MAT 540 : Problem Set 7

Due Thursday, November 7

1 Diagram chasing lemmas via spectral sequences

This problem will ask to reprove some of the diagram chasing lemmas using the two spectral sequences of a double complex. This is circular, because of course the diagram chasing lemmas are used to establish the existence of the spectral sequences. The goal is just to get you used to manipulating spectral sequences on simple examples.

(a). **The $\infty \times \infty$ lemma**: (2 points) Suppose that we have a double complex $X = (X^{n,m}, d_1^{n,m}, d_2^{n,m})$ such that $X^{n,m} = 0$ if $n < 0$ or $m < 0$. Suppose also that the complexes $(X^{\bullet,n}, d_{1,X}^{\bullet,n})$ and $(X^{n,\bullet}, d_{2,X}^{n,\bullet})$ are exact if $n \neq 0$. Using the two spectral sequences of the double complex, prove that we have canonical isomorphisms

$$\mathrm{H}^n(X^{\bullet,0}, d_{1,X}^{\bullet,0}) \simeq \mathrm{H}^n(X^{0,\bullet}, d_{2,X}^{0,\bullet}).$$

(Hint: Both spectral sequences degenerate at the first page.)

(b). **The four lemma**: Consider a commutative diagram with exact rows in $\mathscr{A}$:

(*)

$$\begin{tikzcd}
A' \arrow[r] & B' \arrow[r, "g"] & C' \arrow[r] & D' \\
A \arrow[r] \arrow[u, "u"] & B \arrow[r, "f"'] \arrow[u, "v"] & C \arrow[r] \arrow[u, "w"] & D \arrow[u, "t"]
\end{tikzcd}$$

Suppose that u is surjective and t is injective. We want to show that $f(\operatorname{Ker} v) = \operatorname{Ker} w$ and that $\operatorname{Im} v = g^{-1}(\operatorname{Im} w)$.

(i) (1 point) Show that $\operatorname{Im} v = g^{-1}(\operatorname{Im} w)$ if and only if the morphism $\operatorname{Coker} v \to \operatorname{Coker} w$ induced by g is injective.

We consider the double complex X represented on diagram (*), with the convention that all the objects that don't appear are 0, the object A is in bidegree $(0,0)$, the differential $d_{1,X}$ is horizontal and the differential $d_{2,X}$ is vertical. (So, for example, $X^{3,0} = C$, $X^{1,0} = A'$ and $X^{2,2} = 0$.) Let ${}^{I}E$ and ${}^{II}E$ the two spectral sequences of this double complex.

(ii) (2 points) Show that ${}^{II}E$ degenerates at the second page.

(iii) (1 point) Show that $\mathrm{H}^2(\mathrm{Tot}(X)) = 0$.

(iv) (1 point) Write the first page of ${}^{I}E$.

(v) (1 point) Show that ${}^{I}E$ degenerates at the second page.

(vi) (2 points) Show that $f(\operatorname{Ker} v) = \operatorname{Ker} w$ and that $\operatorname{Im} v = g^{-1}(\operatorname{Im} w)$.

(c). **The long exact sequence of cohomology**: We consider a short exact sequence of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$; to simplify the notation, we will assume that $A^n = B^n = C^n = 0$ for $n < 0$. Consider the following double complex X:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \\
C^0 \arrow[r] \arrow[u] & C^1 \arrow[r] \arrow[u] & C^2 \arrow[r] \arrow[u] & \dots \\
B^0 \arrow[r] \arrow[u] & B^1 \arrow[r] \arrow[u] & B^2 \arrow[r] \arrow[u] & \dots \\
A^0 \arrow[r] \arrow[u] & A^1 \arrow[r] \arrow[u] & A^2 \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

where $X^{0,0} = A$, the differential $d_{1,X}$ (resp. $d_{2,X}$) is represented horizontally (resp. vertically), and $X^{n,m} = 0$ if $n < 0$, $m < 0$ or $m \geq 3$. Let ${}^IE$ and ${}^{II}E$ be the two spectral sequences of X.

(i) (1 point) Show that ${}^IE$ degenerates at the first page and that $\mathrm{H}^n(\mathrm{Tot}(X)) = 0$ for every $n \in \mathbb{Z}$.

(ii) (1 point) Calculate ${}^{II}E_1$.

(iii) (1 point) Show that ${}^{II}E$ degenerates at the third page.

(iv) (2 points) Show that ${}^{II}E_2^{00} = {}^{II}E_\infty^{00}$ and that ${}^{II}E_2^{1q} = {}^{II}E_\infty^{1q}$ for every $q \geq 0$.

(v) (1 point) Show that $d_2^{0q} : {}^{II}E_2^{0q} \to {}^{II}E_2^{2,q-1}$ is an isomorphism for every $q \geq 1$.

(vi) (1 point) Show that we have a long exact sequence

$$\dots \to \mathrm{H}^n(A^\bullet) \to \mathrm{H}^n(B^\bullet) \to \mathrm{H}^n(C^\bullet) \xrightarrow{\delta^n} \mathrm{H}^{n+1}(A^\bullet) \to \mathrm{H}^{n+1}(B^\bullet) \to \dots$$

where δ^n comes from a differential of the spectral sequence ${}^{II}E$.

Solution.

(a). Consider the two spectral sequences ${}^IE$ and ${}^{II}E$ of the double complex X. We have ${}^IE_1^{p,q} = \mathrm{H}^q(X^{p,\bullet}, d_{2,X}^{p,\bullet})$; as all the columns of the double complex are supposed exact except for $X^{0,\bullet}$, this implies that ${}^IE_1^{pq} = 0$ for $p \neq 0$. As ${}^IE_r^{pq}$ is a subquotient of ${}^IE_q^{pq}$ for $r \geq 1$, we deduce that ${}^IE_r^{pq} = 0$ for every $r \geq 1$ and every $p \neq 0$, and in particular $d_r^{pq} : {}^IE_r^{pq} \to {}^IE_r^{p+r,q-r+1}$ is the zero morphism if $r \geq 1$, because its source or target is 0. So the spectral sequence ${}^IE$ degenerates at the first page, and we have ${}^IE_\infty^{pq} = {}^IE_1^{pq}$. Also, as X is a first quadrant double complex, the spectral sequence ${}^IE$ converges to $\mathrm{H}^\bullet(\mathrm{Tot}(X))$, so we get

$$\mathrm{H}^n(\mathrm{Tot}(X)) = {}^IE_\infty^{0,n} = \mathrm{H}^n(X^{0,\bullet}, d_{2,X}^{0,\bullet}).$$

On the other hand, we have ${}^{II}E_1^{pq} = \mathrm{H}^q(X^{\bullet,p}, d_{1,X}^{\bullet,p})$. As all the rows of the double complex are supposed exact except for $X^{\bullet,0}$, this implies that ${}^{II}E_1^{pq} = 0$ if $p \neq 0$. Reasoning as in the first paragraph, we deduce that the spectral sequence ${}^{II}E$ degenerates at the first page, and that we have ${}^{II}E_\infty^{pq} = {}^{II}E_1^{pq}$. As ${}^{II}E$ converges to $\mathrm{H}^\bullet(\mathrm{Tot}(X))$, this gives canonical isomorphisms

$$\mathrm{H}^n(\mathrm{Tot}(X)) = {}^{II}E_\infty^{0,n} = \mathrm{H}^n(X^{\bullet,0}, d_{1,X}^{\bullet,0}).$$

(b). (i) As $\operatorname{Im} w$ is the kernel of the canonical morphism $C' \to \operatorname{Coker} w$, the subobject $g^{-1}(\operatorname{Im} w)$ of B' is the kernel of the morphism $B' \xrightarrow{g} C' \to \operatorname{Coker} w$, which is also equal to the morphism $B' \to \operatorname{Coker} v \to \operatorname{Coker} w$, where the morphism $\operatorname{Coker} v \to \operatorname{Coker} w$ is induced by g. Note also that we always have $\operatorname{Im} v \subset g^{-1}(\operatorname{Im} w)$, because $g \circ v = w \circ f$. So the kernel of the morphism $\operatorname{Coker} v \to \operatorname{Coker} w$ induced by g is $g^{-1}(\operatorname{Im} w)/\operatorname{Im} v$, which gives the result.

(ii) Let us give names to all the morphisms in the diagram:

$$\begin{tikzcd}
A' \arrow[r, "b"] & B' \arrow[r, "g"] & C' \arrow[r, "d"] & D' \\
A \arrow[r, "a"'] \arrow[u, "u"'] & B \arrow[r, "f"'] \arrow[u, "v"'] & C \arrow[r, "c"'] \arrow[u, "w"'] & D \arrow[u, "t"']
\end{tikzcd}$$

As the rows are exact, we have ${}^{II}E_1^{0,0} = \operatorname{Ker} a$, ${}^{II}E^{0,3} = \operatorname{Coker} c$, ${}^{II}E_1^{0,1} = {}^{II}E_1^{0,2} = 0 = {}^{II}E_1^{1,1} = {}^{II}E_1^{1,2} = 0$, ${}^{II}E_1^{1,0} = \operatorname{Ker} b$, ${}^{II}E_1^{1,3} = \operatorname{Coker} d$, and the other ${}^{II}E_1^{p,q}$ are all 0. In other words, the first page of ${}^{II}E$ looks like this:

$$\begin{array}{ccc}
0 & 0 & 0 \\
\operatorname{Coker} c & \operatorname{Coker} d & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\operatorname{Ker} a & \operatorname{Ker} b & 0
\end{array}$$

In particular, for every $r \geq 1$, we have ${}^{II}E_r^{pq} = 0$ if $(p,q) \notin \{(0,0),(3,0),(1,0),(1,3)\}$, so, if $r \geq 2$, every d_r^{pq} has its source or target zero. Hence ${}^{II}E$ degenerates at the second page, and ${}^{II}E_\infty = {}^{II}E_2$.

(iii) As X is a first quadrant double complex, the spectral sequences ${}^{I}E$ and ${}^{II}E$ both converge to the cohomology of $\operatorname{Tot}(X)$. Also, by the calculation in question (ii), for all $(p,q) \in \mathbb{Z}$ such that $p+q=2$, we have ${}^{II}E_\infty^{pq} = {}^{II}E_2^{pq} = 0$. So $\mathrm{H}^2(\operatorname{Tot}(X)) = 0$.

(iv) By definition of ${}^{I}E$, its first page is (where every term that doesn't appear is 0):

$$\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & \operatorname{Coker} v & \operatorname{Coker} w & \operatorname{Coker} t & 0 \\
\operatorname{Ker} u & \operatorname{Ker} v & \operatorname{Ker} w & 0 & 0
\end{array}$$

(v) For every $r \geq 1$, we have ${}^{I}E_r^{pq} = 0$ unless $p \in \{0,1\}$. In particular, if $r \geq 2$, then either the source of the target of d_r^{pq} is 0, so $d_r^{pq} = 0$. This shows that ${}^{I}E$ degenerates at the second page, hence that ${}^{I}E_\infty = {}^{I}E_2$.

(vi) As ${}^{I}E_\infty = {}^{I}E_2$, there exists a filtration on $\mathrm{H}^2(\operatorname{Tot}(X))$ whose quotients are the $\mathrm{E}_2^{p,2-p}$. But we have seen in questino (iii) that $\mathrm{H}^2(\operatorname{Tot}(X)) = 0$, so ${}^{I}E_2^{p,2-p} = 0$ for every $p \in \mathbb{Z}$. On the other hand, by question (iv), we have ${}^{I}E_2^{2,0} = \operatorname{Ker} w/f(\operatorname{Ker} v)$, ${}^{I}E_2^{1,1} = \operatorname{Ker}(g : \operatorname{Coker} v \to \operatorname{Coker} w)$ and ${}^{I}E_2^{0,2} = 0$. This shows that $\operatorname{Ker} w = f(\operatorname{Ker} v)$ and that the morphism $\operatorname{Coker} v \to \operatorname{Coker} w$ induced by g is injective; by question (i), that last fact is equivalent to the fact that $\operatorname{Im} v = g^{-1}(\operatorname{Im} w)$, so we are done.

(c). (i) As all the columns of the complex are exact, we have ${}^{I}E_1^{pq} = 0$ for all $p, q \in \mathbb{Z}$, so the spectral sequence ${}^{I}E$ degenerates at the first page, and we have ${}^{I}E_\infty = {}^{I}E_1 = 0$. Also, as X is a first quadrant double complex, the spectral sequence ${}^{I}E$ converges to $\mathrm{H}^\bullet(\mathrm{Tot}(X))$, so $\mathrm{H}^n(\mathrm{Tot}(X)) = 0$ for every $n \in \mathbb{Z}$.

(ii) Applying the formula for ${}^{II}E_1$, we get that it is equal to:

$$\begin{array}{cccc}
\cdots & \cdots & \cdots & 0 \\
\mathrm{H}^2(A^\bullet) & \mathrm{H}^2(B^\bullet) & \mathrm{H}^2(C^\bullet) & 0 \\
\mathrm{H}^1(A^\bullet) & \mathrm{H}^1(B^\bullet) & \mathrm{H}^1(C^\bullet) & 0 \\
\mathrm{H}^0(A^\bullet) & \mathrm{H}^0(B^\bullet) & \mathrm{H}^0(C^\bullet) & 0
\end{array}$$

In other words, we have ${}^{II}E_1^{pq} = 0$ if $p \notin \{0, 1, 2\}$, ${}^{II}E_1^{0,q} = \mathrm{H}^q(A^\bullet)$, ${}^{II}E_1^{1,q} = \mathrm{H}^q(B^\bullet)$ and ${}^{II}E_1^{2,q} = \mathrm{H}^q(C^\bullet)$.

(iii) By question (ii), we have ${}^{II}E_r^{pq} = 0$ if $r \geq 1$ and $p \notin \{0, 1, 2\}$. So, if $r \geq 3$, then either or target of $d_r^{pq} : {}^{II}E_r^{pq} \to {}^{II}E_e^{p+r,q-r+1}$ is 0, hence $d_r^{pq} = 0$. This shows that ${}^{II}E$ degenerates at the third page.

(iv) For every $r \geq 2$ and every $q \in \mathbb{Z}$, we have ${}^{II}E_r^{1+r,q-r+1} = 0$ and ${}^{II}E_r^{1-r,q+r-1} = 0$, so $d_r^{1,q} : {}^{II}E_r^{1q} \to {}^{II}E_r^{1+r,q-r+1}$ and $d_r^{1-r,q+r-1} : {}^{II}E_r^{1-r,q+r-1} \to {}^{II}E_r^{1,q}$ are both zero, so ${}^{II}E_{r+1}^{1,q} = {}^{II}E_r^{1,q}$. This shows that ${}^{II}E_\infty^{1,q} = {}^{II}E_2^{1,q}$ for every $q \in \mathbb{Z}$.

Also, if $r \geq 2$, we have ${}^{II}E_r^{r,-r+1} = {}^{II}E_2^{-r,r-1} = 0$, so $d_r^{0,0} : {}^{II}E_r^{0,0} \to {}^{II}E_r^{r,-r+1}$ and $d_r^{-r,r-1} : {}^{II}E_r^{-r,r-1} \to {}^{II}E_r^{0,0}$ are both zero, so ${}^{II}E_{r+1}^{0,0} = {}^{II}E_r^{0,0}$. This shows that' ${}^{II}E_\infty^{0,0} = {}^{II}E_2^{0,0}$. (Note that we only used the fact that we have a first quadrant spectral sequence in this paragraph.)

(v) As the spectral sequence ${}^{II}E$ degenerates at the third page and its limit $\mathrm{H}^\bullet(\mathrm{Tot}(X))$ is 0 by question (i), we have ${}^{II}E_3^{pq} = {}^{II}E_\infty^{p,q} = 0$ for all $p, q \in \mathbb{Z}$. As ${}^{II}E_3^{0,q} = \mathrm{Ker}(d_2^{0,q})$ and ${}^{II}E_3^{2,q-1} = \mathrm{Coker}(d_2^{0,q})$, this shows that $d_2^{0,q}$ is an isomorphisms for every $q \in \mathbb{Z}$.

(vi) We have ${}^{II}E_2^{0,0} = {}^{II}E_\infty^{0,0} = 0$ by questions (iv) and (v), so the morphism $d_1^{0,0} : \mathrm{H}^0(A^\bullet) \to \mathrm{H}^0(B^\bullet)$ is injective. Also, for every $q \geq 0$, we have ${}^{II}E_\infty^{1,q} = {}^{II}E_2^{1,q} = 0$, so the sequence

$$\mathrm{H}^q(A^\bullet) \xrightarrow{d_1^{0,q}} \mathrm{H}^q(B^\bullet) \xrightarrow{d_1^{1,q}} \mathrm{H}^q(C^\bullet)$$

is exact. Finally, we have seen in question (v) that, for every $q \geq 1$, the morphism

$$d_2^{0,q} : {}^{II}E_2^{0,q} = \mathrm{Ker}(d_1^{0,q}) \to {}^{II}E_2^{2,q-1} = \mathrm{Coker}(d_1^{1,q-1})$$

is an isomorphism; inverting it, we get an exact sequence

$${}^{II}E_1^{1,q-1} = \mathrm{H}^{q-1}(B^\bullet) \xrightarrow{d_1^{1,q-1}} {}^{II}E_1^{2,q-1} = \mathrm{H}^{q-1}(C^\bullet) \xrightarrow{\delta^{q-1}} {}^{II}E_1^{0,q} = \mathrm{H}^q(A^\bullet) \xrightarrow{d_1^{0,q}} {}^{II}E_1^{1,q} = \mathrm{H}^q(B_\bullet).$$

Putting all these exact sequences together gives the long exact that we wanted.

$\square$

2 Group cohomology

(a). **Cohomology of cyclic groups**: If G is a group, $a \in \mathbb{Z}[G]$ and M is a left $\mathbb{Z}[G]$-module, we denote by $a : M \to M$ the $\mathbb{Z}[C_n]$-linear map $x \longmapsto a \cdot x$. For every $n \geq 1$, we denote by C_n the cyclic group of order n and by σ a generator of C_n, and we write $N = 1+\sigma+\sigma^2+\ldots+\sigma^{n-1}$. We also write $C_\infty = \mathbb{Z}$ and $\sigma = 1 \in C_\infty$. If $n \in \{1, 2, \ldots\} \cup \{\infty\}$, we have a $\mathbb{Z}[C_n]$-linear map $\epsilon : \mathbb{Z}[C_\infty] \to \mathbb{Z}$ sending each element of C_n to $1 \in \mathbb{Z}$.

(i) (2 points) If $n \geq 1$, show that:

$$\ldots \to \mathbb{Z}[C_n] \stackrel{N}{\to} \mathbb{Z}[C_n] \stackrel{\sigma-1}{\to} \mathbb{Z}[C_n] \stackrel{N}{\to} \mathbb{Z}[C_n] \stackrel{\sigma-1}{\to} \mathbb{Z}[C_n] \stackrel{\epsilon}{\to} \mathbb{Z} \to 0$$

is an exact sequence.

(ii) (2 points) If M is a $\mathbb{Z}[C_n]$-module, show that:

$$\mathrm{H}^q(C_n, M) = \begin{cases} M^{C_n} & \text{if } q = 0 \\ M^{C_n}/N \cdot M & \text{if } q \geq 2 \text{ is even} \\ \{x \in M \mid N \cdot x = 0\}/(\sigma - 1) \cdot M & \text{if } q \text{ is odd.} \end{cases}$$

(iii) (1 point) Show that

$$0 \to \mathbb{Z}[C_\infty] \stackrel{\sigma-1}{\to} \mathbb{Z}[C_\infty] \stackrel{\epsilon}{\to} \mathbb{Z} \to 0$$

is an exact sequence.

(iv) (2 points) If M is a $\mathbb{Z}[C_\infty]$-module, show that:

$$\mathrm{H}^q(C_\infty, M) = \begin{cases} \{x \in M \mid \sigma \cdot x = x\} & \text{if } q = 0 \\ M/(\sigma - 1) \cdot M & \text{if } q = 1 \\ 0 & \text{if } q \geq 2. \end{cases}$$

(b). Let n be a integer, and let $G = C_n \rtimes C_2$ be the dihedral group of order $2n$, where the nontrivial element of C_2 acts on C_n by multiplication by -1. Then $K = C_n$ is a normal subgroup of G, and $G/K \simeq C_2$.

(i) (3 points) Show that

$$\mathrm{H}^q(C_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } q \geq 2 \text{ is even} \\ 0 & \text{if } q \text{ is odd,} \end{cases}$$

and show that the nontrivial element of C_2 acts by $(-1)^{q/2}$ on $\mathrm{H}^q(C_n, \mathbb{Z})$ if q is even.

(ii) (2 points) Calculate $\mathrm{H}^p(C_2, \mathrm{H}^q(C_n, \mathbb{Z}))$ for all $p, q \geq 0$.

(iii) (2 points) If n is odd, show that

$$\mathrm{H}^m(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } m = 2 \mod 4 \\ \mathbb{Z}/2n\mathbb{Z} & \text{if } m > 0 \text{ and } m = 0 \mod 4 \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

(c). Let G be a group, and suppose that G has a normal subgroup K such that $G/K \simeq \mathbb{Z}$. Let M be a $\mathbb{Z}[G]$-module.

(i) (1 point) Show that the Hochschild-Serre spectral sequence degenerates at E_2.

(ii) (2 points) We fix a generator σ of G/K and, for every $q \in \mathbb{N}$, we write $\mathrm{H}^q(K,M)^\sigma = \{x \in \mathrm{H}^q(K,M) \mid \sigma(x) = x\}$ and $\mathrm{H}^q(K,M)_\sigma = \mathrm{H}^q(K,M)/(\sigma-1)\cdot H^q(K,M)$.

Show that $\mathrm{H}^0(G,M) = \mathrm{H}^0(K,M)^\sigma$, and that we have short exact sequences

$$0 \to \mathrm{H}^{m-1}(K,M)_\sigma \to \mathrm{H}^m(G,M) \to \mathrm{H}^m(K,M)^\sigma \to 0$$

for every $m \geq 1$.

(d). Let G be a group.

(i) (2 points) If K is a central subgroup of G, show that G/K acts trivially on $\mathrm{H}_*(K,\mathbb{Z})$ and on $\mathrm{H}^*(K,\mathbb{Z})$.

Let σ be an element of infinite order in the center of G, and $K = \langle\sigma\rangle$. Let M be a $\mathbb{Z}[G]$-module. We write $M^\sigma = \{x \in M \mid \sigma\cdot x = x\}$ and $M_\sigma = M/(\sigma-1)\cdot M$.

(ii) (1 point) Show that the Hochschild-Serre spectral sequence calculating $\mathrm{H}^*(G,M)$ degenerates at E_3.

(iii) (2 points) Show that $\mathrm{H}^0(G,M) = \mathrm{H}^0(G/K,M^\sigma)$, and that we have a long exact sequence:

$$\begin{aligned} 0 \to \mathrm{H}^1(G/K,M^\sigma) \to &\mathrm{H}^1(G,M) \to \mathrm{H}^0(G/K,M_\sigma) \to \mathrm{H}^2(G/K,M^\sigma) \\ &\to \mathrm{H}^2(G,M) \to \mathrm{H}^1(G/K,M_\sigma) \to \mathrm{H}^3(G/K,M^\sigma) \to \dots \end{aligned}$$

Solution.

(a). (i) Let $x = \sum_{i=0}^{n-1} a_i\sigma^i \in \mathbb{Z}[C_n]$, with $a_0,\dots,a_{n-1} \in \mathbb{Z}$; we also write $a_n = a_0$ and $a_{-1} = a_{n-1}$. We have $\epsilon(x) = a_0 + a_1 + \ldots + a_{n-1}$, $(\sigma-1)(x) = \sum_{i=0}^{n-1}(a_{i+1}-a_i)\sigma^i$ and $N(x) = (a_0+a_1+\ldots+a_{n-1})\sum_{i=0}^{n-1}\sigma^i$. So

$$\mathrm{Ker}(\sigma-1) = \{x = \sum_{i=0}^{n-1} a_i\sigma^i \mid a_0 = a_1 = \ldots = a_{n-1}\} = \mathrm{Im}(N)$$

and

$$\mathrm{Im}(\sigma-1) = \{x = \sum_{i=0}^{n-1} a_i\sigma^i \mid a_0 + a_1 + \ldots + a_{n-1} = 0\} = \mathrm{Ker}(N) = \mathrm{Ker}(\epsilon).$$

(ii) Question (i) gives a resolution of the trivial $\mathbb{Z}[C_n]$-module $\mathbb{Z}$ by free $\mathbb{Z}[C_n]$-modules, so we can use it to calculate $\mathrm{H}^n(C_n,M) = \mathrm{Ext}^n_{\mathbb{Z}[C_n]}(\mathbb{Z},M)$ by Theorem IV.3.4.1 of the notes . So $\mathrm{H}^n(C_m,M)$ is the cohomology of the following complex (concentrated in degree ≥ 0):

$$\mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n],M) \overset{(\cdot)\circ(\sigma-1)}{\to} \mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n],M) \overset{(\cdot)\circ N}{\to} \mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n],M) \overset{(\cdot)\circ(\sigma-1)}{\to} \dots$$

We have an isomorphism $\mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n],M) \overset{\sim}{\to} M$ sending $u : \mathbb{Z}[C_n] \to M$ to $u(\sigma)$. By isomorphism, the endomorphism1 $(\cdot)\circ(\sigma-1)$ (resp. $(\cdot)\circ N$) of $\mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n],M)$ corresponds to the action of $\sigma-1$ (resp. N) on M. Moreover, as σ generates C_n, we have $\mathrm{Ker}(\sigma-1 : M \to M) = M^{C_n}$. This gives the desired formulas for $\mathrm{H}^q(C_n,M)$.

(iii) Let $x = \sum_{n\in\mathbb{Z}} a_n\sigma^n \in \mathbb{Z}[C_\infty]$, with $a_n = 0$ for $|n|$ big enough. Then $\epsilon(x) = \sum_{n\in\mathbb{Z}} a_n$ and $(\sigma-1)(x) = \sum_{n\in\mathbb{Z}}(a_{n+1}-a_n)\sigma^n$. So

$$\mathrm{Im}(\sigma-1) = \{x = \sum_{n\in\mathbb{Z}} a_n\sigma^n \in \mathbb{Z}[C_\infty] \mid \sum_{n\in\mathbb{Z}} a_n = 0\} = \mathrm{Ker}(\epsilon).$$

On the other hand, we have $x = \sum_{n\in\mathbb{Z}} a_n\sigma^n \in \mathrm{Ker}(\sigma-1)$ if and only if $a_n = a_{n+1}$ for every $n \in \mathbb{Z}$, i.e. if and only if all the a_n are equal; as we must have $a_n = 0$ for $|n|$ big enough, this forces all the a_n to be 0. So $\mathrm{Ker}(\sigma-1) = \{0\}$.

(iv) This is exactly the same proof as in question (ii), except that we wrote $\mathrm{Ker}(\sigma - 1 : M \to M)$ as $\{x \in M \mid \sigma\cdot x = x\}$ instead of M^{C_∞}. (These are just two ways of writing the same object.)

(b). (i) We apply the formulas of question (a)(ii). As C_n acts trivially on $\mathbb{Z}$, we have $\mathbb{Z}^{C_n} = \mathbb{Z}$, and N acts on $\mathbb{Z}$ by multiplicatoion by $\sum_{i=0}^{n-1} 1 = n$, so $N\cdot\mathbb{Z} = n\mathbb{Z}$ and $\{x \in \mathbb{Z} \mid N\cdot x = 0\} = \{0\}$. This immediately gives the desired formula for $\mathrm{H}^q(C_n,\mathbb{Z})$.

Let τ be the nontrivial element of C_2. Then, if we make C_n act on it via $(g,x) \longmapsto (\tau g\tau^{-1})\cdot x$, the resolution of (a)(i) is isomorphic to the following (exact) complex of $\mathbb{Z}[C_n]$-modules:

$$(*) \quad \ldots \to \mathbb{Z}[C_n] \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{-\sigma+1} \mathbb{Z}[C_n] \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{-\sigma+1} \mathbb{Z}[C_n] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

To calculate the action of τ on $\mathrm{H}^\bullet(C_n,\mathbb{Z})$, we need to extend the action of τ on $\mathbb{Z}$ (which is given by $\mathrm{id}_\mathbb{Z}$) to a morphism between the resolution of (a)(i) and (*). Here is a possibility:

$$\begin{tikzcd}
\ldots \arrow[r] & \mathbb{Z}[C_n] \arrow[r, "\sigma-1"] \arrow[d, "(-\sigma)^3"'] & \mathbb{Z}[C_n] \arrow[r, "N"] \arrow[d, "(-\sigma)^2"'] & \mathbb{Z}[C_n] \arrow[r, "\sigma-1"] \arrow[d, "(-\sigma)^2"'] & \mathbb{Z}[C_n] \arrow[r, "N"] \arrow[d, "-\sigma"'] & \mathbb{Z}[C_n] \arrow[r, "\sigma-1"] \arrow[d, "-\sigma"'] & \mathbb{Z}[C_n] \arrow[r, "\epsilon"] \arrow[d, "1"'] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}_\mathbb{Z}"'] & 0 \\
\ldots \arrow[r] & \mathbb{Z}[C_n] \arrow[r, "-\sigma+1"] & \mathbb{Z}[C_n] \arrow[r, "N"] & \mathbb{Z}[C_n] \arrow[r, "-\sigma+1"] & \mathbb{Z}[C_n] \arrow[r, "N"] & \mathbb{Z}[C_n] \arrow[r, "-\sigma+1"] & \mathbb{Z}[C_n] \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

So, on $\mathrm{H}^{2i}(C_n,\mathbb{Z})$, the action of τ is given by $(-\sigma)^i$; as $\mathrm{H}^{2i}(C_n,\mathbb{Z})$ is a quotient $\mathbb{Z}^{C_n} = \mathbb{Z}$, where C_n acts trivially, the action of $(-\sigma)^i$ is given by multiplication by $(-1)^i$.

(ii) We apply the formulas of (a)(ii) for $n = 2$. If q is odd, then $\mathrm{H}^q(C_n,\mathbb{Z}) = 0$, so $\mathrm{H}^p(C_2,\mathrm{H}^q(C_n,\mathbb{Z})) = 0$ for every $p \geq 0$. If $q = 0$, then $\mathrm{H}^q(C_n,\mathbb{Z}) = \mathbb{Z}$ with the trivial action of C_2, so, by question (i),

$$\mathrm{H}^p(C_2,\mathrm{H}^0(C_n,\mathbb{Z})) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } p \geq 2 \text{ is even} \\ 0 & \text{if } p \text{ is odd.} \end{cases}$$

Suppose that $q \geq 2$ is even and write $q = 2i$. Then $\mathrm{H}^q(C_n,\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ and the nontrivial element τ of C_2 acts by $(-1)^i$ on $\mathrm{H}^q(C_n,\mathbb{Z})$. We use the formula of (a)(ii), and we distinguish four cases:

(1) <u>i is even and n is odd</u>: Then $\mathrm{H}^{2i}(C_n,\mathbb{Z})^{C_2} = \mathrm{H}^{2i}(C_n,\mathbb{Z})$, $\mathrm{H}^{2i}(C_n,\mathbb{Z})/(1+\tau)\cdot\mathrm{H}^{2i}(C_n,\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})/2(\mathbb{Z}/n\mathbb{Z}) = 0$ and $\{x \in \mathbb{Z}/n\mathbb{Z} \mid 2x = 0\} = \{0\}$. So

$$\mathrm{H}^p(C_2,\mathrm{H}^{2i}(C_n,\mathbb{Z})) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } p = 0 \\ 0 & \text{if } p \geq 1 \end{cases}$$

(2) <u>i is even and n is even</u>: Then $\mathrm{H}^{2i}(C_n,\mathbb{Z})^{C_2} = \mathrm{H}^{2i}(C_n,\mathbb{Z})$, $\mathrm{H}^{2i}(C_n,\mathbb{Z})/(1+\tau)\cdot \mathrm{H}^{2i}(C_n,\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})/2(\mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/(n/2)\mathbb{Z}$ and $\{x\in\mathbb{Z}/n\mathbb{Z} \mid 2x=0\} = (n/2)\mathbb{Z}/n\mathbb{Z}\simeq\mathbb{Z}/2\mathbb{Z}$. So

$$\mathrm{H}^p(C_2,\mathrm{H}^{2i}(C_n,\mathbb{Z})) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } p=0 \\ \mathbb{Z}/(n/2)\mathbb{Z} & \text{if } p\geq 2 \text{ is even} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } p \text{ is odd.}\end{cases}$$

(3) <u>i is odd and n is odd</u>: Then $\mathrm{H}^{2i}(C_n,\mathbb{Z})^{C_2}=0$, the element $1+\tau$ of $\mathbb{Z}[C_2]$ acts on $\mathrm{H}^n(C_n,\mathbb{Z})$ by 0 and the element $\tau-1$ acts by multiplication by -2. So $\mathrm{H}^p(C_2,\mathrm{H}^{2i}(C_n,\mathbb{Z}))=0$ for every $p\geq 0$.

(4) <u>i is odd and n is even</u>: As in case (3), we have $\mathrm{H}^{2i}(C_n,\mathbb{Z})^{C_2}=0$, the element $1+\tau$ of $\mathbb{Z}[C_2]$ acts on $\mathrm{H}^n(C_n,\mathbb{Z})$ by 0 and the element $\tau-1$ acts by multiplication by -2. So

$$\mathrm{H}^p(C_2,\mathrm{H}^{2i}(C_n,\mathbb{Z})) = \begin{cases} 0 & \text{if } p \text{ is even} \\ \mathbb{Z}/(n/2)\mathbb{Z} & \text{if } p \text{ is odd.}\end{cases}$$

(iii) We use the Hochschild-Serre spectral sequence:

$$E_2^{pq} = \mathrm{H}^p(C_2,\mathrm{H}^q(C_n,\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(G,\mathbb{Z}).$$

By question (ii) (and the fact that n is odd), we have

$$E_2^{pq} = \begin{cases} \mathbb{Z} & \text{if } p=q=0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } q=0 \text{ and } p\geq 2 \text{ is even} \\ \mathbb{Z}/n\mathbb{Z} & \text{if } q\geq 1 \text{ is in } 4\mathbb{N} \text{ and } p=0 \\ 0 & \text{otherwise.}\end{cases}$$

So the second page of the spectral sequence looks like this:

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbb{Z}/n\mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & \dots
\end{array}$$

In particular, for every $r\geq 2$, we have $E_2^{pq}=0$ unless p and q are even; this implies that $d_r : E_r^{pq}\to E_2^{p+r,q-r+1}$ is always 0 (if r is odd, then p and $p+r$ cannot be even at the same time; if r is even, then q and $q-r+1$ cannot be even at the same time). So the spectral sequence degenerates at E_2, and we have $E_\infty^{pq}=E_2^{pq}$.

If m is odd, then $E_\infty^{p,m-p}=0$ for every $p\in\mathbb{Z}$, so $\mathrm{H}^m(G,\mathbb{Z})=0$. If $m=2 \mod 4$, then the only $E_\infty^{p,m-p}$ that is nonzero is $E_\infty^{m,0}=\mathbb{Z}/2\mathbb{Z}$, so $\mathrm{H}^m(G,\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$. Finally, suppose that $m>0$ and $m=0 \mod 4$. Then the only two $E_\infty^{p,m-p}$ that are nonzero are $E_\infty^{0,m}=\mathbb{Z}/n\mathbb{Z}$ and $E_\infty^{m,0}=\mathbb{Z}/2\mathbb{Z}$, so we have an exact sequence

$$0\to\mathbb{Z}/2\mathbb{Z}\to\mathrm{H}^m(G,\mathbb{Z})\to\mathbb{Z}/n\mathbb{Z}\to 0.$$

As $\mathrm{H}^m(G,\mathbb{Z})$ is an abelian group and n is odd, this gives an isomorphism

$$\mathrm{H}^m(G,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}\simeq\mathbb{Z}/2n\mathbb{Z}.$$

(c). (i) We have $E_2^{pq} = \mathrm{H}^p(G/K, \mathrm{H}^q(K, M))$, so, by (a)(iv), we get $E_2^{pq} = 0$ if $p \notin \{0, 1\}$. So $d_r^{pq} = 0$ if $r \geq 2$, and the spectral sequence degenerates at E_2.

(ii) By (a)(iv) again, we have

$$E_\infty^{pq} = E_2^{pq} = \mathrm{H}^p(G/K, \mathrm{H}^q(K, M)) = \begin{cases} \mathrm{H}^q(K, M)^\sigma & \text{if } p = 0 \\ \mathrm{H}^q(K, M)_\sigma & \text{if } p = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $m \in \mathbb{Z}$. Then $\mathrm{H}^m(G, M)$ has a decreasing filtration $\mathrm{Fil}^p\mathrm{H}^m(G, M)$ such that $\mathrm{Fil}^p\mathrm{H}^m(G, M) = 0$ if $p \geq 2$, $\mathrm{Fil}^1\mathrm{H}^m(G, M) = E_\infty^{1,m-1}$, $\mathrm{Fil}^0\mathrm{H}^m(G, M)/\mathrm{Fil}^1\mathrm{H}^m(G, M) = E_\infty^{0,m}$ and $\mathrm{Fil}^p\mathrm{H}^m(G, M) = \mathrm{H}^m(G, M)$ if $p \leq 0$. In other words, we have an exact sequence

$$0 \to E_\infty^{1,m-1} \to \mathrm{H}^m(G, M) \to E_\infty^{0,m} \to 0.$$

Combining this with the formula for E_∞^{pq} gives the result. (If $m = 0$, then $E_\infty^{1,m-1} = 0$, so we get $\mathrm{H}^m(G, M) = E_\infty^{0,0} = \mathrm{H}^0(K, M)^\sigma$.)

(d). (i) The action of G on K by conjugation is trivial, and its action on $\mathbb{Z}$ is also trivial, so G acts trivially on $\mathrm{H}_*(G, \mathbb{Z})$ and $\mathrm{H}^*(G, \mathbb{Z})$.

(ii) We have $E_2^{pq} = \mathrm{H}^p(\mathrm{G}/K, \mathrm{H}^q(K, M))$, so, by (a)(iv), $E_2^{pq} = 0$ if $q \notin \{0, 1\}$. In particular, if $r \geq 3$, then the source or target of $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ is 0 for every choice of $(p, q) \in \mathbb{Z}$, so all teh d_r^{pq} are zero. So the spectral sequence degenerates at E_3.

(iii) By question (i), we have $E_\infty^{pq} = E_3^{pq}$, so $E_\infty^{pq} = 0$ if $q \notin \{0, 1\}$,

$$E_\infty^{m,0} = E_3^{m,0} = \mathrm{Coker}(E_2^{m-2,1} \to E_2^{m,0}) = \mathrm{Coker}(\mathrm{H}^{m-2}(G/K, M_\sigma) \to \mathrm{H}^m(G/K, M^\sigma))$$

and

$$E_\infty^{m-1,1} = E_3^{m-1,1} = \mathrm{Ker}(E_2^{m-1,1} \to E_2^{m+1,0}) = \mathrm{Coker}(\mathrm{H}^{m-1}(G/K, M_\sigma) \to \mathrm{H}^{m+1}(G/K, M^\sigma)).$$

Let $m \in \mathbb{Z}$. Then $\mathrm{H}^m(G, M)$ has a decreasing filtration $\mathrm{Fil}^p\mathrm{H}^m(G, M)$ such that $\mathrm{Fil}^p\mathrm{H}^m(G, M) = 0$ if $p \geq m + 1$, $\mathrm{Fil}^p\mathrm{H}^m(G, M) = \mathrm{H}^m(G, M)$ if $p \leq m - 1$, $\mathrm{Fil}^m\mathrm{H}^m(G, M) = E_\infty^{m,0}$, and $\mathrm{Fil}^{m-1}\mathrm{H}^m(G, M)/\mathrm{Fil}^m\mathrm{H}^m(G, M) = E_\infty^{m-1,1}$. If $m = 0$, then $E_\infty^{m-1,1} = 0$, so we get

$$\mathrm{H}^0(G, M) = E_\infty^{0,0} = \mathrm{H}^0(G/K, M^\sigma).$$

If $m = 1$, we get an exact sequence

$$\mathrm{H}^{m-2}(G/K, M_\sigma) \to \mathrm{H}^m(G/K, M^\sigma) \to \mathrm{H}^m(G, M) \to \mathrm{H}^{m-1}(G/K, M_\sigma) \to \mathrm{H}^{m+1}(G/K, M^\sigma).$$

Putting all these exact sequences together gives the desired long exact sequence.

□

3 Flabby and soft sheaves

Let X be a topological space. If $\mathscr{F}$ is a sheaf on X and Y is a subset of X, we set

$$\mathscr{F}(Y) = \varinjlim_{Y \subset U \in \mathrm{Open}(X)^{\mathrm{op}}} \mathscr{F}(U).$$

If $Y \subset Y'$, we have a map $\mathscr{F}(Y') \to \mathscr{F}(Y)$ induced by the restriction maps of $\mathscr{F}$.

We say that $\mathscr{F}$ is *flabby* (or *flasque*) if, for every open subset U of X, the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective. We say that $\mathscr{F}$ is *soft* if, for every *closed* subset F, the map $\mathscr{F}(X) \to \mathscr{F}(F)$ is surjective.

Let R be a ring. If M is a left R-module and $x \in X$, we write $S_{x,M}$ for the presheaf on X given by $S_{x,M}(U) = M$ if $x \in U$ and $S_{x,M}(U) = 0$ if $x \notin U$, with the obvious restriction maps (equal to 0 or id_M). It is easy to see that this is a sheaf, and we call it the *skryscraper sheaf* at x with value M.

(a). (1 point) Show that any flabby sheaf is soft.

(b). (2 points) Let $d \geq 1$, and let $\mathscr{F}$ be the sheaf $U \longmapsto C^\infty(U, \mathbb{C})$ on $\mathbb{R}^d$. Show that the sheaf $\mathscr{F}$ is soft. [1]

(c). (1 point) For every $x \in X$, show that the functor ${}_R\mathbf{Mod} \to \mathrm{Sh}(\mathscr{F}, R)$, $M \longmapsto S_{x,M}$ is right adjoint to the functor $\mathscr{F} \longmapsto \mathscr{F}_x$.

(d). (2 points) If $(M_x)_{x \in X}$ is a family of R-modules, show that $\prod_{x \in X} S_{x,M_x}$ is a flabby sheaf, and that it is an injective sheaf if every M_x is an injective R-module.

(e). (1 point) For every sheaf of R-modules $\mathscr{F}$ on X, we set $G(\mathscr{F}) = \prod_{x \in X} S_{x,\mathscr{F}_x}$. Show that the canonical morphism $\mathscr{F} \to G(\mathscr{F})$ (sending any $s \in \mathscr{F}(U)$ to the family $(s_x)_{x \in U}$) is injective. [2]

(f). (2 points) Show that sheaves of R-modules on X have a functorial resolution by flabby injective sheaves.

(g). (2 points) Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence in $\mathrm{Sh}(X, R)$, with $\mathscr{F}$ flabby. Show that the sequence $0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X) \to 0$ is exact.

An open cover $(U_i)_{i \in I}$ of X is called *locally finite* if every point of X has a neighborhood that meets only finitely many of the U_i. We say that X is *paracompact* if every open cover of X has a locally finite refinement. We admit the following facts:

(1) A metric space is paracompact.

(2) If X is paracompact and $(U_i)_{i \in I}$ is an open cover of X, then there exists an open cover $(V_i)_{\in I}$ of X such that $\overline{V_i} \subset U_i$ for every $i \in I$. [3]

(h). Suppose that X is a separable metric space. [4] Let $0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0$ be a short exact sequence of sheaves of R-modules on X, with $\mathscr{F}$ soft. The goal of this question is to prove that the sequence $0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X) \to 0$ is exact.

(i) (1 point) Let $s \in \mathscr{H}(X)$. Show that there exists a locally finite open cover $(U_n)_{n \in \mathbb{N}}$ and sections $t_n \in \mathscr{G}(U_n)$ such that $g(t_n) = s_{|U_n}$ for every $n \in \mathbb{N}$.

(ii) (2 points) Take an open cover $(V_n)_{n \in \mathbb{N}}$ of X such that $F_n := \overline{V_n} \subset U_n$ for every $n \in \mathbb{N}$. Prove by induction on n that, for every $n \geq 0$, there exists a section $a_n \in \mathscr{G}(F_0 \cup \ldots \cup F_n)$ such that $g(a_n) = s_{|F_0 \cup \ldots \cup F_n}$.

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[1] More generally, if X is a smooth manifold, then the sheaf Ω_X^k of degree k differential forms on X is soft. As the sequence $0 \to \underline{\mathbb{C}}_X \to \Omega_X^1 \to \Omega_X^2 \to \ldots$ is exact by the Poincaré lemma, this, and the fact that soft sheaves are $\mathrm{H}^0(X, \cdot)$-acyclic, shows that the cohomology of the constant sheaf $\underline{\mathbb{C}}_X$ is isomorphic to the de Rham cohomology of X.

[2] The "G" is for "Godement", who invented this method of constructing flabby resolutions of sheaves.

[3] This follows from the fact that there exists a partition of unity subordinate to $(U_i)_{i \in I}$, which uses the fact that paracompact spaces are normal and Urysohn's lemma.

[4] It would be enough to assume that X is paracompact.

(iii) (1 point) Show that s has a preimage in $\mathscr{G}(X)$.

(i). (3 points) If $\mathscr{F}$ is a flabby sheaf of R-modules on a topological space X, or a soft sheaf of R-modules on a separable metric space X, show that $\mathrm{H}^n(X, \mathscr{F}) = 0$ for every $n \geq 1$. (Hint: Try to adapt the strategy of Problem 5(b) of problem set 6.)

Solution.

(a). Let $\mathscr{F}$ be a flabby sheaf. Let F be a closed subset of X and $s \in \mathscr{F}(F)$. By definition of $\mathscr{F}(F)$, there exists an open subset $U \supset F$ of X and a representative $s' \in \mathscr{F}(U)$ of s. As $\mathscr{F}$ is flabby, there exists $t \in \mathscr{F}(X)$ such that $t_{|X} = s'$, and then $t_{|F} = s$. So $\mathscr{F}$ is soft.

(b). Let F be a closed subset of $\mathbb{R}^d$, and let $s \in \mathscr{F}(F)$. By definition of $\mathscr{F}(F)$, there exists an open subset $V \supset F$ of $\mathbb{R}^d$ and a C^∞ function $f : U \to \mathbb{C}$ representing s.

I claim that there exists a locally finite open cover $(U_i)_{i \in I}$ of $\mathbb{R}^n$ and a subset J of I such that $F \subset \bigcup_{j \in J} U_j \subset U$ and $U_i \cap F = \varnothing$ if $i \in I - J$. Here is a way to prove this claim: For every $x \in F$, choose an open neighborhood U_x of x such that $B_x \subset U$. For every $x \in \mathbb{R}^n - F$, choose an open neighborhood U_x of x such that $U_x \cap F = \varnothing$. As $\mathbb{R}^n$ is paracompact, there exists $I \subset \mathbb{R}^n$ such that $(U_x)_{x \in I}$ is a locally finite open cover of $\mathbb{R}^n$. Let $J = I \cap F$. Then $F \cap (\bigcup_{x \in I - J} U_x) = \varnothing$, so $F \subset \bigcup_{x \in J} U_x \subset U$.

Now choose a C^∞ partition of unity $(\varphi_i)_{i \in I}$ subordinate to the open cover $(U_i)_{i \in I}$, and let $\varphi = \sum_{j \in J} \varphi_j$. Then $\mathrm{supp}(\varphi) \subset \bigcup_{j \ni J} U_j \subset U$ and, if $x \in F$, then $1 = \sum_{i \in I} \varphi_i(x) = \sum_{j \in J} \varphi_j(x) = 1$. Define a function $g : \mathbb{R}^n \to \mathbb{C}$ by $g(x) = \varphi(x) f(x)$ if $x \in U$ and $g(x) = 0$ if $x \notin U$. Then g is C^∞ on U, and $g = 0$ on $\mathbb{R}^n - \mathrm{supp}(\varphi)$. So g is C^∞, i.e. $g \in \mathscr{F}(\mathbb{R}^n)$, and $g_{|F} = s$.

(c). Let $\mathscr{F} \in \mathrm{Ob}(\mathrm{Sh}(X, R))$ and $M \in \mathrm{Ob}({}_R\mathbf{Mod})$. Then we have

$$\mathrm{Hom}_R(\mathscr{F}_x, M) = \mathrm{Hom}_R(\varinjlim_{x \in U} \mathscr{F}(U), M) = \varprojlim_{x \in U} \mathrm{Hom}_R(\mathscr{F}(U), M).$$

On the other hand, a morphism $\mathscr{F} \to S_{x,M}$ is a family $(f_U)_{U \in \mathrm{Open}(X)}$ of morphisms of R-modules $f_U : \mathscr{F}(U) \to S_{x,M}(U)$, with $f_U = 0$ if $x \notin U$ (because then $S_{x,M}(U) = 0$ and $f_U : \mathscr{F}(U) \to M$ if $x \in U$, satisfying the condition that, if $x \in V \subset U$, then, for every $s \in \mathscr{F}(U)$, we have $f_V(s_{|V}) = f_U(s)$. In other words, the family $(f_U)_{U \ni x}$ is an element of $\varprojlim_{x \in U} \mathrm{Hom}_R(\mathscr{F}(U), M) = \mathrm{Hom}_R(\mathscr{F}_x, M)$. This defines an isomorphism $\mathrm{Hom}_{\mathrm{Sh}(X,R)}(\mathscr{F}, S_{x,M}) \simeq \mathrm{Hom}_R(\mathscr{F}_x, M)$, that is clearly functorial in $\mathscr{F}$ and M.

(d). Let $\mathscr{F} = \prod_{x \in X} S_{x, M_x}$. Let U be an open subset of X. Then $\mathscr{F}(U) = \prod_{x \in U} M_x$ and $\mathscr{F}(X) = \prod_{x \in X} M_x$, and the restriction morphism $\mathscr{F}(X) \to \mathscr{F}(U)$ is given by the canonical projection on the factors indexed by $x \in U$, which is clearly surjective. So $\mathscr{F}$ is flabby.

Suppose that M_x is an injective R-module for every $x \in X$. Then, for every $x \in X$, the sheaf $S_{x,M}$ is injective by Lemma II.2.4.4 of the notes and question (c). By Lemma II.2.4.3 of the notes , the sheaf $\prod_{x \in X} S_{x, M_x}$ is also injective.

(e). Denote by $c : \mathscr{F} \to G(\mathscr{F})$ the canonical morphism. Let U be an open subset of X and $s \in \mathscr{F}(U)$ such that $c(s) = 0$. As $c(s) = (s_x)_{x \in U}$, we have $s_x = 0$ for every $x \in U$, so $s = 0$.

(f). As ${}_R\mathbf{Mod}$ is a Grothendieck abelian category, there exists a functor $\Phi : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$ and a morphism of functors $\iota : \mathrm{id}_{{}_R\mathbf{Mod}} \to \Phi$ such that, for every $M \in \mathrm{Ob}({}_R\mathbf{Mod})$, the R-module $\Phi(M)$ is injective and $\iota(M) : M \to \Phi(M)$ is an injective morphism. (See Theorem II.3.2.4 of the notes.)

For every sheaf of R-modules $\mathscr{F}$ on X, let $G'(\mathscr{F}) = \prod_{x\in X} \Phi(\mathscr{F}_x)$ and let $c' : \mathscr{F} \to G'(\mathscr{F})$ be the composition of $c : \mathscr{F} \to G(\mathscr{F})$ and of $\prod_{x\in X} \iota(\mathscr{F}_x) : G(\mathscr{F}) \to G'(\mathscr{F})$. Then G' is a functor and c' is a morphism of functors. Also, the sheaf $G'(\mathscr{F})$ is always injective and flabby by question (d). The construction of the proof of Lemma IV.3.1.2 of the notes gives a functorial resolution of $\mathscr{F}$ by injective and flabby sheaves. [5]

(g). We give names to the morphisms of the sequence:

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0.$$

Let $s \in \mathscr{H}(X)$. We want to show that there exists $t \in \mathscr{G}(X)$ such that $g(t) = s$. We consider the set I of pairs (U, t), where $U \subset X$ is an open subset and $t \in \mathscr{G}(U)$ is such that $g(t) = s_{|U}$. Note that I is not empty, because s admits preimages by g locally on X. We consider the following (partial) order on I: $(U_1, t_1) \le (U_2, t_2)$ if $U_1 \subset U_2$ and $t_1 = t_{2|U_1}$. Let J be a nonempty totally ordered subset of I; for every $i \in J$, let (U_i, t_i) be the corresponding pair. Let $U = \bigcup_{i\in J} U_i$. If $i, j \in J$, I claim that $t_{i|U_i\cap U_j} = t_{j|U_i\cap U_j}$; indeed, we may assume that $i \le j$, and then $U_i \subset U_j$ and $t_i = t_{j|U_i}$. So there exists $t \in \mathscr{G}(U)$ such that $t_{|U_i} = t_i$ for every $i \in J$, and then $(U, t) \ge (u_i, t_i)$ for every $i \in J$. By Zorn's lemma, the set I has a maximal element (U, t). I claim that $U = X$, which finishes the proof. Suppose that $U \ne X$. Then there exists an open subset $V \not\subset U$ of X and $t' \in \mathscr{G}(V)$ such that $g(t') = s_{|V}$. In particular, we have $g(t_{U\cap V} - t'_{U\cap V}) = 0$, so there exists $u \in \mathscr{F}(U\cap V)$ such that $f(u =)t_{U\cap V} - t'_{U\cap V}$. As $\mathscr{F}$ is flabby, there exists $v \in \mathscr{F}(X)$ such that $u = v_{|U\cap V}$. Let $t'' = t' + f(v_{|V})$. Then $t'_{|U\cap V} = t'_{|U\cap V} + f(u) = t_{|U\cap V}$, so there exists $t_1 \in \mathscr{G}(U \cup V)$ such that $t_{1|U} = t$ and $t_{1|V} = t''$. We have $g(t_1)_{|U} = g(t) = s_{|U}$ and $g(t_1)_{|V} = g(t'') = g(t') = s_{|V}$, so $g(t_1) = s_{|U\cup V}$. As $U \subsetneq U \cup V$, this contradicts the maximality of (U, t).

(h).
(i) We can find an open cover $(U_i)_{i\in I}$ of X and sections $t_i \in \mathscr{G}(U_i)$ with $g(t_i) = s_{|U_i}$ for every $i \in I$. As X is paracompact, after replacing the cover $(U_i)_{i\in I}$ by a refinement, we may assume that it is locally finite. As X is a separable metric space, its topology has a countable basis. So we may assume that the cover $(U_i)_{i\in I}$ is countable.

(ii) We take $a_0 = t_{0|F_0}$. Suppose that $n \ge 0$ and that we have found a_n. Let $\Omega \supset F_0 \cup \ldots \cup F_n$ be an open subset of X and $a'_n \in \mathscr{G}(\Omega)$ be a representative of a_n. We have $g(a'_n)_{|U_{n+1}\cap(F_0\cup\ldots\cup F_n)} = s_{|U_{n+1}\cap(F_0\cup\ldots\cup F_n)} = g(t_{n+1})_{|U_{n+1}\cap(F_0\cup\ldots\cup F_n)}$, so, after shrinking Ω, we may assume that $g(a'_n) = g(t_{n+1|\Omega\cap U_{n+1}})$. Then there exists $b \in \mathscr{F}(\Omega \cap U_{n+1})$ such that $f(b) = a'_n - t_{n+1|\Omega\cap U_{n+1}}$. As $\mathscr{F}$ is soft, there exists $b' \in \mathscr{F}(X)$ such that $b'_{|F_0\cup\ldots\cup F_n} = b_{|F_0\cup\ldots\cup F_n}$. After shrinking Ω again, we may assume that $b'_{|\Omega} = b$. Let $t'_{n+1} = t_{n+1} + f(b'_{|U_{n+1}})$. Then $g(t'_{n+1}) = g(t_{n+1}) = s_{|U_{n+1}}$ and $t'_{n+1|\Omega\cap U_{n+1}} = t_{n+1|\Omega\cap U_{n+1}} + f(b_{\Omega\cap U_{n+1}}) = a'_{n|\Omega\cap U_{n+1}}$. So there exists $a'_{n+1} \in \mathscr{G}(U_{n+1} \cup \Omega)$ such that $a'_{n+1|\Omega} = a'_n$ and $a'_{n+1|U_{n+1}} = t_{n+1}$, and we have $g(a'_{n+1}) = s_{|\Omega\cup U_{n+1}}$. We take for $a_{n+1} \in \mathscr{G}(F_0 \cup \ldots \cup F_{n+1})$ the element represented by $a'_{n+1} \in \mathscr{G}(\Omega \cup U_{n+1})$.

(iii) As $X = \bigcup_{n\ge 0} V_n$, the family $(a_{n|V_0\cup\ldots\cup V_n})_{n\ge 0}$ glues to a section $a \in \mathscr{G}(X)$ such that $g(a_{|V_n}) = s_{|V_n}$ for every n, hence $g(a) = s$.

(i). Let $\mathscr{C}$ be the full subcategory of $\mathrm{Sh}(X, R)$ whose objects are flabby sheaves of R-modules on X. Suppose that $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence in $\mathrm{Sh}(X, R)$ with $\mathscr{F}$ and $\mathscr{G}$ flabby. We claim that $\mathscr{H}$ is also flabby. Indeed, let U be an open subset of X. By

[5] Actually, with a little more work we could show that every injective sheaf is flabby, so any functorial resolution of $\mathscr{F}$ by injective sheaves (which exists because $\mathrm{Sh}(X, R)$ is a Grothendieck abelian category) is a resolution by injective and flabby sheaves. But it is simpler to use the functor G.

question (g) (whose proof adapts immediately to show that $\mathscr{G}(U) \to \mathscr{H}(U)$ is surjective), we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}(X) \arrow[r] \arrow[d, "u"'] & \mathscr{G}(X) \arrow[r] \arrow[d, "v"'] & \mathscr{H}(X) \arrow[r] \arrow[d, "w"'] & 0 \\
0 \arrow[r] & \mathscr{F}(U) \arrow[r] & \mathscr{G}(U) \arrow[r] & \mathscr{H}(U) \arrow[r] & 0
\end{tikzcd}$$

where the morphisms u and v are surjective. So w is also surjective.

We show by induction on n that, for every $n \geq 1$ and every $\mathscr{F} \in \mathrm{Ob}(\mathscr{C})$, we have $\mathrm{H}^n(X, \mathscr{F}) = 0$. Suppose that $n = 1$. Let $\mathscr{F}$ be an object of $\mathscr{C}$. Choose a monomorphism $\mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ an injective sheaf. Let $\mathscr{H} = \mathscr{G}/\mathscr{F}$. The long exact sequence of cohomology gives an exact sequence

$$\mathscr{G}(X) \to \mathscr{H}(X) \to \mathrm{H}^1(X, \mathscr{F}) \to \mathrm{H}^1(X, \mathscr{G}).$$

But $\mathrm{H}^1(X, \mathscr{G}) = 0$ because $\mathscr{G}$ is injective and $\mathscr{G}(X) \to \mathscr{H}(X)$ is surjective because $\mathscr{F}$ is flabby (by question (g)), so $\mathrm{H}^1(X, \mathscr{F}) = 0$.

Now suppose the result known for $n \geq 1$, and let us prove it for $n + 1$. Let $\mathscr{F}$ be a flabby sheaf on X. By question (f), there exists a monomorphism $\mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ an injective flabby sheaf. Let $\mathscr{H} = \mathscr{G}/\mathscr{F}$. We have shown that $\mathscr{H}$ is flabby. The long exact sequence of cohomology gives an exact sequence

$$\mathrm{H}^n(X, \mathscr{H}) \to \mathrm{H}^{n+1}(X, \mathscr{F}) \to \mathrm{H}^{n+1}(X, \mathscr{G}).$$

But $\mathrm{H}^{n+1}(X, \mathscr{G}) = 0$ because $\mathscr{G}$ is injective, and $\mathrm{H}^n(X, \mathscr{H}) = 0$ by the induction hypothesis, so $\mathrm{H}^{n+1}(X, \mathscr{F}) = 0$.

The proof for soft sheaves on a separable metric space is exactly the same, once we have proved that a quotient of soft sheaves is soft; this is the same proof as for a quotient of flabby sheaves, using question (h) instead of question (g).

□