \in \{1,\dots,n\},\ (f_i+\wp)(g+\wp)^{-1} \in R\},$$

where $(f_i+\wp)(g+\wp)^{-1}$ is an element of $\mathrm{Frac}(A/\wp)$ (this makes sense because the image of g in $A/\wp$ is nonzero by the first condition).

In particular, we get a canonical map $\mathrm{supp} : \mathrm{Spv}(A) \to \mathrm{Spec}(A)$ that sends a valuation to its kernel or support.

Notation I.2.1.3. If $x \in \mathrm{Spv}(A)$ corresponds to the pair $(\wp, R)$, we write $\wp_x = \wp$, $R_x = R$, $\Gamma_x = \Gamma_R$, $K(x) = \mathrm{Frac}(A/\wp_x)$, and we denote by $|.|_x : A \to \Gamma_x \cup \{0\}$ the composition of $A \to A/\wp_x$ and of the valuation corresponding to R_x on $K(x)$. For $f \in A$, we often write $f(x)$ for the image of f in $A/\wp_x$, and $|f(x)|$ for the image of $f(x)$ in Γ_x.

Proposition I.2.1.4. *Let A be a commutative ring.*

(i) If A is a field, then $\mathrm{Spv}(A) = RZ(A)$ *(as topological spaces).*

(ii) In general, the map $\mathrm{supp} : \mathrm{Spv}(A) \to \mathrm{Spec}(A)$ *is continuous and surjective. For every* $\wp \in \mathrm{Spec}(A)$*, the fiber of this map over* $\wp$ *is isomorphic (as a topological space) to* $RZ(\mathrm{Frac}(A/\wp))$.

Proof. (i) This is a particular case of (ii).

(ii) For every $\wp \in \mathrm{Spec}(A)$, we have $(\wp, \mathrm{Frac}(A/\wp)) \in \mathrm{supp}^{-1}(\wp)$, so supp is surjective. Next, for every $f \in A$, we have

$$\mathrm{supp}^{-1}(D(f)) = \{(\wp, R) \in \mathrm{Spv}(A) \mid f \notin \wp\} = U\left(\frac{f}{f}\right)$$

(where $D(f) = \{\wp \in \mathrm{Spec}(A) \mid f \notin \wp\}$), so supp is continuous. Finally, fix $\wp \in \mathrm{Spec}(A)$. Then $\mathrm{supp}^{-1}(\wp) = \{(\wp, R) \mid R \in \mathrm{Frac}(A/\wp)\}$ is canonically in bijection with $RZ(\mathrm{Frac}(A/\wp))$. Moreover, if $f_1, \ldots, f_n, g \in A$, then the intersection $U\left(\frac{f_1,\ldots,f_n}{g}\right) \cap \mathrm{supp}^{-1}(\wp)$ is empty if $g \in \wp$, and, if $g \notin \wp$, it is equal to

$$\{(\wp, R) \in \mathrm{supp}^{-1}(\wp) \mid (f_1 + \wp)(g + \wp)^{-1}, \ldots, (f_n + \wp)(g + \wp)^{-1} \in R\},$$

which corresponds by the bijection $\mathrm{supp}^{-1}(\wp) \simeq RZ(\mathrm{Frac}(A/\wp))$ to the open subset $U((f_1 + \wp)(g + \wp)^{-1}, \ldots, (f_n + \wp)(g + \wp)^{-1})$. So the bijection $\mathrm{supp}^{-1}(\wp) \simeq RZ(\mathrm{Frac}(A/\wp))$ is a homeomorphism.

□

Remark I.2.1.5. The map $\mathrm{supp} : \mathrm{Spv}(A) \to \mathrm{Spec}(A)$ has an obvious section, sending a prime ideal $\wp$ of A to the pair $(\wp, \mathrm{Frac}(A/\wp))$ (in terms of valuations, this is the composition of the quotient map $A \to A/\wp$ and of the trivial valuation on $A/\wp$). This section is also a continuous map, because the inverse image of the open set $U\left(\frac{f_1,\ldots,f_n}{g}\right)$ is $\{\wp \in \mathrm{Spec}(A) \mid g \notin \wp\}$, which is open in $\mathrm{Spec}(A)$.

Example I.2.1.6. (1) We have (see example I.1.1.12(4)

$$\mathrm{Spv}(\mathbb{Q}) = RZ(\mathbb{Q}) = \{|.|_{\mathrm{triv}},\ |.|_\ell,\ \ell \in \mathbb{Z} \text{ prime}\}.$$

By remark I.1.6.4, the trivial valuation $|.|_{\mathrm{triv}}$ is the generic point of $\mathrm{Spv}(\mathbb{Q})$, and each $|.|_\ell$ is a closed point. In fact, for all $f_1, \ldots, f_n, g \in \mathbb{Z}$ such that $g \neq 0$, we have

$$U\left(\frac{f_1, \ldots, f_n}{g}\right) = \{|.|_{\mathrm{triv}}\} \cup \{|.|_\ell,\ \text{for } \ell \text{ not dividing } g\}.$$

So $\mathrm{Spv}(\mathbb{Q})$ is isomorphic to $\mathrm{Spec}(\mathbb{Z})$ as a topological space.

(2) We have

$$\mathrm{Spv}(\mathbb{Z}) = \mathrm{Spv}(\mathbb{Q}) \cup \{|.|_{\ell,\mathrm{triv}},\ \ell \in \mathbb{Z}\ \mathrm{prime}\},$$

where $|.|_{\ell,\mathrm{triv}}$ is the composition of $\mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z}$ and of the trivial valuation on $\mathbb{Z}/\ell\mathbb{Z}$. Note that $\mathrm{Spv}(\mathbb{Q}) = \mathrm{supp}^{-1}((0))$ and $\{|.|_{\ell,\mathrm{triv}}\} = \mathrm{supp}^{-1}(\ell\mathbb{Z})$. The points $|.|_{\ell,\mathrm{triv}}$ are all closed, we have $\overline{\{|.|_\ell\}} = \{|.|_\ell, |.|_{|ell,\mathrm{triv}}\}$ (so $|.|_\ell$ specializes to $|.|_\ell$), and the point $|.|_{\mathrm{triv}}$ is generic.

Indeed, note that, for x, y in a topological space X, y specializes to x if and only if every open subset of X that contains x also contains y. Let $f_1, \ldots, f_n, g \in \mathbb{Z}$, and let $U = U\left(\frac{f_1,\ldots,f_n}{g}\right)$. As $U = \varnothing$ if $g = 0$, we assume that $g \neq 0$. Then $|.|_{\mathrm{triv}}$ is always in U, $|.|_{\ell,\mathrm{triv}}$ is in U if and only if $g \notin \ell\mathbb{Z}$ and $|.|_\ell$ is in U if and only if $f_1 g^{-1}, \ldots, f_n g^{-1} \in \mathbb{Z}_{(\ell)}$ (which is automatically true if $g \notin \ell\mathbb{Z}$).

Definition I.2.1.7. Let $\varphi : A \to B$ be a morphism of rings. We denote by $\mathrm{Spv}(\varphi)$ the map $\mathrm{Spv}(B) \to \mathrm{Spv}(A)$, $|.| \longmapsto |.| \circ \varphi$.

Note that this is a continuous map, because, for all $f_1, \ldots, f_n, g \in A$, we have

$$\mathrm{Spv}(\varphi)^{-1}\left(U\left(\frac{f_1, \ldots, f_n}{g}\right)\right) = U\left(\frac{\varphi(f_1), \ldots, \varphi(f_n)}{\varphi(g)}\right).$$

(This follows immediately from the definitions.)

Remark I.2.1.8. We get a commutative square

$$\begin{tikzcd} \mathrm{Spv}(B) \arrow[r] \arrow[d] & \mathrm{Spv}(A) \arrow[d] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

This square is cartesian if B is a localization or a quotient of A, but not in general.

The goal of this section is to prove that $\mathrm{Spv}(A)$ is a spectral space (see definition I.2.2.7) and that the continuous maps $\mathrm{Spv}(\varphi)$ are spectral (i.e. quasi-compact).

We will outline the strategy of the proof. First "remember" that a topological space X is called spectral if it satisfies the following conditions :

(i) X is quasi-compact.

(ii) X has a collection of quasi-compact open subsets that is stable by finite intersections and generates its topology.

(iii) X is quasi-separated (see definition I.2.2.1(iv)).

(iv) X is T_0 (see definition I.2.2.4(v)).

(v) X is sober, that is, every closed irreducible subset of X has a unique generic point.

We spelled out all these conditions because they are all important, but they are not independent; indeed, (iii) follows from (ii) (see remark I.2.2.3(1)) and (iv) follows from (v) (see remark I.2.2.5).

The main example of a spectral space (and the reason for the name) is $\mathrm{Spec}(A)$ with its Zariski topology, for A a ring.[2] We quickly indicate how to check conditions (i), (ii) and (v) in this case :

(i) Remember that the closed subsets of $\mathrm{Spec}(A)$ are the $V(I) = \{\wp \in \mathrm{Spec}(A) \mid I \subset \wp\}$, for I an ideal of A (and that we have a canonical isomorphism $V(I) = \mathrm{Spec}(A/I)$). If $(I_j)_{j\in J}$ is a family of ideals of A, then

$$\bigcap_{j\in J} V(I_j) = \varnothing \Leftrightarrow 1 \in \sqrt{\sum_{j\in J} I_j} \Leftrightarrow 1 \in \sum_{j\in J} I_j,$$

and this happens if and only if there is a finite subset J' of J such that $1 \in \sum_{j\in J'} I_j$. So $\mathrm{Spec}(A)$ is quasi-compact.

(ii) We take as our generating family of open subsets the subsets $D(f_1, \dots, f_n) := \mathrm{Spec}(A) - V((f_1, \dots, f_n))$, for $f_1, \dots, f_n \in A$. The fact that $D(f_1, \dots, f_n)$ is quasi-compact is proved in example I.2.2.2. (If $n = 1$, it follows from the obvious isomorphism $D(f) = \mathrm{Spec}(A[f^{-1}])$.)

(v) It is easy to see that a closed subset $V(I)$ of $\mathrm{Spec}(A)$ is irreducible if and only if $\wp := \sqrt{I}$ is a prime ideal of A. Then $V(I) = V(\wp) = \mathrm{Spec}(A/\wp)$ has a unique generic point, given by the prime ideal (0) of $A/\wp$.

In the case of $\mathrm{Spv}(A)$, the quasi-compactness is still not too hard to check directly (see proposition I.1.6.5 for a particular case), but we cannot apply the same strategy afterwards : we want to use the subsets $U\left(\frac{f_1,\dots,f_n}{g}\right)$ as our generating family of quasi-compact open subsets, but they are not isomorphic in general to valuation spectra even when $n = 1$, and neither are the closed subsets of $\mathrm{Spv}(A)$. Instead, we take a more indirect route that uses the constructible topology of $\mathrm{Spv}(A)$.

If X is a quasi-compact and quasi-separated space, the collection of constructible subsets of X is the smallest collection of subsets of X that is stable by finite unions, finite intersections and complements and that contains all quasi-compact open subsets of X. (In general, we need to replace "quasi-compact" with a different condition called "retrocompact", see definitions I.2.2.1(iii) and I.2.3.1). Constructible subsets form the base of a new topology on X, called the constructible topology, in which quasi-compact open subsets of X are open and closed. The constructible topology is Hausdorff and quasi-compact if X is spectral (proposition I.2.4.1). The main technical tool is a partial converse of this result, due to Hochster (theorem I.2.5.1) : If X' is a quasi-compact topological space and if $\mathscr{U}$ is a collection of open and closed subsets of X',

[2] Indeed, Hochster has proved that every spectral space is homeomorphic to the spectrum of a ring, see theorem 6 in section 7 of [13].

it gives a criterion for the topology generated by $\mathscr{U}$ to be spectral (and then the topology of X' will be the corresponding constructible topology).

To apply Hochster's spectrality criterion to $\mathrm{Spv}(A)$, we still need to be able to check its hypotheses. The key point is the following (see the proof of theorem I.2.6.1 for details): Each equivalence class of valuations on A defines a binary relation on A ("divisibility with respect to the corresponding valuation ring"). This gives an injective map from $\mathrm{Spv}(A)$ to the set $\{0,1\}^{A\times A}$ of binary relations on A, and if we put the product topology of $\{0,1\}^{A\times A}$ (which is Hausdorff and quasi-compact by Tychonoff's theorem), then this topology induces the constructible topology on $\mathrm{Spv}(A)$ and every set $U\left(\frac{f_1,\dots,f_n}{g}\right)$ is open and closed.

I.2.2 Some topological notions

First we fix some vocabulary : Let X be a topological space and $\mathscr{U}$ be a collection of open subsets of X. We say that $\mathscr{U}$ is a *base* of the topology of X if every open subset of X is a union of elements of $\mathscr{U}$; we say that $\mathscr{U}$ *generates* the topology of X or that $\mathscr{U}$ is a *subbase* of the topology of X if the collection of finite intersections of elements of $\mathscr{U}$ is a base of the topology (or, in other words, if the topology of X is the coarsest topology for which all the elements of $\mathscr{U}$ are open subsets).

Definition I.2.2.1. (i) We say that a topological space X is *quasi-compact* if every open covering of X has a finite refinement.

(ii) We say that a continuous map $f : X \to Y$ is *quasi-compact* if the inverse image of any quasi-compact open subset of Y is quasi-compact.

(iii) We say that X is *quasi-separated* if the diagonal embedding $X \to X\times X$ is quasi-compact. This means that, for any quasi-compact open subspaces U and V of X, the intersection $U\cap V$ is still quasi-compact.

Example I.2.2.2. Let A be a ring. Then an open subset U of $\mathrm{Spec}(A)$ is quasi-compact if and only if there exists a finitely generated ideal I of A such that $U = \mathrm{Spec}(A) - V(I)$.

Indeed, suppose that $U = \mathrm{Spec}(A) - V(I)$, with I finitely generated, say $I = (f_1,\dots,f_r)$. Let $(I_j)_{j\in J}$ be a family of ideals of A such that $U \cap \bigcap_{j\in J} V(I_j) = \varnothing$, that is, $\bigcap_{j\in J} V(I_j) \subset V(I)$. Then $I \subset \sqrt{\sum_{j\in J} I_j}$, so there exists $N\geq 1$ such that $f_1^N,\dots,f_r^N \in \sum_{j\in J} I_j$. Choose a finite subset J' of J such that $f_1^N,\dots,f_r^N \in \sum_{j\in J'} I_j$. Then $V(I) = V(f_1^N,\dots,f_r^N) \supset \bigcap_{j\in J'} V(I_j)$, so $U\cap \bigcap_{j\in J'} V(I_j) = \varnothing$.

Conversely, suppose that U is quasi-compact, and let I be an ideal of A such that $U = \mathrm{Spec}(A) - V(I)$. We have $U = \bigcup_{f\in I} D(f)$, so, by the quasi-compactness assumption, there exist $f_1,\dots,f_r \in I$ such that $U = \bigcup_{i=1}^r D(f_i) = \mathrm{Spec}(A) - V(f_1,\dots,f_r)$.

Remark I.2.2.3. To check that a continuous map $f : X \to Y$ is quasi-compact, it suffices to check that $f^{-1}(U)$ is quasi-compact for U in a base of quasi-compact open subsets of Y. In particular, if the topology of X has a basis of quasi-compact open subsets which is stable by finite intersections, then X is quasi-separated.

Definition I.2.2.4. Let X be a topological space.

(i) We say that X is *irreducible* if, whenever $X = Y \cup Z$ with Y and Z closed subsets of X, we have $X = Y$ or $X = Z$.

(ii) If $x, y \in X$, we say that x is a *specialization* of y (or that y is a *generization*[3] of x, or that y *specializes* to x) if $x \in \overline{\{y\}}$.

(iii) We say that a point $x \in X$ is *closed* if $\{x\}$ is closed.

(iv) We say that a point $x \in X$ is *generic* if $\{x\}$ is dense in X.

(v) We say that X is a *Kolmogorov space* (or a T_0 *space* if, for all $x \neq y$ in X, there exists an open subset U of X such that $x \in U$ and $y \notin U$, or that $x \notin U$ and $y \in U$.

(vi) We say that X is *sober* if every irreducible closed subset of X has a unique generic point.

Remark I.2.2.5. A topological space X is T_0 if and only every irreducible closed subset of X has at most one generic point.

Indeed, note that X is T_0 if and only, for all $x, y \in X$ such that $x \neq y$, there exists a closed subset of X that contains exactly one of x and y, which means that $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. If X is T_0, let Z be an irreducible closed subset of X and let $x, y \in Z$ be two generic points. Then $x \in \overline{\{y\}} = Z$ and $y \in \overline{\{x\}} = Z$, so $x = y$. Conversely, suppose that every irreducible closed subset of X has at most one generic point, and let $x, y \in X$ such that $x \neq y$. As $\overline{\{x\}}$ and $\overline{\{y\}}$ are irreducible closed and have distinct generic points, we have $\overline{\{x\}} \neq \overline{\{y\}}$. So we have $\overline{\{x\}} \not\subset \overline{\{y\}}$, and then $x \notin \overline{\{y\}}$, or we have $\overline{\{y\}} \not\subset \overline{\{x\}}$, and then $y \notin \overline{\{x\}}$.

Example I.2.2.6. For every ring, $\mathrm{Spv}(A)$ is T_0.

Indeed, let $x, y \in \mathrm{Spv}(A)$ such that $x \neq y$. If A is a field, then $\mathrm{Spv}(A) = RZ(A)$ and x, y correspond to distinct valuation subrings $R_x, R_y \subset K$. We may assume without loss of generality that there exists $a \in R_x - R_y$, and then the open subset $U(a)$ of $RZ(A)$ contains x but not y.

In the general case, we have either $\mathrm{supp}(x) = \mathrm{supp}(y)$ or $\mathrm{supp}(x) \neq \mathrm{supp}(y)$. If $\mathrm{supp}(x) = \mathrm{supp}(y)$, then x and y are both in the same fiber of supp, which is a Riemann-Zariski space, and so, by the first case treated above and proposition I.2.1.4(ii), we can find an open subset of $\mathrm{Spv}(A)$ that contains exactly one of x and y. If $\mathrm{supp}(x) \neq \mathrm{supp}(y)$, as $\mathrm{Spec}(A)$ is T_0, we may assume without loss of generality that there exists an open subset U of $\mathrm{Spec}(A)$ such that $\mathrm{supp}(x) \in U$ and $\mathrm{supp}(y) \notin U$. Then $\mathrm{supp}^{-1}(U)$ is an open subset of $\mathrm{Spv}(A)$ that contains x and not y.

[3] EGA says “générisation”, so I translated it as “generization”, but some English-language references (for example the stacks project) use “generalization”.

Definition I.2.2.7. A topological space X is called *spectral* if it satisfies the following conditions :

(a) X is quasi-compact and quasi-seperated;

(b) the topology of X has a base of quasi-compact open subsets;

(c) X is sober.

We say that X is a *locally spectral space* if it has an open covering by spectral spaces.

Example I.2.2.8. Any affine scheme is a spectral space, and any scheme is a locally spectral space. (In fact, Hochster has proved that these are the only examples, see theorems 6 and 9 of [13].)

Proposition I.2.2.9. *([26] remark 3.13)*

(i) A sober space is T_0.

(ii) A locally spectral space is sober.

(iii) A locally spectral space is spectral if and only if it is quasi-compact and quasi-separated.

(iv) Let X be a sober space. Then every locally closed subspace of X is sober.

(v) Let X be a spectral space. Then every quasi-compact open subspace (resp. every closed subspace) of X is spectral.

(vi) Let X be a locally spectral space. Then every open subspace of X is locally spectral, and the topology of X has a base consisting of spectral open subspaces.

Proof. (i) follows from remark I.2.2.5. (ii) follows from the fact that a space that has a covering by sober open subspaces is sober (see (3) of [25, Lemma 06N9]). (iii) is obvious from the definitions. (iv) is (3) of [25, Lemma 0B31]. (v) is a particular case of [25, Lemma 0902]. (vi) is clear.

□

I.2.3 The constructible topology

We will only define the constructible topology on a quasi-compact and quasi-separated topological space, because that is the only case when we will need it (at least for now). For the general definition, see for example [25, Section 04ZC].

Definition I.2.3.1. Let X be a quasi-compact and quasi-separated topological space and Y be a subset of X.

(i) We say that Y is *constructible* if it is a finite union of subsets of the form $U \cap (X - V)$, with U and V quasi-compact open subsets of X.

(ii) We say that Y is *ind-conctructible* (resp. *pro-constructible*) if it a union (resp. an intersection) of consrtuctible subsets of X.

Any finite union of quasi-compact open subsets is quasi-compact, and, if X is quasi-separated, so is any finite intersection of quasi-compact open subsets. This immediately implies the following lemma.

Lemma I.2.3.2. *Let X be a quasi-compact and quasi-separated topological space. The collection of constructible subsets of X is closed under finite unions, finite intersections and taking complements.*

So we could also have defined the collection of constructible subsets of X as the smallest collection of subsets of X that is stable by finite unions, finite intersections and complements, and contains the quasi-compact open subsets of X.

Definition I.2.3.3. Let X be a quasi-compact and quasi-separated topological space. The *constructible topology* on X is the topology with base the collection of the constructible subsets of X. Equivalently, it is the topology generated by the quasi-compact open subsets of X and their complements.

We denote by X_{cons} the space X equipped with its constructible topology.

It is clear from the definitions that the open (resp. closed) subsets for the constructible topology are the ind-constructible (resp. pro-constructible) subsets of X.

Example I.2.3.4. Let X be a finite T_0 space. Then X is spectral and every subset of X is constructible.

Proof. Every subset of X is quasi-compact (because it is finite), so X is quasi-compact and quasi-separated. As X is T_0, we just need to prove that every closed irreducible subset of X has at least one generic point. Let Z be a closed irreducible subset of X. Then Z is the union of the finite family of closed subsets $(\overline{\{z\}})_{z\in Z}$, so there exists $z \in Z$ such that $Z = \overline{\{z\}}$. Finally, we proved that every subset of X is constructible. It suffices to prove that $\{x\}$ is constructible for every $x \in X$. So let $x \in X$. As X if T_0, for every $y \in X - \{x\}$, we can choose Y_y open or closed (and in particular constructible) such that $x \in Y_y$ and $y \notin Y_y$. So $\{x\} = \bigcap_{y\in X-\{x\}} Y_y$ is constructible.

□

Proposition I.2.3.5. *Let X be a quasi-compact and quasi-separated topological space. Then any constructible subset of X is quasi-compact.*

In particular, if Y is an open (resp. closed) subset of X, then Y is constructible if and only if it is quasi-compact (resp. if and only if $X - Y$ is quasi-compact).

Proof. Let Y be a constructible subsets of X. We write $Y = \bigcup_{i=1}^n (U_i - V_i)$, with $U_1, \ldots, U_n, V_1, \ldots, V_n$ quasi-compact open subsets of X. For every $i \in \{1, \ldots, n\}$, $U_i - V_i$ is a closed subset of the quasi-compact space U_i, so it is also quasi-compact. Hence Y is quasi-compact.

□

I.2.4 The constructible topology on a spectral space

Proposition I.2.4.1. *Let X be a spectral space. Then X_{cons} is Hausdorff, totally disconnected and quasi-compact.*

Proof. (See [25, Lemma 0901].) Let $x, y \in X$ such that $x \neq y$. As X is T_0, we may assume that there exists an open subset U of X such that $x \in U$ and $y \notin U$. As quasi-compact open subsets form a base of the topology of X, we may assume that U is quasi-compact. Then U and $X - U$ are constructible, hence open and closed in the constructible topology, and we have $x \in U$ and $y \in X - U$. So X_{cons} is Hausdorff and totally disconnected.

We show that X is quasi-compact. Let $\mathscr{B}$ be the collection of quasi-compact open subsets of X and of their complements. Then $\mathscr{B}$ generates the topology of X, so, by the Alexander subbase theorem ([25, Lemma 08ZP]), it suffices to show that every covering of X by elements of $\mathscr{B}$ has a finite refinement. As the collection $\mathscr{B}$ is stable by taking complements, it also suffices to show that every subcollection of $\mathscr{B}$ that has the finite intersection property has nontrivial intersection.

So let $\mathscr{B}' \subset \mathscr{B}$, and suppose that $\mathscr{B}'$ has the finite intersection property and that $\bigcap_{B \in \mathscr{B}'} B = \varnothing$. Using Zorn's lemma, we may assume that $\mathscr{B}'$ is maximal among all subcollections that the finite intersection property and empty intersection. Let $\mathscr{B}'' \subset \mathscr{B}'$ be the set of $B \in \mathscr{B}'$ that are closed, and let $Z = \bigcap_{B \in \mathscr{B}''} B$; this is a closed subset of X, and it is not empty because X is quasi-compact.

Suppose first that Z is reducible. Then we can write $Z = Z' \cup Z''$, with $Z', Z'' \neq Z$. In particular, we have $Z' \not\subset Z''$ and $Z'' \not\subset Z'$, so we can find quasi-compact open subsets U' and U'' of X such that $U' \subset X - Z''$, $U'' \subset X - Z'$, $U' \cap Z' \neq \varnothing$ and $U'' \cap Z'' \neq \varnothing$. Let $B' = X - U' \supset Z''$ and $B'' = X - U'' \supset Z'$. We want to show that $\mathscr{B}' \cup \{B'\}$ or $\mathscr{B}' \cup \{B''\}$ has the finite intersection property, which will contradict the maximality of $\mathscr{B}'$. Suppose that this does not hold, then there exist there exist $B'_1, \ldots, B'_n, B''_1, \ldots, B''_m \in \mathscr{B}'$ such that $B' \cap B'_1 \cap \ldots \cap B'_n = B'' \cap B''_1 \cap \ldots \cap B''_m = \varnothing$, then $Z \cap B'_1 \cap \ldots \cap B'_n \cap B''_1 \cap \ldots \cap B''_m = \varnothing$. As $B'_1 \cap \ldots \cap B'_n \cap B''_1 \cap \ldots \cap B''_m$ is quasi-compact and $Z = \bigcap_{B \in \mathscr{B}''} B$ with every $B \in \mathscr{B}''$ closed, we can find $B_1, \ldots, B_r \in \mathscr{B}''$ such that $B_1 \cap \ldots \cap B_r \cap B'_1 \cap \ldots \cap B'_n \cap B''_1 \cap \ldots \cap B''_m = \varnothing$; but this contradicts the fact that $\mathscr{B}'$ has the finite intersection property. So Z cannot be reducible.

Suppose now that Z is irreducible. As X is sober, Z has a unique generic point, say η. Let $U \in \mathscr{B}' - \mathscr{B}''$. Then U is quasi-compact and the family of closed subsets $(B \cap U)_{B \in \mathscr{B}''}$ has the finite intersection property, so their intersection $Z \cap U$ is nonempty; as $Z \cap U$ is an open subset of Z, it contains its generic point η. But then $\eta \in \bigcap_{B\in\mathscr{B}'} B$, which contradicts the fact that $\bigcap_{B\in\mathscr{B}'} B = \varnothing$.

□

Corollary I.2.4.2. *(See proposition 3.23 of [26].) Let X be a spectral space. Then :*

(i) The constructible topology is finer than the original topology on X.

(ii) A subset of X is constructible if and only if it is open and closed in the constructible topology (i.e. if and only if it is both ind-constructible and pro-constructible).

(iii) If U is an open subspace of X, then the map $U_{\mathrm{cons}} \to X_{\mathrm{cons}}$ is open and continuous (i.e. the subspace topology on U induced by the constructible topology on X is the constructible topology on U).

Proof. (i) Every open subset of X is the union of its quasi-compact open subsets, hence is ind-constructible.

(ii) We already know that constructible subsets of X are open and closed in X_{cons}. Conversely, let $Y \subset X$ be open and closed in X_{cons}. Then Y is ind-constructible, so we can write $Y = \bigcup_{i\in I} Y_i$, with the Y_i constructible. Also, by I.2.4.1, Y is quasi-compact for the constructible topology, so there exists a finite subset I' of I such that $Y = \bigcup_{i\in I'} Y_i$, which proves that Y is constructible.

(iii) Let U be an open subset of X. By [25, Lemma 005J], the map $U_{\mathrm{cons}} \to X_{\mathrm{cons}}$ is continuous.

Conversely, let E be a constructible subset of U; we want to show that E is ind-constructible in X. We can write $U = \bigcup_{i\in I} U_i$, with the U_i quasi-compact. For every $i \in I$, the set $E \cap U_i$ is constructible in U_i, hence constructible in X by [25, Lemma 09YD]. So $E = \bigcup_{i\in I}(E \cap U_i)$ is ind-constructible in X.

□

Corollary I.2.4.3. *([25, Lemma 0902].) Let X be a spectral space and $Y \subset X$ be closed in the constructible topology. Then Y is a spectral space for the subspace topology.*

Definition I.2.4.4. A continuous map $f : X \to Y$ of locally spectral spaces is called *spectral* if, for every open spectral subspace V of Y and every open spectral subspace U of $f^{-1}(V)$, the induced map $U \to V$ is quasi-compact.

Proposition I.2.4.5. *(Proposition 3.27 of [26].) Let $f : X \to Y$ be a continuous map of spectral spaces. Then the following are equivalent :*

(i) f is spectral.

(ii) $f : X_{\mathrm{cons}} \to Y_{\mathrm{cons}}$ is continuous.

(iii) f is quasi-compact.

(iv) The inverse image by f of every constructible subset of Y is a constructible subset of X.

If these conditions are satisfied, then $f : X_{\mathrm{cons}} \to Y_{\mathrm{cons}}$ is proper.

Proof. The implications (i)⇒(iii)⇒(iv)⇒(v)⇔(ii) are clear. By proposition I.2.3.5 and corollary I.2.4.2(ii), an open subset of X is quasi-compact if and only if it is open and closed in the constructible topology, so (ii) implies (iii). It remains to show that (iii) implies (i). Suppose that f is quasi-compact. Let $V \subset Y$ and $U \subset f^{-1}(V)$ be open spectral subspaces, and let W be a quasi-compact open subset of V. We want to show that $f^{-1}(W) \cap U$ is quasi-compact, but this follows from the fact that $f^{-1}(W)$ and U are quasi-compact and that X is quasi-separated.

Finally, as X_{cons} and Y_{cons} are Hausdorff and quasi-compact by proposition I.2.4.1, if $f : X_{\mathrm{cons}} \to Y_{\mathrm{cons}}$ is continuous, then it is proper.

□

I.2.5 A criterion for spectrality

The goal of this section is to give a criterion for a topological space to be spectral, due to Hochster.

Theorem I.2.5.1. *(Proposition 3.31 of [26].) Let $X' = (X_0, \mathscr{T}')$ be a quasi-compact topological space, let $\mathscr{U} \subset \mathscr{T}'$ be a collection of open and closed subsets of X', let $\mathscr{T}$ be the topology on X_0 generated by $\mathscr{U}$, and set $X = (X_0, \mathscr{T})$.*

If X is T_0, then it is spectral, every element of $\mathscr{U}$ is a quasi-compact open subset of X, and $X' = X_{\mathrm{cons}}$.

Proof. After replacing $\mathscr{U}$ by the collection of finite intersections of sets of $\mathscr{U}$, we may assume that $\mathscr{U}$ is stable by finite intersections. Note that the topology of X is coarser than the topology of X'. In particular, every quasi-compact subset of X' is also quasi-compact as a subset of X; in particular, X itself is quasi-compact. As elements of $\mathscr{U}$ are closed in X', the previous sentence also applies to them, and we see that they are all quasi-compact as subsets of X. So the topology of X has a basis of quasi-compact open subsets which is stable by finite intersections. Let $\mathscr{T}''$ be the topology on X generated by the quasi-compact open subsets and their complements. By lemma I.2.5.2, it suffices to show that $\mathscr{T}'' = \mathscr{T}'$.

First note that $\mathscr{T}''$ is coarser than $\mathscr{T}'$, because $\mathscr{T}''$ is generated by elements of $\mathscr{U}$ and their complements, and every element of $\mathscr{U}$ is open and closed for $\mathscr{T}'$. We also claim that $\mathscr{T}''$ is Hausdorff : Indeed, let $x, y \in X$ such that $x \neq y$; as X is T_0, we may assume (up to switching x and y) that there exists $U \in \mathscr{U}$ such that $x \in U$ and $y \in X - U$, and U and $X - U$ are both open in $\mathscr{T}''$ by definition of $\mathscr{T}''$. So the identity map from X' to $X'' := (X_0, \mathscr{T}'')$ is a continuous map from a quasi-compact space from a Hausdorff space, which implies that it is a homeomorphism (see [6] chapitre I §9 Nº4 corollaire 2 du théorème 2, p. 63).

□

Lemma I.2.5.2. *(Lemma 3.29 of [26].) Let X be a quasi-compact T_0 space, and suppose that its topology has a basis consisting of quasi-compact open subsets which is stable under finite intersections. Let X' be the topological space with the same underlying set as X, and whose topology is generated by the quasi-compact open subsets of X and their complements. Then the following are equivalent :*

(i) X is spectral.

(ii) X' is Hausdorff and quasi-compact, and its topology has a basis consisting of open and closed subsets.

(iii) X' is quasi-compact.

Also, if these conditions are satisfied, then $X' = X_{\mathrm{cons}}$.

Proof.

(i)⇒(ii) If X is spectral, then it is quasi-compact and quasi-separated, so $X' = X_{\mathrm{cons}}$ by definition of the constructible topology, and the fact that X' is Hausdorff and quasi-compact is proposition I.2.4.1. Also, the topology of X' is generated by the consrtuctible subsets of X, which are open and closed as subsets of X' corollary I.2.4.2(ii).

(ii)⇒(iii) Obvious.

(iii)⇒(i) We already know that X is quasi-separated and has a basis of quasi-compact open subsets. So we just need to show that X is sober. By definition of the topology of X', this topology is finer than the topology of X.

Let Z be a closed irreducible subset of X. As X is T_0, this subset has at most one generic point, and we want to show that it has at least one. Let Z' be the set with the topology induced by the topology of X'. As X' is quasi-compact and Z' is closed in X' (because the topology of X' is finer than that of X), Z' is also quasi-compact. Suppose that Z has no generic point. For every $z \in Z$, we have $\overline{\{z\}} \subsetneq Z$, so there exists an open quasi-compact subset U_z fo Z such that $U_z \cap \overline{\{z\}} = \varnothing$. We have $Z = \bigcup_{z \in Z}(Z \setminus U_z)$, and each $Z \setminus U_z$ is open in Z' by definition of the topology of X'. As Z' is quasi-compact, there exists $z_1, \dots, z_n \in Z$ such that $Z = \bigcup_{i=1}^{n}(Z \setminus U_{z_i})$; but this contradicts the irreducibility of Z.

□

I.2.6 Spectrality of $\mathrm{Spv}(A)$

Theorem I.2.6.1. *(Proposition 4.7 of [26].) Let A be a commutative ring. Then $\mathrm{Spv}(A)$ is spectral. The open subsets*

$$U\left(\frac{f_1,\dots,f_n}{g}\right) = \{|.| \in \mathrm{Spv}(A) \mid |f_1|,\dots,|f_n| \leq |g| \neq 0\},$$

for $f_1,\dots,f_n,g \in A$, are quasi-compact, and they and their complements generate the constructible topology of $\mathrm{Spv}(A)$.

Moreover, if $\varphi : A \to B$ is a morphism of rings, then the induced map $\mathrm{Spv}(\varphi) : \mathrm{Spv}(B) \to \mathrm{Spv}(A)$ is spectral.

Remark I.2.6.2. The continuous map $\mathrm{supp} : \mathrm{Spv}(A) \to \mathrm{Spec}(A)$ is also spectral, since $\mathrm{supp}^{-1}(D(g)) = U\left(\frac{g}{g}\right)$ for every $g \in A$.

Proof. Let $X = \mathrm{Spv}(A)$, and denote its topology by $\mathscr{T}$. We want to apply theorem I.2.5.1 to the family of subsets $U\left(\frac{f_1,\dots,f_n}{g}\right)$, for $f_1,\dots,f_n,g \in A$.

(A) For every $x \in X$, if $(\wp_x, R_x)$ is the pair corresponding to x as in remark I.2.1.2, we define a relation $|_x$ on A by : $f|_x g$ if there exists an element a of R_x such that $a(f+\wp_x) = g+\wp_x$. (In other words, if $|.|_x$ is a valuation in the equivalence class corresponding to x, we have $f|_x g$ if and only $|f|_x \geq |g|_x$.)

This defines a map ρ from X to the set $\{0,1\}^{A\times A}$ of relations on A. It follows directly from the definition that, for every $x \in X$, we have $\mathrm{supp}(x)$ is the set of $f \in A$ such that $0|_x f$; so $\rho(x)$ determines $\mathrm{supp}(x)$.

(B) <u>Claim</u> : The map $\rho : X \to \{0,1\}^{A\times A}$ is injective.

Indeed, let $x, y \in X$ be such that $\rho(x) = \rho(y)$, and let $(\wp_x, R_x)$ and $(\wp_y, R_y)$ be the pairs corresponding to x and y. By the remark at the end of (A), we have $\wp_x = \wp_y$. Let $K = \mathrm{Frac}(A/\wp_x)$. Let $a \in K$. Then we can find $f, g \in A$ such that $a = (f+\wp_x)(g+\wp_x)^{-1}$, and we have :

$$a \in R_x \Leftrightarrow g|_x f \Leftrightarrow g|_y f \Leftrightarrow a \in R_y.$$

So $R_x = R_y$.

(C) <u>Claim</u> : The image of ρ is the set of relations $|$ on A satisfying the following conditions : for all $f, g, h \in A$,

(a) $f|g$ or $g|f$;

(b) if $f|g$ and $g|h$, then $f|h$;

(c) if $f|g$ and $f|h$, then $f|(g+h)$;

(d) if $f|g$, then $fh|gh$;

(e) if $fh|gh$ and $0 \not| h$, then $f|g$;

(f) $0 \not| 1$.

Let's prove this claim. It is easy to check that every element in the image of ρ satisfies (a)-(f).

Conversely, let $|$ be a relation on A satisfying (a)-(f). Note that we have $f|0$ for every $f \in R$ by (d). Let $\wp = \{f \in A \text{ such that } 0|f\}$. Then $\wp$ is an ideal of A by (c) and (d), it does not contain 1 by (f), and it is a prime ideal by (e). Next we note that, for all $f, f' \in A$ such that $f + \wp = f' + \wp$, we have $f|f'$ and $f'|f$; indeed, if $f' - f \in \wp$, then $0|(f' - f)$, so $f|(f' - f)$ by (b) (and the fact that $f|0$), hence $f|f'$ by (c), and similarly $f'|f$. Let's check that $|$ induces a relation on $A/\wp$. Let $f, f', g, g' \in A$ such that $f - f', g - g' \in \wp$. We need to show that $f|g \Rightarrow f'|g'$; but we have just seen that $f'|f$ and $g|g'$, so this follows from (b). Now let R be the set of $a \in \mathrm{Frac}(A/\wp)$ that can be written $a = (f + \wp)(g + \wp)^{-1}$, with $f, g \in A$ and $g|f$; by (d) and (e), this condition is actually independent of the choice of f and g. We see easily that R is a subring of $\mathrm{Frac}(A/\wp)$, and it is a valuation subring thanks to condition (a). So the pair $(\wp, R)$ defines a point x of X, and $\rho(x)$ is equal to $|$ by definition of $\wp$ and R.

(D) We put the discrete topology on $\{0, 1\}$ and the product topology on $\{0, 1\}^{A\times A}$. By Tychonoff's theorem, the space $\{0, 1\}^{A\times A}$ is compact. Let $\mathscr{T}'$ be the topology on X induced by the topology of $\{0, 1\}^{A\times A}$ via ρ, and let $X' = (X, \mathscr{T}')$. Then X' is also compact. Indeed, for $f, g, h \in A$ fixed, each of the conditions (a)-(f) of (X) defines a closed subset of $\{0, 1\}^{A\times A}$, so, by the conclusion (C), the subset $\rho(X)$ of $\{0, 1\}^{A\times A}$ is closed.

(E) Let $f_1, \dots, f_n, g \in A$. Then an element $|$ of $\rho(X)$ is in $\rho\left(U\left(\frac{f_1,\dots,f_n}{g}\right)\right)$ if and only $g|f_i$ for every $i \in \{1, \dots, n\}$ and $0 \not| g$. Each of these conditions defines an open and closed subset on $\{0, 1\}^{A\times A}$ (which is the pullback by one of the projection maps $\{0, 1\}^{A\times A}$ of a subset of $\{0, 1\}$), so $U\left(\frac{f_1,\dots,f_n}{g}\right)$ is open and closed in the topology $\mathscr{T}'$. Also, the space X is T_0 by remark I.2.2.6. Applying theorem I.2.5.1 to X' and to the collection of subsets $U\left(\frac{f_1,\dots,f_n}{g}\right)$, for $f_1, \dots, f_n, g \in A$, we get that X is spectral, that the $U\left(\frac{f_1,\dots,f_n}{g}\right)$ form a basis of quasi-compact open subsets of X, and that $X' = X_{\mathrm{cons}}$.

(F) Finally, we prove the last statement. Let $\varphi : A \to B$ be a morphism of rings. By proposition I.2.4.5, to show that the continuous map $\mathrm{Spv}(f) : \mathrm{Spv}(B) \to \mathrm{Spv}(A)$ is spectral, it suffices to show that it is quasi-compact; but this follows from the fact that

$$\mathrm{Spv}(\varphi)^{-1}\left(U\left(\frac{f_1, \dots, f_n}{g}\right)\right) = U\left(\frac{\varphi(f_1), \dots, \varphi(f_n)}{\varphi(g)}\right)$$

for all $f_1, \dots, f_n, g \in A$.

$\square$

Remark I.2.6.3. It follows from the theorem that the collection of consrtuctible subsets of $\mathrm{Spv}(A)$ is the smallest collection of subsets of $\mathrm{Spv}(A)$ that is stable by finite unions, finite intersections and complements and that contains all subsets of the form $U\left(\frac{f}{g}\right)$, for $f, g \in A$. It is also the smallest collection of subsets of $\mathrm{Spv}(A)$ that is stable by finite unions, finite intersections and complements and that contains all subsets of the form $U'(f,g) := \{|.| \in \mathrm{Spv}(A) \mid |f| \leq |g|\}$, for $f, g \in A$.

Indeed, we have for all $f, g \in A$:

$$U\left(\frac{f}{g}\right) = U'(f,g) \cap (\mathrm{Spv}(A) - U'(g,0))$$

and

$$U'(f,g) = U\left(\frac{f}{g}\right) \cup \left(\mathrm{Spv}(A) - \left(U\left(\frac{0}{g}\right) \cup U\left(\frac{0}{f}\right)\right)\right).$$

I.3 The specialization relation in $\mathrm{Spv}(A)$

The goal of this section is to study the specialization relation in $\mathrm{Spv}(A)$. Here are the main points :

(1) Specialization is an order relation on $\mathrm{Spv}(A)$ (and more generally on any T_0 space), and it tells us which constructible subsets are open or closed.

(2) Specialization in $\mathrm{Spv}(A)$ breaks into two simpler cases, horizontal specialization and vertical specialization. More precisely (see theorem I.3.4.3), if $x \in \mathrm{Spv}(A)$, every specialization of x is a horizontal specialization of a vertical specialization of x; moreover, in many cases (for example if $|A|_x \not\subset \Gamma_{x,\geq 1}$), every specialization of x is also a vertical specialization of a horizontal specialization of x.

(3) We say that a specialization y of x (in $\mathrm{Spv}(A)$) is a vertical specialization if $\wp_y = \wp_x$. So vertical specializations of x are parametrized by valuation subrings of $K(x)$ contained in R_x (i.e. by valuation subrings of of $R_x/\mathfrak{m}_{R_x}$), and vertical generizations of x are parametrized by valuation subrings of $K(x)$ containing R_x (i.e. by prime ideals of R_x). See theorem I.1.4.2.

(4) Horizontal specialization changes the support of a valuation. Here are some facts about it. Let $x \in \mathrm{Spv}(A)$.

   (a) If H is any subgroup of Γ_x, we define a map $|.|_{x|_H} : A \to H \cup \{0\}$ by $|a|_{x|_H} = \begin{cases} |a|_x & \text{if } |a|_x \in H \\ 0 & \text{otherwise.} \end{cases}$ If this is a valuation, it is a specialization of x, and we call this a horizontal specialization of x. Also, $|.|_{x|_H}$ is a valuation if and only if H is a convex subgroup of Γ_x and contains the convex subgroup $c\Gamma_x$ generated by $|A|_x \cap \Gamma_{x,\geq 1}$. (Note that $|.|_{x|_H}$ does *not* determine H in general.)

(b) Horizontal specializations of x are totally ordered (by specialization), and the minimal horizontal specializaton of x is $x_{|c\Gamma_x}$.

(c) A horizontal specialization y of x is uniquely determined by its support, and it is the generic point of $\overline{\{x\}} \cap \operatorname{supp}^{-1}(\wp_y)$.

(d) The possible supports of horizontal specializations of x are the x-convex prime ideals of A (where $\wp \in \operatorname{Spec}(A)$ is x-convex if $0 \leq |a|_x \leq |b|_x$ and $b \in \wp$ implies that $a \in \wp$).

(e) If $|A|_x \not\subset \Gamma_{x,\leq 1}$, then every fiber of $\operatorname{supp} : \overline{\{x\}} \to \operatorname{Spec}(A)$ contains a horizontal specialization of x, which is its generic point.

I.3.1 Specialization in a spectral space

Notation I.3.1.1. Let X be a topological space, and let $x, y \in X$. Remember from definition I.2.2.4(ii) that we say that y specializes to x (or that x is a specialization of y, or that y is a generization of x) if $x \in \overline{\{x\}}$.

If this is the case, we write $y \rightsquigarrow x$ or $x \leftsquigarrow y$.

Lemma I.3.1.2. *If X is T_0, then specialization is an order relation on X.*

Proof. Specialization is clearly reflexive and transitive (without any condition on X), so we just need to check that it is antisymmetric. Let $x, y \in X$ such that $x \rightsquigarrow y$ and $y \rightsquigarrow x$. Then $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$, so x and y are both generic points of the irreducible closed subset $\overline{\{x\}}$. By remark I.2.2.5, this implies that $x = y$.

□

The following result is one of the reasons that it is useful to understand the specialization relation in a spectral space.

Proposition I.3.1.3. *(Proposition 3.30 of [26] and [25, Lemma 0903]). Let X be a spectral space and $Z \subset X$ be a subspace.*

(i) The following are equivalent :

(a) Z is pro-constructible;

(b) Z is spectral, and the inclusion $Z \to X$ is spectral.

(ii) (a) Suppose that Z is pro-constructible. If $x \in \overline{Z}$, then x is the specialization of a point of Z. (In other words, $\overline{Z}$ is the set of specializations of points of Z.)

(b) If Z is pro-constructible, then Z is closed if and only if it is stable under specialization.

(c) *If Z is ind-constructible, then Z is open if and only if it is stable under generization.*

Proof.

(i) Suppose that (b) holds. Then Z is closed in X_{cons} by proposition I.2.4.5, which means that it is pro-constructible.

Suppose that (a) holds. Then Z is spectral by corollary I.2.4.3. We want to show that the inclusion $Z \to X$ is spectral. By proposition I.2.4.5, this is equivalent to the fact that it is quasi-compact. Let U be a quasi-compact open subsets of X. Then $U \cap Z$ is pro-constructible. As X_{cons} is quasi-compact by proposition I.2.4.1, its closed subset $U \cap Z$ is also quasi-compact, and so $U \cap Z$ is quasi-compact in the topology induced by X (which is coarser than the constructible topology).

(ii) Note that (b) follows immediately from (a), and that (c) is just (b) applied to $X - Z$. So we just need to prove (a). Let $x \in \overline{Z}$. Let $(U_i)_{i \in I}$ be the family of all quasi-compact open neighborhoods of x in X. Then $(U_i)_{i \in I}$ is cofinal in the set of all open neighborhoods of x and stable by finite intersection, because X is spectral, and $U_i \cap Z \neq \varnothing$ for every $i \in I$ because $x \in \overline{Z}$. Also, for every $i \in I$, the intersection $Z \cap U_i$ is pro-constructible, hence closed in X_{cons}. As X_{cons} is quasi-compact (proposition I.2.4.1), we have $Z \cap \bigcap_{i \in I} U_i \neq \varnothing$. Let $z \in Z \cap \bigcap_{i \in U} U_i$. Then z is contained in every open neighborhood of x, so $x \in \overline{\{z\}}$.

□

We easily see that this implies the following corollary.

Corollary I.3.1.4. *([25, Lemma 09XU].) Let $f : X \to Y$ be a bijective continuous spectral map of spectral spaces. If generizations (resp. specializations) lift along f, then f is a homeomorphism.*

I.3.2 Vertical specialization

We now fix a commutative ring A.

Remark I.3.2.1. (1) Let $\wp, \wp' \in \mathrm{Spec}(A)$. Then $\wp$ is a specialization of $\wp'$ if and only if $\wp \supset \wp'$. Indeed, the smallest closed subset of $\mathrm{Spec}(A)$ containing $\wp'$ is $V(\wp')$, that is, $\overline{\{\wp'\}} = V(\wp')$.

(2) Let $x, y \in \mathrm{Spv}(A)$. If x is a specialization of y, then $\mathrm{supp}(x) \supset \mathrm{supp}(y)$. (This just follows from the fact that $\mathrm{supp} : \mathrm{Spv}(A) \to \mathrm{Spec}(A)$ is continuous.)

Definition I.3.2.2. Let $x, y \in \mathrm{Spv}(A)$. We say that x is a *vertical specialization* of y (or that y is a *vertical generization* of x) if x is a specialization of y and $\mathrm{supp}(x) = \mathrm{supp}(y)$.

Remember that $\mathrm{supp}^{-1}(\wp) = RZ(\mathrm{Frac}(A/\wp))$ for every $\wp \in \mathrm{Spec}(A)$ (proposition I.2.1.4), and that the specialization relation in $RZ(K)$, for K a field, just corresponds to the inclusion of valuation subrings (remark I.1.6.4). So points (i) and (ii) of the following proposition are a direct consequence of corollary I.1.4.4, and point (ii) follows immediately from theorem I.1.4.2.

Proposition I.3.2.3. *Let $x \in \mathrm{Spv}(A)$. Then :*

(i) There is a canonical order-reversing bijection from the set of vertical generizations of x (ordered by specialization, where we think of the specialization as the "smaller" element) to $\mathrm{Spec}(R_x)$ (ordered by inclusion). If $y = (\wp_x, R_y)$ is a vertical generization of x, then the corresponding element of $\mathrm{Spec}(R_x)$ is $\mathfrak{m}_{R_y}$; conversely, if $\wp$ is a prime ideal of R_x, the corresponding vertical generization of x is $(\wp_x, (R_x)_\wp)$.

(ii) There is a canonical order-preserving bijection from the set of vertical generizations of x (ordered by specialization) to the set of convex subgroups of Γ_x (ordered by inclusion). If $y = (\wp_x, R_y)$ is a vertical generization of x, the corresponding convex subgroup of Γ_x is $\mathrm{Ker}(\Gamma_x \to \Gamma_{R_y})$.

(iii) There is a canonical order-preserving bijection from the set of vertical specializations of x (ordered by specialization) to $\mathrm{Spv}(R_x/\mathfrak{m}_{R_x}) = RZ(R_x/\mathfrak{m}_{R_x})$ (also ordered by specialization).

Notation I.3.2.4. If $x \in \mathrm{Spv}(A)$ and H is a convex subgroup of Γ_x, we denote by x/H the vertical generization of x corrseponding to H.

I.3.3 Horizontal specialization

Definition I.3.3.1. Let $x \in \mathrm{Spv}(A)$. The *characteristic group* of x is the convex subgroup $c\Gamma_x$ of Γ_x generated by $\Gamma_{x,\geq 1} \cap |A|_x$.

Example I.3.3.2. (1) We have $c\Gamma_x = \{1\}$ if and only if $|a|_x \leq 1$ for every $a \in A$. This does not hold in general.

(2) If $A = K$ is a field, then $|K|_x = \{0\} \cup \Gamma_x$, so $c\Gamma_x = \Gamma_x$.

(3) If K is a field, A is a valuation subring of K and $x \in \mathrm{Spv}(A)$ is the corresponding valuation, then $|a|_x \leq 1$ for every $a \in A$, so $c\Gamma_x = \{1\}$.

Definition I.3.3.3. Let $x \in \mathrm{Spv}(A)$ and let H be a subgroup of Γ_x. We define a map $|.|_{x_{|H}} : A \to \Gamma_x \cup \{0\}$ by setting

$$|f|_{x_{|H}} = \begin{cases} |f|_x & \text{if } |f|_x \in H \\ 0 & \text{otherwise.} \end{cases}$$

If $|.|_{x_{|H}}$ is a valuation on A, we denote the corresponding point of $\mathrm{Spv}(A)$ by $x_{|H}$.

Proposition I.3.3.4. *(Remarks 4.15 and 4.16 of [26].) Let* $x \in \mathrm{Spv}(A)$ *and let* H *be a convex subgroup of* Γ_x*. We denote the map* $A \to A/\wp_x \subset \mathrm{Frac}(A/\wp_x)$ *by* π*. Then the following are equivalent :*

(i) $|.|_{x_{|H}}$ *is a valuation on* A*.*

(ii) $H \supset c\Gamma_x$*.*

(iii) If we denote by $\wp_H = \{a \in R_x \mid \forall \delta \in H, |a|_x < \delta\}$ *the corresponding prime ideal of* R_x *(see corollary I.1.4.4), then* $\pi(A) = A/\wp_x \subset R_{x,\wp_H}$*.*

Proof.

(i)⇒(ii) Suppose that $|.|_{x_{|H}}$ is a valuation on A. To show that H contains $c\Gamma_x$, it suffices to show that it contains $\Gamma_{x,\geq 1} \cap |A|_x$. So let $a \in A$ be such that $|a|_x \geq 1$. If $|a|_x = 1$, we have $|a|_x \in H$, so assume that $|a|_x > 1$. Then $|a+1|_x = |a|_x$. If $|a|_x \notin H$, then $0 = |a+1|_{x_{|H}} = \max\{|a|_{x_{|H}}, 1\} = 1$, contradiction. So $|a|_x \in H$.

(ii)⇒(iii) Note that $R_{x,\wp_H}$ is the valuation subring of $\mathrm{Frac}(A/\wp_x)$ corresponding to the composition of $|.|_x : \mathrm{Frac}(A/\wp_x) \to \Gamma_x \cup \{0\}$ and of the quotient map $\Gamma_x \cup \{0\} \to (\Gamma_x/H) \cup \{0\}$, so an element a of $\mathrm{Frac}(A/\wp_x)$ is in $R_{x,\wp_H}$ if and only if there exists $\gamma \in H$ such that $|a|_x \leq \gamma$.

Now suppose that $H \supset c\Gamma_x$, and let $a \in A$. We want to show that there exists $\gamma \in H$ such that $|a|_x \leq \gamma$. If $|a|_x \leq 1$, then we can take $\gamma = 1$. If $|a|_x > 1$, then $|a|_x \in c\Gamma_x \subset H$, so we can take $\gamma = |a|_x$.

(iii)⇒(i) We have seen that (iii) implies that, for every $a \in A$, there exists $\gamma \in H$ such that $|a|_x \leq \gamma$; in particular, if $|a|_x \geq 1$, then $|a|_x \in H$, because H is convex. (This proves (ii).)

We check that $|.|_{x_{|H}}$ satisfies the conditions of definition I.1.1.10(ii). It is clear that $|0|_{x_{|H}} = 0$ and $|1|_{x_{|H}} = 1$. Let $a, b \in A$. The only way we can have $|ab|_{x_{|H}} \neq |a|_{x_{|H}}|b|_{x_{|H}}$ is if $|ab|_x \in H$ but $|a|_x, |b|_x \notin H \cup \{0\}$. So suppose that $|a|_x, |b|_x \notin H \cup \{0\}$. This implies that $|a|_x, |b|_x < 1$, and then $|ab|_x \leq |a|_x \leq 1$ and $|ab|_x \leq |b|_x \leq 1$, so the convexity of H implies that $|ab|_x$ cannot be in H. Finally, we prove that $|a+b|_{x_{|H}} \leq \max\{|a|_{x_{|H}}, |b|_{x_{|H}}\}$. If $|a+b|_x \notin H$, this obviously holds, so suppose that $|a+b|_x \in H$. We may assume that $|a|_x \leq |b|_x$. If $|b|_x \in H$, we are done, so we may assume that $|b|_x \notin H$. But then $|a+b|_x \leq |b|_x \leq 1$ and H is convex, so this impossible.

□

Proposition I.3.3.5. *(Remark 4.16 of [26].) We use the notation of proposition I.3.3.4, and we denote the quotient map* $A \to A/\wp_x$ *by* π*. If the conditions of this proposition hold, then :*

(i) $x_{|H}$ *is a specialization of* x*.*

(ii) $\mathrm{Ker}(x) \subset \mathrm{Ker}(x_{|H}) = \pi^{-1}(\wp_H R_{x,\wp_H}) = \{a \in A \mid \forall \gamma \in H,\ |a|_x < \gamma\}$*, with equality if and only if* $H = \Gamma_x$ *(which is also equivalent to* $\wp_H = 0$*, or to* $x = x_{|H}$*).*

(iii) *$R_x/\wp_H$ is a valuation subring of the field $R_{x,\wp_H}/\wp_H$, and $|.|_{x_{|H}}$ is the composition of $A \xrightarrow{\pi} R_{x,\wp_H} \to R_{x,\wp_H}/\wp_H$ and of the valuation corresponding to $R_x/\wp_H$.*

Note that it is not easy in general to describe the valuation group of $x_{|H}$ (beyond the obvious fact that it is included in H).

Proof. (i) Let $f_1, \dots, f_n, g \in A$ such that $x_{||H} \in U := U\left(\frac{f_1,\dots,f_n}{g}\right)$. We want to show that $x \in U$. We have $|f_i|_{x_{|H}} \leq |g|_{x_{|H}} \neq 0$ for every i, so in particular $|g|_x \in H$ and $|g|_x = |g|_{x_{|H}}$. Suppose that we $|f_i|_x > |g|_x$ for some i. Then $|f_i|_x \notin H$, so $|f_i|_x < 1$ (because $H \supset c\Gamma_x$), so $|g|_x < |f_i|_x < 1$, which implies that $|f_i|_x \in H$ because H is convex, contradiction. So $x \in U$.

(ii) The inclusion $\mathrm{Ker}(x) \subset \mathrm{Ker}(x_{|H})$ is obvious.

As $\wp_H R_{x,\wp_H}$ is the maximal ideal of $R_{x,\wp_H}$ and $R_{x,\wp_H}$ is the valuation subring of $\mathrm{Frac}(A/\wp_x)$ corresponding to the valuation $\mathrm{Frac}(A/\wp_x) \xrightarrow{|.|_x} \Gamma_x \cup \{0\} \to (\Gamma_x/H) \cup \{0\}$, we have

$$\wp_H R_{x,\wp_H} = \{a \in \mathrm{Frac}(A/\wp_x) \mid \forall \gamma \in H,\ |a|_x < \gamma\}.$$

Let $a \in A$. We have $a \in \mathrm{Ker}(x_{|H})$ if and only if $|a|_x \notin H$. As $H \supset c\Gamma_x \supset \Gamma_{x,\geq 1} \cap |A|_x$, this implies that $|a|_x < 1$. So we have $|a|_x \notin H$ if and only if $|a|_x < \gamma$ for every $\gamma \in H$. (Indeed, if $|a|_x \notin H$ and $|a|_x > \gamma$ for some $\gamma \in H$, then $\gamma < |a|_x < 1$, contradicting the convexity of H). This proves the formula for $\mathrm{Ker}(x_{|H})$.

We obviously have $\mathrm{Ker}(x) = \mathrm{Ker}(x_{|H})$ if $\Gamma_x = H$. Conversely, suppose that $\mathrm{Ker}(x) = \mathrm{Ker}(x_{|H})$, and let's prove that $H = \Gamma_x$. Let $\gamma \in \Gamma_x$. Write $\gamma = |a|_x/|b|_x$, with $a, b \in A$. Then $|a|_x, |b|_x \neq 0$, so $|a|_{x,H}, |b|_{x,H} \neq 0$, so $|a|_x, |b|_x \in H$, so $\gamma \in H$.

(iii) In $\mathrm{Frac}(A/\wp_x)$, we have the valuation subring with valuation group Γ_x, and the bigger ring $R_{x,\wp_H}$ with valuation group Γ_x/H and maximal ideal $\wp_H$.. So we are in the situation of theorem I.1.4.2, and we see that $R_x/\wp_H$ is a valuation subring of $R_{x,\wp_H}/\wp_H$ with valuation group $\mathrm{Ker}(\Gamma_x \to \Gamma_x/H) = H$. The second statement also follows easily from this theorem and from (ii).

□

Definition I.3.3.6. Let $x, y \in \mathrm{Spv}(A)$. We say that y is a *horizontal specialization* of x (or that x is a *horizontal generization* of y) if y is the form $x_{|H}$, for H a convex subgroup of Γ_x containing $c\Gamma_x$.

Remark I.3.3.7. Note that x is a horizontal specialization of itself (corresponding to the convex subgroup $H = \Gamma_x$).

We now give a decription of the possible kernels of horizontal specializations of x.

Definition I.3.3.8. Let $x \in \mathrm{Spv}(A)$. A subset T of A is called *x-convex* if for all $a_1, a_2, b \in A$, if $|a_1|_x \leq |b|_x \leq |a_2|_x$ and $a_1, a_2 \in T$, then $b \in T$.

For example, if $0 \in T$, then T is x-convex if and only if, for all $a \in A$ and $b \in T$ such that $|a|_x \leq |b|_x$, we have $a \in T$.

Proposition I.3.3.9. *(Proposition 4.18 of [26].) Let* $x \in \mathrm{Spv}(A)$*. Let* C *be the set of* x*-convex prime ideals of* A*, order by inclusion, and let* S *be the set of horizontal specializations of* x*, ordered by specialization. Then* $y \longmapsto \mathrm{Ker}(y)$ *induces an order-reversing bijection* $S \xrightarrow{\sim} C$*, and the sets* S *and* C *are totally ordered.*

Remark I.3.3.10. So we have a commutative diagram :

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, "\wr"] & \overline{\{x\}} \arrow[r, hook] \arrow[d] & \mathrm{Spv}(A) \arrow[d, "\mathrm{supp}"] \\ C \arrow[r, hook] & \mathrm{Spec}(A/\wp_x) \arrow[r, hook] & \mathrm{Spec}(A) \end{tikzcd}$$

We will see in the next subsection (see corollary I.3.4.5) that, if $|A|_x \not\subset \Gamma_{x,\leq 1}$ (which occurs for example if A contains a subfield on which the valuation $|.|_x$ is nontrivial), then C is actually the image of $\overline{\{x\}}$ in $\mathrm{Spec}(A)$, so every x-convex prime ideal $\mathfrak{q}$ of A lifts to a unique horizontal specialization y of x, and moreover this y is the unique generic point of $\mathrm{supp}^{-1}(\mathfrak{q}) \cap \overline{\{x\}}$.

Proof of proposition I.3.3.9. The kernel of any horizontal specialization of x is x-convex by proposition I.3.3.5(ii), so $\mathrm{supp} : S \to C$ is well-defined, and it is clearly order-reversing.

We have seen in proposition I.3.3.4 that S is in bijection to the set P of prime ideals $\wp$ of R_x such that $A/\wp_x \subset R_{x,\wp}$, and this bijection is order-reversing. As the set of ideals of a valuation ring is totally ordered (proposition I.1.1.4), this implies that S is totally ordered. If we identify S and P, the map $\mathrm{supp} : S \to C$ becomes $\wp \longmapsto \pi^{-1}(\wp R_{x,\wp})$, where $\pi : A \to \mathrm{Frac}(A/\wp_x)$ is the quotient map (see proposition I.3.3.5(ii)).

Now we construct the inverse map. If $\mathfrak{q}$ is a x-convex prime ideal of A, then we clearly have $\mathfrak{q} \supset \wp_x$, and $|a|_x < |b|_x$ for every $a \in \mathfrak{q}$ and every $b \in A - \mathfrak{q}$; in particular, $|\mathfrak{q}|_x \subset \Gamma_{x,<1}$. So, if H is the convex subgroup of Γ_x generated by $|A - \mathfrak{q}|_x$, we have $H \supset c\Gamma_x$. We want to prove that $\mathrm{supp}(x_{|H}) = \mathfrak{q}$, or equivalently that $\mathfrak{q} = \pi^{-1}(\wp R_{x,\wp})$. But we know from proposition I.3.3.5(ii) that $\mathrm{Ker}(x_{|H}) = \{a \in A | \forall \gamma \in H,\ |a|_x < \gamma\}$, and this is equal to $\{a \in A | \forall b \in A - \mathfrak{q},\ |a|_x < |b|_x\}$ by definition of H.

□

Proposition I.3.3.11. *(Proposition 4.4.2 of [9].) Let* $x, y, z \in \mathrm{Spv}(A)$*. If* y *is a horizontal specialization of* x *and* z *is a horizontal specialization of* y*, then* z *is a horizontal specialization of* x*.*

Proof. Let H be a convex subgroup of Γ_x and G be a convex subgroup of Γ_y such that $y = x_{|H}$ and $z = y_{|G}$. In particular, we have $\Gamma_y \subset H$. We denote by G' the smallest convex subgroup of H containing G. Then :

(a) $G = G' \cap \Gamma_y$: Indeed, we obviously have $G \subset G' \cap \Gamma_y$. Conversely, if $\delta \in G'$, then, by definition of G', there exist $\gamma_1, \gamma_2 \in G$ such that $\gamma_1 \leq \delta\gamma_2$. If moreover $\delta \in \Gamma_y$, then this implies that $\delta \in G$, because G is convex in Γ_y.

(b) $G' \supset c\Gamma_x$: As G' is convex, it suffices to show that $G' \supset |A|_x \cap \Gamma_{x,\geq 1}$. Let $a \in A$ such that $|a|_x \geq 1$. Then $|a|_x \in H$, so $|a|_y = |a|_{x_{|H}} = |a|_x \geq 1$, so $|a|_x = |a|_y \in G \subset G'$.

(c) $x_{|G'}$ (which makes sense by (b)) is equal to z : Indeed, this follows immediately from definition I.3.3.3 and from (a).

□

I.3.4 Putting things together

Proposition I.3.4.1. *(Lemma 4.19 of [26].)*

(i) If $x \rightsquigarrow y$ is a horizontal specialization and $y \rightsquigarrow z$ is a vertical specialization, then there exists a vertical specialization $x \rightsquigarrow y'$ that admits z as a horizontal specialization.

$$\begin{tikzcd} x \arrow[r, rightsquigarrow] \arrow[d, dotted, "\exists"'] & y \arrow[d, rightsquigarrow] \\ y' \arrow[r, dotted] & z \end{tikzcd}$$

(ii) If $x \rightsquigarrow y$ is a horizontal specialization and $x \rightsquigarrow y'$ is a vertical specialization, then there exists a unique *horizontal specialization $y' \rightsquigarrow z$ such that z is a vertical specialization of y.*

$$\begin{tikzcd} x \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & y \arrow[d, dotted, "\exists!"] \\ y' \arrow[r, dotted] & z \end{tikzcd}$$

Proof. (i) Let $H \supset c\Gamma_x$ be a convex subgroup of Γ_x such that $y = x_{|H}$, and $\wp_H$ be the corresponding prime ideal of R_x. Then $|.|_y$ is the composition of $A \to A/\wp_x \to R_{x,\wp_H} \to R_{x,\wp_H}/\wp_H R_{x,\wp_H}$ and of the valuation on $R_{x,\wp_H}/\wp_H$ corresponding to the valuation subring $R_x/\wp_H$, so $\wp_y = \mathrm{Ker}(A \to R_{x,\wp_H}/\wp_H)$ and we have an extension of fields $K(y) \subset R_{x,\wp_H}/\wp_H$ such that $R_y = K(y) \cap (R_x/\wp_H)$. Let $R_z \subset R_y$ be the valuation subring of $K(y)$ corresponding to the vertical specialization $y \rightsquigarrow z$. Then, by corollary I.1.2.4, there exists a valuation subring $B \subset R_x/\wp_H$ of $R_{x,\wp_H}/\wp_H$ such that $R_z = K(y) \cap B$. Let B' be the inverse image of B in $R_{x,\wp_H}$. Then B' is a valuation subring of $K(x)$ by theorem I.1.4.2(ii), and $B' \subset R_x$ by definition. Let y' be the valuation on A given by the composition of $A \to A/\wp_x \subset K(x)$ and of $|.|_{B'}$ (so that $\wp_{y'} = \wp_x$ and $R_{y'} = B'$). Then y' is a vertical specialization of x.

It remains to show that z is a horizontal specialization of y'. Let $\mathfrak{q} = R_{y'} \cap \wp_H \in \mathrm{Spec}(R_{y'})$. Then $R_{y',\mathfrak{q}} = R_{x,\wp_H}$ by theorem I.1.4.2(i)(c), so $R_{y',\mathfrak{q}}$ contains the image of $A/\wp_{y'} = A/\wp_x$, so it defines a horizontal specialization of y' by proposition I.3.3.4. By proposition I.3.3.5, this horizontal specialization is the composition of $A \to A/\wp_{y'} \subset R_{y',\mathfrak{q}} = R_{x,\wp_H} \to R_{x,\wp_H}/\wp_H R_{x,\wp_H}$ and of the valuation defined by the valuation subring $R_{y',\mathfrak{q}}/\mathfrak{q} = B$; in other words, it is the valuation z.

(ii) We have $\wp_z = \wp_y$ for every vertical specialization z and y, and horizontal valuations of y' are uniquely determined by their kernel (by proposition I.3.3.9), so z is unique if it exists.

Let H be the convex subgroup of $\Gamma_{y'}$ such that $x = y'/H$ (so we have $\Gamma_x = \Gamma_{y'}/H$), let $G \supset c\Gamma_x$ be a convex subgroup of Γ_x such that $y = x_{|G}$, and denote by G' the inverse image of G in $\Gamma_{y'}$. Then G' is a convex subgroup of $\Gamma_{y'}$, and we have $G' \supset c\Gamma_{y'}$ (indeed, if $a \in A$ is such that $|a|_{y'} \geq 1$, then $|a|_x \geq 1$ because $|a|_x$ is the image of $|a|_{y'}$ by the quotient map $\Gamma_{y'} \to \Gamma_x$, so $|a|_x \in G$, and finally $|a|_{y'} \in G'$). Let $z = y'_{|G'}$. Then it is easy to check (from the formulas for $|.|_z$ and $|.|_y$ given in definition I.3.3.3) that $z = y/H$.

□

Corollary I.3.4.2. *(Corollary 4.20 of [26].) Let* $x \in \mathrm{Spv}(A)$*, and let* $\wp$ *be a generization of* $\wp_x$ *in* $\mathrm{Spec}(A)$ *(i.e.* $\wp \subset \wp_x$*). Then there exists a horizontal generization* y *of* x *such that* $\wp_y = \wp$*.*

Proof. Let R be the localization $(A/\wp)_{\wp/\wp_x}$. Then R is a local ring and $\mathrm{Frac}(R) = \mathrm{Frac}(A/\wp)$, so, by theorem I.1.2.2, there exists a valuation subring B of $\mathrm{Frac}(A/\wp)$ such that $\mathfrak{m}_R = R \cap \mathfrak{m}_B$. Let y' be the corresponding valuation on A (i.e. such that $\wp_{y'} = \wp$ and $R_{y'} = B$). By definition of y', the image of A in $K(y')$ is included in $R_{y'}$, so we can construction a horizontal specialization z of y' using the maximal ideal $\mathfrak{m}$ of $R_{y'}$. The valuation corresponding to z is the composition of the map $A \to A/\wp \to R_{y'} \to R_{y'}/\mathfrak{m}$ and of the trivial valuation on $R_{y'}/\mathfrak{m}$. As $\mathfrak{m}_R = R \cap \mathfrak{m}$ and R also contains the image of A in $A/\wp$, this valuation is also the composition of $A \to R \to R/\mathfrak{m}_R = \mathrm{Frac}(A/\wp_x)$ and of the trivial valuation on $\mathrm{Frac}(A/\wp_x)$. In particular, we have $\wp_z = \wp_x$, so x is a vertical specialization of z. By proposition I.3.4.1(i), there exists a vertical specialization y of y' such that x is a horizontal specialization of y; then $\wp_y = \wp_{y'} = \wp$, so we are done.

□

Theorem I.3.4.3. *(Proposition 4.21 of [26].) Let* $x, y \in \mathrm{Spv}(A)$ *such that* y *is a specialization of* x*. Then :*

(i) There exists a vertical specialization $x \rightsquigarrow x'$ *such that* y *is a horizontal specialization of* x'*.*

(ii) There exists a vertical generization y' *of* y *such that one of the following conditions holds :*

(a) y' *is a horizontal specialization of* x*;*

(b) $|A|_x \subset \Gamma_{x,\leq 1}$, y' *induces the trivial valuation on* $K(y') = K(y)$*, and* $\wp_{y'} = \wp_y$ *contains* $\wp_{x_{|\{1\}}}$.

Note that the condition $|A|_x \subset \Gamma_{x,\leq 1}$ is equivalent to the fact that $c\Gamma_x = \{1\}$.

Proof. (ii) We first assume that $c\Gamma_x = \{1\}$ and that $|A - \wp_y|_x = \{1\}$. Let H be the trivial subgroup of Γ_x. Then we can form the horizontal specialization $x_{|H}$, and we have $\mathrm{Ker}(x_{|H}) = \{a \in A \mid |a|_x < 1\} \subset \wp_y$. Let y' be the composition of the map $A \to A/\wp_y$ and of the trivial valuation on $\mathrm{Frac}(A/\wp_y)$. Then y is a vertical specialization of y' and $\wp_{y'} \supset \mathrm{Ker}(x_{|H})$, so we are done. (And we are in case (b).)

Now assume that $c\Gamma_x \neq \{1\}$ or $|A - \wp_y|_x \neq \{1\}$. We claim that the following relation holds :

$$(*) \qquad \forall a \in A - \wp_y,\ \forall b \in \wp_y,\ |a|_x \leq |b|_x \Rightarrow |a|_x = |b|_x \neq 0.$$

Indeed, the hypothesis that $a \in A - \wp_y$ and $b \in \wp_y$ means that $|b|_y = 0 \leq |a|_y \neq 0$, which implies that $y \in U\left(\frac{b}{a}\right)$. As x is a generization of y, we must then also have $x \in U\left(\frac{b}{a}\right)$, i.e. $|b|_x \leq |a|_x \neq 0$. So, if $|a|_x \leq |b|_x$, we get $|a|_x = |b|_x \neq 0$.

We prove that the ideal $\wp_y$ is x-convex. Let $a, b \in A$ such that $b \in \wp_y$ and $|a|_x \leq |b|_x$; we need to prove that $a \in \wp_y$. Suppose that $a \notin \wp_y$; then, by (*), we have $|a|_x = |b|_x \neq 0$. There are two cases:

(1) If $|A|_x \not\subset \Gamma_{x,\leq 1}$: Then there exists $c \in A$ such that $|c|_x > 1$. We have $|a|_x < |bc|_x$, $a \in A - \wp_y$ and $bc \in \wp_y$, which contradicts (*).

(2) If $|A|_x \subset \Gamma_{x,\leq 1}$ and $|A - \wp_y|_x \neq \{1\}$: Then there exists $c \in A - \wp_y$ such that $|c|_x < 1$. We have $|ac|_x < |b|_x$, $ac \in A - \wp_y$ and $b \in \wp_y$, which again contradicts (*).

So both cases are impossible, and this finishes the proof that $\wp_y$ is x-convex.

By proposition I.3.3.9, there exists a horizontal specialization y' of x such that $\wp_{y'} = \wp_y$. To finish the proof, it suffices to show that y is a vertical specialization of y'; as $\wp_y = \wp_{y'}$, it suffices to show that y is a specialization of y'. So let $f, g \in A$ such that $y \in U\left(\frac{f}{g}\right)$; we want to show that $y' \in U\left(\frac{f}{g}\right)$. We have $|f|_y \leq |g|_y \neq 0$. As x is a generization of y, this implies that $|f|_x \leq |g|_x$. We know that $|a|_{y'} = |a|_x$ or 0 for every $a \in A$, by the formula in definition I.3.3.3. As $\wp_y = \wp_{y'}$, we have $g \notin \wp_{y'}$, so $|g|_{y'} \neq 0$, hence $|g|_{y'} = |g|_x$; so $|f|_{y'} \leq |f|_x \leq |g|_{y'}$, and we are done.

(i) Let y' be the vertical generization of y given by (ii). If we are in case (ii)(a), then we get (i) by proposition I.3.4.1(i). So we may assume that we are in case (ii)(b), that is, that $c\Gamma_x = \{1\}$, that $|.|_{y'}$ is trivial on $K(y)$ and that $\wp_y \supset \mathrm{Ker}(x_{|H})$, where $H = \{1\} \subset \Gamma_x$. By corollary I.3.4.2, there exists a horizontal generization z of y such that $\wp_z = \mathrm{Ker}(x_{|H})$. As $x_{|H}$ induces the trivial valuation on $K(x_{|H})$, it is generic in the fiber $\mathrm{supp}^{-1}(\mathrm{Ker}(x_{|H}))$, and so z is a vertical specialization of $x_{|H}$. By proposition I.3.4.1(i), there exists a vertical specialization x' of x such that z is a horizontal specialization of x'. Finally, by proposition I.3.3.11, y is a horizontal specialization of x'.

$$\begin{tikzcd}
x \arrow[r, rightsquigarrow] \arrow[d, dotted] & x_{|H} \arrow[d, rightsquigarrow] & y' \arrow[d, rightsquigarrow] \\
x' \arrow[r, dotted] \arrow[d, mapsto, "\mathrm{supp}"'] & z \arrow[r, rightsquigarrow] \arrow[d, mapsto, "\mathrm{supp}"'] & y \arrow[d, mapsto, "\mathrm{supp}"'] \\
\wp_x \arrow[r, hook] & \wp_{x_{|H}} \arrow[r, hook] & \wp_y
\end{tikzcd}$$

□

Corollary I.3.4.4. *Let $x \in \mathrm{Spv}(A)$. If $|A|_x \not\subset \Gamma_{x,\leq 1}$, then $x_{|c\Gamma_x}$ has only vertical specializations.*

Proof. Let y be a specialization of $x_{|c\Gamma_x}$. Then y is also a specialization of x, so, by theorem I.3.4.3(ii), there exists a vertical generization y' of y such that y' is a horizontal specialization of x. But $x_{|c\Gamma_x}$ is the minimal horizontal specialization of x (remember that the set of horizontal specializations of x is totally ordered by proposition I.3.3.9), so it is a horizontal specialization of y'. As $\wp_{y'} = \wp_y \supset \wp_{x_{|c\Gamma_x}}$ (because y is a specialization of $x_{|c\Gamma_x}$), we must have $\wp_{y'} = \wp_{x_{|c\Gamma_x}}$, hence $y' = x_{|c\Gamma_x}$ by proposition I.3.3.9 again, so y is a vertical specialization of $x_{|c\Gamma_x}$.

□

Corollary I.3.4.5. *Let $x \in \mathrm{Spv}(A)$. If $|A|_x \not\subset \Gamma_{x,\leq 1}$, then $\mathrm{supp}(\overline{\{x\}}) \subset \mathrm{Spec}(A)$ coincides with the set of x-convex prime ideals $\mathfrak{q}$ of A, and, for every such $\mathfrak{q}$, the intersection $\mathrm{supp}^{-1}(\mathfrak{q}) \cap \overline{\{x\}}$ has a unique generic point, which is the horizontal specialization of x corresponding to $\mathfrak{q}$ by proposition I.3.3.9.*

Proof. Let y be a specialization of x. We want to show that $\wp_y$ is x-convex. By theorem I.3.4.3(ii), there exists a horizontal specialization y' of x such that y is a vertical specialization of y', and then $\wp_y = \wp_{y'}$, so $\wp_y$ is x-convex.

Now let $\mathfrak{q}$ be a x-convex prime ideal of A, and let z be the unique horizontal specialization of x such that $\wp_z = \mathfrak{q}$. and let y be a specialization of x such that $\wp_y = \mathfrak{q}$. Then, by theorem I.3.4.3(ii), there exists a vertical specialization of y' that is also a horizontal specialization of x. As $\wp_{y'} = \mathfrak{q}$, we must have $z = y'$. So z is dense in $\overline{\{x\}} \cap \mathrm{supp}^{-1}(\mathfrak{q})$.

□

Remark I.3.4.6. Let $x \in \mathrm{Spv}(A)$. Suppose that A has a subfield k such that $|.|_x$ is not trivial on k. Then $|A|_x \not\subset \Gamma_{x,\leq 1}$. Indeed, let $a \in k$ be such that $|a|_x \neq 1$. Then either $|a|_x > 1$ and we are done, or $|a|_x < 1$ and then $|a^{-1}|_x > 1$.

I.3.5 Examples

Note the following useful remark to recognize specializations :

Remark I.3.5.1. If $x, y \in \mathrm{Spv}(A)$, then y is a specialization of x if and only if, for every $f, g \in A$:

$$(|f|_y \leq |g|_y \neq 0) \Rightarrow (|f|_x \leq |g|_x \neq 0).$$

Indeed, y is a specialization of x if and only every open subset of $\mathrm{Spv}(A)$ that contains y also contains x, and the sets $U\left(\frac{f}{g}\right)$ generate the topology of $\mathrm{Spv}(A)$.

I.3.5.1 $\underline{A = \mathbb{Q}_\ell[T]}$:

For every $r \in (0, +\infty)$, we define a valuation $|.|_r$ on A by the following formula : if $f = \sum_{n \geq 0} a_n T^n \in A$, then

$$|f|_r = \max_{n \geq 0}\{|a_n|_\ell r^n\}.$$

It is not too hard to prove that

$$|f|_r = \sup_{x \in \overline{\mathbb{Q}}_\ell,\ |x|_\ell \leq r} |f(x)|_\ell.$$

We have $\mathrm{Ker}\,|.|_r = (0)$ and $c\Gamma_{|.|_r} = \Gamma_{|.|_r} = \langle \ell, r \rangle \subset \mathbb{R}_{>0}$. In particular, the valuation $|.|_r$ has no proper horizontal specialization.

<u>Claim</u>: The map $(0, +\infty) \to \mathrm{Spv}(A)$, $r \longmapsto |.|_r$ is continuous exactly at the points of $(0, +\infty) - \ell^{\mathbb{Q}}$.

For every $a \in \overline{\mathbb{Q}}_\ell$, we define a valuation $|.|_a$ on A by $|f|_a = |f(a)|_\ell$. The kernel of this valuation is (P_a), where P_a is the minimal polynomial of a over $\mathbb{Q}_\ell$; its valuation group is the subgroup of $\mathbb{R}_{>0}$ generated by ℓ and $|a|_\ell$ (so it is a subgroup of $\ell^{\mathbb{Q}}$), and we have $c\Gamma_{|.|_a} = \Gamma_{|.|_a}$.

We can also consider the T-valuation on A. Take $\Gamma = \gamma^{\mathbb{Z}}$, with the convention that $\gamma < 1$, and define $|.|_T$ by $f|_T = \gamma^{\mathrm{ord}_0(f)}$, where $\mathrm{ord}_0(f)$ is the order of vanishing of f at 0 (and with the convention that $\mathrm{ord}_0(0) = +\infty$ and $\gamma^{+\infty} = 0$). We have $\mathrm{Ker}\,|.|_T = (0)$, $\Gamma_{|.|_T} = \gamma^{\mathbb{Z}}$ and $c\Gamma_{|.|_T} = \{1\}$. So we can form the horizontal specialization $|.|_{T|\{1\}}$. It sends $f \in A$ to 1 if $\mathrm{ord}_0(f) = 0$ and to 0 otherwise, so it is the trivial valuation on A with support (T).

These are all rank 1 valuations. The rank 0 valuations are all of the form $|.|_{\wp,\mathrm{triv}} : A \to A/\wp \overset{|.|_{\mathrm{triv}}}{\to} \{0, 1\}$, where $\wp \in \mathrm{Spec}(A)$. For example, if $\wp = (0)$, we get the trivial valuation of A, which is the generic point of $\mathrm{Spv}(A)$. We have seen in the previous paragraph that $|.|_{(T),\mathrm{triv}}$ is a horizontal specialization of $|.|_T$ and that $|.|_T$ is a vertical specialization of $|.|_{\mathrm{triv}}$. Does there exist a horizontal specialization of $|.|_{\mathrm{triv}}$ that has $|.|_{(T),\mathrm{triv}}$ as a vertical specialization

?

$$\begin{tikzcd} {|\cdot|_{\mathrm{triv}}} \arrow[r, dotted] \arrow[d, squiggly] & ? \arrow[d, dotted] \\ {|\cdot|_T} \arrow[r, squiggly] & {|\cdot|_{(T),\mathrm{triv}}} \end{tikzcd}$$

Let's construct some rank 2 valuations. Consider the group $\Gamma = \mathbb{R}^+_{>0} \times \{1^-\}^{\mathbb{Z}}$ with the lexicographic order (here "1^-" is just a symbol that we use to denote a generator of the second factor); in other words, we have $r < 1^- < 1$ for every $r \in (0,1)$, where we abbreviate $(r,1)$ to r for every $r \in \mathbb{R}_{>0}$. Let $r \in (0,+\infty)$, set $r^- = r \cdot 1^- \in \Gamma$ (so we have $s < r^- < r$ for every $s \in (0,r)$), and define a valuation $|.|_{r^-}$ on A by the following formula : if $f = \sum_{n\geq 0} a_n T^n \in A$, then

$$|f|_r = \max_{n\geq 0}\{|a_n|_\ell (r^-)^n\}.$$

We have $\mathrm{Ker}\,|.|_{r^-} = (0)$ and $c\Gamma_{|.|_{r^-}} = \Gamma_{|.|_{r^-}} = \langle \ell, r^-\rangle \subset \Gamma$. It is easy to see that $|.|_{r^-}$ is a vertical specialization of $|.|_r$.

I.3.5.2 $\underline{A = \mathbb{Z}[T]}$:

We can restrict all the valuations of the previous example to $\mathbb{Z}[T]$ (and we will use the same notation for them). But note that the groups $c\Gamma_x$ can change.

For example, if $r \in (0,1]$, we now have have $c\Gamma_{|.|_r} = \{1\}$, so we can form the horizontal specialization $|.|'_r$ of $|.|_r$ corresponding to $H = \{1\}$. This is the trivial valuation on A with kernel $\{f \in A \mid |f|_r < 1\}$. If for example $r = 1$, this kernel is ℓA. If $r = \ell^{-1}$, then $\mathrm{Ker}\,|.|'_r$ is equal to

$$\{\sum_{n\geq 0} a_n T^n \mid \forall n \in \mathbb{N},\ |a_n|_\ell < \ell^n\} = (\ell, T).$$

By proposition I.3.4.1(ii), there exists a unique horizontal specialization of $|.|_{r^-}$ that is also a vertical specialization of $|.|'_r$. What is it ?

$$\begin{tikzcd} {|\cdot|_r} \arrow[r, squiggly] \arrow[d, squiggly] & {|\cdot|'_r} \arrow[d, dotted] \\ {|\cdot|_{r^-}} \arrow[r, dotted] & ? \end{tikzcd}$$

Remember that we also have the T-adic valuation $|.|_T : A \to \gamma^{\mathbb{Z}} \cup \{0\}$. We can consider the $\mod \ell$ T-adic valuation $|.|_{\ell,T}$ defined by $|f|_{\ell,T} = \gamma^{\mathrm{ord}_0(f \mod \ell)}$. This is a rank 1 valuation with kernel ℓA. It is a vertical specialization of the trivial valuation with kernel ℓA, which is a horizontal specialization of $|.|_r$ for $r = 1$ (it is equal to $|.|'_1$). So there exists a vertical specialization of $|.|_1$ that has $|.|_{\ell,T}$ as a horizontal specialization. Question : what is this valuation

?

$$\begin{tikzcd}
{|.|_1} \arrow[r, rightsquigarrow] \arrow[d, dotted] & {|.|'_1 = |.|_{(\ell),\mathrm{triv}}} \arrow[d, rightsquigarrow] \\
? \arrow[r, dotted] & {|.|_{\ell,T}}
\end{tikzcd}$$

I.4 Valuations with support conditions

Let A be commutative ring and J be an ideal of A such that $\mathrm{Spec}(A) - V(J)$ is quasi-compact. The goal of this section is to study a subset of $\mathrm{Spv}(A)$ that we will denote by $\mathrm{Spv}(A, J)$. This subset will turn out to be the set of valuations on A that either have support in $V(J)$, or that have support in $\mathrm{Spec}(A) - V(J)$ and have every proper horizontal specialization with support in $V(J)$.

Our reason for studying this subset is that, if A is a topological ring of the type used by Huber (in Huber's terminology, a f-adic ring), and if A^{00} is the set of topologically nilpotent elements of A, then the set $\mathrm{Cont}(A)$ of continuous valuations on A is a closed subset of $\mathrm{Spv}(A, A^{00} \cdot A)$; in fact, it is the subset of valuations x such that $|A^{00}|_x \subset \Gamma_{x,<1}$. So we will be able to deduce properties of $\mathrm{Cont}(A)$ from properties of the $\mathrm{Spv}(A, J)$. (Our reason for studying $\mathrm{Cont}(A)$ is that the space we are really interested in, the adic spectrum $\mathrm{Spa}(A, A^+)$ of an affinoid ring, is a pro-constructible subset of $\mathrm{Cont}(A)$.)

In this section, we will do the following things :

- Give a more explicit definition of $\mathrm{Spv}(A, J)$.
- Prove that $\mathrm{Spv}(A, J)$ is spectral, and give an explicit base of quasi-compact open subsets (they are of the form $\mathrm{Spv}(A, J) \cap U\left(\frac{f_1,\dots,f_n}{g}\right)$ for well-chosen sets $\{f_1, \dots, f_n\} \subset A$).
- Construct a continuous and spectral retraction from $\mathrm{Spv}(A)$ onto $\mathrm{Spv}(A, J)$.

From now on, we fix a ring A an ideal J of A such that $\mathrm{Spec}(A) - V(J)$ is quasi-compact; remember that this last condition is equivalent to the fact that $\sqrt{J}$ is equal to the radical of a finitely generated ideal of A (see remark I.2.2.2).

Here is a summary of the section. In the first subsection, we want to construct a (somewhat explicit) retraction $r : \mathrm{Spv}(A) \to \mathrm{Spv}(A, J)$ such that $r(x)$ is a horizontal specialization of x for evey $x \in \mathrm{Spv}(A)$. Let $x \in \mathrm{Spv}(A)$. There are three possibilities :

(1) If $\mathrm{supp}(x) \in V(J)$ (i.e. if $|J|_x = \{0\}$), then we take $r(x) = x$.

(2) If $\mathrm{supp}(x) \not\in V(J)$ and if x has no horizontal specialization with support in $V(J)$, we take $r(x) = x_{|c\Gamma_x}$ (the minimal horizontal specialization of x; note that this is $\mathrm{Spv}(A, J)$, because it has no proper horizontal specializations). This happens if and only if $|J|_x \cap c\Gamma_x \neq \varnothing$. (See proposition I.4.1.1.)

(3) If $\mathrm{supp}(x) \notin V(J)$ but x has at least one horizontal specialization with support in $V(J)$, we want to take for $r(x)$ the minimal horizontal specialization of x with support in $\mathrm{Spec}(A) - V(J)$. Proposition I.4.1.3 (and its corollaries) shows that such a specialization exists and gives a construction of a convex subgroup H_J of Γ_x such that $r(x) = x_{|H_J}$. This is where we need the hypothesis on $\sqrt{J}$: if $\sqrt{J} = \sqrt{(a_1, \dots, a_n)}$, then H_J is the convex subgroup of Γ_x generated by $\max_{1 \leq i \leq n}\{|a_i|_x\}$.

In the second subsection, we show that $\mathrm{Spv}(A, J)$ is spectral and give an explicit base of quasi-compact open subsets of its topology. This is a pretty straightforward application of Hochster's spectrality criterion (theorem I.2.5.1) and of the spectrality of $\mathrm{Spv}(A)$ (theorem I.2.6.1).

I.4.1 Supports of horizontal specializations

Proposition I.4.1.1. [4] *([9] Lemma 9.1.11 and Remark 9.1.12.) Let $x \in \mathrm{Spv}(A)$. Then the following are equivalent :*

(i) $|J|_x \cap c\Gamma_x \neq \varnothing$.

(ii) $|J|_x \cap \Gamma_{x,\geq 1} \neq \varnothing$.

(iii) Every horizontal specialization of x has support in $\mathrm{Spec}(A) - V(J)$.

(iv) The valuation $x_{|c\Gamma_x}$ has support in $\mathrm{Spec}(A) - V(J)$.

Proof. As $x_{|c\Gamma_x}$ is the horizontal specialization of x with minimal support (for the specialization relation in $\mathrm{Spec}(A)$), (iii) and (iv) are equivalent. Also, as $c\Gamma_x$ contains $|A|_x \cap \Gamma_{x,\geq 1}$, (ii) implies (i).

Let's prove that (i) implies (ii). Assume that (i) holds, and choose $a \in J$ such that $|a|_x \in c\Gamma_x$. If $|a|_x \geq 1$, then (ii) holds, so we assume that $|a|_x < 1$. By definition of $c\Gamma_x$, we can find $b, b' \in A$ such that $|b|_x, |b'|_x \geq 1$ and $|b|_x|b'|_x^{-1} \leq |a|_x < 1$. Then $1 \leq |b|_x \leq |ab'|_x$ and $ab' \in J$, so $|J|_x \cap \Gamma_{x,\geq 1} \neq \varnothing$.

Finally, we prove that (ii) and (iv) are equivalent. If $|J|_x \cap c\Gamma_x \neq \varnothing$, then there exists $a \in J$ such that $|a|_x \in c\Gamma_x$, and then we have $|a|_{x_{|c\Gamma_x}} = |a|_x \neq 0$, so $a \notin \mathrm{supp}(x_{|c\Gamma_x})$, i.e. $\mathrm{supp}(x_{|c\Gamma_x}) \notin V(J)$. Conversely, if $\mathrm{supp}(x_{|c\Gamma_x}) \notin V(J)$, then $J \not\subset \mathrm{supp}(x_{|c\Gamma_x})$, so there exists $a \in J$ such that $|a|_{x_{|c\Gamma_x}} \neq 0$, and then we have $|a|_x = |a|_{x_{|c\Gamma_x}} \in c\Gamma_x$.

∎

We now turn to the case where $|J|_x \cap c\Gamma_x = \varnothing$. The following definition will be useful.

Definition I.4.1.2. If $(\Gamma, \times)$ is a totally ordered abelian group and H is a subgroup of Γ, we say that $\gamma \in \Gamma \cup \{0\}$ is *cofinal for H* if for all $h \in H$ there exists $n \in \mathbb{N}$ such that $\gamma^n < h$.

[4] This result holds for an ideal J of A, without the extra condition on $\sqrt{J}$.

Proposition I.4.1.3. *([9] Proposition 9.1.13.) Let $x \in \mathrm{Spv}(A)$, and suppose that $|J|_x \cap c\Gamma_x = \varnothing$. Then :*

(i) The set of convex subgroups H of Γ_x such that $c\Gamma_x \subset H$ and that every element of $|J|_x$ is cofinal for H is nonempty, and it has a maximal element (for the inclusion), which we will denote by H_J.

(ii) If moreover $|J|_x \neq \{0\}$ (i.e. if $\mathrm{supp}(x) \in \mathrm{Spec}(A) - V(J)$), then $H_J \neq c\Gamma_x$, $|J|_x \cap H_J \neq \varnothing$, and H_J is contained in every convex subgroup H of Γ_x satisfying $|J|_x \cap H \neq \varnothing$. In particular, $x_{|H_J}$ has support in $\mathrm{Spec}(A) - V(J)$, and it is the minimal horizontal specialization of x with that property.

Proof. If $|J|_x = \{0\}$, then the elements of $|J|_x$ are cofinal for every subgroup of Γ_x, so every convex subgroup $H \supset c\Gamma_x$ satisfies the conditions of (i), and we can take $H_J = \Gamma_x$.

From now on, we assume that $|J|_x \neq \{0\}$, i.e. that $\mathrm{supp}(x) \notin V(J)$. By lemma I.4.1.4, none of the statements change if we replace J by its radical, so we may assume that J is finitely generated, say $J = (a_1, \dots, a_n)$. Let $\delta = \max\{|a_i|_x,\ 1 \leq i \leq n\}$. As $|J|_x \neq \{0\}$, we have $\delta \neq 0$, i.e. $\delta \in \Gamma_x$; also, as $|J|_x \cap c\Gamma_x = \varnothing$, we have $\delta \notin c\Gamma_x$ and $\delta < 1$. Let H_J be the convex subgroup of Γ_x generated by δ, that is,

$$H_J = \{\gamma \in \Gamma_x | \exists n \in \mathbb{N}, \delta^n \leq \gamma \leq \delta^{-n}\}.$$

Note that $\delta \in H_J$, and so we have $H_J \cap |J|_x \neq \varnothing$. We will show that this group H_J satisfies the properties of (i) and (ii).

First we show that H_J strictly contains $c\Gamma_x$. As convex subgroups of Γ_x are totally ordered by inclusion (see proposition I.1.3.4), we have $c\Gamma_x \subset H_J$ or $H_J \subset c\Gamma_x$. As $\delta \in H_J - c\Gamma_x$, the second case is impossible, so $c\Gamma_x \subsetneq H_J$.

Next we show that every element of $|J|_x$ is cofinal for H_J. Let $I = \{a \in A \mid |a|_x \text{ is cofinal for } H_J\}$. Note that $a_1, \dots, a_n \in I$. Indeed, for every $i \in \{1, \dots, n\}$, we have $|a_i|_x \leq \delta$, and δ is cofinal for H_J by definition of H_J. By lemma I.4.1.4, I is a radical ideal of A, and in particular $I \supset J$.

We have shown that H_J satisfies the properties of (i). Let's show that it is maximal for these properties. Let H be a subgroup of Γ_x such that every element of $|J|_x$ is cofinal for H. In particular, the generator δ of H_J (which is an element of $J|_x$) is cofinal for H. Let $\gamma \in H$. There exists $n \in \mathbb{N}$ such that $\delta^n < \gamma$; as $\delta < 1$, this means that $\delta^n < \gamma$ for every n big enough. As $\gamma^{-1} \in H$, a similar property holds for γ^{-1}, so we can find $n \in \mathbb{N}$ such that $\delta^n < \gamma$ and $\delta^n < \gamma^{-1}$, and then we have $\delta^n < \gamma < \delta^{-n}$, hence $\gamma \in H_J$.

Finally, we prove (ii). We have already shown that $c\Gamma_x \subsetneq H_J$, and we have $|J|_x \cap H_J \neq \varnothing$ by definition of H_J. Let H be a convex subgroup of Γ_x such that $|J|_x \cap H \neq \varnothing$. As before, using the fact that convex subgroups of Γ_x are totally ordered by inclusion, we see that $c\Gamma_x \subsetneq H$. To prove that $H_J \subset H$, it suffices to show that $\delta \in H$. Let $a \in J$ such that $|a|_x \in H$. We write $a = \sum_{i=1}^n b_i a_i$, with $b_1, \dots, b_n \in A$. Then $|a|_x \leq \max\{|b_i|_x |a_i|_x,\ 1 \leq i \leq n\}$, and we choose

$i \in \{1, \ldots, n\}$ such that $|a|_x \leq |b_i|_x |a_i|_x$; note that $|a_i|_x \leq \delta$ by definition of δ. If $|b_i|_x \leq 1$, then $H \ni |a|_x \leq |a_i|_x \leq \delta < 1$, so $\delta \in H$ because H is convex. If $|b_i|_x \geq 1$, then $|b_i|_x \in c\Gamma_x$, so $H \ni |b_i|_x^{-1}|a|_x \leq |a_i|_x \leq \delta < 1$, and again we deduce that $\delta \in H$.

The last sentence of (ii) follows from the fact that, for any convex subgroup $H \supset c\Gamma_x$ of Γ_x, the valuation $x_{|H}$ has support in $\mathrm{Spec}(A) - V(J)$ if and only if $|J|_x \cap H \neq \varnothing$. (See for example the end of the proof of proposition I.4.1.1.)

□

Lemma I.4.1.4. *Let $x \in \mathrm{Spv}(A)$.*

(i) For every subgroup H of Γ_x, we have : $|J|_x \cap H = \varnothing \Leftrightarrow |\sqrt{J}|_x \cap H = \varnothing$.

(ii) For every subgroup H of Γ_x, the following are equivalent :

(a) every element of $|J|_x$ is cofinal for H;

(b) every element of $|\sqrt{J}|_x$ is cofinal for H.

Proof. (i) We obviously have $|J|_x \cap H \subset |\sqrt{J}|_x \cap H$, so $|J|_x \cap H = \varnothing$ if $|\sqrt{J}|_x \cap H = \varnothing$. Conversely, suppose that $|\sqrt{J}|_x \cap H \neq \varnothing$, and let $a \in \sqrt{J}$ such that $|a|_x \in H$. There exists $N \geq 1$ such that $a^N \in J$, and then $|a|_x^N \in |J|_x \cap H$, so $|J|_x \cap H \neq \varnothing$.

(ii) Obviously (b) implies (a). If (a) holds, let $a \in \sqrt{J}$. Then $a^N \in I$ for some $N \geq 1$. Let $\gamma \in H$. Then there exists $n \in \mathbb{N}$ such that $|a^N|_x^n < \gamma$, i.e. $|a|_x^{nN} < \gamma$. So $|a|_x$ is cofinal for H.

□

Lemma I.4.1.5. *Let $x \in \mathrm{Spv}(A)$, and let H be a subgroup of Γ_x such that $c\Gamma_x \subsetneq H$. Then*

$$I = \{a \in A \mid |a|_x \textit{ is cofinal for } H\}$$

is a radical ideal of A.

Proof. Let $a, b \in I$. As $|a + b|_x \leq \max(|a|_x, |b|_x)$, and as both $|a|_x$ and $|b|_x$ is cofinal for H, so is $|a + b|_x$, hence $a + b \in I$.

Let $a \in I$ and $c \in A$. If $|c|_x \leq 1$, then $|ca|_x \leq |a|_x$, so $|ca|_x$ is cofinal for H, and $ca \in I$. Suppose that $|c|_x > 1$, then $|c|_x \in c\Gamma_x \subset H$. Let $\gamma \in H - c\Gamma_x$. As $c\Gamma_x$ is convex, γ is either smaller than all the elements of $c\Gamma_x$, or bigger than all the elements of $c\Gamma_x$; replacing γ by γ^{-1} if necessary, we may assume that we are in the first case. So $\delta < \gamma^{-1}$ for every $\delta \in c\Gamma_x$, and in particular $|c|_x^n < \gamma^{-1}$ for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $|a|_x^n < \gamma$. Then we have, for every $N \in \mathbb{N}$,

$$|ca|_x^{n+N} = |c|_x^{n+N} |a|_x^n |a|_x^N < \gamma^{-1} |a|_x^n |a|_x^N < |a|_x^N,$$

so $|ca|_x$ is cofinal for H (because $|a|_x$ is), and finally $ca \in I$.

So we have shown that I is an ideal. The fact that I is radical follows immediately from (ii) of lemma I.4.1.4.

□

Definition I.4.1.6. (i) Let $x \in \mathrm{Spv}(A)$. We define a convex subgroup $c\Gamma_x(J)$ of Γ_x by the following formula

$$c\Gamma_x(J) = \begin{cases} H_J & \text{if } |J|_x \cap c\Gamma_x = \varnothing; \\ c\Gamma_x & \text{if } |J|_x \cap c\Gamma_x \neq \varnothing. \end{cases}$$

(Where H_J is defined in (i) of proposition I.4.1.3.)

(ii) We define a map $r : \mathrm{Spv}(A) \to \mathrm{Spv}(A)$ by $r(x) = x_{|c\Gamma_x(J)}$.

Remark I.4.1.7. For $x \in \mathrm{Spv}(A)$, the subgroup $c\Gamma_x(J)$ of Γ_x depends only on $\sqrt{J}$. Hence the map r depends only on $\sqrt{J}$. (We actually showed this at the beginning of the proof of proposition I.4.1.3.)

These objects satisfy the following properties.

Corollary I.4.1.8. *([9] Proposition 9.2.2.) Let $x \in \mathrm{Spv}(A)$. Then :*

(i) $c\Gamma_x(J)$ is a convex subgroup of Γ_x, and $c\Gamma_x \subset c\Gamma_x(J)$.

(ii) $c\Gamma_x(J) = \Gamma_x$ if and only if every proper horizontal specialization of x has support in $V(J)$.

(iii) If $|J|_x \neq \{0\}$, then $c\Gamma_x(J)$ is minimal among all the convex subgroups H of Γ_x such that $H \supset c\Gamma_x$ and $H \cap |J|_x \neq \varnothing$.

(iv) If $|J|_x \cap c\Gamma_x = \varnothing$, then $c\Gamma_x(J)$ is maximal among all the convex subgroups H of Γ_x such that $H \supset c\Gamma_x$ and that every element of $|J|_x$ is cofinal for H.

(v) We have $r(x) = x$ if and only if $c\Gamma_x(J) = \Gamma_x$.

Proof. (i) This follows immediately from the definition of $c\Gamma_x(J)$.

(iii) This is point (ii) of proposition I.4.1.3.

(iv) This is point (i) of proposition I.4.1.3.

(ii) If $|J|_x = \{0\}$, then $c\Gamma_x(J) = H_J = \Gamma_x$ and x has support in $V(J)$ (hence all its specializations also do).

Suppose that $|J|_x \neq \{0\}$. If $c\Gamma_x(J) = \Gamma_x$, let y be a proper horizontal specialization of x, and write $y = x_{|H}$, with $c\Gamma_x \subset H \subsetneq \Gamma_x = c\Gamma_x(J)$; by (iii), we have $H \cap |J|_x = \varnothing$, hence $\mathrm{supp}(y) \notin V(J)$. Conversely, suppose that $\mathrm{supp}(y) \notin V(J)$ for every proper horizontal specialization y of x (in particular, $\mathrm{supp}(x) \notin V(J)$). Then $H \cap |J|_x = \varnothing$ for every proper convex subgroup $H \supset c\Gamma_x$ of Γ_x, so Γ_x is the only convex subgroup of Γ_x containing $c\Gamma_x$ and meeting $|J|_x$, hence $\Gamma_x = c\Gamma_x(J)$ by (iii).

(v) If $c\Gamma_x(J) = \Gamma_x$, then obviously $r(x) = x$. Conversely, suppose that $r(x) = x$. Then $|A|_x \subset c\Gamma_x(J)$. As $|A|_x$ generates Γ_x, this implies that $c\Gamma_x(J) = \Gamma_x$.

□

I.4.2 The subspace $\mathrm{Spv}(A, J)$

Definition I.4.2.1. We define a subset $\mathrm{Spv}(A, J)$ of $\mathrm{Spv}(A)$ by

$$\mathrm{Spv}(A, J) = \{x \in \mathrm{Spv}(A) \mid r(x) = x\} = \{x \in \mathrm{Spv}(A) \mid c\Gamma_x(J) = \Gamma_x\}.$$

Remark I.4.2.2. (1) We have $\mathrm{supp}^{-1}(V(J)) = \{x \in \mathrm{Spv}(A) \mid |J|_x = \{0\}\} \subset \mathrm{Spv}(A, J)$.

(2) If $J = A$, then $\mathrm{Spv}(A, A)$ is the set of $x \in \mathrm{Spv}(A)$ having no proper horizontal specializations. (This follows from corollary I.4.1.8(ii).)

(3) By remark I.4.1.7, we have $\mathrm{Spv}(A, J) = \mathrm{Spv}(A, \sqrt{J})$ (in other words, $\mathrm{Spv}(A, J)$ only depends on $\sqrt{J}$).

(4) By definition, the map r is a retraction from $\mathrm{Spv}(A)$ onto $\mathrm{Spv}(A, J)$ (i.e. $\mathrm{Im}(r) = \mathrm{Spv}(A, J)$ and $r^2 = r$).

Lemma I.4.2.3. *([9] Lemma 9.2.4) Let $a_1, \ldots, a_n \in J$ such that $\sqrt{(a_1, \ldots, a_n)} = \sqrt{J}$, and let $x \in \mathrm{Spv}(A)$. The following are equivalent :*

(i) $x \in \mathrm{Spv}(A, J)$.

(ii) $\Gamma_x = c\Gamma_x$*, or* $|a|_x$ *is cofinal for* Γ_x *for every* $a \in J$.

(iii) $\Gamma_x = c\Gamma_x$*, or* $|a_i|_x$ *is cofinal for* Γ_x *for every* $i \in \{1, \ldots, n\}$.

Proof. If (i) holds and $c\Gamma_x \neq \Gamma_x$, then $c\Gamma_x(J) \neq c\Gamma_x$, so $\Gamma_x = c\Gamma_x(J) = H_J$, and the second part of (ii) holds by proposition I.4.1.1(i). Suppose that (ii) holds. If $c\Gamma_x = \Gamma_x$, then $c\Gamma_x(J) = \Gamma_x$. If every element of $|J|_x$ is cofinal for Γ_x, then $|J|_x \cap \Gamma_{x,\geq 1} = \varnothing$, so $|J|_x \cap c\Gamma_x = \varnothing$ by proposition I.4.1.1. So, by corollary I.4.1.8(iv), $c\Gamma_x(J)$ is maximal among all the convex subgroup $H \supset c\Gamma_x$ of Γ_x such that every element of $|J|_x$ is cofinal for H; as Γ_x itself satisfies these properties by assumption, we have $c\Gamma_x(J) = \Gamma_x$.

As (ii) obviously implies (iii), it remains to show that (iii) implies (ii). Suppose that (iii) holds, and that $\Gamma_x \neq c\Gamma_x$. By lemma I.4.1.5, the set of elements of a such that $|a|_x$ is cofinal for Γ_x is a radical ideal of A, so, if it contains $a_1, \ldots, a_n$, it also contains J.

□

We come to the main theorem of this section.

Theorem I.4.2.4. *(Proposition 9.2.5 of [9], lemma 7.5 of [26].)*

(i) $\mathrm{Spv}(A, J)$ *is a spectral space (for the topology induced by the topology of* $\mathrm{Spv}(A)$*).*

(ii) A base of quasi-compact open subsets for the topology of $\mathrm{Spv}(A, J)$ *is given by the sets*

$$U_J\left(\frac{f_1, \dots, f_n}{g}\right) = \{x \in \mathrm{Spv}(A, J) \mid \forall i \in \{1, \dots, n\},\ |f_i|_x \leq |g|_x \neq 0\},$$

for nonempty finite sets $\{f_1, \dots, f_n\}$ *such that* $J \subset \sqrt{(f_1, \dots, f_n)}$*.*

(iii) The retraction $r : \mathrm{Spv}(A) \to \mathrm{Spv}(A, J)$ *is a continuous and spectral map.*

(iv) If $x \in \mathrm{Spv}(A)$ *has support in* $\mathrm{Spec}(A) - V(J)$*, so does* $r(x)$*.*

Note that the inclusion $\mathrm{Spv}(A, J) \to \mathrm{Spv}(A)$ is not spectral in general.

Proof. We may assume that J is finitely generated.

We proceed in several steps.

(1) Let $\mathscr{U}$ be the family of subsets defined in (ii). First, as the elements of $\mathscr{U}$ are the intersection with $\mathrm{Spv}(A, J)$ of open subsets of $\mathrm{Spv}(A)$, they are all open in $\mathrm{Spv}(A, J)$. Also, if $f_1, \dots, f_n \in A$ are such that $J \subset \sqrt{(f_1, \dots, f_n)}$ and $g \in A$, then clearly

$$U_J\left(\frac{f_1, \dots, f_n}{g}\right) = U_J\left(\frac{f_1, \dots, f_n, g}{g}\right).$$

(2) We show that $\mathscr{U}$ is stable by finite intersections. Let T, T' be two finite subsets of A such that $J \subset \sqrt{(T)}$ and $J \subset \sqrt{(T')}$, and let $g, g' \in A$. We want to show that $U_J\left(\frac{T}{g}\right) \cap U_J\left(\frac{T'}{g'}\right) \in \mathscr{U}$. By (1), we may assume that $g \in T$ and $g' \in T'$. Let $T'' = \{ab,\ a \in T,\ b \in T'\}$. Then $J \subset \sqrt{(T'')}$, and we have

$$U_J\left(\frac{T}{g}\right) \cap U_J\left(\frac{T'}{g'}\right) = U_J\left(\frac{T''}{gg'}\right).$$

(3) We show that $\mathscr{U}$ is a base of the topology of $\mathrm{Spv}(A, J)$. Let $x \in \mathrm{Spv}(A, J)$, and let U be an open neighborhood of x in $\mathrm{Spv}(A)$. We want to find an element of $\mathscr{U}$ that is contained in U. Choose $f_1, \dots, f_n, g \in A$ such that $x \in U\left(\frac{f_1, \dots, f_n}{g}\right) \subset U$.

Suppose that $\Gamma_x = c\Gamma_x$. Then there exists $a \in A$ such that $|g|_x^{-1} \leq |a|_x$, i.e., $|ag|_x \geq 1$, and then

$$x \in U_J\left(\frac{af_1, \dots, af_n, 1}{ag}\right) \subset U\left(\frac{f_1, \dots, f_n}{g}\right).$$

Suppose that $\Gamma_x \neq c\Gamma_x$. Let $a_1, \dots, a_m$ be a set of generators of J. By lemma I.4.2.3, there exists $r \in \mathbb{N}$ such that $|a_i|_x^r < |g|_x$ for every $i \in \{1, \dots, m\}$, and then

$$x \in U_J\left(\frac{f_1, \dots, f_n, a_1^r, \dots, a_m^r}{g}\right) \subset U\left(\frac{f_1, \dots, f_n}{g}\right).$$

(4) Let T be a finite subset of A such that $J \subset \sqrt{(T)}$, and let $g \in A$. We write $V = U_J\left(\frac{T}{g}\right)$ and $W = U\left(\frac{T}{g}\right)$ We claim that $r^{-1}(V) = W$.

We obviously have $V \subset W$. As every point of $r^{-1}(V)$ is a (horizontal) generization of a point of V and as W is open, this implies that $r^{-1}(V) \subset W$. Conversely, let $x \in W$; we want to show that $y := r(x) \in V$. If $|J|_x = \{0\}$, then $c\Gamma_x(J) = \Gamma_x$, so $r(x) = x \in V$. So we may assume that $|J|_x \neq \{0\}$, i.e. that $\mathrm{supp}(x) \notin V(J)$. Then $|J|_x \cap c\Gamma_x(J) \neq \varnothing$ (indeed, if $|J|_x \cap c\Gamma_x = \varnothing$, then $c\Gamma_x(J) = H_J$, and $H_J \cap |J|_x \neq \varnothing$ by definition); in particular, $\mathrm{supp}(y) \notin V(J)$, so (iv) holds. Let $H = c\Gamma_x(J)$, so that $y = x_{|H}$. Suppose that $g \in \mathrm{Ker}(y)$. As $\mathrm{Ker}(y)$ is x-convex, this implies that $a \in \mathrm{Ker}(y)$ for every $a \in T$, so $\mathrm{Ker}(y) \supset \sqrt{(T)} \supset J$, which contradicts the fact that $|J|_x \cap H \neq \varnothing$. So $g \notin \mathrm{Ker}(y)$, and in particular $|g|_y = |g|_x \neq 0$. As $|a|_y \leq |a|_x$ for every $a \in A$, we deduce that $y \in W$, hence that $y \in V = W \cap \mathrm{Spv}(A, J)$.

(5) Let $\mathscr{C}$ be the smallest collection of subsets of $\mathrm{Spv}(A, J)$ that contains $\mathscr{U}$ and is stable by finite unions, finite intersections and complements, and let X' be $\mathrm{Spv}(A, J)$ with the topology generated by $\mathscr{C}$. By (4), for every $Y \in \mathscr{C}$, $r^{-1}(Y)$ is a constructible subset of $\mathrm{Spv}(A)$. Hence $r : \mathrm{Spv}(A)_{\mathrm{cons}} \to X'$ is a continuous map. Since $\mathrm{Spv}(A)_{\mathrm{cons}}$ is quasi-compact (proposition I.2.4.1) and r is surjective, X' is also quasi-compact. By definition of the topology of X', every element of $\mathscr{U}$ is open and closed in X'. Also, $\mathrm{Spv}(A, J)$ is T_0, because it is a subspace of the T_0 space $\mathrm{Spv}(A)$, and $\mathscr{U}$ is a base of the topology of $\mathrm{Spv}(A, J)$ by (3). So Hochster's spectrality criterion (theorem I.2.5.1) implies that $\mathrm{Spv}(A, J)$ is spectral, that $X' = \mathrm{Spv}(A, J)_{\mathrm{cons}}$ and that $\mathscr{U}$ is a base of quasi-compact open subsets of $\mathrm{Spv}(A, J)$. This shows (i) and (ii), and (iii) follows from (4).

□

II Topological rings and continuous valuations

II.1 Topological rings

II.1.1 Definitions and first properties

Definition II.1.1.1. Let A be a topological ring.

(i) We say that A is *non-Archimedean* if 0 has a basis of neighborhoods consisting of subgroups of the underlying additive group of A.

(ii) We say that A is *adic* if there exists an ideal I of A such that $(I^n)_{n\geq 0}$ is a fundamental system of neighborhoods of 0 in A. In that case, we call the topology on A the *I-adic topology* and we say that I is an *ideal of definition*.

If M is a A-module, the topology on M for which $(I^n M)_{n\geq 0}$ is a fundamental system of neighborhoods of 0 is also called the *I-adic topology on M*.

(iii) We say that A if a *f-adic ring* (or a *Huber ring*) if there exists an open subring A_0 of A and a finitely generated ideal I of A_0 such that $(I^n)_{n\geq 0}$ is a fundamental system of neighborhoods of 0 in A_0. In that case, we say that A_0 is a *ring of definition* (for the topology of A), that I is an *ideal of definitin* of A_0 and that (A_0, I) is a *couple of definition*.

(iv) We say that A is a *Tate ring* if it is a f-adic ring and has a topologically nilpotent unit.

Note that A_0 and I in point (iii) are far from unique in general. (See for example corollary II.1.1.8.)

Remark II.1.1.2. (1) Note that we are not assuming that A is separated and/or complete for the I-adic topology.

(2) If I and J are two ideals of A, then J-adic topology on A is finer than the I-adic topology if and only if there exists a positive integer n such that $J^n \subset I$.

Definition II.1.1.3. Let A be a topological ring. A subset E of A is called *bounded* (in A) if for every neighborhood U of 0 in A there exists an open neighborhood V of 0 such that $ax \in U$ for every $a \in E$ and $x \in V$.

Remark II.1.1.4. Let A be a topological ring and A_0 be an open adic subring of A. Then A_0 is bounded in A. Indeed, let I be an ideal of A_0 such that the topology on A_0 is the I-adic topology. As A_0 is open in A, the family $(I^n)_{n\geq 0}$ is a fundamental system of neighborhoods of 0 in A. Let U be an open subset of A such that $0 \in U$. Then there exists $n \geq 0$ such that $I^n \subset U$, so, if we take $V = I^n$, then $ax \in U$ for every $a \in A_0$ and every $x \in V$.

Notation II.1.1.5. Let A be a ring. If U, $U_1, \dots, U_r$ are subsets of A (with $r \geq 2$) and n is a positive integer, we write $U_1 \cdot \ldots \cdot U_r$ for the set of finite sums of products $a_1 \dots a_r$ with $a_i \in U_i$, and $U(n) = \{a_1 \dots a_n,\ a_1, \dots, a_n \in U\}$. If $U_1 = \dots = U_r = U$, we write U^r instead of $U \cdot \ldots \cdot U$.

Proposition II.1.1.6. *(Proposition 6.1 of [26].) Let A be a topological ring. Then the following are equivalent :*

(i) A is a f-adic ring.

(ii) There exists an additive subgroup U of A and a finite subset T of U such that $(U^n)_{n\geq 1}$ is a fundamental system of neighborhoods of 0 in A and such that $T \cdot U = U^2 \subset U$.

The proof rests on the following lemma.

Lemma II.1.1.7. *(Lemma 6.2 of [26].) Let A be a topological ring and A_0 be a subring of A (with the subspace topology). The following are equivalent :*

(a) A is f-adic and A_0 is a ring of definition.

(b) A is f-adic, and A_0 is open in A and adic.

(c) A satisfies condition (ii) of proposition II.1.1.6, and A_0 is open in A and bounded.

Proof. Note that (a) trivially implies (b). Also, (b) implies (c) by remark II.1.1.4. So it remains to show that (c) implies (a).

Let U and T be as in condition (ii) of proposition II.1.1.6, and suppose that A_0 is open and bounded in A. Then there exists a positive integer r such that $U^r \subset A_0$, and we have $T(r) \subset U^r \subset A_0$. Let I be the ideal of A_0 generated by $T(r)$; note that I is finitely generated, because $T(r)$ is finite. For every $n \geq 1$, we have

$$I^n = T(nr)A_0 \supset T(nr)U^r = U^{r+nr},$$

so I^n is an open neighborhood of 0 in A_0. Let U be any open neighborhood of 0 in A_0. As A_0 is bounded and $(U^m)_{m\geq 1}$ is a fundamental system of neighborhoods of 0, there exists a positive integer m such that $U^m A_0 \subset U$, hence $I^m \subset U$. So $(I^n)_{n\geq 1}$ is a fundamental system of neighborhoods of 0 in A_0, i.e. the topology on A_0 is the I-adic topology. As A_0 is open in A, we are done.

□

Proof of proposition II.1.1.6. It is easy to see that (i) implies (ii) (take $U = I$ and take for T a finite system of generators of I).

Conversely, suppose that A satisfies (ii). Let $A_0 = \mathbb{Z} + U$. This is open in A because U is, and it is a subring because $U^n \subset U$ for every $n \geq 1$. If we show that A_0 is bounded, we will be done thanks to lemma II.1.1.7. Let V be a neighborhood of 0 in A. By assumption, there exists a positive integer m such that $U^m \subset V$. As U^m is open and $A_0 \cdot U^m = U^m + U^{m+1} \subset U^m \subset V$, we are done.

□

Corollary II.1.1.8. *(Corollary 6.4 of [26].) Let A be a f-adic ring.*

(i) If A_0 and A_1 are two rings of definition of A, then so are $A_0 \cap A_1$ and $A_0 \cdot A_1$.

(ii) Every open subring of A is f-adic.

(iii) If $B \subset C$ are subrings of A with B bounded and C open, then there exists a ring of definition A_0 of A such that $B \subset A_0 \subset C$.

(iv) A is adic if and only if it is bounded (in itself).

Proof. (i) If A_0 and A_1 are rings of definition, they are open and bounded, hence so are $A_0 \cap A_1$ and $A_0 \cdot A_1$, so $A_0 \cap A_1$ and $A_0 \cdot A_1$ are also rings of definition by lemma II.1.1.7.

(ii) Let B be an open subring of A, and let (A_0, I) be a couple of definition in A. Then there exists a positive integer n such that $I^n \subset B$, and $(B \cap A_0, I^n)$ is a couple of definition in B.

(iii) By (ii), we may assume that $C = A$. Let A_0 be a ring of definition of A. Then $A_0 \cdot B$ is open and bounded, hence is a ring of definition by lemma II.1.1.7.

(iv) If A is bounded, then it is a ring of definition of itself by lemma II.1.1.7, so it is adic. Conversely, if A is adic, then it is bounded by remark II.1.1.4.

□

Remark II.1.1.9. Suppose that A is f-adic and that (A_0, I_0) and (A_1, I_1) are couples of definition. Then we know that $A_0 \cdot A_1$ is a ring of definition, and it is easy to see that $I_0 \cdot A_1$ and $I_1 \cdot A_0$ both are ideals of definition in it. On the other hand, we also know that $A_0 \cap A_1$ is a ring of definition, but there is no reason for $I_0 \cap I_1$ to be an ideal of definition (because we don't know if it is finitely generated).

II.1.2 Boundedness

Recall that bounded subsets of topological rings are introduced in definition II.1.1.3.

Definition II.1.2.1. Let A be a topological ring. We say that a subset E of A is *power-bounded* if the set $\bigcup_{n\geq 1} E(n)$ is bounded, where (as in notation II.1.1.5 $E(n) = \{e_1 \dots e_n,\ e_1, \dots, e_n \in E\}$. We say that E is *topologicall nilpotent* if, for every neighborhood W of 0 in A, there exists a positive integer N such that $E(n) \subset W$ for $n \geq N$.

If E is a singleton $\{a\}$, we say that a is *power-bounded* (resp. *topologically nilpotent*) if E is. [1]

Notation II.1.2.2. Let A be a topological ring. We denote by A^0 the subset of its power-bounded elements, and by A^{00} the subset of its topologically nilpotent elements.

Lemma II.1.2.3. *(Remark 5.26 of [26].) Let A be an adic ring. If $x \in A$, the following are equivalent :*

(i) x is topologically nilpotent.

(ii) There exists an ideal of definition I such that the image of x in A/I is nilpotent.

(iii) There exists an ideal of definition I such that $x \in I$.

In particular, A^{00} is an open radical ideal of A and it is the union of all the ideals of definition. Moreover, A^{00} itself is an ideal of definition if and only if there exists an ideal of definition I such that the nilradical of A/I is nilpotent (and then this condition holds for all ideals of definition).

Proof. We first prove the equivalence of (i), (ii) and (iii), for $x \in A$. It is clear that (iii) implies (i). Suppose that (i) holds. Let I be an ideal of definition of A. As I is a neighborhood of 0, there exists $N \in \mathbb{N}$ such that $x^n \in I$ for $n \geq N$; so the image of x in A/I is nilpotent, and (ii) holds. Finally, suppose that (ii) holds, and let I be an ideal of definition such that $x + I$ is nilpotent in A/I. Let n be a positive integer such that $x^n \in I$, and let $J = I + xA$. Then J is an open ideal of A, $I \subset J$, and $J^n \subset I$, so the I-adic and J-adic topologies on A coincide, which means that J is an ideal of definition; this shows that (iii) holds.

We now prove the rest of the lemma. The fact that A^{00} is the union of all the ideals of definition follows from the equivalence of (i) and (iii); in particular, as ideals of definition are open, A^{00} is also open; it is clear that A^{00} is radical. If A^{00} is an ideal of definition, then there exists an ideal of definition I such that the nilradical of A/I is nilpotent (just take $I = A^{00}$, and observe that the nilradical of A/A^{00} is (0)). Conversely, suppose that there exists an ideal of definition I such that the nilradical of A/I is nilpotent. Then there exists a positive integer n such that $(A^{00})^n \subset I$, so the I-adic and A^{00}-adic topologies on A coincide, i.e., A^{00} is an ideal of definition.

□

Proposition II.1.2.4. *(Corollary 6.4 of [26].) Let A be a f-adic ring. Then the set of power-bounded elements A^0 is an open and integrally closed subring of A, and it is the union of all the rings of definition of A. Moreover, A^{00} is a radical ideal of A^0.*

[1] Note that this agrees with definition I.1.5.3.

Note that A^{00} is not an ideal of A in general.

Remark II.1.2.5. (See proposition 5.30 of [26], or adapt the proof below.) All the assertions remain true for a non-Archimedean ring A, except the fact that A^0 is open and the union of all the rings of definition of A.

Proof of the proposition. By (iii) of corollary II.1.1.8, every bounded subring of A is contained in a ring of definition; by (ii) of lemma II.1.2.6, this implies that every power-bounded element of A is contained in a ring of definition, so A^0 is contained in the union of all the rings of definition of A. Conversely, if A_0 is a ring of definition of A, then it is bounded by remark II.1.1.4, so all its elements are power-bounded by lemma II.1.2.6(ii), so $A_0 \subset A^0$. This proves that A^0 is the union of all the rings of definition of A, so it is a subring of A; as rings of definition are open, A^0 is open.

We show that A^0 is integrally closed in A. Let $a \in A$ be integral over A^0. By the previous paragraph, there exists a ring of definition A_0 such that a is integral over A_0; in particular, A_0 is bounded. So there exists $n \in \mathbb{N}$ such that $A_0[a] = A_0 + A_0 a + \ldots + A_0 a^n$, hence $A_0[a]$ is bounded, which implies that a is power-bounded, i.e., $a \in A^0$.

We prove that A^{00} is a radical ideal of A^0. Let $a, a' \in A^{00}$ and $b \in A^0$. We prove that $a + a', ab \in A^{00}$. Let U be a neighborhood of 0 in A; as A is non-Archimedean, we may assume that U is an additive subgroup of A. Let $V \subset U$ be a neighborhood of 0 such that $b^n V \subset U$ for every $n \geq 1$, and let N be a positive integer such that $a^n, (a')^n \in V$ for $n \geq N$. Then $(a+a')^n \in U$ for $n \geq 2N$ by the binomial formula, and $(ab)^n = b^n a^n \in b^n V \subset U$ for $n \geq N$. It remains to show that A^{00} is a radical ideal of A^0. Let $a \in A^0$, and suppose that we have $a^r \in A^{00}$ for some positive integer r. Let U be a neighborhood of 0 in A, and let V be a neighborhood of 0 such that $a^n V \subset U$ for every $n \geq 1$. Choose a positive integer N such that $(a^r)^n \in V$ for every $n \geq N$. Then we have $a^n \in U$ for every $n \geq rN$. This shows that $a \in A^{00}$.

□

Lemma II.1.2.6. *(Proposition 5.30 of [26].) Let A be a non-Archimedean topological ring.*

(i) Let T be a subset of A, and let T' be the subgroup generated by T. Then T' is bounded (resp. power-bounded, resp. topologically nilpotent) if and only if T is.

(ii) Let T be a subset of A. Then T is power-bounded if and only if the subring generated by T is bounded.

Proof. (i) We prove the non-obvious direction. Suppose that T is bounded. Let U be a neighborhood of 0, and let V be a neighborhood of 0 such that $ax \in U$ for every $a \in T$ and $x \in V$. As A is non-Archimedean, we may assume that U and V are additive subgroups of A, and then we have $T' \cdot V \subset U$. So T' is bounded. The proofs are similar for T power-bounded and T topologically nilpotent.

(ii) Let B be the subring generated by T. Then B is the subgroup generated $\{1\} \cup \bigcup_{n \geq 1} T(n)$, so, by (i), it is bounded if and only if $\{1\} \cup \bigcup_{n \geq 1} T(n)$ is bounded; we see easily that this equivalent to the fact that $\bigcup_{n\geq 1} T(n)$ is bounded, i.e. that T is power-bounded.

□

Note the following useful lemma characterizing open ideals of A.

Lemma II.1.2.7. *Let A be a f-adic ring and J be an ideal of A. Then J is open if and only if $A^{00} \subset \sqrt{J}$.*

Proof. Suppose that J is open. Then it is a neighborhood of 0 in A, so, for every $a \in A^{00}$, we have $a^n \in J$ for n big enough, which means that $A^{00} \subset \sqrt{J}$.

Conversely, suppose that $A^{00} \subset \sqrt{J}$. Let (A_0, I) be a couple of definition of A. Then $I \subset A^{00}$ by lemma II.1.2.3, so $I \subset \sqrt{J}$. Write $I = (a_1, \dots, a_r)$ with $a_1, \dots, a_r \in A_0$, and choose $N \in \mathbb{N}$ such that $a_1^N, \dots, a_r^N \in J$. Then J contains I^{rN}, and I^{rN} is open, so J is open.

□

II.1.3 Bounded sets and continuous maps

A continuous map of f-adic rings does not necessarily send bounded sets to bounded sets. We want to introduce a condition that will guarantee this property.

Definition II.1.3.1. Let A and B be f-adic rings. A morphism of rings $f : A \to B$ is called *adic* if there exist a couple of definition (A_0, I) of A and a ring of definition B_0 of B such that $f(A_0) \subset B_0$ and that $f(I)B_0$ is an ideal of definition of B.

Example II.1.3.2. (1) A continuous, surjective and open morphism of f-adic rings is adic.

(2) Let A be $\mathbb{Q}_\ell$ with the discrete topology and B be $\mathbb{Q}_\ell$ with the topology given by the ℓ-adic valuation. Then the identity $f : A \to B$ is continuous, and A is bounded in A, but $f(A)$ is not bounded in B. By proposition II.1.3.3, this implies that f is not adic.

(3) Let $f : \mathbb{Z}_\ell \to \mathbb{Z}_\ell[[X]]$ be the inclusion, where $\mathbb{Z}_\ell$ has the ℓ-adic topology and $\mathbb{Z}_\ell[[X]]$ has the (ℓ, X)-adic topology. Then f is not adic.

Proposition II.1.3.3. *Let A and B be f-adic rings and $f : A \to B$ be an adic morphism of rings. Then :*

(i) f is continuous.

(ii) If A_0 and B_0 are rings of definition of A and B such that $f(A_0) \subset B_0$, then, for every ideal of definition I of A_0, the ideal $f(I)B_0$ is an ideal of definition of B.

(iii) For every bounded subset E of A, the set $f(E)$ is bounded in B.

Proof. Let (A_0, I) and B_0 be as in definition II.1.3.1, and write $J = f(I)B_0$.

(i) For every $n \geq 1$, we have $f^{-1}(J^n) = f^{-1}(f(I^n)B_0) \supset I^n$. So f is continuous.

(ii) Let (A'_0, I') be a couple of definition of A and B'_0 be a ring of definition of B such that $f(A'_0) \subset B'_0$. We want to show that $J := f(I')B'_0$ is an ideal of definition of B'_0.

(iii) Let U be a neighborhood of 0 in B. We may assume that $U = J^n$ for some $n \geq 1$. Let V be a neighborhood of 0 in A such that $ax \in I^n$ for every $a \in E$ and every $x \in V$; we may assume that $V = I^m$ for some $m \geq 1$. Then $f(E) \cdot f(I)^m = f(E \cdot I^m) \subset f(I^n) \subset J^n$, so $f(E)J^m \subset J^n$.

□

Proposition II.1.3.4. *(Proposition 6.25 of [26].) Let A and B be f-adic rings and $f : A \to B$ be a continuous morphism of rings. Suppose that A is a Tate ring. Then B is a Tate ring, f is adic, and, for every ring of definition B_0 of B, we have $f(A) \cdot B_0 = B$.*

Proof. Let B_0 be a ring of definition of B. By lemma II.1.3.5, we can find a ring of definition A_0 of A such that $f(A_0) \subset B_0$. Let $\varpi \in A$ be a topologically nilpotent unit. After replacing replacing ϖ by some ϖ^r, we may assume that $\varpi \in A_0$. As f is a continuous morphism of rings, $f(\varpi) \in B_0$ is a topologically nilpotent unit of B. In particular, B is a Tate ring. By proposition II.2.5.2, $I := \varpi A_0$ is an ideal of definition of A_0, and $f(I)B_0 = f(\varpi)B_0$ is an ideal of definition of B_0. So f is adic. Also, by the same lemma, we have $B = B_0[f(\varpi)^{-1}]$, so $B = f(A) \cdot B$.

□

Lemma II.1.3.5. *Let $f : A \to B$ be a continuous morphism of f-adic rings. For every ring of definition B_0 of B, there exists a ring of definition A_0 of A such that $f(A_0) \subset B_0$.*

Proof. Let A'_0 and B_0 be rings of definition of A and B. Then $f^{-1}(B_0)$ is an open subring of A and $A'_0 \cap f^{-1}(B_0)$ is a bounded subring, so, by corollary II.1.1.8(iii), there exists a ring of definition A_0 of A such that $A'_0 \cap f^{-1}(B_0) \subset A_0 \subset f^{-1}(B_0)$, and we clearly have $f(A_0) \subset B_0$.

□

II.1.4 Examples

Some of these will be particular cases of constructions that we will see later, I'll add references later for the others.

1. Any ring is a topological ring for the discrete topology. It is f-adic but not Tate.

2. The rings $\mathbb{R}$ and $\mathbb{C}$ (with the usual topology) are topological rings. They are not non-Archimedean. A subset of $\mathbb{C}$ (or $\mathbb{R}$) is bounded if and only if it is bounded in the usual sense. We have
$$\mathbb{C}^0 = \{z \in \mathbb{C} \mid |z| \leq 1\}$$

and

$$\mathbb{C}^{00} = \{z \in \mathbb{C} \mid |z| < 1\}.$$

Note that these are not additive subgroups.

3. $\mathbb{Z}_\ell$ with the ℓ-adic topology is an adic topological ring (any power of $\ell\mathbb{Z}_\ell$ is an ideal of definition), and $\mathbb{Q}_\ell$ with the ℓ-adic topology is an f-adic topological ring (with $\mathbb{Z}_\ell$ as a ring of definition). We have $\mathbb{Q}_\ell^0 = \mathbb{Z}_\ell$ and $\mathbb{Q}_\ell^{00} = \ell\mathbb{Z}_\ell$.

4. $\overline{\mathbb{Q}}_\ell$ (with the topology coming from the unique extension of the ℓ-adic valuation on $\mathbb{Q}_\ell$) is a f-adic ring. We have $\overline{\mathbb{Q}}_\ell^0 = \overline{\mathbb{Z}}_\ell$, the integral closure of $\mathbb{Z}_\ell$ in $\overline{\mathbb{Q}}_\ell$; this is a ring of definition, and it is adic with ideal of definition $\ell\overline{\mathbb{Z}}_\ell$ (for example). Also, $\overline{\mathbb{Q}}_\ell^{00}$ is the maximal ideal of $\overline{\mathbb{Z}}_\ell$; it is not an ideal of definition, because it is equal to its own square (also, it is not finitely generated).

   Note that $\mathbb{Z} + \ell\overline{\mathbb{Z}}_\ell$ is also a ring of definition of $\overline{\mathbb{Q}}_\ell$, because it is open and bounded.

5. Let A be a Noetherian ring and I an ideal of A. The I-adic topology on A is Hausdorff if I is contained in the Jacobson radical of A (for example if A is local and $I \neq A$), or if A is a domain and $I \neq A$.

6. Let A be a ring with the topology induced by a rank 1 valuation $|.|$, and let Γ be the valuation group of $|.|$. Then :

   - a subset E of A is bounded if and only if there exists $\gamma \in \Gamma$ such that $E \subset \{a \in A \mid |a| \leq \gamma\}$.
   - $A^0 = \{a \in A \mid |a| \leq 1\}$;
   - $A^{00} = \{a \in A \mid |a| < 1\}$.

   These statements are all false if $|.|$ has rank ≥ 2. Indeed, in that case Γ has a proper convex subgroup Δ, and an element of A that has valuation in Δ cannot be topologically nilpotent, because there exists $\gamma \in \Gamma$ such that $\gamma < \delta$ for every $\delta \in \Delta$.

7. Let $A = k((t))((u))$, with the valuation $|.|$ corresponding to the valuation subring $R := \{f = \sum_{n\geq 0} a_n u^n \mid a_n \in k((t)) \text{ and } a_0 \in k[[t]]\}$. More explicitly, take $\Gamma = \mathbb{Z} \times \mathbb{Z}$; if $f = \sum_{n\geq r} a_n u^n \in A$ with $a_n \in k((t))$ and $a_r \neq 0$, and if $a_r = \sum_{m\geq s} b_m t^m$ with $b_m \in k$ and $b_s \neq 0$, then $|f| = (-r, -s)$. [2] Note that the valuation topology on A coincides with the topology defined by the u-adic valuation (see example I.1.5.5). So $t \in A$ is not topologically nilpotent, even though it has valuation $< (0, 0)$.

8. Let k be a field with the topology induced by a rank 1 valuation $|.|$. Then $k^0 = \{x \in k \mid |x| \leq 1\}$ is a ring of definition of k (often called the ring of integers of k); it is a local ring with maximal ideal $k^{00} = \{x \in k \mid |x| < 1\}$. A nonzero topologically nilpotent in k, i.e. an element of $k^{00} - \{0\}$ is called a *pseudo-uniformizer*. k is a Tate ring; a ring of definition is k^0, and any pseudo-uniformizer generates an ideal of definition

[2] We put the minus signs so that $|.|$ will be a multiplicative valuation; see remark I.1.1.11.

of k^0.

The *Tate algebra* in n indeterminates over k is the subalgebra $T_n = T_{n,k} = l\langle X_1, \dots, X_n\rangle$ of $k[[X_1, \dots, X_n]]$ whose elements are power series $f = \sum_{\nu\in\mathbb{N}^n} a_\nu X^\nu$ such that $|a_\nu| \to 0$ as $\nu_1 + \dots + \nu_n \to +\infty$ (where $\nu = (\nu_1, \dots, \nu_n)$). The *Gauss norm* $\|.\|$ on T_n is defined by $\|\sum_\nu a_\nu X^\nu\| = \sup_{\nu\in\mathbb{N}^n} |a_\nu|$. With the topology induced by this norm, T_n is a Tate ring, and contains $k[X_1, \dots, X_n]$ as a dense subring. We have $T_n^0 = k^0\langle X_1, \dots, X_n\rangle := T_n \cap k^0[[X_1, \dots, X_n]]$, and this is a ring of definition. Any pseudo-uniformizer of k generates an ideal of definition of T_n^0. Also, $T_n^{00} = k^{00}\langle X_1, \dots, X_n\rangle$.

9. We keep the notation of the previous example, and we suppose that k is complete.[3]

   The algebra $k\langle X_1, \dots, X_n\rangle$ is a Banach k-algebra, it is Noetherian and all its ideals are closed. Alors, if $B^n(\overline{k})$ is the closed unit ball in $\overline{k}^n$ (i.e. $B^n(\overline{k}) = \{(x_1, \dots, x_n) \in \overline{k}^n \mid \forall i \in \{1, \dots, n\},\ |x_i| \leq 1\}$, where we denote by $|.|$ the uniaue extension of the valuation $|.|$ to $\overline{k}$), then a formal power series $f \in k[[X_1, \dots, X_n]]$ is in $k\langle X_1, \dots, X_n\rangle$ if and only if, for every $x \in B^n(\overline{k})$, the series $f(x)$ converges in the completion of $\overline{k}$.

   An *affinoid k-algebra* is a quotient of an algebra $k\langle X_1, \dots, X_n\rangle$. These are also called *topologically finitely generated k-algebras*. . Such an algebra is also a Tate ring, it is Noetherian, and all its ideals are closed. Also, if A is an affinoid k-algebra, then A^0 is a ring of definition of A (i.e. bounded) if and only if A is reduced. (The fact that A is reduced if A^0 is bounded is proved in remark IV.1.1.3, and the converse follows immediately from Theorem 1 of section 6.2.4 of [3].)

   For example, $A = \mathbb{Q}_\ell[T]/(T^2)$ is an affinoid $\mathbb{Q}_\ell$-algebra (note that A is also equal to $\mathbb{Q}_\ell\langle T\rangle/(T^2)$), but $A^0 = \mathbb{Z}_\ell \oplus \mathbb{Q}_\ell T$ is not bounded (neither is $A^{00} = \ell\mathbb{Z}_\ell \oplus \mathbb{Q}_\ell T$).

10. Another important class of examples are perfectoid algebras. These are always Tate rings. For example, $\mathbb{C}_\ell$ is a perfectoid field, and the ℓ-adic completion of $\bigcup_{n\geq 1} \mathbb{C}_\ell\langle X^{1/\ell^n}\rangle$ is a perfectoid $\mathbb{C}_\ell$-algebra.

11. $A = \mathbb{Z}_\ell[[T]]$ with the (ℓ, T)-adic topology is an adic and f-adic ring, but it is not a Tate ring. This type of f-adic ring is also very useful, because their adic spectra will be formal schemes.

[3] Some of the results are still true under weaker conditions.

II.2 Continuous valuations

II.2.1 Definition

Definition II.2.1.1. Let A be a topological ring. We say that a valuation $|.|$ on A is *continuous* if the valuation topology on A is coarser than its original topology.

In other words, a valuation $|.| : A \to \Gamma \cup \{0\}$ is continuous on A if and only if, for every $\gamma \in \Gamma$, the set $\{a \in A \mid |a| < \gamma\}$ is an open subset of A. By remark I.1.5.2(3), if the value group of $|.|$ is not trivial, then $|.|$ is continuous if and only if, for every $\gamma \in \Gamma$, the set $\{a \in A \mid |a| \leq \gamma\}$ is an open subset of A.

Definition II.2.1.2. If A is a topological ring, the set of continuous valuations of A is called the *continuous valuation spectrum of* A and denoted by $\mathrm{Cont}(A)$. We see it as a topological space with the topology induced by the topology of $\mathrm{Spv}(A)$.

II.2.2 Spectrality of the continuous valuation spectrum

Theorem II.2.2.1. *(Theorem 7.10 of [26].) Let A be a f-adic ring. Then*

$$\mathrm{Cont}(A) = \{x \in \mathrm{Spv}(A, A^{00} \cdot A) \mid \forall a \in A^{00},\ |a|_x < 1\}.$$

If I is an ideal of definition of a ring of definition of A, we also have

$$\mathrm{Cont}(A) = \{x \in \mathrm{Spv}(A, I \cdot A) \mid \forall a \in I,\ |a|_x < 1\}.$$

Proof. By lemma II.2.2.2 and remark I.4.2.2(3), we have

$$\{x \in \mathrm{Spv}(A, A^{00} \cdot A) \mid \forall a \in A^{00},\ |a|_x < 1\} = \{x \in \mathrm{Spv}(A, I \cdot A) \mid \forall a \in I,\ |a|_x < 1\}.$$

Let $x \in \mathrm{Cont}(A)$. Let $a \in A^{00}$ and let $\gamma \in \Gamma$. As a is topologically nilpotent and $\{b \in A \mid |b|_x < \gamma\}$ is a neighborhood of 0 in A, there exists a positive integer n such that $|a^n|_x = |a|_x^n < \gamma$. This shows that every element of $|A^{00}|_x$ is cofinal for Γ_x, and so, by lemma I.4.2.3, $x \in \mathrm{Spv}(A, A^{00} \cdot A)$. Also, we have $|a|_x < 1$ for every $a \in A^{00}$ by lemma I.1.5.7.

Conversely, let $x \in \{x \in \mathrm{Spv}(A, A^{00} \cdot A) \mid \forall a \in A^{00},\ |a|_x < 1\}$. If $c\Gamma_x \neq \Gamma_x$, then $|a|_x$ is cofinal for Γ_x for every $a \in A^{00}$ by lemma I.4.2.3. Suppose that $c\Gamma_x = \Gamma_x$. Let $a \in A^{00}$ and $\gamma \in \Gamma_x$. As $\Gamma_x = c\Gamma_x$, there exists $b \in A$ such that $|b|_x \neq 0$ and $|b|_x^{-1} \leq \gamma$. We can find $n \geq 1$ such that $ba^n \in A^{00}$ (because $A^{00}$is open in A), and then $|ba^n|_x < 1$ by the assumption on x, so $|a|_x^n < |b|_x^{-1} \leq \gamma$. So we see again that every element of $|A^{00}|_x$ is cofinal for Γ_x.

We finally show that x is continuous. Write $I = (a_1, \ldots, a_r)$ with $a_1, \ldots, a_r \in A_0$, and set $\delta = \max\{|a_i|_x,\ 1 \leq i \leq n\}$. Let $\gamma \in \Gamma_x$. By the previous paragraph, there exists $n \geq 1$ such that

$\delta^n < \gamma$. As $|a|_x < 1$ for every $a \in I$, this implies that $|a|_x < \delta^n < \gamma$ for every $a \in I^n \cdot I = I^{n+1}$, so the open neighborhood I^{n+1} of 0 is included in $\{a \in A \mid |a|_x < \gamma\}$. This implies that $|.|_x$ is continuous.

□

Lemma II.2.2.2. *Let A be a f-adic ring, and let I be an ideal of definition of a ring of definition of A. Then $\sqrt{A^{00} \cdot A} = \sqrt{I \cdot A}$.*

Also, if $x \in \mathrm{Spv}(A)$, the following are equivalent :

(a) $|a|_x < 1$ *for every* $a \in I$;

(b) $|a|_x < 1$ *for every* $a \in A^{00}$.

Proof. We prove both statements at the same time. Let $x \in \mathrm{Spv}(A)$. Let A_0 be the ring of definition in which I is an ideal of definition. We have $A^{00} \cap A_0 \supset I$ by lemma II.1.2.3. So (b) implies (a), and also $A^{00} \cdot A \supset I \cdot A$, hence $\sqrt{A^{00} \cdot A} \supset \sqrt{I \cdot A}$. Conversely, if $a \in A^{00}$, then there exists $r \geq 1$ such that $a^r \in I$. This shows that (a) implies (b), and also that $A^{00} \cdot A \subset \sqrt{I \cdot A}$, and hence that $\sqrt{A^{00} \cdot A} \subset \sqrt{I \cdot A}$.

□

Corollary II.2.2.3. *For every f-adic ring A, the continuous valuation spectrum $\mathrm{Cont}(A)$ is a spectral space.*

Moreover, the sets

$$U_{\mathrm{cont}}\left(\frac{f_1, \dots, f_n}{g}\right) = \{x \in \mathrm{Cont}(A) \mid \forall i \in \{1, \dots, n\},\ |f_i|_x \leq |g|_x \neq 0\},$$

for $f_1, \dots, f_n, g \in A$ such that $A^{00} \subset \sqrt{(f_1, \dots, f_n)}$, form a base of quasi-compact open subsets of $\mathrm{Cont}(A)$.

Note that the condition $A^{00} \subset \sqrt{(f_1, \dots, f_n)}$ is equivalent to saying that the ideal $(f_1, \dots, f_n)$ is open. (See lemma II.1.2.7.)

Proof. Let $J = A^{00} \cdot A$. We have

$$\mathrm{Cont}(A) = \mathrm{Spv}(A, J) - \bigcup_{g \in A^{00}} U_J\left(\frac{1}{g}\right),$$

so $\mathrm{Cont}(A)$ is a closed subset of $\mathrm{Spv}(A, J)$. As $\sqrt{J}$ is the radical of the ideal of A generated by an ideal of definition of a subring of definition, which is finitely generated, the theorem follows from theorem I.4.2.4 and corollary I.2.4.3.

□

II.2.3 Specializations

The subset $\mathrm{Cont}(A)$ of $\mathrm{Spv}(A)$ is not stable by general specializations (or generizations), but we do have the following result.

Proposition II.2.3.1. *Let A be a f-adic ring and $x \in \mathrm{Cont}(A)$. Then :*

(i) Every horizontal specialization of x is continuous.

(ii) Every vertical generization y of x such that $\Gamma_y \neq \{1\}$ is continuous.

Remark II.2.3.2. If $\Gamma_y = \{1\}$, then y is the trivial valuation with support $\wp_y$, and it is continuous if and only if $\wp_y$ is an open ideal of A. This might or might not be the case in general.

Proof of the proposition. (i) Let y be a horizontal specialization of x. Then we have $|a|_y \leq |a|_x$ for every $a \in A$ (by the formula of definition I.3.3.3). So, if $\gamma \in \Gamma$, the subgroup $\{a \in A \mid |a|_y < \gamma\}$ of $(A,+)$ contains the open subgroup $\{a \in A \mid |a|_x < \gamma\}$; this implies that $\{a \in A \mid |a|_y < \gamma\}$ is open.

(ii) Let y be a vertical generization of x. By proposition I.3.2.3(ii), there exists a convex subgroup H of Γ_x such that $\Gamma_y = \Gamma_x/H$ and $|.|_y$ is the composition of $|.|_x$ and of the quotient map $\pi : \Gamma_x \cup \{0\} \to \Gamma_y \cup \{0\}$. Let $\gamma \in \Gamma_x$. Then

$$\{a \in A \mid |a|_x \leq \gamma\} \subset \{a \in A \mid |a|_y = \pi(|a|_x) \leq \pi(\gamma)\}.$$

As both these sets are additive subgroups of A, and as the smaller one is open, the bigger one is also open. By remark I.1.5.2(3), if $\Gamma_y \neq \{1\}$, this implies that y is a continuous valuation.

□

II.2.4 Analytic points

In this section, A is a f-adic ring.

Definition II.2.4.1. A point $x \in \mathrm{Cont}(A)$ is called *analytic* if $\wp_x$ is not open.

We denote by $\mathrm{Cont}(A)_{\mathrm{an}}$ the subset of analytic points in $\mathrm{Cont}(A)$.

Proposition II.2.4.2. *Let $x \in \mathrm{Cont}(A)$. The following are equivalent :*

(i) x is analytic.

(ii) $|A^{00}|_x \neq \{0\}$.

(iii) For every couple of definition (A_0, I) of A, we have $|I|_x \neq 0$.

(iv) There exists a couple of definition (A_0, I) of A such that we have $|I|_x \neq 0$.

Proof. (ii) implies (iii) because A^{00} contains every ideal of definition of a ring of definition of A, and (iii) obvisouly implies (iv). Suppose that (i) holds. As $\wp_x$ is not open, it cannot contain the open additive subgroup A^{00} of A, so (ii) holds. Suppose that (iv) holds, and let $a \in I$ such that $|a|_x \neq 0$. Then $|a^n|_x \neq 0$ for every $n \geq 1$, so $\wp_x$ does not contain any of the sets I^n, so it cannot be open.

$\square$

Remark II.2.4.3. Remember that

$$\mathrm{Cont}(A) = \{x \in \mathrm{Spv}(A, A^{00} \cdot A) \mid \forall a \in A^{00},\ |a|_x < 1\}$$

by theorem II.2.2.1. So, by proposition II.2.4.2, $\mathrm{Cont}(A)_{\mathrm{an}}$ is the set of points of x of $\mathrm{Spv}(A)$ such that

- the support of x is not in $V(A^{00} \cdot A)$;
- every proper horizontal specialization of x has support in $V(A^{00} \cdot A)$;
- $|a|_x < 1$ for every $a \in A^{00}$.

Remark II.2.4.4. It is easy to show (see lemma 6.6 of [26]) that an ideal $\mathfrak{a}$ of A is open if and only $\sqrt{\mathfrak{a}}$ contains the ideal $A^{00} \cdot A$.

Corollary II.2.4.5. *Let I be an ideal of definition of a ring of definition of A, and let $f_1, \dots, f_n$ be generators of I. Then*

$$\mathrm{Cont}(A)_{\mathrm{an}} = \bigcup_{i=1}^{n} U_{\mathrm{cont}}\left(\frac{f_1, \dots, f_n}{f_i}\right).$$

In particular, $\mathrm{Cont}(A)_{\mathrm{an}}$ *is a quasi-compact open subset of* $\mathrm{Cont}(A)$.

Proof. Let $x \in \mathrm{Cont}(A)$. For $i \in \{1, \dots, n\}$, we have $x \in U_{\mathrm{cont}}\left(\frac{f_1,\dots,f_n}{f_i}\right)$ if and only if $0 \neq |f_i|_x = \max_{1 \leq j \leq n}\{|f_j|_x\}$. So x is in $\bigcup_{i=1}^n U_{\mathrm{cont}}\left(\frac{f_1,\dots,f_n}{f_i}\right)$ if and only if there exists $i \in \{1, \dots, n\}$ such that $|f_i|_x \neq 0$. This is equivalent to the fact that $|I|_x \neq \{0\}$, so it is equivalent to $x \in \mathrm{Cont}(A)_{\mathrm{an}}$ by proposition II.2.4.2.

$\square$

Proposition II.2.4.6. *Let $x \in \mathrm{Cont}(A)_{\mathrm{an}}$. Then x has rank ≥ 1, and the valuation $|.|_x$ on $K(x)$ is microbial.*

Proof. If $x \in \mathrm{Cont}(A)$ and $\Gamma_x = \{1\}$, then $\wp_x = \{a \in A \mid |a|_x < 1\}$ is open, so x cannot be analytic. So analytic points of $\mathrm{Cont}(A)$ must have positive rank.

We prove the second statement. Let $x \in \mathrm{Cont}(A)_{\mathrm{an}}$. By proposition II.2.4.2, there exists $a \in A^{00}$ such that $|a|_x \neq 0$. So the image of a in $\mathrm{Frac}(A/\wp_x) \subset K(x)$ is topologically nilpotent (for the valuation topology on $K(x)$) and invertible. By theorem I.1.5.4, this implies that the valuation $|.|_x$ on $K(x)$ is microbial.

□

Finally, we show that specializations among analytic points are particularly simple.

Proposition II.2.4.7. *Every specialization inside* $\mathrm{Cont}(A)_{\mathrm{an}}$ *is vertical.*

In particular, if A is a Tate ring, then every specialization in $\mathrm{Cont}(A)$ *is vertical.*

The second part follows from the first and from remark II.2.5.7.

Proof. Let $x, y \in \mathrm{Cont}(A)_{\mathrm{an}}$ such that y is a specialization of x. Let y' be the vertical generization of y (in $\mathrm{Spv}(A)$). given by theorem I.3.4.3(ii). If we were in case (b) of theorem I.3.4.3(ii), then we would have $\wp_y \supset \wp_{x_{|\{1\}}} \supset \{a \in A \mid |a|_x < 1\}$; but this would imply that $\wp_y$ is open and contradict the condition $y \in \mathrm{Cont}(A)_{\mathrm{an}}$. So we are in case (a) of theorem I.3.4.3(ii), which means that y' is a horizontal specialization of x. In particular, y' is continuous by proposition II.2.3.1(i), and it is analytic because $\wp_{y'} = \wp_y$ is not open.

Let $H \supset c\Gamma_x$ be a convex subgroup of Γ_x such that $y' = x_{|H}$. We want to show that $H = \Gamma_x$, which will imply that y is a vertical specialization of $x = y'$. Suppose that $H \neq \Gamma_x$. Then we can find $\gamma \in \Gamma_x - H$ such that $\gamma < 1$. Let $a \in A$ such that $|a|_x < \gamma$. Then $|a|_x \notin H$ (otherwise γ would be in H, because H is convex), so $|a|_{y'} = 0$. This shows that $\wp_{y'}$ contains the open subset $\{a \in A \mid |a|_x < \gamma\}$ and contradicts the fact that y' is analytic.

□

Corollary II.2.4.8. *For every $x \in \mathrm{Cont}(A)_{\mathrm{an}}$, the set of generizations of x in $\mathrm{Cont}(A)_{\mathrm{an}}$ is totally ordered and admits an order-preserving bijection with the set of proper convex subgroups of Γ_x. In particular, the continuous rank 1 valuations are exactly the maximal points of $\mathrm{Cont}(A)_{\mathrm{an}}$ for the order given by specialization (i.e. the $x \in \mathrm{Cont}(A)_{\mathrm{an}}$ such that $\overline{\{x\}}$ is an irreducible component of $\mathrm{Cont}(A)_{\mathrm{an}}$).*

Moreover, every $x \in \mathrm{Cont}(A)_{\mathrm{an}}$ has a unique rank 1 generization, which is its maximal generization.

Proof. All generizations of x in $\mathrm{Cont}(A)_{\mathrm{an}}$ is vertical by proposition II.2.4.7. As every nontrivial vertical generization of x is continuous by proposition II.2.3.1(ii), and as vertical generizations of x have the same support as x, we see that generizations of x in $\mathrm{Cont}(A)_{\mathrm{an}}$ are exactly the nontrivial vertical generizations of x. By proposition I.3.2.3, these are in order-preserving bijection with proper convex subgroups of Γ_x, and in order-reversing bijection with the nonzero prime ideals of R_x. In particular, x is maximal if and only Γ_x has no nonzero proper convex subgroups, i.e. if and only if Γ_x has height 1.

To finish the proof, we must show that x has a maximal vertical generization. By proposition II.2.4.6, the valuation $|.|_x$ on $K(x)$ is microbial, so R_x has a prime ideal of height 1. As the ideals of R_x are totally ordered by inclusion, this implies that R_x has a unique prime ideal of height 1. The corresponding generization of x is the maximal generization of x in $\mathrm{Cont}(A)$, and also the unique rank 1 generization of x in $\mathrm{Cont}(A)$.

□

II.2.5 Tate rings

In this section, we gather some results that are specific to Tate rings. In general, Tate rings behave more nicely than general f-adic rings.

Definition II.2.5.1. If A is a Tate ring, a topologically nilpotent unit of A is called a *pseudo-uniformizer*.

Proposition II.2.5.2. *Let A be a Tate ring, let A_0 be a ring of definition of A, and let ϖ be a topologically nilpotent unit of A. Suppose that $\varpi \in A_0$. Then ϖA_0 is an ideal of definition of A_0, and $A = A_0[\varpi^{-1}]$.*

Proof. Let $I = \varpi A_0$. As ϖ is a unit in A, multiplication by ϖ is continuous, so $I^n = \varpi^n A_0$ is an open subset of A_0 for every $n \geq 1$. So we just need to show that every neighborhood of 0 contains some I^n. Let U be an open neighborhood of 0 in A. As A_0 is bounded, there exists an open neighborhood of 0 such that $ax \in U$ for every $a \in A_0$ and every $x \in V$. As ϖ is topologically nilpotent, there exists a positive integer n such that $\varpi^n \in V$. Then we have $I^n = \varpi^n A_0 \subset U$.

We show the last statement. Let $a \in A$. As multiplication by a is continuous and ϖ is topologically nilpotent, 0 is a limit of the sequence $(a\varpi^n)_{n\geq 0}$. So there exists $N \in \mathbb{N}$ such that $a\varpi^n \in A_0$ for every $n \geq N$, and $a \in A_0[\varpi^{-1}]$.

□

Conversely, if a f-adic ring A has a pair of definition (A_0, I) with I principal, then any generator of I is topologically nilpotent in A, so A is a Tate ring if I is generated by a unit of A.

Remark II.2.5.3. (Proposition 6.2.6 of [9].) If A_0 is an adic ring with a principal ideal of definition $I := \varpi A_0$, then $A := A_0[\varpi^{-1}]$ is a Tate ring for the topology for which the image of A_0 is a ring of definition and the image of I an ideal of definition.

Proof. Let $u : A_0 \to A$ be the canonical map. We check that the subgroups $(u(\varpi^n A_0))_{n\geq 0}$ of A satisfy the conditions of lemma II.3.3.8, hence are a fundamental system of neighborhood for a topological ring structure on A. Conditions (a) and (c) of the lemma are clear. Let $a \in A$ and $n \in \mathbb{N}$. We write $a = u(b)u(\varpi)^{-r}$, with $b \in A_0$ and $r \in \mathbb{N}$. Then we have $au(\varpi^{n+r}A_0) \subset u(\varpi^n A_0)$.

□

Example II.2.5.4. Take $A_0 = \mathbb{Z}[X]$ with the X-adic topology. Then $A = A_0[X^{-1}]$ is a Tate ring. Note that this ring does not contain a field.

Corollary II.2.5.5. *Let A be a Tate ring, let A_0 be a ring of definition of A, and let ϖ be a topologically nilpotent unit of A. Then a subset E of A is bounded if and only there exists $n \in \mathbb{Z}$ such that $E \subset \varpi^n A_0$.*

Remember proposition II.1.3.4.

Proposition II.2.5.6. *Let $f : A \to B$ be a continuous ring morphism between f-adic ring. If A is a Tate ring, then so is B, and f is adic.*

Remark II.2.5.7. If A is a Tate ring, then $\mathrm{Cont}_{\mathrm{an}}(A) = \mathrm{Cont}(A)$.

This follows immediately from the definition and from the next lemma.

Lemma II.2.5.8. *Let A be a Tate ring. Then the only open ideal of A is A itself.*

Proof. Let ϖ be a topologically nilpotent unit of A, and let J be an open ideal of A. Then there exists $r \geq 1$ such that $\varpi^r \in J$, so J contains a unit and $J = A$.

□

So proposition II.2.4.7 implies that, if A is a Tate ring, every specialization in $\mathrm{Cont}(A)$ is vertical. Also, corollary II.2.4.8 says that each point of $\mathrm{Cont}(A)$ has a unique rank 1 generization in $\mathrm{Cont}(A)$, which is also its maximal generization in $\mathrm{Cont}(A)$.

Definition II.2.5.9. A *non-Archimedean field* is a topological field K whose topology is given by a rank 1 valuation.

Note that we do not assume that K is complete.

The following result is an immediate consequence of theorem I.1.5.4.

Corollary II.2.5.10. *Let K be a topological field whose topology is given by a valuation. Then K is a non-Archimedean field if and only if it is a Tate ring.*

Corollary II.2.5.11. *Let K be a non-Archimedean field, and let $|.|$ be a rank 1 valuation defining its topology.*

(i) The ring K^0 is local with maximal ideal K^{00}.

(ii) Let $x \in \mathrm{Cont}(A)$. Then $|.|_x$ is microbial and its valuation topology coincides with the original topology of K; moreover, we have $K^{00} \subset R_x \subset K^0$.

(iii) If R is a valuation subring of K such that $K^{00} \subset R \subset K^0$, then the corresponding valuation is continuous.

In other words, we get a canonical bijection $\mathrm{Cont}(K) \xrightarrow{\sim} \mathrm{Spv}(K^0/K^{00})$.

Proof. (ii) It suffices to show that every element of $K^0 - K^{00}$ is invertible in K^0. This follows immediately from the fact that $K^0 = \{a \in K \mid |a| \leq 1\}$, $K^{00} = \{a \in K \mid |a| < 1\}$ and $(K^0)^\times = \{a \in K \mid |a| = 1\}$.

(ii) Any topologically nilpotent unit of K is also topologically nilpotent for the valuation topology, so $|.|_x$ is microbial by theorem I.1.5.4, that is, it admits a rank 1 generization $y \in \mathrm{Cont}(K)$ such that $|.|_x$ and $|.|_y$ define the same topology. We obviously have $K^{00} \subset R_x \subset R_y$, so it suffices to prove the result for y. In that case, the statement is equivalent to the fact that $|.|_y$ and $|.|$ are equivalent. As y has rank 1, we must have $|a|_y \leq 1$ for every power-bounded element $a \in K$, so we have $K^0 \subset R_y$. This means that y is a vertical generization of $|.|$; but, as $|.|_y$ and $|.|$ have the same rank, they must be equivalent.

(iii) The ring K^0 is maximal among all proper valuation subrings of K containing R, so, by corollary I.1.4.4, its maximal ideal K^{00} is a height 1 prime ideal of R. Now theorem I.1.5.4 implies that $|.|_R$ and $|.|$ define the same topology on K, and in particular $|.|_R$ is continuous.

□

II.3 Constructions with f-adic rings

II.3.1 Completions

Remember the following definitions from general topology.

Definition II.3.1.1. Let X be a set. A *filter of subsets of* X (or *filter on* X) is a nonempty family $\mathscr{F}$ of subsets of X that is stable by finite intersection and such that, if $A \in \mathscr{F}$ and $B \supset A$, then $B \in \mathscr{F}$.

If X is a topological space and $x \in X$, we say that x is a *limit* of the filter $\mathscr{F}$ if every neighborhood of x is in $\mathscr{F}$.

In a metric space (or more generally in a first-countable topological space), we can characterize many topological properties using sequences. This does not work in a general topological space, but we can use filters (or their cousins nets) instead, and everything adapts quite easily.

Remark II.3.1.2. A topological space X is Hausdorff if and only if every filter on X has at most one limit.

Remark II.3.1.3. If $(x_n)_{n\geq 0}$ is a sequence in X, the associated filter is

$$\mathscr{F} := \{E \in A \mid \exists n \in \mathbb{N},\ x_m \in E \text{ for } m \geq n\}.$$

Then x is a limit of $(x_n)_{n\geq 0}$ if and only if it is a limit of $\mathscr{F}$.

Remark II.3.1.4. Any f-adic ring is a first-countable topological space, so we are being somewhat pedantic here.

Definition II.3.1.5. Let A be a commutative topological group (for example the additive subgroup of a topological ring).

(i) We say that a filter $\mathscr{F}$ on A is a *Cauchy filter* if, for every neighborhood U of 0, there exists $E \in \mathscr{F}$ such that $x - y \in U$ for all $x, y \in E$.

(ii) We say that A is *complete* if it is Hausdorff and if every Cauchy filter on A has a limit.

Remark II.3.1.6. Note that Bourbaki does not require complete commutative topological groups to be Hausdorff.

Remark II.3.1.7. If $(x_n)_{n\geq 0}$ is a sequence in A, we say that it is a *Cauchy sequence* if, for every neighborhood U of 0, there exists $n \in \mathbb{N}$ such that $x_m - x_p \in U$ for all $m, p \geq n$. So $(x_n)_{n\geq 0}$ is a Cauchy sequence if and only if the associated filter (see remark II.3.1.3) is a Cauchy filter.

Completions of abelian topological groups and topological rings always exist, and they satisfy the obvious universal property. (See for example [6] Chapitre III §3 Nº5 Théorème 2 and §6 Nº5 Théorème 1.) For f-adic rings, these completions take a more explicit form, thanks to the following theorem.

Theorem II.3.1.8. *Let A_0 be a ring and I be an ideal of A_0. For every A_0-module M, we set $\widehat{M} = \varprojlim_{n\geq 0} M/I^n M$ and denote the obvious map $M \to \widehat{M}$ by f.*

Suppose that the ideal I is finitely generated*. Then :*

(i) The abelian group $\widehat{M}$ is Hausdorff and complete for the $f(I)\widehat{M}$-adic topology.

(ii) For every $n \geq 0$, the map f induces an isomorphism $M/I^n M \xrightarrow{\sim} \widehat{M}/f(I)^n \widehat{M}$.

(iii) If A_0 is Noetherian, then $\widehat{A}_0$ is a flat A_0-algebra.

Points (i) and (ii) are proved in [25, Lemma 05GG], and point (iii) in [25, Lemma 00MB]. Note that (i) and (ii) are false in general if I is not finitely generated (see [25, Section 05JA]), and that (iii) is false in general if A_0 is not Noetherian, even for a finitely generated ideal (see [25, Example 0BNU] and [25, Section 0AL8]).

In particular, if A_0 is an adic ring and I is a finitely generated ideal of definition of A_0, then $\widehat{A}_0$ is the completion of A_0. It is easy to see that $\widehat{A}_0$ does not depend on the choice of the ideal of definition (see remark II.1.1.2(2)).

Corollary II.3.1.9. *Let A be a f-adic ring, let (A_0, I) be a couple of definition of A, and set $\widehat{A} = \varprojlim_{n\geq 0} A/I^n$ (as an abelian group; note that we take the quotient of A by the ideal I^n of A_0 and not by the ideal that I^n generates in A). Then :*

(i) *The canonical map $\widehat{A}_0 \to \widehat{A}$ is injective, and the square*

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & \widehat{A}_0 \arrow[d] \\ A \arrow[r] & \widehat{A} \end{tikzcd}$$

is cartesian.

(ii) *If we put the unique topology on $\widehat{A}$ for which $\widehat{A}_0$ is an open subgroup, then the abelian topological group $\widehat{A}$ is complete.*

(iii) *There is a unique ring structure on $\widehat{A}$ that makes the canonical map $A \to \widehat{A}$ continuous, and $\widehat{A}$ is a topological ring.*

(iv) *The ring $\widehat{A}$ is f-adic and $(\widehat{A}_0, I\widehat{A}_0)$ is a couple of definition of $\widehat{A}$. Moreover, the canonical map $A \to \widehat{A}$ is adic.*

(v) *The canonical map $\widehat{A}_0 \otimes_{A_0} A \to \widehat{A}$ is an isomorphism.*

(vi) *If A_0 is Noetherian, then $\widehat{A}$ is a flat A-algebra.*

(vii) *If A_0 is Noetherian and A is a finitely generated A_0-algebra, then $\widehat{A}$ is Noetherian.*

It is easy to see that $\widehat{A}$ does not depend on the choice of the pair of definition (A_0, I).

Proof. (i) For every $n \geq 0$, the map $A_0/I^n \to A/I^n$ is injective. As projective limits are left exact, the morphism $\widehat{A}_0 \to \widehat{A}$ is injective.

Let $i : A \to \widehat{A}$ be the canonical map. To prove the second statement, we must show that $i(A) \cap \widehat{A}_0 = i(A_0)$. The fact that $i(A_0) \subset i(A) \cap \widehat{A}_0$ is obvious. Conversely, let $a \in A$ such that $i(a) \in \widehat{A}_0$. Then, for every $n \geq 1$, there exists $b_n \in A_0$ such that $a \in b_n + I^n$; in other words, a is in the closure of A_0 in A. As A_0 is an open subgroup of A, it is also closed, so $a \in A_0$.

(ii) It is easy to see that $\widehat{A}$ is Hausdorff (because 0 has a Hausdorff neighborhood, i.e. $\widehat{A}_0$). Let $\mathscr{F}$ be a Cauchy filter on $\widehat{A}$. As $\widehat{A}_0$ is a neighborhood of 0 in $\widehat{A}$, there exists $F \in \mathscr{F}$ such that $x - y \in \widehat{A}_0$ for all $x, y \in F$. Let $x_0 \in F$, and define a family $\mathscr{F}_0$ of subsets of $\widehat{A}_0$ by : $G \in \mathscr{F}_0 \Leftrightarrow x_0 + G \in \mathscr{F}$. Then $\mathscr{F}_0$ is not empty because $F - x_0 \in \mathscr{F}_0$, and it is clearly a Cauchy filter on $\widehat{A}_0$. As $\widehat{A}_0$ is complete, $\mathscr{F}_0$ has a limit a, and then $a + x_0$ is a limit of $\mathscr{F}$.

(iii) As A_0 is dense in $\widehat{A}_0$, A is dense in $\widehat{A}$. This implies uniqueness. The existence of the product on $\widehat{A}$ follows from Théorème 1 of [6] Chapitre III §6 №5.

(iv) $\widehat{A}_0$ is an open subring of $\widehat{A}$ by (i), and it has the $I\widehat{A}_0$-adic topology by theorem II.3.1.8, so $\widehat{A}$ is f-adic. The fact that the map $A \to \widehat{A}$ follows immediately from the definition.

(v) This is lemma 1.6 of [14]. Let us explain the proof. Consider the commutative diagram (where all the maps are the obvious ones) :

$$\begin{tikzcd}
A_0 \arrow[rr, "i"] \arrow[dd, hook] & & \widehat{A}_0 \arrow[dl, "g"] \arrow[dd, hook] \\
& \widehat{A}_0 \otimes_{A_0} A \arrow[dr, "j"] & \\
A \arrow[ur, "f"] \arrow[rr, "i"'] & & \widehat{A}
\end{tikzcd}$$

We want to show that the map j is an isomorphism. We will do this by constructing an inverse.

First note that, by proposition II.3.2.1, $\widehat{A}_0 \otimes_{A_0} A$ has a natural structure of f-adic ring and that f, g and j are continuous. Indeed, the maps $A_0 \to \widehat{A}_0$ and $A_0 \to A$ are adic.

We now turn to the construction of an inverse h of $j : \widehat{A}_0 \otimes_{A_0} A \to \widehat{A}$. Let $a \in \widehat{A}$. As $i(A)$ is dense in $\widehat{A}$, we can find $a_0 \in \widehat{A}_0$ and $b \in A$ such that $a = a_0 + i(b)$, and we want to set $h(a) = f(b) + g(a_0)$. We have to check that this does not depend on the choices. Suppose that $a = a_0 + i(b) = a'_0 + i(b')$, with $a_0, a'_0 \in \widehat{A}_0$ and $b, b' \in A$. Then $a_0 - a'_0 = i(b' - b)$, so $b' - b \in A_0$ by (i), and

$$f(b) + g(a_0) = f(b') + f(b - b') + g(a_0) = f(b') + g(i(b - b')) + g(a_0) = f(b') + g(a'_0).$$

So h is well-defined, and it is clear that h is additive and that $f = h \circ i$ and $g = h_{|\widehat{A}_0}$. The last property implies that h is continuous in a neighborhood of 0, hence that h is continuous. As $f = h \circ i$, i has dense image and f is a morphism of rings, h is also a morphism of rings. By construction of h, we have $h \circ j = \mathrm{id}$. Also, if $a \in f(A)$ or $a \in g(\widehat{A}_0)$, then $j(h(a)) = a$, also by construction of h; as h is a morphism or rings, this implies that $j \circ h = \mathrm{id}$.

(vi) and (vii) These follow immediately from (v) (and from theorem II.3.1.8(iii) for (vi)).

□

Definition II.3.1.10. If A is a f-adic ring, the f-adic ring $\widehat{A}$ defined in corollary II.3.1.9 is called the *completion* of A.

Lemma II.3.1.11. *Let A be an abelian topological group and $i : A \to \widehat{A}$ be its completion. Then there is a bijection between the set of open subgroups of A and the set of open subgroups of $\widehat{A}$; it sends an open subgroup G of A to $\overline{i(G)} = \widehat{G}$, and its inverse sends an open subgroup H of $\widehat{A}$ to $i^{-1}(H)$.*

Proof. If Y is a subset of A, then $\overline{i(Y)}$ is canonically isomorphic to the completion of Y by [6] Chapitre II §3 Nº9 corollaire 1 de la proposition 18.

Note also that $\operatorname{Ker} i = \overline{\{0\}}$ and that $i(A)$ is dense in $\widehat{A}$ (for example by [6] Chapitre II §3 N°7 proposition 12). If G is an open subgroup of A, it is also closed, hence contains $\operatorname{Ker} i$, and so we have $G = i^{-1}(\overline{i(G)})$. Conversely, if H is an open subgroup of $\widehat{A}$, then $H \cap i(A)$ is dense in H, so H is the closure of $i(i^{-1}(H))$.

□

Proposition II.3.1.12. *Let A be a f-adic ring.*

(i) We have $\widehat{A}^0 = \widehat{A^0}$ and $\widehat{A}^{00} = \widehat{A^{00}}$. (That is, A^0 (resp. A^{00} is sent to $\widehat{A}^0$ (resp. $\widehat{A}^{00}$) by the bijection of lemma II.3.1.11.)

(ii) If we have open sugroups of G and H of A and $\widehat{A}$ that correspond to each other by the bijection of lemma II.3.1.11, then G is a ring of definition of A if and only if H is a ring of definition of $\widehat{A}$.

(iii) The map $i : A \to \widehat{A}$ induces a bijective map $\operatorname{Cont}(\widehat{A}) \to \operatorname{Cont}(A)$.

In fact, the bijection of (iii) is a homeomorphism, and this is not so obvious and quite important. We will prove this later, after we introduce adic spectra. (See corollary III.4.2.2.)

Proof. (i) Let $i : A \to \widehat{A}$ be the obvious map. By lemma II.3.1.11, a subset E of A is bounded if and only if $i(E)$ is bounded in $\widehat{A}$, and an element $x \in A$ is topologically nilpotent if and only if $i(x) \in \widehat{A}$ is topologically nilpotent. In particular, $i^{-1}(\widehat{A}^0)$ (resp. $i^{-1}(\widehat{A}^{00})$) is contained in A^0 (resp. A^{00}), so $\widehat{A}^0 \subset \widehat{A^0}$ (resp. $\widehat{A}^{00} \subset \widehat{A^{00}}$).

Conversely, as $i(A^0) \subset \widehat{A}^0 \subset \widehat{A^0}$, $\widehat{A}^0$ is dense in $\widehat{A^0}$; but $\widehat{A}^0$ is open in $\widehat{A}$, hence closed, so $\widehat{A}^0 = \widehat{A^0}$. The case of $\widehat{A}^{00}$ is similar.

(ii) As $i(G)$ is dense in H and $i(A)$ is a subring of $\widehat{A}$, H is a subring if and only if G is a subring. Also, we have seen in (i) that G is bounded if $i(G)$ is, so G is a ring of definition if H is. Conversely, suppose that G is a bounded subring of A. Then $i(G)$ is bounded. As $\widehat{A}$ has a fundamental system of open bounded neighborhoods of 0 (for example the powers of an ideal of definition of a ring of definition), we can find an open bounded subgroup $U \subset H$. We have $H = i(G) + U$ because $i(G)$ is dense in U, and so H is bounded.

(iii) Let $|.| : A \to \Gamma \cup \{0\}$ be a continuous valuation, and let $\mathscr{F}$ be a Cauchy filter on A. We claim that :

(a) either, for every $\gamma \in \Gamma$, there exists $F \in \mathscr{F}$ such that $|a| < \gamma$ for every $a \in F$;

(b) otherwise there exists $F \in \mathscr{F}$ such that $|.|$ is constant on F.

Indeed, suppose that (a) does not hold. Then there exists $\gamma_0 \in \Gamma$ such that, for every $F \in \mathscr{F}$, there exists $a \in F$ with $|a| \geq \gamma_0$. As $|.|$ is continuous, the set $\{a \in A \mid |a| < \gamma_0\}$ is an open neighborhood of 0. So, as $\mathscr{F}$ is a Cauchy filter, there exists $F \in \mathscr{F}$ such that $|a - b| < \gamma_0$ for all $a, b \in F$. Fix $a_0 \in F$ such that $|a_0| \geq \gamma_0$. Then, for every $a \in F$, we have $|a - a_0| < \gamma_0 \leq |a_0|$, so the strong triangle inequality implies that $|a| = |a|_0$.

Let $i : A \to \widehat{A}$ be the canonical map. Applying the result of the previous paragraph to $\widehat{A}$ and using the fact that $i(A)$ is dense in $\widehat{A}$, we see that a continuous valuation on $\widehat{A}$ is uniquely determined by its restriction to $i(A)$, so the map $\mathrm{Cont}(\widehat{A}) \to \mathrm{Cont}(A)$ is injective.

We now show that $\mathrm{Cont}(\widehat{A}) \to \mathrm{Cont}(A)$ is surjective. Let $|.| : A \to \Gamma \cup \{0\}$ be a continuous valuation. Let $a \in \operatorname{Ker} i$, and let $\mathscr{F}$ be the filter of neighborhoods of a in A; this is clearly a Cauchy filter. If it satisfies condition (a) above, then $|a| < \gamma$ for every $\gamma \in \Gamma$, so $|a| = 0$. Otherwise, $\mathscr{F}$ satisfies condition (b), so there exists $F \in \mathscr{F}$ such that $|.|$ is constant on F; but we have $a \in F$ and $0 \in F$ (because $i(A)$ is the maximal Hausdorff quotient of A), so again $|a| = 0$. This shows that $|.|$ factors through $i(A)$, so we may assume that i is injective.

We now extend $|.|$ to a map $|.|' : \widehat{A} \to \Gamma \cup \{0\}$. Let $a \in \widehat{A}$. Then there exists a Cauchy filter $\mathscr{F}$ on A that converges to a. If $\mathscr{F}$ satifies condition (a) above, then we set $|a|' = 0$. Otherwise, we choose $F \in \mathscr{F}$ such that $|.|$ is constant on F, and we set $|a|' = |b|$, for any $b \in F$. It is easy to check that this does not depend on the choices and defines a valuation on $\widehat{A}$. We finally show that $|.|'$ is continuous. Let $\gamma \in \Gamma$, and let $G = \{a \in A \mid |a| < \gamma\}$. This is an open subgroup of A and, by the definition of $|.|'$, its closure in $\widehat{A}$ is contained in the group $\{a \in \widehat{A} \mid |a|' < \gamma\}$; so $\{a \in \widehat{A} \mid |a|' < \gamma\}$ is open.

□

II.3.2 Tensor products

We have to be a bit careful with tensor products of f-adic rings, because they don't make sense in general. This corresponds to the fact that fiber products of adic spaces don't always exist, and has a simple geometric explanation, that is given in the remark below. However, if we assume that all the maps are adic, then there is no problem; in particular, we can always define tensor products of Tate rings.

Proposition II.3.2.1. *(See theorem 5.5.4 of [9].) Let $f : A \to B$ and $f : A \to C$ be two* adic *morphisms of f-adic rings. Choose a couple of definition (A_0, I) of A and rings of definition B_0 and C_0 of B and C such that $f(A_0) \subset B_0$ and $g(A_0) \subset C_0$. Let D_0 be the image of $B_0 \otimes_{A_0} C_0$ in $D := B \otimes_A C$, and let J be the ideal of D_0 generated by the image of I.*

We put the J-adic topology on D_0 and equip D with the unique structure of topological group that makes D_0 an open subgroup. Then D is a f-adic ring with couple of definition (D_0, J), and the obvious ring morphisms $u : B \to D$ and $v : C \to D$ are continuous and adic.

Moreover, for every non-Archimedean topological ring D' and every pair of continuous ring morphisms $(u' : B \to D', v' : C \to D')$ such that $u \circ f = v \circ g$, there exists a unique continuous ring morphism $\varphi : D \to D'$ such that $u' = \varphi \circ u$ and $v' = \varphi \circ v$. If D' if f-adic and u' and v' are adic maps, then φ is also an adic map.

Proof. The ideal J of D_0 if of finite type because I is, so the only thing we need to prove to get the first statement is that D is a topological ring. By [6] Chapitre III §6 №3 Remarque, it suffices to prove that the multiplication of D is continuous in a neighborhood of 0 and that the map $x \longmapsto ax$ is continuous for every $a \in D$. The first statement follows from the fact that D_0 is a topological ring, and it suffices to prove the second statement for pure tensors. So let $b \in B$ and $c \in C$, and let U be a neighborhood of 0 in D. We may assume that U is of the form J^r, for some $r \geq 0$. Let $s \geq 0$ such that $b(f(I)B_0)^s \subset (f(I)B_0)^r$ and $c(g(I)C_0)^s \subset (g(I)C_0)^r$. If $x \in (f(I)B_0)^s \otimes_{A_0} (g(I)C_0)^s$, then we have $(b \otimes c)x \in J^r$; as $J^{2s} = (f(I)B_0)^s \otimes_{A_0} (g(I)C_0)^s$, this shows that $(b \otimes c)J^{2s} \subset J^r$.

We now turn to the second statement. By the usual property of the tensor product, there exists a unique morphism of rings $\varphi : D \to D'$ such that $u' = \varphi \circ u$ and $v' = \varphi \circ v$. We want to show that φ is continuous. Let U be an open subgroup of D'. As u' and v' are continuous, there exists $r \geq 1$ such that $u'^{-1}(U) \supset (f(I)B_0)^r$ and $v'^{-1}(U) \supset (f(I)C_0)^r$. Then $\varphi^{-1}(U) \supset J^r$, so $\varphi^{-1}(U)$ is open. The last statement is easy.

□

Example II.3.2.2. (See example 5.5.5 of [9].) Here is an example where things don't work. Take $A = A_0 = \mathbb{Z}_\ell$ with the ℓ-adic topology, $B = B_0 = \mathbb{Z}_\ell[[X]]$ with the (ℓ, X)-adic topology and $C = \mathbb{Q}_\ell \supset C_0 = \mathbb{Z}_\ell$ with the ℓ-adic topology. Note that the obvious map $A \to C$ is adic, but the obvious map $A \to B$ is not. We have $D = \mathbb{Z}_\ell[[X]][\ell^{-1}]$ and $D_0 = \mathbb{Z}_\ell[[X]]$.

Suppose that there is an ideal J of D_0 such that D is f-adic with couple of definition (D_0, J) and such that the canonical maps $B \to D$ and $C \to D$ are continuous. Suppose that $1 \notin J$. In particular, the map $B_0 \to D_0$ is continuous, so J must contain a power of the ideal (ℓ, X). As $\mathbb{Z}_\ell[[X]]$ is a local ring with maximal ideal (ℓ, X), we must also have $J \subset (\ell, X)$. So the topology on D_0 is the (ℓ, X)-adic topology. But then there is no structure of topological ring on D that makes D_0 an open subring; indeed, ℓ is invertible in D, so multiplication by ℓ would have to be a homeomorphism, and this not possible because ℓD_0 is not an open subset of D_0 (for example because it contains no power of X even though X is topologically nilpotent for the (ℓ, X)-adic topology).

So we must have $1 \in J$, which means that the only open subsets of D_0 are $\varnothing$ and D_0. But then there can be no topological ring structure on D that makes D_0 an open subring, for the same reason as before : ℓ is invertible in D, so ℓD_0 would have to be an open subset of D_0.

Remark II.3.2.3. We keep the notation of example II.3.2.2. We temporarily write $\mathrm{Spa}(R)$ for "the affinoid adic space of R", even though that is not quite correct because we need an extra piece of data to define this space. The geometric interpretation of the previous example is that $\mathrm{Spa}(\mathbb{Z}_\ell[[X]][\ell^{-1}])$, if it made sense, would be the fiber product $\mathrm{Spa}(\mathbb{Z}_\ell[[X]]) \times_{\mathrm{Spa}(\mathbb{Z}_\ell)} \mathrm{Spa}(\mathbb{Q}_\ell)$, that is, the generci fiber of of the "formal affine line" $\mathrm{Spa}(\mathbb{Z}_\ell[[X]])$. But this generic fiber should be the open unit disc, which is not an affinoid space.

II.3.3 Rings of polynomials

If A is a f-adic ring, we want to define topologies on rings of polynomials over A that make them f-adic rings. The model is the Tate algebra over a non-Archimedean field k. For example, if $k = \mathbb{Q}_\ell$, we defined in section II.1.4 the Tate algebra $T_n = k\langle X_1, \dots, X_n\rangle$ and saw (or at least claimed) that it is complete and contains the polynomial ring $k[X_1, \dots, X_n]$. So another way to define T_n would be to say that it is the completion of $k[X_1, \dots, X_n]$ for the Gauss norm. Remember also that T_n is the ring of convergent power series on the closed unit ball in $\mathbb{C}_\ell^n$. Of course, the choice of 1 as the radius of the ball was arbitrary, and in general we will want to allow a different radius for each indeterminate (this makes a real difference if the radius is not in $\ell^{\mathbb{Q}}$). This will modify the norm on $k[X_1, \dots, X_n]$ for which we take the completion. In the case of a general f-adic ring A, it does not make sense to talk of radii in $\mathbb{R}_{\geq 0}$, and they will be replaced by the family $(T_i)_{i\in I}$ in the next proposition.

We start with the case of general non-Archimedean rings and specialize to the case of f-adic rings at the end.

Proposition II.3.3.1. *(Remark 5.47 of [26].) Let A be a non-Archimedean topological ring, let $X = (X_i)_{i\in I}$ be a family of indeterminates (not necessarily finite) and let $T = (T_i)_{i\in I}$ be a family of subsets of A. Suppose that, for every $i \in I$, every $n \in \mathbb{N}$ and every neighborhood U of 0 in A, the subgroup $T_i^n U$ is open.* [4]

For every function $\nu : I \to \mathbb{N}$ with finite support, we set $T^\nu = \prod_{i\in I} T_i^{\nu(i)}$. For every open subgroup U of A, we set

$$U_{[X,T]} = \{ \sum_{\nu\in\mathbb{N}^{(I)}} a_\nu X^\nu \in A[(X_i)_{i\in I}] \mid a_\nu \in T^\nu U \text{ for all } \nu \in \mathbb{N}^{(I)}\}.$$

Then :

(i) For every open neighborhood U of 0 in A and every $\nu \in \mathbb{N}^{(I)}$, $T^\nu U$ is an open subgroup of A.

(ii) There is a unique structure of topological ring on $A[X] := A[(X_i)_{i\in I}]$ for which the subgroups $U_{[X,T]}$, for U running through the open subgroups of A, form a fundamental system of neighborhoods of 0.

We denote the resulting topological ring by $A[X]_T$.

(iii) The inclusion $\iota : A \to A[X]_T$ is continuous and the set $\{\iota(t)X_i, i \in I,\ t \in T_i\}$ is power-bounded.

(iv) For every non-Archimedean topological ring B, every continuous ring morphism $f : A \to B$ and every family $(x_i)_{i\in I}$ of elements of B such that $\{f(t)x_i, i \in I,\ t \in T_i\}$ is power-bounded, there exists a unique continuous ring morphism $g : A[X]_T \to B$ such that $f = g \circ \iota$ and $g(X_i) = x_i$ for every $i \in I$.

[4] Where we denote by $T_i^n U$ the subgroup generated by products of n elements of T_i and of one element of U.

Proof. (i) Let $\nu \in \mathbb{N}^{(I)}$. Let $i_1, \dots, i_r$ be the elements of I on which ν takes nonzero values, and let U be an open neighborhood of 0 in A. We know that $T_i^n \circ V$ is an open subgroup of A for every $i \in I$, every $n \geq 0$ and every open subgroup V of A, so

$$T^n \circ U = T_{i_1}^{\nu_{i_1}} \cdot \ldots \cdot T_{i_r}^{\nu_{i_r}} \cdot U$$

is an open subgroup of A.

(ii) If U and V are two open subgroups of A, then $U_{[X,T]} \cap V_{[X,T]} = (U \cap V)_{[X,T]}$ and $U_{[X,T]} \cdot V_{[X,T]} = (U \cdot V)_{[X,T]}$. So the family of the $U_{[X,T]}$, for U an open subgroup of A, satisfies all the conditions of lemma II.3.3.8.

(iii) The map ι is continuous by definition on the topology on $A[X]_T$. Let E be the subgroup generated by all the products $\iota(t_1)X_{i_1} \dots \iota(t_n)X_{i_n}$, for $n \in \mathbb{N}$, $i_1, \dots, i_n \in I$ and $t_{i_s} \in T_{i_s}$, $1 \leq s \leq n$. We want to show that E is bounded in $A[X]_T$. Let G be an open subgroup of $A[X]_T$. Then there exists an open subgroup U of A such that $U_{[X,T]} \subset G$. But we have $E \cdot U_{[X,T]} = U_{[X,T]}$, so $E \cdot U_{[X,T]} \subset G$.

(iv) By the universal property of the polynomial ring, there exists a unique morphism of rings $g : A[X]_T \to B$ such that $g \circ i = f$ and that $g(X_i) = x_i$ for every $i \in I$. So we just need to show that this g is continuous. Let E be the subgroup of B generated by all the products $f(t_1)X_{i_1} \dots f(t_n)X_{i_n}$, for $n \in \mathbb{N}$, $i_1, \dots, i_n \in I$ and $t_{i_s} \in T_{i_s}$, $1 \leq s \leq n$. We know that E is bounded. Let H be an open subgroup of B. Then there exists an open subgroup G of B such that $E \circ G \subset H$. As f is continuous, $U := f^{-1}(G)$ is an open subgroup of A. Then $U_{[X,T]} \subset g^{-1}(H)$; as $g^{-1}(H)$ is a subgroup of $A[X]_T$, this implies that it is open.

□

Proposition II.3.3.2. *(Remark 5.38 of [26].) Let A, X and T be as in proposition II.3.3.1 (with the same condition on T). We denote by $A[[T]]$ the formal power series ring $A[[(T_i)_{i \in I}]]$. Then :*

(i) The set

$$A\langle X\rangle_T := \{ \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu X^\nu \in A[[X]] \mid \text{for all open subgroups } U \text{ of } A,\ a_\nu \in T^\nu U \text{ for almost all } \nu\}$$

(here "almost all" means "all but a finite number") is a subring of $A[[X]]$.

(ii) There is a unique structure of topological ring on $A\langle X\rangle_T$ for which the subgroups

$$U_{\langle X,T\rangle} = \{ \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu T^\nu \in A\langle X\rangle_T \mid a_\nu \in T^\nu U \text{ for all } \nu \in \mathbb{N}^{(I)}\},$$

for U running through all the open subgroups of A, form a fundmental system of neighborhoods of 0.

Proof. (i) First, it is easy to see that $A\langle X\rangle_T$ is stable by multiplication by all the elements of A and all the X_i, $i \in I$: Let $f = \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu X^\nu \in A\langle X\rangle_T$, let $a \in A$ and $i \in I$. Let U be an open subgroup of A.

(a) Choose an open subgroup V of A such that $aV \subset U$. Then $a_\nu T^\nu \cdot V$ for almost all ν, so $(aa_\nu) \in T^\nu \cdot U$ for almost all ν.

(b) By assumption, $T_i \cdot U$ is an open subgroup of A, so $a_\nu \in T^\nu \cdot T_i \cdot U$ for almost all ν.

So $A\langle X\rangle_T$ is a $A[X]$-submodule of $A[[X]]$.

Let $f, g \in A\langle X\rangle_T$. We want to check that $fg \in A\langle X\rangle_T$. Write $fg = \sum_\nu c_\nu X^\nu$. Let U be an open subgroup of A, and choose an open subgroup V of A such that $V \cdot V \subset U$. As $f, g \in A\langle X\rangle_T$, we can write $f = f_0 + f_1$ and $g = g_0 + g_1$, with $f_0, g_0 \in A[X]$ and $f_1, g_1 \in V_{\langle X,T\rangle}$. If $f_0g_0 + f_1g_0 + f_0g_1 = \sum_\nu a_\nu X^\nu$ and $f_1g_1 = \sum_\nu b_\nu X^\nu$, then $c_\nu = a_\nu + b_\nu$. We now that $f_0g_0 + f_1g_0 + f_0g_1$ by the first paragraph, so $a_\nu \in T^\nu \cdot U$ for almost all ν. On the other hand, $b_\nu \in T^\nu \cdot U$ for every ν by the choice of V. So $c_\nu \in T^\nu \cdot U$ for almost all ν.

(ii) Again, we have $U_{\langle X,T\rangle} \cap V_{\langle X,T\rangle} = (U \cap V)_{\langle X,T\rangle}$ and $U_{\langle X,T\rangle} \cap V_{\langle X,T\rangle} \subset (U \cdot V)_{\langle X,T\rangle}$ for all open subgroups $U, V \subset A$, so lemma II.3.3.8 applies.

□

Proposition II.3.3.3. *(See proposition 5.49 of [26].) Let A, X and T be as in propositions II.3.3.1 and II.3.3.2. Then :*

(i) $A[X]_T$ is a dense subring of $A\langle X\rangle_T$, and the topology on $A[X]_T$ is the one induced by the topology on $A\langle X\rangle_T$.

(ii) If A is Hausdorff and T_i is bounded for every $i \in I$, the topological rings $A[X]_T$ and $A\langle X\rangle_T$ are Hausdorff.

(iii) If A is complete and T_i is bounded for every $i \in I$, the topological ring $A\langle X\rangle_T$ is complete (so it is the completion of $A[X]_T$).

Proof. (i) As we already noted in the proof of proposition II.3.3.2(i), for every open subgroup U of A, every element of $A\langle X\rangle_T$ is the sum of a polynomial and of an element of $U_{\langle X,T\rangle}$ (by the very definition of $A\langle X\rangle_T$). So $A[X]_T$ is dense in $A\langle X\rangle_T$. The second statement just follows from the fact that $A[X]_T \cap U_{\langle X,T\rangle} = U_{[X,T]}$ for every open subgroup U of A.

(ii) By (i), it suffices to show that $A\langle X\rangle_T$ is Hausdorff. Let $\nu \in \mathbb{N}^{(I)}$. As all the T_i are bounded, T^ν is bounded. So the intersection of all the $T^\nu \cdot U$, for $U \subset A$ an open subgroup, is equal to the intersection of all the open neighborhoods of 0 in A, i.e. $\{0\}$ because A is Hausdorff. This shows that $\bigcap_U U_{\langle X,T\rangle} = \{0\}$.

(iii) It suffices to show that $A\langle X\rangle_T$ is complete. If E is a subset of $A\langle X\rangle_T$, let $E_\nu \subset A$ be the set of all the ν-coefficients of elements of E. Let $\mathscr{F}$ be a Cauchy filter on $A\langle X\rangle_T$. For every $\nu \in \mathbb{N}^{(I)}$, let $\mathscr{F}_\nu = \{E_\nu,\ E \in \mathscr{F}\}$. As each T^ν is bounded, all the $\mathscr{F}_\nu$ are Cauchy filters on A, so they converge because A is complete. Let a_ν be the limit of $\mathscr{F}_\nu$. We want to show that $f := \sum_{\nu\in\mathbb{N}^{(I)}} a_\nu X^\nu \in A[[X]]$ is in $A\langle X\rangle_T$, and that $\mathscr{F}$ converges to f.

Let U be an open subgroup of A. Choose $E \in \mathscr{F}$ such that $g - h \in U_{\langle X,T\rangle}$ for all $g, h \in E$. In other words, if $g = \sum_{\nu\in\mathbb{N}^{(I)}} g_\nu X^\nu$ and $h = \sum_{\nu\in\mathbb{N}^{(I)}} h_\nu X^\nu$ are in E, we have $g_\nu - h_\nu \in T^\nu \circ U$ for every ν; fixing ν and going to the limit on h, we get that $g_\nu - a_\nu \in T^\nu \circ U$; in particular, this implies that $a_\nu \in T^\nu \circ U$ for almost every ν. As U was arbitrary, this shows first that $f \in A\langle X\rangle_T$, and then that, for every open subgroup U of A, we can find $E \in \mathscr{F}$ such that $g - f \in U_{\langle X,T\rangle}$ for every $g \in E$, i.e. that f is the limit of $\mathscr{F}$.

□

Corollary II.3.3.4. *Suppose that A is complete and T_i is bounded for every $i \in I$. For every complete non-Archimedean topological ring B, every continuous ring morphism $f : A \to B$ and every family $(x_i)_{i\in I}$ of elements of B such that $\{f(t)x_i, i \in I,\ t \in T_i\}$ is power-bounded, there exists a unique continuous ring morphism $g : A\langle X\rangle_T \to B$ such that $f = g \circ \iota$ and $g(X_i) = x_i$ for every $i \in I$.*

Proof. We already know that there exists a unique continuous ring morphism $g : A[X]_T \to B$ satisfying the two conditiions of the statement (by proposition II.3.3.1(iv)). By [6] Chapitre III §3 N°5 corollaire de la proposition 8, g has a unique extension to a continuous morphism of topological groups from $A\langle X\rangle_T$ to B, and this extension is clearly a morphism of rings.

□

We now specialize to the case of interest of us, i.e. that of f-adic rings.

First we note the following useful fact.

Lemma II.3.3.5. *(Lemma 6.20 of [26].) Let A be a f-adic ring and T be a subset of A. If T generates an open ideal of A, then, for any open subgroup U of A and any $n \geq 0$, the subgroup $T^n \cdot U$ is open.*

Proof. Let U and n be as in the statement, and let (A_0, I) be a couple of definition of A. By assumption, the ideal J of A generated by T is open, so it contains some power of I. Hence J^n also contains a power of I. After changing the ideal of definition, we may assume that $I \subset J^n = T^n \cdot A$. Let L be a finite set of generators of I, and let M be a finite set such that $L \subset T^n \cdot M$. As M is finite, it is bounded, so we can find an integer $r \in \mathbb{N}$ such that $M \cdot I^r \subset U$. Then we have $I^{r+1} = L \cdot I^r \subset T^n \cdot M \cdot I^r \subset T^n \cdot U$, so $T^n \cdot U$ is open.

□

Proposition II.3.3.6. *(Proposition 6.21 of [26].) Let A be a non-Archimedean topological ring, let $X = (X_\lambda)_{\lambda\in L}$ be a family of indeterminates (not necessarily finite) and let $T = (T_\lambda)_{\lambda\in L}$ be a family of subsets of A.*

Let (A_0, I) be a couple of definition of A. Suppose that, for every $\lambda \in L$, the subset T_λ generates an open ideal of A. Then :

(i) The ring $A[X]_T$ is f-adic, with couple of definition $(A_{0[X,T]}, I_{[X,T]})$. In particular, the canonical map $A \to A[X]_T$ is adic. Moreover, if A is a Tate ring, then $A[X]_T$ is also a Tate ring.

(ii) *Suppose that the family of indeterminates $(X_\lambda)_{\lambda\in L}$ is* finite. [5] *Then the ring $A\langle X\rangle_T$ is f-adic, with couple of definition $(A_{0\langle X,T\rangle}, I_{\langle X,T\rangle})$, and the canonical map $A \to A\langle X\rangle_T$ is adic. Moreover, if A is a Tate ring, then $A\langle X\rangle_T$ is also a Tate ring.*

Proof. (i) It is clear that $A_{0[X,T]}$ is an open subring of $A[X]_T$, and that $I_{[X,T]} = I \cdot A_{0[X,T]}$ is a finitely generated ideal of $A_{0[X,T]}$. As $(I_{[X,T]})^n = (I^n)_{[X,T]}$ for every $n \geq 1$, we see that the topology on $A_{0[X,T]}$ is the $I_{[X,T]}$-adic topology, so $A[X]_T$ is adic. The map $A \to A[X]_T$ is clearly adic, and the last statement is also clear.

(ii) First note that $A_{0\langle X,T\rangle}$ is an open subring of $A\langle X\rangle_T$.

Let J be an ideal of definition of A_0; we claim that $J_{\langle X,T\rangle} = JA_{0\langle X,T\rangle}$. The inclusion $J_{\langle X,T\rangle} \supset JA_{0\langle X,T\rangle}$ is clear. Conversely, let $f = \sum_\nu a_\nu X^\nu \in J_{\langle X,T\rangle}$. Let $x_1, \dots, x_r \in A_0$ be generators of J. We write $\mathbb{N}^L = \bigcup_{k=1}^{+\infty} N_k$, with all the N_k finite and with $a_\nu \in T^\nu \cdot J^k$ for every $k \in \mathbb{N}$ and every $\nu \in N_k$. If $k \in \mathbb{N}$ and $a_\nu \in N_k$, we write $a_\nu = \sum_{i=1}^r x_i a_{\nu,i}$, with the $a_{\nu,i} \in T^\nu \cdot I^{k-1}$. Let $f_i = \sum_{\nu\in\mathbb{N}^L} a_{\nu,i} X^\nu$, for $1 \leq i \leq s$. Then $f_1, \dots, f_s \in A_{0\langle X,T\rangle}$, and so $f = x_1 f_1 + \dots x_s f_s \in JA_{0\langle X,T\rangle}$.

In particular, we have $I_{\langle X,T\rangle} = IA_{0\langle X,T\rangle}$. Hence, for every $n \geq 1$, applying the previous paragraph to the ideal of definition I^n of A_0, we get that $(I_{\langle X,T\rangle})^n = I^n A_{0\langle X,T\rangle} = (I^n)_{\langle X,T\rangle}$. This shows that the topology on $A_{0\langle X,T\rangle}$ is the $IA_{0\langle X,T\rangle}$-adic topology, and so $A\langle X\rangle_T$ is f-adic. The map $A \to A\langle X\rangle_T$ is clearly adic, and the last statement is also clear.

□

Example II.3.3.7. Take $A = \mathbb{Z}_\ell$, $I = \{1\}$ and $T_1 = \{\ell\}$. We get a f-adic ring

$$\mathbb{Z}_\ell\langle X\rangle_T = \{\sum_{n\geq 0} a_n X^n \in \mathbb{Z}_\ell[[X]] \mid \ell^{-n} a_n \to 0 \text{ as } n \to +\infty\},$$

with ring of definition

$$\mathbb{Z}_{\ell\langle X,T\rangle} = \{\sum_{n\geq 0} a_n X^n \in \mathbb{Z}_\ell\langle X\rangle_T \mid \forall n \in \mathbb{N},\ \ell^{-n} a_n \in \mathbb{Z}_\ell\}.$$

The topology on $\mathbb{Z}_{\ell\langle X,T\rangle}$ is the $\ell\mathbb{Z}_{\ell\langle X,T\rangle}$-adic topology. Note in particular that $\mathbb{Z}_\ell\langle X\rangle_T$ is strictly bigger than $\mathbb{Z}_{\ell\langle X,T\rangle}$, and that the ring $\mathbb{Z}_\ell\langle X\rangle_T$ is not adic, even though $\mathbb{Z}_\ell$ is. For example, ℓ is topologically nilpotent in $\mathbb{Z}_\ell\langle X\rangle_T$, but ℓX is not, because $\ell^n X^n \notin \ell\mathbb{Z}_{\ell\langle X,T\rangle}$ for $n \geq 0$. (This could not happen in an adic ring, in which all elements are power-bounded.)

Lemma II.3.3.8. *(Remark 5.24 of [26].) Let A be a ring and $\mathscr{G}$ be a set of additive subgroups of A. Then requiring $\mathscr{G}$ to be a fundamental system of neighborhoods of 0 makes A a topological ring if and only the following conditions hold :*

(a) *For all $G, G' \in \mathscr{G}$, there exists $H \in \mathscr{G}$ such that $H \subset G \cap G'$.*

[5] We would get the same conclusion if we assumed instead that A is Hausdorff and that all the T_λ are bounded.

(b) For every $a \in A$ and every $G \in \mathscr{G}$, there exists $H \in \mathscr{G}$ such that $aH \subset G$.

(c) For every $G \in \mathscr{G}$, there exists $H \in \mathscr{G}$ such that $H \cdot H \subset G$.

Proof. The conditions are obviously necessary. Conversely, suppose that they hold. Condition (a) says that $\mathscr{G}$ does give a fundamental system of neighborhoods for a topological group structure on A (see [6] Chapitre III §1 N°2 proposition 1). Conditions (b) and (c) now say that A is a topological ring (see [6] Chapitre III §6 N°3 Remarque).

□

Notation II.3.3.9. If $T_i = \{1\}$ for every $i \in I$, we write $A[X]_T = A[X]$ and $A\langle X\rangle_T = A\langle X\rangle$.

II.3.4 Localizations

Proposition II.3.4.1. *(Proposition 5.51 of [26].) Let A be a non-Archimedean topological ring and $T = (T_i)_{i\in I}$ be a family of subsets of A satisfying the condition of proposition II.3.3.1. Let $S = (s_i)_{i\in I}$ be a family of elements of A, and denote by R the multiplicative subset of A generated by $\{s_i,\ i \in I\}$.*

Then there exists a unique non-Archimedean topological ring structure on $R^{-1}A$, making it into a topological ring that we will denote by $A\left(\frac{T}{S}\right) = A\left(\frac{T_i}{s_i} \mid i \in I\right)$, satisfying the following properties :

(i) The canonical morphism $\varphi : A \to A\left(\frac{T}{S}\right)$ is continuous and the set $\{\frac{\varphi(t)}{\varphi(s_i)},\ i \in I,\ t \in T_i\}$ is power-bounded in $A\left(\frac{T}{S}\right)$.

(ii) For every non-Archimedean topological ring B and every continuous map $f : A \to B$ such that $f(s_i)$ is invertible in B for every $i \in I$ and that the set $\{\frac{f(t)}{f(s_i)},\ i \in I,\ t \in T_i\}$ is power-bounded in B, there exists a unique continuous ring morphism $g : A\left(\frac{T}{S}\right) \to B$ such that $f = g\varphi$.

Proof. Let D be the subring of $R^{-1}A$ generated by all the $\frac{\varphi(t)}{\varphi(s_i)}$, for $i \in I$ and $t \in T_i$. Then the family of subsets $D \cdot \varphi(U) \subset R^{-1}A$, for U an open subgroup of A, satisfies the conditions of lemma II.3.3.8, which implies that there is a unique structure of topological ring on $R^{-1}A$ that makes this family a fundamental system of neighborhoods of 0. It is clear that $\varphi : A \to R^{-1}A$ is continuous for this topology, because $\varphi^{-1}(D \cdot \varphi(U)) \supset U$.

As D is a subring of $R^{-1}A$, and as it is bounded by definition of the topology, the se t$\{\frac{\varphi(t)}{\varphi(s_i)},\ i \in I,\ t \in T_i\}$ is power-bounded.

We check that $\varphi : A \to R^{-1}A$ satisfies the universal property of (ii) (which will also imply uniqueness). Let $f : A \to B$ be as in (ii). By the universal property of the localization, there exists a unique ring morphism $g : R^{-1}A \to B$ such that $f = g \circ \varphi$, so we just need to check that this g is continuous. Let E be the subring of B generated by the power-bounded set $\{\frac{f(t)}{f(s_i)},\ i \in I,\ t \in T_i\}$; then E is bounded by lemma II.1.2.6(ii), and we have $g(D) \subset E$. Let U be an open subgroup of B. As E is bounded, there exists an open subgroup V of B such that $E \cdot V \subset U$. As f is continuous, $W := f^{-1}(V)$ is an open subgroup of A. As $g(\varphi(W) \cdot D) = f(W) \cdot f(D) \subset V \cdot E \subset U$, $g^{-1}(U)$ contains the open subgroup $\varphi(W) \cdot D$, so it is open.

□

Remark II.3.4.2. (Remark 5.52 of [26].)

(1) We have
$$A\left(\frac{T_i}{s_i} \mid i \in I\right) = A\left(\frac{T_i \cup \{s_i\}}{s_i} \mid i \in I\right),$$
so we can always assume that $s_i \in T_i$ for every $i \in I$.

(2) Let J be the ideal of $A[X]_T$ generated by the set $\{1 - s_i X_i,\ i \in I\}$. Then $A[X]_T/J$, with the quotient topology, satifies the same universal property as $A\left(\frac{T}{S}\right)$, so we have a canonical isomorphism
$$A[X]_T/J = A\left(\tfrac{T}{S}\right).$$

Proposition II.3.4.3. *(Proposition 6.21 of [26].) Let A, $T = (T_i)_{i \in I}$ and $S = (s_i)_{i \in I}$ be as in proposition II.3.4.1, and suppose that A is f-adic.*

Then $A\left(\frac{T}{S}\right)$ is also f-adic, and the canonical map $A \to A\left(\frac{T}{S}\right)$ is adic.

If $I = \{1\}$ is a singleton and $T_1 = \{t_1, \ldots, t_n\}$ is finite, we also write $A\left(\frac{T}{S}\right) = A\left(\frac{t_1, \ldots, t_n}{s_0}\right)$.

Proof. This follows immediately from remark II.3.4.2 and from proposition II.3.3.6(i).

□

Remark II.3.4.4. Let us give an explicit ring of definition of $B := A\left(\frac{T}{S}\right)$ (the notation is that of proposition of II.3.4.3). Let A_0 be a ring of definition of A. Then $A_{0[X,T]}$ is a ring of definition of $A[X]_T$, so its image by the surjective map $A[X]_T \to B$ is a ring of definition of B. By definition, we have
$$A_{0[X,T]} = \{\sum_{\nu \in \mathbb{N}^{(I)}} a_\nu X^\nu \in A[(X_i)_{i \in I}] \mid a_\nu \in T^\nu A_0 \text{ for all } \nu \in \mathbb{N}^{(I)}\}.$$
So its image B_0 in B is the A_0-submodule of B generated by the sets $\prod_{i \in I} T_i^{\nu_i} s_i^{-\nu_i}$, for $(\nu_i)_{i \in I} \in \mathbb{N}^I$. In other words, B_0 is the A_0-subalgebra of B generated by the elements $t s_i^{-1}$, for $i \in I$ and $t \in T_i$.

In particular, if $I = \{1\}$ is a singleton, $s = s_1$ and $T_1 = \{t_1, \ldots, t_n\}$, then $B_0 = A_0[t_1 s^{-1}, \ldots, t_n s^{-1}]$.

Definition II.3.4.5. If A is a f-adic ring and T and S are as in proposition II.3.4.1, we denote the completion of $A\left(\frac{T}{S}\right)$ by $A\left\langle\frac{T}{S}\right\rangle$. This is also a f-adic ring, and the canonical map $A \to A\langle\frac{T}{S}\rangle$ is adic.

If A is complete, we can also see $A\langle\frac{T}{S}\rangle$ as the quotient of $A\langle X\rangle_T$ by the closure of the ideal generated by $\{1 - s_i X_i,\ i \in I\}$.

Putting the universal properties of proposition II.3.4.1 and of the completion together, we get the following result :

Proposition II.3.4.6. *Let A be a f-adic ring and T and S be as in proposition II.3.4.1. Then, for every* complete *non-Archimedean topological ring B and every continuous map $f : A \to B$ such that $f(s_i)$ is invertible in B for every $i \in I$ and that the set $\{\frac{f(t)}{f(s_i)},\ i \in I,\ t \in T_i\}$ is power-bounded in B, there exists a unique continuous ring morphism $g : A\langle\frac{T}{S}\rangle \to B$ such that $f = g\varphi$.*

Example II.3.4.7. Take $A = A_0 = \mathbb{Z}_\ell[[u]]$, with ideal of definition $J = (\ell, X)$. Take $I = \{1\}$ and $T_1 = \{\ell, u\}$.

Let $B = A[X]_T$ and $B_0 = A_{[X,T]}$. Note that B_0 is strictly contained in B, because a polynomial $\sum_{n\geq 0} a_n X^n$ is in B_0 if and only if $a_n \in (\ell, u)^n A$ for every $n \geq 0$, so for example $X \in B - B_0$. We have $\widehat{B} = A\langle X\rangle_T$ and $\widehat{B}_0 = A_{\langle X,T\rangle}$, and again $\widehat{B}_0 \subsetneq \widehat{B}$.

We now consider the localizations $A\left(\frac{\ell,u}{\ell}\right)$ and $A\left(\frac{\ell,u}{u}\right)$ as f-adic rings, and in particular we want to write down rings of definition. As rings, we have

$$A\left(\tfrac{T}{s}\right) = \mathbb{Z}_\ell[[u]][\ell^{-1}]$$

and

$$A\left(\tfrac{T}{s}\right) = \mathbb{Z}_\ell[[u]][u^{-1}].$$

We get rings of definition by using the description of remark II.3.4.4. As A is a ring of definition of itself, this remark shows that $A[\frac{u}{\ell}]$ is a ring of definition of $A\left(\frac{\ell,u}{\ell}\right)$ and $A[\frac{\ell}{u}]$ is a ring of definition of $A\left(\frac{\ell,u}{u}\right)$. Note that ℓ is not invertible in $A[\frac{u}{\ell}]$, even though it is of course invertible in $A\left(\frac{\ell,u}{\ell}\right)$; similarly, u is not invertible in the ring of definition $A[\frac{\ell}{u}]$ of $A\left(\frac{\ell,u}{u}\right)$. Note also that the completed localizations are not adic rings, even though we started from an adic ring.

In $A\langle\frac{\ell,u}{\ell}\rangle$, a ring of definition is $A_{\langle X,T\rangle}/(1-\ell X)A_{\langle X,T\rangle}$, which is isomorphic to

$$A\langle\frac{u}{\ell}\rangle := \{\sum_{n\geq 0} a_n \left(\frac{u}{\ell}\right)^n,\ a_n \in A,\ a_n \to 0 \text{ as } n \to +\infty\}.$$

II.4 The Banach open mapping theorem

The reference for this section is Henkel's note [12], where Henkel explains how to adapt the proof of [7] chapitre I §3 N°3 Théorème 1.

II.4.1 Statement and proof of the theorem

Theorem II.4.1.1. *(Theorem 1.6 of [12]) Let A be a topological ring that has a sequence of units converging to 0. Let M and N be Hausdorff topological A-modules that have countable topological systems of open neighborhoods of 0, and let $u : M \to N$ be a continuous A-linear map. Suppose that M is complete. Then the following properties are equivalent :*

(i) N is complete and u is surjective;

(ii) N is complete and $u(M)$ is open in N;

(iii) for every neighborhood U of 0 in M, $\overline{u(U)}$ is a neighborhood of 0 in N;

(iv) u is open.

Proof. Note that the conditions on M and N imply that their topology is defined by translation-invariant metrics (see theorem II.4.1.7).

Condition (i) clearly implies (ii), and (iii) implies (iv) by theorem II.4.1.7 and lemma II.4.1.4.

We show that (ii) implies (iii). Choose a sequence of units $(a_n)_{n\geq 0}$ of A that converges to 0. Let U be a neighborhood of 0 in M, and let V be a neighborhood of 0 in M such that $V - V \subset U$. By lemma II.4.1.2, we have $M = \bigcup_{n\geq 0} a_n^{-1} \cdot V$, so

$$u(M) = \bigcup_{n\geq 0} a_n^{-1} \cdot u(V) \subset \bigcup_{n\geq 0} a_n^{-1} \cdot \overline{u(V)}.$$

As the elements a_n are units, multiplication by a_n and a_n^{-1} is a homeomorphism of N. In particular, all the sets $a_n^{-1} \cdot \overline{u(V)}$ are closed in N. As $u(M)$ is open, the Baire category theorem implies that at least one of the $a_n^{-1} \cdot \overline{u(V)}$ has nonempty interior, so $\overline{u(V)}$ has nonempty interior. Let y be an interiot point of $\overline{u(V)}$. Then $0 = y - y$ is an interior point of $\overline{u(V)} - \overline{u(V)} \subset \overline{u(V - V)} \subset \overline{u(U)}$, which means that $\overline{u(U)}$ is a neighborhood of 0, as desired.

Finally, we show that (iv) implies (i). If u is open, then $u(M)$ is open, and this implies that u is surjective by lemma II.4.1.3. As u is open, it induces an isomorphism of topological A-modules $M/\operatorname{Ker}(u) \xrightarrow{\sim} N$ (i.e. an isomorphism of A-modules that is also a homeomorphism). As N is Hausdorff, $\operatorname{Ker}(u)$ is a closed subgroup of M by [6] chapitre III §2 Nº6 proposition 18, and then $M/\operatorname{Ker}(u)$ is complete by [6] chapitre IX §3 Nº1 proposition 4.

□

Lemma II.4.1.2. *(Lemma 1.7 of [12].) Let A be a topological ring, let $(a_i)_{i\in I}$ be a family of elements of A whose closure contains 0, let M be a topological A-module, and let U be a neighborhood of 0 in M. Then, for every $x \in M$, there exists $i \in I$ such that $a_i \cdot x \in U$.*

Proof. Let $x \in M$. As the action map $A \times M \to M$ is continuous, there exists a neighborhood V of 0 in A such that $V \cdot x \subset U$. If $i \in I$ is such that $a_i \in V$, we have $a_i \cdot x \in U$.

□

Lemma II.4.1.3. *(Lemma 1.13 of [12].) Let A be a topological ring in which each neighborhood of 0 contains a unit, let N be a topological A-module, and let N' be a A-submodule of N. Then $N' = N$ if and only if N' contains a nonempty open subset of N.*

Proof. Suppose that N' contains a nonempty open subset U of N. Translating U by an element of N', we may assume that $0 \in U$. Applying lemma II.4.1.2 to the family of all units of A, we get $N = \bigcup_{a \in A^\times} a^{-1} \cdot U \subset N'$.

□

Lemma II.4.1.4. *(Proposition 1.12 of [12].) Let M and N be commutative topological groups whose topology comes from a translation-invariant metric, and let $u : M \to N$ be a continuous morphism of groups such that, for every neighborhood U of 0 in M, $\overline{u(U)}$ is a neighborhood of 0 in N. Suppose that M is complete. Then u is open.*

Proof. We fix translation-invariant metrics giving the topologies of M and N, and denote by $B_M(x,r)$ (resp. $B_N(x,r)$) the open ball with center x and radius r for both the metric on M (resp. N).

The hypothesis says that, for every $r > 0$, there exists $\rho(r) > 0$ such that $B_N(0,\rho(r)) \subset \overline{u(B_M(0,r))}$. Using the fact that the metrics are translation-invariant, we easily get $B_N(u(x),\rho(r)) \subset \overline{u(B_M(x,r))}$ for every $x \in M$ and every $r > 0$.

Fix $r > 0$ and $a > r$. We want to show that $B_N(0,\rho(r)) \subset u(B_M(0,a))$. (This will clearly finish the proof.) The argument is that of [7] chapitre I §3 N°3 lemme 2. Choose a sequence $(r_n)_{n \geq 1}$ of positive real numbers such that $r_1 = r$ and $\sum_{n \geq 1} r_n = a$, and a sequence of positive real numbers $(\rho_n)_{n \geq 1}$ such that $\rho_n \leq \rho(r_n)$ and that $\lim_{n \to +\infty} \rho_n = 0$. Let $y \in B_N(0,r)$. We want to show that $y \in u(B_M(0,a))$. We define a sequence $(x_n)_{n \geq 0}$ of elements of M such that $x_n \in B_M(x_{n-1},r_n)$ and $u(x_n) \in B_N(y,\rho_{n+1})$ for $n \geq 1$ in the following way :

- $x_0 = 0$;
- if $n \geq 1$ and $x_0, \dots, x_{n-1}$ have been chosen to satisfy the two required conditions, then we have $y \in B_N(u(x_{n-1}),\rho_n) \subset \overline{u(B_M(x_{n-1},r_n))}$, so $B_N(y,\rho_{n+2}) \cap \overline{u(B_M(x_{n-1},r_n))}$ is not empty, and we choose x_n in this set.

The sequence $(x_n)_{n \geq 1}$ is a Cauchy sequence because $\sum_{n \geq 1} r_n$ converges, so $\sum_{n \geq N} r_n$ tends to 0 as $N \to +\infty$. As M is complete, $(x_n)_{n \geq 1}$ has a limit x. We have $x \in B_M(0,a)$ by the triangle inequality, and $u(x) = \lim_{n \to +\infty} u(x_n) = y$ because u is continuous. So we are done.

□

Corollary II.4.1.5. *(Theorem 1.17 of [12].) Let A be a topological ring that has a sequence of units converging to 0. Let M and N be topological A-modules such that M is finitely generated, Hausdorff, complete and has a countable fundamental system of open neighborhoods of 0. Then any A-linear map $u : M \to N$ is continuous.*

Proof. Let $\pi : A^n \to M$ be a surjective A-linear map. Then the $u \circ \pi : A^n \to N$ is given by the formula

$$(u \circ \pi)(a_1, \dots, a_n) = \sum_{i=1}^{n} a_i \cdot u(\pi(e_i)),$$

where $(e_1, \dots, e_n)$ is the canonical basis of A^n, so it is continuous. Similarly, π is continuous. By theorem II.4.1.1, π is open, so, for open subset U of N, the subset $u^{-1}(U) = \pi((u \circ \pi)^{-1}(U))$ of M is open. This shows that u is continuous.

□

Corollary II.4.1.6. *Let A be a complete Tate ring, let M and N be topological A-modules, and let $u : M \to N$ be a surjective A-linear map. Suppose that M and N are quotients of finite free A-modules by closed submodules. Then u is open.*

Proof. If ϖ is a topologically nilpotent unit of A, then $(\varpi^n)_{n \geq 0}$ is a sequence of units of 0 converging to 0. Also, as A has a countable system of neighborhoods of 0, its topology comes from a translation-invariant metric by theorem II.4.1.7. By [6] chapitre IX §3 N°1 proposition 4, the A-modules M and N are Hausdorff, complete and metrizable. Also, u is continuous by corollary II.4.1.5. So we can apply theorem II.4.1.1 to get the conclusion.

□

Theorem II.4.1.7. *([6] chapitre IX §3 N°1 propositions 1 and 2) Let G be a commutative topological group. Then the topology of G is given by a translation-invariant pseudometric if and only if G has a countable fundamental system of neighborhoods of* 0.

II.4.2 Applications

Proposition II.4.2.1. *(Proposition 2.11 of [12].) Let A be a complete Tate ring, and let M be a complete topological A-module that has a countable fundamental system of neighborhoods of* 0. *Then the following conditions are equivalent :*

(i) M is a Noetherian A-module;

(ii) every A-submodule of M is closed.

In particular, the ring A is Noetherian if and only if every ideal of A is closed.

Proposition II.4.2.2. *(Theorems 1.17 and 2.12 of [12].) Let A be a complete Noetherian Tate ring. Then :*

(i) Every finitely generated A-module has a unique topology that makes it a Hausdorff complete topological A-module having a countable fundamental system of neighborhoods of 0.

(ii) Let M and N be finitely generated A-modules endowed with the topology of (i). If $u : M \to N$ is a A-linear map, then u is continuous, $\operatorname{Im} u$ *is closed in N and $u : M \to \operatorname{Im} u$ is open.*

We call the topology of point (i) of the proposition the *canonical topology* on M.

III The adic spectrum

In all this chapter, A is a f-adic ring.

III.1 Rings of integral elements

We will be interested in subsets of $\mathrm{Cont}(A)$ defined by conditions of the type $|a|_x \leq 1$, for a in some fixed subset Σ of A. We will see in this section that we can only assume that Σ is an open an integrally closed subring of A, and that the case of most interest is when $\Sigma \subset A^0$.

Remark III.1.1. Let $x \in \mathrm{Cont}(A)$ and let $a \in A^0$ (i.e. a is power-bounded). If x is a rank 1 valuation, then we necessarily have $|a|_x \leq 1$. In general, this is not true. See the next example.

Example III.1.2. Let k a field, and let $A = k((t))((u))$, with the rank 2 valuation x of example I.1.4.3 (so we have $R_x = \{f = \sum_{n \geq 0} f_n t^n \in k((t))[[u]] \mid f_0 \in k[[u]]\}$). Let y be the u-adic valuation on A. Then we have seen in example I.1.5.5 that $|.|_x$ and $|.|_y$ define the same topology on A. If we put this topology on A, then x and y are in $\mathrm{Cont}(A)$. As y has rank 1, we have

$$A^0 = \{f \in A \mid |f|_x \leq 1\} = k((t))[[u]]$$

and

$$A^{00} = \{f \in A \mid |f|_x < 1\} = uA^0.$$

If a is any element of $A^0 - R_x$ (for example $a = \frac{1}{t}$), then a is power-bounded but $|a|_x > 1$.

Definition III.1.3. Let Σ be a subset of A. We write

$$\mathrm{Spa}(A, \Sigma) = \{x \in \mathrm{Cont}(A) \mid \forall a \in \Sigma,\ |a|_x \leq 1\}.$$

Note that we obviously have $\mathrm{Spa}(A, \Sigma) \supset \mathrm{Spa}(A, \Sigma')$ if $\Sigma \subset \Sigma'$.

Proposition III.1.4. *(Lemma 3.3 of [14].) Let Σ be a subset of A. We denote by A_Σ the smallest open and integrally closed subring of A containing Σ. Then :*

(i) $\mathrm{Spa}(A, \Sigma)$ *is a pro-constructible subset of* $\mathrm{Cont}(A)$*;*

(ii) we have

$$A_\Sigma = \{f \in A \mid \forall x \in \mathrm{Spa}(A, \Sigma), |f|_x \leq 1\}$$

and $\mathrm{Spa}(A, \Sigma) = \mathrm{Spa}(A, A_\Sigma)$.

Proof. (i) If $a \in A$, then

$$\mathrm{Spa}(A, \{a\}) = \{x \in \mathrm{Cont}(A) \mid |a|_x \leq 1\} = U_{\mathrm{cont}}\left(\frac{1, a}{1}\right)$$

is a quasi-compact open subset of $\mathrm{Cont}(A)$, and in particular it is constructible. So $\mathrm{Spa}(A, \Sigma) = \bigcap_{a \in \Sigma} \mathrm{Spa}(A, \{a\})$ is pro-constructible.

(ii) Let

$$A' = \{f \in A \mid \forall x \in \mathrm{Spa}(A, \Sigma), |f|_x \leq 1\}.$$

Then A' is clearly a subring of A, and it is open because it contains the open subgroup A^{00}. We claim that A' is integrally closed in A. Indeed, let $f \in A$, and suppose that we have an equation $f^n + a_1 f^{n-1} + \ldots + a_n = 0$, with $n \geq 1$ and $a_1, \ldots, a_n \in A'$. Then, for every $x \in \mathrm{Spa}(A, \Sigma)$, we have

$$|f|_x^n \leq \max_{1 \leq i \leq n} |a_i|_x |f|_x^{n-1} \leq \max(1, |f|_x, \ldots, |f|_x^{n-1}),$$

and this is only possible if $|f|_x \leq 1$.

As A' contains Σ, it also contains A_Σ. Note also that $\mathrm{Spa}(A, \Sigma) = \mathrm{Spa}(A, A')$ by definition of A'. So it just remains to show that $A' \subset A_\Sigma$.

So suppose that we have an element $a \in A - A_\Sigma$. We want to construct a continuous valuation x on A such that $|a|_x > 1$. Consider the element a^{-1} of $A_\Sigma[a^{-1}] \subset A[a^{-1}]$; this is not a unit, because otherwise a would be an element of $A_\Sigma[a^{-1}]$, so it would be integral over A_Σ, which is impossible because A_Σ is integrally closed in A and $a \notin A_\Sigma$. So there exists a prime ideal $\wp$ of $A_\Sigma[a^{-1}]$ such that $a^{-1} \in \wp$. Let $\mathfrak{q} \subset \wp$ be a minimal prime ideal of $A_\Sigma[a^{-1}]$. Then $B := (A_\Sigma[a^{-1}]/\mathfrak{q})_{\wp/\mathfrak{q}}$ is a local subring of the field $K := \mathrm{Frac}(A_\Sigma[a^{-1}/\mathfrak{q}])$, so, by theorem I.1.2.2(i), there exists a valuation subring $R \supset B$ of K such that $\mathfrak{m}_B = B \cap \mathfrak{m}_R$. This valuation subring defines a valuation $|.|_R$ on K, hence a valuation $|.|$ on $A_\Sigma[a^{-1}]$ via the obvious map $A_\Sigma[a^{-1}] \to A_\Sigma[a^{-1}]/\mathfrak{q} \subset K$, and we have $\mathrm{Ker}\,|.| = \mathfrak{q}$. Also, by the choice of R, we have $|f| \leq 1$ for every $f \in A_\Sigma[a^{-1}]$ and $|f| < 1$ for every $f \in \wp$, and in particular $|a^{-1}| < 1$.

Let $S = A_\Sigma[a^{-1}] - \mathfrak{q}$. As $S^{-1}A_\Sigma[a^{-1}]$ is flat over $A_\Sigma[a^{-1}]$, the map $S^{-1}A_\Sigma[a^{-1}] \to S^{-1}A[a^{-1}]$ is injective, so $S^{-a}A[a^{-1}] \neq \{0\}$, and there exists a prime ideal $\mathfrak{q}'$ of $A[a^{-1}]$ such that $\mathfrak{q} \cap S = \varnothing$. In particular, we have $\mathfrak{q}' \cap A_\Sigma[a^{-1}] \subset \mathfrak{q}$, which implies that $\mathfrak{q}' \cap A_\Sigma[a^{-1}] = \mathfrak{q}$ because $\mathfrak{q}$ is a minimal prime ideal, and we get a field extension $K \subset K' := \mathrm{Frac}(A[a^{-1}]/\mathfrak{q}')$. By proposition I.1.2.3, there exists a valuation subring R' of K' such that $R' \cap K = R$. Let $v \in \mathrm{Spv}(A)$ correspond to the composition of $A \to A[a^{-1}]/\mathfrak{q}' \subset K'$ and of $|.|_{R'}$. Then we have $|f|_v \leq 1$ for every $f \in A_\Sigma$ and $|a|_v > 1$. Note also that $|f|_v < 1$ for every $f \in A^{00}$. Indeed, if $f \in A^{00}$, then there exists an integer $r \geq 1$ such that $f^r a \in A_\Sigma$ (because A_Σ is an open subring of A), and then $|f|_v^r |a|_v = |f^r a|_v \leq 1$ and $|a|_v > 1$, which implies that $|f|_v < 1$.

We would be done if the valuation v was continuous, but this has no reason to be true. So let $w = v_{|c\Gamma_v}$ be the minimal horizontal specialization of v. We obviously have

$w \in \mathrm{Spv}(A, A^{00} \cdot A)$. Also, $|f|_w \leq |f|_v$ for every $f \in A$, so we have $|f|_w \leq 1$ for every $f \in A_\Sigma$ and $|f|_w \leq |f|_v < 1$ for every $f \in A^{00}$. In particular, $v \in \mathrm{Cont}(A)$ by theorem II.2.2.1. Also, we have $|a|_v > 1$, so $|a|_v \in c\Gamma_v$, hence $|a|_w = |a|_v > 1$. Finally, we have found $w \in \mathrm{Cont}(A)$ such that $w \in \mathrm{Spa}(A, A_\Sigma)$ and $|a|_w > 1$, which means that $a \notin A'$.

□

Proposition III.1.5. *(Lemma 3.3 of [14].) Let $A^+ \subset A$ be an open and integrally closed subring.*

(i) If $A^+ \subset A^0$, then $\mathrm{Spa}(A, A^+)$ is dense in $\mathrm{Cont}(A)$ and contains all the trivial valuations in $\mathrm{Cont}(A)$ and all the rank 1 points of $\mathrm{Cont}(A)$. More precisely, every $x \in \mathrm{Cont}(A)$ is a vertical specialization of a point y of $\mathrm{Spa}(A, A^+)$, and we can choose y to be of rank 1 if x is analytic.

(ii) Suppose that A is a Tate ring and has a Noetherian ring of definition. If $\mathrm{Spa}(A, A^+)$ is dense in $\mathrm{Cont}(A)$, then $A^+ \subset A^0$.

Proof. (i) We first prove that the continuous trivial valuations and the continuous rank 1 valuations are in $\mathrm{Spa}(A, A^+)$. Let $x \in \mathrm{Cont}(A)$. If $\Gamma_x = \{1\}$, then obviously $|a|_x \leq 1$ for every A^+, so $x \in \mathrm{Spa}(A, A^+)$. If x has rank 1, then $|a|_x \leq 1$ for every $x \in A^0 \supset A^+$, so again $x \in \mathrm{Spa}(A, A^+)$.

Now we prove the last sentence. Let $x \in \mathrm{Cont}(A)$. If $\mathrm{supp}(x)$ is open, then every vertical generization (or specialization) of x is continuous, and in particular the maximal vertical generization $y = x/\Gamma_x$ of x is in $\mathrm{Cont}(A)$. As $|f|_y \in \{0, 1\}$ for every $f \in A$, we clearly have $y \in \mathrm{Spa}(A, A^+)$.

Suppose that $\mathrm{supp}(x)$ is not open, i.e. that $x \in \mathrm{Cont}(A)_{\mathrm{an}}$. By corollary II.2.4.8, x has a rank 1 generization $y \in \mathrm{Cont}(A)_{\mathrm{an}}$. We have already seen that such a y has to be in $\mathrm{Spa}(A, A^+)$.

(ii) Suppose that $A^+ \not\subset A^0$, and choose $a \in A^+ - A^0$. By proposition III.1.4(ii), there exists $x \in \mathrm{Spa}(A, A^0)$ such that $|a|_x > 1$. Let L be as in lemma III.1.6, and let $S = \{x \in L \mid |a|_x > 1\}$; by what we just wrote, we have $S \neq \varnothing$. As S is the intersection of L and of a constructible subset of $\mathrm{Spv}(A)$, lemma III.1.6 implies that there exists a maximal element $y \in \mathrm{Cont}(A)$ such that $|a|_y > 1$. In particular, y is not in $\mathrm{Spa}(A, A^+)$ and, as it is maximal, it cannot be a proper specialization of a point of $\mathrm{Spa}(A, A^+)$. As $\mathrm{Spa}(A, A^+)$ is pro-constructible in $\mathrm{Cont}(A)$, proposition I.3.1.3(ii) implies that y is not in the closure of $\mathrm{Spa}(A, A^+)$. So $\mathrm{Spa}(A, A^+)$ is not dense in $\mathrm{Cont}(A)$.

□

Lemma III.1.6. *Let A be a Tate ring which has a Noetherian ring of definition. Let*

$$L = \{x \in \mathrm{Spv}(A) \mid \forall a \in A^0,\ |a|_x \leq 1 \textit{ and } \forall a \in A^{00},\ |a|_x < 1\},$$

and let $\mathrm{Cont}(A)_{\max}$ be the set of maximal points of $\mathrm{Cont}(A)$ (for the order given by specialization).

Then $\mathrm{Cont}(A)_{\max}$ *is the set of rank* 1 *points of* $\mathrm{Cont}(A)$*,* $\mathrm{Cont}(A)_{\max} \subset L$*, and* $\mathrm{Cont}(A)_{\max}$ *is dense in* L *for the constructible topology of* $\mathrm{Spv}(A)$*.*

Proof. As A is a Tate ring, we have $\mathrm{Cont}(A) = \mathrm{Cont}(A)_{\mathrm{an}}$ (see remark II.2.5.7). So, by II.2.4.8, $\mathrm{Cont}(A)_{\max}$ is the set of rank 1 valuations in $\mathrm{Cont}(A)$, and in particular $\mathrm{Cont}(A)_{\max} \subset L$.

Fix a pair of definition (A_0, I) of A with A_0 Noetherian, and let ϖ be a topologically nilpotent unit in A. To show that $\mathrm{Cont}(A)_{\max}$ is dense in L for the constructible topology, we must show that every ind-constructible subset of $\mathrm{Spv}(A)$ that intersects L also intersects $\mathrm{Cont}(A)_{\mathrm{an}}$. So let E be a ind-constructible subset of $\mathrm{Spv}(A)$ such that $E \cap L \neq \varnothing$. It suffices to treat the case where there exist $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_m, d_1, \dots, d_m \in A$ such that

$$E = \{x \in \mathrm{Spv}(A) \mid \forall i \in \{1, \dots, n\},\ |a_i|_x \leq |b_i|_x \text{ and } \forall j \in \{1, \dots, m\},\ |c_j|_x < |d_j|_x\}$$

(these sets form a base of the constructible topology). Let $x \in E \cap L$, and denote the canonical map $A \to A/\wp_x \subset K(x)$ by f. We may assume that there exists $r \in \{0, \dots, n\}$ such that $f(b_1), \dots, f(b_r) \neq 0$ and $f(b_{r+1}) = \dots = f(b_n) = 0$. Let B be the subring of $K(x)$ generated by $f(A_0)$, the $\frac{f(a_i)}{f(b_i)}$ for $1 \leq i \leq r$ and the $\frac{f(c_j)}{f(d_j)}$ for $1 \leq j \leq m$. Then B is Noetherian, $B \subset R_x$ and $B \cap \mathfrak{m}_{R_x}$ contains the $\frac{f(c_j)}{f(d_j)}$ for $1 \leq j \leq m$ and $f(I)$ (because $I \subset A^{00}$). As the prime ideal $\wp_x$ is not open (because A is Tate, so its only open ideal is A itself), it does not contain I, so $f(I) \neq \{0\}$ and so $\wp := B \cap \mathfrak{m}_{R_x} \neq \{0\}$. The Noetherian local ring $B_\wp$ is not a field, and K is a finitely generated extension of $\mathrm{Frac}(B_\wp)$ (it is generated by $f(\varpi^{-1})$, by proposition II.2.5.2), so, by EGA II 7.1.7, there exists a discrete valuation subring R of K such that $B_\wp \subset R$ and $\mathfrak{m}_R \cap B_\wp = \wp B_\wp$. Let y be the corresponding valuation on A (i.e. the composition of f and of $|.|_R$). We have $\wp_y = \wp_x$, and $R_y = R$ contains B, so $|a_i|_y \leq |b_i|_y$ for $1 \leq i \leq r$, $|a_i|_y = |b_i|_y = 0$ for $r + 1 \leq i \leq n$, $|c_j|_y < |d_j|_y$ for $1 \leq j \leq m$ and $|f|_y \leq 1$ for every $f \in A_0$. In particular, $y \in E$. We want to show that $y \in \mathrm{Cont}(A)_{\max}$. As y has rank 1, it suffices to show that y is continuous by the first paragraph of the proof. As y is discrete, it is continuous if and only if $\{a \in A \mid |a|_y \leq 1\}$ is open; but this subring contains A_0 and A_0 is open, so we are done.

□

Propositions III.1.4 and III.1.5 suggest that it reasonable to consider the sets $\mathrm{Spa}(A, \Sigma)$ when Σ is an open and integrally closed subring of A contained in A^0 (because then $\mathrm{Spa}(A, \Sigma)$ is dense in $\mathrm{Cont}(A)$ and determines Σ). We give a special name to these rings.

Definition III.1.7. A *ring of integral elements* in A is an open and integrally closed subring A^+ of A such that $A^+ \subset A^0$. We also say that (A, A^+) is an *affinoid ring* (in Huber's terminology) or a *Huber pair* (in other people's terminology). A *morphism of Huber pairs* $\varphi : (A, A^+) \to (B, B^+)$ is a continuous ring morphism $\varphi : A \to B$ such that $\varphi(A^+) \subset B^+$; it is called *adic* if the morphism $A \to B$ is adic.

We say that the Huber pair (A, A^+) is Tate (resp. adic, resp. complete) if A is.

Example III.1.8. (1) The biggest ring of integral elements is A^0, and the smallest one in the integral closure of $\mathbb{Z} \cdot 1 + A^{00}$. More generally, if $A' \subset A^0$ is an open subring of A, then its integral closure (in A) is a ring of integral elements.

(2) If K is a topological field whose topology is defined by a valuation x, then, for any vertical specialization y of x, R_y is a ring of integral elements in K. Indeed, R_y is integrally closed in K because it is a valuation subring, and it contains the maximal ideal of R_x by theorem I.1.4.2(i), so it is open.

For example, if we take $K = k((t))((u))$ with the topology given by the u-adic valuation, then $\{f \in k((t))[[u]] \mid f(0) \in k[[t]]\}$ is a ring of integral elements in K.

III.2 The adic spectrum of a Huber pair

Definition III.2.1. Let (A, A^+) be a Huber pair. Then its *adic spectrum* is the topological space $\mathrm{Spa}(A, A^+)$.

A *rational subset* of $\mathrm{Spa}(A, A^+)$ is a subset of the form

$$R\left(\frac{f_1, \dots, f_n}{g}\right) = \{x \in \mathrm{Spa}(A, A^+) \mid \forall i \in \{1, \dots, n\},\ |f_i|_x \le |g|_x \ne 0\},$$

with $f_1, \dots, f_n, g \in A$ such that $f_1, \dots, f_n$ generate an open ideal of A.

We also write $\mathrm{Spa}(A, A^+)_{\mathrm{an}} = \mathrm{Spa}(A, A^+) \cap \mathrm{Cont}(A)_{\mathrm{an}}$ for the set of analytic points of $\mathrm{Spa}(A, A^+)$.

Remark III.2.0.1. Let $f_1, \dots, f_n, g, f_1', \dots, f_m', g' \in A$ such that the ideals $(f_1, \dots, f_n)$ and $(f_1', \dots, f_m')$ are open. Then

Example III.2.2. If A^+ is the integral closure of $\mathbb{Z} \cdot 1 + A^{00}$, then $\mathrm{Spa}(A, A^+) = \mathrm{Cont}(A)$.

Remark III.2.3. Suppose that (A, A^+) is a Huber pair, with A a Tate ring, and let $R\left(\frac{f_1, \dots, f_n}{g}\right)$ be a rational subset of $\mathrm{Spa}(A, A^+)$. As the only open ideal of A is A itself, there exist $a_1, \dots, a_n \in A$ such that $a_1 f_1 + \ldots + a_n f_n = 1$. In particular, for every $x \in \mathrm{Spv}(A)$, we have $1 = |1|_x = \max_{1 \le i \le n} |a_i|_x |f_i|_x$, so there exists $i \in \{1, \dots, n\}$ such that $|f_i|_x \ne 0$. So we have

$$R\left(\frac{f_1, \dots, f_n}{g}\right) = \{x \in \mathrm{Spa}(A, A^+) \mid \forall i \in \{1, \dots, n\},\ |f_i|_x \le |g|_x\}.$$

(That is, we can delete the condition "$|g|_x \ne 0$" from the definition, because it follows from the other conditions.)

Corollary III.2.4. *Let (A, A^+) be a Huber pair. Then $\mathrm{Spa}(A, A^+)$ is a spectral space, and the rational subsets are open quasi-compact and form a base of the topology of $\mathrm{Spa}(A, A^+)$.*

Proof. By proposition III.1.4(i), $\mathrm{Spa}(A, A^+)$ is a pro-constructible subset of $\mathrm{Cont}(A)$. By corollary II.2.2.3, $\mathrm{Cont}(A)$ is spectral, and it has base of quasi-compact open subsets given by the

$$U_{\mathrm{cont}}\left(\frac{f_1, \ldots, f_n}{g}\right) = \{x \in \mathrm{Cont}(A) \mid \forall i \in \{1, \ldots, n\},\ |f_i|_x \leq |g|_x \neq 0\},$$

for $f_1, \ldots, f_n, g \in A$ such that $f_1, \ldots, f_n$ generate an open ideal of A. By proposition I.3.1.3(i), $\mathrm{Spa}(A, A^+)$ is spectral, and the inclusion $\mathrm{Spa}(A, A^+) \to \mathrm{Cont}(A)$ is spectral (i.e. quasi-compact). The result follows from this and from the fact that $R\left(\frac{f_1,\ldots,f_n}{g}\right) = \mathrm{Spa}(A, A^+) \cap U_{\mathrm{cont}}\left(\frac{f_1,\ldots,f_n}{g}\right)$.

□

Let $\varphi : (A, A^+) \to (B, B^+)$ be a morphism of Huber pairs. Then the continuous map $\mathrm{Spv}(B) \to \mathrm{Spv}(A)$ restricts to a continuous map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$, which we will denote by $\mathrm{Spa}(\varphi)$. The basic properties of these maps are given in the next proposition.

Proposition III.2.5. *(Proposition 3.8 of [14].) Let $\varphi : (A, A^+) \to (B, B^+)$ be a morphism of Huber pairs. Then*

(i) If $x \in \mathrm{Spa}(B, B^+)$ is not analytic, then $\mathrm{Spa}(\varphi)(x)$ is not analytic.

(ii) If φ is adic, then $\mathrm{Spa}(\varphi)$ sends $\mathrm{Spa}(B, B^+)_{\mathrm{an}}$ to $\mathrm{Spa}(A, A^+)_{\mathrm{an}}$.

(iii) If B is complete and $\mathrm{Spa}(\varphi)$ sends $\mathrm{Spa}(B, B^+)_{\mathrm{an}}$ to $\mathrm{Spa}(A, A^+)_{\mathrm{an}}$, then φ is adic.

(iv) If φ is adic, then the inverse image by $\mathrm{Spa}(\varphi)$ of any rational domain of $\mathrm{Spa}(A, A^+)$ is a rational domain of $\mathrm{Spa}(B, B^+)$. In particular, $\mathrm{Spa}(\varphi)$ is spectral.

Proof. We write $f = \mathrm{Spa}(\varphi)$.

(i) We have $\mathrm{supp}(f(x)) = \varphi^{-1}(\mathrm{supp}(x))$, so $\mathrm{supp}(f(x))$ is open if $\mathrm{supp}(x)$ is open.

(ii) Let (A_0, I) be a couple of definition of A and B_0 be a ring of definition of B such that $\varphi(A_0) \subset B_0$ and $\varphi(I)B_0$ is an ideal of definition of B_0. Let $x \in \mathrm{Spa}(B, B^+)$. If $f(x)$ is not analytic, then $\mathrm{supp}(f(x)) = \varphi^{-1}(\mathrm{supp}(x))$ is an open prime ideal of A, so it contains I, so $\mathrm{supp}(x)$ contains $f(I)$, which implies that $\mathrm{supp}(x)$ is an open ideal of B and that x is not analytic.

(iii) Suppose that φ is not adic. Choose a couple of definition (A_0, I) and (B_0, J) of A and B such that $\varphi(A_0) \subset B_0$ and $\varphi(I) \subset J$. As φ is not adic, we have $\sqrt{\varphi(I)B_0} \neq \sqrt{J}$, so there exists a prime ideal $\wp$ of B_0 such that $\varphi(I) \subset \wp$ and $J \not\subset \wp$. Since B_0 is J-adically complete, J is contained in the Jacobson radical of B_0 by [5] Chapitre III §2 N°13 lemma 3 (the idea is that every $a \in J$ is topologically nilpotent, so $1 - a$ is invertible with inverse $\sum_{n \geq 0} a^n$). So there exists a prime ideal $\mathfrak{q}$ of B_0 containing both J and $\wp$ (take for example any maximal ideal of B_0). Let R be a valuation subring of $\mathrm{Frac}(B_0/\wp)$ dominating the local subring $(B_0/\wp)_{\mathfrak{q}/\wp}$, and let x be the corresponding valuation on B_0; in particular, we have $\mathrm{supp}(x) = \wp$ and $|a|_x < 1$ for every $a \in J$. Let $r : \mathrm{Spv}(B_0) \to \mathrm{Spv}(B_0, J)$ be the retraction introduced in definition I.4.1.6. As $J \not\subset \mathrm{supp}(x)$, we have $J \not\subset \mathrm{supp}(r(x))$ by theorem I.4.2.4(iv). Also, as $r(x)$ is a horizontal specialization of x, we have $|a|_{r(x)} \leq |a|_x < 1$ for every $a \in J$. So $r(x) \in \mathrm{Cont}(B_0)$ by theorem II.2.2.1. Let y be the unique point of $\mathrm{Cont}(B)_{\mathrm{an}}$ such that the restriction of $|.|_y$ to B_0 is $|.|_{r(x)}$ (see lemma III.2.6). Let $z \in \mathrm{Cont}(B)_{\mathrm{an}}$ be the unique rank 1 vertical generization of y (see corollary II.2.4.8). Then $z \in \mathrm{Spa}(B, B^+)$ because z has rank 1. On the other hand, $\mathrm{supp}(z) = \mathrm{supp}(y) \supset \mathrm{supp}(r(x)) \supset \mathrm{supp}(x) = \wp$, so $\mathrm{supp}(f(z)) = \varphi^{-1}(\wp) \supset I$ is open and so $f(z)$ is not analytic.

(iv) Let $f_1, \ldots, f_n, g \in A$ such that $f_1, \ldots, f_n$ generate an open ideal $\mathfrak{a}$ of A. Then $\mathfrak{a}$ contains any ideal of definition of a ring of definition of B, and, as φ is adic, $\varphi(\mathfrak{a}) = (\varphi(f_1), \ldots, \varphi(f_n))$ is an open ideal of B. As we clearly have

$$f^{-1}\left(R\left(\frac{f_1,\ldots,f_n}{g}\right)\right) = R\left(\frac{\varphi(f_1),\ldots,\varphi(f_n)}{\varphi(g)}\right),$$

this proves the result.

□

Lemma III.2.6. *(Lemma 3.7 of [14], lemma 7.44 of [26].) Let B be an open subring of A. Remember that B is f-adic by corollary II.1.1.8(ii). We consider the commutative square*

$$\begin{array}{ccc} \mathrm{Spv}(A) & \xrightarrow{g} & \mathrm{Spv}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow[f]{} & \mathrm{Spec}(B) \end{array}$$

where the horizontal maps are induced by the inclusion $B \subset A$.

(i) If $T \subset \mathrm{Spec}(B)$ is the subset of open prime ideals, then $f^{-1}(T) \subset \mathrm{Spec}(A)$ is the subset of open prime ideals, and f induces a homeomorphism $\mathrm{Spec}(A) - f^{-1}(T) \xrightarrow{\sim} \mathrm{Spec}(B) - T$.

(ii) $\mathrm{Cont}(A) = g^{-1}(\mathrm{Cont}(B))$.

(iii) The restriction of g to $\mathrm{Cont}(A)_{\mathrm{an}}$ induces a homeomorphism $\mathrm{Cont}(A)_{\mathrm{an}} \xrightarrow{\sim} \mathrm{Cont}(B)_{\mathrm{an}}$, and, for every $x \in \mathrm{Cont}(A)_{\mathrm{an}}$, the canonical injection $\Gamma_{g(x)} \to \Gamma_x$ is an isomorphism.

Proof. (i) Let $\wp \in \mathrm{Spec}(A)$. As B is an open subring of A, $\wp$ is open in A if and only if $\wp \cap B$ is open (in B or A). This proves the first statement.

Let $\mathfrak{q} \in \mathrm{Spec}(B) - T$. Then $\mathfrak{q}$ does not contains B^{00}; we fix $s \in B^{00} - \mathfrak{q}$. As B is open in A, for every $a \in A$, we have $s^n a \in B$ for n big enough. So the injective ring morphism $B_s \to A_s$ is also surjective, and the map $\mathrm{Spec}(B_s) \to \mathrm{Spec}(A_s)$ is a homeomorphism. As $\mathrm{Spec}(B) - T = \bigcup_{s \in B^{00}} \mathrm{Spec}(B_s)$ and $\mathrm{Spec}(A) - f^{-1}(T) = \bigcup_{s \in B^{00}} \mathrm{Spec}(A_s)$ (as B is open in A, so is B^{00}, so it cannot be contained in a non-open prime ideal of A), we get the second statement.

(iii) If $x \in \operatorname{Cont}(A)_{\mathrm{an}}$, then $g(x) \in \operatorname{Cont}(B)$, and the support of $g(x)$ is not open by (i), so $g(x) \in \operatorname{Cont}(B)_{\mathrm{an}}$.

Let $x \in \operatorname{Cont}(A)_{\mathrm{an}}$ and $y \in \operatorname{Spv}(A)$ such that $g(x) = g(y)$ (that is, such that $|.|_x$ and $|.|_y$ restrict to the same valuation on B). As x is analytic and B^{00} is an open subgroup of A (hence not contained in $\operatorname{supp}(x)$), there exists $b \in B^{00}$ such that $|b|_x = |b|_y \neq 0$. Let $a \in A$. Then there exists $n \geq 1$ such that $b^n a \in B$, and we get $|a|_x = |b^n a|_x |b|_x^{-n} = |b^n a|_y |b|_y^{-n} = |a|_y$. So $x = y$.

Let $x \in \operatorname{Cont}(B)_{\mathrm{an}}$. By (i), there exists a non-open prime ideal $\wp$ of A such that $\wp \cap B = \wp_x$. By proposition I.1.2.3, we can extend the valuation $|.|_x$ on $K(x) = \operatorname{Frac}(B/\wp_x)$ to a valuation on $\operatorname{Frac}(A/\wp)$; by composing with $A \to A/\wp$, we get an element y of $\operatorname{Spv}(A)$ such that $g(y) = x$. Also, it follows from the proof of (i) that $B_{\wp_x} \to A_\wp$ is an isomorphism, so $K(x) \to K(y)$ is an isomorphism and the injection $\Gamma_x \to \Gamma_y$ is an isomorphism. Also, for every $\gamma \in \Gamma_y$, the group $\{a \in A \mid |a|_y < \gamma\}$ contains the open subgroup $\{b \in B \mid |a|_x < \gamma\}$, so it is open; this shows that $y \in \operatorname{Cont}(A)$. Finally, we have constructed an element $y \in \operatorname{Cont}(A)_{\mathrm{an}}$ such that $g(y) = x$.

We have shown that $g : \operatorname{Cont}(A)_{\mathrm{an}} \to \operatorname{Cont}(B)_{\mathrm{an}}$ is bijective and continuous. Also, if R is a rational subset of $\operatorname{Cont}(B)$, then $g(R)$ is a rational subset of $\operatorname{Cont}(A)$ (because, if $f_1, \ldots, f_n \in B$ generate an open ideal of B, they also generate an open ideal of A); so g is open, hence $g : \operatorname{Cont}(A)_{\mathrm{an}} \to \operatorname{Cont}(B)_{\mathrm{an}}$ is a homeomorphism.

(ii) If $x \in \operatorname{Cont}(A)$, then we clearly have $g(x) \in \operatorname{Cont}(B)$. Let $x \in \operatorname{Spv}(A)$ such that $y := g(x) \in \operatorname{Cont}(B)$. We want to show that $x \in \operatorname{Cont}(A)$. If $y \in \operatorname{Cont}(B)_{\mathrm{an}}$, then there exists $x' \in \operatorname{Cont}(A)_{\mathrm{an}}$ such that $g(x') = y$ (by (ii)), and we have in the beginning of the proof of (ii) that this implies that $x = x' \in \operatorname{Cont}(A)_{\mathrm{an}}$. If $y \notin \operatorname{Cont}(B)_{\mathrm{an}}$, then $\operatorname{supp}(y)$ is open, so $\operatorname{supp}(x) \supset \operatorname{supp}(y)$ is also open, and x is continuous (because $\{a \in A \mid |a|_x < \gamma\}$ contains $\operatorname{supp}(x)$, hence is open, for every $\gamma \in \Gamma_x$).

□

III.3 Perturbing the equations of a rational domain

The goal of this section is to show that a rational domain $R\left(\frac{f_1,\ldots,f_n}{g}\right)$ is not affected by a small perturbation of $f_1, \ldots, f_n, g$. This result is very important in many parts of the theory (for example the proof that rational domain are preserved when we complete the Huber pair or, if it is a perfectoid Huber pair, when we tilt it).

Theorem III.3.1. *(Lemma 3.10 of [14].) Let A be a complete f-adic ring and let $f_1, \ldots, f_n, g$ be elements of A such that $f_1, \ldots, f_n$ generate an open ideal of A. Then there exists a neighborhood V of 0 in A such that, for all $f'_1, \ldots, f'_n, g' \in A$, if $f'_i \in f_i + U$ for $i \in \{1, \ldots, n\}$ and $g' \in g + U$,* *then the ideal of A generated by $f'_1, \ldots, f'_n$ is open and*

$$U_{\mathrm{cont}}\left(\frac{f_1, \ldots, f_n}{g}\right) = U_{\mathrm{cont}}\left(\frac{f'_1, \ldots, f'_n}{g'}\right).$$

Proof. Let $U = U_{\mathrm{cont}}\left(\frac{f_1,\ldots,f_n}{g}\right)$.

Let A_0 be a ring of definition of A, and let J be the ideal of A generated by $f_1, \ldots, f_n$. As $A_0 \cap J$ is open, it contains an ideal of definition I of A_0. Choose generators $a_1, \ldots, a_r$ of I. By lemma III.3.2, if $a'_1 \in a_1 + I, \ldots, a'_r \in a_r + I^2$, then $I = a'_1 A_0 + \ldots + a'_r A_0$. For $j \in \{, \ldots, r\}$, write $a_j = \sum_{i=1}^n a_{ij} f_i$, with $a_{1j}, \ldots, a_{nj} \in A$. Let W be a neighborhood of 0 in A such that, for every $b \in W$ and every $(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, r\}$, we have $a_{ij} b \in I^2$. Then, if $f'_1, \ldots, f'_n \in A$ are such that $f'_i \in f_i + W$ for every i, the elements $\sum_{i=1}^n a_{ij} f'_i$, $1 \leq j \leq r$, of A are in A_0 and generate the ideal I; in particular, the ideal of A generated by $f'_1, \ldots, f'_n$ contains I, so it is open.

Set $f_0 = g$. For every $i \in \{0, \ldots, n\}$, we write $U_i = U_{\mathrm{cont}}\left(\frac{f_0,\ldots,f_n}{f_i}\right)$; note that $U_0 = U$, that $U_0, \ldots, U_n$ are open quasi-compact, and that, for every $i \in \{0, \ldots, n\}$ and every $x \in U_i$, we have $|f_i|_x \neq 0$. By lemma III.3.3, for every $i \in \{0, \ldots, n\}$, there exists a neighborhood V_i of 0 in A such that, for every $x \in U_i$ and every $f \in V_i$, we have $|f|_x < |f_i|_x$.

Let $V = A^{00} \cap W \cap V_0 \cap \ldots \cap V_n$. This is a neighborhood of 0 in A. Let $f'_1, \ldots, f'_n, g \in A$ such that $f'_i \in f_i + V$ for every $i \in \{0, \ldots, n\}$ and $g' \in g + V$. We have already seen that $f'_1, \ldots, f'_n$ generate an open ideal of A. Let $U' = U_{\mathrm{cont}}\left(\frac{f'_1,\ldots,f'_n}{g'}\right)$. We want to show that $U_0 = U'$. Write $f'_0 = g'$.

We first prove that $U_0 \subset U'$. Let $x \in U_0$. For $i \in \{0, \ldots, n\}$, we have $f'_i - f_i \in V_0$, so $|f'_i - f_i|_x < |f_0|_x$. In particular, taking $i = 0$ we get $|f'_0|_x = |f_0|_x$. If $i \in \{1, \ldots, n\}$, then $|f_i|_x \leq |f_0|_x = |f'_0|_x$ and $|f'_i - f_i|_x < |f_0|_x$ imply that $|f'_i|_x \leq \max(|f_i|_x, |f'_i - f_i|_x) \leq |f'_0|_x$. So $x \in U'$.

Conversely, we prove that $U' \subset U_0$. Let $x \in \mathrm{Cont}(A) - U_0$. If $|f_i|_x = 0$ for every $i \in \{0, \ldots, n\}$, then $\mathrm{supp}(x) \supset (f_1, \ldots, f_n)$ is open, so it contains A^{00}, so $f'_0 - f_0 \in \mathrm{supp}(x)$, so $|f'_0|_x = 0$ and $x \notin U'$. From now on, we assume that there exists $i \in \{0, \ldots, n\}$ such that $|f_i|_x \neq 0$. Choose $j \in \{0, \ldots, n\}$ such that $|f_j|_x = \max_{0 \leq i \leq n} |f_i|_x$; in particular, we have $|f_j|_x \neq 0$. As $x \notin U_0$, we have $|f_0|_x < |f_j|_x$. Note also that $x \in U_j$. For every $i \in \{0, \ldots, n\}$, we have $f'_i - f_i \in V_j$, so $|f'_i - f_i|_x < |f_j|_x$. In particular, $|f'_j - f_j|_x < |f_j|_x$, so $|f_j|_x = |f'_j|_x$. On the other hand, $|f'_0|_x \leq \max(|f_0|_x, |f'_0 - f_0|_x) < |f_j|_x = |f'_j|_x$, so $x \notin U'$.

□

Lemma III.3.2. *Let A_0 be a* complete *adic ring, let I be an ideal of definition of A_0, and suppose that we have elements $a_1, \ldots, a_r \in I$ such that $I = (a_1, \ldots, a_r)$. Then, for all $a'_1, \ldots, a'_r \in A$ such that $a'_i - a_i \in I^2$ for every $i \in \{1, \ldots, r\}$, we have $I = (a'_1, \ldots, a'_r)$.*

Proof. As $I^2 \subset I$, we have $a'_1, \dots, a'_r \in I$, so $(a'_1, \dots, a'_r) \subset I$. Consider the A-linear map $u' : A^r \to I$, $(b_1, \dots, b_r) \longmapsto b_1 a'_1 + \dots + b_r a'_r$. We want to show that u' is surjective. For every $n \geq 0$, we have $u'(I^n A^r) \subset I^{n+1}$. So, by [5] Chapitre III §2 N°8 corollaire 2 du théorème 1, it suffices to show that, for every $n \geq 0$, the map

$$\mathrm{gr}_n(u') : (I^n A^r)/(I^{n+1} A^r) \to I^{n+1}/I^{n+2}$$

induced by u' is surjective. But we have $\mathrm{gr}_n(u') = \mathrm{gr}_n(u)$, where u is the map $A^r \to I$, $(b_1, \dots, b_r) \longmapsto b_1 a_1 + \dots + b_r a_r$, and all the $\mathrm{gr}_n(u)$ are surjective because $I = (a_1, \dots, a_r)$.

□

Lemma III.3.3. *(Lemma 3.11 of [14].) Let (A, A^+) be a Huber pair, X be quasi-compact subset of $\mathrm{Spa}(A, A^+)$ and s be an element of A such that $|s|_x \neq 0$ for every $x \in X$. Then there exists a neighborhood V of 0 in A such that, for every $x \in X$ and $a \in U$, we have $|a|_x < |s|_x$.*

In particular, there exists a finite subset T of A such that the ideal $T \cdot A$ is open and that, for every $t \in T$, we have $|t|_x < |s|_x$.

Proof. Let T be a finite subset of A^{00} such that $T \circ A^{00}$ is open (for example a set of generators of an ideal of definition of a ring of definition of A). For every $n \geq 1$, the group $T^n \circ A^{00}$ is open by lemma II.3.3.5, so

$$X_n = \{x \in \mathrm{Spa}(A, A^+) \mid \forall t \in T^n,\ |t|_x \leq |s|_x \neq 0\}$$

is a rational subset of $\mathrm{Spa}(A, A^+)$, and in particular it is open and quasi-compact. Also, as every element of T is topologically nilpotent and T is finite, for every $x \in \mathrm{Spa}(A, A^+)$ such that $|s|_x \neq 0$, we have $x \in X_n$ for n big enough. In particular, we have $X \subset \bigcup_{n \geq 1} X_n$. But X is quasi-compact, so there exists $n \geq 1$ such that $X \subset X_n$. If we take $V = T^n \cdot A^{00}$, then V is an open subgroup of A and $|a|_x < |s|_x$ for every $x \in X_n \supset X$.

We prove the second statement. Let (A_0, I) be a couple of definition of A. As V is open, it contains some power of I. Replacing I by this power, we may assume that $I \subset V$. Then we can take for T a finite set of generators of I.

□

III.4 First properties of the adic spectrum

III.4.1 Quotients

Notation III.4.1.1. Let (A, A^+) be a Huber pair, and let $\mathfrak{a}$ be an ideal of A. The quotient Huber pair $(A, A^+)/\mathfrak{a} = (A/\mathfrak{a}, (A/\mathfrak{a})^+)$ is defined by taking $(A/\mathfrak{a})^+$ to be the integral closure of $A^+/(A^+ \cap \mathfrak{a})$ in $A/\mathfrak{a}$.

Proposition III.4.1.2. *Let (A, A^+) be a Huber pair, and let $\mathfrak{a}$ be an ideal of A. We denote the canonical map $(A, A^+) \to (A, A^+)/\mathfrak{a}$ by φ. Then $\mathrm{Spa}(\varphi)$ induces a homeomorphism from $\mathrm{Spa}((A, A^+)/\mathfrak{a})$ to the closed subset*

$$\mathrm{Spa}(A, A^+) \cap \mathrm{supp}^{-1}(V(\mathfrak{a})) = \{x \in \mathrm{Spa}(A, A^+) \mid \mathrm{supp}(x) \supset \mathfrak{a}\}.$$

A closed subset of $\mathrm{Spa}(A, A^+)$ as in the proposition is called a *Zariski closed subset of* $\mathrm{Spa}(A, A^+)$. Not every closed subset of $\mathrm{Spa}(A, A^+)$ is Zariski closed.

Proof. It is easy to see that $\mathrm{Spa}(\varphi)$ is injective with image $\mathrm{Spa}(A, A^+) \cap \mathrm{supp}^{-1}(V(\mathfrak{a}))$. Also, if $f_1, \dots, f_n, g$ are such that $f_1, \dots, f_n$ generate an open ideal J of A and if $\overline{f}_1, \dots, \overline{f}_n, \overline{g}$ are their images in $A/\mathfrak{a}$, then $\overline{f}_1, \dots, \overline{f}_n$ generate the ideal $(J + \mathfrak{a})/\mathfrak{a}$ of $A/\mathfrak{a}$, which is clearly open, and the image by $\mathrm{Spa}(\varphi)$ of $R\left(\frac{\overline{f}_1, \dots, \overline{f}_n}{\overline{g}}\right)$ is $R\left(\frac{f_1, \dots, f_n}{g}\right) \cap \mathrm{supp}^{-1}(V(\mathfrak{a}))$. This finishes the proof.

□

III.4.2 Spa and completion

Definition III.4.2.1. Let (A, A^+) be a Huber pair. Its *completion* is the pair $(\widehat{A}, \widehat{A}^+)$, where $\widehat{A}^+$ is the closure of the image of A^+ in $\widehat{A}$.

By lemma III.4.2.3, the completion of a Huber pair is a Huber pair.

Corollary III.4.2.2. *(Proposition 3.9 of [14].) Let (A, A^+) be a Huber pair. Then the canonical map $\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+)$ is a homeomorphism, and a subset of $\mathrm{Spa}(\widehat{A}, \widehat{A}^+)$ is a rational domain if and only if its image in $\mathrm{Spa}(A, A^+)$ is a rational domain.*

Proof. Let $\varphi : A \to \widehat{A}$ be the canonical map. By proposition II.3.1.12(iii), φ induces a bijection $\mathrm{Cont}(\widehat{A}) \to \mathrm{Cont}(A)$. It follows immediately from the definition of $\widehat{A}^+$ that φ induces a morphism of Huber pairs $(A, A^+) \to (\widehat{A}, \widehat{A}^+)$ and that $\mathrm{Spa}(\varphi) : \mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+)$ is a bijection. Also, as φ is adic, the inverse image by $\mathrm{Spa}(\varphi)$ of a rational domain of $\mathrm{Spa}(A, A^+)$ is a rational domain by proposition III.2.5(iv). It remains to show that $\mathrm{Spa}(\varphi)$ maps rational subsets to rational subsets.

Let R be a rational subset of $\mathrm{Spa}(\widehat{A}, \widehat{A}^+)$. As $\varphi(A)$ is dense in $\widehat{A}$, by theorem III.3.1, there exist $f_1, \dots, f_n, g \in A$ such that $\varphi(f_1), \dots, \varphi(f_n)$ generate an open ideal of $\widehat{A}$ and

$$R = R\left(\frac{\varphi(f_1), \dots, \varphi(f_n)}{\varphi(g)}\right).$$

We would be done if we knew that $f_1, \dots, f_n$ generate an open ideal of A, but we don't. On the other hand, as $\mathrm{Spa}(\varphi)(R)$ is a quasi-compact subset of $\mathrm{Spa}(A, A^+)$ and $|g|_x \neq 0$ for every

$x \in \mathrm{Spa}(\varphi)(R)$, we know by lemma III.3.3 that there exists $f'_1, \dots, f'_m \in A$ generating an open ideal of A and such that $|f'_j|_x < |g|_x$ for every $j \in \{1, \dots, m\}$. Then

$$\mathrm{Spa}(\varphi)(R) = R\left(\frac{f_1, \dots, f_n, f'_1, \dots, f'_m}{g}\right).$$

□

Lemma III.4.2.3. *Let A be a f-adic ring. If we have open sugroups of G and H of A and $\widehat{A}$ that correspond to each other by the bijection of lemma II.3.1.11, then G is a ring of integral elements of A if and only if H is a ring of integral elements of $\widehat{A}$.*

Proof. Let $i : A \to \widehat{A}$ be the canonical map. As $i(A)$ is dense in $\widehat{A}$, it is easy to see that G is a subring of A if and only if H is a subring of $\widehat{A}$. By proposition II.3.1.12(i), we have $G \subset A^0$ if and only $H \subset \widehat{A}^0$.

Suppose that G is an open and integrally closed subring of A. We want to prove that H is integrally closed in $\widehat{A}$. Let $x \in \widehat{A}$ be integral over H, and write $x^d + a_1 x^{d-1} + \ldots + a_d = 0$, with $d \geq 1$ and $a_1, \dots, a_d \in H$. As H is an open neighborhood of 0 in $\widehat{A}$, we can find $x' \in A$ and $a'_1, \dots, a'_d \in G$ such that $x - i(x') \in H$ and $(x^d + a_1 x^{d-1} + \ldots + a_d) - (i(x')^d + i(a'_1) i(x')^{d-1} + \ldots + i(a'_d)) \in H$. But then $x'^d + a'_1 x'^{d-1} + \ldots + a_d \in G$, so $x' \in G$ as G is integrally closed, and $x = (x - i(x')) + i(x') \in H$.

Conversely, suppose that H is an open and integrally closed subring of $\widehat{A}$. Then, if $x \in A$ is integral over G, then $i(x)$ is integral over H, so $i(x) \in H$, so $x \in G$. Hence G is integrally closed.

□

III.4.3 Rational domains and localizations

Notation III.4.3.1. Let (A, A^+) be a Huber pair, let $X = (X_i)_{i \in I}$ be a family of indeterminates, and let $T = (T_i)_{i \in I}$ be a family of subsets of A such that T_i generates an open ideal of A for every $i \in I$.

(1) Remember that $A[X]_T$ is f-adic by proposition II.3.3.6. We denote by $A[X]_T^+$ the integral closure in $A[X]_T$ of the open subring $A^+_{[X,T]}$. So we get a Huber pair $(A[X]_T, A[X]_T^+)$.

(2) If A is complete and I is finite, we denote by $A\langle X\rangle_T^+$ the integral closure in $A\langle X\rangle_T$ of the open subring $A^+_{\langle X,T\rangle}$. We get a complete Huber pair $(A\langle X\rangle_T, A\langle X\rangle_T^+)$.

Now suppose that I is a singleton, so T is a subset of A that generates an open ideal, and let $s \in A$.

(3) Denote by $A\left(\frac{T}{s}\right)^+$ the integral closure in $A\left(\frac{T}{s}\right)$ of the A^+-subalgebra generated by all the ts^{-1}, for $t \in T$ (this A^+-algebra is open in $A\left(\frac{T}{s}\right)$ by definition of the topology in the proof of proposition II.3.4.1). Then $(A\left(\frac{T}{s}\right), A\left(\frac{T}{s}\right)^+)$ is a Huber pair.

It is also canonically isomorphic to the Huber pair $(A[X]_T, A[X]_T^+)/(1 - sX)$ (see III.4.1.1).

(4) By combining (3) and definition III.4.2.1, we get a Huber pair $(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$.

Corollary III.4.3.2. *Let (A, A^+) be a Huber pair, and let $f_1, \dots, f_n, g \in A$ such that $f_1, \dots, f_n$ generate an open ideal of A. Let $\varphi : (A, A^+) \to (A\left(\frac{f_1,\dots,f_n}{g}\right), A\left(\frac{f_1,\dots,f_n}{g}\right)^+)$ be the canonical map.*

Then $\mathrm{Spa}(\varphi)$ induces a homeomorphism $\mathrm{Spa}(A\left(\frac{f_1,\dots,f_n}{g}\right), A\left(\frac{f_1,\dots,f_n}{g}\right)^+) \overset{\sim}{\to} R\left(\frac{f_1,\dots,f_n}{g}\right)$, and a subset R of $\mathrm{Spa}(A\left(\frac{f_1,\dots,f_n}{g}\right), A\left(\frac{f_1,\dots,f_n}{g}\right)^+)$ is a rational domain if and only if $\mathrm{Spa}(\varphi)(R)$ is a rational domain (in $\mathrm{Spa}(A, A^+)$).

Proof. We write $f = \mathrm{Spa}(\varphi)$, $(B, B^+) = (A\left(\frac{f_1,\dots,f_n}{g}\right), A\left(\frac{f_1,\dots,f_n}{g}\right)^+)$ and $U = R\left(\frac{f_1,\dots,f_n}{g}\right)$. As φ is spectral, the inverse image by f of a rational domain of $\mathrm{Spa}(A, A^+)$ is a rational domain of $\mathrm{Spa}(B, B^+)$ (proposition III.2.5(iv)). So it suffices to prove that f sends rational domain to rational domains and induces a bijection from $\mathrm{Spa}(B, B^+)$ to U.

As the underlying ring of B is just $A[g^{-1}]$, if we have two valuations on B that coincide on the image of A, then they coincide on B. So f is injective.

Let $x \in \mathrm{Spa}(B, B^+)$, and let $y = f(x)$. Then $|g|_x = |g|_y \neq 0$ because g is invertible in B. For every $i \in \{1, \dots, n\}$, we have $f_i g^{-1} \in B^+$, so $|f_i g^{-1}|_x \leq 1$, and $|f_i|_y = |f_i|_x \leq |g|_x = |g|_y \neq 0$. So $y \in U$.

Let $y \in U$. Then $|g|_y \neq 0$, so $|.|_y$ extends to a valuation on $A[g^{-1}]$, hence gives a point x of $\mathrm{Spv}(B)$; note that $\Gamma_x = \Gamma_y$. We want to show that $x \in \mathrm{Spa}(B, B^+)$. Let D be the subring of B generated by $f_1 g^{-1}, \dots, f_n g^{-1}$; as $y \in U$, we have $|b|_x \leq 1$ for every $b \in D$. Let $\gamma \in \Gamma_x$. Then $V := \{a \in A \mid |a|_y < \gamma\}$ is an open subgroup of A, and $V \cdot D \subset \{b \in B \mid |b|_x < \gamma\}$; as $V \cdot D$ is open by definition of the topology on B (see the proof of proposition II.3.4.1), this shows that x is continuous. Also, B^+ is the integral closure of $A^+ \cdot D$, and $|b|_x \leq 1$ for every $b \in A^+ \cdot D$, so $x \in \mathrm{Spa}(B, B^+)$ (see proposition III.1.4(ii)).

It remains to show that f sends rational domains of $\mathrm{Spa}(B, B^+)$ to rational domains of $\mathrm{Spa}(A, A^+)$. Let $t_1, \dots, t_m, s \in B$ such that $t_1, \dots, t_m$ generate an open ideal of B, and let $E = R\left(\frac{t_1,\dots,t_m}{s}\right) \subset \mathrm{Spa}(B, B^+)$. After multiplying $t_1, \dots, t_m, s$ by a high enough power of g (which does not affect the condition on $(t_1, \dots, t_m)$ because g is a unit in B), we may assume that $t_1, \dots, t_m, s \in A$. Of course, we don't know that $t_1, \dots, t_m$ generate an open ideal, because it has no reason to be true. But, as $g(E) \subset \mathrm{Spa}(A, A^+)$ is quasi-compact and $|s|_x \neq 0$ for every $x \in g(E)$, we can find by lemma III.3.3 elements $t'_1, \dots, t'_p \in A$ generating an open ideal and such that $|t'_j|_x < |s|_x$ for every $j \in \{1, \dots, p\}$. Then

$$g(E) = R\left(\frac{t_1, \dots, t_m, t'_1, \dots, t'_p}{s}\right),$$

so $g(E)$ is a rational domain of $\mathrm{Spa}(A, A^+)$.

□

Using corollary III.4.2.2, we also get :

Corollary III.4.3.3. *We can replace* $(A\left(\frac{f_1,\dots,f_n}{g}\right), A\left(\frac{f_1,\dots,f_n}{g}\right)^+)$ *with* $(A\langle\frac{f_1,\dots,f_n}{g}\rangle, A\langle\frac{f_1,\dots,f_n}{g}\rangle^+)$ *in the statement of corollary III.4.3.2.*

III.4.4 Non-emptiness

In this section, we give a crierion for $\mathrm{Spa}(A, A^+)$ to be non-empty, and some consequences.

Proposition III.4.4.1. *Let* (A, A^+) *be a Huber pair.*

(i) The following are equivalent :

  (a) $\mathrm{Spa}(A, A^+) = \varnothing$*;*

  (b) $\mathrm{Cont}(A) = \varnothing$*;*

  (c) $A/\overline{\{0\}} = \{0\}$.

(ii) The following are equivalent :

  (a) $\mathrm{Spa}(A, A^+)_{\mathrm{an}} = \varnothing$*;*

  (b) $\mathrm{Cont}(A)_{\mathrm{an}} = \varnothing$*;*

  (c) $A/\overline{\{0\}}$ *has the discrete topology.*

Proof. Note that the support of a continuous valuation x is always a closed prime ideal of A (because it is the intersection of the open and closed subgroups $\{a \in A \mid |a|_x < \gamma\}$, for $\gamma \in \Gamma_x$). In particular, for every $x \in \mathrm{Cont}(A)$, we have $\overline{\{0\}} \subset \mathrm{supp}(x)$.

(ii) If (c) holds, then $\overline{\{0\}}$ is open, so $\mathrm{supp}(x)$ is open for every $x \in \mathrm{Cont}(A)$, and so (b) holds. Also, $\mathrm{Spa}(A, A^+)_{\mathrm{an}}$ is dense in $\mathrm{Cont}(A)_{\mathrm{an}}$ by proposition III.1.5(i), so (a) and (b) are equivalent.

Suppose that (b) holds. We want to prove (c). Let (A_0, I) be a couple of definition of A.

We claim that, if $\wp \subset \mathfrak{q}$ are prime ideals of A_0 and $I \subset \mathfrak{q}$, then $I \subset \wp$. Indeed, assume that $I \not\subset \wp$. Let $x \in \mathrm{Spv}(A_0)$ such that $\mathrm{supp}(x) = \wp$ and that R_x dominates the local subring

$(A_0/\wp)_{\mathfrak{q}/\wp}$ of $K(x) = \mathrm{Frac}(A_{0,x}/\wp)$. Let $r : \mathrm{Spv}(A_0) \to \mathrm{Spv}(A_0, I)$ be the retraction of definition I.4.1.6. Then $r(x) \in \mathrm{Spv}(A_0, I)$ and $|a|_{r(x)} < 1$ for every $a \in \mathfrak{q} \supset I$, so $r(x) \in \mathrm{Cont}(A_0)$ by theorem II.2.2.1. As $I \not\subset \mathrm{supp}(x)$, we have $I \not\subset \mathrm{supp}(r(x))$ by theorem I.4.2.4(iv), so $\mathrm{supp}(r(x))$ is not open, which means that $r(x)$ is analytic. By lemma III.2.6(iii), x extends to an analytic point y of $\mathrm{Cont}(A)$, which contradicts the assumption that $\mathrm{Cont}(A)_{\mathrm{an}} = \varnothing$.

Now we prove that (c) holds. Let $S = 1 + I$, let $B = S^{-1}A_0$ and let $\varphi : A_0 \to B$ be the canonical map. Then $\varphi(I)$ is contained in the Jacobson radical of B, so, if $\wp \in \mathrm{Spec}(B)$, then we can find $\mathfrak{q} \supset \wp$ in $\mathrm{Spec}(B)$ such that $\varphi(I) \subset \mathfrak{q}$ (just take $\mathfrak{q}$ to be a maximal ideal containing $\wp$). By the claim, this implies that every prime ideal of B contains $\varphi(I)$, i.e. that $\varphi(I)$ is contained in the nilradical of B. As I is finitely generated, there exists $n \geq 1$ such that $\varphi(I)^n = \{0\}$. So there exists $a \in I$ such that $(1 + a)I^n = \{0\}$ (in A_0 this time). This implies that $I^n \subset I^{n+1}$, hence that $I^n = I^{n+r}$ for every $r \geq 0$. So $\overline{\{0\}} = I^n$ and the topology on $A/\overline{\{0\}}$ is discrete.

(i) We again have that (a) and (b) are equivalent because $\mathrm{Spa}(A, A^+)$ is dense in $\mathrm{Cont}(A)$ (proposition III.1.5). Also, (c) clearly implies (b). Assume that (b) holds. Then, by (ii), the topology on $A/\overline{\{0\}}$ is discrete. If $\overline{\{0\}} \neq A$, then we can find a prime ideal $\wp$ of A containing $\overline{\{0\}}$, and then $\wp$ is open and the trivial valuation with support $\wp$ is an element of $\mathrm{Cont}(A)$, contradicting (b). So $A = \overline{\{0\}}$.

□

Corollary III.4.4.2. *Let (A, A^+) be a* complete *Huber pair. If $\mathrm{Spa}(A, A^+)_{\mathrm{an}} = \varnothing$, then A is a discrete ring.*

Corollary III.4.4.3. *Let (A, A^+) be a complete Huber pair, and let T be a subset of A. Then the following are equivalent :*

(a) The ideal generated by T is A.

(b) For every $x \in \mathrm{Spa}(A, A^+)$, there exists $t \in T$ such that $|t|_x \neq 0$.

If these conditions are satisfied and T is finite, then $(R\left(\frac{T}{t}\right))_{t \in T}$ is an open covering of $\mathrm{Spa}(A, A^+)$.

Proof. If (a) holds, we can write $1 = a_1t_1 + \ldots + a_nt_n$ with $a_1, \ldots, a_n \in A$ and $t_1, \ldots, t_n \in T$. For every $x \in \mathrm{Spa}(A, A^+)$, we have $1 = |1|_x \leq \max_{1 \leq i \leq n} |a_i|_x |t_i|_x$, so there exists $i \in \{1, \ldots, n\}$ such that $|t_i|_x \neq 0$.

Suppose that (a) does not holds, and let $\mathfrak{m}$ be a maximal ideal of A such that $T \subset \mathfrak{m}$. By lemma III.4.4.5, $\mathfrak{m}$ is a closed ideal of A, hence $A/\mathfrak{m}$ is Hausdorff, so $\mathrm{Spa}((A, A^+)/\mathfrak{m}) \neq \varnothing$ by proposition III.4.4.1(i). As $\mathrm{Spa}((A, A^+)/\mathfrak{m}) \simeq \mathrm{Spa}(A, A^+) \cap \mathrm{supp}(V(\mathfrak{m}))$ by proposition III.4.1.2, there exists $x \in \mathrm{Spa}(A, A^+)$ such that $\mathrm{supp}(x) = \mathfrak{m}$. As $T \subset \mathfrak{m}$, we have $|t|_x = 0$ for every $t \in T$. So (b) does not hold.

We prove the last statement. Let $x \in \mathrm{Spa}(A, A^+)$. Choose $t_0 \in T$ such that $|t_0|_x = \max_{t \in T} |t|_x$. Then $|t_0|_x \neq 0$ by condition (b), so $x \in R\left(\frac{T}{t_0}\right)$.

□

Corollary III.4.4.4. *Let (A, A^+) be a Huber pair, and let $f \in A$.*

(i) We have $f \in A^+$ if and only if $|f|_x \leq 1$ for every $x \in \mathrm{Spa}(A, A^+)$.

(ii) If A is complete*, we have $f \in A^\times$ if and only if $|f|_x \neq 0$ for every $x \in \mathrm{Spa}(A, A^+)$.*

(iii) If A is a Tate ring, then f is topologically nilpotent if and only if $|f|_x^n \to 0$ as $n \to +\infty$ [1] for every $x \in \mathrm{Spa}(A, A^+)$.

Proof. Point (i) is just proposition III.1.4(ii), and point (ii) is corollary III.4.4.3 applied to $T = \{f\}$.

We prove (iii). Assume that $f \in A^{00}$, and let $x \in \mathrm{Spa}(A, A^+)$. Let $\gamma \in \Gamma_x$. As 0 is a limit of $(f^n)_{n \geq 0}$, there exists $n \in \mathbb{N}$ such that, for every $m \geq n$, f^m is in the open neighborhood $\{a \in A \mid |a|_x < \gamma\}$ of 0. Conversely, suppose that $|f|_x^n \to 0$ for every $x \in \mathrm{Spa}(A, A^+)$. Let $\varpi \in A$ be a topologically nilpotent unit. We have $\mathrm{Spa}(A, A^+) = \bigcup_{n \geq 0} R\left(\frac{f^n, \varpi}{\varpi}\right)$ by hypothesis. As $\mathrm{Spa}(A, A^+)$ is quasi-compact, there exists $n \in \mathbb{N}$ such that $\mathrm{Spa}(A, A^+) = R\left(\frac{f^n, \varpi}{\varpi}\right)$. This means that $|f^n/\varpi|_x \leq 1$ for every $x \in \mathrm{Spa}(A, A^+)$, hence $f^n \in \varpi A^+$ by (i). As $A^+ \subset A^0$, we get that $f^n = \varpi a$ for some power-bounded $a \in A$, and so f is topologically nilpotent.

□

Lemma III.4.4.5. *Let A be a complete f-adic ring. Then $A^\times$ is open in A and every maximal ideal of A is closed.*

Proof. For every $a \in A^{00}$, we have $1 - a \in A^\times$ (because $\sum_{n \geq 0} a^n$ is an inverse of a, see [5] Chapitre III §2 №13 lemma 3). So $1 + A^{00} \subset A^\times$. As A^{00} is open, this implies that $A^\times$ is open in A (if $a \in A^\times$, then multiplication by a is a homeomorphism of A, so $a(1 + A^{00})$ is an open neighborhood of a).

Let $\mathfrak{m}$ be a maximal ideal of A. Then $\mathfrak{m}$ is contained in the closed subset $A - A^\times$, so its closure is also contained in $A - A^\times$, and in particular it does not contains 1. But the closure of $\mathfrak{m}$ is an ideal of A and $\mathfrak{m}$ is maximal, so $\mathfrak{m}$ is equal to its closure.

□

Example III.4.4.6. Let k be a non-Archimedean field whose topology is given by the rank 1 valuation $|.|$, and let $(A, A^+) = (k[X], k^0[X])$, where $k[X] = k[X]_{\{1\}}$. Let $\varpi \in k$ such that $0 < |\varpi| < 1$. Then $f = 1 + \varpi X \in A$ satisfies $|f|_x \neq 0$ for every $x \in \mathrm{Spa}(A, A^+)$ (see section III.5.2), but $f \notin A^\times$. So some hypothesis is necessary in corollary III.4.4.4(ii). (We can get away with less than completeness, see [1] proposition 7.3.10(6).)

[1] That is, for every $\gamma \in \Gamma_x$, there exists $n \in \mathbb{N}$ such that $|f|_x^m < \gamma$ for $m \geq n$.

III.5 Examples

III.5.1 Completed residue fields and adic Points

Definition III.5.1.1. Let (A, A^+) be a Huber pair and let $x \in \mathrm{Spa}(A, A^+)$. Remember that we denote by $K(x)$ the fraction field of $A/\operatorname{supp}(x)$. If x is not analytic, we put the discrete topology on $K(x)$ and we set $\kappa(x) = K(x)$ and $K(x)^+ = \kappa(x)^+ = R_x$. If x is analytic, we put the topology defined by the valuation $|.|_x$ on $K(x)$, we denote by $\kappa(x)$ the completion of $K(x)$ and by $\kappa(x)^+$ the completion of $K(x)^+ := R_x$.

We call $\kappa(x)$ the *completed residue field* of $\mathrm{Spa}(A, A^+)$ at x.

We also denote by $|.|_x$ the valuation induced by $|.|_x$ on $K(x)$ (resp. $\kappa(x)$). It is a continuous valuation, with valuation subring $K(x)^+$ (resp. $\kappa(x)^+$), and it defines the topology of $K(x)$ and $\kappa(x)$ if x is analytic.

Note that $(\kappa(x), \kappa(x)^+)$ is a Huber pair, and that $\kappa(x)^+$ is a valuation subring of $\kappa(x)$.

Proposition III.5.1.2. *Let $x \in \mathrm{Spa}(A, A^+)$. Then :*

(i) x is analytic if and only if $\kappa(x)$ is microbial.

(iii) The map $\mathrm{Spa}(\kappa(x), \kappa(x)^+) \to \mathrm{Spa}(A, A^+)$ (coming from the canonical map $(A, A^+) \to (\kappa(x), \kappa(x)^+)$) induces a homeomorphism between $\mathrm{Spa}(\kappa(x), \kappa(x)^+)$ and the set of vertical generizations of x.

Proof. (i) If x is not analytic, then $\kappa(x)$ is discrete, so is is not microbial. Conversely, suppose that x is analytic. Then, by corollary II.2.4.8, R_x has a prime ideal of height 1, so $K(x)$ is microbial, and so is its completion $\kappa(x)$.

(ii) Suppose that x is analytic. Then, by corollary III.4.2.2, the map $\mathrm{Spa}(\kappa(x), \kappa(x)^+) \to \mathrm{Spa}(K(x), R_x)$ is a homeomorphism. So, in both cases, we have to show that the map $\mathrm{Spa}(K(x), R_x) \to \mathrm{Spa}(A, A^+)$ induces a homeomorphism from $\mathrm{Spa}(K(x), R_x)$ to the set of vertical generizations of x in $\mathrm{Spa}(A, A^+)$, where we put the valuation topology on $K(x)$ to define $\mathrm{Spa}(K(x), R_x)$. We already know that the set of valuations rings of R of $K(x)$ such that $R_x \subset R$ (i.e. $RZ(K(x), R_x)$) is homeomorphic to the set of vertical generizations of x in $\mathrm{Spv}(A)$, and every vertical generization of x is automatically ≤ 1 on A^+, so we just need to check that, if $R \in RZ(K(x), R_x)$ corresponds to y, then $|.|_R$ is continuous if and only if $|.|_y$ is. As $|.|_R$ and $|.|_y$ have the same valuation group and as vertical generizations with nontrivial valuation group of a continuous valuation are automatically continuous (proposition II.2.3.1(ii)), the only nontrivial case is the case $R = K(x)$. In that case, $|.|_y$ is not continuous because $\{a \in A \mid |a|_y < 1\} = \operatorname{supp}(x)$ is not open in A, and neither is $|.|_R$ because $\{a \in K(x) \mid |a|_R < 1\} = \{0\}$ is not open in $K(x)$.

If x is not analytic, then $\kappa(x)$ has the discrete topology, so $\mathrm{Spa}(\kappa(x), \kappa(x)^+) = RZ(K(x), R_x)$. On the other hand, every vertical generization of x is continuous because it has open kernel. So we also get the result.

□

Here is a good reason to use the completed residue field $\kappa(x)$ instead of $K(x)$.

Proposition III.5.1.3. *Let $U := R\left(\frac{T}{s}\right)$ be a rational domain of $X := \mathrm{Spa}(A, A^+)$, and let $f : Y := \mathrm{Spa}(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+) \xrightarrow{\sim} U$ be the homeomorphism of corollary III.4.3.2. We fix a point $y \in Y$ and let $x = f(y)$. The canonical morphism $\varphi : (A, A^+) \to (A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$ induces a morphism $(K(x), K(x)^+) \to (K(y), K(y)^+)$, and this gives an isomorphism $(\kappa(x), \kappa(x)^+) \xrightarrow{\sim} (\kappa(y), \kappa(y)^+)$ on the completions of these Huber pairs.*

Note that the inclusion $K(x) \subset K(y)$ is strict in general.

Proof. We have $|.|_x = |.|_y \circ \varphi$, so $\varphi^{-1}(\mathrm{supp}(y)) = \mathrm{supp}(x)$, which gives the first statement, and also the fact that $K(x)^+$ is the inverse image of $K(y)^+$ in $K(x)$. To prove the second statement, we must show that the image of $K(x)$ in $K(y)$ is dense.

First note that, as the canonical map $A \to A\langle\frac{T}{s}\rangle$ is adic, x is analytic if and only if y is analytic (see proposition III.2.5).

As $|s|_x = |s|_y \neq 0$, $K(x)$ is also the fraction field of $A[s^{-1}]/(A[s^{-1}] \cap \mathrm{supp}(y))$. Also, we know that $A[s^{-1}] = A\left(\frac{T}{s}\right)$ is dense in $A' := A\langle\frac{T}{s}\rangle$ (by definition of the second ring), and that the valuation topology on A induced by $|.|_y$ is weaker than the original topology of A'; so $A[s^{-1}]$ is dense in A' for the valuation topology, and this implies that $K(x)$ is dense in $K(y)$ is x and y are analytic.

If x and y are not analytic, then the open subgroup $\mathrm{supp}(y)$ of A' is the completion of $\mathrm{supp}(y) \cap A[s^{-1}]$, and the morphism of discrete rings $A[s^{-1}]/(A[s^{-1}] \cap \mathrm{supp}(y)) \to A'/\mathrm{supp}(y)$ is an isomorphism. So the morphism $K(x) \to K(y)$ is an isomorphism.

□

Definition III.5.1.4. An *affinoid field* is a pair (k, k^+), where k is a *complete* non-Archimedean field and $k^+ \subset k^0$ is an open valuation subring of k.

Example III.5.1.5. If (A, A^+) is a Huber pair and $x \in \mathrm{Spa}(A, A^+)_{\mathrm{an}}$, then $(\kappa(x), \kappa(x)^+)$ is an affinoid field.

Remark III.5.1.6. Let (k, k^+) be an affinoid field. Then k^+ is a ring of integral elements and a ring of definition of k. In particular, (k, k^+) is a Huber pair.

Indeed, the open subring k^+ of k is also integrally closed because it is a valuation subring (see proposition I.1.2.1(i)). So it is a ring of integral elements. On the other hand, k^+ is open and bounded in k (it is bounded because it is contained in k^0, and k^0 is bounded because the topology of k is defined by a rank 1 valuation), so it is a ring of definition by lemma II.1.1.7.

Note also that the topology defined by the valuation corresponding to k^+ is equal to the original topology on k. Indeed, the maximal ideal k^{00} is a height 1 prime ideal in k^+, so we can apply theorem I.1.5.4.

Definition III.5.1.7. An *adic Point* is the adic spectrum of an affinoid field.

Note that an adic Point is not a point in general. In fact :

Lemma III.5.1.8. *Let (k, k^+) be an affinoid field. Then $\operatorname{Spa}(k, k^+)$ is totally ordered by specialization. It has a unique closed point (the minimal element) corresponding to k^+, and a unique generic point (the maximal element) corresponding to k^0.*

Proof. By remark III.5.1.6, any valuation subring $R \subset k^0$ of k defines a continuous valuation on k, this valuation is in $\operatorname{Spa}(k, k^+)$ if and only if $k^+ \subset R$. So $\operatorname{Spa}(k, k^+)$ is in order-reversing bijection with the set of valuation subrings R of k such that $k^+ \subset R \subset k^0$ (a locally closed subset of $RZ(k)$). By theorem I.1.4.2, it is also in order-preserving bijection with the set of prime ideals $\wp$ of k^+ such that $\wp \subset k^{00}$. (Note that k^{00} is the prime ideal of k^+ corresponding to the valuation ring k^0, see theorem I.1.5.4.) As the set of ideals of k^+ is totally ordered by inclusion, this shows the first statement. The second statement is clear.

□

Proposition III.5.1.9. *Let (A, A^+) be a Huber pair. Consider the following sets :*

(a) Σ is the set of maps of Huber pairs $\varphi : (A, A^+) \to (k, k^+)$, where (k, k^+) is an affinoid field and $\operatorname{Frac}(\varphi(A))$ is dense in k, modulo the following equivalence relation : $\varphi_1 : (A, A^+) \to (k_1, k_1^+)$ and $\varphi_2 : (A, A^+) \to (k_2, k_2^+)$ are equivalent if there exists an isomorphism of Huber pairs $u : (k_1, k_1^+) \xrightarrow{\sim} (k_2, k_2^+)$ such that $u \circ \varphi_1 = \varphi_2$.

(b) Σ' is the set of equivalence classes of continuous valuations $x \in \operatorname{Spa}(A, A^+)$ such that $\kappa(x)$ is microbial.

Then the map $\Sigma' \to \Sigma$ sending x to the canonical map $(A, A^+) \to (\kappa(x), \kappa(x)^+)$ and the map $\Sigma \to \operatorname{Spa}(A, A^+)_{\mathrm{an}}$ sending $\varphi : (A, A^+) \to (k, k^+)$ to the image of the closed point of $\operatorname{Spa}(k, k^+)$ by $\operatorname{Spa}(\varphi)$ are both well-defined and bijective.

Note that we have a similar statement for an affine scheme $\operatorname{Spec}(A)$, where we use actual points (i.e. $\operatorname{Spec}(k)$) and we get all the points of $\operatorname{Spec}(A)$.

Proof. If $x \in \operatorname{Spa}(A, A^+)$ and $\kappa(x)$ is microbial, then $(\kappa(x), \kappa(x)^+)$ is an affinoid field; so the first map is well-defined. We show that it is a bijection by constructing an inverse. Let $\varphi : (A, A^+) \to (k, k^+)$ be an element of Σ. Then composing the valuation on k corresponding to k^+ with φ gives a continuous valuation x on A such that $\operatorname{supp}(x) = \operatorname{Ker}(\varphi)$ and

$A^+/\operatorname{supp}(x) \subset R_x$; note that x is the image of the closed point of $\operatorname{Spa}(k, k^+)$. We get a map $K(x) \to k$ such that $R_x = K(x) \cap k^+$, and that has dense image by assumption, hence induces an isomorphism $(\kappa(x), \kappa(x)^+) \xrightarrow{\sim} (k, k^+)$. It is easy to check that this defined an inverse of the map $\Sigma' \to \Sigma$.

Now we consider the second map. By the first paragraph, we just need to show that an element $x \in \operatorname{Spa}(A, A^+)$ is analytic if and only if $\kappa(x)$ is microbial; but this is proposition III.5.1.2(i).

$\square$

III.5.2 The closed unit ball

Let k be a complete and algebraically closed non-Archimedean field, and denote its rank 1 valuation by $|.| : k \to \mathbb{R}_{\geq 0}$. Remember that k^{00} is the maximal ideal of k^0 (this is true for any non-Archimedean field). We denote the residue field k^0/k^{00} by κ; it is also algebraically closed.

Let $A = k\langle t\rangle$ (see II.3.3.9), and let $A^+ = A^0 = k^0\langle t\rangle$. The points of $X := \operatorname{Spa}(A, A^+)$ are usually divided into 5 types :

(1) Classical points : Let $x \in k^0$ (i.e. a point of the closed unit disk in k). Then the map $A \to \mathbb{R}_{\geq 0}$, $f \longmapsto |f(x)|$ is an element of $\operatorname{Spa}(A, A^+)$, and its support is the maximal ideal $(t - x)$ of A. We will often denote this point by x.

Note that every maximal ideal of A is of the form $(t - x)$ for $x \in k^0$ (cf. [4] section 2.2 corollary 13), so classical points are in bijection with the maximal spectrum of A.

(2),(3) Let $r \in [0, 1]$ and let $x \in k^0$. Let x_r be the point of $\operatorname{Spv}(A)$ corresponding to the valuation

$$f = \sum_{n\geq 0} a_n(t - x)^n \longmapsto \sup_{n\geq 0} |a_n|r^n = \sup_{\substack{y\in k^0,\\ |y-x|\leq r}} |f(y)|.$$

Then $x_r \in \operatorname{Spa}(A, A^+)$, and it only depends on $D(x, r) := \{y \in k^0 \mid |x - y| \leq r\}$. If $r = 0$ then $x_r = x$ is a classical point, and if $r = 1$ then x_r is independent of x and is called the *Gauss norm*.

If $x \in k^0$ is fixed, then the map $[0, 1] \to \operatorname{Spa}(A, A^+)$, $r \longmapsto x_r$ is continuous precisely at the points of $[0, 1] - |k^\times|$.

If $r \in |k^\times|$, then we say that the pointn x_r is of type (2); otherwise, we say that it is of type (3).

(4) Let $D_1 \supset D_2 \supset \ldots$ be an infinite sequence of closed disks in k^0 such that $\bigcap_{n\geq 1} D_n = \varnothing$. [2] Then the valuation

$$f \longmapsto \inf_{n\geq 1} \sup_{x\in D_n} |f(x)|$$

[2] Such sequences exist if and only if k is not spherically complete. For example, $\mathbb{C}_\ell = \widehat{\overline{\mathbb{Q}}}_\ell$ is not spherically complete.

defines a rank 1 point of X, which is not of type (1), (2) or (3).

(5) Rank 2 valuations : Let $x \in k^0$ and $r \in (0,1]$. We denote by $\Gamma_{<r}$ the abelian group $\mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, with the unique order such that $r' < \gamma < r$ for every $r' < r$. (With the notation of section I.3.5.1, this is the group Γ and $\gamma = r^-$.) Denote by $x_{<r}$ the point of $\mathrm{Spv}(A)$ corresponding to the valuation

$$f = \sum_{n \geq 0} a_n (t-x)^n \longmapsto \max_{n \geq 0} |a_n| \gamma^n \in \Gamma_{<r} \cup \{0\}.$$

Then $x_{<r}$ is a point of $\mathrm{Spa}(A, A^+)$, and it only depends on $D^0(x,r) := \{y \in k^0 \mid |x-y| < r\}$.

Similarly, if $r \in (0,1)$, let $\Gamma_{>r}$ the abelian group $\mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, with the unique order such that $r' > \gamma > r$ for every $r' > r$. Denote by $x_{>r}$ the point of $\mathrm{Spv}(A)$ corresponding to the valuation

$$f = \sum_{n \geq 0} a_n (t-x)^n \longmapsto \max_{n \geq 0} |a_n| \gamma^n \in \Gamma_{>r} \cup \{0\}.$$

Then $x_{>r}$ is a point of $\mathrm{Spa}(A, A^+)$, and it only depends on $D(x,r)$. (So $x_{>r} = x'_{>r}$ if $x_r = x'_r$.)

If $r \notin |k^\times|$, then $x_{<r} = x_{>r} = x_r$. But if $r \in |k^\times|$, we get two new points of $\mathrm{Spa}(A, A^+)$, which are called points of type (5).

If we think of $\mathrm{Spa}(A, A^+)$ as a tree, then points of (1) are end points, points of type (2) and (3) are points on the limbs of the tree (and type (2) points are exactly the branching points), points of type (4) are “dead ends”. Points of type (5) are in the closure of points of type (2) (so they are less easy to visualize).

Points of type (1), (3), (4) and (5) are closed. If $x \in k^0$ and $r \in |k|^\times \cap (0,1]$, then the closure of the corresponding point of type (2) x_r is $\{x_r, x_{<r}, x_{>r}\}$ (where $x_{>r}$ appears only if $r < 1$).

Remark III.5.2.1. We could also have defined a point $x_{>1}$ of $\mathrm{Spv}(A)$, for $x \in k^0$. This is a continuous valuation on A, but it is not a point of $\mathrm{Spa}(A, A^+)$, because it is not ≤ 1 on A^+. In fact, if A'^+ is the integral closure of $k^0 + A^{00}$ in A, then $\mathrm{Spa}(A, A'^+) = \mathrm{Spa}(A, A^+) \cup \{x_{>1}\}$.

Remark III.5.2.2. We get the Berkovich space of A by identifying the points of type (5) $x_{<r}$ and $x_{>r}$ with x_r. Note that this is a Hausdorff space. In general, if k is a complete non-Archimedean field, the Berkovich space of an affinoid k-algebra A is the maximal Hausdorff quotient of $\mathrm{Spa}(A, A^0)$.

III.5.3 Formal schemes

Let A be an adic ring with a finitely generated ideal of definition I. Let

$$X = \mathrm{Spa}(A, A) = \{x \in \mathrm{Cont}(A) \mid \forall a \in A,\ |a|_x \leq 1\}.$$

Remember that a valuation is called trivial if it has rank 0, i.e. if its value group is the trivial group. A trivial valuation is continuous if and only if it has open support, so the subset X_{triv} of trivial valuation is in bijection with the set of open prime ideal ideals of A, i.e. $\mathrm{Spf}(A)$. It is easy to see that this is a homeomorphism.

We also have a retraction $\mathrm{Spa}(A,A) \to \mathrm{Spa}(A,A)_{\mathrm{triv}}$ given by $x \longmapsto x_{|c\Gamma_x}$, and it is a spectral map. If $\mathscr{O}_X$ is the structure presheaf of X to be defined shortly, then we have an isomorphism of locally ringed spaces $\mathrm{Spf}(A) \simeq (X_{\mathrm{triv}}, r_*\mathscr{O}_X)$.

This will give a fully faithful functor from the category of Noetherian formal affine schemes to the category of adic spaces.

III.6 The structure presheaf

In this section, (A, A^+) is a Huber pair.

III.6.1 Universal property of rational domains

Proposition III.6.1.1. *(Lemma 8.1 and proposition 8.2 of [26].) Let T be a finite subset of A such that the ideal $T \cdot A$ is open, let $s \in A$, and let $U = R\left(\frac{T}{s}\right) \subset \mathrm{Spa}(A, A^+)$.*

(i) The canonical map $\iota : (A, A^+) \to (A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$ induces a spectral homeomorphism $\mathrm{Spa}(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+) \xrightarrow{\sim} U$ sending rational domains to rational domains.

(ii) For every continuous morphism $\varphi : (A, A^+) \to (B, B^+)$ to a complete *Huber pair such that $\mathrm{Spa}\,\varphi : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors through U, there is a unique continuous ring morphism $\psi : A\langle\frac{T}{s}\rangle \to B$ such that $\psi \circ \iota = \varphi$, and we have $\psi(A\langle\frac{T}{s}\rangle^+) \subset B^+$.*

(iii) Let T' be another finite subset of A such that the ideal $T' \cdot A$ is open, let $s' \in A$, and let $U' = R\left(\frac{T'}{s'}\right) \subset \mathrm{Spa}(A, A^+)$. If $U' \subset U$, then there exists a unique continuous ring morphism $\rho : A\langle\frac{T}{s}\rangle \to A\langle\frac{T'}{s'}\rangle$ such that $\iota' = \rho \circ \iota$, where $\iota' : A \to A\langle\frac{T'}{s'}\rangle$ is the canonical map.

Proof. (i) This is corollary III.4.3.3.

(ii) The assumption on $\mathrm{Spa}(\varphi)$ means that, for every $x \in \mathrm{Spa}(B, B^+)$ and every $t \in T$, we have $|\varphi(t)|_x \leq |\varphi(s)|_x \neq 0$. By points (ii) and (i) of corollary III.4.4.4, this implies that $\varphi(s) \in B^\times$, and then that $\varphi(t)\varphi(s)^{-1} \in B^+$ for every $t \in T$. In particular, $\varphi(t)\varphi(s)^{-1}$ is power-bounded for every $t \in T$, so the existence and uniqueness of ψ follow from the universal property of $A\langle\frac{T}{s}\rangle$ (proposition II.3.4.6), and the fact that ψ preserves the rings of integral elements follows immediately from the fact that $\psi(t)\psi(s)^{-1} \in B^+$ for every $t \in T$.

(iii) This follows immediately from (i) and (ii).

$\square$

Corollary III.6.1.2. *Let $T, T' \subset A$ be finite subsets such that the ideals $T \cdot A$ and $T' \cdot A$ are open, and let $s, s' \in A$. If $R\left(\frac{T}{s}\right) = R\left(\frac{T'}{s'}\right)$, then there is a canonical isomorphism of Huber pairs $(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+) \xrightarrow{\sim} (A\langle\frac{T'}{s'}\rangle, A\langle\frac{T'}{s'}\rangle^+)$ making the following diagram commute :*

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A\langle\frac{T}{s}\rangle \arrow[d, "\wr"] \\ & A\langle\frac{T'}{s'}\rangle \end{tikzcd}$$

In other words, the rational domain $R\left(\frac{T}{s}\right)$ uniquely determines the Huber pair $(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$ as a Huber pair over (A, A^+).

III.6.2 Definition of the structure presheaf

Definition III.6.2.1. Let (A, A^+) be a Huber pair, and let $X = \mathrm{Spa}(A, X^+)$. The structure presheaf $\mathscr{O}_X$ on X is the presheaf with values in the category of complete topological rings and continuous ring morphisms defined by the following formulas :

- if $U = R\left(\frac{T}{s}\right)$ is a rational domain of X, then $\mathscr{O}_X(U) = A\langle\frac{T}{s}\rangle$;
- if U is an arbitrary open subset of X, then

  $$\mathscr{O}_X(U) = \varprojlim_{U' \subset U} \mathscr{O}_X(U'),$$

  where U' ranges over rational domains of X contained in U and the transition maps are given by proposition III.6.1.1(iii), and where we put the projective limit topology on $\mathscr{O}_X(U)$.

Note that the definition of $\mathscr{O}_X(U)$ for U a rational domain makes sense because U determines the topological A-algebra $A\langle\frac{T}{s}\rangle$ by corollary III.6.1.2. Also, the presheaf does take its value in the category of complete topological rings, because a projective limit of complete topological rings is complete by [6] Chapitre II §3 Nº9 corollaires 1 et 2 de la proposition 18.

Remark III.6.2.2. In particular, we have $\mathscr{O}_X(X) = \widehat{A}$.

Definition III.6.2.3. We use the notation of definition III.6.2.1. We define a subpresheaf $\mathscr{O}_X^+$ of $\mathscr{O}_X$ by the formula

$$\mathscr{O}_X^+(U) = \{f \in \mathscr{O}_X(U) \mid \forall x \in U,\ |f|_x \leq 1\},$$

for every open subset U of X.

This is also a presheaf of complete topological rings.

Lemma III.6.2.4. *If $U = R\left(\frac{T}{s}\right)$ is a rational domain, then*

$$(\mathscr{O}_X(U), \mathscr{O}_X^+(U)) = (A\langle \tfrac{T}{s}\rangle, A\langle \tfrac{T}{s}\rangle^+).$$

Proof. The formula for $\mathscr{O}_X(U)$ is just its definition. For $\mathscr{O}_X^+(U)$, we use the fact that $U = \mathrm{Spa}(A\langle \frac{T}{s}\rangle, A\langle \frac{T}{s}\rangle^+)$ (corollary III.4.3.2) and proposition III.1.4(ii).

□

Let $\varphi : (A, A^+) \to (B, B^+)$ be a morphism of Huber pairs, and let $f = \mathrm{Spa}(\varphi) : Y := \mathrm{Spa}(B, B^+) \to X := \mathrm{Spa}(A, A^+)$. If $U \subset X$ and $V \subset Y$ are rational domains such that $f(V) \subset U$, then proposition III.6.1.1 (and III.4.3.2) gives a continuous ring morphism $\mathscr{O}_X(U) \to \mathscr{O}_Y(V)$. So, if $U \subset X$ is an open subset, we get a morphism of rings $f_U^\flat : \mathscr{O}_X(U) \to \mathscr{O}_Y(f^{-1}(U)) = f_*\mathscr{O}_X(U)$, and this is clearly a morphism of presheaves. It also follows immediately from the definitions that $f^\flat$ sends $\mathscr{O}_X^+$ to $f_*\mathscr{O}_Y^+$.

Lemma III.6.2.5. *Let $T \subset A$ is a finite subset generating an open ideal, $s \in A$, and let $\varphi : (A, A^+) \to (A\langle \frac{T}{s}\rangle, A\langle \frac{T}{s}\rangle^+)$ be the obvious morphism. We get as before a continuous spectral map $f : U := \mathrm{Spa}(A\langle \frac{T}{s}\rangle, A\langle \frac{T}{s}\rangle^+) \to X := \mathrm{Spa}(A, A^+)$ and a morphism of presheaves $f^\flat : \mathscr{O}_X \to f_*\mathscr{O}_U$.*

Then, for every open subset V of X such that $V \subset f(U)$, the map $f_V^\flat : \mathscr{O}_X(V) \to \mathscr{O}_U(f^{-1}(V))$ is an isomorphism.

Proof. We know that f is a homeomorphism from U to $R\left(\frac{T}{s}\right)$ by corollary III.4.3.2, and that an open subset $V \subset f(U)$ of X is a rational domain if and only if $f^{-1}(V)$ is a rational domain of U. Moreover, if V is a rational domain, it is easy to see that $f_V^\flat : \mathscr{O}_X(V) \to \mathscr{O}_U(f^{-1}(X))$ is an isomorphism (using the explicit formulas for these rings). The lemma follows immediately from this.

□

III.6.3 Stalks

Let (A, A^+) be a Huber pair, and let $X = \mathrm{Spa}(A, A^+)$. If $x \in X$, we consider the stalk

$$\mathscr{O}_{X,x} = \varinjlim_{U \ni x \text{ open}} \mathscr{O}_X(U) = \varinjlim_{U \ni x \text{ rational}} \mathscr{O}_X(U)$$

(the equality follows from the fact that rational domains are a base of the topology of X) as an abstract ring without a topology. For every rational domain $U \ni x$ of X, the valuation $|.|_x$ extends to a unique continuous valuation on $\mathscr{O}_X(U)$ (by corollary III.4.3.2). So we get a valuation $|.|_x : \mathscr{O}_{X,x} \to \Gamma_x \cup \{0\}$.

Proposition III.6.3.1. *(i) The ring $\mathscr{O}_{X,x}$ is local, with maximal ideal $\mathfrak{m}_x := \{f \in \mathscr{O}_{X,x} \mid |f|_x = 0\}$.*

We denote by $k(x)$ the residue field of $\mathscr{O}_{X,x}$, and we still write $|.|_x$ for the valuation induced on $k(x)$ by $|.|_x$. Let $k(x)^+$ be the valuation subring of $k(x)$.

(ii) The stalk $\mathscr{O}^+_{X,x}$ of $\mathscr{O}^+_X$ at x is given by the formula

$$\mathscr{O}^+_{X,x} = \{f \in \mathscr{O}_{X,x} \mid |f|_x \leq 1\}.$$

In other words, $\mathscr{O}^+_{X,x}$ is the inverse image of $k(x)^+$ in $\mathscr{O}_{X,x}$.

(iii) The ring $\mathscr{O}^+_{X,x}$ is also local, with maximal ideal $\mathfrak{m}^+_x := \{f \in \mathscr{O}_{X,x} \mid |f|_x < 1\}$. In particular, we have a canonical isomorphism between the residue fields of $\mathscr{O}^+_{X,x}$ and $k(x)^+$.

(iv) Let $u : A \to \mathscr{O}_{X,x}$ be the morphism coming from the restriction morphisms $A \to \widehat{A} = \mathscr{O}_X(X) \to \mathscr{O}_X(U)$, for $U \ni x$ an open subset of X. Then we have $|.|_x \circ u = |.|_x$, so u gives a morphism $(K(x), K(x)^+) \to (k(x), k(x)^+)$, which induces an isomorphism on the completions. In other words, the completion of $(k(x), k(x)^+)$ is canonically isomorphic to $(\kappa(x), \kappa(x)^+)$.

(v) If $\varphi : (A, A^+) \to (B, B^+)$ is a morphism of Huber pair and y is a point of $Y := \mathrm{Spa}(B, B^+)$ such that $\mathrm{Spa}(\varphi)(y) = x$, then the morphism of rings $\mathrm{Spa}(\varphi)^\flat_x : \mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$ induced by $\mathrm{Spa}(\varphi)^\flat$ is such that $|.|_x \circ \mathrm{Spa}(\varphi)^\flat_x = |.|_y$. In particular, $\mathrm{Spa}(\varphi)^\flat_x$ is a morphism of local rings, it sends $\mathscr{O}^+_{X,x}$ to $\mathscr{O}^+_{Y,y}$ and also induces a morphism of local rings $\mathscr{O}^+_{X,x} \to \mathscr{O}^+_{Y,y}$.

Proof. (i) It is clear that $\mathfrak{m}_x$ is an ideal of $\mathscr{O}_{X,x}$, so it suffices to show that every element of $\mathscr{O}_{X,x} - \mathfrak{m}_x$ is a unit. Let $U \ni x$ be an open subset of X and let $f \in \mathscr{O}_X(U)$ such that $|f|_x \neq 0$. We want to show that the image of f in $\mathscr{O}_{X,x}$ is invertible. After shrinking U, we may assume that U is a rational domain of X. Then we have $U = \mathrm{Spa}(\mathscr{O}_X(U), \mathscr{O}_X(U)^+)$ and x defines a continuous valuation on $\mathscr{O}_X(U)$. As $|f|_x \neq 0$, there exists by lemma III.3.3 a finite subset T of $\mathscr{O}_X(U)$ such that the ideal $T \cdot \mathscr{O}_X(U)$ is open and that $|t|_x < |f|_x$ for every $t \in T$. Let V be the rational domain $R\left(\frac{T}{f}\right)$ of $\mathrm{Spa}(\mathscr{O}_X(U), \mathscr{O}_X(U)^+)$. Then $x \in V$, V is open in X, and f is invertible in $\mathscr{O}_X(V) = \mathscr{O}_U(V)$.

(ii) We have an injective morphism $\mathscr{O}^+_{X,x} \to \mathscr{O}_{X,x}$ induced by the injections $\mathscr{O}_X(U)^+ \to \mathscr{O}_X(U)$, and it is obvious that its image is contained in $\{f \in \mathscr{O}^+_{X,x} \mid |f|_x \leq 1\}$ (because, for every open subset U of X, $\mathscr{O}_X(U)^+ = \{f \in \mathscr{O}_X(U) \mid \forall y \in U,\ |f|_y \leq 1\}$). Conversely, let $U \ni X$ be an open subset of X, and let $f \in \mathscr{O}_X(U)$ such that $|f|_x \leq 1$. We want to show that the image of f in $\mathscr{O}_{X,x}$ is in $\mathscr{O}^+_{X,x}$. By lemma III.6.3.2, the set $V := \{y \in U \mid |f|_y \leq 1\}$ is an open subset of X, and we obviously have $x \in X$ and $f_{|V} \in \mathscr{O}_X(V)^+$; this implies the desired result.

(iii) As $\mathfrak{m}^+_x$ is clearly an ideal of $\mathscr{O}^+_{X,x}$, it suffices to show that every element of $\mathscr{O}^+_{X,x} - \mathfrak{m}^+_x$ is invertible. Let $f \in \mathscr{O}^+_{X,x}$, and suppose that $f \notin \mathfrak{m}_x$, i.e. that $|f|_x = 1$. Then $f \in \mathscr{O}^\times_{X,x}$ by (i), and $|f^{-1}|_x = 1$, so $f^{-1} \in \mathscr{O}^+_{X,x}$.

(iv) The fact that $|.|_x \circ u = |.|_x$ follows immediately from the definition of the valuation $|.|_x$ on $\mathscr{O}_{X,x}$. By (i), we have

$$k(x) = \varinjlim_{U \ni x} \mathscr{O}_X(U)/\{f \in \mathscr{O}_X(U) \mid |f|_x = 0\}$$

and

$$k(x)^+ = \varinjlim_{U \ni x} \mathscr{O}_X(U)^+/\{f \in \mathscr{O}_X(U)^+ \mid |f|_x = 0\},$$

where U runs through all rational domains of X containing x. By proposition III.5.1.3, if $U' \subset U$ are two rational domains of X containing x, then the restriction maps

$$\mathscr{O}_X(U)/\{f \in \mathscr{O}_X(U) \mid |f|_x = 0\} \to \mathscr{O}_X(U')/\{f \in \mathscr{O}_X(U') \mid |f|_x = 0\}$$

and

$$\mathscr{O}_X(U)^+/\{f \in \mathscr{O}_X(U)^+ \mid |f|_x = 0\} \to \mathscr{O}_X(U')^+/\{f \in \mathscr{O}_X(U')^+ \mid |f|_x = 0\}$$

have dense image. So the image of $K(x)$ (resp. $K(x)^+$) in $k(x)$ (resp. $k(x)^+$) is dense, which implies the result.

(v) The fact that $|.|_x \circ \mathrm{Spa}(\varphi)^\flat_x = |.|_y$ follows immediately from the definitions, and the other statements follow from this.

□

Lemma III.6.3.2. *(Remark 8.12 of [26].) Let U be an open subset of $X := \mathrm{Spa}(A, A^+)$ and $f, g \in \mathscr{O}_X(U)$. Then $V := \{x \in U \mid |f|_x \leq |g|_x \neq 0\}$ is an open subset of X.*

Proof. We know that U is a union of rational domains of X, and it suffices to show that the intersection of V with each of these rational domains is open. So we may assume that U is a rational domain, $U = R\left(\frac{T}{s}\right) = \mathrm{Spa}(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$. It also suffices to show that V is open in U, so we may assume that $U = X$, i.e. $f, g \in A$. Then V is open by definition of the topology on $\mathrm{Spa}(A, A^+)$ as the topology induced by that of $\mathrm{Spv}(A)$. [3]

□

If A is a Tate ring, then we can show that the categories of finite étale covers of $\mathscr{O}^+_{X,x}$ and $k(x)^+$ are equivalent. First we need a definition.

Definition III.6.3.3. (See [25, Definition 09XE].) Let R be a ring and I be an ideal of R. We say that the pair (R, I) is *henselian* (or *I-adically henselian*) if :

[3]Note that V is not a rational domain in general, because we did not assume that f generates an open ideal of A.

(a) I is contained in the Jacobson radical of A;

(b) for any monic polynomial $f \in A[X]$ and any factorization $\overline{f} = g_0 h_0$ in $A/I[X]$, where $\overline{f}$ is the image of f in $A/I[X]$, if g_0 and h_0 are monic and generate the unit ideal of $A/I[X]$, then there exists a factorization $f = gh$ in $A[X]$ with g, h monic and such that $g_0 = g \mod I$ and $h_0 = g \mod I$.

If R is local and I is its maximal ideal, we also say that R is henselian.

Theorem III.6.3.4. *([25, Lemma 0ALJ].) Let R be a ring and I be an ideal of R. If R is I-adically complete, then the pair (R, I) is henselian.*

Theorem III.6.3.5. *([25, Lemma 09XI].) Let R be a ring and I be an ideal of R. The following are equivalent :*

(i) The pair (R, I) is henselian.

(ii) For every étale map of rings $R \to R'$ and every map of R-algebras $\sigma : R' \to R/I$, there exists an map of R-algebras $R' \to R$ lifting σ.

$$\begin{tikzcd} R' \arrow[dr, "\sigma"] \arrow[d, dotted, bend left] \\ R \arrow[u] \arrow[r] & R/I \end{tikzcd}$$

(iii) For any finite R-algebra S, the map $S \to S/IS$ induces a bijection on idempotent elements.

(iv) For any integral R-algebra S, the map $S \to S/IS$ induces a bijection on idempotent elements.

(v) I is contained in the Jacobson radical of R and every monic polynomial $f \in R[X]$ of the form

$$f(X) = X^n(X-1) + a_n X^n + \ldots + a_1 X + a_0,$$

with $a_0, \ldots, a_n \in I$ and $n \geq 1$, has a root in $1 + I$.

Moreover, if these conditions hold, then the root in point (v) is unique.

Theorem III.6.3.6. *([25, Lemma 09ZL].) Let (R, I) be a henselian pair. Then the functor $S \longmapsto S/IS$ induces an equivalence between the category of finite étale R-algebras and the category of finite étale R/I-algebras.*

Proposition III.6.3.7. *(See proposition 7.5.5 of [1].) We use the notation of proposition III.6.3.1, and we suppose that A is a Tate ring. Let $\varpi \in A$ be a topologically nilpotent element.*

(i) The ring $\mathscr{O}^+_{X,x}$ is ϖ-adically henselian, and the map $\mathscr{O}^+_{X,x} \to k(x)^+$ induces an isomorphism on ϖ-adic completions.

(ii) *The pairs* $(\mathscr{O}_{X,x}, \mathfrak{m}_x)$ *and* $(\mathscr{O}^+_{X,x}, \mathfrak{m}_x)$ *are henselian.*

Proof. (i) An inductive limit of henselian pairs is henselian by [25, Lemma 0A04]. So it suffices to show that $\mathscr{O}_X(U)^+$ is ϖ-adically henselian for every rational domain U of X. Let U be a rational domain of X. Then $(B, B^+) := (\mathscr{O}_X(U), \mathscr{O}_X(U)^+)$ is a complete Huber pair by lemma III.6.2.4, and it is a Tate pair because A is a Tate ring. Let $B_0 \subset B^+$ be a ring of definition of B. By proposition II.2.5.2, ϖB_0 is an ideal of definition of B_0, so B_0 is ϖ-adically complete, hence ϖ-adically henselian by theorem III.6.3.4. As B^+ is the union of all the rings of definition of B contained in it (by corollary II.1.1.8(iii)), anotehr application of [25, Lemma 0A04] shows that B^+ is ϖ-adically henselian.

We prove the second statement. Note that $\mathfrak{m}_x \subset \mathscr{O}^+_{X,x}$ and that $\mathscr{O}^+_{X,x}/\mathfrak{m}_x = k(x)^+$. Also, we have $\varpi\mathfrak{m}_x = \mathfrak{m}_x$ because ϖ is a unit in A, so $\mathfrak{m}_x = \varpi^n\mathfrak{m}_x \subset \varpi^n\mathscr{O}^+_{X,x}$ for every $n \in \mathbb{N}$. This implies that the map $\mathscr{O}^+_{X,x} \to k(x)^+$ induces an isomorphism on ϖ-adic completions.

(ii) As $\mathfrak{m}_x = \varpi\mathfrak{m}_x \subset \varpi\mathscr{O}^+_{X,x}$, the fact that $(\mathscr{O}^+_{X,x}, \mathfrak{m}_x)$ is henselian follows from the first statement of (i) and from [25, Lemma 0DYD].

To prove that $(\mathscr{O}_{X,x}, \mathfrak{m}_x)$ is henselian, it suffices to note that $\mathfrak{m}_x$ is contained in the Jacobson readical of $\mathscr{O}_{X,x}$, and that it satisfies the property of theorem III.6.3.5(v) (because it does in $\mathscr{O}^+_{X,x}$).

□

III.6.4 The category $\mathscr{V}^{\mathrm{pre}}$

Definition III.6.4.1. We denote by $\mathscr{V}^{\mathrm{pre}}$ the category of triples $(X, \mathscr{O}_X, (|.|_x)_{x\in X})$, where :

- X is a topological space;
- $\mathscr{O}_X$ is a presheaf of complete topological rings on X such that, for every $x \in X$, the stalk $\mathscr{O}_{X,x}$ (seen as an abstract ring) is a local ring;
- for every $x \in X$, $|.|_x$ is an equivalence class of valuations on $\mathscr{O}_{X,x}$ whose support is equal to the maximal ideal of $\mathscr{O}_{X,x}$.

A morphism $(X, \mathscr{O}_X, (|.|_x)_{x\in X}) \to (Y, \mathscr{O}_Y, (|.|_y)_{y\in Y})$ is a pair $(f, f^\flat)$, where $f : X \to Y$ is a continuous map and $f^\flat : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is a morphism of presheaves of topological rings such that, for every $x \in X$, the morphism $f^\flat_x : \mathscr{O}_{X,x} \to \mathscr{O}_{Y,f(x)}$ induced by $f^\flat$ is compatible with the valuations (i.e. $|.|_{f(x)} \circ f^\flat_x = |.|_x$). Note that this implies that $f^\flat_x$ is a local morphism.

Example III.6.4.2. If (A, A^+) is a Huber pair, then $\mathrm{Spa}(A, A^+)$ is an object of $\mathscr{V}^{\mathrm{pre}}$, and any morphism of Huber pairs $\varphi : (A, A^+) \to (B, B^+)$ induces a morphism $\mathrm{Spa}(\varphi) : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ in $\mathscr{V}^{\mathrm{pre}}$.

Corollary III.6.4.3. *Let* (A, A^+) *be a Huber pair, and let* $\varphi : (A, A^+) \to (\widehat{A}, \widehat{A}^+)$ *be the canonical morphism. Then* $\mathrm{Spa}(\varphi)$ *is an isomorphism in* $\mathscr{V}^{\mathrm{pre}}$.

Proof. By corollary III.4.2.2, the map $\mathrm{Spa}(\varphi) : \mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+)$ is a homeomorphism preserving rational domains. Also, if $T \subset A$ is a finite subset such that $T \cdot A$ is open and if $s \in A$, we have $(A\langle \frac{T}{s}\rangle, A\langle \frac{T}{s}\rangle^+) = (\widehat{A}\langle \frac{T}{s}\rangle, \widehat{A}\langle \frac{T}{s}\rangle^+)$. This implies the result.

$\square$

Proposition III.6.4.4. *Let (A, A^+) and (B, B^+) be Huber pairs, and suppose that B is* complete. *Then $\varphi \longmapsto \mathrm{Spa}(\varphi)$ induces a bijection*

$$\mathrm{Hom}((A, A^+), (B, B^+)) \to \mathrm{Hom}_{\mathscr{V}^{\mathrm{pre}}}(\mathrm{Spa}(B, B^+), \mathrm{Spa}(A, A^+)),$$

where the first Hom *is taken in the category of Huber pairs. The inverse of this bijections sends a morphism $(f, f^\flat) : Y := \mathrm{Spa}(B, B^+) \to X := \mathrm{Spa}(A, A^+))$ to the morphism*

$$(A, A^+) \to (\widehat{A}, \widehat{A}^+) = (\mathscr{O}_X(X), \mathscr{O}_X(X)^+) \overset{f^\flat_X}{\to} (\mathscr{O}_Y(Y), \mathscr{O}_Y(Y)^+) = (B, B^+).$$

Proof. If $\varphi : (A, A^+) \to (B, B^+)$ is a morphism of Huber pairs, then the composition of $\mathrm{Spa}(\varphi)^\flat$ and of the canonical map $A \to \widehat{A}$ is φ by definition of $\mathrm{Spa}(\varphi)^\flat$.

Conversely, let $(f, f^\flat) : Y \to X$ be a morphism in $\mathscr{V}^{\mathrm{pre}}$, and let $\varphi : A \to B$ be the composition of $f^\flat_X$ and of $A \to \widehat{A}$. We want to show that $(f, f^\flat) = (\mathrm{Spa}(\varphi), \mathrm{Spa}(\varphi)^\flat)$. Let $U = R\left(\frac{T}{s}\right)$ be a rational domain in X, and let $V = f^{-1}(U)$. We have

$$V = \{y \in Y \mid \forall t \in T, |t|_{f(y)} \leq |s|_{f(y)} \neq 0\} = \{y \in Y \mid \forall t \in T, |\varphi(t)|_y \leq |\varphi(s)|_y \neq 0\},$$

where the second equality comes from the fact that $|a|_{f(y)} = |f^\flat_y(a)|_y = |\varphi(a)|_y$ for every $a \in A$. If W is a quasi-compact open subset of V, then, by lemma III.3.3, we can find a finite subset T_W of B generating an open ideal of B and such that $|t|_y \leq |\varphi(s)|_y$ for every $y \in W$ and $t \in T_W$. Then we have $W \subset W' := R\left(\frac{\varphi(T), T_W}{\varphi(s)}\right) \subset V$, so the map $f^\flat_U : \mathscr{O}_X(U) \to \mathscr{O}_Y(W)$ factors through $\mathscr{O}_Y(W') = B\langle \frac{\varphi(T), T'_W}{\varphi(s)}\rangle$. We know that $f^\flat_U$ is equal to φ on the image of A, that φ is continuous and that the rings $A\langle \frac{T}{s}\rangle$ and $B\langle \frac{\varphi(T), T'_W}{\varphi(s)}\rangle$ are completions of localizations of A and B, so this implies that $f^\flat_U : \mathscr{O}_X(U) \to \mathscr{O}_X(W)$ is equal to $\mathrm{Spa}(\varphi)^\flat_U$. Going to the limit on $W \subset V$ open quasi-compact, we see that $f^\flat_U : \mathscr{O}_X(U) \to \mathscr{O}_Y(V)$ is also equal to $\mathrm{Spa}(\varphi)^\flat$. This implies the analogous statement for an arbitrary open subset U of X by the definition of the presheaf $\mathscr{O}_X$.

$\square$

III.6.5 Adic spaces

Definition III.6.5.1. An *open immersion* in $\mathscr{V}^{\mathrm{pre}}$ is a morphism $(f, f^\flat) : (X, \mathscr{O}_X, (|.|_x)_{x \in X}) \to (Y, \mathscr{O}_Y, (|.|_y)_{y \in Y})$ such that $f : X \to Y$ is a homeomorphism onto an open subset U of Y and that the induced morphism $(X, \mathscr{O}_X, (|.|_x)_{x \in X}) \to (U, \mathscr{O}_{Y|U}, (|.|_y)_{y \in U})$ is an isomorphism in $\mathscr{V}^{\mathrm{pre}}$.

Example III.6.5.2. Let (A, A^+) be a Huber pair. If $T \subset A$ generates an open ideal and $s \in A$, and if $\varphi : (A, A^+) \to (A\langle \frac{T}{s} \rangle, A\langle \frac{T}{s} \rangle^+)$ is the obvious map, then $\mathrm{Spa}(\varphi)$ is an open immersion in $\mathscr{V}^{\mathrm{pre}}$ by lemma III.6.2.5.

Definition III.6.5.3. We denote by $\mathscr{V}$ the full subcategory of $\mathscr{V}^{\mathrm{pre}}$ whose objects are the triples $(X, \mathscr{O}_X, (|.|_x)_{x \in X})$ such that $\mathscr{O}_X$ is a sheaf.

An *affinoid adic space* is an object of $\mathscr{V}$ that is isomorphic to $\mathrm{Spa}(A, A^+)$, for (A, A^+) a Huber pair.

An *adic space* is an object $(X, \mathscr{O}_X, (|.|_x)_{x \in X})$ of $\mathscr{V}$ such that there exists an open covering $(U_i)_{i \in I}$ such that, for every $i \in I$, the triple $(U_i, \mathscr{O}_{X|U_i}, (|.|_x)_{x \in U_i})$ such that is an affinoid adic space.

A *morphism of adic spaces* is a morphism of $\mathscr{V}$.

The next natural question is : which Huber pairs give rise to affinoid adic spaces ? (It is not true that the structural presheaf of $\mathrm{Spa}(A, A^+)$ is always a sheaf.) We will give some criteria in the next chapter.

IV When is the structure presheaf a sheaf ?

The goal of this chapter is to give sufficient conditions on the Huber pair (A, A^+) for $X = \mathrm{Spa}(A, A^+)$ to be an adic space, i.e. for the structure presheaf $\mathscr{O}_X$ to be a sheaf. As we will see, these conditions also imply that the cohomology of $\mathscr{O}_X$ on any rational domain of X is concentrated in degree 0, as we would expect from the cohomology of the structural sheaf of an affinoid adic space. [1]

IV.1 The main theorem

IV.1.1 Statement

Before we can state the main theorem of this chapter, we need some definitions.

Definition IV.1.1.1. Let A be a Tate ring. We say that A is *strongly Noetherian* if $\widehat{A}\langle X_1, \ldots, X_n\rangle$ [2] is Noetherian for every $n \geq 0$.

Definition IV.1.1.2. A non-Archimedean topological ring A is called *uniform* if A^0 is bounded in A.

Note that, if A is f-adic, this is equivalent to the fact that A^0 is a ring of definition of A.

Remark IV.1.1.3. Any Hausdorff uniform Tate ring is reduced. Indeed, let A be a Tate ring, and let $\varpi \in A$ be a topologically nilpotent unit. Suppose that A is uniform, so that A^0 is a ring of definition of A. Let $a \in A$ be a nilpotent element. For every $n \in \mathbb{N}$, the element $\varpi^{-n}a$ is nilpotent, hence power-bounded, so $a \in \varpi^n A^0$. As the topology of A^0 is the ϖA^0-adic topology by proposition II.2.5.2, and as A^0 is Hausdorff by hypothesis, we have $\bigcap_{n\geq 0} \varpi^n A^0 = \{0\}$. So $a = 0$.

Definition IV.1.1.4. Let (A, A^+) be a Huber pair. Then we say that (A, A^+) is *stably uniform* if, for every rational subset U of $\mathrm{Spa}(A, A^+)$, the f-adic ring $\mathscr{O}_{\mathrm{Spa}(A,A^+)}(U)$ is uniform.

[1] Note however that, unlike the case of schemes, there is no cohomological characterization of affinoid adic spaces.
[2] See notation II.3.3.9.

Theorem IV.1.1.5. *(Theorem 2.2 of [15], theorem 8.27 of [26], theorem 7 of [8].) Let (A, A^+) be a Huber pair, and let $X = \mathrm{Spa}(A, A^+)$. Suppose that (A, A^+) satisfies one of the following conditions :*

(a) the completion $\widehat{A}$ is discrete;

(b) the completion $\widehat{A}$ has a Noetherian ring of definition;

(c) A is a strongly Noetherian Tate ring;

(d) the Huber pair (A, A^+) is Tate and stably uniform.

Then $\mathscr{O}_X$ is a sheaf, and, for every rational domain U of X and every $i \geq 1$, we have $\mathrm{H}^i(U, \mathscr{O}_X) = 0$.

See section IV.3 for the proof in cases (c) and (d).

Remark IV.1.1.6. Case (a) of the theorem applies to discrete rings A, and we get that $\mathrm{Spv}(A)$ and Riemann-Zariski spaces are adic spaces.

Case (b) applies for example to a complete Noetherian adic ring A, and gives (with some more work) a fully faithful embedding of the category of locally Noetherian formal schemes over $\mathrm{Spf}(A)$ into the category of adic spaces over $\mathrm{Spa}(A, A)$. Note that adic rings are not Tate, so we cannot apply (c) or (d).

Case (c) applies for example to affinoid algebras over a complete non-Archimedean field k, and gives a fully faithful embedding of the category of rigid analytic varieties over k into the category of adic spaces over $\mathrm{Spa}(k, k^0)$. Note that these affinoid algebras (even k itself) do not have a Noetherian ring of definition unless the valuation defining the topology of k is discrete, so we cannot apply case (b) in general.

Finally, case (d) typically applies to perfectoid algebras. Note that, if k is a complete non-Archimedean field and A is an affinoid k-algebra, then A is uniform if and only if it is reduced (see section II.1.4 for a reference); in particular, case (d) is not sufficient if we are interested in non-reduced rigid analytic varieties.

IV.1.2 Examples of strongly Noetherian Tate rings

Proposition IV.1.2.1. *(Proposition 6.29 of [26].) Let $\varphi : A \to B$ be a morphisms of ring. Suppose that A and B are f-adic rings, and that B is complete. Then the following conditions are equivalent :*

(i) There exists a positive integer n, finite subsets $T_1, \dots, T_n$ such that $T_i \cdot A$ is open in A for every i and a surjective continuous open A-algebra morphism

$$\pi : A\langle X_1, \dots, X_n\rangle_{T_1,\dots,T_n} \to B.$$

(ii) *The morphism φ is adic, there exists a finite subset M of B such that the A-subalgebra $\varphi(A)[M]$ of B is dense in B, and there exist rings of definition A_0 of A and B_0 of B and a finite subset N of B_0 such that $\varphi(A_0) \subset B_0$ and that $\varphi(A_0)[N]$ is dense in B_0.*

(iii) *There exists rings of definition A_0 of A and B_0 of B such that :*

  (a) *$\varphi(A_0) \subset B_0$;*

  (b) *B is finitely generated (as an algebra) over $\varphi(A) \cdot B_0$;*

  (c) *there exists a surjective continuous open A_0-algebra morphism $\widehat{A}_0\langle X_1, \ldots, X_n\rangle \to B_0$, for some $n \in \mathbb{N}$.*

(iv) *For every open subring A_0 of A, there exists an open subring B_0 of B such that conditions (a), (b) and (c) of (iii) hold.*

Definition IV.1.2.2. If a morphism $\varphi : A \to B$ satisfies the equivalent conditions of proposition IV.1.2.1, we say that the A-algebra B is *topologically of finite type*.

Proposition IV.1.2.3. *(Propositions 6.33, 6.35 and 6.36 of [26].) Let A be a strongly Noetherian f-adic ring, and let B be a A-algebra that is topologically of finite type (so B is a complete f-adic ring). Then :*

(i) *B is strongly Noetherian (in particular, it is Noetherian).*

(ii) *If A has a Noetherian ring of definition, so does B.*

Theorem IV.1.2.4. *(Theorem 1 of section 5.2.6 of [3].) Any complete non-Archimedean field is strongly Noetherian.*

IV.1.3 Examples of stably uniform Tate rings

We fix a prime number ℓ.

Definition IV.1.3.1. We say that a ring A is *of characteristic ℓ* if $\ell \cdot 1_A = 0$. If A is of characteristic ℓ, then the map $\mathrm{Frob}_A : A \to A$, $a \longmapsto a^\ell$ is a ring endormophism called the *Frobenius endomorphism* of A; we say that A is *perfect* (resp. *semiperfect*) if Frob_A is bijective (resp. injective).

Notation IV.1.3.2. If A is perfect, we often write $a \longmapsto a^{1/\ell}$ for the inverse of Frob_A.

Remark IV.1.3.3. Unfortunately, there is another definition of perfect and semiperfect rings (for example, a not necessarily commutative ring R is called left perfect if every left module has a projective cover); it is totally unrelated to the previous definition. We will only use definition IV.1.3.1 in these notes.

Remark IV.1.3.4. Let A be a ring of characteristic ℓ. If A is reduced, then $\mathrm{Ker}(\mathrm{Frob}_A) = \{0\}$, so A is perfect if and only if it is semiperfect.

Theorem IV.1.3.5. *(Lemma 7.1.6 of [1].) Let A be a complete Tate ring of characteristic ℓ. Suppose that A is perfect. Then A has a perfect ring of definition and is stably uniform.*

We will see later that these rings are exactly the perfectoid Tate rings of characteristic ℓ.

Proof. Suppose that we have shown that A has a perfect ring of definition. By lemmas IV.1.3.8 and IV.1.3.9, for every finite subset T of A generating an open ideal and every $s \in A$, the ring $A\langle \frac{T}{s} \rangle$ is also a complete and perfect Tate ring. So it suffices to prove that A has a perfect ring of definition and is uniform.

Let A_0 be a ring of definition of A and $\varpi \in A$ be a topologically nilpotent unit. For every $n \geq 1$, let $A_n = A_0^{1/\ell^n}$, and let $A_\infty = \bigcup_{n \geq 0} A_n$. Then A_∞ is a perfect subring of A and $A_0 \subset A_\infty \subset A^0$.

We first show that A_∞ is bounded, hence a ring of definition of A. Note that $\mathrm{Frob}_A : A \to A$ is continuous and surjective, and it is A-linear if we put the obvious A-module structure on its source and the A-module structure given by $a \cdot b = a^\ell b$ ($a, b \in A$) on its target. So, by the Banach open mapping theorem (see corollary II.4.1.6), Frob_A is open. In particular, the subring $\mathrm{Frob}_A(A_0)$ of A is open, so there exists $r \in \mathbb{N}$ such that $\varpi^r A_0 \subset \mathrm{Frob}_A(A_0)$. Applying Frob_A^{-1}, we see that we have $s \in \mathbb{N}$ such that $\varpi^s A_1 \subset A_0$ (any $s \geq r\ell^{-1}$ will do). As in the proof of lemma IV.1.3.8, this implies that $\varpi^{s+s/\ell+\ldots+s/\ell^{n-1}} A_n \subset A_0$ for every $n \geq 1$, hence that $\varpi^{2s} A_\infty \subset A_0$. So A_∞ is bounded.

Now we show that $\varpi A^0 \subset A_\infty$, which will imply that A^0 is bounded. Let $a \in A^0$. As a is power-bounded, there exists $r \in \mathbb{N}$ such that $\{\varpi^r a^n,\ n \in \mathbb{N}\} \subset A_\infty$. As A_∞ is closed in A under taking ℓth roots, this implies that $\varpi^{r/\ell^n} a \in A_\infty$ for every $n \geq 0$. In particular, taking n such that $r \leq \ell^n$, we get $\varpi a \in A_\infty$.

□

Lemma IV.1.3.6. *Let A be a ring of characteristic ℓ and $S \subset A$ be a multiplicative system. If A is perfect (resp. semiperfect), so is $S^{-1}A$.*

Proof. Let $B = S^{-1}A$. Suppose that A is semiperfect. Let $b \in B$, and write $b = as^{-1}$, with $a \in A$ and $s \in S$. As A is semiperfect, we can find $c, t \in A$ such that $c^\ell = a$ and $t^\ell = s$. Then $ct^{\ell-1}s^{-1} \in B$, and $(ct^{\ell-1}s^{-1})^\ell = b$. So B is semiperfect.

We now assume that A is perfect, and we want to show that $\mathrm{Ker}(\mathrm{Frob}_B) = \{0\}$. So let $a \in A$ and $s \in S$ such that $(as^{-1})^\ell = 0$ in B. This means that there exists $t \in S$ such that $ta^\ell = 0$ in A. Then $ta \in \mathrm{Ker}(\mathrm{Frob}_A)$, so $ta = 0$ in A, so $as^{-1} = 0$ in B.

□

Lemma IV.1.3.7. *Let A be a topological ring, N be a positive integer and T be a finite subset of A. Then T is power-bounded if and only the set $\{t^N,\ t \in T\}$ is power-bounded.*

Proof. Write $T' = \{t^N,\ t \in T\}$.

Remember that T is power-bounded is and only if the set $\bigcup_{n\geq 1} T(n)$ is bounded, where, for every $n \geq 1$, $T(n) = \{t_1 \dots t_n,\ t_1, \dots, t_n \in T\}$. As $\bigcup_{n\geq 1} T(n) \supset \bigcup_{n\geq 1}(T')(n)$, T' is power-bounded if T is. On the other hand, we have

$$\bigcup_{n\geq 1} T(n) = \bigcup_{(m_t)_{t\in T}\in\{0,\dots,N-1\}^T} \left(\prod_{t\in T} t^{n_t/N}\right)\left(\bigcup_{n\geq 1}(T')(n)\right),$$

so $\bigcup_{n\geq 1} T(n)$ is a finite union of translates of $\bigcup_{n\geq 1}(T')(n)$. This shows that T is power-bounded if T' is.

□

Lemma IV.1.3.8. *Let A be a f-adic ring, T be a finite subset of A that generates an open ideal and $s \in A$. Suppose that A is perfect. Then we have a canonical isomorphism of A-algebras $A\left(\frac{T}{s}\right) = A\left(\frac{T^{1/\ell}}{s^{1/\ell}}\right)$.*

If moreover A is a Tate ring and has a perfect ring of definition, then $A\left(\frac{T}{s}\right)$ also has a perfect ring of definition.

Proof. Write $B = A\left(\frac{T}{s}\right)$. As an abstract ring, the f-adic ring B is isomorphic to $A[s^{-1}]$. So, by lemma IV.1.3.6, we know that B is perfect.

The first statement follows immediately from the universal property of the localization (see proposition II.3.4.1) and from lemma IV.1.3.7 : the topological rings $A\left(\frac{T}{s}\right)$ and $A\left(\frac{T^{1/\ell}}{s^{1/\ell}}\right)$ satisfy the same universal property.

Let A_0 be a ring of definition of A, and let B_0 be the A_0-subalgebra of B generated by the elements ts^{-1}, $t \in T$. Then B_0 is a ring of definition of B by remark II.3.4.4.

Suppose that A is a Tate ring, let $\varpi \in A$ be a topologically nilpotent unit, and suppose that A_0 is perfect. Then $B_0^{1/\ell}$ is the A_0-subalgebra of B generated by the $t^{1/\ell}s^{-1/\ell}$, $t \in T$, so it is also a ring of definition of B by the first statement of the lemma and the previous paragraph. In particular, $B_0^{1/\ell}$ is bounded, so there exists a positive integer r such that $\varpi^r B_0^{1/\ell} \subset B_0$. An easy induction on n then shows that $\varpi^{r+r/\ell+\dots+r/\ell^{n-1}} B_0^{1/\ell^n} \subset B_0$ for every $n \geq 1$. In particular, we have $\varpi^2 B_0^{1/\ell^n} \subset B_0$ for every $n \geq 1$. Let $B_0' = \bigcup_{n\geq 1} B_0^{1/\ell^n}$. This is a perfect subring of B, it is open because it contains B_0, and it is bounded because $\varpi^{2r} B_0' \subset B_0$. So B_0' is a perfect ring of definition of B, and we are done.

□

Lemma IV.1.3.9. *Let A be a Tate ring with a perfect ring of definition. Then A and $\widehat{A}$ are perfect, and $\widehat{A}$ also has a perfect ring of definition.*

Proof. Let A_0 be a perfect ring of definition of A, and let $\varpi \in A_0$ a topologically nilpotent unit. Then $A = A_0[\varpi^{-1}]$ by proposition II.2.5.2, so A is perfect by lemma IV.1.3.6.

Let $\widehat{A}_0 = \varprojlim_{n\geq 1} A_0/\varpi^n A_0$. As $\widehat{A} = \widehat{A}_0[\varpi^{-1}]$ by (v) of corollary II.3.1.9, it suffices to prove that $\widehat{A}_0$ is perfect. As $\varpi^{1/\ell}$ is also a topologically nilpotent element of A_0, the canonical map $\varprojlim_{n\geq 1} A_0/\varpi^{n\ell} A_0 \to \widehat{A}_0$ (sending $(x_n) \in \varprojlim_{n\geq 1} A_0/\varpi^{n/\ell} A_0$ to $(x_{\ell n})_{n\geq 1}$) is an isomorphism, and this implies immediately that $\widehat{A}_0$ is perfect.

□

IV.2 Some preliminary results

In this section, we fix a Huber pair (A, A^+) with A a Tate ring, and we set $X = \mathrm{Spa}(A, A^+)$.

IV.2.1 Strictness and completion

Definition IV.2.1.1. ([6] chapitre III §2 Nº8 définition 1.) Let $\varphi : M \to M'$ be a continuous morphism of topological groups. We say that φ is *strict* if the following two topologies on $\varphi(M)$ coincide :

- the quotient topology given by the isomorphism $\varphi(M) \simeq M/\operatorname{Ker}\varphi$;
- the subspace topology given by the inclusion $\varphi(M) \subset M'$.

Proposition IV.2.1.2. *([5] chapitre III §2 Nº12 lemme 2.) Let M, M', M'' be abelian topological groups that have countable fundamental systems of neighborhoods of 0, and let $\varphi : M \to M'$ and $\psi : M' \to M''$ be continuous group morphisms. Suppose that :*

- *the sequence $M \overset{\varphi}{\to} M' \overset{\psi}{\to} M''$ is exact as a sequence of abstract group, i.e. $\operatorname{Ker}\psi = \operatorname{Im}\varphi$;*
- *the morphisms φ and ψ are strict.*

Then the sequence

$$\widehat{M} \overset{\widehat{\varphi}}{\to} \widehat{M'} \overset{\widehat{\psi}}{\to} \widehat{M''}$$

is exact and $\widehat{\varphi}$ and $\widehat{\psi}$ are strict.

IV.2.2 The Čech complex for a special cover

Let $t \in A$. We consider the rational domains $U = R\left(\frac{1,t}{1}\right) = \{x \in X \mid |t|_x \leq 1\}$ and $V = R\left(\frac{1}{t}\right) = \{x \in X \mid |t|_x \geq 1\}$. Note that $U \cap V = \{x \in X \mid |t|_x = 1\}$ is the rational domain $R\left(\frac{1,t,t^2}{t}\right)$.

Let $B = A\left(\frac{1,t}{1}\right)$, $C = A\left(\frac{1}{t}\right)$ and $D = A\left(\frac{1,t,t^2}{t}\right)$. We have canonical adic maps $\varphi_B : A \to B$, $\varphi_C : A \to C$, $\psi_B : B \to D$ and $\psi_C : C \to D$ such that $\psi_B \circ \varphi_B = \psi_C \circ \varphi_C$ is the canonical map from A to D. Note that φ_B and ψ_C are continuous and bijective maps of topological rings, but they are not homeomorphisms in general. For example, B is just A as an abstract ring, but with a topology that makes t power-bounded.

We denote the map $\varphi_B \oplus \varphi_C : A \to B \oplus C$ by ε and the map $\psi_B - \psi_C : B \oplus C \to D$ by δ. Then $A \xrightarrow{\varepsilon} B \oplus C \xrightarrow{\delta} D$ is a complex, whose completion is the Čech complex $\mathscr{O}_X(X) \to \mathscr{O}_X(U) \oplus \mathscr{O}_X(V) \to \mathscr{O}_X(U \cap V)$ of the open cover (U, V) of X. We want to know when this Čech complex is exact. This is the goal of the following proposition.

Proposition IV.2.2.1. *(See Lemma 2 of [8] and the discussion preceding it.)*

(i) The complex $0 \to A \xrightarrow{\varepsilon} B \oplus C \xrightarrow{\delta} D \to 0$ *is exact as a complex of abstract commutative groups.*

(ii) The map $\delta : B \oplus C \to D$ *is strict.*

(iii) The map $\mathscr{O}_X(U) \oplus \mathscr{O}_X(V) \to \mathscr{O}(U \cap V)$ *sending* (f, g) *to* $f_{|U\cap V} - g_{|U\cap V}$ *(i.e. the completion of* δ*) is surjective.*

(iv) The following conditions are equivalent :

(a) the map $\varepsilon : A \to B \oplus C$ *is strict;*

(b) the complex $0 \to \mathscr{O}_X(X) \to \mathscr{O}_X(U) \oplus \mathscr{O}_X(V) \to \mathscr{O}_X(U \cap V)$ *is exact;*

(c) the complex $0 \to \mathscr{O}_X(X) \to \mathscr{O}_X(U) \oplus \mathscr{O}_X(V) \to \mathscr{O}_X(U \cap V) \to 0$ *is exact;*

(d) there exists rings of definition $A_0 \subset A$, $B_0 \subset B$ *and* $C_0 \subset C$, *a topologically nilpotent unit* $\varpi \in A$ *and* $n \in \mathbb{N}$ *such that* $\varpi^n(\varphi_B^{-1}(B_0) \cap \varphi_C^{-1}(C_0)) \subset A_0$.

Remember that a morphism of topological groups $u : G \to H$ is called *strict* if the quotient topology on $u(G) \simeq G/\operatorname{Ker} u$ coincides with the subspace topology induced by the topology of H.

Proof. Let A_0 be a ring of definition of A, and let $\varpi \in A$ be a topologically nilpotent unit such that $\varpi \in A_0$. Then, by remark II.3.4.2, $A_0[t]$ is a ring of definition of B, $A_0[t^{-1}]$ is a ring of definition of C and $A_0[t, t^{-1}]$ is a ring of definition of D. Also, as the f-adic rings A, B, C and D are Tate, the topology on these rings of definition is the ϖ-adic topology by proposition II.2.5.2.

(i) The fact that ε is injective and δ surjective follows from the fact that φ_B and ψ_C are isomorphisms of abstract rings. Let $(b, c) \in B \oplus C$ such that $\delta(b, c) = 0$, i.e. $b - c = 0$ in $A[t^{-1}]$. We have $b \in A$, so $(b, c) = \varepsilon(b)$.

(ii) The map δ is surjective by (i), so we just need to check that δ is open. This follows from the obvious fact that $\delta(\varpi^n A_0[t] \oplus \varpi^n A_0[t^{-1}]) = \varpi^n A_0[t, t^{-1}]$ for every $n \in \mathbb{N}$.

(iii) This follows from (ii) and from [5] chapitre III §2 №12 lemme 2.

(iv) The map ε is strict if and only, for all rings of definition $B_0 \subset B$ and $C_0 \subset C_0$, the groups $\varpi^n(\varphi_B^{-1}(B_0) \cap \varphi_C^{-1}(C_0)) = \varepsilon^{-1}(\varpi^n B_0 \oplus \varpi^n C_0)$, $n \geq 0$, form a fundamental system of neighborhoods of 0 in A. This shows that (a) and (d) are equivalent. Also, (c) obviously implies (b), and (b) implies (c) by (iii).

Suppose that (a) holds. Then all the maps in the exact sequence of (i) are strict, so its completion is still exact by [5] chapitre III §2 Nº12 lemme 2. This shows that (a) implies (c).

It remains to show that (c) implies (d). Fix rings of definition $B_0 \subset B$ and $C_0 \subset C$ such that $A_0 \subset \varphi_B^{-1}(B_0) \cap \varphi_C^{-1}(C_0) := A_0'$, and let A' be the ring A with the topology for which $(\varpi^n A_0')_{n \geq 0}$ is a fundamental system of neighborhoods of 0. (This does define a structure of topological ring on A' by lemma II.3.3.8.) The identity $A \to A'$ is continuous (because A_0' is an open subring of A), and (d) is equivalent to the fact that it is an open map. Note that A' is isomorphic as a topological ring to $\varepsilon(A)$ with the subspace topology, so the obvious sequence $0 \to A' \to B \oplus C \to D \to 0$ is exact and all the maps in it are strict. Using [5] chapitre III §2 Nº12 lemme 2 again, we see that the sequence $0 \to \widehat{A'} \to \widehat{B} \oplus \widehat{C} \to \widehat{D} \to 0$ is exact. As we are assuming that $0 \to \widehat{A'} \to \widehat{B} \oplus \widehat{C} \to \widehat{D} \to 0$ is exact, this implies that the canonical map $\widehat{A} \to \widehat{A'}$ is bijective. By the open mapping theorem (theorem II.4.1.1), the map $\widehat{A} \to \widehat{A'}$ is open. Using lemma II.3.1.11, we see that this implies the openness of $A \to A'$, hence condition (d).

□

IV.2.3 Refining coverings

In this section, we fix a Huber pair (A, A^+), with A a f-adic ring. We want show the existence of enough manageable covers of $X = \mathrm{Spa}(A, A^+)$.

Definition IV.2.3.1. (i) Let $t_1, \ldots, t_n \in A$ generating the ideal (1) of A. The *standard rational covering* of X generated by $t_1, \ldots, t_n$ is the covering $(U_i)_{1 \leq i \leq n}$, where $U_i = R\left(\frac{t_1, \ldots, t_n}{t_i}\right)$. We say that this standard rational covering is *generated by units* if $t_1, \ldots, t_n \in A^\times$.

(ii) Let $t_1, \ldots, t_n \in A$. For every $I \subset \{1, \ldots, n\}$, let

$$V_I = \{x \in X \mid |t_i|_x \leq 1 \ \forall i \in I \text{ and } |t_i|_x \geq 1 \ \forall i \notin I\};$$

note that V_I is the rational domain $R\left(\frac{T_I}{s_I}\right)$, where $s_I = \prod_{j \notin I} t_i$ and

$$T_I = \{1\} \cup \{t_i s_I, \ i \in I\} \cup \{\prod_{j \notin I \cup \{i\}} t_i, \ i \notin I\}.$$

The family $(V_I)_{I \subset \{1, \ldots, n\}}$ is an open covering of X, which we call the *standard Laurent covering* generated by $t_1, \ldots, t_n$. Again, we say that the covering is *generated by units* if $t_1, \ldots, t_n \in A^\times$.

(iii) A *simple Laurent covering* is a standard Laurent covering generated one element.

(iv) A *rational covering* of X is a covering of X by rational domains.

Note that the covering of proposition IV.2.2.1 is a simple Laurent. Note also that we diverge from the vocabulary of [8] when defining rational coverings and follow [19] instead.

Remark IV.2.3.2. Note that, if $t_1, \dots, t_n \in A$ are such that $(t_1, \dots, t_n) = A$, the standard rational covering generated by $t_1, \dots, t_n$ is a covering of X. Indeed, we have

$$\bigcup_{i=1}^{n} R\left(\tfrac{t_1,\dots,t_n}{t_i}\right) = \bigcup_{i=1}^{n}\{x \in X | \max_{1\le j\le n} |t_j|_x = |t_i|_x \neq 0\} = \{x \in X \mid \max_{1\le j\le n} |t_j|_x \neq 0\},$$

which is also the set of $x \in X$ such that at least one of the $|t_i|_x$ is nonzero; but this is all of X, because $1 \in \sum_{i=1}^n At_i$ (and $|1|_x = 1 \neq 0$ for every $x \in X$). The same observation also shows that, for every $i \in \{1, \dots, n\}$,

$$R\left(\frac{t_1,\dots,t_n}{t_i}\right) = \{x \in X \mid \forall j \in \{1,\dots,n\},\ |t_j|_x \le |t_i|_x\}.$$

(If we have $|t_j|_x \le |t_i|_x$ for every j and $|t_i|_x = 0$, then $|t_1|_x = \ldots = |t_n|_x = 0$, and we have just seen that this implies $|1|_x = 0$, which is impossible.)

Proposition IV.2.3.3. *(Lemma 8 of [8] and lemma 2.4.19 of [19].)*

(i) For every open covering $\mathscr{U}$ of X, there exists a rational covering refining $\mathscr{U}$. If moreover A is complete, then, for every open covering $\mathscr{U}$ of X, there exists a standard rational covering $\mathscr{V}$ of X refining $\mathscr{U}$.

(ii) Suppose that A is a Tate ring. Then, for every standard rational covering $\mathscr{U}$ of X, there exists a standard Laurent covering $\mathscr{V}$ of X such that for every $V \in \mathscr{V}$, the covering $(V \cap U)_{U\in\mathscr{U}}$ of V is a standard rational covering generated by units.

(iii) For every standard rational covering generated by units $\mathscr{U}$ of X, there exists a standard Laurent covering $\mathscr{V}$ of X generated by units refining $\mathscr{U}$.

Proof. (i) This is lemma 2.6 of [15]. First we construct a rational covering refining $\mathscr{U}$. Let $x \in X$, and let $y = x_{|c\Gamma_x}$ (so y is the minimal horizontal specialization of x). Note that y is still in $\mathrm{Spa}(A, A^+)$: y is continuous by proposition II.2.3.1(i), and, for every $a \in A^+$, we have $|a|_y \le |a|_x \le 1$. Let $U \in \mathscr{U}$ such that $y \in U$. As y has no proper horizontal specialization, we have $y \in \mathrm{Spv}(A, A)$ (see remark I.4.2.2(2)). So, by point (3) in the proof of theorem I.4.2.4, there exists a finite subset T_x of A such that $T_x \cdot A = A$ and $s_x \in A$ such that $y \in R\left(\frac{T_x}{s_x}\right) \subset U$; after replacing T_x by $T_x \cup \{s_x\}$, we may assume that $s_x \in T_x$. As x is a generization of y, we also have $x \in R\left(\frac{T_x}{s_x}\right)$. So $X = \bigcup_{x\in X} R\left(\frac{T_x}{s_x}\right)$. As X is quasi-compact, we can find $x_1, \dots, x_n \in X$ such that $X = \bigcup_{x\in X} R\left(\frac{T_{x_i}}{s_{x_i}}\right)$. This is a finite rational covering of X refining $\mathscr{U}$.

We now assume that A is complete and refine this rational covering forther to a standard rational covering. We write $T_{x_i} = T_i$ and $s_{x_i} = s_i$. Let

$$T = \{t_1 \dots t_n,\ t_i \in T_i\ \forall i \in \{1, \dots, n\}\}$$

and

$$S = \{t_1 \dots t_n,\ t_i \in T_i\ \forall i \in \{1, \dots, n\} \text{ and } \exists i \in \{1, \dots, n\} \text{ such that } t_i = s_i\}.$$

As each $T_i \cdot A = A$ for every i, we have $S \cdot A = s_1 A + \dots s_n A$. As X is the union of the $R\left(\frac{T_i}{s_i}\right)$, for every $x \in X$, there exists $i \in \{1, \dots, n\}$ such that $|s_i|_x \neq 0$. By corollary III.4.4.3, this implies that $S \cdot A = A$.

We want to show that the standard rational covering generated by S refines the covering $(R\left(\frac{T_i}{s_i}\right))_{1 \leq i \leq n}$, hence $\mathscr{U}$. Let $s \in S$, and write $s = t_1 \dots t_n$, with $t_i \in T_i$. Pick $j \in \{1, \dots, n\}$ such that $t_j = s_j$. We claim that $R\left(\frac{S}{s}\right) \subset R\left(\frac{T_j}{s_j}\right)$. Indeed, let $x \in R\left(\frac{S}{s}\right)$, and let $t \in T_j$. Then $|t_1 \dots t_{i-1} t t_{i+1} \dots t_n|_x \leq |s|_x \neq 0$, and this implies $|t|_x \leq |s_j|_x \neq 0$.

(ii) Let $t_1, \dots, t_n \in A$ such that $(t_1, \dots, t_n)$, and let $(U_1, \dots, U_n)$ be the standard rational covering generated by $t_1, \dots, t_n$. Let $\varpi \in A$ be a topologically nilpotent unit. We choose $a_1, \dots, a_n \in A$ such that $a_1 t_1 + \dots + a_n t_n = 1$. As A^+ is open, there exists $N \in \mathbb{N}$ such that $\varpi^N a_i \in A^+$ for every $i \in \{1, \dots, n\}$. Then, if $x \in X = \operatorname{Spa}(A, A^+)$ and we have $|\varpi^N a_i|_x \leq 1$ for every i, so $|\varpi^N|_x = |\varpi^N a_1 t_1 + \dots + \varpi^N a_n t_n|_x \leq \max_{1 \leq i \leq n} |t_i|_x$, and finally $|\varpi^{N+1}|_x < \max_{1 \leq i \leq n} |t_i|_x$ (because $|\varpi|_x < 1$). Let $(V_I)_{I \subset \{1, \dots, n\}}$ be the standard Laurent covering generated by $\varpi^{-(N+1)} t_1, \dots, \varpi^{-(N+1)} t_n$. We will show that this covering works.

Let $I \subset \{1, \dots, n\}$. We have

$$V_I = \{x \in X \mid |t_i|_x \leq |\varpi^{N+1}|_x\ \forall i \in I \text{ and } |t_i|_x \geq |\varpi^{N+1}|_x\ \forall i \notin I\}.$$

In particular, by the choice of N, $V_{\{1, \dots, n\}} = \varnothing$. Suppose that $I \subsetneq \{1, \dots, n\}$. By the description of V_I as a rational subset in definition IV.2.3.1, we have $t_i \in \mathscr{O}_X(V_I)^\times$ for $i \notin I$. If $x \in V_I$,

$$\max_{i \in I} |t_i|_x \leq |\varpi^{N+1}|_x < \max_{1 \leq i \leq n} |t_i|_x,$$

so

$$\max_{1 \leq i \leq n} |t_i|_x = \max_{i \notin I} |t_i|_x.$$

In particular, $V_I \cap U_i = \varnothing$ if $i \in I$, and $V_I \cap U_i = V_I \cap R\left(\frac{t_j,\ j \notin I}{t_i}\right)$ if $i \notin I$. This shows the statement.

(iii) Let $t_1, \dots, t_n \in A^\times$, and let $(U_1, \dots, U_n)$ be the standard rational covering generated by $t_1, \dots, t_n$. Let $I = \{(i, j) \in \{1, \dots, n\} \mid i < j\}$, and let $t_{(i,j)} = t_i t_j^{-1}$ for $(i, j) \in I$.

We claim that the standard Laurent covering generated by the family $(t_{(i,j)})_{(i,j)\in I}$ works. Denote this covering by $(V_J)_{J\subset I}$. For $J \subset I$, we have

$$V_J = \{x \in X \mid |t_i|_x \leq |t_j|_x \text{ if } (i,j) \in J \text{ and } |t_i|_x \geq |t_j|_x \text{ if } (i,j) \not\in J\}.$$

Choose a finite sequence $(i_1, \ldots, i_r)$ of elements of $\{1, \ldots, n\}$ such that $(i_s, i_{s+1}) \in J$ for $1 \leq s \leq r-1$ and of maximal length for that property. (Such a chain exists, because the condition implies that $i_1 < \ldots < i_r$, so we must have $r \leq n$.) Then, if $i \in \{1, \ldots, n\} - \{i_r\}$, we cannot have $(i_r, i) \in J$ because this would contradict the maximality of $(i_1, \ldots, i_r)$, so $|t_i|_x \leq |t_{i_r}|_x$ for every $x \in V_J$. This shows that $V_J \subset U_{i_r}$.

□

Corollary IV.2.3.4. *(Proposition 2.4.20 of [19].) Suppose that A is a complete Tate ring. Let $\mathscr{P}$ be a property of rational coverings of rational domains of X. Suppose that $\mathscr{P}$ satisfies the following conditions :*

(a) $\mathscr{P}$ is local, i.e. if it holds for a refinement of a covering then it also holds for the original covering.

(b) $\mathscr{P}$ is transitive : let U be a rational domain of X, $(U_i)_{i\in I}$ be a rational covering of U and $(U_{ij})_{j\in J_i}$ be a rational covering of U_i for every $i \in I$; if $\mathscr{P}$ holds for the covering $(U_i)_{i\in I}$ of U and for each covering $(U_{ij})_{j\in J_i}$ of U_i, $i \in I$, then it holds for the covering $(U_{ij})_{i\in I, j\in J_i}$ of U.

(c) $\mathscr{P}$ holds for every simple Laurent covering of a rational domain of X.

Then $\mathscr{P}$ holds for any rational covering of every rational domain of X.

We will see examples of properties $\mathscr{P}$ satisfies (a), (b) and (c) in corollary IV.3.2.1.

Proof. (1) If U is a rational domain of X and $\mathscr{U}$ is a standard Laurent covering of U of $\mathscr{O}_X(U)$, then $\mathscr{P}$ holds for $\mathscr{U}$: We prove this by induction on the number n of elements generating the Laurent covering. If $n = 1$, this is condition (c). Suppose that $n \geq 2$ and that we know the result for $n-1$ (and for every rational domain of X). Let $t_1, \ldots, t_n \in \mathscr{O}_X(U)$, and consider the standard Laurent covering $\mathscr{U} = (V_I)_{I\subset\{1,\ldots,n\}}$ of U that they generate. We also set $W = \{x \in U \mid ||t_1|_x \leq 1\}$ and $W' = \{x \in U \mid ||t_1|_x \geq 1\}$; this is a simple Laurent covering of U, so $\mathscr{P}$ holds for this covering by (c). Then $(W \cap V_I)_{1\in I}$ (resp. $(W' \cap V_I)_{i\not\in I}$) is the standard Laurent covering of W (resp. W') generated by $t_2, \ldots, t_n$. By the assumption hypothesis, $\mathscr{P}$ holds for these two coverings, so it holds for $\mathscr{U}$ by condition (b).

(2) By (1), property (a) and proposition IV.2.3.3(iii), $\mathscr{P}$ holds for any standard rational covering generated by units of a rational domain of X.

(3) Let $\mathscr{U}$ be a standard rational covering of a rational domain U of X. By proposition IV.2.3.3(ii), there exists a standard Laurent covering $\mathscr{V}$ of U such that, for every $V \in \mathscr{V}$,

$(V \cap U)_{U \in \mathscr{U}}$ is a standard rational covering generated by units of V. By (2), property $\mathscr{P}$ holds for each covering $(V \cap U)_{U \in \mathscr{U}}$, and by (1), it holds for $\mathscr{V}$. So, by property (b), $\mathscr{P}$ holds for $\mathscr{U}$.

(4) Let $\mathscr{U}$ be a rational covering of a rational domain U of X. By proposition IV.2.3.3(i), there exists a standard rational covering $\mathscr{V}$ of U refining $\mathscr{U}$. Property $\mathscr{P}$ holds for $\mathscr{V}$ by (3), so it holds for $\mathscr{U}$ by (a).

□

IV.3 Proof of theorem IV.1.1.5 in cases (a), (c) and (d)

In this section, we fix a Huber pair (A, A^+) such that A is a Tate ring, and we write $X = \mathrm{Spa}(A, A^+)$. We also fix a ring of definition A_0 of A and a topologically nilpotent unit ϖ of A such that $\varpi \in A_0$.

IV.3.1 A local criterion for power-boundedness

Proposition IV.3.1.1. *(Lemma 3 of [8].) Let $t_1, \ldots, t_n \in A$, and suppose that the ideal $(t_1, \ldots, t_n)$ is A itself. For every $i \in \{1, \ldots, n\}$, let $\varphi_i : A \to A_i := A\left(\frac{t_1, \ldots, t_n}{t_i}\right)$ be the canonical map. Let $A_{i,0} = A_0[\frac{t_1}{t_i}, \ldots, \frac{t_n}{t_0}] \subset A_i$; this is a ring of definition of A_i.*

Then

$$A^0 \supset \bigcap_{1 \leq i \leq n} \varphi_i^{-1}(A_{i,0}).$$

In other words, an element $a \in A$ such that $\varphi_i(a) \in A_{i,0}$ for every i is power-bounded.

Proof. Let $a \in \bigcap_{1 \leq i \leq n} \varphi_i^{-1}(A_{i,0})$. For each $i \in \{1, \ldots, n\}$, the image of a in $A[t_i^{-1}]$ is in the subring $A_0[t_1 t_i^{-1}, \ldots, t_n t_i^{-1}]$. Choose a homogeneous polynomial $f_i \in A_0[X_1, \ldots, X_n]$ such that $a_i = t_i^{-\deg(f_i)} f_i(t_1, \ldots, t_n)$ in $A[t_i^{-1}]$. Then we can find $c_i \in \mathbb{N}$ such that $t_i^{c_i}(t_i^{\deg(f_i)} a - f_i(t_1, \ldots, t_n)) = 0$ in A. If we set $g_i = X_i^{c_i} f_i \in A_0[X_1, \ldots, X_n]$, then g_i is homogeneous of degree $d_i := c_i + \deg(f_i)$ and $t_i^{d_i} a - g_i(t_1, \ldots, t_n) = 0$ in A for every $i \in \{1, \ldots, n\}$.

Let $N = d_1 + \ldots + d_n$, and choose $A \in \mathbb{N}$ such that $\varpi^A t_i \in A_0$ for every $i \in \{1, \ldots, n\}$. We show by induction on m that $\varpi^{NA} h(t_1, \ldots, t_n) a^m \in A_0$ for every homogeneous polynomial $h \in A_0[X_1, \ldots, X_n]$ of degree N and every $m \in \mathbb{N}$. The statement is clear if $m = 0$, because then $\varpi^{NA} h(t_1, \ldots, t_n) r^m = \varpi^{NA} h(t_1, \ldots, t_n)$ is a polynomial in $\varpi^A t_1, \ldots, \varpi^A t_n$ with coefficients in A_0. Suppose that $m \geq 1$ and that we know the result for $m - 1$. It suffices to prove the statement for h a monomial of degree N, i.e. $h = X_1^{e_1} \ldots X_n^{e_n}$ with $e_1 + \ldots + e_n = N$. Since $N = d_1 + \ldots + d_n$, there is at least one i such that $e_i \geq d_i$, and we may assume that $i = 1$. Then

$$\varpi^{NA} t_1^{e_1} \ldots t_n^{e_n} a^m = \varpi^{NA} t_1^{e_1 - d_1} t_2^{e_2} \ldots t_n^{e_n} g_1(t_1, \ldots, t_n) a^{m-1},$$

and the right hand side is in A_0 by the induction hypothesis.

Now we show that $a \in A^0$. Choose $a_1, \dots, a_n \in A$ such that $a_1t_1 + \dots + a_nt_n = 1$, and choose $B \in \mathbb{N}$ such that $\varpi^B a_i \in A_0$ for every $i \in \{1, \dots, n\}$. Then $h = (\varpi^B a_1 X_1 + \dots + \varpi^B a_n X_n)^N \in A_0[X_1, \dots, X_n]$ is homogeneous of degree N, so, by the previous paragraph, $\varpi^{NA} h(t_1, \dots, t_n) a^m = \varpi^{N(A+B)} a^m$ is in A_0 for every $m \in \mathbb{N}$. This shows that $\{a^m,\ m \in \mathbb{N}\}$ is bounded.

□

Corollary IV.3.1.2. *We keep the notation of proposition IV.3.1.1, and we suppose that A is uniform. Then the morphism $\varphi : A \to \prod_{i=1}^n A_i$, $a \longmapsto (\varphi_i(a))_{1 \leq i \leq n}$ is strict.*

Proof. As A is uniform, we may assume that $A_0 = A^0$. The subspace topology on $\varphi(A)$ has the sets $\varphi(A) \cap (\prod_{i=1}^n \varpi^N A_{i,0})$, $N \in \mathbb{N}$, as a fundamental system of neighborhoods of 0, and the quotient topology has the sets $\varphi(\varpi^N A_0)$ as a fundamental system of neighborhoods of 0. We already know that the quotient topology is finer than the subspace topology because φ is continuous. On the other hand, proposition IV.3.1.1 says that $\varphi(\varpi^N A_0) \supset \varphi(A) \cap (\prod_{i=1}^n \varpi^N A_{i,0})$ for every $N \in \mathbb{N}$, so the subspace topology is finer than the quotient topology, and we are done.

□

Corollary IV.3.1.3. *(Corollary 4 of [8].) Suppose that A is uniform. Let $t \in A$, and consider the open cover $(U = R\left(\frac{1,t}{1}\right), V = R\left(\frac{1}{t}\right)$ of X. Then the Čech complex*

$$0 \to \mathscr{O}_X(X) \to \mathscr{O}_X(U) \oplus \mathscr{O}_X(V) \to \mathscr{O}_X(U \cap V) \to 0$$

is exact.

Proof. We are in the situation of proposition IV.2.2.1, so it suffices to check that condition (iv)(d) of this proposition holds. Applying proposition IV.3.1.1 with $t_1 = 1$ and $t_2 = t$ shows that, with the notation of proposition IV.2.2.1, $\varphi_B^{-1}(B_0) \cap \varphi_C^{-1}(C_0) \supset A^0$, where $B_0 = A_0[t]$ and $C_0 = A_0[t^{-1}]$. As A is uniform, A^0 is bounded, so there exists $n \in \mathbb{N}$ such that $\varpi^n A^0 \subset A_0$. This shows condition (iv)(d) of proposition IV.2.2.1.

□

IV.3.2 Calculation of the Čech cohomology

If $\mathscr{F}$ is a presheaf of abelian groups on X, U is an open subset of X and $\mathscr{U}$ is an open covering of U, we denote by $\check{\mathrm{C}}^\bullet(\mathscr{U}, \mathscr{F})$ the associated Čech complex and by $\check{\mathrm{H}}^i(\mathscr{U}, \mathscr{F})$ its cohomology, i.e., the Čech cohomology groups of $\mathscr{F}$ on U for the covering $\mathscr{U}$. (See [25, Definition 01EF].)

Corollary IV.3.2.1. *(Proposition 2.4.21 of [19].) Let $\mathscr{F}$ be a presheaf of abelian groups on X. Consider the following property $\mathscr{P}$ of rational coverings $\mathscr{U}$ of a rational domain U of*

X : for every rational domain V of U, if $V \cap \mathscr{U} = (V \cap W)_{W \in \mathscr{U}}$, then the canonical map $\mathscr{F}(V) \to \check{H}^0(V \cap \mathscr{U}, \mathscr{F}$ is an isomorphism and $\check{H}^i(\mathscr{V}, \mathscr{F}) = 0$ for $i \geq 1$.

Then property $\mathscr{P}$ satisfies conditions (a) and (b) of corollary IV.2.3.4, so in particular, it holds for every rational covering of every rational domain of X if and only if it satisfies property (c) of that corollary.

Proof. We check (a). Suppose that $\mathscr{U} = (U_i)_{i \in I}$ and $\mathscr{V} = (V_j)_{j \in J}$ are rational coverings of a ratinal domain U of X, that $\mathscr{V}$ refines $\mathscr{U}$, and that $\mathscr{P}$ holds for $\mathscr{V}$. We want to apply corollary IV.3.2.3 to show that $\check{H}^\bullet(\mathscr{U}, \mathscr{O}_X) \simeq \check{H}^\bullet(\mathscr{V}, \mathscr{O}_X)$ (which will imply that $\mathscr{P}$ holds for $\mathscr{U}$), so we need to check that the hypotheses of this corollary hold. For all $i_0, \ldots, i_p$, the fact that the map

$$\mathscr{O}_X(U_{i_0} \cap \ldots \cap U_{i_p}) \to \check{C}^\bullet(U_{i_0} \cap \ldots \cap U_{i_p} \cap \mathscr{V}, \mathscr{O}_X)$$

is a quasi-isomorphism follows immediately from the fact that property $\mathscr{P}$ holds for $\mathscr{V}$. Let $j_0, \ldots, j_q$, and let $V' = V_{j_0} \cap \ldots \cap V_{j_q}$. We want to show that the map $\mathscr{O}_X(V') \to \check{C}^\bullet(V' \cap \mathscr{U}, \mathscr{O}_X)$ is a quasi-isomorphism. But this follows from proposition IV.3.2.4 and from the fact that the coverings $V' \cap \mathscr{U}$ and $\{V'\}$ of V' refine each other.

We now check (b). Let U be a rational domain of X, consider a rational covering $\mathscr{U} = (U_i)_{i \in I}$ of U and a rational covering $\mathscr{U}_i = (U_{ij})_{j \in J_i}$ of U_i for every $i \in I$. Suppose that $\mathscr{P}$ holds for $\mathscr{U}$ and for every covering $\mathscr{U}_i$, $i \in I$. We want to show that it holds for the covering $\mathscr{V} = (U_{ij})_{i \in I, j \in J_i}$ of U. First note that, for every $i \in I$, the covering $\mathscr{U}_i$ of U_i refines $U_i \cap \mathscr{V}$. As $\mathscr{P}$ satisfies condition (a), this implies that $\mathscr{P}$ holds for $U_i \cap \mathscr{V}$. As before, we want to apply corollary IV.3.2.3 to show that $\check{H}^\bullet(\mathscr{U}, \mathscr{O}_X) \simeq \check{H}^\bullet(\mathscr{V}, \mathscr{O}_X)$ (which will imply that $\mathscr{P}$ holds for $\mathscr{V}$). By the beginning of the paragraph, we already know that, for all $i_0, \ldots, i_p \in I$, the map $\mathscr{F}(U_{i_0} \cap \ldots \cap U_{i_p}) \to \check{C}^\bullet(U_{i_0} \cap \ldots \cap U_{i_p} \cap \mathscr{V}, \mathscr{F})$ is a quasi-isomorphism. Moreover, if $V \in \mathscr{V}$, then the coverings $V \cap \mathscr{U}$ and $\{V\}$ of V refine each other, so they have isomorphic Čech complexes by proposition IV.3.2.4. So the hypotheses of corollary IV.3.2.3 are satisfied.

□

In the end of this section, we give some auxiliary results that are used in the proof of corollary IV.3.2.1. We need a way to compare Čech cohomology for two different covers. This is done in section 8.1.4 of [3]. We review their construction.

Let X be a topological space, $\mathscr{F}$ be a presheaf of abelian groups on X and $\mathscr{U} = (U_i)_{i \in I}$, $\mathscr{V} = (V_j)_{j \in J}$ be two open coverings of X. [3] For all $i_0, \ldots, i_p \in I$ (resp. $j_0, \ldots, j_q \in J$), we write $U_{i_0, \ldots, i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ (resp. $V_{j_0, \ldots, j_q} = V_{j_0} \cap \ldots \cap V_{j_q}$). We define a double complex $\check{C}^{\bullet,\bullet}(\mathscr{U}, \mathscr{V}; \mathscr{F})$ by

$$\check{C}^{p,q}(\mathscr{U}, \mathscr{V}; \mathscr{F}) = \prod_{\substack{i_0, \ldots, i_p \in I \\ j_0, \ldots, j_q \in J}} \mathscr{F}(U_{i_0, \ldots, i_p} \cap V_{j_0, \ldots, j_q}),$$

[3] This would work just as well for X a site and $\mathscr{F}$ a presheaf with values in an abelian category.

with differentials

$$'d^{p,q} : \check{\mathrm{C}}^{p,q}(\mathscr{U}, \mathscr{V}; \mathscr{F}) \to \check{\mathrm{C}}^{p+1,q}(\mathscr{U}, \mathscr{V}; \mathscr{F})$$
$$''d^{p,q} : \check{\mathrm{C}}^{p,q}(\mathscr{U}, \mathscr{V}; \mathscr{F}) \to \check{\mathrm{C}}^{p,q+1}(\mathscr{U}, \mathscr{V}; \mathscr{F})$$

such that, if $f \in \check{\mathrm{C}}^{p,q}(\mathscr{U}, \mathscr{V}; \mathscr{F})$, then the $(i_0, \dots, i_{p+1}, j_0, \dots, j_q)$-component of $'d^{p,q}(f)$ is given by

$$\sum_{r=0}^{p+1} (-1)^{r+q} f_{i_0,\dots,i_{r-1},i_{r+1},\dots,i_p,j_0,\dots,j_q|U_{i_0,\dots,i_{p+1}} \cap V_{j_0,\dots,j_q}}$$

and the $(i_0, \dots, i_p, j_0, \dots, j_{q+1})$-component of $''d^{p,q}(f)$ is given by

$$\sum_{s=0}^{q+1} (-1)^{r+q} f_{i_0,\dots,i_p,j_0,\dots,j_{s-1},j_{s+1},\dots,j_q|U_{i_0,\dots,i_p} \cap V_{j_0,\dots,j_{q+1}}}.$$

We also denote by $\check{\mathrm{C}}^{\bullet}(\mathscr{U}, \mathscr{V}; \mathscr{F})$ the associated simple complex.

The obvious maps

$$F(V_{j_0,\dots,j_q}) \to \check{\mathrm{C}}^0(V_{j_0,\dots,j_q} \cap \mathscr{U}, F)$$

induce a morphism of complexes

$$\check{\mathrm{C}}^{\bullet}(\mathscr{V}, \mathscr{F}) \to \check{\mathrm{C}}^{\bullet}(\mathscr{U}, \mathscr{V}; \mathscr{F}).$$

The following result is Lemma 1 of [3] 8.1.4.

Proposition IV.3.2.2. *Suppose that, for all $j_0, \dots, j_q \in J$, the obvious morphism*

$$\mathscr{F}(V_{j_0,\dots,j_q}) \to \check{C}^{\bullet}(V_{j_0,\dots,j_q} \cap \mathscr{U}, \mathscr{F})$$

is a quasi-isomorphism. Then the morphism $\check{C}^{\bullet}(\mathscr{V}, \mathscr{F}) \to \check{C}^{\bullet}(\mathscr{U}, \mathscr{V}; \mathscr{F})$ defined above is a quasi-isomorphism.

Proof. This is a general result for double complexes. See for example corollary 12.5.5 of [17].

□

We obviously have a similar result if we switch the roles of $\mathscr{U}$ and $\mathscr{V}$, so we get the following corollary.

Corollary IV.3.2.3. *([3] 8.2 Theorem 2.) Assume, for $i_0, \dots, i_p \in I$ (resp. $j_0, \dots, j_q \in J$), the obvious morphism*

$$\mathscr{F}(U_{i_0,\dots,i_p}) \to \check{C}^{\bullet}(U_{i_0,\dots,i_p} \cap \mathscr{V}, \mathscr{F})$$
$$(\textit{resp.} \qquad \mathscr{F}(V_{j_0,\dots,j_q}) \to \check{C}^{\bullet}(V_{j_0,\dots,j_q} \cap \mathscr{U}, \mathscr{F}))$$

is a quasi-isomorphism. Then the two morphisms

$$\check{C}^\bullet(\mathscr{V},\mathscr{F}) \to \check{C}^\bullet(\mathscr{U},\mathscr{V};\mathscr{F}) \leftarrow \check{C}^\bullet(\mathscr{U},\mathscr{F})$$

defined above are quasi-isomorphisms.

In particular, we get canonical isomorphisms $\check{H}^r(\mathscr{U},\mathscr{F}) \simeq \check{H}^r(\mathscr{V},\mathscr{F})$, for all $r \in \mathbb{N}$.

We note another and simpler way to compare the Čech complexes under extra hypotheses. Suppose that $\mathscr{V}$ refines $\mathscr{U}$. For every $j \in J$, we choose $c(j) \in I$ such that $V_j \subset U_{c(j)}$. Then we get restriction morphisms

$$\mathscr{F}(U_{c(j_0),\dots,c(j_q)}) \to \mathscr{F}(V_{j_0,\dots,j_q}),$$

for all $j_0,\dots,j_q \in J$, and these induce a morphism of complexes

$$\check{\mathrm{C}}^\bullet(\mathscr{U},\mathscr{F}) \to \check{\mathrm{C}}^\bullet(\mathscr{V},\mathscr{F}).$$

This morphism of complexes depends on the choice of $c : J \to I$, but the maps it induces on cohomology do not (see for example [25, Section 09UY] for details). This immediately implies the following result.

Proposition IV.3.2.4. *If $\mathscr{U}$ refines $\mathscr{V}$ and $\mathscr{V}$ refines $\mathscr{U}$, then the maps $\check{C}^\bullet(\mathscr{V},\mathscr{F}) \to \check{C}^\bullet(\mathscr{U},\mathscr{F})$ and $\check{C}^\bullet(\mathscr{U},\mathscr{F}) \to \check{C}^\bullet(\mathscr{V},\mathscr{F})$ are quasi-isomorphisms quasi-inverse of each other.*

IV.3.3 The strongly Noetherian case

In this section, we explain what happens in the strongly Noetherian case. For now, we assume that A is a Tate ring.

Definition IV.3.3.1. Let M be a finitely generated A-module, endowed with its canonical topology (see proposition II.4.2.2). We denote by $M\langle X\rangle$ the $A\langle X\rangle$-submodule of $M[[X]]$ of elements $f = \sum_{\nu\geq 0} m_\nu X^\nu$ such that, for every neighborhood U of 0 in M, we have $m_\nu \in U$ for all but finitely many ν.

Proposition IV.3.3.2. *([26] remark 8.28 and lemma 8.30) Suppose that A is complete and Noetherian.*

(i) For every finitely generated A-module M, if we put the canonical topology on M, then the morphism

$$M \otimes_A A\langle X\rangle \to M\langle X\rangle, \quad m\otimes a \longmapsto ma$$

is an isomorphism of $A\langle X\rangle$-modules.

(ii) The ring $A\langle X\rangle$ is faithfully flat over A.

(iii) For every $f \in A$, the ring $A\langle X\rangle/(f-X)$ and $A\langle X\rangle/(1-fX)$ are flat over A.

Corollary IV.3.3.3. *(Proposition 8.29 of [26].) Suppose that A is strongly Noetherian, and let $U \subset V$ be two rational domains of $X = \mathrm{Spa}(A, A^+)$. Then the restriction map $\mathscr{O}_X(V) \to \mathscr{O}_X(U)$ is flat.*

Corollary IV.3.3.4. *(Corollary 8.31 of [26].) Suppose that A is strongly Noetherian, and let $(U_i)_{1 \leq i \leq n}$ be a finite rational covering of X. Then the morphism*

$$\mathscr{O}_X(X) \to \prod_{i=1}^{n} \mathscr{O}_X(U_i)$$

is faithfully flat.

Corollary IV.3.3.5. *(Lemma 8.32 of [26].) Suppose that A is strongly Noetherian. Then, for every simple Laurent covering (U, V) of X, the sequence*

$$0 \to \mathscr{O}_X(X) \to \mathscr{O}_X(U) \oplus \mathscr{O}_X(V) \to \mathscr{O}_X(U \cap V) \to 0$$

of proposition IV.2.2.1 is exact.

IV.3.4 Cases (c) and (d)

We now finish the proof of theorem IV.1.1.5 in cases (c) and (d). We assume that A is a Tate ring and fix a topologically nilpotent unit ϖ of A. By corollary III.6.4.3, we can (and will) assume that A is complete.

Consider property $\mathscr{P}$ of corollary IV.3.2.1 for the presheaf $\mathscr{O}_X$. If A is stably uniform (resp. strongly Noetherian), then, by corollary IV.3.1.3 (resp. IV.3.3.5), $\mathscr{P}$ holds for every simple Laurent covering of every rational domain of X. By corollary IV.3.2.1, this implies that $\mathscr{P}$ holds for every rational covering of every rational domain of X. Let $\mathscr{B}$ be the set of rational domains of X; this is a base of the topology of X, and we have just shown that $\mathscr{O}_X$ is a sheaf of abelian groups, hence of abstract rings, on $\mathscr{B}$ (i.e. it satisfies the sheaf condition for every covering of a rational domain by rational domains).

We want to prove that $\mathscr{O}_X$ is a sheaf of topological rings. We use the criterion of EGA I chapitre 0 (3.2.2); if we combine it with the observations of EGA I chapitre 0 (3.1.4), it says that it suffices to check that, for every $U \in \mathscr{B}$ and every rational covering $(U_i)_{i \in I}$ of U, the sequence

$$(*) \qquad 0 \to \mathscr{O}_X(U) \to \prod_{i \in I} \mathscr{O}_X(U_i) \to \prod_{i,j \in I} \mathscr{O}_X(U_i \cap U_j)$$

is exact (as a sequence of abelian groups) and the morphism

$$\varphi : \mathscr{O}_X(U) \to \prod_{i \in I} \mathscr{O}_X(U_i)$$

is strict. We already know that (*) is exact.

We show the strictness of φ. For A stably uniform, this is corollary IV.3.1.2, but this does not work for A stronly Noetherian. Here is an argument that works in both cases : First, as X is quasi-compact, we may assume that I is finite. By the exactness of (*), the image of φ is the kernel of the continuous map $\prod_{i\in I}\mathscr{O}_X(U_i)\to\prod_{i,j\in I}\mathscr{O}_X(U_i\cap U_j)$, so it is closed in $\prod_{i\in I}\mathscr{O}_X(U_i)$, and in particular it is a complete $\mathscr{O}_X(U)$-module. So the fact that $\varphi:\mathscr{O}_X(U)\to\operatorname{Im}\varphi$ is open just follows from the open mapping theorem (theorem II.4.1.1).

Finally, we need to prove that $\mathrm{H}^i(U,\mathscr{O}_X)=0$ for every rational domain U of X and every $i\geq 1$. But this follows immediately from the similar property for Čech cohomology (with respect to rational coverings) and from [25, Lemma 01EW].

Let us outline the proof given in that referece. We see $\mathscr{O}_X$ as a sheaf of abstract rings in this paragraph. Consider the category $\mathscr{A}$ of sheaves of (abstract) abelian groups $\mathscr{F}$ such that, for every rational domain U of X and every rational covering $\mathscr{U}$ of U, the canonical map $\mathscr{F}(U)\to\check{\mathrm{C}}^\bullet(\mathscr{U},\mathscr{F})$ is a quasi-isomorphism (i.e. the augmented Čech complex associated to $\mathscr{U}$ is acyclic). By the beginning of the proof, $\mathscr{O}_X$ is an object of $\mathscr{A}$. We claim that, for every object $\mathscr{F}$ of $\mathscr{A}$, every rational domain U of X and every $i\geq 1$, we have $\mathrm{H}^i(U,\mathscr{F})=0$. We prove this claim by induction on i. So suppose that $i\geq 1$ and that we know the claim for every $j<i$. Let $\mathscr{F}$ be an object of $\mathscr{A}$. Choose an injective morphism of $\mathscr{F}$ into an injective abelian sheaf $\mathscr{I}$. Let $\mathscr{U}$ be a rational covering of a rational domain U of X. Note that $\mathscr{I}_{|U}$ is still injective as an abelian sheaf on U. As the forgetful functor from sheaves to presheaves admits a lft adjoint (the sheafification functor), it preserves injective objects, so $\mathscr{I}_{|U}$ is still injective when seen as a presheaf of abelian groups on U. As the $\check{\mathrm{H}}^i(\mathscr{U},.)$ are the right derived functors of $\check{\mathrm{H}}^0(\mathscr{U},.)$ on the category of presheaves of abelian groups on U (see [25, Lemma 01EN]), we have $\check{\mathrm{H}}^i(\mathscr{U},\mathscr{I})=0$ for every $i\geq 1$. In particular, $\mathscr{I}$ is an object of $\mathscr{A}$.

Let $\mathscr{G}$ be the cokernel of the map $\mathscr{F}\to\mathscr{I}$. It is easy to see (cf. [25, Lemma 01EU]) that the vanishing of $\check{\mathrm{H}}^1(\mathscr{U},\mathscr{I})$ for every rational covering of a rational domain of X implies that, for every rational domain U of X, the $\mathscr{I}(U)\to\mathscr{G}(U)$ is surjective. Also, using the long exact sequence of Čech cohomology coming from the exact sequence of presheaves $0\to\mathscr{F}\to\mathscr{I}\to\mathscr{G}\to 0$, and using the fact that $\mathscr{F}$ and $\mathscr{I}$ are in $\mathscr{A}$, we see that $\mathscr{G}$ is also an object of $\mathscr{A}$.

Fix a rational domain U of X and consider the long exact sequence of cohomology groups :

$$0\to\mathscr{F}(U)\to\mathscr{I}(U)\to\mathscr{G}(U)\to\mathrm{H}^1(U,\mathscr{F})\to\mathrm{H}^1(U,\mathscr{I})\to\mathrm{H}^1(U,\mathscr{G})\to\dots$$

$$\dots\to\mathrm{H}^{i-1}(U,\mathscr{G})\to\mathrm{H}^i(U,\mathscr{F})\to\mathrm{H}^i(U,\mathscr{I})\to\mathrm{H}^i(U,\mathscr{G})\to\dots$$

As $\mathscr{I}$ is injective, we have $\mathrm{H}^p(U,\mathscr{I})=0$ for every $p\geq 1$. So, if $i\geq 2$, then the exact sequence above and the induction hypothesis applied to $\mathscr{G}$ imply that $\mathrm{H}^i(U,\mathscr{F})=0$. Suppose that $i=1$. As $\mathscr{I}(U)\to\mathscr{G}(U)$, the map $\mathrm{H}^1(U,\mathscr{F})\to\mathrm{H}^1(U,\mathscr{I})=0$ is injective, so we also get $\mathrm{H}^1(U,\mathscr{F})=0$.

IV.3.5 Case (a)

We explain how to prove theorem IV.1.1.5 in case (a), i.e., when the topology on $\widehat{A}$ is discrete. Again, by corollary III.6.4.3, we can (and will) assume that A is complete, so that A is a discrete ring. Then A^+ can be any integrally closed subring of A, and

$$\mathrm{Spa}(A, A^+) = \{x \in \mathrm{Spv}(A) \mid \forall a \in A^+,\ |a|_x \leq 1\}.$$

Remember that we have a continuous and spectral map $\mathrm{supp} : \mathrm{Spv}(A) \to \mathrm{Spec}(A)$, $x \longmapsto \mathrm{Ker}(|.|_x)$. We also denote by supp its restriction to the subset $\mathrm{Spa}(A, A^+)$. Note that the map $\mathrm{supp} : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A)$ is surjective because, for every $\wp \in \mathrm{Spec}(A)$, the trivial valuation with support $\wp$ is in $\mathrm{Spa}(A, A^+)$. For the same reason, for every finite subset T of A and every $s \in A$, the image of $R\left(\frac{T}{s}\right) \subset \mathrm{Spa}(A, A^+)$ by supp is the principal open subset $D(s)$ of $\mathrm{Spec}(A)$.

Let $T \subset A$ be a finite subset and $s \in A$. Then $A\left(\frac{T}{s}\right)$ is the ring $A[s^{-1}]$ with the discrete topology, so $\mathscr{O}_X(R\left(\frac{T}{s}\right)) = A[s^{-1}]$ with the discrete topology. Also, as $\mathrm{supp}(R\left(\frac{T}{s}\right)) = D(s)$ is open, we have

$$(\mathrm{supp}^* \mathscr{O}_{\mathrm{Spec}(A)})(R\left(\tfrac{T}{s}\right)) = \mathscr{O}_{\mathrm{Spec}(A)}(D(s)) = \mathscr{O}_X(R\left(\tfrac{T}{s}\right)).$$

This shows that the presheaves $\mathscr{O}_X$ and $\mathrm{supp}^* \mathscr{O}_{\mathrm{Spec}(A)}$ coincide on the family of rational domain of X, and in particular that $\mathscr{O}_X$ is a sheaf on this family and that its augmented Čech complex for any rational covering of a rational domain is exact.

We now get the result by applying the criterion of EGA I chapitre 0 (3.2.2) (to show that $\mathscr{O}_X$ is a sheaf) and [25, Lemma 01EU] (to show that its higher cohomology vanishes on rational domains) as in section IV.3.4.

V Perfectoid algebras

In this chapter, we will study the main example of stably uniform Tate rings, i.e. perfectoid Tate rings. The main references are Scholze's papers [22] and [23], as well as Fontaine's Bourbaki seminar [10] and Bhatt's notes [1].

In all this chapter, we fix a prime number ℓ.

V.1 Perfectoid Tate rings

V.1.1 Definition and basic properties

Definition V.1.1.1. (Section 1.1 of [10], definition 3.1 of [23]) Let A be a Tate ring. We say that A is *perfectoid* if A is complete and uniform, and if there exists a pseudo-uniformizer ϖ of A such that

(a) ϖ^ℓ divides ℓ in A^0;

(b) the Frobenius map $\mathrm{Frob} : A^0/\varpi \to A^0/\varpi^\ell$, $a \longmapsto a^\ell$ is bijective.

If A is a perfectoid Tate ring and a field, we say that A is a *perfectoid field*.

Note that condition (a) implies that A^0/ϖ is a ring of characteristic ℓ, so the Frobenius map in (b) is a morphism of rings.

Remark V.1.1.2. It follows from proposition V.1.1.3 that, if A is a perfectoid Tate ring, then every pseudo-uniformizer ϖ of A such that ϖ^ℓ divides ℓ in A^0 satisfies condition (b) of definition V.1.1.1.

Note also that, if A is perfectoid of characteristic 0, then ℓ is topologically nilpotent (because it is a multiple in A^0 of some uniformizer), so, if ℓ is invertible in A, then the topology on A^0 is the ℓ-adic topology and $A = A^0[\frac{1}{\ell}]$ (by proposition II.2.5.2).

Finally, remember that perfectoid Tate ring, like all complete uniform Tate rings, are reduced. (See remark IV.1.1.3.)

Proposition V.1.1.3. *(Lemma 3.9 of [2].) Let A be a Tate ring, and let ϖ be a pseudo-uniformizer of A such that ϖ^ℓ divides ℓ in A^0.*

(i) The map $\mathrm{Frob} : A^0/\varpi \to A^0/\varpi^\ell$, $a \longmapsto a^\ell$ *is injective.*

(ii) If A is complete and uniform, then the following conditions are equivalent :

(a) every element of $A^0/\ell\varpi A^0$ is a ℓth power;

(b) every element of $A^0/\ell A^0$ is a ℓth power;

(c) every element of $A^0/\varpi^\ell A^0$ is a ℓth power.

Moreover, if these conditions holds, then there exist units u and v in A^0 such that $u\varpi$ and $v\ell$ admit compatible systems of ℓ-power roots in A^0.

Proof. (i) Let $a \in A^0$ such that $a^\ell \in \varpi^\ell A^0$. Then $(a\varpi^{-1})^\ell \in A^0$, so $a\varpi^{-1} \in A^0$, and $a \in \varpi A^0$.

(ii) As ϖ^ℓ divides ℓ and ℓ divides $\ell\varpi$ in A^0, it is clear that (a) implies (b) and (b) implies (c). We want to show that (c) implies (a).

Suppose that (c) holds, and let $a \in A^0$. By lemma V.1.1.5, there exists a sequence $(a_n)_{n\geq 0}$ of elements of A^0 such that $a = \sum_{n\geq 0} a_n^\ell \varpi^{\ell n}$. By lemma V.1.1.6, this implies that $a - (\sum_{n\geq 0} a_n\varpi^n)^\ell \in \varpi\ell A^0$, which gives (a).

We prove the last statement. By lemma V.1.1.7 (applied to A^0 and to $A^0/\pi\ell A^0$), the canonical map $\varprojlim_{a\longmapsto a^\ell} A^0 \to \varprojlim_{a\longmapsto a^\ell} A^0/\varpi\ell A^0$ is an isomorphism. In particular, we can find $\omega = (\omega^{(n)}) \in \varprojlim_{a\longmapsto a^\ell}$ such that $\omega^{(0)} = \varpi$ mod. $\varpi\ell A^0$ (resp. $\omega^{(0)} = \ell$ mod. $\varpi\ell A^0$). In other words, there exists $a \in A^0$ such that $\omega^{(0)} = \varpi(1+\ell a)$ (resp. $\omega^{(0)} = \ell(1+\varpi a)$). The claim now follows from the fact that, for every $a \in A^0$, $1+\varpi a$ and $1+\ell a$ are units in A^0 (because ϖa and ℓa are topologically nilpotent).

□

The following corollary is an immediate consequence of the proposition, but it is convenient when we want to prove that a Tate ring is perfectoid.

Corollary V.1.1.4. *Let A be a complete uniform Tate ring. Then the following conditions are equivalent :*

(i) A is a perfectoid;

(ii) every element of $A^0/\ell A^0$ is a ℓth power, and A has a pseudo-uniformizer ϖ such that ϖ^ℓ divides ℓ in A^0.

Lemma V.1.1.5. *We use the notation of proposition V.1.1.3. Suppose that every element of $A^0/\varpi^\ell A^0$ is a ℓth power, and let $a \in A^0$.*

Then there exists a sequence $(a_n)_{n\geq 0}$ of elements of A^0 such that, for every $n \in \mathbb{N}$, $a - \sum_{i=0}^n a_i^\ell \varpi^{\ell i} \in \varpi^{\ell(n+1)} A^0$.

Proof. We construct the elements a_n by induction on n. The assumption immediately implies that there exists $a_0 \in A^0$ such that $a - a_0^\ell \in \varpi^\ell A^0$. Suppose that $n \geq 1$ and that we have found $a_0, \ldots, a_{n-1}$ such that $a - \sum_{i=0}^{n-1} a_i^\ell \varpi^{\ell i} \in \varpi^{\ell n} A^0$. Let $b \in A^0$ such that $a - \sum_{i=0}^{n-1} a_i^\ell \varpi^{\ell i} = \varpi^{\ell n} b$, and choose $a_n \in A^0$ such that $b - a_n^\ell \in \varpi^\ell A^0$. Then $a - \sum_{i=0}^{n} a_i^\ell \varpi^{\ell i} \in \varpi^{\ell(n+1)} A^0$.

□

The following lemma is an easy consequence of the binomial formula.

Lemma V.1.1.6. *We use the notation of proposition V.1.1.3. For all $a, b \in A^0$, we have $(a + \varpi b)^\ell - a^\ell - (\varpi b)^\ell \in \varpi \ell A^0$.*

Lemma V.1.1.7. *(Lemma 3.4(i) of [22].) Let S be a ring and $\varpi \in S$. Suppose that S is ϖ-adically complete (and Hausdorff) and that ϖ divides ℓ in S. Then the canonical map*

$$\varprojlim_{a \longmapsto a^\ell} S \to \varprojlim_{a \longmapsto a^\ell} S/\varpi S$$

is an isomorphism of topological monoids (where the monoid operations are given by the multiplications of the rings).

Proof. Let $S_1 = \varprojlim_{a \longmapsto a^\ell} S$ and $S_2 = \varprojlim_{a \longmapsto a^\ell} S/\varpi S$. We prove the statement by constructing a continuous multiplicative inverse of the canonical map $S_1 \to S_2$.

First, we construct a continuous multiplicative map $\alpha : S_2 \to S$ such that $\alpha((\overline{s}_n)_{n \geq 0}) = \overline{s_0}$ mod. πS. Let $(\overline{s}_n)_{n \geq 0} \in S_2$. Choose representatives $s_n \in S$ of the $\overline{s}_n$. We claim that :

(i) $\lim_{n \to +\infty} s_n^{\ell^n}$ exists;

(ii) the limit in (i) is independent of the choice of the representatives s_n.

To prove (i), note that, for every $n \in \mathbb{N}$, we have $s_{n+1}^\ell - s_n \in \varpi S$. Applying lemma V.1.1.6 repeatedly and using the fact that ϖ divides ℓ, we deduce that $s_{n+1}^{\ell^{n+1}} - s_n^{\ell^n} \in \varpi^{n+1} S$. This implies that $(s_n^{\ell^n})_{n \geq 0}$ is a Cauchy sequence in S, so it has a limit. To prove (ii), choose some other lifts s'_n of the $\overline{s}_n$. Then, for every $n \in \mathbb{N}$, we have $s'_n - s_n \in \varpi S$, so as before we get $(s'_n)^{\ell^n} - s_n^{\ell^n} \in \varpi^{n+1} S$. This implies that $\lim_{n \to +\infty} s_n^{\ell^n} = \lim_{n \to +\infty} (s'_n)^{\ell^n}$.

By claims (i) and (ii), we can define a map $\alpha : S_2 \to S$ by sending $(\overline{s}_n)_{n \geq 0}$ to $\lim_{n \to +\infty} s_n^{\ell^n}$, where the $s_n \in S$ are any lifts of the $\overline{s}_n$. It is clear that α is multiplicative, and it is also easy to see that it is continuous.

Note that, if $\overline{s} = (\overline{s}_n)_{n \geq 0} \in S_2$, then it has a canonical ℓth root, which is its shift $\overline{s}' = (\overline{s}_{n+1})_{n \geq 0}$, and we have $\alpha(\overline{s}')^\ell = \alpha(\overline{s})$ by definition of α.

We now get the desired map $S_2 \to S_1$ by sending $(\overline{s}_n)_{n \geq 0} \in S_2$ to the sequence

$$(\alpha((\overline{s}_{r+n})_{n \geq 0}))_{r \geq 0},$$

which is clearly an element of S_1.

□

We now look at two particular cases : perfectoid fields and perfectoid rings of characteristic ℓ.

Proposition V.1.1.8. *(Proposition 3.5 of [23].) Let A be a Tate ring of characteristic ℓ. Then the following are equivalent :*

(i) A is perfectoid.

(ii) A is complete and perfect.

Proof. Suppose that A is complete and perfect. Then it is also uniform by theorem IV.1.3.5. So, by corollary V.1.1.4, it suffices to check that every element of A^0 is a ℓth power, which follows immediately from the fact that A is perfect.

Conversely, suppose that A is perfectoid. Then it is complete by assumption, and we want to show that it is perfect. As A is a localization of A^0, it suffices to show that A^0 is perfect. As $\ell = 0$ in A, we have $A^0/\ell A^0 = A^0$, so the conclusion follows from proposition V.1.1.3(ii).

□

For fields, we have the following result of Kedlaya.

Theorem V.1.1.9. *(Theorem 4.2 of [18].) Let A be a perfectoid Tate ring that is also a field. Then the topology of A is given by a rank 1 valuation; in other words, A is a complete non-Archimedean field.*

This implies that the definition of perfectoid field given here is equivalent to their original definition (definition 3.1 of [22]).

Proposition V.1.1.10. *Let K be a complete topological field. Then the following conditions are equivalent :*

(i) K is a perfectoid field;

(ii) the topology of K is given by a rank 1 valuation $|.|$ satisfying the following conditions :

(a) $|.|$ is not discrete (i.e., its valuation group is not isomorphic to $\mathbb{Z}$);

(b) $|\ell| < 1$;

and the ℓth power map on $K^0/\ell K^0$ is surjective.

Proof. Suppose that K is perfectoid. Then we know that its topology is given by a rank 1 valuation $|.|$ by theorem V.1.1.9. By corollary V.1.1.4, the ℓth power map on $K^0/\ell K^0$ is surjective, and K has a pseudo-uniformizer ϖ such that ϖ^ℓ divides ℓ in K^0. In particular, ℓ is topologically nilpotent in K, so $|\ell| < 1$.

Now suppose that K satisfies the conditions of (ii). Then $K^0 = \{a \in K \mid |a| \leq 1\}$, so K^0 is bounded in K, i.e. K is uniform. We check that K satisfies the conditions of corollary

V.1.1.4(ii). The first condition is part of the assumption on K. For the second condition, note that, as $|.|$ is not discrete, there exists $\varpi \in K - \{0\}$ such that $|\varpi|^\ell \leq |\ell|$. In particular, $|\varpi| < 1$, so ϖ is topologically unipotent, hence a pseudo-uniformizer of K; moreover, as $|\ell\varpi^{-\ell}| \leq 1$, we have $\ell\varpi^{-\ell} \in K^0$, which means that ϖ^ℓ divides ℓ in K^0.

□

Example V.1.1.11. (See 3.3, 3.4 of [23].)

(1) The field $\mathbb{Q}_\ell$ is not perfectoid because its topology is given by a discrete valuation.

(2) The field $\mathbb{C}_\ell$ is perfectoid; more generally, any algebraically closed complete non-archimedean field is perfectoid.

(3) Let $\mathbb{Q}_\ell^{\mathrm{cycl}}$ be the completion of $\mathbb{Q}_\ell[\mu^{1/\ell^\infty}] := \bigcup_{n\geq 0} \mathbb{Q}_\ell[\mu^{1/\ell^n}]$ for the unique valuation $|.|$ extending the ℓ-adic valuation on $\mathbb{Q}_\ell$, and let $\mathbb{Z}_\ell^{\mathrm{cycl}} = (\mathbb{Q}_\ell^{\mathrm{cycl}})^0$. Then $\mathbb{Q}_\ell^{\mathrm{cycl}}$ is perfectoid. Indeed, it is complete non-Archimedean. For every $r \geq 1$, the cyclotomic extension $\mathbb{Q}_\ell[\mu_{1/\ell^r}]/\mathbb{Q}_\ell$ is of degree $\ell^{r-1}(\ell-1)$, and, if ω is a primitive ℓ^rth root of 1, then $N_{\mathbb{Q}_\ell[\omega]/\mathbb{Q}_\ell}(1-\omega) = \ell$, hence $|\omega|^{\ell^{r-1}\ell} = |\ell|$. This shows that $|.|$ is not a discrete valuation. We obviously have $|\ell| < 1$. Finally, let $\overline{a} \in \mathbb{Z}_\ell^{\mathrm{cycl}}/\ell\mathbb{Z}_\ell^{\mathrm{cycl}}$. We can find a lift a of $\overline{a}$ in the ring of integers of $\mathbb{Q}_\ell[\mu_{1/\ell^n}]$ for some $n \geq 0$. Pick a primitive ℓ^{n+1}th root of unity ω. Then we can write $a = \sum_{i=0}^{\ell^n-1} a_i\omega^{\ell i}$, with the a_i in $\mathbb{Z}_\ell$. Then, if $b = \sum_{i=0}^{\ell^n-1} a_i\omega^i$, then $b^\ell = a$ modulo $\ell\mathbb{Z}_\ell^{\mathrm{cycl}}$.

(4) Similarly, the completion L of $\mathbb{Q}_\ell[\ell^{1/\ell^\infty}] := \bigcup_{n\geq 0} \mathbb{Q}_\ell[\ell^{1/\ell^n}]$ for the unique valuation $|.|$ extending the ℓ-adic valuation on $\mathbb{Q}_\ell$ is a perfectoid field. Indeed, K is complete and non-Archimedean, and the valuation group of $|.|$ is not isomorphic to $\mathbb{Z}$ because its element $|\ell| < 1$ is divisible by ℓ^r for every $r \in \mathbb{N}$. Also, we have $K^0/\ell K^0 = \mathbb{F}_\ell$, so every element of $K^0/\ell K^0$ is a ℓth power.

(5) As in (3), we can show that the completion $\mathbb{F}_\ell((t^{1/\ell^\infty}))$ of $\bigcup_{n\geq 0} \mathbb{F}_\ell((t^{1/\ell^n}))$ for the unique valuation extending the t-adic valuation on $\mathbb{F}_\ell((t))$ is a perfectoid field. This construction still makes sense if we replace $\mathbb{F}_\ell$ by any ring A, and it gives a perfectoid field if A is a perfect field of characteristic ℓ.

(6) Let K be a perfectoid field of characteristic 0. If A^0 is the ℓ-adic completion of $\bigcup_{n\geq 0} K^0[T^{1/\ell^n}]$, then $K\langle T^{1/\ell^\infty}\rangle := A^0[\frac{1}{\ell}]$ is a perfectoid Tate ring. (See proposition V.1.2.8.) Note that, if $n \geq 0$, then the Gauss norms on $K\langle T^{1/\ell^{n+1}}\rangle$ and on its subring $K\langle T^{1/\ell^n}\rangle$ coincide, so we get a norm on $\bigcup_{n\geq 0} K\langle T^{1/\ell^n}\rangle$. Then $K\langle T^{1/\ell^\infty}\rangle$ is the completion of $\bigcup_{n\geq 0} K\langle T^{1/\ell^n}\rangle$ for this norm.

Note that the construction of $A\langle T^{1/\ell^\infty}\rangle$ makes sense for any f-adic ring A.

(6) Finally, we give an example of a perfectoid Tate alegbra that does not contain a field. First, consider the f-adic ring $A_0 = \mathbb{Z}_\ell^{\mathrm{cycl}}[[T^{1/\ell^\infty}]]$ (see (3) and (5)); this is an adic ring, with the T-adic topology. Let $B = A\langle(\ell/T)^{1/\ell^\infty}\rangle := A\langle X^{1/\ell^\infty}\rangle/(\ell - XT)$. This is still an adic ring with the T-adic topology. Finally, the f-adic ring $B[\frac{1}{T}]$ is a Tate ring, and it is perfectoid.

A pseudo-uniformizer satisfying the conditions of definition V.1.1.1 is $\varpi = T^{1/\ell}$ (note that $\varpi^\ell = T$ divides ℓ in B). The ring $B[\frac{1}{T}]$ doesn't contain a field, because ℓ is nonzero and not invertible in it.

V.1.2 Tilting

Proposition V.1.2.1. *(Section 1.3 of [10] and lemma 3.10 of [23].) Let A be a perfectoid Tate ring. We consider the set*

$$A^\flat = \varprojlim_{a \longmapsto a^\ell} A := \{(a^{(n)}) \in A^{\mathbb{N}} \mid \forall n \in \mathbb{N},\ (a^{(n+1)})^\ell = a^{(n)}\},$$

with the projective limit topology, the pointwise multiplication and the addition defined by $(a^{(n)}) + (b^{(n)}) = (c^{(n)})$, with

$$c^{(n)} = \lim_{r \to +\infty} (a^{(n+r)} + b^{(n+r)})^{\ell^r}.$$

We denote the map $A^\flat \to A$, $(a^{(n)}) \longmapsto a^{(0)}$ by $f \longmapsto f^\sharp$.

Then :

(i) The addition is well-defined (i.e. the limit above always exists) and makes $A^\flat$ into a perfectoid Tate ring of characteristic ℓ.

(ii) The subring $A^{\flat 0}$ of power-bounded elements in $A^\flat$ is given by

$$A^{\flat 0} = \varprojlim_{a \longmapsto a^\ell} A^0.$$

If ϖ is a pseudo-uniformizer of A that divides ℓ in A^0, then the canonical map

$$A^{\flat 0} \to \varprojlim_{a \longmapsto a^\ell} A^0/\varpi$$

is an isomorphism of topological rings.

(iii) There exists a pseudo-uniformizer ϖ of A such that ϖ^ℓ divides ℓ in A^0 and that ϖ is in the image of the map $(.)^\sharp : A^\flat \to A$. Moreover, if $\varpi^\flat$ is an element of $A^\flat$ such that $\varpi = (\varpi^\flat)^\sharp$, then $\varpi^\flat$ is a pseudo-uniformizer of $A^\flat$, the map $f \longmapsto f^\sharp$ induces a ring isomorphism $A^{\flat 0}/\varpi^\flat \simeq A^0/\varpi$, and $A^\flat = A^{\flat 0}[\frac{1}{\varpi^\flat}]$.

Note that, if A is a perfectoid ring of characteristic ℓ, then $\mathrm{Frob}_A : A \to A$ is an isomorphism of topological rings (by the open mapping theorem, i.e. theorem II.4.1.1), so $A^\flat$ is canonically isomorphic to A (via the map $(.)^\sharp$).

Proof. (1) Note that the image of the map $(.)^\sharp : A^\flat \to A$ is the set of elements of A that have compatible systems of ℓth power roots. So, by proposition V.1.1.3, there exists a pseudo-uniformizer ϖ of A such that ϖ^ℓ divides ℓ in A^0 and that ϖ is in the image of $(.)^\sharp$. Choose $\varpi^\flat \in A^\flat$ such that $(\varpi^\flat)^\sharp$.

(2) We show that the addition of $A^\flat$ is well-defined on elements of $A^{\flat 0}$. Let $(a^{(n)})$, $(b^{(n)})$ be elements of $A^{\flat 0}$. Fix $n \in \mathbb{N}$. For every $r \in \mathbb{N}$, we have

$$(a^{(n+r+1)} + b^{(n+r+1)})^\ell = (a^{(n+r+1)})^\ell + (b^{(n+r+1)})^\ell = a^{(n+r)} + b^{(n+r)} \mod \ell A^0 \subset \varpi A^0,$$

so repeated applications of lemma V.1.1.6 give

$$(a^{(n+r+1)} + b^{(n+r+1)})^{\ell^{r+1}} == (a^{(n+r)} + b^{(n+r)})^{\ell^r} \mod \varpi^{r+1} A^0.$$

This implies that $((a^{(n+r)} + b^{(n+r)})^{\ell^r})_{r\geq 0}$ is a Cauchy sequence, so it admits a limit $c^{(n)}$ in A^0. The fact that $(c^{(n+1)})^\ell = c^{(n)}$ for every n (i.e. that $(c^{(n)})$ is an element of $A^\flat$) follows immediately from the definition of $c^{(n)}$.

(3) By lemma V.1.1.7, the canonical map

$$A^{\flat 0} = \varprojlim_{a \longmapsto a^\ell} A^0 \to \varprojlim_{a \longmapsto a^\ell} A^0/\varpi A^0$$

is an isomorphism of topological monoids. We show that it is also compatible with addition. Let $(a^{(n)})$, $(b^{(n)})$ be elements of $A^{\flat 0}$, and let $(c^{(n)}) = (a^{(n)}) + (b^{(n)})$. Fix $n \in \mathbb{N}$. Then

$$c^{(n)} = \lim_{r \to +\infty} (a^{(n+r)} + b^{(n+r)})^{\ell^r}.$$

But we have seen in the proof of (2) that the sequence $((a^{(n+r)} + b^{(n+r)})^{\ell^r})_{r\geq 0}$ is constant modulo ϖA^0, so $c^{(n)} = a^{(n)} + b^{(n)}$ modulo ϖA^0. This shows the claim. In particular, we get that $A^{\flat 0}$ is a non-Archimedean topological ring of characteristic ℓ.

(4) Write $\varpi^\flat = (\varpi^{1/\ell^n})_{n\geq 0}$. (In other words, we choose a compatible system $(\varpi^{1/\ell^n})_{n\geq 0}$ of ℓth power roots of ϖ.) For every $a = (a^{(n)}) \in A^\flat$ and every $r \in \mathbb{N}$, we have $(\varpi^\flat)^r a = (\varpi^{r/\ell^n} a^{(n)})_{n\geq 0}$ by definition of the multiplication of $A^\flat$. In particular, taking $r = \ell$, we see that the isomorphism os topological rings $A^{\flat 0} \overset{\sim}{\to} \varprojlim_{a \longmapsto a^\ell} A^0/\varpi A^0$ induces an isomorphism of topological rings

$$A^{\flat 0}/(\varpi^\flat)^\ell A^{\flat 0} \overset{\sim}{\to} \varprojlim_{a \longmapsto a^\ell} A^0/\varpi^{\ell^{m-n}} A^0 \simeq A^0/\varpi^\ell A^0,$$

where the last isomorphism follows from condition (b) in definition V.1.1.1, applied to the pseudo-uniformizers ϖ^{1/ℓ^n}, $n \geq 0$. As the square

$$\begin{tikzcd} A^{\flat 0}/\varpi^\flat A^{\flat 0} \arrow[r, "(.)^\sharp", "\sim"'] \arrow[d, "\mathrm{Frob}"'] & A^0/\varpi A^0 \arrow[d, "\mathrm{Frob}"] \\ A^{\flat 0}/(\varpi^\flat)^\ell A^{\flat 0} \arrow[r, "(.)^\sharp", "\sim"'] & A^0/\varpi^\ell A^0 \end{tikzcd}$$

is clearly commutative, this shows that $\mathrm{Frob} : A^{\flat 0}/\varpi^\flat A^{\flat 0} \to A^{\flat 0}/(\varpi^\flat)^\ell A^{\flat 0}$ is bijective.

(5) We show that $A^{\flat 0}$ is $\varpi^\flat$-adically Hausdorff and complete. It suffices to show that the $\varpi^\flat$-adic topology on $A^{\flat 0}$ is finer than the product topology coming from the isomorphism $A^{\flat 0} \xrightarrow{\sim} \varprojlim_{a \longmapsto a^\ell} A^0/\varpi A^0$. But, by the formula of (4) for $(\varpi^\flat)^r(a^{(n)})$, if $m \in \mathbb{N}$, then the projections of $(\varpi\flat)^{\ell^m} A^{\flat 0}$ on the last m factors of $\prod_{n\geq 0} A^0/\varpi A^0$ are 0, which implies the desired result.

(6) We show that addition is well-defined on $A^\flat$. Let $(a^{(n)}) \in A^\flat$. We choose $N \in \mathbb{N}$ such that $\varpi^N a^{(0)} \in A^0$. Then, for every $n \geq 0$, $(\varpi^{N/\ell^n} a^{(n)})^{\ell^n} = \varpi^N a^{(0)} \in A^0$, so $\varpi^{N/\ell^n} a^{(n)} \in A^0$. This shows that $(\varpi^\flat)^N(a^{(n)}) \in A^{\flat 0}$.

Let $(b^{(n)})$ be another element of $A^\flat$. Up to increasing N, we may assume that $\varpi^{N/\ell^n} b^{(n)} \in A^0$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. By (2), we know that the sequence $(\varpi^{N/\ell^n}(a^{(n+r)} + b^{(n+r)})^{\ell^r})_{r\geq 0}$ converges in A^0. This implies immediately that the sequence $((a^{(n+r)} + b^{(n+r)})^{\ell^r})_{r\geq 0}$ converges in A.

(7) Using the fact that $A^{\flat 0}$ is a topological ring and the calculation of (6), it is easy to check that $A^\flat$ is a topological ring and that $A^\flat = A^{\flat 0}[\frac{1}{\varpi^\flat}]$. By remark II.2.5.3, $A^\flat$ is a Tate ring having $A^{\flat 0}$ as a ring of definition.

(8) We show that $(A^\flat)^0 = A^{\flat 0}$. (This implies that $A^\flat$ is perfectoid and finishes the proof of the proposition.) As $A^{\flat 0}$ is a bounded subring of $A^\flat$, we clearly have $A^{\flat 0} \subset (A^\flat)^0$. Conversely, let $a = (a^{(n)}) \in (A^\flat)^0$. As a is power-bounded, there exists $N \in \mathbb{N}$ such that $(\varpi^\flat)^N a^n \in A^{\flat 0}$ for every $n \geq 0$. This implies that $\varpi^N (a^{(0)})^n \in A^0$ for every $n \geq 0$, i.e. that $a^{(0)}$ is power-bounded. As $(a^{(n)})^{\ell^n} = a^{(n)}$ for every $n \in \mathbb{N}$, $a^{(n)}$ is also power-bounded, and we have shown that $a \in A^{\flat 0}$.

$\square$

Definition V.1.2.2. The perfectoid Tate ring $A^\flat$ of proposition V.1.2.1 is called the *tilt* of A.

Remark V.1.2.3. Let A and A' be perfectoid Tate algebras, with A' of characteristic ℓ, and let ϖ and ϖ' be pseudo-uniformizers of A and A' such that ϖ^ℓ divides ℓ in A^0 and that $\varpi = (\varpi^\flat)^\sharp$ for some pseudo-uniformizer $\varpi^\flat$ of $A^\flat$.

Then any isomorphism $\varphi : A^0/\varpi^\ell A^0 \xrightarrow{\sim} A'^0/\varpi'^\ell A'^0$ sending ϖ to ϖ' induces an isomorphism $A^\flat \xrightarrow{\sim} A'$.

Indeed, by the second formula in proposition V.1.2.1(ii), we have $A^{\flat 0} = \varprojlim_{a \longmapsto a^\ell} A^0/\varpi^\ell A^0$ and $A'^{\flat 0} = \varprojlim_{a \longmapsto a^\ell} A'^0/\varpi'^\ell A'^0$, so φ induces an isomorphism $\psi : A^{\flat 0} \xrightarrow{\sim} A'^{\flat 0} = A'^0$, and $\psi(\varpi^\flat) = \varpi'$ modulo $\varpi'^\ell A'^0$. This implies that $\psi(\varpi^\flat)$ is also a pseudo-uniformizer of A'^0, so ψ extends to an isomorphism $A^\flat \xrightarrow{\sim} A'$.

Remark V.1.2.4. Let A be a perfectoid Tate ring. Then the map $(.)^\sharp : A^{\flat 0} \to A^0$ induces an isomorphism of rings $A^{\flat 0}/A^{\flat 00} \xrightarrow{\sim} A^0/A^{00}$.

Proof. First note that A^{00} (resp. $A^{\flat 00}$) is an ideal of A^0 (resp. $A^{\flat 0}$), so the statement makes sense. By proposition V.1.2.1, we can choose a pseudo-uniformizer $\varpi^\flat$ of $A^\flat$ such that $\varpi := (\varpi^\flat)^\sharp$ is a pseudo-unfiformizer of A, and then $(.)^\sharp$ induces an isomorphism $A^{\flat 0}/\varpi^\flat A^{\flat 0} \xrightarrow{\sim} A^0/\varpi A^0$. We have $A^{00} \supset \varpi A^0$, and, by lemma II.1.2.3, $A^{00}/\varpi A^0$ is the nilradical of $A^0/\varpi A^0$. We have a similar result for $A^\flat$, and this implies the statement.

$\square$

Proposition V.1.2.5. *Let A be a perfectoid Tate ring. If A^+ is a ring of integral elements in A (i.e. an open and integrally closed subring of A contained in A^0), then $A^{\flat +} := \varprojlim_{a \longmapsto a^\ell} A^+$ is a ring of integral elements in $A^\flat$, and, for every pseudo-uniformizer $\varpi^\flat$ of $A^\flat$ such that $\varpi := (\varpi^\flat)^\sharp$ is a pseudo-uniformizer of A, the isomorphism $A^{\flat 0}/\varpi^\flat \simeq A^0/\varpi$ sends $A^{\flat +}/\varpi^\flat$ to A^+/ϖ.*

Moreover, this induces a bijection between rings of integral elements in A and $A^\flat$.

Proof. Choose pseudo-uniformizers ϖ and $\varpi^\flat$ of A and $A^\flat$ such that $\varpi = (\varpi^\flat)^\sharp$.

If A^+ is a ring of integral elements in A, then $\varpi A^0 \subset A^+ \subset A^0$ (the first inclusion comes from the fact that $\varpi A^0 \subset A^{00}$), and $A^+/\varpi A^0$ is an integrally closed subring of $A^0/\varpi A^0$. Conversely, if S is an integrally closed subring of $A^0/\varpi A^0$, then its inverse image in A^0 is an open and integrally closed subring of A^0, i.e. a ring of integral elements in A. So rings of integral elements in A are in natural bijection with integrally closed subrings of $A^0/\varpi A^0$. We have a similar result for $A^\flat$. As $A^0/\varpi A^0 \simeq A^{\flat 0}/\varpi^\flat A^{\flat 0}$, this gives a bijection betweem rings of integral elements in A and $A^\flat$.

It remains to show that this bijection is given by the formula of the proposition. Let A^+ be a ring of integral elements in A, and let $A^{\flat +}$ be the corresponding ring of integral elements in $A^\flat$. Let $a = (a^{(n)}) \in A^{\flat 0} = \varprojlim_{a \longmapsto a^\ell} A^0$. We want to check that $a \in A^{\flat +}$ if and only if $a^{(n)} \in A^+$ for every $n \in \mathbb{N}$. First, as A^+ is integrally closed in A^0, we have $a^{(n)} \in A^+$ for every $n \in \mathbb{N}$ if and only if $a^{(0)} \in A^+$. Then, as $\varpi A^0 \subset A^+$, we have $a^{(0)} \in A^+$ if and only if $a^{(0)} + \varpi A^0 \in A^+/\varpi A^0$. But $a^{(0)} + \varpi A^0$ is the image of $a + \varpi^\flat A^{\flat 0}$ by the isomorphism $A^0/\varpi A^0 \simeq A^{\flat 0}/\varpi^\flat A^{\flat 0}$, and this isomrophism sends $A^+/\varpi A^0$ to $A^{\flat +}\varpi^\flat A^{\flat 0}$, so we get the desired equivalence.

$\square$

Definition V.1.2.6. If A is a perfectoid Tate ring and $A^+ \subset A$ is a ring of integral elements, we say that (A, A^+) is a *perfectoid Huber pair* and we call $(A^\flat, A^\flat)$ its *tilt*.

Proposition V.1.2.7. *(Proposition 3.6 of [22].) Let A be a perfectoid Tate ring. Then, for every continuous valuation $|.| : A \to \Gamma \cup \{0\}$ on A, the map $|.|^\flat : A^\flat \to \Gamma \cup \{0\}$, $a \longmapsto |a^\sharp|$ is a continuous valuation on $A^\flat$.*

Moreover, if $A = K$ is a perfectoid field, then this induces a bijection $\mathrm{Cont}(K) \xrightarrow{\sim} \mathrm{Cont}(K^\flat)$.

Proof. Let $|.| : A \to \Gamma \cup \{0\}$ be a continuous valuation on A. Then $|.|^\flat$ obviously satisfies all the properties of a valuation on $A^\flat$, except maybe for the strong triangle inequality. We check this last property. Let $a = (a^{(n)})$ and $b = (b^{(n)})$ be elements of $A^\flat$. By definition of the addition on $A^\flat$, we have

$$(a + b)^\sharp = \lim_{n \to +\infty} (a^{(n)} + b^{(n)})^{\ell n} = \lim_{n \to +\infty} ((a^{1/\ell^n})^\sharp + (b^{1/\ell^n})^\sharp)^{\ell n},$$

so

$$|a+b|^\flat = |(a+b)^\sharp| \leq \sup_{n\in\mathbb{N}}(\max(|(a^{1/\ell^n})^\sharp|^{\ell^n}, |(b^{1/\ell^n})^\sharp|^{\ell^n})) = \max(|a|^\flat, |b|^\flat).$$

We show that $|.|^\flat$ is continuous. Let $\gamma \in \Gamma$. By definition $|.|^\flat$, $\{a \in A^\flat \mid |a|^\flat < \gamma\}$ is the inverse image by the continuous map $(.)^\sharp : A^\flat \to A$ of the open subset $\{a \in A \mid |a| < \gamma\}$ of A, so it is an open subset of $A^\flat$.

Now assume that K is a perfectoid field. We want to show that $|.| \longmapsto |.|^\flat$ induces a bijection $\mathrm{Cont}(K) \overset{\sim}{\to} \mathrm{Cont}(K^\flat)$. By corollary II.2.5.11, $\mathrm{Cont}(K)$ is canonically in bijection with the set of valuation subrings K^+ of K such that $K^{00} \subset K^+ \subset K^0$, hence with the set of valuation subrings of the field K^0/K^{00}. We have a similar statement for $K^\flat$. So the result follows from remark V.1.2.4.

□

We fnish with some examples of tilting.

Proposition V.1.2.8. *(Proposition 5.20 of [22].) Let K be a perfectoid field, and let ϖ be a pseudo-uniformizer of K and K^+ be a ring of integral elements in K. As in example V.1.1.11(6), we denote by $K^+\langle X^{1/\ell^\infty}\rangle$ the ϖ-adic completion of the ring $\bigcup_{n\geq 0} K^+[X^{1/\ell^n}]$, and we set $K\langle X^{1/\ell^\infty}\rangle = K^+\langle X^{1/\ell^\infty}\rangle[\frac{1}{\varpi}]$ (note that this last ring does not depend on the choice of K^+).*

Then $K\langle X^{1/\ell^\infty}\rangle$ is perfectoid, with ring of power-bounded elements $(K\langle X^{1/\ell^\infty}\rangle)^0 = K^0\langle X^{1/\ell^\infty}\rangle$, and its tilt is canonically isomorphic to $K^\flat\langle X^{1/\ell^\infty}\rangle$.

Moreover, $K^+\langle X^{1/\ell^\infty}\rangle$ is a ring of integral elements in $K\langle X^{1/\ell^\infty}\rangle$ and the tilt of $(K\langle X^{1/\ell^\infty}\rangle, K^+\langle X^{1/\ell^\infty}\rangle)$ is $(K^\flat\langle X^{1/\ell^\infty}\rangle, K^{\flat +}\langle X^{1/\ell^\infty}\rangle)$.

Proof. Set $A = K\langle X^{1/\ell^\infty}\rangle$ and $A^+ = K^+\langle X^{1/\ell^\infty}\rangle$. We know that A is a complete Tate ring and that $A_0 := K^0\langle X^{1/\ell^\infty}\rangle$ is a ring of definition of A by remark II.2.5.3. In particular, we have $A_0 \subset A^0$. Also, if we set $A'_0 = \bigcup_{n\geq 0} K^0[X^{1/\ell^n}]$ with the ϖ-adic topology and $A' = A'_0[\frac{1}{\varpi}]$, then, by the same remark, A' is a Tate ring with ring of definition A'_0 and, by corollary II.3.1.9(v), A is the completion of A'. So, by proposition II.3.1.12(i), to show that $A^0 = A_0$, it suffices to show that $(A')^0 = A'_0$. Let $f \in (A')^0$. Then there exists $n \in \mathbb{N}$ such that $f \in K^0[X^{1/\ell^n}][\frac{1}{\varpi}] = K[X^{1/\ell^n}]$, and we want to show that f is in $K^0[X^{1/\ell^n}]$. Without loss of generality, we may assume that $n = 0$, so that $f \in K[X]$. Denote by $|.|$ the rank 1 valuation giving the topology of K. Suppose that $f \notin K^0[X]$, write $f = \sum_{s\geq 0} a_s X^s$, and let r be the smallest integer such that $|a_r| = \max_{s\geq 0}|a_s|$. Then $|a_s| \leq |a_r|$ for every s, and $|a_s| < |a_r|$ if $s < r$; also, for every $n \in \mathbb{Z}$, we have $|n \cdot 1_K| \leq 1$. So, if $n \in \mathbb{N}$, then the coefficient α of X^{rn} in f^n satisfies $|\alpha| = |a_r|^n$. This shows that the set $\{f^n,\ n \in \mathbb{N}\}$ is not bounded, so f is not power-bounded.

We have shown that A is uniform. As K is perfectoid, we may assume that $\varpi \in K^+$ and that ϖ^ℓ divides ℓ in K^+; then ϖ is a pseudo-uniformizer of A, and ϖ^ℓ divides ℓ in A^+, and in particular in $A^0 \supset A^+$. It remains to show that every element of $A^0/\varpi A^0$ is a ℓth power. But this is obvious, because $A^0/\varpi A^0 = \bigcup_{n\geq 0}(K^0/\varpi K^0)[X^{1/\ell^n}]$.

We may assume that $\varpi = (\varpi^\flat)^\sharp$, with $\varpi^\flat \in K^{\flat+}$ a pseudo-uniformizer of $K^\flat$. Then we have a canonical isomorphism $A^+/\varpi^\ell A^+ \simeq A^{\flat+}/(\varpi^\flat)^\ell A^{\flat+}$. By remark V.1.2.3, this extends to an isomorphism $A^\flat \overset{\sim}{\to} K^\flat\langle X^{1/\ell^\infty}\rangle$ sending $A^{\flat+}$ to $K^{\flat+}\langle X^{1/\ell^\infty}\rangle$.

□

Proposition V.1.2.9. *(Lemma 5.21 of [22].) Let A be a perfectoid Tate algebra. Then A is a perfectoid field if and only if $A^\flat$ is a perfectoid field.*

Proof. Let $a = (a^{(n)}) \in A^\flat = \varprojlim_{c \mapsto c^\ell} A$. As A is reduced (see remark IV.1.1.3), $a = 0$ if and only if $a^{(0)} = 0$. On the other hand, as $(a^{(n)})^{\ell^n} = a^{(0)}$ for every $n \in \mathbb{N}$, we have $a \in (A^\flat)^\times$ if and only if $a^{(0)} \in A^\times$. This shows that $A^\flat$ is a field if and ony if A is a field. By theorem V.1.1.9, this finishes the proof.

□

Example V.1.2.10. Let L be the perfectoid field of example V.1.1.11(4). Then $(\mathbb{Q}_\ell^{\mathrm{cycl}})^\flat = L^\flat = \mathbb{F}_\ell((t^{1/\ell^\infty}))$. (We can prove this using remark V.1.2.3.)

V.1.3 Witt vectors

We will explain Fontaine's approach to untilting using Witt vectors, so we need a few reminders about their construction and properties. A reference for this is [24] chapitre II §6.

Consider the *Witt polynomials* $W_0, W_1, \ldots \in \mathbb{Z}[X_0, X_1, \ldots]$, defined by

$$W_m = \sum_{n=0}^{m} X_n^{\ell^{m-n}} \ell^n = X_0^{\ell^m} + \ell X_1^{\ell^{m-1}} + \ldots + \ell^{m-1} X_{m-1}^\ell + \ell^m X_m.$$

Theorem V.1.3.1. *([24] chapitre II §6, Théorème 5.) For every $\Phi \in \mathbb{Z}[X, Y]$, there exists a unique sequence $(\varphi_n)_{n\geq 0}$ of elements of the polynomial ring $\mathbb{Z}[X_n, Y_n, n \in \mathbb{N}]$ such that, for every $m \in \mathbb{N}$:*

$$W_m(\varphi_0, \varphi_1, \ldots) = \Phi(W_m(X_0, X_1, \ldots), W_m(Y_0, Y_1, \ldots)).$$

Applying this theorem to the polynomials $X + Y$ and XY, we get sequences of polynomials $(S_n)_{n\geq 0}$ and $(P_n)_{n\geq 0}$.

Theorem V.1.3.2. *([24] chapitre II §6, Théorème 6.) Let A be a commutative ring. We write $W(A) = A^{\mathbb{N}}$, and define an addition and multiplication on $W(A)$ by*

$$a + b = (S_0(a, b), S_1(a, b), \ldots)$$

and

$$a \cdot b = (P_0(a,b), P_1(a,b), \ldots),$$

for $a, b \in A^{\mathbb{N}}$. *Then this makes* $W(A)$ *into a commutative ring, called the ring of* Witt vectors *of* A.

We denote by W_* the map $W(A) \to A^{\mathbb{N}}$ sending a to the sequence $(W_0(a), W_1(a), \ldots)$. The idea is that this map is an isomorphism of rings if ℓ is invertible in A, and that we can reduce to this case by functoriality.

The ring morphism $W_m : W(A) \to A$ is called the mth *ghost component map*.

Definition V.1.3.3. ([24] chapitre II §5.) A *strict ℓ-ring* is a commutative ring A such that ℓ is not a zero divisor in A, A is complete (and Hausdorff) for the ℓ-adic topology and the ring $A/\ell A$ is perfect.

Proposition V.1.3.4. *([24] chapitre II §5 proposition 8.) If A is a strict ℓ-ring, there exists a unique multiplicative section of the reduction map $A \to A/\ell A$.*

We denote this section by $a \longmapsto [a]$ and call it the *Teichmüller representative*.

Theorem V.1.3.5. *([24] chapitre II §5 théorème 5 and §6 théorème 7.) Let A be a perfect ring of characteristic ℓ. Then there exists a unique (up to unique isomorphism) strict ℓ-ring B such that $B/\ell B = A$, and this ring is canonically isomorphic to $W(A)$.*

Moreover, the isomorphism from $W(A)$ to B sends $a = (a_n)_{n \geq 0}$ to $\sum_{m \geq 0} [a_m^{1/\ell^m}] \ell^m$.

In particular, if A is a perfect ring of characteristic ℓ, then we have a Teichmüller representative $[.] : A \to W(A)$, and every element of $W(A)$ can be written as $\sum_{n \geq 0} [a_n] \ell^n$, with the $a_n \in A$ uniquely determined.

V.1.4 Untilting

We will use the ring of Witt vectors to "untilt" a perfectoid Tate ring of characteristic ℓ. First we need a definition.

Definition V.1.4.1. (Section 1.4 of [10] and definition 3.15 of [23].) Let (A, A^+) be a perfectoid Huber pair of characteristic ℓ. An ideal I of $W(A^+)$ is called *primitive of degree* 1 if it is generated by an element ξ of the form $\xi = \ell + [\varpi]\alpha$, where ϖ is a pseudo-uniformizer of A and $\alpha \in W(A^+)$. We also say that the element ξ of $W(A^+)$ is *primitive of degree* 1.

Lemma V.1.4.2. *(Lemma 3.16 of [23].) Any element ξ of $W(A^+)$ that is primitive of degree 1 is torsionfree in $W(A^+)$.*

Proof. Write $\xi = \ell + [\varpi]\alpha$, where ϖ is a pseudo-uniformizer of A and $\alpha \in W(A^+)$. Let $b \in W(A^+)$ such that $b\xi = 0$. We want to show that $b = 0$. By theorem V.1.3.5, we can write $b = \sum_{n\geq 0}[a_n]\ell^n$, with the $a_n \in A^+$ uniquely determined, and it suffices to show that all the a_n are 0. We show by induction on r that $a_n \in \varpi^r A^+$ for every $n, r \in \mathbb{N}$, which implies that $a_n = 0$ because A^+ is ϖ-adically separated. The result is obvious for $r = 0$. Suppose that $r \geq 1$ and that we have $a_n \in \varpi^{r-1}A^+$ for every n. Then we can write $a_n = \varpi^{r-1}a'_n$ with $a'_n \in A^+$. Let $b' = \sum_{n\geq 0}[a'_n]\ell^n \in W(R^+)$. Then we have $0 = b\xi = [\varpi]^{r-1}b'\xi$, so $b'\xi = 0$ because $[\varpi]$ is not a zero divisor in $W(A^+)$. Reducing the euqlity $(\ell + [\varpi]\alpha)b' = 0$ modulo $[\varpi]$ gives $\sum_{n\geq 0}[a'_n]\ell^{n+1} \in [\varpi]W(A^+)$, hence $a'_n \in \varpi A^+$ for every $n \in \mathbb{N}$, and we are done.

□

Theorem V.1.4.3. *(Lemma 3.14 of [23].) Let (A, A^+) be a perfectoid Huber pair, and let $(A^\flat, A^{\flat+})$ be its tilt.*

(i) The map

$$\theta : \begin{cases} W(A^{\flat+}) & \to & A^+ \\ \sum_{n\geq 0}[a_n]\ell^n & \longmapsto & \sum_{n\geq 0} a_n^\sharp \ell^n \end{cases}$$

is a surjective morphism of rings.

(ii) The kernel of θ is primitive of degree 1.

Note that, if $\mathrm{char}(A) = \ell$ (so that $A^\flat = A$), then $\mathrm{Ker}\,\theta$ is the ideal generated by ℓ (which is primitive of degree 1).

Proof. Choose a pseudo-uniformizer $\varpi^\flat$ of $A^\flat$ such that $\varpi := (\varpi^\flat)^\sharp$ is a pseudo-uniformizer of A and that ϖ^ℓ divides ℓ in A^+. (We have seen that we can choose $\varpi^\flat$ and ϖ satisfying all these properties, except that $\varpi^\flat$ may not be in $A^{\flat+}$ and ϖ^ℓ only divides ℓ in A^0. But, as $A^{\flat+}$ and A^+ are open and $\varpi^\flat$ and $\ell\varpi^{1-\ell}$ are topologically nilpotent (for the second one, because it is in ϖA^0), there exists some integer $N \geq 1$ such that $(\varpi^\flat)^N \in A^{\flat+}$ and $\ell^N\varpi^{N-N\ell} \in A^+$, and we get the last property if we replace $\varpi^\flat$ by $(\varpi^\flat)^N$.)

(i) We first check that θ is a morphism of rings. It suffices to check that its composition with each projection $A^+ \to A^+/\varpi^m A^+$ is a morphism of rings. Fix $m \geq 1$. Remember that we have the mth ghost component map $W_m : W(A^+) \to A^+$, sending $(a_n)_{n\geq 0} \in W(A^+)$ to $\sum_{n=0}^m a_n^{\ell^{m-n}}\ell^n$. The composition $W(A^+) \xrightarrow{W_m} A^+ \to A^+/\varpi^m A^+$ sends a family $(a_n)_{n\geq 0}$ of elements of ϖA^+ to 0, hence it factors through a morphism of rings $W(A^+/\varpi A^+) \to A^+/\varpi^m A^+$. We can compose this with the map $A^{\flat+} = \varprojlim A^+ \to A^+/\varpi A^+$, $(a^{(n)}) \longmapsto a^{(m)} + \varpi A^+$ to get a morphism of rings $\theta' : W(A^{\flat+}) \to W(A^+/\varpi A^+) \to A^+/\varpi^m A^+$. We claim that θ' is the composition of θ with the projection $A^+ \to A^+/\varpi^m A^+$. Indeed, let $a = (a_n)_{n\geq 0} \in W(A^{\flat+})$, and write $a_n = (a_n^{(i)})_{i\geq 0}$. Then $a = \sum_{n\geq 0}[a_n^{1/\ell^n}]\ell^n$, so

$$\theta(a) = \sum_{n\geq 0}(a_n^{1/\ell^n})^\sharp \ell^n = \sum_{n\geq 0} a_n^{(n)}\ell^n.$$

On the other hand, the image of a in $W(A^+/\varpi A^+)$ is the sequence $(a_n^{(m)}+\varpi A^+)_{n\geq 0}$. A lift of this in $W(A^+)$ is $(a_n^{(m)})_{n\geq 0}$. Applying the map $W_m : W(A^+) \to A^+/\varpi^m A^+$, we get

$$\theta'(a) = \sum_{n=0}^{m} (a_n^{(m)})^{\ell^{m-n}} \ell^n = \sum_{n=0}^{m} a_n^{(n)} \ell^n.$$

Now we show that θ is surjective. We have $\theta([\varpi^\flat]) = \varpi$, and the map $W(A^{\flat+})/[\varpi^\flat]W(A^{\flat+}) \to A^+/\varpi A^+$ induces by θ sends $\sum_{n\geq 0}[a_n]\ell^n$ to $(a_0)^\sharp + \varpi A^+$, so it is the composition of the canonical maps $W(A^{\flat+}) \to A^{\flat+}$ and $A^{\flat+} \to A^{\flat+}/\varpi^\flat A^{\flat+} \simeq A^+/\varpi A^+$, and in particular it is surjective. As $W(A^{\flat+})$ is $[\varpi^\flat]$-adically complete (because, for every $r \in \mathbb{N}$, $[\varpi^\flat]^r W(A^{\flat+}) = \{\sum_{n\geq 0}[a_n]\ell^n,\ a_n \in (\varpi^\flat)^r W^{\flat+})$ and A^+ is ϖ-adically complete, this implies that θ is surjective by [25, Lemma 0315](1).

(ii) First we show that there exists $f \in \varpi^\flat A^{\flat+}$ such that $f^\sharp = \ell$ modulo $\ell\varpi A^+$. Indeed, as ϖ^ℓ divides ℓ in A^+, $\alpha := \ell\varpi^{-1} \in A^+$. As every element of $A^+/\ell A^+$ is a ℓth power, there exists $\beta \in A^{\flat+}$ such that $\beta^\sharp = \alpha$ modulo ℓA^+. Take $f = \varpi^\flat\beta$. Then $f^\sharp = \varpi\beta^\sharp$ is equal to $\varpi\alpha = \ell$ modulo $\ell\varpi A^+$.

By the claim proved in the previous paragraph the surjectivity of θ, we can write

$$\ell = f^\sharp + \ell(\varpi^\flat)^\sharp \sum_{n\geq 0} a_n^\sharp \ell^n$$

for some $f \in \varpi^\flat A^{\flat+}$ and $a_n \in A^{\sharp+}$. Let

$$\xi = \ell - [f] - [\varpi^\flat]\sum_{n\geq 0}[a_n]\ell^{n+1} \in W(A^{\flat+}).$$

Then ξ is primitive of degree 1 (because $\varpi^\flat$ divides f in $A^{\flat+}$, so $[\varpi^\flat]$ divides $[f]$ in $W(A^{\flat+})$), and $\theta(\xi) = 0$ by the choice of f and the a_n.

It remains to show that $\mathrm{Ker}(\theta) = \xi W(A^{\flat+})$. Note that $\xi \in \ell + [\varpi^\flat]W(A^\flat+)$, so $W(A^{\flat+})/(\xi,[\varpi^\flat]) = W(A^{\flat+})/(\ell,[\varpi^\flat]) \simeq A^{\flat+}/\varpi^\flat A^{\flat+}$, and the map $W(A^{\flat+})/(\xi,[\varpi^\flat]) \to A^+/\varpi A^+$ induced by θ is the map $(.)^\sharp : A^{\flat+}/\varpi^\flat A^{\flat+} \to A^+/\varpi A^+$, which is an isomorphism. To conclude that θ induces an isomorphism, it suffices (by lemma V.1.4.4) to check that $W(A^{\flat+})/(\xi)$ is $[\varpi^\flat]$-adically complete.

To show that $W(A^{\flat+})/(\xi)$ is $[\varpi^\flat]$-adically complete, consider the short exact sequence of $W(A^{\flat+})$-modules

$$0 \to \xi W(A^{\flat+}) \to W(A^{\flat+}) \to W(A^{\flat+})/(\xi) \to 0.$$

As ξ is not a zero divisor (lemma V.1.4.2), the $W(A^{\flat+})$-module $\xi W(A^{\flat+})$ is flat, and so, by lemma [25, Lemma 0315](3), the sequence of $[\varpi^\flat]$-adic completions is still exact. But the first two modules are $[\varpi^\flat]$-adically complete, so the third must also be $[\varpi^\flat]$-adically complete.

□

Lemma V.1.4.4. *Let R be a ring, let $\varpi \in R$, and let $\theta : M \to N$ be a morphism of R-modules. Suppose that M and N are ϖ-adically complete and Hausdorff, that N is ϖ-torsionfree, and that θ induces an isomorphism $M/\varpi M \xrightarrow{\sim} N/\varpi N$. Then θ is an isomorphism.*

Proof. We already know that θ is surjective, by [25, Lemma 0315](1). To show that θ is injective, it suffices to show that, for every $n \geq 1$, the morphism $M/\varpi^n M \to N/\varpi^n N$ induces by θ is injective. We show this by induction on n. The case $n = 1$ is the assumption, so suppose that $n \geq 2$ and that we know the result for every $n' < n$. We have a commutative diagram with exact rows :

$$\begin{tikzcd} & M/\varpi^{n-1}M \arrow[r, "\varpi\cdot(.)"] \arrow[d, "\theta"'] & M/\varpi^n M \arrow[r] \arrow[d, "\theta"'] & M/\varpi M \arrow[r] \arrow[d, "\theta"'] & 0 \\ 0 \arrow[r] & N/\varpi^{n-1}N \arrow[r, "\varpi\cdot(.)"] & N/\varpi^n N \arrow[r] & N/\varpi N \arrow[r] & 0 \end{tikzcd}$$

Indeed, the map $N/\varpi^{n-1}N \to N/\varpi^n N$, $x \longmapsto \varpi x$ is injective because N is ϖ-torsionfree. By the induction hypothesis, the first and third vertical maps are injective. So the injectivity of the middle vertical arrow follows from the five lemma (or an easy diagram chase).

□

Corollary V.1.4.5. *(Theorem 3.17 of [23], see also proposition 1.1 of [10].) There is an equivalence of categories between :*

(a) perfectoid Huber pairs (S, S^+);

(b) triples (R, R^+, I), where (R, R^+) is a perfectoid Huber pair of characteristic ℓ and $I \subset W(R^+)$ is an ideal that is primitive of degree 1.

This equivalence is given by the functors that send a pair (S, S^+) as in (a) to $(S^\flat, S^{\flat+}, \mathrm{Ker}(\theta : W(S^{\flat+}) \to S^+))$, and a triple (R, R^+, I) as in (b) to $((W(R^+)/I)[\frac{1}{[\varpi]}], W(R^+)/I)$, where ϖ is any pseudo-uniformizer of R.

Corollary V.1.4.6. *(See théorème 1.2 of [10].) Fix a perfectoid Huber pair (A, A^+). Then there is an equivalence of categories between perfectoid Huber pairs over (A, A^+) and over $(A^\flat, A^{\flat+})$.*

This equivalence is given by the functor that send a morphism of perfectoid Huber pairs $(A, A^+) \to (B, B^+)$ to its tilt $(A^\flat, A^{\flat+}) \to (B^\flat, B^{\flat+})$. In the other direction, suppose that we have a morhism of perfectoid Huber pairs $(A^\flat, A^{\flat+}) \to (B^\flat, B^{\flat+})$, then we get a morphism $\alpha : W(A^{\flat+}) \to W(B^{\flat+})$. By the definition of a primitive ideal of degree 1, if $I = \mathrm{Ker}(\theta : W(A^{\flat+}) \to A^+)$, then $J := \alpha(I)$ is a primitive ideal of degree 1 of $W(B^{\flat+})$, so we get a morphism of perfectoid Huber pairs

$$(A, A^+) \simeq ((W(A^{\flat+})/I)[\tfrac{1}{[\varpi^\flat]}], W(A^{\flat+})/I) \to ((W(B^{\flat+})/J)[\tfrac{1}{[\varpi^\flat]}], W(B^{\flat+})/J)$$

(where $\varpi^\flat$ is any pseudo-uniformizer of $A^\flat$).

V.1.5 A little bit of almost mathematics

We will introduce as little almost mathematics as possible. For a complete treatment, see [11].

Definition V.1.5.1. Let (A, A^+) be a perfectoid Huber pair.

(i) We say that a A^+-module M is *almost zero* if $A^{00} \cdot M = 0$.

(ii) We say that a morphism of A^+-modules $u : M \to N$ is *almost injective* (resp. *almost surjective*) if $\ker u$ (resp. $\operatorname{Coker} u$) is almost zero; we say that it is *almost an isomorphism* if it is both almost injective and almost surjective.

(iii) If $M \subset N$ are A^+-modules, we say that they are *almost equal* if N/M is almost zero.

Example V.1.5.2. A^+ and A^0 are almost equal.

Remark V.1.5.3. If $u : M \to N$ is almost injective (resp. almost surjective, resp. almost an isomorphism), then $u[\frac{1}{\varpi}] : M[\frac{1}{\varpi}] \to N[\frac{1}{\varpi}]$ is injective (resp. surjective, resp. an isomorphism).

Proposition V.1.5.4. *Let (A, A^+) be a perfectoid Huber pair and M be a A^+-module. Choose a pseudo-uniformizer ϖ of A that has a compatible system $(\varpi^{1/\ell^n})_{n \geq 0}$ of ℓth power roots. (In other words, ϖ is in the image of $(.)^\sharp : A^\flat \to A$.)*

Then M is almost zero if and only if, for every $x \in M$ and for every $n \in \mathbb{N}$, we have $\varpi^{1/\ell^n} x = 0$.

Proof. We have $\varpi^{1/\ell^n} \in A^{00}$ for every $n \in \mathbb{N}$, so, if M is almost zero, then $\varpi^{1/\ell^n} x = 0$ for every $x \in M$ and every $n \in \mathbb{N}$.

Conversely, suppose that, for every $x \in M$ and every $n \in \mathbb{N}$, we have $\varpi^{1/\ell^n} x = 0$. Let $a \in A^{00}$. As ϖA^+ is open in A, there exists $n \in \mathbb{N}$ such that $a^{\ell^n} \in \varpi A^+$. Then $(\varpi^{-1/\ell^n} a)^{\ell^n} \in A^+$, so $\varpi^{-1/\ell^n} a \in A^+$ because A^+ is integrally closed in A. So, for every $x \in M$, $ax = \varpi^{1/\ell^n}((\varpi^{-1/\ell^n} a)x) = 0$.

$\square$

Corollary V.1.5.5. *If $(A, A^+) \to (B, B^+)$ is a morphism of perfectoid Huber pairs and M is a B^+-module, then M is almost zero as a B^+-module if and only if it is almost zero as a A^+-module.*

In particular, for perfectoid Huber pairs over a perfectoid Huber pair (K, K^0) with K a field, we recover the same definition as in definition 4.1 of [22].

Definition V.1.5.6. Let (A, A^+) be a perfectoid Huber pair and M be a A^+-module. We set $M_* = \operatorname{Hom}_{A^+}(A^{00}, M)$. This is also a A^+-module, and it is called the module of *almost elements* of M.

By the general properties of Hom, the functor $M \longmapsto M_*$ commutes with projective limits (in particular, it is left exact).

Remark V.1.5.7. If M_1, M_2, N are A^+-modules and $\mu : M_1 \times M_2 \to N$ is a A^+-bilinear map, then we have a canonical A^+-linear map $M_{1*} \otimes_{A^+} M_{2*} \to N_*$, induced by the A^+-bilinear map

$$\begin{cases} \mathrm{Hom}_{A^+}(A^{00}, M_1) \times \mathrm{Hom}_{A^+}(A^{00}, M_2) & \to & \mathrm{Hom}_{A^+}(A^{00}, N) \\ (f, g) & \longmapsto & (a \longmapsto \mu(f(a) \otimes g(a))). \end{cases}$$

In particular, if B^+ is a A^+-algebra and M is a B^+-module, then $(B^+)_*$ is a A^+-algebra and M_* is a $(B^+)_*$-module. Moreover, if M_1, M_2, N are B^+-modules and the map $\mu : M_1 \times M_2 \to N$ is actually B^+-bilinear, then it is easy to see from the construction given above that the induced map $M_{1*} \times M_{2*} \to N_*$ is $(B^+)_*$-bilinear; so, if for example C^+ is a B^+-algebra, then $(C^+)_*$ is a $(B^+)_*$-algebra.

Proposition V.1.5.8. *Let (A, A^+) be a perfectoid Huber pair and M be a A^+-module.*

(i) The canonical map $M \to M_*$, $x \longmapsto (a \longmapsto ax)$ is almost an isomorphism.

(ii) Choose a pseudo-uniformizer ϖ of A that has a compatible system $(\varpi^{1/\ell^n})_{n\geq 0}$ of ℓth power roots. If M is ϖ-torsionfree (so that the canonical map $M \to A \otimes_{A^+} M$ is injective), then

$$M_* = \{x \in A \otimes_{A^+} M \mid \forall n \in \mathbb{N},\ \varpi^{1/\ell^n} x \in M\} = \{x \in A \otimes_{A^+} M \mid \forall a \in A^{00},\ ax \in M\}.$$

(iii) If M is almost zero, then $M_* = 0$.

(iv) If $u : M \to N$ is an almost injective (resp. surjective) map of A^+-modules, then the map $u_* : M_* \to N_*$ is injective (resp. surjective).

Proof. (i) This follows from the exact sequence

$$0 \to \mathrm{Hom}_{A^+}(A^+/A^{00}, M) \to M \to M_* \to \mathrm{Ext}^1_{A^+}(A^+/A^{00}, M).$$

(ii) Let $v : M_* \to M$ be evaluation at ϖ. If left multiplication by ϖ is an isomorphism on M, then the map $\varpi^{-1}v : M_* \to M$ is an inverse of the canonical map $M \to M_*$, so $M = M_*$.

Now suppose that M is only ϖ-torsionfree. As $(A \otimes_{A^+} M)_* = A \otimes_{A^+} M$ by the previous paragraph and $M \to (A \otimes_{A^+} M)$ is injective by hypothesis, we get injections $M \subset M_* \subset A \otimes_{A^+} M$. Let $M' = \{x \in A \otimes_{A^+} M \mid \forall n \in \mathbb{N},\ \varpi^{1/\ell^n} x \in M\}$ and $M'' = \{x \in A \otimes_{A^+} M \mid \forall a \in A^{00},\ ax \in M\}$. We obviously have $M'' \subset M'$. Conversely, if $x \in M'$, then $(M + A^+x)/M$ is almost zero by proposition V.1.5.4, so $x \in M''$. This shows that $M' = M''$. Now we show that $M_* = M''$. If $x \in M''$, then the map $A^{00} \to M$, $a \longmapsto ax$ is in M_*, and its image by $\varpi^{-1}v$ is x. So $M'' \subset M_*$. Conversely, let $c : A^{00} \to M$ be an element of M_*, and let $x = \varpi^{-1}c(\varpi) \in A \otimes_{A^+} M$. Then, for every $n \in \mathbb{N}$, we have

$$\varpi^{1/\ell^n} x = \varpi^{1/\ell^n} \varpi^{-1} c(\varpi^{1-1/\ell^n} \varpi^{1/\ell^n}) = \varpi^{-1+1/\ell^n} \varpi^{1-1/\ell^n} c(\varpi^{1/\ell}) = c(\varpi^{1/\ell^n}) \in M,$$

so $x \in M' = M''$.

(iii) Let $u : A^{00} \to M$ be an element of M_*. Let $a \in A^{00}$. Then we can find $n \in \mathbb{N}$ such that $\varpi^{-1/\ell^n} a \in A^{00}$ (just choose n such that $a^{\ell^n} \in \varpi A^{00}$), so $u(a) = \varpi^{1/\ell^n} u(\varpi^{-1/\ell^n} a) = 0$. This shows that $u = 0$.

(iv) Suppose that u is almost injective. As the functor $(.)_*$ is right exact, the kernel of $u_* : M_* \to N_*$ is equal to the image of $(\operatorname{Ker} u)_* \to M_*$. But $(\operatorname{Ker} u)_* = 0$ by (iii), so u_* is injective.

Suppose that u is almost surjective. Then, applying the right exact functor $(.)_*$ to the exact sequence $M \to N \to \operatorname{Coker} u \to 0$, we get an exact sequence $M_* \to N_* \to (\operatorname{Coker} u)_* \to 0$. But we know that $(\operatorname{Coker} u)_* = 0$ by (iii), so u_* is surjective.

□

Proposition V.1.5.9. *(See lemma 4.4.1 of [1].) Let (A, A^+) be a perfectoid Huber pair and M be a A^+-module. Choose a pseudo-uniformizer ϖ of A that has a compatible system $(\varpi^{1/\ell^n})_{n\geq 0}$ of ℓth power roots.*

Let M be a A^+-module that is ϖ-torsionfree and ϖ-adically separated and complete. Then the A^+-module M_* is also ϖ-torsionfree and ϖ-adically separated and complete.

Moreover, for every $a \in A^+$ such that M is a-torsionfree, the canonical morphism $aM_* \to (aM)_*$ is an isomorphism, and the canonical morphism $M_*/aM_* \to (M/aM)_*$ is injective, with image equal to that of $(M/\varpi^{1/\ell^n} aM)_*$ for every $n \in \mathbb{N}$.

Proof. First we show that M_* is ϖ-torsionfree. This follows from the fact that the functor $(.)_*$ is left exact, so $\operatorname{Ker}(\varpi : M_* \to M_*) = (\operatorname{Ker}(\varpi : M \to M))_* = 0$.

Let $a \in A^+$ such that M is a-torsionfree. Then, applying $(.)_*$ to the exact sequence $0 \to M \overset{a\cdot(.)}{\to} M \to M/aM \to 0$, we get an exact sequence

$$0 \to M_* \overset{a\cdot(.)}{\to} M_* \to (M/aM)_* \to \operatorname{Ext}^1_{A^+}(A^{00}, M).$$

This shows in particular that $aM_* = (aM)_*$ and that $M_*/aM_* \to (M/aM)_*$ is injective. Let $n \in \mathbb{N}$, and let $\varepsilon = \varpi^{1/\ell^n}$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "a\cdot(.)"] \arrow[d] & M \arrow[r] \arrow[d] & M/aM \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M/\varepsilon M \arrow[r, "a\cdot(.)"'] & M/\varepsilon aM \arrow[r] & M/aM \arrow[r] & 0
\end{tikzcd}$$

Applying the functor $(.)_*$, we get a commutative diagram with exact rows

$$\begin{tikzcd} M_* \arrow[r, "a\cdot(.)"] \arrow[d] & M_* \arrow[r, "b_1"] \arrow[d] & (M/aM)_* \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{A^+}(A^{00}, M) \arrow[d, "c"] \\ (M/\varepsilon M)_* \arrow[r, "a\cdot(.)"'] & (M/\varepsilon aM)_* \arrow[r, "b_2"'] & (M/aM)_* \arrow[r] & \mathrm{Ext}^1_{A^+}(A^{00}, M/\varepsilon M) \end{tikzcd}$$

We want to show that the maps b_1 and b_2 have the same image. It suffices to prove that the map c is injective, which is equivalent to the fact that multiplication by ε is equal to zero on $\mathrm{Ext}^1_{A^+}(A^{00}, M)$. Applying the functor $\mathrm{Hom}_{A^+}(., M)$ to the exact sequence $0 \to A^{00} \to A^+ \to A^+/A^{00} \to 0$, we get an isomorphism $\mathrm{Ext}^1_{A^+}(A^{00}, M) \simeq \mathrm{Ext}^2_{A^+}(A^+/A^{00}, M)$, which shows that $\mathrm{Ext}^1_{A^+}(A^{00}, M)$ is almost zero and implies the desired result.

We finally prove that A_* is ϖ-adically separated and complete. By the assumption on M, we have $M \overset{\sim}{\to} \varprojlim M/\varpi^n M$. As the functor $(.)_*$ commutes with projective limits, the canonical map $M_* \to \varprojlim (M/\varpi^n M)_*$ is an isomorphism. But this map factors as $M_* \to \varprojlim M_*/\varpi^n M_* \to \varprojlim (M/\varpi^n M)_*$, and the second map is injective because $M_*/\varpi^n M_* \to (M/\varpi^n M)_*$ is injective for every $n \in \mathbb{N}$. So the canonical map $M_* \to \varprojlim M_*/\varpi^n M_*$ is also an isomorphism. (We could also have used the previous paragraph to show directly that $\varprojlim M_*/\varpi^n M_* \to \varprojlim (M/\varpi^n M)_*$ is an isomorphism.)

□

Proposition V.1.5.10. *(See proposition 5.2.6 and theorem 6.2.5 of [1].) Let (A, A^+) be a perfectoid Huber pair, and choose a pseudo-uniformizer ϖ of A that has a compatible system $(\varpi^{1/\ell^n})_{n\geq 0}$ of ℓth power roots and that divides ℓ in A^+. Let $\varphi : A \to B$ be a morphism of f-adic rings and B_0 be an open subring of B containing $\varphi(A^+)$.*

Suppose that :

(a) B is complete;

(b) B_0 is $\varphi(\varpi)$-adically separated and complete;

(c) the map $B_0/\varphi(\varpi^{1/\ell}) \to B_0/\varphi(\varpi)$, $b \longmapsto b^\ell$ is almost an isomorphism.

Then B is perfectoid, B_0 is a ring of definition of B and $B^0 = (B_0)_*$.

Proof. We know that B is a Tate ring and that $\varphi(\varpi)$ is a pseudo-uniformizer of B (see proposition II.1.3.4). To simplify the notation, we will write ϖ instead of $\varphi(\varpi)$. By lemma V.1.5.11, B_0 is a ring of definition of B.

We write $B' = (B_0)_*$. We know that B' is ϖ-adically separated and complete by proposition V.1.5.9, and that

$$B' = \{b \in B \mid \forall n \in \mathbb{N},\ \varpi^{1/\ell^n} b \in B_0\}$$

by proposition V.1.5.8(ii). In particular, lemma V.1.5.11 implies that B' is a bounded subring of B, i.e. a ring of definition. By proposition V.1.5.8(iv), the map $(B_0/\varpi^{1/\ell}B_0)_* \to (B_0/\varpi B_0)_*$ induced by $b \longmapsto b^\ell$ is injective, so the map $B'/\varpi^{1/\ell}B' \to B'/\varpi B'$, $b \longmapsto b^\ell$ is also injective, because $B'/\varpi^{1/\ell}B'$ (resp. $B'/\varpi B'$) injects in $(B_0/\varpi^{1/\ell}B_0)_*$ (resp. $(B_0/\varpi B_0)_*$) by proposition V.1.5.9.

We prove that B' is closed by taking ℓth roots. Let $b \in B$ such that $b^\ell \in B'$. We want to prove that $b \in B'$. Choose an integer $r \in \mathbb{N}$ such that $\varpi^{r/\ell}b \in B'$ (this is possible because $B = B'[1/\varpi]$). If $r = 0$, then $b \in B'$ and we are done. Suppose that $r \geq 1$. We have $(\varpi^{r/\ell}b)^\ell = \varpi^r b^\ell \in \varpi^r B' \subset \varpi B'$. By the injectivity of the map $B'/\varpi^{1/\ell}B' \to B'/\varpi B'$, $b \longmapsto b^\ell$, this implies that $\varpi^{r/\ell}b \in \varpi^{1/\ell}B'$, hence that $\varpi^{(r-1)/\ell}b \in B'$. If $r-1 \geq 1$, we can apply this process again to show that $\varpi^{(r-2)/\ell}b \in B'$, etc. In the end, we get that $b \in B'$.

Note that the previous paragraph implies that $B^{00} \subset B'$. Indeed, if $b \in B^{00}$, then there exists $n \in \mathbb{N}$ such that $b^{\ell^n} \in B'$ (because B' is open), so $b \in B'$ by what we just proved.

We show that the map $B'/\varpi^{1/\ell}B' \to B'/\varpi B'$, $b \longmapsto b^\ell$ is surjective. Let $b \in B'$. As the map $B_0/\varpi^{1/\ell}B_0 \to B_0/\varpi B_0$, $c \longmapsto c^\ell$ is surjective, there exists $c_0 \in B_0$ such that $\varpi^{1/\ell}b \in c^\ell + \varpi B_0$. Let $d = \varpi^{-1/\ell^2}c$. Then $d^\ell = \varpi^{-1/\ell}c^\ell \in b + \varpi^{1-1/\ell}B_0 \subset B'$, so, as B' in closed under taking ℓth roots, $d \in B'$. Write $b = d^\ell + \varpi^{1-1/\ell}b'$, with $b' \in B_0$. Then there exists $d' \in B_0$ such that $\varpi^{1-1/\ell}b' \in (d')^\ell + \varpi B_0$, and we finally get $b \in (d+d')^\ell + \varpi B'$.

To finish the proof, we just need to show that $B' = B^0$. The inclusion $B' \subset B^0$ follows from the fact that B' is bounded and from proposition II.1.2.4. Conversely, let $b \in B^0$. Then, for every $n \in \mathbb{N}$, we have $\varpi^{1/\ell^n}b \in B^{00} \subset B'$; so $b \in B'$.

□

Lemma V.1.5.11. *Let A be a complete Tate ring and ϖ be a pseudo-uniformizer of A. Let B be an open subring of A that is ϖ-adically separated and complete. Then B is bounded in A.*

Proof. By corollary II.1.1.8(iii), we can choose a ring of definition A_0 of A such that $A_0 \subset B$. We may assume, after replacing ϖ by a power, that ϖ is in A_0. Note that ϖ is not a zero divisor in B, because it is invertible in A. By remark II.2.5.3, the ring $A' = B[\frac{1}{\varpi}]$, with the topology for which B is a ring of definition, is a complete Tate ring with pseudo-uniformizer ϖ. Of course, as a ring, A' is canonically isomorphic to A. Moreover, the obvious isomorphism $A \xrightarrow{\sim} A'$ is continuous, because B is open in A. By the open mapping theorem (theorem II.4.1.1), this map is open, which means that $A \xrightarrow{\sim} A'$ is an isomorphism of topological rings, hence that B is a ring of definition of A.

□

Using the proposition, we can (almost) generalize proposition V.1.2.8.

Corollary V.1.5.12. . *Let (A, A^+) be a perfectoid Huber pair, and let $\varpi \in A^+$ be a pseudo-uniformizer of A. We denote by $A^+\langle X^{1/\ell^\infty}\rangle$ the ϖ-adic completion of the ring $\bigcup_{n\geq 0} A^+[X^{1/\ell^n}]$, and we set $A\langle X^{1/\ell^\infty}\rangle = A^+\langle X^{1/\ell^\infty}\rangle[\frac{1}{\varpi}]$.*

Then $A\langle X^{1/\ell^\infty}\rangle$ is perfectoid, with ring of power-bounded elements $(A\langle X^{1/\ell^\infty}\rangle)^0$ almost equal to $A^+\langle X^{1/\ell^\infty}\rangle$, and its tilt is canonically isomorphic to $A^\flat\langle X^{1/\ell^\infty}\rangle$. Moreover, $A^+\langle X^{1/\ell^\infty}\rangle$ is a ring of integral elements in $A\langle X^{1/\ell^\infty}\rangle$ and the tilt of $(A\langle X^{1/\ell^\infty}\rangle, A^+\langle X^{1/\ell^\infty}\rangle)$ is $(A^\flat\langle X^{1/\ell^\infty}\rangle, A^{\flat+}\langle X^{1/\ell^\infty}\rangle)$.

Proof. We may assume that there is a pseudo-uniformizer $\varpi^\flat \in A^{\flat+}$ of $A^\flat$ such that $\varpi = (\varpi^\flat)^\sharp$, and that ϖ^ℓ divides ℓ in A^+. Then we may apply proposition V.1.5.10 to $B = A\langle X^{1/\ell^\infty}\rangle$ and to its open subring $B_0 = A^+\langle X^{1/\ell^\infty}\rangle$, and we get that B is perfectoid and $B^0 = (B_0)_*$.

As $B_0/\varpi^\ell B_0 = \bigcup_{n\geq 0}(A^+/\varpi^\ell A^+)[X^{1/\ell^n}]$, we have a canonical isomorphism $B_0/\varpi^\ell B_0 \simeq \bigcup_{n\geq 0} A^{\flat+}/(\varpi^\flat)^\ell A^{\flat+}[X^{1/\ell^n}]$. By remark V.1.2.3, this extends to an isomorphism $B^\flat \xrightarrow{\sim} A^\flat\langle X^{1/\ell^\infty}\rangle$ sending $B^{\flat+}$ to $A^{\flat+}\langle X^{1/\ell^\infty}\rangle$.

□

Corollary V.1.5.13. *(See lemma 9.2.3 of [1].) Let (A, A^+) be a perfectoid Huber pair of characteristic ℓ, let ϖ be a pseudo-uniformizer of A, let $f_1, \dots, f_n, g \in A^+$ with $f_n = \varpi^N$ ($N \in \mathbb{N}$) and $U = R\left(\frac{f_1,\dots,f_n}{g}\right)$. Then :*

(i) Let $A^+\langle\left(\frac{f_i}{g}\right)^{1/\ell^\infty}\rangle$ be the ϖ-adic completion of the subring $A^+[\left(\frac{f_i}{g}\right)^{1/\ell^\infty}]$ of $A[\frac{1}{g}]$. Then $A^+\langle\left(\frac{f_i}{g}\right)^{1/\ell^\infty}\rangle$ is ϖ-torsionfree, ϖ-adically separated and complete, and perfect.

(ii) The canonical A^+-algebra map $\psi : A^+[X_i^{1/\ell^\infty}] \to A^+[\left(\frac{f_i}{g}\right)^{1/\ell^\infty}]$ sending X_i^{1/ℓ^m} to $\left(\frac{f_i}{g}\right)^{1/\ell^m}$ is surjective with kernel containing and almost equal to $I := (g^{1/\ell^m}X_i^{1/\ell^m} - f_i^{1/\ell^m},\ 1 \leq i \leq n,\ m \in \mathbb{N})$.

(iii) Let $X = \operatorname{Spa}(A, A^+)$. The Tate ring $\mathscr{O}_X(U)$ is perfectoid, and $\mathscr{O}_X(U)^0 = A^+\langle\left(\frac{f_i}{g}\right)^{1/\ell^\infty}\rangle_*$ (in particular, these subrings of $\mathscr{O}_X(U)$ are almost equal).

Proof. Remember that A and A^+ are perfect (see proposition V.1.1.8).

(i) This ring is the ϖ-adic completion of the perfect and ϖ-torsionfree ring $A^+[\left(\frac{f_i}{g}\right)^{1/\ell^\infty}]$.

(ii) It is clear that ψ is surjective and that $I \subset \operatorname{Ker}\psi$. We have $A^+[X_i^{1/\ell^\infty}][\frac{1}{\varpi}] = A[X_i^{1/\ell^\infty}]$ and $A^+[\left(\frac{f_i}{g}\right)^{1/\ell^\infty}][\frac{1}{\varpi}] = A[\frac{1}{g}]$ (because $f_n = \varpi^n$).

We claim that $\operatorname{Ker}\psi[\frac{1}{\varpi}] = I[\frac{1}{\varpi}]$. Indeed, let $f \in \operatorname{Ker}\psi[\frac{1}{\varpi}]$. We may assume that $f \in A[X_1, \dots, X_n]$, and we want to show that $f \in (gX_i - f_i) = (gX_1 - f_1, \dots, gX_{n-1} - f_{n-1}, \varpi^{-N}gX_n - 1)$. We have an isomorphism $A[X_1, \dots, X_n]/(\varpi^{-N}gX_n - 1) \simeq A[g^{-1}][X_1, \dots, X_{n-1}]$ and the ideal $(gX_i - f_i)$ to $I' := (X_1 - f_1g^{-1}, \dots, X_{n-1} - f_{n-1}g^{-1})$. This ideal I' is clearly the kernel of the map $A[g^{-1}][X_1,\dots,X_{n-1}] \to A[g^{-1}]$ induced by ψ, so the image of f in $A[g^{-1}][X_1,\dots,X_{n-1}]$ is in I', which is what we wanted.

Now let $B = A^+[X_i^{1/\ell^\infty}]/I$. It is clear from the definition of I that B is perfect, and ψ induces a surjective map $\overline{\psi} : B \to A^+[\left(\frac{f_i}{g}\right)^{1/\ell^\infty}]$ that is an isomorphism after inverting ϖ. In particular, $\mathrm{Ker}(\overline{\psi})$ is ϖ^∞-torsion, so it is almost zero by lemma V.1.5.14, and we are done.

(iii) By (i) and proposition V.1.1.3(i), the open subring $A^+\langle\left(\frac{f_i}{g}\right)^{1/\ell^\infty}\rangle$ of $\mathscr{O}_X(U)$ satisfies the conditions of proposition V.1.5.10. The result follows immediately from this proposition.

□

Lemma V.1.5.14. *Let (A, A^+) be a perfectoid Huber pair of characteristic ℓ and B a perfect A^+-algebra. Then the ϖ^∞-torsion in B is almost zero.*

Proof. Let $N = \{b \in B \mid \exists n \in \mathbb{Z}_{\geq 1},\ \varpi^n b = 0\}$. We want to show that N is almost zero. Let $b \in N$, and let $n \in \mathbb{Z}_{\geq 1}$ such that $\varpi^n b = 0$. Let $r \in \mathbb{N}$. As B is perfect, there exists $c \in B$ such that $b = c^{\ell^r}$, and then $(\varpi^{n/\ell^r} c)^{\ell^r} = 0$, so, using the fact that B is perfect again, $\varpi^{n/\ell^r} c = 0$, which finally gives $\varpi^{n/\ell^r} b = 0$. By proposition V.1.5.4, this implies that N is almost 0.

□

Corollary V.1.5.15. *(Lemma 9.2.5 of [1].) Let (A, A^+) be a perfectoid Huber pair, and $X = \mathrm{Spa}(A, A^+)$. Choose a pseudo-uniformizer ϖ of A that is of the form $(\varpi^\flat)^\sharp$ for $\varpi^\flat$ a pseudo-uniformizer of $A^\flat$, and such that ϖ divides ℓ in A^+. Let $f_1,\dots,f_n,g \in A^+$ such that $f_n = \varpi^N$ for some $N \in \mathbb{N}$. Suppose that we have $a_1,\dots,a_n,b \in A^\flat$ with $a_i^\sharp = f_i$, $b^\sharp = g$ and $a_n = (\varpi^\flat)^N$. Let $U = R\left(\frac{f_1,\dots,f_n}{g}\right) \subset X$ and $U^\flat = R\left(\frac{a_1,\dots,a_n}{b}\right) \subset X^\flat$, so that U is the preimage of $U^\flat$ by the map $X \to X^\flat$, $x \longmapsto x^\flat$. Then :*

(i) Let $B_0 := A^+\langle\left(\frac{f_i}{g}\right)^{1/\ell^\infty}\rangle$ be the ϖ-adic completion of the subring $A^+[\left(\frac{f_i}{g}\right)^{1/\ell^\infty}]$ of $A[\frac{1}{g}]$. Then B_0 is ϖ-torsionfree, ϖ-adically separated and complete, and the map $B_0/\varpi^{1/\ell}B_0 \to B_0/\varpi B_0$, $b \longmapsto b^\ell$ is bijective.

(ii) The canonical A^+-algebra map $\psi : A^+[X_i^{1/\ell^\infty}] \to A^+[\left(\frac{f_i}{g}\right)^{1/\ell^\infty}]$ sending X_i^{1/ℓ^m} to $\left(\frac{f_i}{g}\right)^{1/\ell^m}$ is surjective with kernel containing and almost equal to $I := (g^{1/\ell^m} X_i^{1/\ell^m} - f_i^{1/\ell^m},\ 1 \leq i \leq n,\ m \in \mathbb{N})$.

(iii) The Tate ring $\mathscr{O}_X(U)$ is perfectoid, and $\mathscr{O}_X(U)^0 = A^+\langle\left(\frac{f_i}{g}\right)^{1/\ell^\infty}\rangle_*$ (in particular, these subrings of $\mathscr{O}_X(U)$ are almost equal).

(iv) The tilt of $(\mathscr{O}_X(U), \mathscr{O}_X(U)^+)$ is canonically isomorphic to $(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}_{X^\flat}(U^\flat)^+)$.

Proof. We show (i). The ring B_0 is ϖ-torsionfree and ϖ-adically separated and complete by definition. Also, if $B_0' := A^{\flat+}\langle\left(\frac{a_i}{b}\right)^{1/\ell^\infty}\rangle$, then we have $B_0/\varpi B_0 \simeq B_0'/\varpi B_0'$, and B_0' is perfect; this gives the last statement of (i).

It is clear that ψ is surjective and $I \subset \operatorname{Ker}\psi$. Let $P_0 = A^+[X_i^{1/\ell^\infty}]/I$ and $a_0 : P_0 \to A^+[\left(\frac{f_i}{g}\right)^{1/\ell^\infty}]$ be the map induced by ψ. By definition of $\mathscr{O}_X^+(U)$ (and lemma IV.1.3.8), we have a canonical morphism $b_0 : A^+[\left(\frac{f_i}{g}\right)^{1/\ell^\infty}] \to \mathscr{O}_X^+(U)$. Let (S, S^+) be the perfectoid Huber pair over (A, A^+) that we get by untilting the perfectoid pair $(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}^+_{X^\flat}(U^\flat))$ over $(A^\flat, A^{\flat+})$ (see corollary V.1.4.6). If $i \in \{1, \ldots, n\}$, then b divides a_i in $\mathscr{O}^+_{X^\flat}(U^\flat)$, so $b^\sharp = g$ divides $a_i^\sharp = f_i$ in S^+, so the map $\operatorname{Spa}(S, S^+) \to \operatorname{Spa}(A, A^+)$ factors through U, and the universal property of U (see proposition III.6.1.1(ii)) implies that the map $(A, A^+) \to (S, S^+)$ extends to a map $(\mathscr{O}_X(U), \mathscr{O}_X^+(U)) \to (S, S^+)$. We denote the map $\mathscr{O}_X^+(U) \to S^+$ by c, and we write $d_0 = c \circ b_0$. The maps $a_0 : P_0 \to A^+[\left(\frac{f_i}{g}\right)^{1/\ell^\infty}]$ and $d_0 : A^+[\left(\frac{f_i}{g}\right)^{1/\ell^\infty}] \to S^+$ extends by continuity to maps $a : P \to B_0 = A^+\langle\left(\frac{f_i}{g}\right)^{1/\ell^\infty}\rangle$ and $d : B_0 \to S^+$ between the ϖ-adic completions (we use the fact that S^+ is complete). As a_0 is surjective, a is also surjective (see [25, Lemma 0315](2)). By corollary V.1.5.13, the map $d_0 \circ a_0$ modulo ϖ is almost an isomorphism. By lemma V.1.5.16, $\operatorname{Ker} a_0$ is almost zero, which proves (ii).

Moreover, as $d_0 \circ a_0$ and $d \circ a$ are equal modulo ϖ, $d \circ a$ modulo ϖ is also almost an isomorphism. On the other hand, as d modulo ϖ is surjective, so is d (see [25, Lemma 0315](1)), so $d \circ a$ is also surjective. By lemma V.1.5.16 again, we get that $d \circ a$ is almost an isomorphism. As a is surjective and $\operatorname{Ker} a \subset \operatorname{Ker}(d \circ a)$, a is also almost an isomorphism, hence so is d. In particular (see remark V.1.5.3), the map $d[\frac{1}{\varpi}] : B_0[\frac{1}{\varpi}] \to \mathscr{O}_X(U) \to S$ is an isomorphism. As the first map is injective by definition of B_0, both maps are isomorphisms. Now (iii) follows immediately from proposition V.1.5.10.

We finally prove (iv). We have already seen that we have a canonical map $(\mathscr{O}_X(U), \mathscr{O}_X^+(U)) \to (S, S^+)$, and we just proved that $\mathscr{O}_X(U) \to S$ is an isomorphism; this isomorphism then identifies $\mathscr{O}_X^+(U)$ to a subring of S^+. So it suffices to construct a continuous morphism of A^+-algebras $S^+ \to \mathscr{O}_X^+(U)$. Let $(A^\flat, A^{\flat+}) \to (T, T^+)$ be the tilt of the morphism $(A, A^+) \to (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$. As $\mathscr{O}_X^+(U) \supset B_0$, we have $\frac{f_i}{g} \in \mathscr{O}_X^+(U)$ for every i, so $\frac{a_i}{b} \in T^+$ for every i. (This is clear on the formula for the untilting given after corollary V.1.4.6.) So $\operatorname{Spa}(T, T^+) \to \operatorname{Spa}(A^\flat, A^{\flat+})$ factors through $U^\flat$, and the universal property of $U^\flat$ gives a morphism $(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}^+_{X^\flat}(U^\flat)) \to (T, T^+)$, which untilts to a morphism $(S, S^+) \to (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$.

□

Lemma V.1.5.16. *We use the notation of corollary V.1.5.15. Let $\alpha : M \to N$ be a A^+-module map. Suppose that α is almost surjective, that the induced map $M/\varpi M \to N/\varpi N$ is almost an isomorphism, that M is ϖ-adically separated and that N is ϖ-torsionfree. That α is almost an isomorphism.*

Proof. We want to check that $\mathrm{Coker}\,\alpha$ and $\mathrm{Ker}\,\alpha$ are almost zero. This is true for $\mathrm{Coker}\,\alpha$ by assumption. Let $L = \mathrm{Ker}\,\alpha$. As N is ϖ-torsionfree, $L/\varpi L$ is the kernel of the map $M/\varpi M \to N/\varpi N$ induced by α, so it is almost zero. This implies that, for every $n \in \mathbb{N}$, the A^+-module $L/\varpi^n L$ is almost zero. Let $x \in L$. If $r \in \mathbb{N}$, then the image of $\varpi^{1/\ell^r} x$ in $L/\varpi^n L$ is almost zero for every $n \in \mathbb{N}$, which means that $\varpi^{1/\ell^r} x \in \bigcap_{n \geq 0} \varpi^n L$. As M is ϖ-adically separated, we have $\bigcap_{n\geq 0} \varpi^n L \subset \bigcap_{n \geq 0} \varpi^n M = 0$, so $\varpi^{1/\ell^r} x = 0$. This shows that L is almost zero.

□

V.1.6 Tilting and the adic spectrum

Let (A, A^+) be a perfectoid Huber pair. By proposition V.1.2.7, we have a map $(.)^\flat : \mathrm{Cont}(A) \to \mathrm{Cont}(A^\flat)$ sending a continuous valuation $|.|$ on A to the continuous valuation $a \longmapsto |a|^\flat = |a^\sharp|$ on $A^\flat$. By the formula for $A^{\flat +}$ in proposition V.1.2.5, this map sends $\mathrm{Spa}(A, A^+)$ to $\mathrm{Spa}(A, A^{\flat +})$.

Theorem V.1.6.1. *(See corollary 6.7(ii),(iii) of [22], theorem 3.12 of [23].) Let (A, A^+) be a perfectoid Huber pair. Then the map $X := \mathrm{Spa}(A, A^+) \to X^\flat := \mathrm{Spa}(A^\flat, A^{\flat +})$, $x \longmapsto x^\flat$ is a homeomorphism identifying rational domains of X and $X^\flat$, and, for every rational domain U of X, the Huber pair $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ is perfectoid with tilt $(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}^+_{X^\flat}(U^\flat))$.*

Moreover, for every $x \in X$, the completed residue field $(\kappa(x), \kappa(x)^+)$ is perfectoid, with tilt $(\kappa(x^\flat), \kappa(x^\flat)^+)$.

Applying theorem IV.1.1.5, we immediately get the following corollary.

Corollary V.1.6.2. *Let (A, A^+) be a perfectoid Huber pair, and let $X = \mathrm{Spa}(A, A^+)$. Then $\mathscr{O}_X$ is a sheaf, and, for every rational domain U of X and every $i \geq 1$, we have $\mathrm{H}^i(U, \mathscr{O}_X) = 0$.*

Proof of theorem V.1.6.1. We choose pseudo-uniformizers ϖ and $\varpi^\flat$ of A and $A^\flat$ such that $\varpi = (\varpi^\flat)^\sharp$ and that ϖ divides ℓ in A^+. We write f for the map $(.)^\flat : X \to X^\flat$.

(1) Let $t_1, \dots, t_n, s \in A^\flat$ such that $(t_1, \dots, t_n) = A^\flat$, and let $V = R\left(\frac{t_1,\dots,t_n}{s}\right)$. By lemma V.1.6.6, V does not change if we add a power of $\varpi^\flat$ to $t_1, \dots, t_n$, so we may assume that $t_n = (\varpi^\flat)^N$ for some $N \in \mathbb{N}$. Then $t_n^\sharp = \varpi^N$, so $(t_1^\sharp, \dots, t_n^\sharp) = A$, and we clearly have $f^{-1}(V) = R\left(\frac{t_1^\sharp,\dots,t_n^\sharp}{s^\sharp}\right)$. In particular, f is continuous.

(2) We show that f is surjective. Let $y \in X^\flat$, and let $(L, L^+) = (\kappa(y), \kappa(y)^+)$. Then L is a complete non-Archimedean field (because $X^\flat$ only has analytic points). By definition, L is the completion of $K(x) := \mathrm{Frac}(A^\flat/\mathrm{supp}(x))$. As $A^\flat$ is perfect and $\mathrm{supp}(x)$ is a prime ideal of $A^\flat$, the quotient $A^\flat/\mathrm{supp}(x)$ is a perfect ring, so $K(x)$ is also perfect. As $K(x)^0$ is a perfect ring of definition, we can apply lemma IV.1.3.9, which implies that L is perfect.

So, by proposition V.1.1.8, L is a perfectoid field. We get a morphism of perfectoid Huber pairs $(A^\flat, A^{\flat+}) \to (L, L^+)$. By corollary V.1.4.6, this untilts to a morphism of perfectoid Huber pairs $(A, A^+) \to (K, K^+)$, and K is a perfectoid field by proposition V.1.2.9. Let x be the image by $\mathrm{Spa}(K, K^+) \to \mathrm{Spa}(A, A^+)$ of the unique closed point of $\mathrm{Spa}(K, K^+)$. Then $|.|_x$ is the composition of $(A, A^+) \to (K, K^+)$ and of the rank 1 valuation $|.|$ giving the topology of K, so $|.|_{x^\flat}$ is the composition of $(A^\flat, A^{\flat+}) \to (L, L^+)$ and of $|.|^\flat$. But $|.|^\flat$ is a continuous rank 1 valuation on L, so, by corollary II.2.5.11, it has to be equivalent to the continuous rank 1 valuation defining the topology of L, which means that $x^\flat = y$.

(3) Let $f_1, \ldots, f_n, f_0 \in A$ such that $(f_1, \ldots, f_n) = A$, and let $U = R\left(\frac{f_1, \ldots, f_n}{f_0}\right)$. Again by lemma V.1.6.6, we may assume without changing U that $f_n = \varpi^N$ for some $N \in \mathbb{N}$. By corollary V.1.6.5 with ε some fixed number in $(0, 1)$, there exist elements $t_0, \ldots, t_{n-1}, s$ of $A^\flat$ such that, for every $i \in \{0, \ldots, n-1\}$ and every $x \in X$, we have

$$|f_i - t_i^\sharp|_x \leq |\varpi|_x^{1-\varepsilon} \max(|f_i|_x, |\varpi|_x^{N+1}).$$

In particular, we get

$$\max(|f_i|_x, |\varpi|_x^N) = \max(|t_i^\sharp|_x, |\varpi|_x^N),$$

and, if $|\varpi^N|_x \leq |f_0|_x$ (for example if $x \in U$) of if $|\varpi^N|_x \leq |t_0^\sharp|_x$, then $|f_0|_x = |t_0^\sharp|_x$. It is easy to deduce from these inequalities that $R\left(\frac{t_1^\sharp, \ldots, t_n^\sharp}{t_0^\sharp}\right) = U$, so that $U = f^{-1}(R\left(\frac{t_1, \ldots, t_n}{t_0}\right))$.

(4) Let $x, y \in X$ with $x \neq y$. As X is T_0, we may assume that there exists a rational domain U of X such that $x \in U$ and $y \notin U$. By (2), there exists a rational domain V of $X^\flat$ such that $U = f^{-1}(V)$, and then we have $f(x) \in V$ and $f(y) \notin V$, and in particular $f(x) \neq f(y)$. So f is injective.

(5) The statement about $\mathscr{O}_X(U)$ and its tilt (for $U \subset X$ a rational domain) follows immediately from corollary V.1.5.15. (Multiplying all the equations of a rational domain by the same power of ϖ doesn't chage the rational domain, so we may always assume that all these equations are in A^+.)

(6) Finally, the last statement was proved in (2).

$\square$

Remark V.1.6.3. In remark 9.2.8 of [1], Bhatt gave a proof that f is a homeomorphism that does not use teh approximation lemma (corollary V.1.6.5). We give it here for fun. Step (1) and (2) are as in the proof of the theorem above, and then the next steps are :

(3') As in (5) of the proof above, by corollary V.1.5.15, if $V \subset X^\flat$ is a rational domain and $U = f^{-1}(V)$, then $\mathscr{O}_X(U)$ is perfectoid and $(\mathscr{O}_{X^\flat}(V), \mathscr{O}_{X^\flat}^+(V))$ is the tilt of $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$. As, as f is surjective, we have $V = f(U)$.

(4') Let $f \in A^+$, and let $U = R\left(\frac{f, \varpi}{\varpi}\right)$. Choose any $g \in A^{\flat+}$ such that $g^\sharp = f$ modulo ϖA^+ (such a g always exists), and let $V = R\left(\frac{g, \varpi^\flat}{\varpi^\flat}\right)$. Then we have $U = f^{-1}(V)$. Indeed, if

$x \in X$, then $x \in U$ if and only if $|f|_x \leq |\varpi|_x$; by the strong triangle inequality, this is equivalent to $|g^\sharp|_x \leq |(\varpi^\flat)^\sharp|_x$, i.e. to $|g|_{f(x)} \leq |\varpi^\flat|_{f(x)}$, i.e. to $f(x) \in V$.

(5') Let $f \in A^+$. For $n \in \mathbb{Z}_{\geq 1}$, we set $U_n = R\left(\frac{f,\varpi^n}{\varpi^n}\right)$. We claim that, for every $n \geq 1$, there exists a rational subset V_n of $X^\flat$ such that $U_n = f^{-1}(V_n)$. We prove this by induction on n. If $n = 1$, this is (4'). Suppose the result known for some $n \geq 1$. Then, applying (3') to U_n shows that $V_n = \mathrm{Spa}(\mathscr{O}_X(U_n)^\flat, \mathscr{O}_X^+(U_n)^\flat)$. As U_{n+1} is the rational domain $R\left(\frac{f\varpi^{-n},\varpi}{\varpi}\right)$ of U_n, we can apply (4') to get a rational domain V_{n+1} of V_n such that $U_{n+1} = f^{-1}(V_{n+1})$. Finally, by corollary III.4.3.2, V_{n+1} is also a rational domain in $X^\flat$, so we have proved the claim for U_{n+1}.

(6') Let $f \in A^+$ and $g \in A^{\flat +}$ such that $f = g^\sharp$ modulo ϖA^+, and let $\varepsilon \in \mathbb{Z}[\frac{1}{\ell}] \cap \mathbb{R}_{>0}$. Then, if $U = R\left(\frac{\varpi^{1-\varepsilon}}{f}\right)$ and $V = R\left(\frac{(\varpi^\flat)^{1-\varepsilon}}{g}\right)$, we have $U = f^{-1}(V)$. Indeed, let $x \in X$. Then $x \in U$ if and only if $|\varpi|_x \leq |\varpi|_x^\varepsilon |f|_x$, and again the strong triangle inequality implies that this is equivalent to $|\varpi|_x \leq |\varpi|_x^\varepsilon |g^\sharp|_x$, i.e. to $f(x) \in V$.

(7') Let $f \in A^+$, $n \in \mathbb{Z}_{\geq 1}$ and $c \in \mathbb{Z}[\frac{1}{\ell}] \cap (0,1)$. We claim that the rational domain $U = R\left(\frac{\varpi^{nc}}{f}\right)$ is the preimage by f of a quasi-compact open subset of $X^\flat$. Indeed, write $U = \bigcap_{r \geq 1}^n U_r$, where

$$U_r = \{x \in X \mid |\varpi^{nc-(r-1)c}|_x \leq |f|_x \leq |\varpi^{nc-rc}|_x\} \subset U'_r = \{x \in X \mid ||f|_x \leq |\varpi^{nc-rc}|_x\}.$$

The subsets U_r and U'_r are rational domains of X, and, by (5'), there exists a rational domain V'_r of $X^\flat$ such that $U'_r = f^{-1}(V'_r)$. To show that U_r is the preimage of a rational domain of $X^\flat$, it suffices (thanks to (3')) to show that it is the preimage of a rational domain V_r of V'_r. Let $g = \frac{f}{\varpi^{nc-rc}}$; then $g \in \mathscr{O}_X(U'_r)$, and $U_r = \{x \in U'_r \mid |\varpi|_x^c \leq |g|_x\}$, so the existence of such a V_r follows from (6'). Finally, U is the preimage of the quasi-compact open subset $\bigcup_{r=1}^n V_r$ of $X^\flat$.

(8') Let $f, g \in A^+$ and $N \in \mathbb{N}$, and let $U = R\left(\frac{f,\varpi^N}{g}\right)$. We claim that U is the preimage of a quasi-compact open subset of $X^\flat$. Let $U' = R\left(\frac{\varpi^N}{g}\right)$. By (7'), there exists a quasi-compact open subset V' of $X^\flat$ such that $U' = f^{-1}(V')$. Write $V' = \bigcup_{i \in I} V'_i$, with I finite and the V'_i rational domains of $X^\flat$, and let $U'_i = f^{-1}(V'_i)$; note that the U'_i are rational domains of X by (1). Let $h = \frac{f\varpi^N}{g}$. Then $h \in \mathscr{O}_X(U')$, so we can take its image in each $\mathscr{O}_X(U'_i)$. Let U_i be the rational domain $R\left(\frac{h,\varpi^N}{\varpi^N}\right)$ in U'_i; by (5'), there exists a rational domain V_i of V'_i such that $U_i = f^{-1}(V_i)$. By corollary III.4.3.2, U_i (resp. V_i) is a rational domain in X (resp. $X^\flat$). Also, we clearly have $U = \bigcup_{i \in I} U_i$, so U is the preimage of the quasi-compact open subset $\bigcup_{i \in I} V_i$ of $X^\flat$.

(9') Let $f_1, \ldots, f_n, g \in A$ such that $(f_1, \ldots, f_n) = A$, and let $U = R\left(\frac{f_1,\ldots,f_n}{g}\right)$. After multiplying $f_1, \ldots, f_n, g$ by the same power of ϖ, we may assume (without changing U) that $f_1, \ldots, f_n, g \in A^+$. Also, by lemma V.1.6.6, we may assume that f_n is of the form ϖ^N.

Then $U = \bigcap_{i=1}^{n-1} U_i$, where $U_i = R\left(\frac{f_i, \varpi^N}{g}\right)$. By (8'), there exists quasi-compact open subsets $V_1, \dots, V_{n-1}$ of $X^\flat$ such that $U_i = f^{-1}(V_i)$ for $i \in \{1, \dots, n-1\}$. Then $V = \bigcap_{i=1}^{n-1} V_i$ is a quasi-compact open subset of $X^\flat$, and $U = f^{-1}(V)$.

(10') Let $x, y \in X$ with $x \neq y$. As X is T_0, we may assume that there exists a rational domain U of X such that $x \in U$ and $y \notin U$. By (2), there exists a quasi-compact open subset V of $X^\flat$ such that $U = f^{-1}(V)$, and then we have $f(x) \in V$ and $f(y) \notin V$, and in particular $f(x) \neq f(y)$. So f is injective.

The main technical ingredient is the following approximation lemma.

Proposition V.1.6.4. *(Lemma 6.5 of [22].) Let (K, K^+) be a perfectoid Huber pair. We do* not *assume that K is a field.* [1] *Let $(A, A^+) = (K\langle X_1^{1/\ell^\infty}, \dots, X_n^{1/\ell^n}\rangle, K^+\langle X_1^{1/\ell^\infty}, \dots, X_n^{1/\ell^n}\rangle)$, and let $f \in A^+$ be a homogeneous element of degree $d \in \mathbb{Z}[\frac{1}{\ell}]$. For every $c \in \mathbb{R}_{\geq 0}$ and every $\varepsilon \in \mathbb{R}_{>0}$, there exists $g_{c,\varepsilon} \in A^{+\flat} = K^{+\flat}\langle X_1^{1/\ell^\infty}, \dots, X_n^{1/\ell^\infty}\rangle$ (see corollary V.1.5.12) homogeneous of degree d such that, for every $x \in X = \mathrm{Spa}(A, A^+)$,*

$$|f - g_{c,\varepsilon}^\sharp|_x \leq |\varpi|_x^{1-\varepsilon} \max(|f|_x, |\varpi|_x^c).$$

Proof. Fix f and $\varepsilon \in \mathbb{Z}[\frac{1}{\ell}] \cap (0,1)$. We also fix $a \in \mathbb{Z}[\frac{1}{\ell}]$ such that $0 < a < \varepsilon$. If $c \leq c'$, then the result for c' implies the result for c. So it suffices to prove the result for $c = ar$, with $r \in \mathbb{N}$. We show by induction on r that, for every $r \in \mathbb{N}$, there exists $\varepsilon_r > 0$ and $g_r \in K^{+\flat}\langle X_1^{1/\ell^\infty}, \dots, X_n^{1/\ell^\infty}\rangle$ homogeneous of degree d such that, for every $x \in X$,

$$|f - g_r^\sharp|_x \leq |\varpi|_x^{1-\varepsilon+\varepsilon_r} \max(|f|_x, |\varpi|_x^{ar}).$$

(This implies the desired inequality, because $|\varpi|_x < 1$ for every $x \in X$.)

If $r = 0$, we take $\varepsilon_0 = 0$ and take for g_0 any element of $A^{\flat +}$ such that $g_0^\sharp = f$ modulo ϖA^+.

Suppose that $r \geq 0$ and that we have found ε_r and g_r. Decreasing ε_r only makes the inequality more true, so we may assume that $\varepsilon_r \in \mathbb{Z}[\frac{1}{\ell}]$ and $\varepsilon_r \leq \varepsilon - a$. Let $X^\flat = \mathrm{Spa}(A^\flat, A^{\flat +})$, and let $U_r^\flat \subset X^\flat$ be the rational domain

$$R\left(\frac{g_r, (\varpi^\flat)^{ra}}{(\varpi^\flat)^{ra}}\right) = \{x \in X^\flat \mid |g_r|_x \leq |\varpi^\flat|_x^{ra}\}.$$

Then $U_r = f^{-1}(U_r^\flat)$ is the rational domain $R\left(\frac{g_r^\sharp, \varpi^{ra}}{\varpi^{ra}}\right) = R\left(\frac{f, \varpi^{ra}}{\varpi^{ra}}\right)$ of X. Let $h = f - g_r^\sharp$. By the condition on g_r, we have $h \in \varpi^{ra+1-\varepsilon+\varepsilon_r}\mathscr{O}_X^+(U_r)$. By corollary V.1.5.15(iii), the subrings $\mathscr{O}_X(U_r)^\circ$ and $A^+\langle\left(\frac{g_r^\sharp}{\varpi^{ra}}\right)^{1/\ell^\infty}\rangle$ of $\mathscr{O}_X(U_r)$ are almost equal, so $\varpi^{-ra-1+\varepsilon-\varepsilon_r}h$ is almost an element of the second one, which means that we can find $\varepsilon_{r+1} \in \mathbb{Z}[\frac{1}{\ell}]$ such that $0 < \varepsilon_{r+1} < \varepsilon_r$ and that $\varpi^{-ra-1+\varepsilon-\varepsilon_{r+1}}h$ is in $A^+\langle\left(\frac{g_r^\sharp}{\varpi^{ra}}\right)^{1/\ell^\infty}\rangle$. As h and $g_r^\sharp$ are homogeneous of degree d, $\varpi^{-ra-1+\varepsilon-\varepsilon_{r+1}}h$ is in the ϖ-completion of $\bigoplus_{i \in \mathbb{Z}[\frac{1}{\ell}] \cap [0,1]} \left(\frac{g_r^\sharp}{\varpi^{ra}}\right)^i A_{d-di}^+$, where

[1] Lemma 6.5 of [22] makes this assumption, but the proof seems to work in general.

A^+_{d-di} is the set of elements of A^+ that are homogeneous of degree $d - di$. So we can write h as a convergent sum $\varpi^{ra+1-\varepsilon+\varepsilon_{r+1}} \sum_{i\in\mathbb{Z}[\frac{1}{\ell}]\cap[0,1]} \left(\frac{g_r^\sharp}{\varpi^{ra}}\right)^i h_i$, where $h_i \in A^+_{d-di}$ (this means that $I := \{i \in \mathbb{Z}[\frac{1}{\ell}] \cap [0,1] \mid h_i \neq 0\}$ is finite and countable, and that $h_{\varphi(n)} \to 0$ as $n \to 0$ for every bijection $\varphi : \mathbb{N} \xrightarrow{\sim} I$). For every i, choose $s_i \in A^{\flat+}_{d-di}$ such that $h_i - s_i^\sharp \in \varpi A^+$. We may choose the s_i such that the sum $s := \sum_{i\in\mathbb{Z}[\frac{1}{\ell}]\cap[0,1]} \left(\frac{g_r}{(\varpi^\flat)^{ra}}\right)^i s_i$ converges, and we set $g_{r+1} = g_r + (\varpi^\flat)^{ra+1-\varepsilon+\varepsilon_{r+1}} s$. We claim that, for every $x \in X$, we have

$$|f - g_{r+1}^\sharp|_x \leq |\varpi|_x^{1-\varepsilon+\varepsilon_{r+1}} \max(|f|_x, |\varpi|_x^{a(r+1)}).$$

Let $x \in X$. First assume that $|f|_x > |\varpi|_x^{ar}$. Then, by the induction hypothesis, we have $|g_r^\sharp|_x = |f_r|_x > |\varpi|_x^{ra}$, so it suffices to show that, for every $i \in \mathbb{Z}[\frac{1}{\ell}] \cap [0,1]$, we have

$$\left| \left((\varpi^\flat)^{ra+1-\varepsilon+\varepsilon_{r+1}} \left(\frac{g_r}{(\varpi^\flat)^{ra}}\right)^i s_i \right)^\sharp \right|_x \leq |\varpi|_x^{1-\varepsilon+\varepsilon_{r+1}} |f|_x.$$

As $|s_i^\sharp|_x \leq 1$, this follows from $|(g_r^i)^\sharp|_x \leq |f|_x$, which holds because $i \in [0,1]$. Now we assume that $|f|_x \leq |\varpi|_x^{ar}$, which implies that $|g_r^\sharp|_x \leq |\varpi|_x^{ar}$. We claim that then

$$|f - g_{r+1}^\sharp|_x \leq |\varpi|_x^{a(r+1)+1-\varepsilon+\varepsilon_{r+1}},$$

which clearly implies the result. As $ar+1 > a(r+1)+1-\varepsilon+\varepsilon_{r+1}$ (because $\varepsilon_{r+1} < \varepsilon_r < \varepsilon - a$), it is enough to show that $f - g_{r+1}^\sharp \in \varpi^{ar+1}\mathscr{O}_X^+(U_r)$. Note that

$$\frac{g_{r+1}}{(\varpi^\flat)^{ar}} = \frac{g_r}{(\varpi^\flat)^{ar}} + \sum_{i\in\mathbb{Z}[\frac{1}{\ell}]\cap[0,1]} (\varpi^\flat)^{1-\varepsilon+\varepsilon_{r+1}} \left(\frac{g_r}{(\varpi^\flat)^{ar}}\right)^i s_i,$$

and that all the terms of the sum in the right hand side are in $\mathscr{O}^+_{X^\flat}(U_r^\flat)$. So we get an equality

$$\frac{g_{r+1}^\sharp}{\varpi^{ar}} = \frac{g_r^\sharp}{\varpi^{ar}} + \sum_{i\in\mathbb{Z}[\frac{1}{\ell}]\cap[0,1]} \varpi^{1-\varepsilon+\varepsilon_{r+1}} \left(\frac{g_r^\sharp}{\varpi^{ar}}\right)^i h_i$$

in $\mathscr{O}_X^+(U_r)$ modulo ϖ. If we multiply this equality by ϖ^{ar}, we get $f - g_{r+1}^\sharp = f - g_r^\sharp - h$ modulo $\varpi^{1+ra}\mathscr{O}_X^+(U_r)$. By the choice of h, $f - g_r^\sharp - h \in \varpi^{1+ra}\mathscr{O}_X^+(U_r)$, so $f - g_{r+1}^\sharp \in \varpi^{1+ra}\mathscr{O}_X^+(U_r)$, and this implies the desired inequality.

□

Corollary V.1.6.5. *(See corollary 6.7(i) of [22].) Let (A, A^+) be a perfectoid Huber pair, and let $X = \mathrm{Spa}(A, A^+)$. Fix a pseudo-uniformizer $\varpi \in A^+$ of A such that $\varpi = (\varpi^\flat)^\sharp$ for $\varpi^\flat \in A^{\flat+}$. Let $c \in \mathbb{R}_{\geq 0}$, $\varepsilon > 0$ and $f \in A$. Then there exists $g_{c,\varepsilon} \in A^\flat$ such that, for every $x \in X$, we have*

$$|f - g_{c,\varepsilon}^\sharp|_x \leq |\varpi|_x^{1-\varepsilon} \max(|f|_x, |\varpi|_x^c).$$

Note that, if we take $\varepsilon < 1$, then the inequality of the proposition implies that, for every $x \in X$, we have

$$\max(|f|_x, |\varpi|_x^N) = \max(|g_{c,\varepsilon}^\sharp|_x, |\varpi|_x^N).$$

Proof. After increasing c (and multiplying f by a power of ϖ), we may assume that $c \in \mathbb{N}$ and $f \in A^+$. As $A^{\flat+}/\varpi^\flat A^{\flat+} = A^+/\varpi A^+$, we can find $g_0, \ldots, g_c \in A^{\flat+}$ and $f_{c+1} \in A^+$ such that $f = g_0^\sharp + \varpi g_1^\sharp + \ldots + \varpi^c g_c^\sharp + \varpi^{c+1} f_{c+1}$. It suffices to treat the case where $f_{c+1} = 0$. Consider the continuous ring morphism $\psi : B := A\langle T_0^{1/\ell^\infty}, \ldots, T_c^{1/\ell^\infty}\rangle \to A$ sending T_i^{1/ℓ^m} to $(g_i^{1/\ell^m})^\sharp$; this morphism sends B^+ to A^+. By assumption, we have $f = psi(f')$, where $f' = T_0 + \varpi T_1 + \ldots + \varpi^c T_c$. By proposition V.1.6.4, there exists $g' \in A^{+\flat}\langle T_0^{1/\ell^\infty}, \ldots, T_c^{1/\ell^\infty}\rangle$ homogeneous of degree 1 such that, for every $y \in \mathrm{Spa}(B, B^+)$, we have

$$|f' - (g')^\sharp|_y \leq |\varpi|_y^{1-\varepsilon} \max(|f'|_y, |\varpi|_y^c).$$

So, if $g = \psi^\flat(g')$, then $g^\sharp = \psi((g')^\sharp)$, and we have

$$|f - g_{c,\varepsilon}^\sharp|_x \leq |\varpi|_x^{1-\varepsilon} \max(|f|_x, |\varpi|_x^c)$$

for every $x \in \mathrm{Spa}(A, A^+)$.

□

Lemma V.1.6.6. *Let (A, A^+) be a Huber pair with A a Tate ring, let ϖ be a pseudo-uniformizer of A, and let $f_1, \ldots, f_n, g \in A$ such that $(f_1, \ldots, f_n) = A$. Then there exists $N \in \mathbb{N}$ such that*

$$R\left(\frac{f_1, \ldots, f_n}{g}\right) = R\left(\frac{f_1, \ldots, f_n, \varpi^N}{g}\right).$$

Proof. Write $\sum_{i=1}^n a_i f_i = 1$, with $a_1, \ldots, a_n \in A$. As A^+ is open in A, there exists $N \in \mathbb{N}$ such that $\varpi^N a_i \in A^+$ for every $i \in \{1, \ldots, n\}$. Then, if $x \in R\left(\frac{f_1, \ldots, f_n}{g}\right)$, we have

$$|\varpi^N|_x = |\sum_{i=1}^n \varpi^N a_i f_i|_x \leq \max_{1 \leq i \leq n} |f_i|_x \leq |g|_x,$$

hence $x \in R\left(\frac{f_1, \ldots, f_n, \varpi^N}{g}\right)$. The other inclusion is obvious.

□

V.2 Perfectoid spaces

Definition V.2.1. (Definition 6.15 of [22].) A *perfectoid space* is an adic space (definition III.6.5.3) that is locally isomorphic to $\mathrm{Spa}(A, A^+)$, for (A, A^+) a perfectoid Huber pair. We say that X is *affinoid perfectoid* if $X = \mathrm{Spa}(A, A^+)$ for (A, A^+) a perfectoid Huber pair. A morphism of perfectoid spaces is a morphism of adic spaces.

By theorem V.1.6.1, for every perfectoid Huber pair (A, A^+), the space $\mathrm{Spa}(A, A^+)$ is an affinoid perfectoid space (i.e. its structure presheaf is a sheaf). Theorem V.1.6.1 also has the following corollary.

Corollary V.2.2. *(Proposition 6.17 of [22].) Every perfectoid space X has a tilt $X^\flat$, which is a perfectoid space over $\mathbb{F}_\ell$ with an isomorphism of topological spaces $(.)^\flat : X \to X^\flat$, such that, for every affinoid perfectoid subspace U of X, if $U^\flat$ is the image of U in $X^\flat$, then the tilt of the pair $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ is $(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}_{X^\flat}^+(U^\flat))$.*

Moreover, tilting induces an equivalence between the categories of perfectoid spaces over X and over $X^\flat$.

Unlike general adic spaces, perfectoid spaces admit fiber products.

Proposition V.2.0.1. *(Proposition 6.18 of [22].) Let $X \to Z$ and $Y \to Z$ be two morphism of perfectoid spaces. Then the fiber product $X \times_Z Y$ exists in the category of adic spaces, and it is a perfectoid space.*

Proof. We may assume that X, Y and Z are perfectoid affinoid, so $X = \mathrm{Spa}(A, A^+)$, $Y = \mathrm{Spa}(B, B^+)$ and $Z = \mathrm{Spa}(C, C^+)$. As the maps $C \to A$ and $C \to B$ are automatically adic (because A, B and C are Tate rings, see proposition II.1.3.4), the completed tensor product $D := \widehat{A \otimes_C B}$ exists and is a Tate ring (see proposition II.3.2.1 and corollary II.3.1.9). We take for D^+ the completion of the integral closure of the image of $A^+ \otimes_{C^+} B^+$ in D. Then $\mathrm{Spa}(D, D^+)$ is the fiber product of X and Y over Z in the category of adic spaces, and it remains to prove that (D, D^+) is a perfectoid pair. See the proof of proposition 6.18 of [22] for details.

□

V.3 The almost purity theorem

V.3.1 Statement

Definition V.3.1.1. (Definition 7.1 of [22].)

(i) A morphism $(A, A^+) \to (B, B^+)$ of Huber pairs is called *finite étale* if B is a finite étale A-algebra and has the corresponding canonical topology (see proposition II.4.2.2) and if B^+ is the integral closure of A^+ in A.

(ii) A morphism $f : X \to Y$ of adic spaces is called *finite étale* if there is a cover of Y by open affinoid subsets V such that $f^{-1}(V)$ is affinoid and the morphism $(\mathscr{O}_Y(V), \mathscr{O}_Y^+(V)) \to (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ is finite étale.

(iii) A morphism $f : X \to Y$ of adic spaces is called *étale* if, for every $x \in X$, there exists open neighborhoods U and V of x and $f(x)$ and a commutative diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "f_{|U}"'] & W \arrow[d, "p"] \\ & V \end{tikzcd}$$

where j is an open embedding and p is finite étale.

Remark V.3.1.2. The definition of an étale morphism that we give here is not Huber's definition (Huber's definition is modelled on the definition for schemes, see definition 1.6.5 of [16]). However, it is equivalent to it for adic spaces that are locally of finite type over a non-archimedean field (for example for adic spaces coming from rigid analytic varieties), by lemma 2.2.8 of [16], and it also gives a reasonable notion for adic spaces that are locally adic spectra of perfectoid pairs.

The main result of this section is the following :

Theorem V.3.1.3. *(Theorem 7.9 of [22].) Let (A, A^+) be a perfectoid Huber pair, and let $X = \mathrm{Spa}(A, A^+)$.*

(i) The functor $Y \longmapsto \mathscr{O}_Y(Y)$ induces an equivalence between the category of finite étale maps of adic spaces $Y \to X$ and the category of finite étale maps of rings $A \to B$.

(ii) If $A \to B$ is a finite étale map of rings and B^+ is the integral closure of A in B, then B (with its canonical topology) is perfectoid and (B, B^+) is a perfectoid Huber pair.

(iii) Tilting induces an equivalence between the categories of finite étale Huber pairs over (A, A^+) and over $(A^\flat, A^{\flat+})$.

This theorem will be proved in section V.3.7. It has an immediate corollary for étale sites of perfectoid spaces.

Definition V.3.1.4. Let X be a perfectoid space. Then the *étale site* $X_{\text{ét}}$ of X is the category of étale morphisms of perfectoid spaces $Y \to X$, with the Grothendieck topology for which coverings are families $(u : Y_i \to Y)_{i \in I}$ such that $Y = \bigcup_{i \in I} u_i(Y_i)$.

Corollary V.3.1.5. *If X is a perfectoid space, then tilting induces an isomorphism of sites $X_{\text{ét}} \simeq X^\flat_{\text{ét}}$, and this isomorphism is functorial in X.*

If $(A, A^+) \to (B, B^+)$ is a finite étale map of perfectoid Huber pairs, we will need a way to describe the map of rings of integral elements $A^+ \to B^+$. This map is not étale, but we will see that it is almost étale, for the correct definition of "almost".

Definition V.3.1.6. (See sections 4.2 and 4.3 of [1].) Let (A, A^+) be a perfectoid Huber pair, let B^+ be a A^+-algebra, and let M be a B^+-module.

(i) We say that M is *almost of finite presentation* (or *almost finitely presented*) if, for every $a \in A^{00}$, there exists a finitely presented B^+-module M_a and a morphism of B^+-modules $M_a \to M$ whose kernel and cokernel are killed by a.

(ii) We say that M is *almost projective* if, for every B^+-module N and every integer $i \geq 1$, the B^+-module $\mathrm{Ext}^i_{B^+}(M, N)$ is almost zero.

(iii) Suppose that M is a B^+-algebra. We say that it is *almost finite étale* (over B^+) if it is almost of finite presentation, almost projective and if there exists $e \in (M \otimes_{B^+} M)_*$ such that $e^2 = e$, $\mu_*(e) = 1$ and $\ker(\mu)_* \cdot e = 0$, where $\mu : M \otimes_{B^+} M \to M$ is the multiplication map.

Remember that, by remark V.1.5.7, M_* is a $(B^+)_*$-algebra in the situation of (iii).

Remark V.3.1.7. (1) As in proposition V.1.5.4, we can require the conditions of definition V.3.1.6(i) only for fractional power of a well-chosen pseudo-uniformizer of A.

(2) We can define the abelian category of almost B^+-modules by taking the quotient of the category of B^+-modules by the Serre subcategory of almost zero B^+-modules, and many of the "almost" notions have a natural interpretation in this category. However, a B^+-module that is almost projective is in general *not* a projective object of the category of almost B^+-modules; for example, B^+ itself is almost projective, but it is not a projective almost B^+-modules. (The category of almost modules has tensor products and internal Homs defined in the obvious way, and almost projectivity can be defined in the usual way using the internal Hom functor.)

Remark V.3.1.8. The definition of almost finite étale maps is motivated by the following result in ordinary commutative algebra : Let $R \to S$ be a locally free map of rings (i.e. let S be a flat R-algebra that is finitely presented as a R-module). Then $R \to S$ is étale if and only if the map of rings $\mu : S \otimes_R S \to S$, $a \otimes b \longmapsto ab$ has a section, i.e. if and only there exists an idempotent $e \in S \otimes_R S$ such that $\mu(e) = 1$ and $(\mathrm{Ker}\,\mu)e = 0$.

Most of theorem V.3.1.3 will follow from the next result, which is slitghly more precise.

Theorem V.3.1.9. *Let $\varphi : (A, A^+) \to (B, B^+)$ be a morphism of Huber pairs, with (A, A^+) perfectoid. Choose a pseudo-uniformizer $\varpi \in A^+$ of A that is of the form $\varpi = (\varpi^\flat)^\sharp$, with $\varpi^\flat \in A^{flat+}$ a pseudo-uniformizer of $A^\flat$.*

(i) If φ is finite étale, then B is also perfectoid.

(ii) Suppose that B is perfectoid. Then the following conditions are equivalent :

(a) φ is finite étale;

(b) the A^+-algebra B^+ is almost finite étale;

(c) the $A^+/\varpi A^+$-algebra $B^+/\varpi B^+$ is almost finite étale.

This theorem will be proved in section V.3.7.

V.3.2 The trace map and the trace pairing

Let $R \to S$ be a finite locally free morphism of rings (i.e. S is a flat R-module of finite presentation, or equivalently a finitely generated projective R-module). For every R-linear endomorphism u of S, we can define its *trace* $\mathrm{Tr}(u) \in R$ in the following way : There exists an affine covering $(\mathrm{Spec}(R_i))_{i \in I}$ of $\mathrm{Spec}(R)$ such that $S \otimes_R R_i$ is a free R_i-module of finite rank for every $i \in I$. If $i \in I$, then u defines a R_i-linear endomorphism u_i of $S \otimes_R R_i$, so we can define $\mathrm{Tr}(u_i) \in R_i$, and these elements glue to a global section $\mathrm{Tr}(u)$ of $\mathscr{O}_{\mathrm{Spec}(R)}$.

If $a \in S$, left multiplication by a on S defines a R-linear endormophism on S, that we denote by m_a. We get a R-linear morphism $\mathrm{Tr}_{S/R} : S \to R$, $a \longmapsto \mathrm{Tr}(m_a)$.

Finally, the *trace pairing* is the R-linear morphism $\mathrm{Tr} : S \otimes_R S \to R$, $(a, b) \longmapsto \mathrm{Tr}_{S/R}(ab)$.

Remember the following "well-known" result.

Theorem V.3.2.1. *Let $R \to S$ be a finite locally free morphism of rings. Then the following conditions are equivalent :*

(i) The morphism $R \to S$ is étale.

(ii) The trace pairing $\mathrm{Tr} : S \otimes_R S \to R$ is nondegenerate, i.e. the R-linear morphism $S \to \mathrm{Hom}_R(S, R)$ that it defines by adjunction is an isomorphism (of R-modules).

(iii) There exists an idempotent $e \in S \otimes_R S$ such that $\mu(e) = 1$ and $(\mathrm{Ker}\,\mu)e = 0$, where $\mu : S \otimes_R S \to S$ is the multiplication. In other words, there exists an isomorphism of S-algebras $S \otimes_R S \simeq S \times S'$ (that does not preserve unit elements) such that $\mu : S \otimes_R S \to S$ corresponds to the first projection.

We will also need the following results.

Lemma V.3.2.2. *Let $R \to S$ be a finite locally free morphism of rings, and let R^+ be a subring of R that is integrally closed in R and such that $R = R^+[\frac{1}{\varpi}]$, for some $\varpi \in R^+$. Let S^+ be the integral closure of R^+ in S. Then $S = S^+[\frac{1}{\varpi}]$.*

Proof. Note that the rank of S as a R-module is a locally constant function on $\mathrm{Spec}(R)$. As $\mathrm{Spec}(R)$ is quasi-compact, we can write $\mathrm{Spec}(R)$ as a finite disjoint union of open and closed subschemes over which the rank of S is constant, and it suffices to prove the result over each of this subschemes. So we may assume that S has constant R-rank, say n.

Let $a \in S$, and let $f = \sum_{r=0}^{n} (-1)^r \mathrm{Tr}(\wedge^r m_a) T^r \in R[T]$ be the characteristic polynomial of m_a, where $\wedge^r m_a : \bigwedge_R^r S \to \bigwedge_R^r S$ is the rth exterior power of m_a. By the Cayley-Hamilton theorem, we have $f(a) = 0$, so $a \in S^+$ if $\mathrm{Tr}(\wedge^r m_a) \in R^+$ for $0 \leq r \leq n$. In general, as $R = R^+[\frac{1}{\varpi}]$, we may find $m \in \mathbb{N}$ such that $\varpi^{mr} \mathrm{Tr}(\wedge^r m_a) = \mathrm{Tr}(\wedge^r m_{\varpi^m a}) =\in R^+$ for every $r \in \{0, \ldots, n\}$, so $\varpi^m a \in S^+$, and we are done.

□

Lemma V.3.2.3. *Let $R \to S$ be a finite étale map of rings, and let $e \in S \otimes_R S$ be an idempotent as in theorem V.3.2.1(iii). Write $e = \sum_{i=1}^n a_i \otimes b_i$, with n a positive integer and $a_i, b_i \in S$. Then, for every $c \in S$, we have*

$$\sum_{i=1}^n a_i \mathrm{Tr}_{S/R}(cb_i) = \sum_{i=1}^n b_i \mathrm{Tr}_{S/R}(ca_i) = c.$$

Proof. Let $\mu : S \otimes_R S \to S$ be the multiplication, and let $S' = \mathrm{Ker}\,\mu$. As $(\mathrm{Ker}\,\mu)e = 0$, we have $(a \otimes 1)e = (1 \otimes a)e$ (in $S \otimes_R S$) for every $a \in S$. We get an isomorphism of S-algebras $S \times S' \xrightarrow{\sim} S \otimes_R S$ sending (a, b) to $(a \otimes 1)e + b$, whose inverse sends $x \in S \otimes_R S$ to $(\mu(x), x - (\mu(x) \otimes 1)e)$. Note that $\sum_{i=1}^n b_i \mathrm{Tr}_{S/R}(ca_i) = \mathrm{Tr}_{S \otimes_R S/S}((c \otimes 1)e)$, where we see $S \otimes_R S$ as a S-algebra using the map $b \longmapsto b \otimes 1$. As the trace is additive on direct products of finite étale S-algebras, we see that $\mathrm{Tr}_{S \otimes_R S/S}((c \otimes 1)e) = \mathrm{Tr}_{S/S}(c) + \mathrm{Tr}_{S'/S}(0) = c$. We prove the other equality in the same way (this time by using the map $S \to S \otimes_R S$, $b \longmapsto 1 \otimes b$).

□

V.3.3 From almost finite étale to finite étale

In this section, we fix a perfectoid Huber pair (A, A^+), and a pseudo-uniformizer $\varpi \in A^+$ of A that has a compatible system $(\varpi^{1/\ell^n})_{n \geq 0}$ of ℓth power roots and such that ϖ divides ℓ in A^+.

Proposition V.3.3.1. *We have a fully faithful functor from the category of almost finite étale A^+-algebras B_0 (with morphisms taken up to almost equality) to the category of finite étale (A, A^+) pairs (B, B^+), defined by sending B_0 to the pair $(B := B_0[\frac{1}{\varpi}], B^+)$, where B^+ is the integral closure of A^+ in B. Moreover, every pair (B, B^+) in the essential image of this functor is perfectoid.*

Proof. The functor is well-defined by lemma V.3.3.3. Suppose that B_0 is an almost finite étale A^+-algebra, with image (B, B^+). By lemma V.3.3.3 again, we have $(B_0)_* = B^0$, so we can almost recover B_0 from (B, B^+). This shows that the functor is fully faithful.

□

Lemma V.3.3.2. *If M is an almost finitely presented A^+-module, then $M[\frac{1}{\varpi}]$ is a finitely presented A-module. If moreover M is almost projective, then $M[\frac{1}{\varpi}]$ is a projective A-module, and M is almost ϖ-torsionfree (i.e. the submodule $M[\varpi^\infty]$ of ϖ-power torsion elements of M is almost zero).*

Proof. (1) We show that $M[\frac{1}{\varpi}]$ is finitely presented as a A-module. By assumption, there exists a finitely presented A^+-module N and a morphism of A^+-modules $N \to M$ whose kernel and cokernel are ϖ-torsion. So we get an isomorphism $N[\frac{1}{\varpi}] \simeq M[\frac{1}{\varpi}]$, and $N[\frac{1}{\varpi}]$ is a finitely presented A-module.

(2) We show that the A-module $M[\frac{1}{\varpi}]$ is projective if M is almost projective. (See the proof of lemma 2.4.15 of [11].)

Let $\varepsilon \in A^{00}$. As M is almost finitely presented, there exists $\varphi : (A^+)^n \to M$ such that $\varepsilon \operatorname{Coker}(\varphi) = 0$. Let $M' = \operatorname{Im}(\varphi)$, $\psi : A^n \to M'$ be the induced surjection and $j : M' \to M$ be the inclusion. Then we have $\varepsilon(M/M') = 0$, so there exists a A^+-linear map $\gamma : M \to M'$ such that $j \circ \gamma : M \to M$ is multiplication by ε. We have an exact sequence

$$\operatorname{Hom}_{A^+}(M, (A^+)^n) \xrightarrow{\psi^*} \operatorname{Hom}_{A^+}(M, M') \to \operatorname{Ext}^1_{A^+}(M, \operatorname{Ker}\psi).$$

As M is almost projective, the third term is almost zero, so ψ^* is almost surjective. Let $\delta \in A^{00}$. Then $\delta\gamma \in \operatorname{Hom}_{A^+}(M, M')$ is in the image of ψ, so there exists $u : M \to (A^+)^n$ such that $\delta\gamma = \psi \circ u$. In particular, composing by the inclusion $j : M' \to M$, we see that $(\delta\varepsilon)\mathrm{id}_M = \varphi \circ u$. If we take $\varepsilon = \varpi^{1/\ell}$ and $\delta = \varpi^{(\ell-1)/\ell}$, we see that we can find maps $v : (A^+)^n \to M$ and $u : M \to (A^+)^n$ such that $v \circ u = \varpi \mathrm{id}_M$. But then $\varpi^{-1}v[\frac{1}{\varpi}]$ is a section of $u[\frac{1}{\varpi}]$, so $M[\frac{1}{\varpi}]$ is a direct factor of A^n, hence it is a projective A-module.

(3) We show that M is almost ϖ-torsionfree if M is almost projective. Let $N = M[\varpi^\infty]$. Let $r \in \mathbb{N}$. In (2), we have shown that we can find maps $v : (A^+)^n \to M$ and $u : M \to (A^+)^n$ such that $v \circ u = \varpi^{1/\ell^r}\mathrm{id}_M$. If $x \in N$, then $u(x) = 0$ because A^+ is ϖ-torsionfree, so $\varpi^{1/\ell^r}x = v(u(x)) = 0$. As r was arbitrary, this shows that N is almost zero.

□

Lemma V.3.3.3. *If B_0 is an almost finite étale A^+-algebra, then $B := B_0[\frac{1}{\varpi}]$ is a finite étale A-algebra, it is perfectoid for its canonical topology, $(B_0)_* = B^0$, and the integral closure B^+ of A^+ in B is a ring of integral elements of B.*

Proof. We first reduce to the case where B_0 is ϖ-torsionfree. Let $J = B_0[\varpi^\infty]$ be the ideal of ϖ-power torsion elements of B_0. It suffices to show that J is almost zero. But this is the last statement of lemma V.3.3.2.

We already know that B is a finitely presented projective A-module by lemma V.3.3.2. We denote the multiplication map on B_0 by $\mu : B_0 \otimes_{A^+} B_0 \to B_0$. As B_0 is almost finite étale over A^+, there exists an idempotent $e \in (B_0 \otimes_{A^+} B_0)_*$ such that $\mu_*(e) = 1$ and $(\operatorname{Ker}\mu_*)e = 0$. Note that $B \otimes_A B = (B_0 \otimes_{A^+} B_0)[\frac{1}{\varpi}] = (B_0 \otimes_{A^+} B_0)_*[\frac{1}{\varpi}]$, and that the multiplication $\nu : B \otimes_A B \to B$ is equal to $\mu[\frac{1}{\varpi}]$ and to $\mu_*[\frac{1}{\varpi}]$. Let f be the image of e by the obvious map $(B_0 \otimes_{A^+} B_0)_* \to B \otimes_A B$. Then f is an idempotent, $\nu(f) = 1$ and $(\operatorname{Ker}\nu)f = 0$. In particular, the map $\nu : B \otimes_A B \to B$ has a section, so it is flat. This means that the map $A \to B$ is weakly étale, and we have already seen that it is of finite presentation. The fact that B is an étale A-algebra now follows from [25, Lemma 0CKP].

In particular, we can put the canonical topology on B (as a A-module), which makes B into a complete topological A-algebra. Let $u : M \to B_0$ be a A^+-module map with ϖ-torsion kernel and cokernel, and such that M is a finitely generated A^+-module. We may assume that M is

ϖ-torsionfree. Then u induces an isomorphism $M[\frac{1}{\varpi}] \xrightarrow{\sim} B$, and M with its ϖ-adic topology is an open subgroup of $M[\frac{1}{\varpi}]$. As $\varpi^{-1}B_0 \subset u(M) \subset B_0$, this shows that B_0 is open and bounded in B. Also, as $B_0 \subset \varpi^{-1}u(M)$, B_0 also has the ϖ-adic topology. So B is a complete Tate ring.

By lemma V.3.3.4, the map $a \longmapsto a$ is an almost isomorphism from $B_0/\varpi^{1/\ell}B_0$ to $B_0/\varpi B_0$. So we can apply proposition V.1.5.10 to conclude that B is perfectoid and that $B^0 = (B_0)_*$.

It remains to show that B^+ is an open subring of B (we already know that $B^+ \subset B^0$, since $B_0 \subset B^0$ and B_0 contains the image of A^+ in B). We have seen that there exists a finitely generated A^+-submodule M of B such that $M \subset B_0 \subset \varpi^{-1}M$. Let $b \in B_0$. Then $\varpi bM \subset M$, so ϖb is integral over A^+. This shows that $\varpi B_0 \subset B^+$, hence that B^+ is open.

□

Lemma V.3.3.4. *(Lemma 4.3.8 of [1].)* [2] *Let $R \to S$ be a weakly étale map of $\mathbb{F}_p$-algebras; this means that the maps $R \to S$ and $S \otimes_R S \to S$, $a \otimes b \longmapsto ab$ are both flat. Then the diagram*

$$\begin{tikzcd} R \arrow[r, "\mathrm{Frob}_R"] \arrow[d] & R \arrow[d] \\ S \arrow[r, "\mathrm{Frob}_S"'] & S \end{tikzcd}$$

(where Frob_R and Frob_S are the absolute Frobenius maps $a \longmapsto a^p$) is a pushout square of rings. In particular, if R is perfect, so is S.

V.3.4 The positive characteristic case

Proposition V.3.4.1. *(Proposition 4.3.4 of [1].) Let (A, A^+) be a perfectoid Huber pair of characteristic ℓ, and let $\varpi \in A^+$ be a pseudo-uniformizer of A. Let $\eta : A^+ \to B_0$ be an integral map with B_0 perfect. Suppose that the induced map $\eta[\frac{1}{\varpi}] : A \to B := B_0[\frac{1}{\varpi}]$ is finite étale. Then η is almost finite étale.*

Proof. Let $J = B_0[\varpi^\infty]$ be the ideal of ϖ-power torsion elements of B_0. We claim that J is almost zero. Indeed, let $b \in J$, and let $n \in \mathbb{N}$ such that $\varpi^n b = 0$. Then, for every $r \in \mathbb{N}$, we have $(\varpi^{n/\ell^r} b^{1/\ell^r})^{\ell^r} = 0$, hence $\varpi^{n/\ell^r} b^{1/\ell^r} = 0$ because B_0 is perfect, hence $\varpi^{n/\ell^r} b = 0$. By proposition V.1.5.4, this implies that J is almost zero. So replacing B_0 by B_0/J affects neither the hypothesis nor the conclusion, and we may assume that $J = 0$, i.e. that B'_0 is ϖ-torsionfree.

Now let B' be the integral closure of B_0 in B. We show that B' is almost equal to B_0, which will allow us to replace B_0 by B'. Let $f \in B'$. Then the B_0-module of B spanned by $f^{\mathbb{N}}$ is finitely generated, so there exists $r \in \mathbb{N}$ such that $\varpi^r f^n \in B_0$ for every $n \in \mathbb{N}$. As B_0 is perfect, this implies that $\varpi^{r/\ell^n} f \in B_0$ for every $n \in \mathbb{N}$, so $f \in (B_0)_*$ by proposition V.1.5.8(ii). So we

[2] Almostify this lemma.

have shown that $B_0 \subset B' \subset (B_0)_*$, and the fact that B_0 and B' are almost equal follows from proposition V.1.5.8(i).

By the previous two paragraphs, we may assume that B_0 is ϖ-torsionfree and integrally closed in B. As $A \to B$ is finite étale by assumption, we can find an idempotent $e \in B \otimes_A B = (B_0 \otimes_{A^+} B_0)[\frac{1}{\varpi}]$ such that $\mu(e) = 1$ and $(\operatorname{Ker}\mu)e = 0$, where $\mu : B \otimes_A B \to B$ is the multiplication map (see remark V.3.1.8). Choose $r \in \mathbb{N}$ such that $\varpi^r e \in B_0 \otimes_{A^+} B_0$. As e is idempotent, we have $e^{\ell^n} = e$ for every $n \in \mathbb{N}$, hence $e = e^{1/\ell^n}$ because $B \otimes_A B$ is perfect (by lemma V.3.3.4). As $B_0 \otimes_{A^+} B_0$ is also perfect (by the same lemma) and injects in $B \otimes_A B$, we see that $(\varpi^r e)^{1/\ell^n} = \varpi^{r/\ell^n} e \in B_0 \otimes_{A+} B_0$ for every $n \in \mathbb{N}$, so $e \in (B_0 \otimes_{A^+} B_0)_*$ by proposition V.1.5.8(ii).

It remains to show that B_0 is almost finitely presented and almost projective over A^+. Let $n \in \mathbb{N}$. We just proved that $\varpi^{1/\ell^n} e \in B_0 \otimes_{A^+} B_0$, so we can write $\varpi^{1/\ell^n} e = \sum_{i\in I} a_i \otimes b_i$, with I finite and $a_i, b_i \in B_0$. Consider the maps $\alpha : B \to A^I$ and $\beta : A^I \to B$ defined by $\alpha(b) = (\operatorname{Tr}_{B/A}(ba_i))_{i\in I}$ and $\beta((c_i)_{i\in I}) = \sum_{i\in I} c_i b_i$. We claim that $\beta\circ\alpha$ is equal to multiplication by ϖ^{1/ℓ^n}. Indeed, let $b \in B$. Then

$$\beta(\alpha(b)) = \sum_{i\in I} b_i \operatorname{Tr}_{B/A}(ba_i) = \varpi^{1/\ell^n} \operatorname{Tr}((b\otimes 1)e),$$

so the claim follows from lemma V.3.2.3.

Moreover, as B_0 is integrally closed in B (and A^+ is integrally closed in A), the map $\operatorname{Tr}_{B/A} : B \to A$ sends B_0 to A^+ as B_0, so α sends B_0 to $(A^+)^I$. It is clear that β sends $(A^+)^I$ to B_0. As B_0 and A^+ are ϖ-torsionfree, the restrictions to α and β define maps $\alpha_0 : B_0 \to (A^+)^I$ and $\beta_0 : (A^+)^I \to B_0$ such that $\beta_0\circ\alpha_0$ is equal to multiplication by ϖ^{1/ℓ^n}. Consider the sequence

$$(A^+)^I \overset{\gamma_0}{\to} (A^+)^I \overset{\beta_0}{\to} B_0,$$

where $\gamma_0 = \varpi^{1/\ell^m}\mathrm{id} - \alpha_0 \circ \beta_0$. Then $\beta_0 \circ \gamma_0 = 0$, so we get a map $\operatorname{Coker}(\gamma_0) \to B_0$ whose kernel and cokernel are killed by ϖ^{1/ℓ^m}.[3] As $\operatorname{Coker}(\gamma_0)$ is clearly finitely presented, this shows that B_0 is an almost finitely presented A^+-module. Let N be another A^+-module, and let $i \geq 1$ be an integer. Then multiplication by ϖ^{1/ℓ^m} on $\operatorname{Ext}^i_{A^+}(B_0, N)$ factors as $\operatorname{Ext}^i_{A^+}(B_0, N) \overset{\beta_0^*}{\to} \operatorname{Ext}^i_{A^+}((A^+)^I, N) \overset{\alpha_0^*}{\to} \operatorname{Ext}^i_{A^+}(B_0, N)$. As $\operatorname{Ext}^i_{A^+}((A^+)^I, N) = 0$, this shows that multiplication by ϖ^{1/ℓ^m} is 0 on $\operatorname{Ext}^i_{A^+}(B_0, N)$. Hence $\operatorname{Ext}^i_{A^+}(B_0, N)$ is almost zero.

□

Corollary V.3.4.2. *(Theorem 4.3.6 of [1].) Let (A, A^+) be a perfectoid Huber pair of characteristic ℓ, and let $\varpi \in A^+$ be a pseudo-uniformizer of A. Then the functor of proposition V.3.3.1 is an equivalence of categories.*

In other words, we have an equivalence of categories from the category of finite étale A-algebras to the category of almost finite étale A^+-algebras (where we identity two maps that are

[3] Check.

almost equal). This equivalence is given by sending a finite étale map $A \to B$ to the integral closure B^+ of A^+ in B, and by sending an almost finite étale A^+-algebra B_0 to the A-algebra $B_0[\frac{1}{\varpi}]$.

In particular, every finite étale A-algebra is perfectoid for the canonical topology.

Proof. Let $\mathscr{C}$ (resp. $\mathscr{C}'$) be the category of finite étale A-algebras (resp. almost finite étale A^+-algebras with almost equal maps identified). By lemma V.3.3.3, the formula $B_0 \longmapsto B_0[\frac{1}{\varpi}]$ define a functor from $\Psi : \mathscr{C}' \to \mathscr{C}$, and every A-algebra in this essential image of this functor is perfectoid.

Let B be a finite étale A-algebra, and let B^+ be the integral closure of A^+ in B. By lemma V.3.2.2, we have $B = B^+[\frac{1}{\varpi}]$. By proposition V.3.4.1, B^+ is an almost finite étale A^+-algebra. So sending B to B^+ defines a functor $\Phi : \mathscr{C} \to \mathscr{C}'$.

It remains to show that the functors Φ and Ψ are mutually quasi-inverse. If B is a finite étale A-algebra and $B^+ = \Phi(B)$, then we have already seen that $B = B^+[\frac{1}{\varpi}] = \Psi(B^+)$. Conversely, let B_0 be an almost finite étale A^+-algebra, let $B = B_0[\frac{1}{\varpi}]$, and let B^+ be the integral closure of A^+ in B. We claim that $(B_0)_* = B^+_*$, which will give a functorial isomorphism between B_0 and B^+ in $\mathscr{C}'$. But this is proved in lemma V.3.3.3.

□

V.3.5 Deforming almost finite étale extensions

The main result of this section is the following :

Theorem V.3.5.1. *(Theorem 5.3.27 of [11].) Let (A, A^+) be a perfectoid pair, and let $\varpi \in A^+$ be a pseudo-uniformizer of A. Then reduction modulo ϖ induces an equivalence of category from the category of almost finite étale A^+-algebras to the category of almost finite étale $A^+/\varpi A^+$-algebras.*

The following definition is temporary. Eventually, we will prove that it is equivalent to definition V.3.1.1(i).

Definition V.3.5.2. Let (A, A^+) be a perfectoid Huber pair. A pair (B, B^+) is called *strongly finite étale* if :

(i) B^+ is an almost finite étale A^+-algebra;

(ii) B^+ is the integral closure of A^+ in B.

Corollary V.3.5.3. *Let (A, A^+) be a perfectoid Huber pair. Any strongly finite étale pair over (A, A^+) is also perfectoid, and tilting induces an equivalence between the categories of strongly finite étale pairs over (A, A^+) and of finite étale pairs over $(A^\flat, A^{\flat+})$.*

Proof. Let $\varpi^\flat \in A^{\flat+}$ be a pseudo-uniformizer of $A^\flat$ such that $\varpi := (\varpi^\flat)^\sharp$ is a pseudo-uniformizer of A dividing ℓ in A^+.

By proposition V.3.3.1, every strongly finite étale pair over (A, A^+) is perfectoid, and the category of strongly finite étale pairs over (A, A^+) is equivalent to the category of almost finite étale A^+-algebras; we have a similar result for $(A^\flat, A^{\flat+})$. By corollary V.3.4.2, every finite étale pair over $(A^\flat, A^{\flat+})$ is strongly finite étale. Finally, by theorem V.3.5.1, the category of almost étale A^+-algebras (resp. $A^{\flat+}$-algebras) is equivalent to the category of almost finite étale $A^+/\varpi A^+$-algebras (resp. $A^{\flat+}/\varpi^\flat A^{\flat+}$-algebras). So we get an equivalence between the category of strongly finite étale pairs over (A, A^+) and the category of finite étale pairs over $(A^\flat, A^{\flat+})$, and this is the tilting equivalence by definition of tilting and by remark V.1.2.3.

□

V.3.6 The case of perfectoid fields

Proposition V.3.6.1. *(Proposition 3.2.10 of [1], proof due to Kedlaya.) Let K be a perfectoid field. If $K^\flat$ is algebraically closed, then so is K.*

Remember that we know that $K^\flat$ is a perfectoid field by proposition V.1.2.9.

Proof. We may assume that K has characteristic 0. Let $f(T) \in K^0[T]$ be a monic polynomial of degree $d \geq 1$. We want to find a root of $f(T)$ in K^0. We will construct by induction a sequence $(x_n)_{n\geq 0}$ of elements of K^0 such that, forr every $n \in \mathbb{N}$:

(a) $|f(x_n)| \leq |\ell|^n$;

(b) if $n \geq 1$, then $|x_n - x_{n-1}| \leq |\ell|^{(n-1)/d}$.

Condition (b) shows that $(x_n)_{n\geq 0}$ converges to some $x \in K^0$, and then (a) shows that $f(x) = 0$.

We set $x_0 = 0$. Suppose that $n \geq 0$ and that we have constructed $x_0, \dots, x_n$ satisfying (a) and (b). We want to construct x_{n+1} satisfying the same properties. Write $f(T + x_n) = \sum_{i=0}^d b_i T^i$, with $b_0, \dots, b_d \in K^0$. We have $b_d = 1$ by assumption. If $b_0 = 0$, then $f(x_n) = 0$, so we may take $x_m = x_n$ for every $m \geq n + 1$. From now on, we assume that $b_0 \neq 0$. Let

$$c = \min\{|\tfrac{b_0}{b_j}|^{\frac{1}{j}},\ 1 \leq j \leq d,\ b_j \neq 0\}.$$

As $b_d \neq 0$, we have $c \leq |b_0|^{\frac{1}{d}} \leq 1$. Let $|.|$ be the rank 1 valuation giving the topology of K. The value groups of $|K^\times|$ and $|K^\flat|_\flat$ are canonically isomorphic by construction of $|.|_\flat$. [4] As $K^\flat$ is algebraically closed, $|K^\times| \simeq |(K^\flat)^\times|_\flat$ is a $\mathbb{Q}$-vector space, so there exists $u \in K^\times$ such that $c = |u|$. As $c \leq 1$, we have $u \in K^0$. By definition of c, we have $\frac{b_i}{b_0} u^i \in K^0$ for every i, and there exists at least one $i \geq 1$ such that $\frac{b_i}{b_0} u^i \notin K^{00}$, i.e. such that $\frac{b_i}{b_0} u^i$ is a unit in K^0.

[4] Add a lemma ?

Choose $t \in K^{\flat 0}$ such that $|t^\sharp| = |\ell|$. It is easy to see that $K^{\flat 0}/tK^{\flat 0} \simeq K^0/\ell K^0$. [5] Consider any lift $g(T) \in K^{\flat 0}[T]$ of the image of the polynomial $\sum_{i=0}^d \frac{b_i}{b_0} u^i T^i$ in $(K^0/\ell K^0)[T] \simeq (K^{\flat 0}/tK^{\flat 0})[T]$. By lemma V.3.6.2, there exists a unit y in $K^{\flat 0}$ such that $g(y) = 0$.

We set $x_{n+1} = x_n + uy^\sharp$. We have to check that this x_{n+1} satisfies conditions (a) and (b). Firs, we have

$$f(x_{n+1}) = f(uy^\sharp + x_n) = \sum_{i=0}^d b_i u^i (y^i)^\sharp = b_0 \left(\sum_{i=0}^d \frac{b_i}{b_0} u^i (y^i)^\sharp \right).$$

As y is a root of $g(T)$, the sum between parentheses is equal to 0 modulo ℓ. So

$$|f(x_{n+1})| \leq |b_0||\ell| = |f(x_n)||\ell| \leq |\ell|^n |\ell| = |\ell|^{n+1}.$$

This proves (a). Moreover, as y is a unit (so $|y^\sharp| = 1$), we have

$$|x_{n+1} - x_n| = |uy^\sharp| = |u| = c \leq |b_0|^{\frac{1}{d}} = |f(x_n)|^{\frac{1}{d}} \leq |\ell|^{\frac{n}{d}}.$$

This proves (b).

□

Lemma V.3.6.2. *(Lemma 3.2.11 of [1].) Let K be a complete and algebraically closed non-archimedean field, and $R = K^0$. Let $f(T) \in R[T]$ be a polynomial of degree $e \geq 1$ such that the constant coefficient and at least one other coefficient of $f(T)$ are units in R. Then $f(T)$ has a root which is a unit of R.*

Proof. Let $\mathfrak{m} = K^{00}$ be the maximal ideal of R and $k = R/\mathfrak{m}$ be its residue field. By the hypothesis, the image of $f(T)$ in $k[T]$ is a polynomial of degree ≥ 1 whose constant coefficient is a unit, so we can a pseudo-uniformizer ϖ of K such that the image of $f(T)$ in $R/\varpi R[T]$ is a polynomial of degree ≥ 1 whose leading term and constant term are units in $R/\varpi R$. We also write $f : R[T] \to R[T]$ for multiplication by $f(T)$. Then the induced map $R/\varpi R[T] \to R/\varpi R[T]$ is finite free of degree ≥ 2, so, if $R\langle T\rangle$ is the ϖ-completion of $R[T]$ and $\widehat{f} : R\langle T\rangle \to R\langle T\rangle$ is the extesion of f by continuity, the map $\widehat{f}$ is also finite free of degree ≥ 2. In particular, the ring $S := R\langle T\rangle/(f(T))$ is a finite free R-algebra of dimension ≥ 2. As R is a henselian local ring, we can write $S \simeq \prod_{i \in I} S_i$, where I is finite and each S_i is a finite free local R-algebra. Reducing modulo $\mathfrak{m}$, we get $k[T]/(f(T)) \simeq \prod_{i \in I} S_i/\mathfrak{m}$, with the $S_i/\mathfrak{m}$ local. As the constant term of $f(T)$ modulo $\mathfrak{m}$ is nonzero, at least one of the roots of $f(T)$ modulo $\mathfrak{m}$ is a unit in k. So the map $k[T]/(f(T)) \overset{\sim}{\to} \prod_{i \in I} S_i/\mathfrak{m}$ sends T to a unit in at least one of the residue fields of the $S_i/\mathfrak{m}$; as the S_i are local, the map $R[T]/(f(T))$ sends T to a unit in at least one of the S_i, say S_{i_0}. As S_{i_0} is a finite free R-module and K is algebraically closed, the ring $S_{i_0,red}[\frac{1}{\varpi}]$ is isomorphic to a nonempty product of copies of K. Projecting on one of the copies gives a morphism $S_{i_0} \to K$. As S_{i_0} is integral over R and R is integrally closed in K, this map factors through a

[5] Add a lemma ?

map $S_{i_0} \to R$. So we have produced maps $R[T]/(f(T)) \to S_{i_0} \to R$; the first one sends T to a unit in S_{i_0}, so the composition sends T to a unit in R. The image of T by this composition is the desired root of $f(T)$.

□

Corollary V.3.6.3. *(See theorems 3.2.8 and 6.2.10 of [1].) Let K be a perfectoid field. Then :*

(i) Any finite separable field extension of K is perfectoid.

(ii) If L/K is a finite separable field extension, then L/K and $L^\flat/K^\flat$ have the same degree.

(iii) Tilting induces an equivalence between the categories of finite separable field extensions of K and of $K^\flat$.

Proof. Let K_{fet} (resp. $K^\flat_{fet}$) be the category of finite separable field extensions of K (resp. $K^\flat$). By corollary V.3.5.3, untilting induces a fully faithful functor $(.)^\sharp : K^\flat_{fet} \to K_{fet}$ that preserves degrees, and it suffices to show that this functor is essentially surjective. As $(.)^\sharp$ is fully faithful, it preserves automorphism groups, and so it preserves Galois extensions. By the main theorem of Galois theory, if $L/K^\flat$ is a finite Galois extension, then $(.)^\sharp$ induces a bijection between subextensions of $L/K^\sharp$ and of $L^\sharp/K$. So it suffices to show that every finite separable extension of K embeds in some $L^\sharp$, for $L/K^\flat$ a finite Galois extension.

Let $C = \widehat{\overline{K^\flat}}$ be the completion of an algebraic closure of $K^\flat$. This is an algebraically closed perfectoid extension of $K^\flat$, and its untilt $C^\sharp$ over K is a perfectoid extension of K, that is also algebraically closed by proposition V.3.6.2. As C is the filtered inductive limit in the category of perfectoid $K^\flat$-algebras of all the finite separable extensions L of $K^\flat$ contained in C. Let $C_0 \subset C^\sharp$ be the union of all the $L^\sharp$, for $L/K^\flat$ a finite separable extension contained in C. Then C_0 is clearly algebraic over K, and it is dense in $C^\sharp$. (By construction of the untilting functor.) By Krasner's lemma (see for example section 25.2 of [20]), C_0 is algebraically closed if and only if $C^\sharp$ is, so C_0 is algebraically closed. This implies that every finite separable field extension of K embeds into C_0, as desired.

□

V.3.7 Proof of theorems V.3.1.3 and V.3.1.9

V.3.7.1 Proof of theorem V.3.1.3 from theorem V.3.1.9

Suppose that theorem V.3.1.9 holds. This immediately implies (ii) and (iii) of theorem V.3.1.3.

We show (i). Let $X = \operatorname{Spa}(A, A^+)$, and let $Y \to X$ be a finite étale map of adic spaces. By definition, there is a finite cover $X = \bigcup_{i=1}^n \operatorname{Spa}(A_i, A_i^+)$ by rational domains such that, for every i, $Y \times_X \operatorname{Spa}(A_i, A_i^+) \simeq \operatorname{Spa}(B_i, B_i^+)$ with $(A_i, A_i^+) \to (B_i, B_i^+)$ a finite étale map of Huber pairs. By theorem V.3.1.9, for every i, the pair (B_i, B_i^+) is perfectoid and B_i^+ is almost finite étale over A^+. In particular, Y is a perfectoid space, and tilting the situation gives a finite étale map $Y^\flat \to X^\flat := \mathrm{Spa}(A^\flat, A^{\flat+})$. In other words, we may assume that A has characteristic ℓ. Then we finish as in the proof of proposition 7.6 of [22] : We can write (A, A^+) as the completion of an inductive limit of perfections $(A', A'^+)^{perf}$ of strongly Noetherian Huber pairs, the finite étale morphism $Y \to X$ descend to a finite étale morphism to Spa of one of these $(A', A'^+)^{perf}$ which comes from a finite étale morphism to $\mathrm{Spa}(A', A'^+)$ by the topological invariance of the étale site, this morphism comes from a finite étale A'-algebra by example 1.6.6(ii) of [16], and we pull this back to (A, A^+) using lemma 7.3(iv) of [22].

V.3.7.2 Proof of theorem V.3.1.9

We already know that (i) holds in characteristic ℓ by corollary V.3.4.2, and for perfectoid field by corollary V.3.6.3. In (ii), we know that (b) and (c) are equivalent by theorem V.3.5.1, and that (b) implies (a) by proposition V.3.3.1; we also know that (a) implies (b) in characteristic ℓ or if A is a field by corollaries V.3.4.2 and V.3.6.3.

Let B be a finite étale A-algebra, and let B^+ be the integral closure of A^+ in B. We still need to show that B is perfectoid and that B^+ is almost finite étale over A^+. Consider the morphism of adic spaces $f : Y := \mathrm{Spa}(B, B^+) \to X := \mathrm{Spa}(A, A^+)$. Let $x \in X$. Then the finite étale map $Y \times_X \mathrm{Spa}(\kappa(x), \kappa(x)^+)$ comes from a finite étale perfectoid pair over $(\kappa(x), \kappa(x)^+)$, which corresponds to a finite étale perfectoid pair over $(\kappa(x)^\flat, \kappa(x)^{\flat+}) = (\kappa(x^\flat), \kappa(x^\flat)^+)$ (see theorem V.1.6.1).

On the other hand, $(\kappa(x^\flat), \kappa(x^\flat)^+)$ is the completion of $\varinjlim_{U^\flat \ni x^\flat} (\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}^+_{X^\flat}(U^\flat))$, where we take the limit over rational domains $U^\flat$ of $X^\flat$ such that $x^\flat \in U^\flat$, by proposition III.6.3.1 and proposition III.6.3.7(i). As taking the category of finite étale covers commutes with filtered colimits and with completions (more precisely, see corollary 10.0.5 of [1]), the finite étale pair over $(\kappa(x^\flat), \kappa(x^\flat)^+)$ from the previous paragraph extends over a neighborhood of $x^\flat$. In other words (and using theorem V.1.6.1), we can find a rational domain V of X containing x such that the finite étale $\mathscr{O}_X(V)$-algebra $B_V := B \otimes_A \mathscr{O}_X(V)$ is perfectoid, and that the integral closure of $\mathscr{O}^+_X(V)$ in B_V is almost finite étale over $\mathscr{O}^+_X(V)$.

As X is quasi-compact, this gives a finite rational convering $(V_i)_{i\in I}$ of X such that each V_i satisfies the two properties above. In othert words, $U_i := Y \times_X V_i$ is of the form $\mathrm{Spa}(B_i, B_i^+)$ with $S_i := B \otimes_A \mathscr{O}_X(V_i)$ a finite étale perfectoid $\mathscr{O}_X(V_i)$-algebra and B_i^+ almost finite étale over $\mathscr{O}^+_X(V_i)$. For all $i, j \in I$, we have $U_i \times_{V_j} (V_i \cap V_j) \simeq U_j \times_{V_i} (V_i \cap V_j)$, because both sides are isomorphic to the unique finite étale cover of $V_i \cap V_j$ corresponding to the finite étale $\mathscr{O}_X(V_i \cap V_j)$-algebra $B \otimes_A \mathscr{O}_X(V_i \cap V_j)$. So we can glue the U_i to get a perfectoid space Y and a map $Y \to X$ that is locally of the form $\mathrm{Spa}(B', B'^+) \to \mathrm{Spa}(A', A'^+)$, for $(A', A'^+) \to (B', B'^+)$ a strongly finite étale map of Huber pairs. As in the proof of (i) of theorem V.3.1.3 in the previous subsection, we deduce that Y is of the form $\mathrm{Spa}(C, C^+)$, for $(A, A^+) \to (C, C^+)$ strongly finite étale. It remains to show that the A-algebras B and C are isomorphic. This follows from the fact that they define isomorphic coherent $\mathscr{O}_X$-modules, which is clear from the definition of Y.

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