MAT 540 : Problem Set 9

Due Thursday, November 21

1 Abelian subcatgeories of triangulated categories

Let $\mathscr{D}$ be a triangulated category. We denote the shift functors by $X \longmapsto X[1]$, and we write triangles as $X \to Y \to Z \to X[1]$ or $X \to Y \to Z \overset{+1}{\to}$. For every $X, Y \in \mathrm{Ob}(\mathscr{D})$ and every $n \in \mathbb{Z}$, we write $\mathrm{Hom}^n_{\mathscr{D}}(X, Y) = \mathrm{Hom}(X, Y[n])$.

(a). Let $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} X[1]$ and $X' \overset{u'}{\to} Y' \overset{v'}{\to} Z' \overset{w'}{\to} X'[1]$ be two distinguished triangles of $\mathscr{D}$, and let $g : Y \to Y'$ be a morphism.

(i) (2 points) Show that the following conditions are equivalent:

(1) $v' \circ g \circ u = 0$;

(2) there exists $f : X \to X'$ such that $u' \circ f = g \circ u$;

(3) there exists $h : Z \to Z'$ such that $h \circ v = v' \circ g$;

(4) there exist $f : X \to X'$ and $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles.

(ii) (1 point) Suppose that the conditions (i) hold and that $\mathrm{Hom}^{-1}_{\mathscr{D}}(X, Z') = 0$. Show that the morphisms f and h of (i)(2) and (i)(3) are unique.

(b). Let $\mathscr{C}$ be a full subcategory of $\mathscr{D}$, and suppose that $\mathrm{Hom}^n(X, Y) = 0$ if $X, Y \in \mathrm{Ob}(\mathscr{C})$ and $n < 0$.

(i) (2 points) Let $f : X \to Y$ be a morphism of $\mathscr{C}$. Take a distinguished triangle $X \overset{f}{\to} Y \to S \overset{+1}{\to}$ in $\mathscr{D}$, and suppose that we have a distinguished triangle $N[1] \to S \to C \overset{+1}{\to}$ with $N, C \in \mathrm{Ob}(\mathscr{C})$. In particular, we get morphisms $\alpha : N[1] \to S \to X[1]$ and $\beta : Y \to S \to X$.

Show that $\alpha[-1] : N \to X$ is a kernel of f and that $\beta : Y \to C$ is a cokernel of f.

We say that a morphism f of $\mathscr{C}$ is *admissible* if there exist distinguished triangles satisfying the conditions of (i). We say that a sequence of morphisms of $\mathscr{C}$ $0 \to X \overset{f}{\to} Y \overset{g}{\to} Z \to 0$ is an *admissible short exact sequence* if there exists a distinguished triangle $X \overset{f}{\to} Y \overset{g}{\to} Z \overset{+1}{\to}$ in $\mathscr{D}$.

(ii) (2 points) Suppose that $\mathscr{C}$ as a zero object. If $X \overset{f}{\to} Y \overset{g}{\to} Z \overset{+1}{\to}$ is a distinguished triangle in $\mathscr{D}$ with $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, show that f and g are admissible, that f is a kernel of g and that g is a cokernel of f.

(iii) (2 points) If $f : X \to Y$ is an admissible monomorphism (resp. epimorphism) in $\mathscr{C}$

and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ is a distinguished triangle in $\mathscr{D}$, show that Z (resp. $Z[-1]$) is isomorphic to an object of $\mathscr{C}$ and $Z = \mathrm{Coker}(f)$ (resp. $Z = \mathrm{Ker}(f)$).

(iv) (4 points) Suppose that every morphism of $\mathscr{C}$ is admissible and $\mathscr{C}$ is an additive subcategory of $\mathscr{D}$. Show that $\mathscr{C}$ is an abelian category and that every short exact sequence in $\mathscr{C}$ is admissible.

(v) (3 points) Suppose that $\mathscr{C}$ is an abelian category and that every short exact sequence in $\mathscr{C}$ is admissible. Show that every morphism of $\mathscr{C}$ is admissible.

2 t-structures

We use the convention of problem 1. A *t-structure* on $\mathscr{D}$ is the date of two full subcategories $\mathscr{D}^{\leq 0}$ and $\mathscr{D}^{\geq 0}$ such that (with the convention that $\mathscr{D}^{\leq n} = \mathscr{D}^{\leq 0}[-n]$ and $\mathscr{D}^{\geq n} = \mathscr{D}^{\geq 0}[-n]$);

(0) If $X \in \mathrm{Ob}(\mathscr{D})$ is isomorphic to an object of $\mathscr{D}^{\leq 0}$ (resp. $\mathscr{D}^{\geq 0}$), then X is in $\mathscr{D}^{\leq 0}$ (resp. $\mathscr{D}^{\geq 0}$).

(1) For every $X \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$ and every $Y \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$, we have $\mathrm{Hom}(X, Y) = 0$.

(2) We have $\mathscr{D}^{\leq 0} \subset \mathscr{D}^{\leq 1}$ and $\mathscr{D}^{\geq 0} \supset \mathscr{D}^{\geq 1}$.

(3) For every $X \in \mathrm{Ob}(\mathscr{D})$, there exists a distinguished triangle $A \to X \to B \xrightarrow{+1}$ with $A \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$ and $B \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$.

We fix a t-structure $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ on $\mathscr{D}$.

(a). (1 point) Show that the distinguished triangle of condition (3) is unique up to unique isomorphism.

(b). (3 points) For every $n \in \mathbb{Z}$, show that the inclusion functor $\mathscr{D}^{\leq n} \subset \mathscr{D}$ has a right adjoint $\tau^{\leq n}$ and the inclusion functor $\mathscr{D}^{\geq n} \subset \mathscr{D}$ has a left adjoint $\tau^{\geq n}$. (Hint: It suffice to treat the case $n = 0$.)

(c). (2 points) For every $n \in \mathbb{Z}$, show that there is a unique morphism $\delta : \tau^{\geq n+1}X \to (\tau^{\leq n}X)[1]$ such that the triangle $\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \xrightarrow{\delta} (\tau^{\leq n}X)[1]$, where the other two morphisms are given by the counit and unit of the adjunctions of (b).

(d). (3 points) Let $a, b \in \mathbb{Z}$ such that $a \leq b$, and let $X \in \mathrm{Ob}(\mathscr{D})$. Show that there exists a unique morphism $\alpha : \tau^{\geq a}\tau^{\leq b}X \to \tau^{\leq b}\tau^{\geq a}X$ such that the following diagram commutes:

$$\begin{tikzcd}
\tau^{\leq b}X \arrow[r] \arrow[d] & X \arrow[r] & \tau^{\geq a}X \\
\tau^{\geq a}\tau^{\leq b}X \arrow[rr, "\alpha"'] & & \tau^{\leq b}\tau^{\geq a}X \arrow[u]
\end{tikzcd}$$

(where all the other morphisms are counit or unit morphisms of the adjunctions of (b)), and that α is an isomorphism. (Hint: Apply the octahedral axiom to $\tau_{\leq a}X \xrightarrow{f} \tau_{\leq b}X \xrightarrow{g} X$.)

(e). (1 points) If $a, b \in \mathbb{Z}$ are such that $a \leq b$, show that, for every $X \in \mathrm{Ob}(\mathscr{D})$, we have $\tau^{\geq a}\tau^{\leq b}X \in \mathrm{Ob}(\mathscr{D}^{\leq a}) \cap \mathrm{Ob}(\mathscr{D}^{\geq b})$.

Let $\mathscr{C} = \mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq 0}$; that is, $\mathscr{C}$ is the full subcategory of $\mathscr{D}$ such that $\mathrm{Ob}(\mathscr{C}) = \mathrm{Ob}(\mathscr{D}^{\leq 0}) \cap \mathrm{Ob}(\mathscr{D}^{\geq 0})$. We denote the functor $\tau^{\leq 0}\tau^{\geq 0} : \mathscr{D} \to \mathscr{C}$ by H^0. The category $\mathscr{C}$ is called the *heart* or *core* of the t-structure.

(f). (1 point) Show that $\mathscr{C}$ is an abelian category.

(g). (2 points) Show that, if $X \to Y \to Z \overset{+1}{\to}$ is a distinguished triangle in $\mathscr{D}$ such that $X, Z \in \mathrm{Ob}(\mathscr{C})$, then Y is also in $\mathscr{C}$.

(h). The goal of this question is to show that the functor $\mathrm{H}^0 : \mathscr{D} \to \mathscr{C}$ is a cohomological functor. Let $X \to Y \to Z \overset{+1}{\to}$ be a distinguished triangle in $\mathscr{D}$.

(i) (2 points) If $X, Y, Z \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$, show that the sequence $\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z) \to 0$ is exact in $\mathscr{C}$. (Hint: A sequence of morphisms $A \to B \to C \to 0$ in an abelian category $\mathscr{A}$ is exact if and only if, for every object D of $\mathscr{A}$, the sequence of abelian groups $\mathrm{Hom}_{\mathscr{A}}(D, A) \to \mathrm{Hom}_{\mathscr{A}}(D, B) \to \mathrm{Hom}_{\mathscr{A}}(D, C) \to 0$ is exact.)

(ii) (2 points) If $X \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$, show that the sequence $\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z) \to 0$ is exact in $\mathscr{C}$. (Hint: Construct a distinguished triangle $X \to \tau^{\leq 0} Y \to \tau^{\leq 0} Z \overset{+1}{\to}$.)

(iii) (1 point) If $Z \in \mathrm{Ob}(\mathscr{D}^{\geq 0})$, show that the sequence $0 \to \mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z)$ is exact in $\mathscr{C}$.

(iv) (2 points) In general, show that the sequence $\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z)$ is exact in $\mathscr{C}$.

3 The canonical t-structure

Let $\mathscr{A}$ be an abelian category.

(a). (2 points) Let $n \in \mathbb{Z}$. If $X \in \mathrm{Ob}(\mathrm{D}^{\leq n}(\mathscr{A}))$ and $Y \in \mathrm{Ob}(\mathrm{D}^{\geq n+1}(\mathscr{A}))$, show that $\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, Y) = 0$.

(b). (3 points) Show that $(\mathscr{D}^{\leq 0}(\mathscr{A}), \mathscr{D}^{\geq 0}(\mathscr{A}))$ is a t-structure on $\mathrm{D}(\mathscr{A})$, that its heart is equivalent to $\mathscr{A}$, and that the associated functor $\mathrm{H}^0 : \mathscr{D}(\mathscr{A}) \to \mathscr{A}$ is the 0th cohomology functor.

4 Torsion

Let $\mathscr{D} = \mathscr{D}(\mathbf{Ab})$, and let

$$^*\mathrm{D}^{\leq 0} = \{X \in \mathscr{D} \mid \mathrm{H}^i(X) = 0 \text{ for } i > 1, \text{ and } \mathrm{H}^1(X) \text{ is torsion}\}$$

and

$$^*\mathrm{D}^{\geq 0} = \{X \in \mathscr{D} \mid \mathrm{H}^i(X) = 0 \text{ for } i < 0, \text{ and } \mathrm{H}^0(X) \text{ is torsionfree}\}.$$

Let $\mathscr{C} = {}^*\mathrm{D}^{\leq 0} \cap {}^*\mathrm{D}^{\geq 0}$.

(a). Show that $({}^*\mathrm{D}^{\leq 0}, {}^*\mathrm{D}^{\geq 0})$ is a t-structure on $\mathscr{D}$.(2 points for condition (1), 1 for condition (2) and 2 for condition (3))

(b). Let $f : A \to B$ be a morphism of torsionfree abelian groups. We can see A and B as objects of $\mathscr{C}$ (concentrated in degree 0), and then f is also a morphism of $\mathscr{C}$.

(i) (2 points) Show that f is a monomorphism in $\mathscr{C}$ if and only if f is injective (and $\mathbf{Ab}$) and $B/f(A)$ is torsionfree.

(ii) (1 point) Show that f is an epimorphism in $\mathscr{C}$ if and only if $f \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective.

(iii) (3 points) Calculate the kernel, the cokernel and the image of f in $\mathscr{C}$.

(c). (1 points) For every $n \geq 1$, show that $\mathrm{Ext}^1_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$.

(d). (1 point) If A and B are finitely generated abelian groups, show that $\mathrm{Ext}^n_{\mathbf{Ab}}(A,B)=0$ for every $n\geq 2$. [1]

(e). (2 points) Let $X\in\mathrm{Ob}(\mathscr{C})$. Suppose that $\mathrm{H}^i(X)$ is a finitely generated abelian groups for every $i\in\mathbb{Z}$. If $\mathrm{Hom}_{\mathscr{D}}(X,\mathbb{Z})=0$, show that $X=0$.

(f). (1 point) Give an example of a $X\in\mathrm{Ob}(\mathscr{C})$ nonzero such that $\mathrm{Hom}_{\mathscr{D}}(X,\mathbb{Z})=0$.

(g). (2 points) Let $X\in\mathrm{Ob}(\mathscr{D})$. If $X\in\mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$ (resp $X\in\mathrm{Ob}({}^*\mathrm{D}^{\geq 0})$), show that $R\,\mathrm{Hom}_{\mathbf{Ab}}(X,\mathbb{Z})$ is in $\mathrm{D}^{\geq 0}(\mathbf{Ab})$ (resp. $\mathrm{D}^{\leq 0}(\mathbf{Ab})$).

(h). (3 points) Let X be a complex of finitely generated abelian groups. If $R\,\mathrm{Hom}_{\mathbf{Ab}}(X,\mathbb{Z})$ is in $\mathrm{D}^{\geq 0}(\mathbf{Ab})$ (resp. $\mathrm{D}^{\leq 0}(\mathbf{Ab})$), show that $X\in\mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$ (resp $X\in\mathrm{Ob}({}^*\mathrm{D}^{\geq 0})$).

5 Weights

Let $\mathscr{A}$ be an abelian category. Suppose that we have a family $(\mathscr{A}_n)_{n\in\mathbb{Z}}$ of full abelian subcategories of $\mathscr{A}$ such that:

(1) If $n\neq m$, then $\mathrm{Hom}_{\mathscr{A}}(A,B)=0$ for any $A\in\mathrm{Ob}(\mathscr{A}_n)$ and $B\in\mathrm{Ob}(\mathscr{A}_m)$.

(2) Any object A of $\mathscr{A}$ has a *weight filtration*, that is, an increasing filtration $\mathrm{Fil}_\bullet A$ such that $\mathrm{Fil}_n A=0$ for $n<<0$, $\mathrm{Fil}_n A=A$ for $n>>0$ and $\mathrm{Fil}_n A/\mathrm{Fil}_{n+1}A\in\mathrm{Ob}(\mathscr{A}_n)$ for every $n\in\mathbb{Z}$.

For every $n\in\mathbb{Z}$, we denote by $\mathscr{A}_{\leq n}$ (resp. $\mathscr{A}_{\geq n}$) the full subcategory of $\mathscr{A}$ whose objects are the $A\in\mathrm{Ob}(\mathscr{A})$ having a weight filtration $\mathrm{Fil}_\bullet A$ such that $\mathrm{Fil}_n A=A$ (resp. $\mathrm{Fil}_n A=0$).

(a). (1 point) If $A\in\mathrm{Ob}(\mathscr{A}_{\leq n})$ and $B\in\mathrm{Ob}(\mathscr{A}_{\geq n+1})$, show that $\mathrm{Hom}_{\mathscr{A}}(A,B)=0$.

(b). (2 points) Show that the inclusion functor $\mathscr{A}_{\leq n}\subset\mathscr{A}$ has a right adjoint ${}^w\tau^{\leq n}$, and that the inclusion functor $\mathscr{A}_{\geq n}\subset\mathscr{A}$ has a left adjoint ${}^w\tau^{\geq n}$.

(c). (2 points) If $A\in\mathrm{Ob}(\mathscr{A}_{\leq n})$ and $B\in\mathrm{Ob}(\mathscr{A}_{\geq n+1})$, show that $\mathrm{Ext}^i_{\mathscr{A}}(B,A)=0$ for every $i\in\mathbb{Z}$.

(d). (4 points) Define two full subcategories ${}^w\mathrm{D}^{\leq n}$ and ${}^w\mathrm{D}^{\geq n}$ of $\mathrm{D}^b(\mathscr{A})$ by:

$$\mathrm{Ob}({}^w\mathrm{D}^{\leq n})=\{X\in\mathrm{Ob}(\mathrm{D}^b(\mathscr{A}))\mid\forall i\in\mathbb{Z},\ \mathrm{H}^i(X)\in\mathscr{A}_{\leq n}\}$$

and

$$\mathrm{Ob}({}^w\mathrm{D}^{\geq n+1})=\{X\in\mathrm{Ob}(\mathrm{D}^b(\mathscr{A}))\mid\forall i\in\mathbb{Z},\ \mathrm{H}^i(X)\in\mathscr{A}_{\geq n+1}\}.$$

Show that $({}^w\mathrm{D}^{\leq n},{}^w\mathrm{D}^{\geq n+1})$ is a t-structure on $\mathrm{D}^b(\mathscr{A})$, and that the heart of this t-structure is $\{0\}$.

[1] This actually holds for any abelian groups.