MAT 540 : Problem Set 4

Due Thursday, October 10

1 Cartesian and cocartesian squares

(a). (4 points) Consider a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

in an abelian category $\mathscr{A}$. Consider the morphisms $u = \begin{pmatrix} f \\ g \end{pmatrix} : A \to B \oplus C$ and $v = \begin{pmatrix} h & -k \end{pmatrix} : B \oplus C \to D$.

Prove that the following statements are equivalent :

(i) The canonical morphism $A \to B \times_D C$ is an epimorphism.

(ii) The canonical morphism $B \sqcup_A C \to D$ is a monomorphism.

(iii) The complex $A \overset{u}{\to} B \oplus C \overset{v}{\to} D$ is exact.

(b). (2 points) Let $A \overset{g}{\to} B \overset{f}{\to} C$ be morphisms in $\mathscr{A}$. Show that $g^{-1}(\operatorname{Ker} f) = \operatorname{Ker}(f \circ g)$.

(c). (2 points) Keep the notation of the previous question, and suppose that g is surjective. Show that $g(\operatorname{Ker}(f \circ g)) = \operatorname{Ker} f$.

(d). (2 points) Keep the notation and assumptions of the previous question. If $u : D \to B$ is a morphism such that $f \circ u = 0$, show that there exists a commutative diagram

$$\begin{tikzcd} D' \arrow[r, "g'"] \arrow[d, "u'"'] & D \arrow[d, "u"] & \\ C \arrow[r, "g"'] & B \arrow[r, "f"'] & A \end{tikzcd}$$

such that g' is surjective and $f \circ g \circ u' = 0$.

2 A random fact

(2 points)

Let $\mathscr{A}$ be an abelian category and $f : B \to A$ be a morphism of $\mathscr{A}$. Show that, for every object C of $\mathscr{A}$, the morphism $\operatorname{Hom}_{\mathscr{A}}(\operatorname{Im}(f), C) \to \operatorname{Hom}_{\mathscr{A}}(B, C)$ (induced by $B \to \operatorname{Im}(f)$)

induces an isomorphism

$$\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}(f), C) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}\, f, C)).$$

3 More sheaves on an abelian category

We use the notation of problem 6 of problem set 3 : We fix an abelian category, and we denote by Sh the category of sheaves of abelian groups on $\mathscr{A}$ for the canonical topology. It is a full subcategory of the catgeory of presheaves $\mathrm{PSh} = \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Ab})$. Both Sh and PSh are abelian categories, and the forgetful functor $\mathrm{Sh} \to \mathrm{PSh}$ is left exact but not exact; this functor admits a left adjoint $F \longmapsto F^{\mathrm{sh}}$, which is exact.

(a). (2 points) Let $f : A \to B$ be a surjective morphism in $\mathscr{A}$. Show that, for every morphism $u : C \to B$, there exists a commutative square

$$\begin{tikzcd} C' \arrow[r, "f'"] \arrow[d] & C \arrow[d, "u"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

with $f' : C' \to C$ surjective.

(b). (3 points) Show that the Yoneda embedding $h_{\mathscr{A}} : \mathscr{A} \to \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Set})$, $A \longmapsto \mathrm{Hom}_{\mathscr{A}}(\cdot, A)$, factors as $\mathscr{A} \xrightarrow{h} \mathrm{Sh} \xrightarrow{\mathrm{For}} \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Set})$, where For is the forgetful functor and h is a fully faithful left exact additive functor.

(c). (2 points) Show that the functor $h : \mathscr{A} \to \mathrm{Sh}$ is exact.

4 Other embedding theorems

If we weaken the assumptions in Morita's theorem, we can still get interesting results. There are many variants, we will prove two here.

Let $\mathscr{A}$ be an abelian category, Q an object of $\mathscr{A}$ and $R = \mathrm{End}_{\mathscr{A}}(Q)$. As explained in the paragraph before Theorem II.3.1.6 of the notes, we can see the functor $\mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$ as an additive left exact functor from $\mathscr{A}$ to $\mathbf{Mod}_R$.

Note that we are *not* assuming that Q is projective for now.

We assume that $\mathscr{A}$ admits all small colimits. From now on, we assume that $\mathscr{A}$ admits all small colimits. (You can mostly ignore the smallness condition. It basically means that you can take colimits indexed by all sets that are built out of sets like $\mathrm{Hom}_{\mathscr{A}}(A, B)$. The rigorous way to say it is that $\mathscr{A}$ is a $\mathscr{W}$-category, with $\mathscr{W}$ a universe, and that it admits limits indexed by $\mathscr{W}$-small categories.)

(a). (3 points) Show that, for every right R-module M, the functor $\mathscr{A} \to \mathbf{Set}$, $A \longmapsto \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathscr{A}}(Q, A))$ is representable. We denote a pair representing this functor by $(M \otimes_R Q, \eta(M))$.

(b). (1 point) Show that the functor $G = \mathrm{Hom}_{\mathscr{A}}(Q, \cdot) : \mathscr{A} \to \mathbf{Mod}_R$ admits a left adjoint F.

(c). (2 points) If M is a free right R-module, show that $\eta(M) : M \to G(F(M))$ is injective, and that it is bijective if M is also finitely generated.

(d). Let $\mathscr{A}' \subset \mathscr{A}$ be a full subcategory of $\mathscr{A}$ that is stable by taking finite limits and finite colimits.

(i) (1 point) Show that $\mathscr{A}'$ is an abelian category and that the inclusion functor $\mathscr{A}' \to \mathscr{A}$ is exact.

From now on, we assume that the category $\mathscr{A}'$ is small. Suppose that Q is a generator of $\mathscr{A}$. For every object A of $\mathscr{A}$, consider the surjective morphism $q_A : \bigoplus_{\mathrm{Hom}_{\mathscr{A}}(Q,A)} Q \to A$ of Proposition II.3.1.3(i)(e) of the notes. Let $P = \bigoplus_{A \in \mathrm{Ob}(\mathscr{A}')} \bigoplus_{\mathrm{Hom}_{\mathscr{A}}(Q,A)} Q$; for every $A \in \mathrm{Ob}(\mathscr{A}')$, we have a surjective morphism $p_A : P \to A$, which is given by q_A on the summand of P indexed by A and by 0 on the other summands. Let $S = \mathrm{End}_{\mathscr{A}}(P)$, and consider the functor $G' = \mathrm{Hom}_{\mathscr{A}}(P, \cdot) : \mathscr{A} \to \mathbf{Mod}_S$.

(ii) (2 points) Show that G' is faithful, and that it is exact if Q is projective.

(iii) (3 points) If Q is projective, show that the restriction of G' to $\mathscr{A}'$ is fully faithful.

From now on, we also assume that small filtrant colimits are exact in $\mathscr{A}$ and that Q is a generator of $\mathscr{A}$. [1] We do not assume that Q is projective.

(e). The goal of this question is to show that G is fully faithful. Let $\mathscr{C}$ be the full subcategory of $\mathscr{A}$ whose objects are finite direct sums of copies of P, and $\mathscr{D}$ the full subcategory of $\mathbf{Mod}_R$ whose objects are finitely generated free R-modules. We denote by $h : \mathscr{A} \to \mathrm{PSh}(\mathscr{C})$ the functor $A \longmapsto \mathrm{Hom}_{\mathscr{A}}(\cdot, A)_{|\mathscr{C}}$, and by $h' : \mathbf{Mod}_R \to \mathrm{PSh}(\mathscr{D})$ the functor $M \longmapsto \mathrm{Hom}_R(\cdot, M)_{|\mathscr{D}}$.

(i) (1 point) Show that G induces an equivalence of categories $\mathscr{C} \to \mathscr{D}$.

(ii) (1 points) Show that $h' : \mathbf{Mod}_R \to \mathrm{PSh}(\mathscr{D})$ is fully faithful.

(iii) (1 points) Assuming that $h : \mathscr{A} \to \mathrm{PSh}(\mathscr{C})$ is fully faithful, show that $G : \mathscr{A} \to \mathbf{Mod}_R$ is fully faithful.

(iv) (3 points) Show that h is left exact and faithful, and that, for any morphism f of $\mathscr{A}$, if $h(f)$ is surjective, then f is surjective.

(v) (2 points) For any object B of $\mathscr{C}$, any morphism $f : B \to A$ in $\mathscr{A}$ and any object C of $\mathscr{A}$, show that the map

$$\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}\, f, C) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathrm{Im}(h(B) \to h(A)), h(C))$$

is an isomorphism. (Hint : problem 2.)

Let A be an object of $\mathscr{A}$. Denote by $\mathscr{C}/A$ the category of pairs (B, f), where $B \in \mathrm{Ob}(\mathscr{C})$ and $f : B \to A$ is a morphism of $\mathscr{A}$; a morphism $u : (B, f) \to (B', f')$ is a morphism $u : B \to B'$ such that $f = f' \circ u$.

Let I be the set of finite subsets of $\mathrm{Ob}(\mathscr{C}/A)$, ordered by inclusion; the corresponding category is clearly filtrant. Define a functor $\xi : I \to \mathscr{C}$ by sending a finite set $J = \{(B_1, f_1), \ldots, (B_n, f_n)\}$ to $B_1 \oplus \ldots \oplus B_n$; note that $\xi(J)$ comes with a morphism to A, given by $\begin{pmatrix} f_1 & \ldots & f_n \end{pmatrix}$.

(vi) (1 point) Show that the canonical morphism $\varinjlim_{J \in I} h(\xi(J)) \to h(A)$ is an epimorphism.

(vii) (1 point) Show that the canonical morphism $\varinjlim_{J \in I} \mathrm{Im}(h(\xi(J)) \to h(A)) \to h(A)$ is an isomorphism.

[1] In other words, $\mathscr{A}$ is a Grothendieck abelian category.

(viii) (2 points) Show that the canonical morphism $\varinjlim_{J\in I} \xi(J) \to A$ is an epimorphism.

(ix) (1 point) Show that the canonical morphism $\varinjlim_{J\in I} \mathrm{Im}(\xi(J) \to A) \to A$ is an isomorphism.

(x) (2 points) For C another object of $\mathscr{A}$, show that $\mathrm{Hom}_{\mathscr{A}}(A,C) \xrightarrow{h} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(A), h(C))$ is bijective.

(f). The goal of this question is to show that F is exact.

(i) (2 points) Show that it suffices to prove that F preserves injections.

Let $f : M \to N$ be a morphism in $\mathbf{Mod}_R$.

(ii) (2 points) Suppose that M is finitely generated free and that N is free. Show that the composition of $\eta(\mathrm{Ker}\, f) : \mathrm{Ker}\, f \to G(F(\mathrm{Ker}\, f))$ and of the canonical morphism $G(F(\mathrm{Ker}\, f)) \to \mathrm{Ker}(G(F(f)))$ is an isomorphism.
Hint : Use the commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\eta(M)"'] & N \arrow[d, "\eta(N)"] \\ G(F(M)) \arrow[r, "G(F(f))"'] & G(F(N)) \end{tikzcd}$$

(iii) (3 points) Suppose that M is finitely generated, that N is free and that f is injective. Show that $F(f)$ is injective. (Hint : question 1(c).)

(iv) (1 point) Suppose that N is free and that f is injective. Show that $F(f)$ is injective. (Hint : M is the union of its finitely generated submodules.)

(v) (1 point) Suppose that f is injective. Show that we can find a commutative diagram with exact rows :

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & K \arrow[r] & N' \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

such that f' is injective and N' is free.

(vi) (3 points) Suppose that f is injective. Applying F to the diagram of (v) and using 1(a), show that $F(f)$ is injective.

In conclusion, here are our embedding results so far :

(1) If $\mathscr{A}$ admits small colimits and a projective generator, we have shown that every small full abelian subcategory $\mathscr{A}'$ of $\mathscr{A}$ such that $\mathscr{A}' \subset \mathscr{A}$ is exact admits a fully faithful exact functor into a category of modules over some ring.

(2) If $\mathscr{A}$ is a Grothendieck abelian category (it admits small colimits, small filtrant colimits are exact, and $\mathscr{A}$ has a generator), then we have shown that $\mathscr{A}$ admits a fully faithful left exact functor into a category of modules over a ring, with an exact left adjoint. This is known as the Gabriel-Popescu embedding theorem.

(3) We also have Morita's theorem (Theorem II.3.1.6 of the notes) : If $\mathscr{A}$ admits small colimits and has a projective generator P such that the functor $\mathrm{Hom}_{\mathscr{A}}(P, \cdot)$ commutes with small direct sums, then $\mathscr{A}$ is equivalent to a category of modules over a ring.