MAT 540 : Problem Set 8

Due Thursday, November 14

1 Right multiplicative systems

Let $\mathscr{C}$ be a category and W be a set of morphisms of $\mathscr{C}$. Let $\mathscr{I}$ be a full subcategory of $\mathscr{C}$ and $W_{\mathscr{I}}$ be the set of morphisms of $\mathscr{I}$ that are in W. Suppose that W is a right multiplicative system and that, for every $s : X \to Y$ in W such that $X \in \mathrm{Ob}(\mathscr{I})$, there exists a morphism $f : Y \to Z$ with $Z \in \mathrm{Ob}(\mathscr{I})$ and $f \circ s \in W$.

Show that $W_{\mathscr{I}}$ is a right multiplicative system. (1 point for (S1)+(S2), 1 point each for (S3) and (S4))

Solution. Conditions (S1) and (S2) of Definition V.2.2.1 of the notes are clear. We check condition (S3). Let $f : X \to Y$ and $s : X \to X'$ be morphisms of $\mathscr{I}$ such that $s \in W$. Then there exist a morphism $g : X' \to Y'$ in $\mathscr{C}$ and a morphism $t : Y \to Y'$ in W such that $t \circ f = g \circ s$. Moreover, by the hypotheses of the proposition, there exists $h : Y' \to Y''$, with $Y'' \in \mathrm{Ob}(\mathscr{I})$, such that $h \circ t \in W$. As $\mathscr{I}$ is a full subcategory of $\mathscr{C}$, we get a commutative diagram in $\mathscr{I}$:

$$\begin{array}{ccc} X' & \xrightarrow{h \circ g} & Y'' \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle h \circ t} \\ X & \xrightarrow[f]{} & Y \end{array}$$

We now check condition (S4). Let $f, g : X \to Y$ be two morphisms of $\mathscr{I}$, and let $s : X' \to X$ be a morphism of $W_{\mathscr{I}}$ such that $f \circ s = g \circ s$. As W is a right multuplicative system, there exists $t : Y \to Y'$ in W such that $t \circ f = t \circ g$. Take $h : Y' \to Y''$ such that $Y'' \in \mathrm{Ob}(\mathscr{I})$ and $h \circ t \in W$. Then $h \circ t \in W_{\mathscr{I}}$, and we have $(h \circ t) \circ f = (h \circ t) \circ g$.

□

2 Isomorphisms in triangulated categories

(4 points)

Let $(\mathscr{D}, T)$ be a triangulated category, and let $f : X \to Y$ be a morphism of $\mathscr{D}$. Show that f is an isomorphism if and only if there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ with $Z = 0$.

Solution. Suppose that f is an isomorphism. By (TR2), there exists a distinguished triangle

$X \xrightarrow{f} Y \to Z \to T(X)$. By (TR4), the commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle f^{-1}} \\ X & \xrightarrow[\mathrm{id}_X]{} & X \end{array}$$

can be completed to a morphism of distinguished triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle g} & & \\ X & \xrightarrow[\mathrm{id}_X]{} & X & \longrightarrow & 0 & \longrightarrow & T(X) \end{array}$$

By Corollary V.1.1.12 of the notes, the morphism g is an isomorphism, so $Z = 0$.

Conversely, suppose that there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ with $Z = 0$. Then, for every object W of $\mathscr{D}$, applying $\mathrm{Hom}_{\mathscr{D}}(W, \cdot)$ to the triangle $X \to Y \to Z \to T(X)$ and using Proposition V.1.1.11(ii) of the notes shows that $f_* : \mathrm{Hom}_{\mathscr{D}}(W, X) \to \mathrm{Hom}_{\mathscr{D}}(W, Y)$ is an isomorphism. By the Yoneda lemma (Corollary I.3.2.9 of the notes), the morphism f is an isomorphism.

□

3 Null systems

Let $(\mathscr{D}, T)$ be a triangulated. Remember that a null system in $\mathscr{D}$ is a set $\mathscr{N}$ of objects of $\mathscr{D}$ such that:

(N1) $0 \in \mathscr{N}$;

(N2) for every $X \in \mathrm{Ob}(\mathscr{C})$, we have $X \in \mathscr{N}$ if and only if $T(X) \in \mathscr{N}$;

(N3) if $X \to Y \to Z \to T(X)$ is a distinguished triangle and if $X, Y \in \mathscr{N}$, then $Z \in \mathscr{N}$.

We fix a null system $\mathscr{N}$, and we denote by $W_{\mathscr{N}}$ the set of morphisms $f : X \to Y$ in $\mathscr{D}$ such that there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ with $Z \in \mathscr{N}$.

(a). (1 point) If $X \in \mathscr{N}$ and Y is isomorphic to X, show that $Y \in \mathscr{N}$.

(b). (1 point) Show that $W_{\mathscr{N}}$ contains all the isomorphisms of $\mathscr{D}$.

(c). (2 points) Show that $W_{\mathscr{N}}$ is stable by composition.

(d). (4 points) Show that $W_{\mathscr{N}}$ satisfies conditions (S3) and (S4) of Definition V.2.2.1 of the notes.

(e). (2 points) Show that $W_{\mathscr{N}}$ is also a left multiplicative system.

Solution.

(a). Let $f : X \to Y$ be an isomorphism. By problem 2, the triangle $X \xrightarrow{f} Y \to 0 \to T(X)$ is distinguished. By axiom (TR3), the triangle $0 \to X \xrightarrow{f} Y \to T(0) = 0$ is also distinguished and so, by (N1) and (N3), we have $Y \in \mathscr{N}$.

(b). This follows immediately from problem 2 and from (N0).

(c). Let $f : X \to Y$ and $g : Y \to Z$ be in $W_{\mathscr{N}}$. Choose distinguished triangles $X \overset{f}{\to} Y \to Z' \to T(X)$ and $Y \overset{g}{\to} Z \to Z' \to T(Y)$ with $Z', Y' \in \mathscr{N}$. Let $X \overset{g \circ f}{\to} Y \to Z' \to T(X)$ be a distinguished triangle. By the octahedral axiom (axiom (TR5)), there exists a distinguished triangle $Z' \to Y \to X' \to T(X')$. By (N3), we have $X' \in \mathscr{N}$, and so $g \circ f \in W_{\mathscr{N}}$.

(d). We show condition (S3). Let $f : X \to Y$ and $s : X \to X'$ be morphisms in $\mathscr{D}$ with $s \in W_{\mathscr{N}}$. By the definition of $W_{\mathscr{N}}$ and axioms (TR3) and (N2), we can find a distinguished triangle $Z \overset{h}{\to} X \to X' \to T(Z)$ with $Z \in \mathscr{N}$. By (TR2), we can find a diatinguished triangle $Z \overset{f \circ h}{\to} Y \overset{t}{\to} Y' \to T(Z)$, and $t \in W_{\mathscr{N}}$ by (TR3) and (N2). Finally, by (TR4), we can complete the commutative diagram

$$\begin{array}{ccccccc}
Z & \xrightarrow{h} & X & \xrightarrow{s} & X' & \longrightarrow & T(Z) \\
\downarrow{\scriptstyle \mathrm{id}_Z} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathrm{id}_{T(Z)}} \\
Z & \xrightarrow[f \circ h]{} & Y & \xrightarrow[t]{} & Y' & \longrightarrow & T(Z)
\end{array}$$

In other words, we can find a morphism $g : X' \to Y'$ such that $g \circ s = t \circ f$. This finishes the proof of (S3).

We show condition (S4). Let $f, g : X \to Y$ be two morphisms of $\mathscr{D}$, and suppose that there exists $s : X' \to X$ such that $f \circ s = g \circ s$ and $s \in W_{\mathscr{N}}$. If $h = f - g$, then we have $h \circ s = 0$. Choose a distinguished triangle $X' \overset{s}{\to} X \overset{u}{\to} Z \to T(X')$ with $Z \in \mathscr{N}$. Applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(\cdot, Y)$ to this distinguished triangle, we get an exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(Z, Y) \to \mathrm{Hom}_{\mathscr{D}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(X, X').$$

As the image $h \circ s$ of $h \in \mathrm{Hom}_{\mathscr{D}}(X, Y)$ by the second morphism of this sequence is 0, there exists $k \in \mathrm{Hom}_{\mathscr{D}}(Z, Y)$ such that $h = k \circ u$. Consider a distinguished triangle $Z \overset{k}{\to} Y \overset{t}{\to} Y' \to T(Z)$. As $Z \in \mathscr{N}$, we have $t \in W_{\mathscr{N}}$. Also, as $t \circ k = 0$ (by Proposition V.1.1.11(i) of the notes), we have $t \circ h = 0$, so $t \circ f = t \circ g$.

(e). We know that $\mathscr{D}^{\mathrm{op}}$ is also a triangulated category, and $\mathscr{N}^{\mathrm{op}} = \{X \in \mathrm{Ob}(\mathscr{D}^{\mathrm{op}}) \mid X \in \mathscr{N}\}$ is a null system in $\mathscr{D}^{\mathrm{op}}$; indeed, axioms (N1) and (N2) obviously hold, and axiom (N3) for $\mathscr{N}^{\mathrm{op}}$ follows from (N3) for $\mathscr{N}$ thanks to (TR3) and (N2). Also, again thanks to (TR3) and (N2), the set of morphisms $W_{\mathscr{N}^{\mathrm{op}}}$ determined by $\mathscr{N}^{\mathrm{op}}$ is equal to $(W_{\mathscr{N}})^{\mathrm{op}}$. So, by question (d), the set $(W_{\mathscr{N}})^{\mathrm{op}}$ is a right mutliplicative system. But this is equivalent to the fact that $W_{\mathscr{N}}$ is a left multiplicative system.

□

4 Localization of functors

Let $\mathscr{C}$ be a category, let W be a set of morphisms of $\mathscr{C}$, and let $\mathscr{I}$ be a full subcategory of $\mathscr{C}$; denote by $W_{\mathscr{I}}$ the set of morphisms of $\mathscr{I}$ that are in W. We fix a localization $Q : \mathscr{C} \to \mathscr{C}[W^{-1}]$ of $\mathscr{C}$ by W, and we denote by $\iota : \mathscr{I} \to \mathscr{C}$ the inclusion functor. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Suppose that:

(a) W is a right multiplicative system;

(b) for every $X \in \mathrm{Ob}(\mathscr{C})$, there exists a morphism $s : X \to Y$ in W such that $Y \in \mathrm{Ob}(\mathscr{I})$;

(c) for every $s \in W_{\mathscr{I}}$, the morphism $F(s)$ is an isomorphism.

Show that, for every functor $G : \mathscr{C}[W^{-1}] \to \mathscr{D}$, the map

$$\alpha : \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{D})}(F, G \circ Q) \to \mathrm{Hom}_{\mathrm{Func}(\mathscr{I},\mathscr{D})}(F \circ \iota, G \circ Q \circ \iota)$$

induced by composition on the right by ι is bijective. (2 points for injectivity, 3 points for surjectivity)

Solution. Let $u_1, u_2 : F \to G \circ Q$ be morphism of functors such that $\alpha(u_1) = \alpha(u_2)$. Let $X \in \mathrm{Ob}(\mathscr{C})$, and choose a morphism $s : X \to X'$ such that $X' \in \mathrm{Ob}(\mathscr{I})$. Then we have commutative diagrams

$$\begin{array}{ccc} F(X) & \xrightarrow{u_1(X)} & G \circ Q(X) \\ {\scriptstyle F(s)}\big\downarrow & & \big\downarrow \wr\, {\scriptstyle G\circ Q(s)} \\ F(X') & \xrightarrow[u_1(X')]{} & G \circ Q(X') \end{array} \qquad \text{and} \qquad \begin{array}{ccc} F(X) & \xrightarrow{u_2(X)} & G \circ Q(X) \\ {\scriptstyle F(s)}\big\downarrow & & \big\downarrow \wr\, {\scriptstyle G\circ Q(s)} \\ F(X') & \xrightarrow[u_2(X')]{} & G \circ Q(X') \end{array}$$

and $u_1(X') = u_2(X')$ because $X' \in \mathrm{Ob}(\mathscr{I})$, so $u_1(X) = u_2(X)$. This shows that $u_1 = u_2$, and hence that α is injective.

We show that α is surjective. Let $v : F \circ \iota \to G \circ Q \circ \iota$ be a morphism of fucntors. Let $X \in \mathrm{Ob}(\mathscr{C})$, and let $s : X \to X'$ be a morphism of W such that $X' \in \mathrm{Ob}(\mathscr{I})$. Then $G \circ Q(s)$ is an isomorphism, and we set $u(X) = (G \circ Q(s))^{-1} \circ v(X') \circ F(s)$. We must check that this does not depend on the choice of s. Let $s' : X \to X''$ be another morphism of W such that $X'' \in \mathrm{Ob}(\mathscr{I})$. By condition (S3), we can find a commutative square

$$\begin{array}{ccc} X & \xrightarrow{s'} & X'' \\ {\scriptstyle s}\big\downarrow & & \big\downarrow {\scriptstyle t} \\ X' & \xrightarrow[t']{} & Y \end{array}$$

with $t \in W$. After composing with a morphism $Y \to Y'$ in W such that $Y' \in \mathrm{Ob}(\mathscr{I})$, we may assume that $Y \in \mathrm{Ob}(\mathscr{I})$. The images of s, t and s' by $G \circ Q$ are isomorphisms, so $G \circ Q(t')$ is also an isomorphism. As v is a morphism of functors, we have

$$\begin{aligned} (G \circ Q(s'))^{-1} \circ v(X'') \circ F(s') &= (G \circ Q(s'))^{-1} \circ (G \circ Q(t))^{-1} \circ v(Y) \circ F(t) \circ F(s') \\ &= (G \circ Q(s))^{-1} \circ (G \circ Q(t'))^{-1} \circ v(Y) \circ F(t') \circ F(s) \\ &= (G \circ Q(s))^{-1} \circ v(X') \circ F(s). \end{aligned}$$

So $u(X)$ is well-defined. It remains to show that teh family $(u(X))_{X \in \mathrm{Ob}(\mathscr{C})}$ is a morphism of functors from F to $G \circ Q$. Let $f : X \to Y$ be a morphism of $\mathscr{C}$. We choose morphisms $s : X \to X'$ and $t : Y \to Y'$ un W such that $X', Y' \in \mathrm{Ob}(\mathscr{I})$. By condition (S3), we can find morphisms $f' : X' \to Z$ and $s' : Y' \to Z$ such that $s' \in W$ and that $s' \circ t \circ f = f' \circ s$. After composing s' and g by a morphism $Z \to Z'$ in W such that $Z' \in \mathrm{Ob}(\mathscr{I})$, we may assume that $Z \in \mathrm{Ob}(\mathscr{I})$. Then, using the fact that v is a morphism of functors and the definition of u, we get

$$\begin{aligned} (G \circ Q(f)) \circ u(X) &= (G \circ Q(f))(G \circ Q(s))^{-1} \circ v(X') \circ F(s) \\ &= (G \circ Q(t))^{-1} \circ (G \circ Q(s'))^{-1} \circ (G \circ Q(g)) \circ v(X') \circ F(s) \\ &= (G \circ Q(t))^{-1} \circ (G \circ Q(s'))^{-1} \circ v(Z) \circ F(g) \circ F(s) \\ &= (G \circ Q(t))^{-1} \circ (G \circ Q(s'))^{-1} \circ v(Z) \circ F(s') \circ F(t) \circ F(f) \\ &= (G \circ Q(t))^{-1} \circ v(Y') \circ F(t) \circ F(f) \\ &= u(Y) \circ F(f). \end{aligned}$$

This shows that u is a morphism of functors.

□

5 Localization of a triangulated category

Let $(\mathscr{D}, T)$ be a triangulated category, let $\mathscr{N}$ be a null system in $\mathscr{D}$, and let $W = W_{\mathscr{N}}$ be the corresponding multiplicative system. (See problem 3.) We write $Q : \mathscr{D} \to \mathscr{D}/\mathscr{N}$ for $Q : \mathscr{D} \to \mathscr{D}[W^{-1}]$.

(a). (1 point) Show that there exists an auto-equivalence $T_{\mathscr{N}} : \mathscr{D}/\mathscr{N} \to \mathscr{D}/\mathscr{N}$ such that $T_{\mathscr{N}} \circ Q \simeq Q \circ T$.

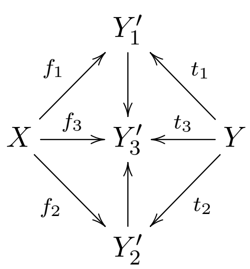
We say that a triangle in $\mathscr{D}/\mathscr{N}$ is distinguished if it is isomorphic to the image by Q of a distinguished triangle of $\mathscr{D}$. Axiom (TR0) of Definition V.1.1.4 of the notes is obvious.

(b). (5 points: 1 point per axiom) Show that axioms (TR1)-(TR5) also hold.

Solution.

(a). The functor T preserves $W_{\mathscr{N}}$ (by (TR3) and (N2)), so the functor $\mathscr{D} \xrightarrow{T} \mathscr{D} \xrightarrow{Q} \mathscr{D}/\mathscr{N}$ sends elements to $W_{\mathscr{N}}$ to isomorphisms, so it factors through a factor $T_{\mathscr{N}} : \mathscr{D}/\mathscr{N} \to \mathscr{D}/\mathscr{N}$.

If we want to justify the construction of $T_{\mathscr{N}}$ in Theorem V.3.1.4 of the notes, we can say this: If $X \in \mathrm{Ob}(\mathscr{D}/\mathscr{N}) = \mathrm{Ob}(\mathscr{D})$, we set $T_{\mathscr{N}}(X) = T(X)$. Let $u : X \to Y$ be a morphism of $\mathscr{D}$, and chose a diagram $X \xrightarrow{f} Y' \xleftarrow{t} Y$ in $\mathscr{D}$ representing u, with $t \in W_{\mathscr{N}}$. We take $T_{\mathscr{N}}(u)$ to be the morphism from $T_{\mathscr{N}}(X)$ to $T_{\mathscr{N}}(Y)$ represented by the diagram $T(X) \xrightarrow{T(f)} T(Y') \xleftarrow{T(t)} T(Y)$. This makes sense because $T(t) \in Z_{\mathscr{N}}$, by (TR3) and (N2). If we choose two representatives $X \xrightarrow{f_1} Y'_1 \xleftarrow{t_1} Y$ and $X \xrightarrow{f_2} Y'_2 \xleftarrow{t_2} Y$ of u, then we have a commutative diagram



with $t_3 \in W_{\mathscr{N}}$. Then applying T gives a commutative diagram

$$\begin{array}{ccccc} & & T(Y_1') & & \\ & \overset{T(f_1)}{\nearrow} & \downarrow & \overset{T(t_1)}{\nwarrow} & \\ T(X) & \xrightarrow{T(f_3)} & T(Y_3') & \xleftarrow{T(t_3)} & T(Y) \\ & \underset{T(f_2)}{\searrow} & \uparrow & \underset{T(t_2)}{\swarrow} & \\ & & T(Y_2') & & \end{array}$$

so $\begin{array}{ccccc} & & T(Y_1') & & \\ & \overset{T(f_1)}{\nearrow} & & \overset{T(t_1)}{\nwarrow} & \\ T(X) & & & & T(Y) \end{array}$ and $\begin{array}{ccccc} & & T(Y_2') & & \\ & \overset{T(f_2)}{\nearrow} & & \overset{T(t_2)}{\nwarrow} & \\ T(X) & & & & T(Y) \end{array}$ represent the same morphism from $T(X)$ to $T(Y)$ in $\mathscr{D}/\mathscr{N}$. So $T_{\mathscr{N}}$ is well-define, and it is easy to see that it is a functor.

(b)(TR1) Let $X \in \operatorname{Ob}(\mathscr{D}/\mathscr{N})$. Then the triangle $X \overset{\mathrm{id}_X}{\to} X \to 0 \to T_{\mathscr{N}}(X)$ in $\mathscr{D}/\mathscr{N}$ is isomorphic to the image by Q of the distinguished triangle $X \overset{\mathrm{id}_X}{\to} X \to 0 \to T(X)$ in $\mathscr{D}$, so it is distinguished.

(TR2) Let $u : X \to Y$ be a morphism in $\mathscr{D}/\mathscr{N}$, and choose morphisms $f : X \to Y'$ and $s : Y \to Y'$ in $\mathscr{D}$ such that $s \in W_{\mathscr{N}}$ and $u = Q(s)^{-1} \circ Q(f)$. Choose a distinguished triangle $X \overset{f}{\to} Y' \overset{g}{\to} Z \to T(X)$ in $\mathscr{D}$. Then the triangle $X \overset{u}{\to} Y \overset{Q(g \circ s)}{\to} Z \to T_{\mathscr{N}}(X)$ in $\mathscr{D}/\mathscr{N}$ is isomorphic to the image by Q of $X \overset{f}{\to} Y' \overset{g}{\to} Z \to T(X)$, so it is distinguished.

(TR3) Let $X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} T_{\mathscr{N}}(X)$ be a triangle in $\mathscr{D}/\mathscr{N}$. If it isn distinguished, then it is isomorphic to the image by Q of a distinguished triangle $X' \overset{f'}{\to} Y' \overset{g'}{\to} Z \overset{h'}{\to} T(X')$ in $\mathscr{D}$, and then the triangle $Y \overset{g}{\to} Z \overset{h}{\to} T(X) \overset{-T_{\mathscr{N}}(f)}{\to} T_{\mathscr{N}}(Y)$ is isomorphic to the image by Q of $Y' \overset{g'}{\to} Z \overset{h'}{\to} T(X') \overset{-T(f')}{\to} T(Y')$, hence it is also distinguished. The proof of the converse is similar.

(TR4) Consider a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T_{\mathscr{N}}(X) \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & & & \downarrow{\scriptstyle T(u)} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & T_{\mathscr{N}}(X') \end{array}$$

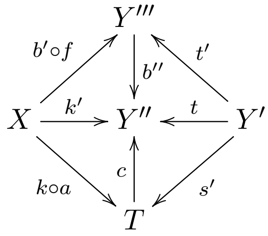
in $\mathscr{D}/\mathscr{N}$, where the rows are distinguished triangles. By the definition of distinguished triangles in $\mathscr{D}/\mathscr{N}$, we may assume that f, g, h, f', g', h' are morphisms of $\mathscr{D}$. We write $u = Q(s)^{-1} \circ Q(a)$, where $a : X \to X''$ and $s : X' \to X''$ are morphisms of $\mathscr{D}$ such that $s \in W_{\mathscr{N}}$. As $W_{\mathscr{N}}$ is a multiplicative system, we can find a commutative square

$$\begin{array}{ccc} X'' & \xrightarrow{k} & T \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s'} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

with $s' \in W_{\mathscr{N}}$. Write $v = Q(t')^{-1} \circ Q(b')$, with $b' : Y \to Y'''$ and $t' : Y' \to Y''$ are morphisms of $\mathscr{D}$ such that $t' \in W_{\mathscr{N}}$. Then

$$Q(s')^{-1} \circ Q(k \circ a) = Q(f') \circ Q(s)^{-1} \circ Q(s) = Q(f') \circ u = v \circ Q(f) = Q(t')^{-1} \circ Q(b' \circ f),$$

so, by the description of the Hom in the localization after Definition V.2.2.3 of the notes, there exists a commutative diagram



with $t \in W_{\mathscr{N}}$. Let $b = b'' \circ b' : Y \to Y''$. Then

$$Q(t)^{-1} \circ Q(b) = Q(t')^{-1} \circ Q(b') = v.$$

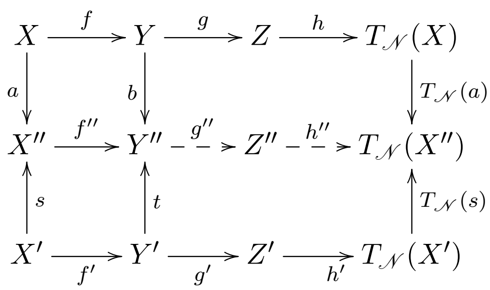
Let $f'' = c \circ k : X'' \to Y''$. Then

$$f'' \circ a = c \circ k \circ a = k' = b'' \circ b' \circ f = b \circ f$$

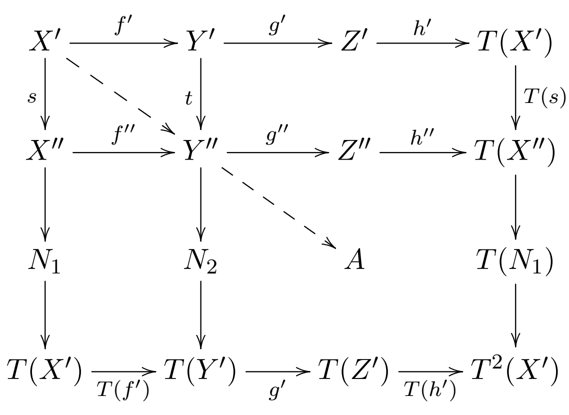
and

$$t \circ f' = c \circ s' \circ f' = c \circ k \circ s'' = f'' \circ s'',$$

so we have constructed a commutative diagram in $\mathscr{D}$:



and, by axiom (TR2), we can extend $f'' : X'' \to Y''$ to a distinguished triangle $X'' \overset{f''}{\to} Y'' \overset{g''}{\to} Z'' \overset{h''}{\to} T(X'')$. Completing s and t to distinguished triangles, we get a commutative diagram where the first two rows and columns are distinguished triangles and $N_1, N_2 \in \mathscr{N}$:



We also complete $t \circ f' = f'' \circ s$ to a distinguished triangle $X' \to Y'' \to A \to T(X')$. By the octahedral axiom (TR5) applied to the triangles based on $(f', t, t \circ f')$ and on $(s, f'', f''circs)$, we have distinguished triangles

$$Z' \to A \to N_2 \to T(Z')$$

and

$$N_1 \to A \to Z'' \to T(N_1).$$

Applying the octahedral axiom again for the morphisms $Z' \to A$, $A \to Z''$ and their composition, we get a commutative diagram where the rows and the third columns are distinguished triangles:

$$\begin{array}{ccccccc}
Z' & \longrightarrow & A & \longrightarrow & N_2 & \longrightarrow & T(Z') \\
\| & & \downarrow & & \downarrow & & \| \\
Z' & \longrightarrow & Z'' & \longrightarrow & N_3 & \longrightarrow & T(Z') \\
\downarrow & & \| & & \downarrow & & \downarrow \\
A & \longrightarrow & Z'' & \longrightarrow & T(N_1) & \longrightarrow & T(A) \\
 & & & & \downarrow & & \\
 & & & & T(N_2) & &
\end{array}$$

In particular, we have $N_3 \in \mathscr{N}$. So we have completed the commutative square

$$\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\
X'' & \xrightarrow[f'']{} & Y''
\end{array}$$

to a morphism of triangles

$$\begin{array}{ccccccc}
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X') \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow & & \downarrow{\scriptstyle T(s)} \\
X'' & \xrightarrow[f'']{} & Y'' & \xrightarrow[g'']{} & Z'' & \xrightarrow[h'']{} & T(X'')
\end{array}$$

such that the morphism $Z' \to Z''$ is in $W_{\mathscr{N}}$. Moreover, by (TR4), we can complete the commutative square

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
X'' & \xrightarrow[f'']{} & Y''
\end{array}$$

to a morphism of triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow & & \downarrow{\scriptstyle T(a)} \\
X'' & \xrightarrow[f'']{} & Y'' & \xrightarrow[g'']{} & Z'' & \xrightarrow[h'']{} & T(X'')
\end{array}$$

So we have constructed a commutative diagram in $\mathscr{D}$ whose rows are distinguished triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow & & \downarrow{\scriptstyle T(a)} \\ X'' & \xrightarrow[f'']{} & Y'' & \xrightarrow[g'']{} & Z'' & \xrightarrow[h'']{} & T(X'') \\ \uparrow{\scriptstyle s} & & {\scriptstyle t}\uparrow & & \uparrow & & {\scriptstyle T(s)}\uparrow \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X') \end{array}$$

and such that s, t and the morphism $Z' \to Z''$ are in $W_{\mathscr{N}}$. Taking the image of this by Q, we get a morphism of distinguished triangles in $\mathscr{D}/\mathscr{N}$ extending the pair (u, v).

(TR5) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathscr{D}/\mathscr{N}$. After replacing Y and Z by isomorphic objects, we may assume that f and g are morphisms of $\mathscr{D}$. Applying (TR5) in $\mathscr{D}$ to distinguished triangles based on the morphisms $(f, g, g \circ f)$ and taking the image of the resulting diagram by Q, we get (TR5) in $\mathscr{D}/\mathscr{N}$.

□

6 More group cohomology

The description of group cohomology in Subsection IV.3.5 of the notes can be useful in this problem.

We define elements u, v, r and s of the symmetric group $\mathfrak{S}_4$ by $u = (12)(34)$, $v = (14)(23)$, $r = (123)$ and $s = (13)$. The Klein four group is the normal subgroup K of $\mathfrak{S}_4$ generated by u and v.

Let k be a field of characteristic 2.

(a). (2 points) Show that $\mathfrak{S}_4 / K \simeq \mathfrak{S}_3$.

(b). (2 points) Show that there is a unique representation $\tau : \mathfrak{S}_4 \to \mathrm{GL}_2(k)$ such that $\tau(u) = \tau(v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\tau(r) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\tau(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let $M = M_2(k)$, with the action of $\mathfrak{S}_4$ given by $g \cdot A = \tau(g)A\tau(g)^{-1}$, for $g \in \mathfrak{S}_4$ and $A \in M_2(k)$. We identify $\mathfrak{S}_3$ with the subgroup of $\mathfrak{S}_4$ generated by r and s. We have a short exact sequence of groups

$$1 \to \mathbb{Z}/3\mathbb{Z} \to \mathfrak{S}_3 \to \mathbb{Z}/2\mathbb{Z} \to 1,$$

where the generator $1 \in \mathbb{Z}/3\mathbb{Z}$ is sent to $r \in \mathfrak{S}_3$.

(c). (2 points) If N is any representation of $\mathbb{Z}/3\mathbb{Z}$ on a k-vector space, show that $\mathrm{H}^p(\mathbb{Z}/3\mathbb{Z}, N) = 0$ for every $p \geq 1$. (You might find Remark IV.3.5.1 of the notes useful.)

(d). (1 point) If N is any representation of $\mathfrak{S}_3$ on a k-vector space, show that we have canonical isomorphisms $\mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, N^{\mathbb{Z}/3\mathbb{Z}}) \xrightarrow{\sim} \mathrm{H}^p(\mathfrak{S}_3, N)$ for every $p \geq 0$.

(e). (2 points) Show that $\mathrm{H}^p(\mathfrak{S}_3, M) = 0$ for every $p \geq 1$.

(f). (1 point) Show that we have canonical isomorphisms

$$\mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, \mathrm{H}^1(K, M)^{\mathbb{Z}/3\mathbb{Z}}) \xrightarrow{\sim} \mathrm{H}^p(\mathfrak{S}_3, \mathrm{H}^1(K, M)),$$

for every $p \geq 0$.

(g). (1 points) Show that $\mathrm{H}^1(K, M) = \mathrm{Hom}_{\mathbf{Grp}}(K, M)$, and that the action of $\mathfrak{S}_3$ on $\mathrm{H}^1(K, M)$ is given by $(g \cdot \varphi)(x) = g \cdot \varphi(g^{-1}xg)$, if $g \in \mathfrak{S}_3$, $x \in K$ and $\varphi \in \mathrm{H}^1(K, M)$.

(h). (3 points) Show that $\mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^1(K, M))$ is a 1-dimensional k-vector space, and that $\mathrm{H}^p(\mathfrak{S}_3, \mathrm{H}^1(K, M)) = 0$ if $p \geq 1$.

(i). (2 points) Show that we have canonical isomorphisms $\mathrm{H}^1(\mathfrak{S}_4, M) \xrightarrow{\sim} \mathrm{H}^1(K, M)^{\mathfrak{S}_3}$ and $\mathrm{H}^2(\mathfrak{S}_4, M) \xrightarrow{\sim} \mathrm{H}^2(K, M)^{\mathfrak{S}_3}$.

(j). (3 points) Let N be a k-vector space with trivial action of K. Show that the map $Z^2(K, N) \to N^3$ sending a 2-cocycle $\eta : K^2 \to N$ to $(\eta(u, u) - \eta(1, 1), \eta(v, v) - \eta(1, 1), \eta(uv, uv) - \eta(1, 1))$ induces an isomorphism $\mathrm{H}^2(K, N) \xrightarrow{\sim} N^3$.

(k). (2 points) Show that $\mathrm{H}^2(\mathfrak{S}_4, M)$ is a 2-dimensional k-vector space.

Solution.

(a). We have $K = \langle u, v \rangle = \{1, u, v, uv\}$, with $uv = (13)(24)$, so the elements of K are 1 and the permutation in $\mathfrak{S}_4$ that are the product of two transpositions with disjoint supports. This implies that K is a normal subgroup of $\mathfrak{S}_4$. Also, it is easy to see that the subgroup H of $\mathfrak{S}_4$ generated by r and s is equal to the group $\{\sigma \in \mathfrak{S}_4 \mid \sigma(4) = 4\}$, which is isomorphic to $\mathfrak{S}_3$. We have $H \cap K = \{1\}$, so the composition $H \subset \mathfrak{S}_4 \to \mathfrak{S}_4 / K$ is injective; as $|H| = 6 = 24/4 = |\mathfrak{S}_4 / K|$, this composition is an isomorphism, so $\mathfrak{S}_3 \simeq H \xrightarrow{\sim} \mathfrak{S}_4 / K$.

(b). The uniqueness of τ follows from the fact that the set $\{u, v, r, s\}$ generates $\mathfrak{S}_4$.

Let us show the existence of τ. Consider the bijection $\mathbb{F}_2^2 - \{0\} \simeq \{1, 2, 3\}$ sending $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to 3, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to 1 and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to 2. This induces an injective morphism of groups $\psi : \mathrm{GL}_2(\mathbb{F}_2) \to \mathfrak{S}_3$ sending $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to s and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ to r. As $|\mathrm{GL}_2(\mathbb{F}_2)| = 6 = |\mathfrak{S}_3|$, the morphism ψ is an automorphism, and we get the representation $\tau : \mathfrak{S}_4 \to \mathrm{GL}_2(k)$ as the composition

$$\mathfrak{S}_4 \to \mathfrak{S}_4 / K \simeq \mathfrak{S}_3 \overset{\psi^{-1}}{\to} \mathrm{GL}_2(\mathbb{F}_2) \subset \mathrm{GL}_2(k).$$

(c). Let Γ be any finite group of odd order. We will show that, for every $k[\Gamma]$-module N and any $p \geq 1$, we have $\mathrm{H}^p(\Gamma, N) = 0$. By Remark IV.3.5.1 of the notes, we can calculate $\mathrm{H}^p(\Gamma, N)$ as a derived functor on the category $\mathscr{A} = {}_{k[\Gamma]}\mathbf{Mod}$. We claim that the abelian category $\mathscr{A}$ is semisimple (that is, every short exact sequence splits), which implies that every additive functor on $\mathscr{A}$ is exact, hence has trivial higher derived functors.

The semisimplicity of $\mathscr{A}$ follows from Maschke's theorem, whose proof in this case goes like so: Let $0 \to N_1 \overset{u}{\to} N_2 \overset{v}{\to} N_3 \to 0$ be an exact sequence of left $k[\Gamma]$-modules. As k is a field, there exists a k-linear map $w_0 : N_3 \to N_2$ such that $v \circ w_0 = \mathrm{id}_{N_3}$. Define $z : N_3 \to N_2$ by

$$w(x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \cdot w_0(\gamma^{-1} \cdot x),$$

where we use the fact that $|\Gamma|$ is odd to see that it is invertible in k. Then an easy calculation shows that w is $k[\Gamma]$-linear and $w \circ v = \mathrm{id}_{N_3}$.

(d). Consider the Hochschild-Serre spectral sequence for the extension

$1 \to \mathbb{Z}/3 \to \mathfrak{S}_3 \to \mathbb{Z}/2\mathbb{Z} \to 1$ and the $k[\mathfrak{S}_3]$-module N:

$$E_2^{pq} = \mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, \mathrm{H}^q(\mathbb{Z}/3\mathbb{Z}, N)) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{S}_3, N).$$

By question (c), we have $E_2^{pq} = 0$ if $q \neq 0$, so the spectral sequence degenerates at the second page, and $E_\infty^{pq} = E_2^{pq}$. So, for every $p \geq 0$, we get an isomorphism

$$\mathrm{H}^p(\mathfrak{S}_3, N) \simeq E_\infty^{p,0} = E_2^{p,0} = \mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, N^{\mathbb{Z}/3\mathbb{Z}}).$$

(e). We use the formula of question (d). By definition of the action of $\mathfrak{S}_4$ on M, the k-vector space $M^{\mathbb{Z}/3\mathbb{Z}}$ is the centralizer of $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ in $M_2(k)$, that is, the space $\left\{\begin{pmatrix} a & b \\ b & a+b \end{pmatrix}, \ a, b \in k\right\}$, with the action of the nontrivial element s of $\mathbb{Z}/2\mathbb{Z}$ given by conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If $a, b \in k$, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & a+b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & b \\ b & a \end{pmatrix}.$$

So we get

$$M^{\mathfrak{S}_3} = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ a \in k\right\},$$

$$(1+s) \cdot M^{\mathbb{Z}/3\mathbb{Z}} = (s-1) \cdot M^{\mathbb{Z}/3\mathbb{Z}} = \left\{\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \ b \in k\right\}$$

(remember that $2 = 0$ in k), and

$$\{x \in M^{\mathbb{Z}/3\mathbb{Z}} \mid (1+s) \cdot x = 0\} = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ a \in k\right\}.$$

By question 2(a)(ii) of problem set 7 , we get $\mathrm{H}^p(\mathfrak{S}_3, M) = \mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, M^{\mathbb{Z}/3\mathbb{Z}}) = 0$ if $p \geq 1$, and $\mathrm{H}^0(\mathfrak{S}_3, M) = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ a \in k\right\}$.

(f). Apply (d) to the $k[\mathfrak{S}_3]$-module $\mathrm{H}^1(K, M)$, where the action of $\mathfrak{S}_3$ comes from the isomorphism $\mathfrak{S}_3 \simeq \mathfrak{S}_4 / K$ of (a).

(g). We use the description of $\mathrm{H}^1(K, M)$ given in Subsection IV.3.5 of the notes. As K acts trivially on M, a remark in this subsection gives $\mathrm{H}^1(K, M) = Z^1(K, M) = \mathrm{Hom}_{\mathbf{Grp}}(K, M)$. Moreover, if we make $G = \mathfrak{S}_4$ act on $\mathbb{Z}^{K^{n+1}}$ via its action by diagonal conjugation on K^{n+1}, then the unnormalized bar resolution $X^\bullet \to \mathbb{Z}$ of $\mathbb{Z}$ as a left $\mathbb{Z}[K]$-module is G-equivariant. So we get actions of G on the groups $C^n(K, M)$ that preserve the subgroups $Z^n(K, M)$ and $B^n(K, M)$, and induce the action of G on $\mathrm{H}^n(K, M)$. By definition of the action of G on $X^\bullet$, the action of G on $C^n(K, M) \xrightarrow{\sim} \mathscr{F}(K^n, M)$ (the set of functions from K^n to M) is given by $(g \cdot \eta)(k_1, \ldots, k_n) = g \cdot \eta(g^{-1}k_1 g, \ldots, g^{-1}k_n g)$, for $g \in G$, $\eta : K^n \to M$ and $k_1, \ldots, k_n \in K$. This implies in particular the second statement of (g).

(h). We have $r^{-1}ur = uv$ and $r^{-1}vr = u$, so, by (g), we have an isomorphism $\mathrm{H}^1(K, M) = \mathrm{Hom}_{\mathbf{Grp}}(K, M) \xrightarrow{\sim} M^2$ sending $c : K \to M$ to $(c(u), c(v))$, and the action of $r \in G$ on $\mathrm{H}^1(K, M)$ corresponding to the following action on M^2: if $x, y \in M$, then $r \cdot (x, y) = (\tau(r)(x+y)\tau(r)^{-1}, \tau(r)x\tau(x)^{-1})$. If $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we have

$$\tau(r)x\tau(r)^{-1} = \begin{pmatrix} c+d & c \\ a+b+c+d & a+c \end{pmatrix}.$$

So a straightforward calculation shows that

$$\mathrm{H}^1(K,M)^{\mathbb{Z}/3\mathbb{Z}} \xrightarrow{\sim} \left\{(x,y)\in M^2 \mid \exists a,b\in k \text{ with } x=\begin{pmatrix} a & b \\ a+b & a\end{pmatrix} \text{ and } y=\begin{pmatrix} b & a+b \\ a & b\end{pmatrix}\right\}.$$

Moreover, we have $sus=v$ and $svs=u$, so the action of $s\in G$ on $\mathrm{H}^1(K,M)$ corresponds to the following action on M^2: if $x,y\in M$, then $s\cdot(x,y)=(\tau(s)y\tau(s),\tau(s)x\tau(s))$. If $x=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, then we have

$$\tau(s)x\tau(s)=\begin{pmatrix} d & c \\ b & a\end{pmatrix}.$$

So, if $N=\mathrm{H}^1(K,M)^{\mathbb{Z}/3\mathbb{Z}}$, we have

$$\begin{aligned} N^{\mathbb{Z}/2\mathbb{Z}} = \{n\in N \mid (1-s)\cdot n=0\} &= \{n\in N\mid (1+s)\cdot n=0\} \\ &= (1-s)\cdot N=(1+s)\cdot N \\ &= \left\{\left(\begin{pmatrix} a & a \\ 0 & a\end{pmatrix},\begin{pmatrix} a & 0 \\ a & a\end{pmatrix}\right),\ a\in k\right\}. \end{aligned}$$

By question (f) and question 2(a)(ii) of problem set 7 , we get

$$\mathrm{H}^0(\mathfrak{S}_3,\mathrm{H}^1(K,M))=\left\{\left(\begin{pmatrix} a & a \\ 0 & a\end{pmatrix},\begin{pmatrix} a & 0 \\ a & a\end{pmatrix}\right),\ a\in k\right\}$$

and, if $p\geq 1$, then

$$\mathrm{H}^p(\mathfrak{S}_3,\mathrm{H}^1(K,M))=0.$$

(i). Consider the Hochschild-Serre spectral sequence for the extension $1\to K\to\mathfrak{S}_4\to\mathfrak{S}_3\to 1$ and the $k[\mathfrak{S}_4]$-module M:

$$E_2^{pq}=\mathrm{H}^p(\mathfrak{S}_3,\mathrm{H}^q(K,M))\Rightarrow \mathrm{H}^{p+q}(\mathfrak{S}_4,M).$$

By questions (e) and (h), we have $E_2^{pq}=0$ if $q\in\{0,1\}$ and $p\neq 0$. So the second page of the spectral sequence looks like this:

$$\begin{matrix} \mathrm{H}^0(\mathfrak{S}_3,\mathrm{H}^2(K,M)) & \mathrm{H}^1(\mathfrak{S}_3,\mathrm{H}^2(K,M)) & \mathrm{H}^2(\mathfrak{S}_3,\mathrm{H}^2(K,M)) & \mathrm{H}^3(\mathfrak{S}_3,\mathrm{H}^2(K,M)) \\ \mathrm{H}^0(\mathfrak{S}_3,\mathrm{H}^1(K,M)) & 0 & 0 & 0 \\ \mathrm{H}^0(\mathfrak{S}_3,\mathrm{H}^0(K,M)) & 0 & 0 & 0 \end{matrix}$$

In particular, if $r\geq 2$ and $q\in\{0,1,2\}$, then $d_r^{0,q}:E_r^{0,q}\to E_r^{r,q-r+1}$ is zero, because $E_r^{r,q-r+1}=0$, hence $E_{r+1}^{0,q}=E_r^{0,q}$. So we get $E_\infty^{0,q}=E_2^{0,q}$ if $q\in\{0,1,2\}$, and $E_\infty^{1,0}=E_\infty^{1,1}=E_\infty^{2,0}=0$ (because the corresponding E_2 terms are 0). This gives isomorphisms

$$\mathrm{H}^0(\mathfrak{S}_4,M)\xrightarrow{\sim} E_\infty^{0,0}=\mathrm{H}^0(\mathfrak{S}_3,\mathrm{H}^0(K,M)),$$
$$\mathrm{H}^1(\mathfrak{S}_4,M)\xrightarrow{\sim} E_\infty^{0,1}=\mathrm{H}^0(\mathfrak{S}_3,\mathrm{H}^1(K,M)),$$

and

$$\mathrm{H}^2(\mathfrak{S}_4,M)\xrightarrow{\sim} E_\infty^{0,2}=\mathrm{H}^0(\mathfrak{S}_3,\mathrm{H}^2(K,M)).$$

(j). Let $\eta \in C^2(K,N)$. As K acts trivially on N, the function η is a 2-cocycle if and only if, for all $g_1, g_2, g_3 \in K$, we have

$$0 = \eta(g_2, g_3) - \eta(g_1 g_2, g_3) + \eta(g_1, g_2 g_3) - \eta(g_1, g_2).$$

As N is a k-vector space and k has characteristic 2, this relation can also be written as

$$0 = \eta(g_2, g_3) + \eta(g_1 g_2, g_3) + \eta(g_1, g_2 g_3) + \eta(g_1, g_2). \tag{*}$$

Also, the function η is a 2-coboundary if and only if there exists a function $c : K \to M$ such that $\eta = d^1(c)$, that is, for all $g_1, g_2 \in K$,

$$\eta(g_1, g_2) = c(g_1) + c(g_2) + c(g_1 g_2). \tag{**}$$

Let $\eta \in Z^2(K,M)$. Taking $g_1 = g_2 = 1$ in equation (*), we get, for every $g \in K$, $\eta(1,1) = \eta(1,g)$. Similarly, taking $g_2 = g_3 = 1$ in (*), we get, for every $g \in K$, $\eta(1,1) = \eta(g,1)$. Taking (g_1, g_2, g_3) equal to (u,v,uv), (v,u,uv), (u,uv,v), (v,uv,u), (uv,u,v) and (uv,v,u), we get the following six relations:

$$\eta(u,v) + \eta(v,uv) = \eta(u,u) + \eta(uv,uv) \tag{1}$$

$$\eta(v,u) + \eta(u,uv) = \eta(v,v) + \eta(uv,uv) \tag{2}$$

$$\eta(u,uv) + \eta(uv,v) = \eta(u,u) + \eta(v,v) \tag{3}$$

$$\eta(v,uv) + \eta(uv,u) = \eta(u,u) + \eta(v,v) \tag{4}$$

$$\eta(uv,u) + \eta(u,v) = \eta(v,v) + \eta(uv,uv) \tag{5}$$

$$\eta(uv,v) + \eta(v,u) = \eta(u,u) + \eta(uv,uv) \tag{6}$$

Taking (g_1, g_2, g_3) equal to (u,u,v), (v,v,u) and (uv,uv,u), (and using the fact that $\eta(1,g) = \eta(g,1) = \eta(1,1)$ for every $g \in K$), we get the following three relations:

$$\eta(u,v) + \eta(u,uv) = \eta(u,u) + \eta(1,1) \tag{7}$$

$$\eta(v,u) + \eta(v,uv) = \eta(v,v) + \eta(1,1) \tag{8}$$

$$\eta(uv,v) + \eta(uv,u) = \eta(uv,uv) + \eta(1,1) \tag{9}$$

Let $\alpha : C^2(K,N^3) \to N^3$ be the morphism sending $\eta : K^2 \to N$ to $(\eta(u,u) - \eta(1,1), \eta(v,v) - \eta(1,1), \eta(uv,uv) - \eta(1,1))$. We claim that $(\operatorname{Ker} \alpha) \cap Z^2(K,N) = B^2(K,N)$.

Suppose first that $\eta \in B^2(K,N)$, and write $\eta = d^1(c)$, with $c : K \to N$. Taking $g_1 = g_2$ in (**) and using the fact that every element of K is of order 1 or 2, we get, for every $g \in K$, $\eta(g,g) = c(1)$. Hence $\eta(g,g) = \eta(1,1)$ for every $g \in K$, so $\alpha(\eta) = 0$.

Conversely, let $\eta \in Z^2(K,N)$ such that $\alpha(\eta) = 0$. Then $\eta(u,u) = \eta(v,v) = \eta(uv,uv) = \eta(1,1)$, so equations (1)-(6) imply that

$\eta(u,v) = \eta(v,uv) = \eta(uv,u)$ and $\eta(v,u) = \eta(uv,v) = \eta(u,uv)$, and then equation (7) implies that $\eta(u,v) = \eta(u,uv)$, so we finally get

$$\eta(u,v) = \eta(v,uv) = \eta(uv,u) = \eta(v,u) = \eta(uv,v) = \eta(u,uv).$$

Define $c : K \to N$ by $c(u) = c(v) = 0$, $c(1) = \eta(1,1)$ and $c(uv) = \eta(u,v)$. Then it is easy to check that $\eta = d^1(c)$, so $\eta \in B^2(K,M)$.

To finish the proof, we need to show that α induces a surjection $Z^2(K,N) \to N^3$. Let $(x,y,z) \in N^3$. We want to find $\eta \in Z^2(K,N)$ such that $\alpha(\eta) = (x,y,z)$. As we can always translate η by an element of $B^2(K,N)$ without changing $\alpha(\eta)$, we may take $\eta(1,1) = \eta(u,v) = 0$. Then we must have $\eta(u,u) = x$, $\eta(v,v) = y$ and $\eta(uv,uv) = z$, and equations (1)-(9) imply that

$$\eta(v,uv) = x + z$$

$$\eta(uv,u) = y + z$$

$$\eta(u,uv) = x$$

$$\eta(uv,v) = y$$

$$\eta(v,u) = x + y + z$$

Also, if η is a 2-cocyle, we must have $\eta(1,g) = \eta(g,1) = \eta(1,1) = 0$ for every $g \in K$. This determines the values of η on all of K^2, and it is easy to check that the function η that we defined is indeed a 2-cocycle.

(k). We know that $\mathrm{H}^2(\mathfrak{S}_4, M) \simeq \mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^2(K,M))$ by question (i), so we need to calculate the action of $\mathfrak{S}_3$ on $\mathrm{H}^2(K,M)$; we will use the isomorphism $\alpha : \mathrm{H}^2(K,M) \xrightarrow{\sim} M^3$ of question (j). By the proof of question (g), an element $g \in \mathfrak{S}_4$ acts on a 2-cocycle $\eta \in Z^2(K,M)$ by $(g \cdot \eta)(k_1,k_2) = g \cdot \eta(g^{-1}k_1 g, g^{-1}k_2 g)$. Let $\eta \in Z^2(K,M)$, and let $(x,y,z) = \alpha(\eta)$. We have $sus = v$, $svs = u$, $s(uv)s = uv$, $r^{-1}ur = uv$, $r^{-1}vr = u$ and $r^{-1}(uv)r = v$, so

$$\alpha(s \cdot \eta) = (s \cdot y, s \cdot x, s \cdot z)$$

and

$$\alpha(r \cdot \eta) = (r \cdot z, r \cdot x, r \cdot y).$$

So η represents an element of $\mathrm{H}^2(K,M)^{\mathfrak{S}_3}$ if and only if $s \cdot y = x$, $s \cdot x = y$, $s \cdot z = z$, $r \cdot z = x$, $r \cdot x = y$ and $r \cdot y = z$. We already calculate the action of r and s on M in the solution of question (h). The relation $s \cdot z = z$ is equivalent to the fact that $z = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, for $a, b \in k$. Then we get

$$x = r \cdot z = \begin{pmatrix} a+b & b \\ 0 & a+b \end{pmatrix}$$

and

$$y = r \cdot x = \begin{pmatrix} a+b & 0 \\ b & a+b \end{pmatrix}.$$

We have $z = r \cdot y$ because $r^3 = 1$, and it is clear that $x = s \cdot y$ and $y = s \cdot x$. So the k-vector space

$$\mathrm{H}^2(K,M)^{\mathfrak{S}_3} \simeq \left\{ \left(\begin{pmatrix} a+b & b \\ 0 & a+b \end{pmatrix}, \begin{pmatrix} a+b & 0 \\ b & a+b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right),\ a, b \in k \right\}$$

is 2-dimensional.

□