MAT 540 : Problem Set 4

Due Thursday, October 10

1 Cartesian and cocartesian squares

(a). (4 points) Consider a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

in an abelian category $\mathscr{A}$. Consider the morphisms $u = \begin{pmatrix} f \\ g \end{pmatrix} : A \to B \oplus C$ and $v = \begin{pmatrix} h & -k \end{pmatrix} : B \oplus C \to D$.

Prove that the following statements are equivalent :

(i) The canonical morphism $A \to B \times_D C$ is an epimorphism.

(ii) The canonical morphism $B \sqcup_A C \to D$ is a monomorphism.

(iii) The complex $A \xrightarrow{u} B \oplus C \xrightarrow{v} D$ is exact.

(b). (2 points) Let $A \xrightarrow{g} B \xrightarrow{f} C$ be morphisms in $\mathscr{A}$. Show that $g^{-1}(\operatorname{Ker} f) = \operatorname{Ker}(f \circ g)$.

(c). (2 points) Keep the notation of the previous question, and suppose that g is surjective. Show that $g(\operatorname{Ker}(f \circ g)) = \operatorname{Ker} f$.

(d). (2 points) Keep the notation and assumptions of the previous question. If $u : D \to B$ is a morphism such that $f \circ u = 0$, show that there exists a commutative diagram

$$\begin{tikzcd} D' \arrow[r, "g'"] \arrow[d, "u'"'] & D \arrow[d, "u"] & \\ C \arrow[r, "g"'] & B \arrow[r, "f"'] & A \end{tikzcd}$$

such that g' is surjective and $f \circ g \circ u' = 0$.

Solution.

(a). We claim that the canonical morphism $B \times_D C \to B \times C = B \oplus C$ identifies $B \times_D C$ to $\operatorname{Ker} v$. Indeed, we have, for every $E \in \operatorname{Ob}(\mathscr{A})$,

$$\begin{aligned} \operatorname{Hom}_{\mathscr{A}}(E, B \times_D C) &= \{(w_1, w_2) \in \operatorname{Hom}_{\mathscr{A}}(E, B) \times \operatorname{Hom}_{\mathscr{A}}(E, C) \mid h \circ w_1 = k \circ w_2\} \\ &= \{w \in \operatorname{Hom}_{\mathscr{A}}(E, B \times C) \mid v \circ w = 0\} \\ &= \operatorname{Hom}(E, \operatorname{Ker} v). \end{aligned}$$

So (i) is equivalent to the fact that the morphism $A \to \operatorname{Ker} v$ induced by u is an epimorphism, which implies that the canonical morphism $\operatorname{Im}(u) \to \operatorname{Ker}(v)$ is an epimorphism, hence an isomorphism (because it is automatically injective), which is (iii).

We prove that (ii) and (iii) are equivalent. Let $\iota : B \oplus C \to B \oplus C$ be the morphism with matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that $\iota \circ \iota = \mathrm{id}_{B \oplus C}$, and in particular ι is an isomorphism. Let $u' = \iota \circ u$ and $v' = v \circ \iota$. Then $v' \circ u' = v \circ u = 0$, and we have a commutative square

$$\begin{tikzcd} \operatorname{Im}(u) \arrow[r] \arrow[d, "\iota"', "\sim"] & \operatorname{Ker}(v) \arrow[d, "\iota"', "\sim"] \\ \operatorname{Im}(u') \arrow[r] & \operatorname{Ker}(v') \end{tikzcd}$$

so (iii) is equivalent to the condition that the complex $(*)$ $A \overset{u'}{\to} B \oplus C \overset{v'}{\to} D$ be exact. We claim that the canonical morphism $B \oplus C \to B \sqcup_A C$ identifies $B \sqcup_A C$ to $\operatorname{Coker}(u')$. Indeed, for every $E \in \operatorname{Ob}(\mathscr{A})$,

$$\begin{aligned} \operatorname{Hom}_{\mathscr{A}}(\operatorname{Coker} u', E) &= \{w \in \operatorname{Hom}_{\mathscr{A}}(B \oplus C, E) \mid w \circ u' = 0\} \\ &= \{(w_1, w_2) \in \operatorname{Hom}_{\mathscr{A}}(B, E) \times \operatorname{Hom}_{\mathscr{A}}(C, E) \mid w_1 \circ f = w_2 \circ g\} \\ &= \operatorname{Hom}_{\mathscr{A}}(B \sqcup_A C, E). \end{aligned}$$

So (ii) is equivalent to the injectivity of the morphism $\operatorname{Coker}(u') \to D$ induced by v'; if $p : B \oplus C \to \operatorname{Coker}(u')$ is the canonical surjection, this means that (ii) is equivalent to the fact that, for every object E of $\mathscr{A}$, we have

$$\begin{aligned} \{f \in \operatorname{Hom}_{\mathscr{A}}(E, B \oplus C) \mid v' \circ f = 0\} &= \{f \in \operatorname{Hom}_{\mathscr{A}}(E, \mathrm{B} \oplus C) \mid p \circ f = 0\} \\ &= \{f \in \operatorname{Hom}_{\mathscr{A}}(E, \operatorname{Im}(u')\}, \end{aligned}$$

where the last equality is because $\operatorname{Im}(u') = \operatorname{Ker}(p)$. This shows that condition (ii) is equivalent to $\operatorname{Ker}(v') = \operatorname{Im}(u')$, that is, to the exactness of the complex (*).

(b). By definition of $g^{-1}(\operatorname{Ker} f)$, we have a commutative diagram, where the square is cartesian :

$$\begin{tikzcd} g^{-1}(\operatorname{Ker} f) \arrow[r, "g'"] \arrow[d, "j"'] & \operatorname{Ker} f \arrow[d, "i"] & \\ A \arrow[r, "g"'] & B \arrow[r, "f"'] & C \end{tikzcd}$$

In particular, $f \circ g \circ j = f \circ i \circ g' = 0$, so $g^{-1}(\operatorname{Ker} f) \subset \operatorname{Ker}(f \circ g)$. Conversely, let C be an object of $\mathscr{A}$ and $h : C \to A$ a morphism such that $f \circ g \circ h = 0$. Then we have a unique morphism $h' : C \to \operatorname{Ker} f$ such that $i \circ h' = g \circ h$, and this in turns defines a unique morphism $k : C \to g^{-1}(\operatorname{Ker} f)$ such that $j \circ k = h$ and $g' \circ k = h'$. Applying this to the inclusion $\operatorname{Ker}(f \circ g) \to A$, we see that this inclusion factors through $g^{-1}(\operatorname{Ker} f) \to A$, that is, $\operatorname{Ker}(f \circ g) \subset g^{-1}(\operatorname{Ker} f)$.

(c). By (b), it suffices to show that $g(g^{-1}(\operatorname{Ker} f))$. In fact, this is true for any subobject of B, so let $B' \subset B$. We want to show that the morphism $g^{-1}(B') \to B'$ induced by g is surjective. By definition of $g^{-1}(B')$, we have a cartesian square

$$\begin{tikzcd} g^{-1}(B') \arrow[r] \arrow[d] & B' \arrow[d] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

and then the conclusion follows from Corollary II.2.1.16 of the notes.

(d). Let $B' = \mathrm{Im}(u) \subset B$. The morphism u factors as $D \xrightarrow{v} B' \xrightarrow{i} B$ with v surjective and i injective, and we take $D' = g^{-1}(B') \times_{B'} D$. We have a commutative diagram where both squares are cartesian

$$\begin{tikzcd}
D' \arrow[r, "g'"] \arrow[d, "a"'] & D \arrow[d, "v"] \\
g^{-1}(B') \arrow[r] \arrow[d, "b"'] & B' \arrow[d, "i"] \\
C \arrow[r, "g"'] & B
\end{tikzcd}$$

and we take $u' = b \circ a$. The morphism g' is surjective by Corollary II.2.1.16 of the notes, and $f \circ g \circ u' = f \circ i \circ v \circ g' = f \circ u \circ g' = 0$.

□

2 A random fact

(2 points)

Let $\mathscr{A}$ be an abelian category and $f : B \to A$ be a morphism of $\mathscr{A}$. Show that, for every object C of $\mathscr{A}$, the morphism $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}(f), C) \to \mathrm{Hom}_{\mathscr{A}}(B, C)$ (induced by $B \to \mathrm{Im}(f)$) induces an isomorphism

$$\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}(f), C) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}\, f, C)).$$

Solution. The statement is saying two things :

(1) The map $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}(f), C) \to \mathrm{Hom}_{\mathscr{A}}(B, C)$ is injective; this follows from the fact that $B \to \mathrm{Im}(f)$ is surjective.

(2) A morphism $g : B \to C$ factors through the quotient $\mathrm{Im}(f)$ of B if and only if $g(\mathrm{Ker}\, f) = 0$, or, in other words, the morphism $B \to \mathrm{Im}(f)$ is the cokernel of the morphism $\mathrm{Ker}(f) \to B$. This follows from the fact that $\mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f)$. (Remember that $\mathrm{Coim}(f)$ is by definition the cokernel of the morphism $\mathrm{Ker}(f) \to B$.)

□

3 More sheaves on an abelian category

We use the notation of problem 6 of problem set 3 : We fix an abelian category, and we denote by Sh the category of sheaves of abelian groups on $\mathscr{A}$ for the canonical topology. It is a full subcategory of the catgeory of presheaves $\mathrm{PSh} = \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Ab})$. Both Sh and PSh are abelian categories, and the forgetful functor $\mathrm{Sh} \to \mathrm{PSh}$ is left exact but not exact; this functor admits a left adjoint $F \longmapsto F^{\mathrm{sh}}$, which is exact.

(a). (2 points) Let $f : A \to B$ be a surjective morphism in $\mathscr{A}$. Show that, for every morphism

$u : C \to B$, there exists a commutative square

$$\begin{tikzcd} C' \arrow[r, "f'"] \arrow[d] & C \arrow[d, "u"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

with $f' : C' \to C$ surjective.

(b). (3 points) Show that the Yoneda embedding $h_{\mathscr{A}} : \mathscr{A} \to \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Set})$, $A \longmapsto \mathrm{Hom}_{\mathscr{A}}(\cdot, A)$, factors as $\mathscr{A} \xrightarrow{h} \mathrm{Sh} \xrightarrow{\mathrm{For}} \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Set})$, where For is the forgetful functor and h is a fully faithful left exact additive functor.

(c). (2 points) Show that the functor $h : \mathscr{A} \to \mathrm{Sh}$ is exact.

Solution.

(a). We take $C' = A \times_B C$ and $f' : C' \to C$ equal to the second projection. The surjectivity of f' follows from Corollary II.2.1.16 of the notes.

(b). For every $A \in \mathrm{Ob}(\mathscr{A})$, the functor $\mathrm{Hom}_{\mathscr{A}}(\cdot, A) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ factors through the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$, so we can see $\mathrm{Hom}_{\mathscr{A}}(\cdot, A)$ as an object of PSh; also, if $f : A \to B$ is a morphism, then $f^* : \mathrm{Hom}_{\mathscr{A}}(\cdot, B) \to \mathrm{Hom}_{\mathscr{A}}(\cdot, A)$ is a morphism of presheaves of abelian groups (and not just of presheaves of sets), because composition is bilinear. So the Yoneda embedding factors as $\mathscr{A} \xrightarrow{h'} \mathrm{PSh} \xrightarrow{\mathrm{For}} \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Set})$, where For is the forgetful functor. The functor h' is additive and left exact because $\mathrm{Hom}_{\mathscr{A}}(\cdot, \cdot)$ is additive and left exact in both variables (and in particular the second). Also, for every $A \in \mathrm{Ob}(\mathscr{A})$, the representable presheaf $\mathrm{Hom}_{\mathscr{A}}(\cdot, A)$ is a sheaf for the canonical topology by problem 6(c) of problem set 3, so we get the factorization of the statement. Finally, the functor h is left exact because the sheafification functor $\mathrm{PSh} \to \mathrm{Sh}$ is exact and isomorphic to the identity functor on Sh, so any complex of sheaves $0 \to F_1 \to F_2 \to F_3$ that is exact in PSh is also exact in Sh.

(c). By question (b) and Lemma II.2.3.2 of the notes, it suffices to show that h sends surjections to surjections. Let $f : A \to B$ be a surjective morphism, and let $C \in \mathrm{Ob}(\mathscr{A})$. Let $u : C \to B$ be an element of $h_B(C)$. Choose a commutative diagram as in question (a). Then $f' : C' \to C$ is a covering family for the canonical topology of $\mathscr{A}$, and the morphism $u' : C' \to A$ gives an element of $h_A(C')$ whose image by $f^* : h_A(C') \to h_B(C')$ is $f \circ u' = f'^*(u)$. This shows that $f^* : h_B \to h_A$ is surjective in the category Sh.

□

4 Other embedding theorems

If we weaken the assumptions in Morita's theorem, we can still get interesting results. There are many variants, we will prove two here.

Let $\mathscr{A}$ be an abelian category, Q an object of $\mathscr{A}$ and $R = \mathrm{End}_{\mathscr{A}}(Q)$. As explained in the paragraph before Theorem II.3.1.6 of the notes, we can see the functor $\mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$ as an additive left exact functor from $\mathscr{A}$ to $\mathbf{Mod}_R$.

Note that we are *not* assuming that Q is projective for now.

We assume that $\mathscr{A}$ admits all small colimits. From now on, we assume that $\mathscr{A}$ admits all small colimits. (You can mostly ignore the smallness condition. It basically means that you can take colimits indexed by all sets that are built out of sets like $\mathrm{Hom}_{\mathscr{A}}(A, B)$. The rigorous way to say it is that $\mathscr{A}$ is a $\mathscr{W}$-category, with $\mathscr{W}$ a universe, and that it admits limits indexed by $\mathscr{W}$-small categories.)

(a). (3 points) Show that, for every right R-module M, the functor $\mathscr{A} \to \mathbf{Set}$, $A \longmapsto \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathscr{A}}(Q, A))$ is representable. We denote a pair representing this functor by $(M \otimes_R Q, \eta(M))$.

(b). (1 point) Show that the functor $G = \mathrm{Hom}_{\mathscr{A}}(Q, \cdot) : \mathscr{A} \to \mathbf{Mod}_R$ admits a left adjoint F.

(c). (2 points) If M is a free right R-module, show that $\eta(M) : M \to G(F(M))$ is injective, and that it is bijective if M is also finitely generated.

(d). Let $\mathscr{A}' \subset \mathscr{A}$ be a full subcategory of $\mathscr{A}$ that is stable by taking finite limits and finite colimits.

(i) (1 point) Show that $\mathscr{A}'$ is an abelian category and that the inclusion functor $\mathscr{A}' \to \mathscr{A}$ is exact.

From now on, we assume that the category $\mathscr{A}'$ is small. Suppose that Q is a generator of $\mathscr{A}$. For every object A of $\mathscr{A}$, consider the surjective morphism $q_A : \bigoplus_{\mathrm{Hom}_{\mathscr{A}}(Q,A)} Q \to A$ of Proposition II.3.1.3(i)(e) of the notes. Let $P = \bigoplus_{A \in \mathrm{Ob}(\mathscr{A}')} \bigoplus_{\mathrm{Hom}_{\mathscr{A}}(Q,A)} Q$; for every $A \in \mathrm{Ob}(\mathscr{A}')$, we have a surjective morphism $p_A : P \to A$, which is given by q_A on the summand of P indexed by A and by 0 on the other summands. Let $S = \mathrm{End}_{\mathscr{A}}(P)$, and consider the functor $G' = \mathrm{Hom}_{\mathscr{A}}(P, \cdot) : \mathscr{A} \to \mathbf{Mod}_S$.

(ii) (2 points) Show that G' is faithful, and that it is exact if Q is projective.

(iii) (3 points) If Q is projective, show that the restriction of G' to $\mathscr{A}'$ is fully faithful.

From now on, we also assume that small filtrant colimits are exact in $\mathscr{A}$ and that Q is a generator of $\mathscr{A}$. [1] We do not assume that Q is projective.

(e). The goal of this question is to show that G is fully faithful. Let $\mathscr{C}$ be the full subcategory of $\mathscr{A}$ whose objects are finite direct sums of copies of P, and $\mathscr{D}$ the full subcategory of $\mathbf{Mod}_R$ whose objects are finitely generated free R-modules. We denote by $h : \mathscr{A} \to \mathrm{PSh}(\mathscr{C})$ the functor $A \longmapsto \mathrm{Hom}_{\mathscr{A}}(\cdot, A)_{|\mathscr{C}}$, and by $h' : \mathbf{Mod}_R \to \mathrm{PSh}(\mathscr{D})$ the functor $M \longmapsto \mathrm{Hom}_R(\cdot, M)_{|\mathscr{D}}$.

(i) (1 point) Show that G induces an equivalence of categories $\mathscr{C} \to \mathscr{D}$.

(ii) (1 points) Show that $h' : \mathbf{Mod}_R \to \mathrm{PSh}(\mathscr{D})$ is fully faithful.

(iii) (1 points) Assuming that $h : \mathscr{A} \to \mathrm{PSh}(\mathscr{C})$ is fully faithful, show that $G : \mathscr{A} \to \mathbf{Mod}_R$ is fully faithful.

(iv) (3 points) Show that h is left exact and faithful, and that, for any morphism f of $\mathscr{A}$, if $h(f)$ is surjective, then f is surjective.

(v) (2 points) For any object B of $\mathscr{C}$, any morphism $f : B \to A$ in $\mathscr{A}$ and any object C of $\mathscr{A}$, show that the map

$$\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}\, f, C) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathrm{Im}(h(B) \to h(A)), h(C))$$

is an isomorphism. (Hint : problem 2.)

[1] In other words, $\mathscr{A}$ is a Grothendieck abelian category.

Let A be an object of $\mathscr{A}$. Denote by $\mathscr{C}/A$ the category of pairs (B, f), where $B \in \mathrm{Ob}(\mathscr{C})$ and $f : B \to A$ is a morphism of $\mathscr{A}$; a morphism $u : (B, f) \to (B', f')$ is a morphism $u : B \to B'$ such that $f = f' \circ u$.

Let I be the set of finite subsets of $\mathrm{Ob}(\mathscr{C}/A)$, ordered by inclusion; the corresponding category is clearly filtrant. Define a functor $\xi : I \to \mathscr{C}$ by sending a finite set $J = \{(B_1, f_1), \ldots, (B_n, f_n)\}$ to $B_1 \oplus \ldots \oplus B_n$; note that $\xi(J)$ comes with a morphism to A, given by $\begin{pmatrix} f_1 & \ldots & f_n \end{pmatrix}$.

(vi) (1 point) Show that the canonical morphism $\varinjlim_{J\in I} h(\xi(J)) \to h(A)$ is an epimorphism.

(vii) (1 point) Show that the canonical morphism $\varinjlim_{J\in I} \mathrm{Im}(h(\xi(J)) \to h(A)) \to h(A)$ is an isomorphism.

(viii) (2 points) Show that the canonical morphism $\varinjlim_{J\in I} \xi(J) \to A$ is an epimorphism.

(ix) (1 point) Show that the canonical morphism $\varinjlim_{J\in I} \mathrm{Im}(\xi(J) \to A) \to A$ is an isomorphism.

(x) (2 points) For C another object of $\mathscr{A}$, show that $\mathrm{Hom}_{\mathscr{A}}(A, C) \xrightarrow{h} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(A), h(C))$ is bijective.

(f). The goal of this question is to show that F is exact.

(i) (2 points) Show that it suffices to prove that F preserves injections.

Let $f : M \to N$ be a morphism in $\mathbf{Mod}_R$.

(ii) (2 points) Suppose that M is finitely generated free and that N is free. Show that the composition of $\eta(\mathrm{Ker}\, f) : \mathrm{Ker}\, f \to G(F(\mathrm{Ker}\, f))$ and of the canonical morphism $G(F(\mathrm{Ker}\, f)) \to \mathrm{Ker}(G(F(f)))$ is an isomorphism.
<u>Hint</u> : Use the commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\eta(M)"'] & N \arrow[d, "\eta(N)"] \\ G(F(M)) \arrow[r, "G(F(f))"'] & G(F(N)) \end{tikzcd}$$

(iii) (3 points) Suppose that M is finitely generated, that N is free and that f is injective. Show that $F(f)$ is injective. (Hint : question 1(c).)

(iv) (1 point) Suppose that N is free and that f is injective. Show that $F(f)$ is injective. (Hint : M is the union of its finitely generated submodules.)

(v) (1 point) Suppose that f is injective. Show that we can find a commutative diagram with exact rows :

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & K \arrow[r] & N' \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

such that f' is injective and N' is free.

(vi) (3 points) Suppose that f is injective. Applying F to the diagram of (v) and using 1(a), show that $F(f)$ is injective.

In conclusion, here are our embedding results so far :

(1) If $\mathscr{A}$ admits small colimits and a projective generator, we have shown that every small full abelian subcategory $\mathscr{A}'$ of $\mathscr{A}$ such that $\mathscr{A}' \subset \mathscr{A}$ is exact admits a fully faithful exact functor into a category of modules over some ring.

(2) If $\mathscr{A}$ is a Grothendieck abelian category (it admits small colimits, small filtrant colimits are exact, and $\mathscr{A}$ has a generator), then we have shown that $\mathscr{A}$ admits a fully faithful left exact functor into a category of modules over a ring, with an exact left adjoint. This is known as the Gabriel-Popescu embedding theorem.

(3) We also have Morita's theorem (Theorem II.3.1.6 of the notes) : If $\mathscr{A}$ admits small colimits and has a projective generator P such that the functor $\mathrm{Hom}_{\mathscr{A}}(P, \cdot)$ commutes with small direct sums, then $\mathscr{A}$ is equivalent to a category of modules over a ring.

Solution.

(a). This is similar to what happens in the proof of Theorem II.3.1.6 of the notes, with a few changes to reflect the fact that $\eta(M)$ is not an isomorphism anymore. We denote by $\Phi_M : \mathscr{A} \to \mathbf{Set}$ the functor $A \longmapsto \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathscr{A}}(Q, A))$.

(1) If $M = R$, then the functor $\Phi_M : A \longmapsto \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathscr{A}}(Q, A)) \simeq \mathrm{Hom}_{\mathscr{A}}(Q, A)$ is representable by Q, and the morphism $\eta(M) \in \mathrm{Hom}_R(R, \mathrm{Hom}_{\mathscr{A}}(Q, Q)) = \mathrm{Hom}_R(R, R)$ is the identity of R.

(2) If $M = R^{(X)}$ with X a set, then we have isomorphisms of functors

$$\Phi_M = \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathscr{A}}(Q, \cdot)) \simeq \prod_X \mathrm{Hom}_R(R, \mathrm{Hom}_{\mathscr{A}}(Q, \cdot)) \simeq \prod_X \mathrm{Hom}_{\mathscr{A}}(Q, \cdot) \simeq \mathrm{Hom}_{\mathscr{A}}(Q^{(X)}, \cdot),$$

so the functor Φ_M is representable by $Q^{(X)}$, and the morphism $\eta(M) \in \mathrm{Hom}_R(R^{(X)}, \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)}))$ is the canonical morphism $R^{(X)} = \mathrm{Hom}_{\mathscr{A}}(Q, Q)^{(X)} \to \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)})$ of Subsection I.5.4.2 of the notes (which might not be an isomrophism).

(3) In general, we chose an exact sequence $R^{(X)} \xrightarrow{u} R^{(Y)} \to M \to 0$, with X and Y sets. This induces morphisms of functors $\Phi_M \to \Phi_{R^{(Y)}} \xrightarrow{u^*} \Phi_{R^{(X)}}$, and the second of these comes from a morphism $f : Q^{(X)} \to Q^{(Y)}$ between the objects representing $\Phi_{R^{(X)}}$ and $\Phi_{R^{(Y)}}$ such that the following diagram commutes :

(*)
$$\begin{tikzcd} R^{(X)} \arrow[r, "\eta(R^{(X)})"] \arrow[d, "u"'] & \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)}) \arrow[d, "\mathrm{Hom}_{\mathscr{A}}(Q,f)"] \\ R^{(Y)} \arrow[r, "\eta(R^{(Y)})"] & \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(Y)}) \end{tikzcd}$$

Let $B = \mathrm{Coker}\, f$. By Subsection I.5.4.2 of the notes, there is a canonical morphism $M = \mathrm{Coker}(G(f)) \to G(\mathrm{Coker}\, f) = G(B)B$, which might not be an isomorphism. This induces a morphism of functors $\mathrm{Hom}_{\mathscr{A}}(B, \cdot) \to \mathrm{Hom}_R(G(B), G(\cdot)) \to \mathrm{Hom}_R(M, G(\cdot)) = \Phi_M$. To show that this morphism is an isomorphism, we use, as in the proof of Theorem II.3.1.6 of the notes, that we have a commutative diagram with exact columns for every $A \in \mathrm{Ob}(\mathscr{A})$:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Phi_M(A) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{A}}(B, A) \arrow[d] \\
\Phi_{R^{(Y)}}(A) \arrow[r, "\sim"] \arrow[d, "u^*(A)"'] & \mathrm{Hom}_{\mathscr{A}}(Q^{(Y)}, A) \arrow[d, "\mathrm{Hom}_{\mathscr{A}}(f,A)"] \\
\Phi_{R^{(X)}}(A) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathscr{A}}(Q^{(X)}, A)
\end{tikzcd}$$

(The fact that the columns are exact only uses the left exact of the Hom functors.) We get the morphism $\eta(M) : M \to \mathrm{Hom}_{\mathscr{A}}(Q, B)$ by taking the morphism between the cokernels of the vertical maps in the commutative square (*).

(b). This follows from (a) and from Proposition I.4.7 of the notes; in fact, we have $F(M) = M \otimes_R Q$. Also, by the proof of that proposition, the morphisms $\eta(M) : M \to \mathrm{Hom}_{\mathscr{A}}(Q, M \otimes_R Q) = G(F(M))$ define a morphism of functors $\mathrm{id}_{\mathbf{Mod}_R} \to G \circ F$, which is the unit of the adjunction.

(c). If $M = R^{(X)}$ with X a set, we saw in the solution of (a) that $M \otimes_X Q = Q^{(X)}$ and that $\eta(M) : R^{(X)} \to \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)})$ is the canonical morphism $\mathrm{Hom}_{\mathscr{A}}(Q, Q)^{(X)} \to \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)})$. If X is finite, this morphism is an isomorphism because $\mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$, being an additive functor, commutes with finite coproducts. In general, we claim that $\eta(M)$ is injective. Let $Q^X = \prod_X Q$; we have a family of morphisms $(q_x : Q^{(X)} \to Q)_{x \in X}$, such that the composition of q_x with the morphism $Q \to Q^{(X)}$ corresponding to $y \in X$ is id_Q if $y = x$, and 0 if $y \neq x$. This gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(Q, Q)^{(X)} \arrow[r, "\eta(M)"] \arrow[d, "(1)"'] & \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)}) \arrow[d] \\
\mathrm{Hom}_{\mathscr{A}}(Q, Q)^X \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathscr{A}}(Q, Q^X)
\end{tikzcd}$$

where the map (1) is the inclusion of the direct sum into the direct product (in the category of abelian groups), hence an injection. So $\eta(M)$ is injective.

(d). (i) The category $\mathscr{A}'$ is clearly preadditive. It is additive, because finite products in $\mathscr{A}$ of objects of $\mathscr{A}'$ are in $\mathscr{A}'$ by hypothesis, and they are finite products in $\mathscr{A}'$ by the fullness of $\mathscr{A}'$. For the same reason, every morphism in $\mathscr{A}'$ has a kernel and a cokernel, which are its kernel and its cokernel in $\mathscr{A}$. If f is a morphism in $\mathscr{A}'$, then the canonical morphism from its coimage to its image in $\mathscr{A}'$ is the same asthe canonical morphism from its image to its coimage in $\mathscr{A}$, so it is an isomorphism. This shows that $\mathscr{A}'$ is an abelian. We have seen in the construction of finite products, kernels and cokernels in $\mathscr{A}'$ that the inclusion functor from $\mathscr{A}'$ to $\mathscr{A}$ commutes with these, so it commutes with finite limits and colimits, so it is exact.

(ii) By the universal property of the direct sum, the functor G' is isomorphism to $\prod_{A \in \mathrm{Ob}(\mathscr{A})} \prod_{\mathrm{Hom}_{\mathscr{A}}(Q,A)} \mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$. As Q is a generator, the functor $\mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$ is faithful; so G' is also faithful.

If Q is projective, then, by Lemma II.2.4.3 of the notes, P is also projective, and then the functor $\mathrm{Hom}_{\mathscr{A}}(P, \cdot)$ is exact.

(iii) Let A and B be objects of $\mathscr{A}'$, and let $u : G'(A) \to G'(B)$ be a morphism of right S-modules. We want to show that there exists $g \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ such that $G'(g) = u$. By construction of P, we have surjective morphisms $p_A : P \to A$ and $p_B : P \to B$. As G' is exact, we get a diagram with exact rows

$$\begin{tikzcd} S \arrow[r, "G'(p_A)"] \arrow[d, dashed, "v"] & \mathrm{Hom}_{\mathscr{A}}(P, A) \arrow[r] \arrow[d, "u"] & 0 \\ S \arrow[r, "G'(p_B)"] & \mathrm{Hom}_{\mathscr{A}}(P, B) \arrow[r] & 0 \end{tikzcd}$$

As S is a projective in $\mathbf{Mod}_S$, there exists a morphism $v : S \to S$ making the diagram. This morphism is of the form $g \longmapsto f \circ g$, with $f = v(1) \in S = \mathrm{Hom}_{\mathscr{A}}(P, P)$. Consider the diagram with exact rows :

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(p_A) \arrow[r, "i"] & P \arrow[r, "p_A"] \arrow[d, "f"'] & A \arrow[r] \arrow[d, dashed, "g"] & 0 \\ & & P \arrow[r, "p_B"'] & B \arrow[r] & 0 \end{tikzcd}$$

To show that there exists a morphism $g : A \to B$ making this diagram commute, it suffices to show that $p_B \circ f \circ i = 0$. As G' is faithful, it suffices to show that $G'(p_B) \circ G'(f) \circ G'(i) = 0$; as $G'(p_B) \circ G'(f) = G'(p_B) \circ u = v \circ G'(p_A)$, we have $G'(p_B) \circ G'(f) \circ G'(i) = v \circ G'(p_A \circ i) = 0$.

To finish the proof, it suffices to prove that $G'(g) = u$. We know that $G'(g) \circ G'(p_A) = G'(p_B) \circ v = u \circ G'(p_A)$, so the equality $G'(g) = u$ follows from the fact that $G'(p_A)$ is surjective.

(e). The idea of this seemingly strange procedure is that we are showing that teh subcategory $\mathscr{C}$ (resp. $\mathscr{D}$), that contains a generator, "generates" $\mathscr{A}$ (resp. $\mathbf{Mod}_R$) in some precise sense (this notion is called being *strictly generating*, see Definition 5.3.1 of Kashiwara-Schapira); so the equivalence $\mathscr{C} \to \mathscr{D}$ of (i) will extend to a fully faithful functor $\mathscr{A} \to \mathbf{Mod}_R$. The proof of this fact is a specialization to our case of the proof of Theorem 5.3.6 of Kashiwara-Schapira.

(i) If X is a finite set, then the canonical morphism $R^{(X)} = \mathrm{Hom}_{\mathscr{A}}(Q, Q)^{(X)} \to G(Q^{(X)}) = \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)})$ is an isomorphism. Just as in the second paragraph of the proof of Theorem II.3.1.6 of the notes, we deduce that, if X and Y are finite sets, then the map $G : \mathrm{Hom}_{\mathscr{A}}(Q^{(X)}, Q^{(Y)}) \to \mathrm{Hom}_R(R^{(X)}, R^{(Y)})$ is bijective. (As Y is finite, we only use the fact that additive functors commute with finite direct sums, and so we don't need Q to have the extra property of that theorem.)

We have just shown that the restriction of the functor G to $\mathscr{C}$ is fully faithful, and that its essential image is $\mathscr{D}$. So G induces an equivalence of categories from $\mathscr{C}$ to $\mathscr{D}$ by Corollary I.2.3.9 of the notes.

(ii) Let $H : \mathrm{PSh}(\mathscr{D}) \to \mathbf{Mod}_R$ be the functor sending a presheaf $\mathscr{F}$ to $\mathscr{F}(R)$ (and a morphism $u : \mathscr{F} \to \mathscr{G}$ of preasheaves to $u(R) : \mathscr{F}(R) \to \mathscr{G}(R)$).

For every right R-module M, we have a canonical isomorphism $H(h'(M)) = \mathrm{Hom}_R(R, M) \overset{\sim}{\to} M$, $u \longmapsto u(1)$. This defines an isomorphism of functors $H \circ h' \overset{\sim}{\to} \mathrm{id}_{\mathbf{Mod}_R}$. Let M and N be right R-modules. Then we get a sequence of morphisms of abelian groups

$$\mathrm{Hom}_R(M, N) \overset{h'}{\to} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{D})}(h'(M), h'(N)) \overset{H}{\to} \mathrm{Hom}_R(H(h'(M)), H(h'(N)) \simeq \mathrm{Hom}_R(M, N),$$

whose composition is equal to $\mathrm{id}_{\mathrm{Hom}_R(M,N)}$. So the first map is injective. Also, as the functors $\mathrm{Hom}_{\mathscr{A}}(\cdot, M)$ and $\mathrm{Hom}_{\mathscr{A}}(\cdot, N)$ are additive, the presheaves $h'(M)$ $h'(N)$ commute with finite direct sums, so they are determined by their sections on R; this shows that the second map in the sequence above is also injective; as it is surjective, it must be bijective, and this implies that $h' : \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{D})}(h'(M), h'(N))$ is bijective.

(iii) We have a diagram of categories and functors

$$\begin{tikzcd} \mathscr{A} \arrow[r, "G"] \arrow[d, "h"'] & \mathbf{Mod}_R \arrow[d, "h'"] \\ \mathrm{PSh}(\mathscr{C}) & \mathrm{PSh}(\mathscr{D}) \arrow[l, "\Phi"'] \end{tikzcd}$$

where Φ is the equivalence of categories induced by the equivalence $\mathscr{C} \to \mathscr{D}$ of (i). This diagram is not necessarily commutative, but we have an isomorphism of functors $\Phi \circ h \circ G \simeq h'$. We already know that Φ and h' are fully faithful, so, if h is fully faithful, we can conclude that G is fully faithful.

(iv) The functor h is left exact because $A \longmapsto \mathrm{Hom}_{\mathscr{A}}(\cdot, A)$ is.

Let $f : A \to B$ be a morphism in $\mathscr{A}$ such that $h(f) = 0$. Then the R-linear map $h(A)(Q) = \mathrm{Hom}_{\mathscr{A}}(Q, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathscr{A}}(Q, B) = h(B)(Q)$ is 0; in other words, we have $G(f) = 0$. As G is faithful (by Proposition II.3.1.3 of the notes, we get that $f = 0$. So h is faithful.

Let $f : A \to B$ be a morphism in $\mathscr{A}$ such that $h(f)$ is surjective. Let $g_1, g_2 : B \to C$ be two morphisms such that $g_1 \circ f = g_2 \circ f$. Then $h(g_1) \circ h(f) = h(g_2) \circ h(f)$, so $h(g_1) = h(g_2)$ by the surjectivity of $h(f)$; as h is faithul, this implies that $g_1 = g_2$. So f is an epimorphism.

(v) By problem 2 and the left exactness of h, we have isomorphisms

$$\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}\, f, C) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}\, f, C))$$

and

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathrm{Im}(h(B) \to h(A)), h(C)) \\ &\qquad \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(B), h(C)) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathrm{Ker}(h(f)), h(C))) \\ &\qquad \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(B), h(C)) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(\mathrm{Ker}(f)), h(C))). \end{aligned}$$

By these isormophisms, the map that we are trying to understand corresponds to the map

$$\begin{aligned} &u : \mathrm{Ker}(\mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}\, f, C)) \\ &\qquad \to \mathrm{Ker}(\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(B), h(C)) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(\mathrm{Ker}(f)), h(C))) \end{aligned}$$

induced by h. As $h : \mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(B), h(C))$ is an isomorphism (by Yoneda's lemma, applied to the representable presheaf $h(B)$ on $\mathscr{C}$) and the map $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}\, f, C) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(\mathrm{Ker}\, f), h(C))$ is injective (because h is faithful), the map u is bijective.

(vi) Let C be an object of $\mathscr{C}$. Then applying the morphism $\varinjlim_{J \in I} h(\xi(J)) \to h(A)$ to C gives the map

$$\varinjlim_{J \in I} \mathrm{Hom}_{\mathscr{A}}(C, \xi(J)) \to \mathrm{Hom}_{\mathscr{A}}(C, A).$$

Saying that this is surjective means that every morphism $f : C \to A$ factors as $C \to \xi(J) \to A$ for $J \in I$, which is true : just take $J = \{(C, f)\}$ and $C \to \xi(J)$ equal to id_C. So $\varinjlim_{J\in I} h(\xi(J))(C) \to h(A)(C)$ is surjective for every $C \in \mathrm{Ob}(\mathscr{C})$, which implies that $\varinjlim_{J\in I} h(\xi(J)) \to h(A)$ is an epimorphism.

(vii) As I is filtrant, we have $\mathrm{Im}(\varinjlim_{J\in I} h(\xi(J)) \to h(A)) = \varinjlim_{J\in I} \mathrm{Im}(h(\xi(J)) \to h(A))$. So the result follows immediately from (vi).

(viii) Note that $\varinjlim_{J\in I} h(\xi(J)) \to h(A)$ factors as $\varinjlim_{J\in I} h(\xi(J)) \to h(\varinjlim_{J\in I} \xi(J)) \to h(A)$, where the first morphism is that of Subsection I.5.4.2 of the notes and the second is the image by h of the canonical morphism $\varinjlim_{J\in I} \xi(J) \to A$. By (vi), the second morphism is an epimorphism, so, by (iv), the morphism $\varinjlim_{J\in I} \xi(J) \to A$ is an epimorphism.

(ix) As in (vii), this follows immediately from (viii) and from the fact that I is filtrant.

(x) The map $h : \mathrm{Hom}_{\mathscr{A}}(A, C) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(A), h(C))$ is equal to the composition

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{A}}(A, C) &\overset{\sim}{\to} \mathrm{Hom}_{\mathscr{A}}(\varinjlim_{J\in I} \mathrm{Im}(\xi(J) \to A), C) \text{ by (ix)} \\
&\simeq \varprojlim_{J\in I^{\mathrm{op}}} \mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}(\xi(J) \to A), C) \\
&\overset{\sim}{\to} \varprojlim_{J\in I^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathrm{Im}(h(\xi(J)) \to h(A)), h(C)) \text{ by (v)} \\
&\simeq \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\varinjlim_{J\in I} \mathrm{Im}(h(\xi(J)) \to h(A)), h(C)) \\
&\simeq \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(A), h(C)) \text{ by (vii).}
\end{aligned}$$

(f). (i) We already know that F is right exact, because it is a left adjoint (Proposition II.2.3.3 of the notes.) So the statement follows from Lemma II.2.3.2 of the notes.

(ii) We have a commutative diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\, f \arrow[r] \arrow[d, "\eta(\mathrm{Ker}\, f)"'] & M \arrow[r, "f"] \arrow[dd, "\eta(M)"'] & N \arrow[dd, "\eta(N)"'] \\
& G(F(\mathrm{Ker}\, f)) \arrow[d] & & \\
0 \arrow[r] & \mathrm{Ker}(G(F(f))) \arrow[r] & G(F(M)) \arrow[r, "G(F(f))"'] & G(F(N))
\end{tikzcd}$$

where the unmarked vertical one is the canonical morphism. So the result follows from a diagram chase in $\mathbf{Mod}_R$.

(iii) As M is finitely generated, there exists a surjective R-linear map $g : M' \to M$, with M' free of finite type. As F is right exact, the morphism $F(g)$ is also surjective. So, by question 1(c), we have $\mathrm{Ker}(F(f)) = F(g)(\mathrm{Ker}(F(f \circ g)))$. Hence, to prove that $\mathrm{Ker}(F(f)) = 0$, it suffices to show that the composition $\mathrm{Ker}(F(f \circ g)) \to F(M') \overset{F(g)}{\to} F(M)$ is 0. As G is conservative, it suffices to prove this after applying G, and as G is left exact, it suffices to prove that the composition $\mathrm{Ker}(G(F(f \circ g))) \to G(F(M')) \overset{G(F(g))}{\to} G(F(M))$ is 0. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(f\circ g) \arrow[r, "u"] \arrow[d, "(1)"'] & M' \arrow[r, "g"] \arrow[d, "\eta(M')"] & M \arrow[d, "\eta(M)"] \\
\mathrm{Ker}(G(F(f\circ g))) \arrow[r, "v"'] & G(F(M')) \arrow[r, "G(F(g))"'] & G(F(M))
\end{tikzcd}$$

We know that $g \circ u = 0$ because $0 = \mathrm{Ker}\, f = g(\mathrm{Ker}(f \circ g))$ by question 1(c), so $\eta(M) \circ g \circ u = 0$. As the morphism (1) is surjective by (ii), this implies that $G(F(g)) \circ v = 0$, as desired.

(iv) Let I be the set of all the finitely generated submodules of M; for $i \in I$, we denote the corresponding submodule by M_i. Then I is filtrant, and $M = \varinjlim_{i\in I} M_i$. As F is a left adjoint, the canonical morphism $\varinjlim_{i\in I} F(M_i) \to F(M)$ is an isomorphism by Proposition I.5.4.3 of the notes, and $F(f)$ corresponds to $\varinjlim_{i\in I} F(f_{|M_i})$ by this isomorphism. For each $i \in I$, the morphism $F(f_{|M_i})$ is injective by (iii). As filtrant colimits are exact in $\mathbf{Mod}_R$ by Corollary II.2.3.4 of the notes, this implies that $F(f)$ is also injective.

(v) Let $g : N' \to N$ be a surjective R-linear map with N' free, let $M' = N' \times_N M$ and let $f' : M' \to M$ and $g' : N' \to N$ be the two projections. The morphism g' is surjective by Corollary II.2.1.16 of the notes. As f is injective, so is f' (it is true and easy to prove that in any abelian category the pullback of an injective morphism is injective, but in the category of R-modules it is immediate). Let $i : K = \mathrm{Ker}\, g \to N'$ and $i' : \mathrm{Ker}(g') \to M'$ be the canonical injections. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(g') \arrow[r, "i'"] \arrow[d, dashed, "u"'] & M' \arrow[r, "g'"] \arrow[d, "f'"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & K \arrow[r, "i"'] & N' \arrow[r, "g"'] & N \arrow[r] & 0
\end{tikzcd}$$

As $g \circ f' \circ i' = f \circ g \circ i' = 0$, there exists a unique morphism $u : \mathrm{Ker}(g') \to K$ such that $i \circ u = f' \circ i'$. We want to show that u is an isomorphism. As f' is injective, the map $f' \circ i'$ is injective, hence u is also injective. To prove that u is surjective, we can do a bit of diagram chasing : Let $x \in K$. Then $g(i(x)) = 0 = f(0)$, so $y = (i(x), 0) \in N' \times M$ is actually in M', and we have $f'(y) = i(x)$ and $g'(y) = 0$. In particular, there exists $z \in \mathrm{Ker}(g')$ such that $y = i'(z)$. As $i(u(z)) = f'(i'(z)) = i(x)$ and i is injective, we get that $x = u(z)$.

(vi) Applying F to the diagram of (v) gives a commutative diagram with exact rows

$$\begin{tikzcd}
F(K) \arrow[r] \arrow[d, equal] & F(M') \arrow[r] \arrow[d, "F(f')"'] \arrow[dr, phantom, "(*)"] & F(M) \arrow[r] \arrow[d, "F(f)"'] & 0 \\
F(K) \arrow[r] & F(N') \arrow[r] & F(N) \arrow[r] & 0
\end{tikzcd}$$

By (iv), the map $F(f')$ is injective. Also, as F is a left adjoint, it commutes with colimits, so the square (*) is cocartesian. By question 1(a), the morphism $F(M') \to F(N') \times_{F(N)} F(M)$ is surjective. As $F(f')$ is injective, this morphism is also injective, so it is an isomorphism; in other words, the square (*) is cartesian. By Corollary II.2.1.16 of the notes, the morphism $F(f)$ is injective.

$\square$