MAT 540 : Problem Set 3

Due Friday, October 4

1. Free preadditive and additive categories. (extra credit)

Remember that **Cat** is the category of category (the objects of **Cat** are categories, and the morphisms of **Cat** are functors). Let **PreAdd** be the category whose objects are preadditive categories and whose morphisms are additive functors; let **Add** be the full subcategory of **PreAdd** whose objects are additive categories. We have a (faithful) forgetful functor For : **PreAdd** $\to$ **Cat**; we also denote the inclusion functor from **Add** to **PreAdd** by F.

(a). (2 points) Show that For has a left adjoint, that we will denote by $\mathscr{C} \longmapsto \mathbb{Z}[\mathscr{C}]$.

(b). (4 points) Show that F has a left adjoint, that we will denote by $\mathscr{C} \longmapsto \mathscr{C}^{\oplus}$. (Hint : If $\mathscr{C}$ is preadditive, consider the category $\mathscr{C}^{\oplus}$ whose objects are 0 and finite sequences $(X_1, \ldots, X_n)$ of objects of $\mathscr{C}$, where a morphism from $(X_1, \ldots, X_n)$ to $(Y_1, \ldots, Y_m)$ is a $m \times n$ matrix of morphisms $X_i \to Y_j$, and where the only from 0 to any object and from any object to 0 is 0.)

Solution. One subtlety is that the categories **Cat**, **PreAdd** and **Add** are actually 2-categories, so the Homs in these categories are themselves categories, and it is not reasonable in general to expect the adjunction isomorphism to be an isomorphism of categories; it it much more natural to require it to be an equivalence that is natural in both its entries in the appropriate sense. There are several natural ways to make this precise, and it can quickly become extremely painful. See for example `https://ncatlab.org/nlab/show/2-adjunction` for a discussion and further references.

(a). Let $\mathscr{C}$ be a category. We defined the category $\mathbb{Z}[\mathscr{C}]$ in the following way :

- $\mathrm{Ob}(\mathbb{Z}[\mathscr{C}]) = \mathrm{Ob}(\mathscr{C})$;
- for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{Hom}_{\mathbb{Z}[\mathscr{C}]}(X, Y) = \mathbb{Z}^{(\mathrm{Hom}_{\mathscr{C}}(X,Y))}$;
- the composition law of $\mathbb{Z}[\mathscr{C}]$ is deduced from that of $\mathscr{C}$ by bilinearity.

Note that $\mathscr{C}$ is naturally a subcategory of $\mathbb{Z}[\mathscr{C}]$.

This construction is functorial in $\mathscr{C}$, that is, any functor $F : \mathscr{C} \to \mathscr{D}$ defines in an obvious way a functor $\mathbb{Z}[F] : \mathbb{Z}[\mathscr{C}] \to \mathbb{Z}[\mathscr{D}]$, and we have $\mathbb{Z}[G \circ F] = \mathbb{Z}[G] \circ \mathbb{Z}[F]$. (In other words, $\mathscr{C} \longmapsto \mathbb{Z}[\mathscr{C}]$ is a strict 2-functor, see `https://ncatlab.org/nlab/show/strict+2-functor`). Let $\mathscr{C}$ be a category, $\mathscr{D}$ be a preadditive category and F be a functor. Then there is an obvious additive functor $\alpha(\mathscr{C}, \mathscr{D})(F) : \mathbb{Z}[\mathscr{C}] \to \mathscr{D}$; it is equal to F on the objects of $\mathbb{Z}[\mathscr{C}]$ and equal to the unique extension of F by linearity on the groups of morphisms. Also, any morphism $u : F \to G$ of functors $\mathscr{C} \to \mathscr{D}$ gives rise to a morphism of additive functors $\alpha(u) : \alpha(\mathscr{C}, \mathscr{D})(F) \to \alpha(\mathscr{C}, \mathscr{D})(G)$. This defines a functor $\alpha(\mathscr{C}, \mathscr{D}) : \mathrm{Func}(\mathscr{C}, \mathscr{D}) \to \mathrm{Func}_{\mathrm{add}}(\mathbb{Z}[\mathscr{C}], \mathscr{D})$, that is natural in $\mathscr{C}$ in $\mathscr{D}$. In this case, the functor $\alpha(\mathscr{C},\mathscr{D})$ is actually an isomorphism of categories. Indeed, if $G:\mathbb{Z}[\mathscr{C}]\to\mathscr{D}$ is an functor, then its restriction F to the subcategory $\mathscr{C}$ of $\mathbb{Z}[\mathscr{C}]$ is a functor $\mathscr{C}\to\mathscr{D}$ such that $G=\alpha(\mathscr{C},\mathscr{D})(F)$.

(b). Let $\mathscr{C}$ be a preadditive category. We show that the preadditive category $\mathscr{C}^{\oplus}$ defined in the problem is additive. It has a zero object by construction, so it suffices to show that the product of two objects always exists. Let $X=(X_1,\dots,X_n)$ and $Y=(Y_1,\dots,Y_m)$ be two objects of $\mathscr{C}^{\oplus}$. Let $Z=(X_1,\dots,X_n,Y_1,\dots,Y_m)$ and $p:Z\to X$, $q:Z\to Y$ be the morphisms given by the matrices $\begin{pmatrix}I_n & 0_{n,m}\end{pmatrix}$ and $\begin{pmatrix}0_{m,n} & I_m\end{pmatrix}$, where $I_n=\begin{pmatrix}\mathrm{id}_{X_1} & & 0\\ & \ddots & \\ 0 & & \mathrm{id}_{X_n}\end{pmatrix}$, $I_m=\begin{pmatrix}\mathrm{id}_{Y_1} & & 0\\ & \ddots & \\ 0 & & \mathrm{id}_{Y_m}\end{pmatrix}$ and $0_{n,m}$ (resp. $0_{m,n}$) is a $n\times m$ (resp. $m\times n$) matrix with all its entries equal to 0. We claim that this makes Z into the product of X and Y. Indeed, we have morphisms $i:X\to Z$ and $j:Y\to Z$ with matrices $\begin{pmatrix}I_n\\ 0_{m,n}\end{pmatrix}$ and $\begin{pmatrix}0_{n,m}\\ I_m\end{pmatrix}$ respectively, and it is easy to check the conditions of Proposition III.1.1.6(iii) of the notes.

Note that we have an obvious inclusion $\mathscr{C}\subset\mathscr{C}^{\oplus}$, which is fully faithful. If $\mathscr{C}$ is additive, then $(X_1,\dots,X_n)$ is isomorphic to $X_1\oplus\dots\oplus X_n$ for all $X_1,\dots,X_n\in\mathrm{Ob}(\mathscr{C})$ (and the zero object of $\mathscr{C}^{\oplus}$ is isomorphic to the zero object of $\mathscr{C}$), so the inclusion $\mathscr{C}\subset\mathscr{C}^{\oplus}$ is essentially surjective in this case, hence an equivalence of categories.

If $\mathscr{C}$ is a preadditive category, $\mathscr{D}$ is an additive category and $F:\mathscr{C}\to\mathscr{D}$ is an additive functor, then we can extend F to an additive functor $\alpha(\mathscr{C},\mathscr{D})(F):\mathscr{C}^{\mathrm{op}}\to\mathscr{D}$. However, this requires the choice of a particular direct sum for every finite family of objects of $\mathscr{D}$, so this construction is not unique (just unique up to unique isomorphism); in particular, if F is obtained by restriction from an additive functor $G:\mathscr{C}^{\oplus}\to\mathscr{D}$, we can only say that $\alpha(\mathscr{C},\mathscr{D})(F)$ and G are isomorphic (and the isomorphism between them is unique). So we still get a functor $\alpha(\mathscr{C},\mathscr{D}):\mathrm{Func}_{\mathrm{add}}(\mathscr{C},\mathscr{D})\to\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\oplus},\mathscr{D})$ (natural in $\mathscr{C}$ and $\mathscr{D}$), but it is an equivalence of categories, not an isomorphism.

$\square$

2. Pseudo-abelian completion. Let $\mathscr{C}$ be an additive category. If X is an object of $\mathscr{C}$, an endomorphism $p\in\mathrm{End}_{\mathscr{C}}(X)$ is called a *projector* or *idempotent* if $p\circ p=p$. A *pseudo-abelian* (or *Karoubian*) category is a preadditive category in which every projector has a kernel.

(a). (3 points) Let $\mathscr{C}$ be a category and $p\in\mathrm{End}_{\mathscr{C}}(X)$ be a projector. Show that :

- $\mathrm{Ker}(p,\mathrm{id}_X)$ exists if and only if $\mathrm{Coker}(p,\mathrm{id}_X)$ exists;
- if $u:Y\to X$ is a kernel of (p,id_X) and $v:X\to Z$ is a cokernel of (p,id_X), then there exists a unique morphism $f:Z\to Y$ such that $u\circ f\circ v=p$, and this morphism f is an isomorphism.

(b). (3 points) If $\mathscr{C}$ is a pseudo-abelian category, show that every projector has a kernel, a cokernel, a coimage and an image and that, if $p\in\mathrm{End}_{\mathscr{C}}(X)$ is a projector, then the canonical morphisms $\mathrm{Ker}(p)\to X$ and $\mathrm{Im}(p)\to X$ make X into a coproduct of $(\mathrm{Ker}(p),\mathrm{Im}(p))$. (In other words, the coproduct $\mathrm{Ker}(p)\oplus\mathrm{Im}(p)$ exists, and it is canonically isomorphic to X.)

(c). (3 points) Let $\mathscr{C}$ be a category. Its *pseudo-abelian completion* (or *Karoubi envelope*) is the category $\mathrm{kar}(\mathscr{C})$ defined by :

- $\mathrm{Ob}(\mathrm{kar}(\mathscr{C})) = \{(X,p) \mid X \in \mathrm{Ob}(\mathscr{C}),\ p \in \mathrm{End}_{\mathscr{C}}(X) \text{ is a projector}\}$;
- $\mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((X,p),(Y,q)) = \{f \in \mathrm{Hom}_{\mathscr{C}}(X,Y) \mid q \circ f = f \circ p = f\}$;
- the composition is given by that of $\mathscr{C}$, and the identity morphism of (X,p) is p.

Show that $\mathrm{kar}(\mathscr{C})$ is a pseudo-abelian category, and that the functor $\mathscr{C} \to \mathrm{kar}(\mathscr{C})$ sending X to (X, id_X) is additive and fully faithful.

(d). (2 points) If $\mathscr{C}$ is an additive category, show that $\mathrm{kar}(\mathscr{C})$ is also additive.

(e). (3 points) Let **PseuAb** be the full subcategory of **PreAdd** (see problem 1) whose objects are pseudo-abelian categories. Show that the inclusion functor **PseuAb** $\to$ **PreAdd** has a left adjoint.

Solution.

(a). We prove the first statement. Let $u : Y \to X$ be a kernel of (p, id_X). Consider the morphism $p : X \to X$. As $p \circ p = p = p \circ \mathrm{id}_X$, there exists a unique morphism $v : X \to Y$ such that $p = u \circ v$. We claim that $v : X \to Y$ is the cokernel of (p, id_X). First, note that $u \circ v \circ p = p \circ p = p = u \circ v \circ \mathrm{id}_X$; as u is a monomorphism by Lemma II.1.3.3 of the notes, we get that $v \circ p = v \circ \mathrm{id}_X$. Also, we have $u \circ v \circ u = p \circ u = u$, so $v \circ u = \mathrm{id}_Y$, again because u is a monomorphism. Let $v' : X \to Y'$ be a morphism such that $v' \circ p = v'$. Let $w = v' \circ u : Y \to Y'$. Then $w \circ v = v' \circ u \circ v = v' \circ p = v'$. Let $w' : Y \to Y'$ be another morphism such that $w' \circ v = v'$; then $w' = w' \circ v \circ u = v' \circ u = w$. This shows that $v : X \to Y$ is indeed a cokernel of (p, id_X).

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, dashed, "w"'] & X \arrow[r, "p"] \arrow[dr, "\mathrm{id}_X"] & X \\
Y' & X \arrow[l, "v'"] \arrow[u, "p"'] \arrow[ul, "v"] & X
\end{tikzcd}$$

Conversely, if $v : X \to Y$ is a cokernel of (p, id_X), then applying the previous paragraph to $\mathscr{C}^{\mathrm{op}}$ shows that (p, id_X) has a kernel.

We prove the second statement. Let $u : Y \to X$ be a kernel of (p, id_X) and $v : X \to Z$ be a cokernel of (p, id_X). By the first paragraph of the proof, there exists a morphism $v' : X \to Y$ such that v' is a cokernel of (p, id_X), $v' \circ u = \mathrm{id}_Y$ and $u \circ v' = p$. By the uniqueness of the cokernel, there exists a unique morphism $f : Z \to Y$ such that $f \circ v = v'$, and this morphism is an isomorphism. Finally, as u is a monomorphism, the condition $f \circ v = v'$ is equivalent to $u \circ f \circ v = u \circ v' = p$.

(b). Let $p \in \mathrm{End}_{\mathscr{C}}(X)$ be a projector. Then $q = \mathrm{id}_X - p$ is also a projector, because $q \circ q = \mathrm{id}_X - p - p + p \circ p = q$. As $\mathrm{Ker}(p) = \mathrm{Ker}(q, \mathrm{id}_X)$ and $\mathrm{Coker}(p) = \mathrm{Coker}(q, \mathrm{id}_X)$, question (a) implies that p has a cokernel, and that we may assume that $\mathrm{Ker}(p) = \mathrm{Coker}(p)$ (as objects of $\mathscr{C}$). Let $Y = \mathrm{Ker}(p)$, and let $u : Y \to X$ and $v : X \to Y$ be the kernel and cokernel morphisms. We saw in the solution of (a) that $u \circ v = q = \mathrm{id}_X - p$ and $v \circ u = \mathrm{id}_Y$. Similarly, let $Z = \mathrm{Ker}(q)$, and let $a : Z \to X$ and $b : X \to Z$ be the kernel and the cokernel morphisms; we have $b \circ a = \mathrm{id}_Z$ and $a \circ b = p$. We claim that $a : Z \to X$ is the kernel of $v : X \to Y$, that is, the image of p. As a is the kernel of q, we have $q \circ a = 0$, that is, $p \circ a = a$; so $v \circ a = v \circ p \circ a = 0$. Let $a' : Z' \to X$ be a morphism such that $v \circ a' = 0$. Then $q \circ a' = u \circ v \circ a' = 0$, so there exists a unique morphism $c : Z' \to Z$ such that $a \circ c = a'$. This finishes the proof that $a : Z \to X$ is the image of p. A similar proof (actually, the same proof in $\mathscr{C}^{\mathrm{op}}$) shows that $q : X \to Z$ is the coimage of p. Note in particular that the canonical morphism $\mathrm{Coim}(p) \to \mathrm{Im}(p)$ is an isomorphism.

It remains to show that X is the coproduct of $(u : Y \to X, a : Z \to X)$. Let $u' : Y \to X'$ and $a' : Z \to X'$ be morphisms. We must show that there exists a unique morphism $f : X \to X'$ such that $f \circ u = u'$ and $f \circ a = a'$. Take $f = u' \circ v + a' \circ b$. Then $f \circ u = u' \circ v \circ u + a \circ b \circ v = u' \circ \mathrm{id}_Y = u'$ (the fact that $b \circ v = 0$ follows from the previous paragraph applied to q, which shows that $v : Y \to X$ is the kernel of b); similarly, $f \circ a = u' \circ v \circ a + a' \circ b \circ a = a' \circ \mathrm{id}_Z = a'$. Let $f' : X \to X'$ be another morphism such that $f' \circ u = u'$ and $f' \circ a = a'$. Then

$$f' = f' \circ (p + q) = f' \circ (a \circ b + u \circ v) = a' \circ b + u' \circ v = .$$

(c). We first show that $\mathrm{kar}(\mathscr{C})$ is a pseudo-abelian category. First, $\mathrm{kar}(\mathscr{C})$ is clearly a preadditive category, because $\mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((X,p),(Y,q))$ is a subgroup of $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ for all $(X,p),(Y,q) \in \mathrm{Ob}(\mathrm{kar}(\mathscr{C}))$.

Let (X,p) be an object of $\mathrm{kar}(\mathscr{C})$, and let $f \in \mathrm{End}_{\mathrm{kar}(\mathscr{C})}((X,p))$ be a projector. We need to show that f has a kernel. By definition of the morphisms and composition in $\mathrm{kar}(\mathscr{C})$, f is an endomorphism of X in $\mathscr{C}$ such that $p \circ f = f \circ p = f$, and such that $f \circ f = f$. Let $g = p - f = \mathrm{id}_{(X,p)} - f \in \mathrm{End}_{\mathrm{kar}(\mathscr{C})}((X,p))$. Then $g \in \mathrm{End}_{\mathscr{C}}(X)$ and $g \circ g = p \circ p - p \circ f - f \circ p + f \circ f = p - f = g$, so (X,g) is an object of $\mathrm{kar}(\mathscr{C})$, and $g \in \mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((X,g),(X,p))$. We claim that $g : (X,g) \to (X,p)$ is the kernel of f. First, we have $g \circ f = p \circ f - f \circ f = 0$. Let (Y,q) be another object of $\mathrm{kar}(\mathscr{C})$, and let $u : (Y,q) \to (X,p)$ be a morphism such that $f \circ u = 0$. So $u \in \mathrm{Hom}_{\mathscr{C}}(Y,X)$ and $u \circ q = p \circ u = u$. Then have $g \circ u = p \circ u - f \circ u = p \circ u = q \circ u = u$, so u also define a morphism from (Y,q) to (X,g) in $\mathrm{kar}(\mathscr{C})$, and the following diagram commutes:

$$\begin{tikzcd}
(X,g) \arrow[r, "g"] & (X,p) \arrow[r, "f"] & (X,p) \\
& (Y,q) \arrow[ul, "u"] \arrow[u, "u"'] \arrow[ur, "0"'] &
\end{tikzcd}$$

Suppose that $v : (Y,q) \to (X,g)$ is another morphism (in $\mathrm{kar}(\mathscr{C})$) such that $g \circ v = u$. Then $v \in \mathrm{Hom}_{\mathscr{C}}(Y,X)$ and $g \circ v = v \circ q = v$, so we get $v = u$.

The last statement is clear.

(d). The object $(0, \mathrm{id}_0)$ of $\mathrm{kar}(\mathscr{C})$ is initial and final, so it suffices to show that the product of two objects of $\mathrm{kar}(\mathscr{C})$ always exists. (We'll get finite products by an obvious induction.) Let (X,p) and (Y,q) be two objects of $\mathrm{kar}(\mathscr{C})$, let $Z = X \oplus Y$ and $r = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in \mathrm{End}_{\mathscr{C}}(Z)$. Then r is clearly a projector, so (Z,r) is an object of $\mathrm{kar}(\mathscr{C})$. For every object (T,s) of $\mathrm{kar}(\mathscr{C})$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((T,s),(Z,r)) &= \{f \in \mathrm{Hom}_{\mathscr{C}}(T,Z) \mid r \circ f = f \circ s = f\} \\
&= \{(f_1,f_2) \in \mathrm{Hom}_{\mathscr{C}}(T,X) \times \mathrm{Hom}_{\mathscr{C}}(T,Y) \mid p \circ f_1 = f_1 \circ s = f_1 \\
&\qquad \text{and } q \circ f_2 = f_2 \circ s = f_2\} \\
&\simeq \mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((T,s),(X,p)) \times \mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((T,s),(Y,q)),
\end{aligned}$$

so (Z,r) is the product of (X,p) and (Y,q) in $\mathrm{kar}(\mathscr{C})$.

(e). We denote by $\Phi : \mathbf{PseuAb} \to \mathbf{PreAdd}$ the inclusion functor. Note that, if $\mathscr{C}$ is a preadditive category, then the construction of $\mathrm{kar}(\mathscr{C})$ is functorial in $\mathscr{C}$ and the functor $\eta(\mathscr{C}) : \mathscr{C} \to \mathrm{kar}(\mathscr{C})$ defined in (c) gives a morphism of functors $\mathrm{id}_{\mathbf{PseuAb}} \to \Phi \circ \mathrm{kar}$. Indeed, if $F : \mathscr{C} \to \mathscr{D}$ is an additive functor between preadditive categories, then we get a commutative diagram of functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\eta(\mathscr{C})"] \arrow[d, "F"'] & \mathrm{kar}(\mathscr{C}) \arrow[d, "\mathrm{kar}(F)"] \\ \mathscr{D} \arrow[r, "\eta(\mathscr{D})"'] & \mathrm{kar}(\mathscr{D}) \end{tikzcd}$$

by taking $\mathrm{kar}(F)$ to be the functor sending $(X, p) \in \mathrm{Ob}(\mathrm{kar}(\mathscr{C}))$ to $(F(X), F(p)) \in \mathrm{Ob}(\mathrm{kar}(\mathscr{D}))$ and sending $f \in \mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((X, p), (Y, q))$ to $F(f)$ (we obviously have $F(f) \in \mathrm{Hom}_{\mathrm{kar}(\mathscr{D})}((F(X), F(p)), (F(Y), F(q))))$.

We claim that (kar, Φ) is a pair of adjoint functors. We already constructed a candidate unit morphism $\eta : \mathrm{id}_{\mathbf{PseuAb}} \to \Phi \circ \mathrm{kar}$. Let $\mathscr{C}$ be a pseudo-abelian category. We claim that $\eta(\mathscr{C}) : \mathscr{C} \to \mathrm{kar}(\mathscr{C})$ is an equivalence of categories. We already know that it is fully faithful, so it suffices to show that it is essentially surjective. Let (X, p) be an object of $\mathrm{kar}(\mathscr{C})$, and let $a : Z \to X$ be the image of X in $\mathscr{C}$. We claim that $a \in \mathrm{Hom}_{\mathrm{kar}\mathscr{C}}((Z, \mathrm{id}_Z), (X, p))$, and that it is an isomorphism. The first statement just says that $p \circ a = a$, and we proved it in (b). For the second statement, remember that we also know by (b) that Z is the cokernel of $\mathrm{id}_X - p$, and let $b : X \to Z$ be the corresponding cokernel morphism. Then $b \circ p = b$ by (a), so $b \in \mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((X, p), (Z, \mathrm{id}_Z))$, and we have seen in (a) that $b \circ a = \mathrm{id}_Z$ and $a \circ b = p = \mathrm{id}_{(X,p)}$.

Now let $\mathscr{C}$ be a preadditive category and $\mathscr{D}$ be a pseudo-abelian category. Then we have functors, clearly functorial in $\mathscr{C}$ and $\mathscr{D}$:

$$\mathrm{Func}(\mathscr{C}, \mathscr{D}) \xrightarrow{\mathrm{kar}} \mathrm{Func}(\mathrm{kar}(\mathscr{C}), \mathrm{kar}(\mathscr{D})) \xleftarrow{\eta(\mathscr{D}) \circ (\cdot)} \mathrm{Func}(\mathrm{kar}(\mathscr{C}), \mathscr{D}).$$

Also, the functor on the right is an equivalence of categories. To construct a quasi-inverse of this equivalence, we need a quasi-inverse of $\eta(\mathscr{D})$ that is functorial in $\mathscr{D}$. It would be painful to show by hand that we can choose a quasi-inverse of $\eta(\mathscr{D})$ in a way that is (weakly) natural in $\mathscr{D}$, unless we have the good idea of using a left (or right) adjoint of $\eta(\mathscr{D})$ as quasi-inverse, and then things are slightly less annoying. Still, the functors Φ and kar are only adjoint in the sense of 2-categories. See the discussion in the solution of problem 1.

$\square$

3. Torsionfree abelian groups (extra credit)

Let $\mathbf{Ab}_{\mathrm{tf}}$ be the full subcategory of $\mathbf{Ab}$ whose objects are torsionfree abelian groups.

(a). (2 points) Give formulas for kernels, cokernels, images and coimages in $\mathbf{Ab}_{\mathrm{tf}}$.

(b). (2 points) Show that the inclusion functor $\iota : \mathbf{Ab}_{\mathrm{tf}} \to \mathbf{Ab}$ admits a left adjoint κ, and give this left adjoint.

Solution.

(a). Let $f : A \to B$ be a morphism of groups, with A and B torsionfree. Let $C = \{a \in A \mid f(a) = 0\}$ and $D = \{b \in B \mid \exists n \in \mathbb{Z} - \{0\}$ and $a \in A$ such that $f(a) = nb\}$. (This subgroup D is called the *saturation* of $f(A)$ in B.)

We claim that $i : C \to A$ is the kernel of $\mathbf{Ab}_{\mathrm{tf}}$. Indeed, C is torsionfree because it is a subgroup of A, and, as $\mathbf{Ab}_{\mathrm{tf}}$ is a full subcategory of $\mathbf{Ab}$, we have, for every torsionfree abelian group G,

$$\mathrm{Hom}_{\mathbf{Ab}_{\mathrm{tf}}}(G,C) = \{u \in \mathrm{Hom}_{\mathbf{Ab}_{\mathrm{tf}}}(G,A) \mid f \circ u = 0\}.$$

We show that B/D is torsionfree. Let x be a torsion element of B/D, and let $b \in B$ be a lift of x. Then there exists $n \in \mathbb{Z} - \{0\}$ such that $nb \in D$, and it is obvious on the definition of D that this implies that $b \in D$, hence that $x = 0$. We claim that $B \to B/D$ is the cokernel of f in $\mathbf{Ab}_{\mathrm{tf}}$. Let $p : B \to B/D$ be the canonical projection. As D contains all the $f(a)$, for $a \in A$, we clearly have $p \circ f = 0$. Let $g : B \to G$ be a morphism in $\mathbf{Ab}_{\mathrm{tf}}$ such that $g \circ f = 0$. Let $b \in D$; then there exists $a \in A$ and $n \in \mathbb{Z} - \{0\}$ such that $nb = f(a)$, so $ng(b) = g(f(a)) = 0$, so $g(b) = 0$ because G is torsionfree; this shows that $\mathrm{Ker}\, g \supset D$, so there is a unique morphism $h : B/D \to G$ such that $g = h \circ p$.

To find the image and coimage of f, we use their definitions, as well as the description of kernels and cokernels that we just obtained. The image of f is the kernel of the cokernel of f, so it is equal to D. The coimage of f is the cokernel of the kernel of f, so it is equal to the quotient A/C', where $C' = \{a \in A \mid \exists n \in \mathbb{Z} - \{0\}, na \in C\}$. Note that, in general, $f(A) := \{f(a),\ a \in A\}$ is neither the image nor the coimage of f.

(b). We define a functor $\kappa : \mathbf{Ab} \to \mathbf{Ab}_{\mathrm{tf}}$ by $\kappa(A) = A/A_{\mathrm{tor}}$, where A_{tor} is the torsion subgroup of A. If $f : A \to B$ is a morphism of abelian groups, then $f(A_{\mathrm{tor}}) \subset B_{\mathrm{tor}}$, so f induces a morphism $\kappa(f) : A/A_{\mathrm{tor}} \to B/B_{\mathrm{tor}}$. This clearly defines a functor $\mathbf{Ab}_{\mathrm{tf}} \to \mathbf{Ab}$. Let A be an abelian group and B a torsionfree abelian group. Then every group morphism $A \to B$ factors uniquely through A/A_{tor}, so we get a bijection

$$\mathrm{Hom}_{\mathbf{Ab}}(A, \iota(B)) \simeq \mathrm{Hom}_{\mathbf{Ab}}(A/A_{\mathrm{tor}}, B) = \mathrm{Hom}_{\mathbf{Ab}_{\mathrm{tf}}}(\kappa(A), B),$$

and this bijection is clearly an isomorphism of functors.

□

4. Filtered R-modules Let R be a ring, and let $\mathrm{Fil}({}_R\mathbf{Mod})$ be the category of filtered R-modules $(M, \mathrm{Fil}_* M)$ (see Example II.1.4.3 of the notes) such that $M = \bigcup_{n\in\mathbb{Z}} \mathrm{Fil}_n M$. [1]

(a). (2 points) Give formulas for kernels, cokernels, images and coimages in $\mathrm{Fil}({}_R\mathbf{Mod})$.

(b). (2 points) Let $\iota : \mathrm{Fil}({}_R\mathbf{Mod}) \to \mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod})$ be the functor sending a filtered R-module $(M, \mathrm{Fil}_* M)$ to the functor $\mathbb{Z} \to {}_R\mathbf{Mod}$, $n \longmapsto \mathrm{Fil}_n M$. Show that ι is fully faithful.

(c). (3 points) Show that ι has a left adjoint κ, and give a formula for κ.

(d). (2 points) Show that every object of the abelian category $\mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod})$ is isomorphic to the cokernel of a morphism between objects in the essential image of ι.

Solution.

(a). Let $f : (M, \mathrm{Fil}_* M) \to (N, \mathrm{Fil}_* N)$ be a morphism of filtered R-modules. Let $M' = \{x \in M \mid f(x) = 0\}$, with the filtration $\mathrm{Fil}_* M'$ defined by $\mathrm{Fil}_n M' = M' \cap \mathrm{Fil}_n M$. Let $N' = N/f(M)$, with the filtration $\mathrm{Fil}_* N'$ defined by $\mathrm{Fil}_n N' = (\mathrm{Fil}_n N + f(M))/f(M)$ (that is, $\mathrm{Fil}_n N'$ is the image of $\mathrm{Fil}_n N$ by the quotient map $N \to N'$).

The inclusion $u : M' \to M$ is a morphism in $\mathrm{Fil}({}_R\mathbf{Mod})$, by definition of the filtration on M'. We claim that $(M', \mathrm{Fil}_* M')$ is the kernel of f in $\mathrm{Fil}({}_R\mathbf{Mod})$. First, we have

[1] We say that the filtration is *exhaustive*.

$f \circ u = 0$. Let $g : (M'', \mathrm{Fil}_* M'') \to (M, \mathrm{Fil}_* M)$ be a morphism such that $f \circ g = 0$. As the functor $\mathrm{Fil}({}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$ that forgets the filtration is faithful, there is at most one morphism $h : (M'', \mathrm{Fil}_* M'') \to (M', \mathrm{Fil}_* M')$ such that $g = u \circ h$. Also, as M' is the kernel of f in ${}_R\mathbf{Mod}$, there exists $h : M'' \to M'$ such that $g = u \circ f$, and it suffices to check that this h is compatible with the filtrations. Let $n \in \mathbb{Z}$. Then $g(\mathrm{Fil}_n M'') \subset \mathrm{Fil}_n M$, so $h(\mathrm{Fil}_n M'') = M' \cap g(\mathrm{Fil}_n M'') \subset M' \cap \mathrm{Fil}_n M = \mathrm{Fil}_n M'$.

The quotient map $p : N \to N'$ is a morphism in $\mathrm{Fil}({}_R\mathbf{Mod})$, by definition of $\mathrm{Fil}_* N'$. We claim that $(N', \mathrm{Fil}_* N')$ is the cokernel of f. Let $g : (N, \mathrm{Fil}_* N) \to (N'', \mathrm{Fil}_* N'')$ be a morphism such that $g \circ f = 0$. As in the previous paragraph, it suffices to prove that the unique morphism of R-modules $h : N' \to N''$ such that $h \circ p = g$ (given by the fact that N' is the cokernel of f in ${}_R\mathbf{Mod}$) is compatible with the filtrations. Let $n \in \mathbb{Z}$. Then $\mathrm{Fil}_n\, N' = p(\mathrm{Fil}_n N)$, so $h(\mathrm{Fil}_n N') = g(\mathrm{Fil}_n N) \subset \mathrm{Fil}_n N''$.

We can now calculate the image and coimage of f using our formulas for the kernel and cokernel of a morphism of $\mathrm{Fil}({}_R\mathbf{Mod})$. The image of f is the kernel of $p : (N, \mathrm{Fil}_* N) \to (N', \mathrm{Fil}_* N')$, so it is the submodule $f(M)$ of N, with the filtration given by $\mathrm{Fil}_n f(M) = f(M) \cap \mathrm{Fil}_n N$. The coimage is the cokernel of $u : (M', \mathrm{Fil}_* M') \to (M, \mathrm{Fil}_* M)$, so it is the R-module $M/M' \simeq f(M)$, with the filtration image of that of M by the quotient map $M \to M/M'$. Note that, even though the image and coimage of f have the same underlying R-module, their filtrations are different in general.

(b). If $(M, \mathrm{Fil}_* M)$ is a filtered R-module and $F = \iota(M, \mathrm{Fil}_* M)$ is the associated functor $\mathbb{Z} \to {}_R\mathbf{Mod}$, then $\varinjlim_{\mathbb{Z}} F = \varinjlim_{n \in \mathbb{Z}} \mathrm{Fil}_n M = \bigcup_{n \in \mathbb{Z}} \mathrm{Fil}_n M$; if $(M, \mathrm{Fil}_* M)$ is an object of $\mathrm{Fil}({}_R\mathbf{Mod})$, this is equal to M.

Let $f, g : (M, \mathrm{Fil}_* M) \to (N, \mathrm{Fil}_* N)$ be two morphisms in $\mathrm{Fil}({}_R\mathbf{Mod})$ such that $\iota(f) = \iota(g)$. Then $\varinjlim \iota(f) = \varinjlim \iota(g)$ as morphisms from $M = \varinjlim_{n \in \mathbb{Z}} \mathrm{Fil}_n M \to \varinjlim_{n \in \mathbb{Z}} \mathrm{Fil}_n N = N$; as the first of these morphisms is equal to f (because its restriction to each $\mathrm{Fil}_n M$ is equal to f) and the second is equal to g (same reason), we get that $f = g$. So the functor ι is faithful.

Let $(M, \mathrm{Fil}_* M)$, $(N, \mathrm{Fil}_* N)$ be objects of $\mathrm{Fil}({}_R\mathbf{Mod})$, and let $\alpha : \iota(M, \mathrm{Fil}_* M) \to \iota(N, \mathrm{Fil}_* N)$ be a morphism of functors. Then $f = \varinjlim \alpha$ is a morphism of R-modules from M to N. We claim that f is actually a morphism of filtered R-modules. Indeed, let $n \in \mathbb{Z}$. Then we have a commutative diagram

$$\begin{tikzcd}
\iota(M, \mathrm{Fil}_* M)(n) \arrow[r, equal] & \mathrm{Fil}_n M \arrow[r, "\alpha(n)"] \arrow[d] & \mathrm{Fil}_n N \arrow[r, equal] \arrow[d] & \iota(N, \mathrm{Fil}_* N)(n) \\
 & M \arrow[r, "f"'] & N &
\end{tikzcd}$$

which shows that $f(\mathrm{Fil}_n M) \subset \mathrm{Fil}_n N$, and also that $\iota(f) = \alpha$. So the functor ι is full.

We can also calculate the essential image of ι. We claim that it is the subcategory of functors $F : \mathbb{Z} \to {}_R\mathbf{Mod}$ such that $F(u)$ is injective for every morphism u of $\mathbb{Z}$. First, the functors $\iota(M, \mathrm{Fil}_* M)$ clearly satisfy this conditions, because the mrophisms $F(u)$ are the inclusion $\mathrm{Fil}_n M \subset \mathrm{Fil}_m M$ for $n \leq m$. Conversely, suppose that F satisfies the condition above. Let $M = \varinjlim F$. For every $n \in \mathbb{Z}$, the morphism $F(n)$ is the colimit of the injections $F(n) \to F(m)$, $m \geq n$, so it is injective because filtrant colimits are exact in ${}_R\mathbf{Mod}$; let $\mathrm{Fil}_n M \subset M$ be its image. We have $M = \bigcup_{n \in \mathbb{Z}} \mathrm{Fil}_n M$ because $M = \varinjlim_{n \in \mathbb{Z}} F(n)$, and the isomorphisms $F(n) \xrightarrow{\sim} \mathrm{Fil}_n M$ induce an isomorphism of functors $F \xrightarrow{\sim} \iota(M, \mathrm{Fil}_* M)$.

(c). We define $\kappa : \mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod}) \to \mathrm{Fil}({}_R\mathbf{Mod})$ in the following way : If $F : \mathbb{Z} \to {}_R\mathbf{Mod}$

is a functor, we set $\kappa(F) = (M, \mathrm{Fil}_* M)$, where $M = \varinjlim F$ and $\mathrm{Fil}_n M$ is the image of the canonical morphism $F(n) \to M$. If $\alpha : F \to G$ is a morphism of functors, we get a morphism of R-modules $f : \varinjlim \alpha : M = \varinjlim F \to N = \varinjlim G$, and this is a morphism of filtered R-modules because we have commutative squares

$$\begin{tikzcd} \mathrm{Fil}(n) \arrow[r, "\alpha(n)"] \arrow[d] & G(n) \arrow[d] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

We claim that κ is left adjoint to ι. First we construct a morphism of functors $\eta : \mathrm{id}_{\mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod})} \to \iota \circ \kappa$. Let $F : \mathbb{Z} \to {}_R\mathbf{Mod}$ be a functor, let $(M, \mathrm{Fil}_* M) = \kappa(F)$, and let $G = \iota(\kappa(F))$. By construction of $\kappa(F)$, we have a morphism of R-modules $F(n) \to \mathrm{Fil}_n M = G(n)$ for every $n \in \mathbb{Z}$, and these morphisms define a morphism of functors $\eta(F) : F \to G$. The fact that the morphisms $\eta(F)$ define a morphism of functors is immediate. Also note that, if $(M, \mathrm{Fil}_* M)$ is an object of $\mathrm{Fil}({}_R\mathbf{Mod})$ and $F = \iota(M, \mathrm{Fil}_* M)$, then $M = \varinjlim F$ and $F(n) = \mathrm{Fil}_n M$ for every $n \in \mathbb{Z}$, so $\kappa(F) = (M, \mathrm{Fil}_* M)$. This gives an isomorphism of functors $\varepsilon : \kappa \circ \iota \xrightarrow{\sim} \mathrm{id}_{\mathrm{Fil}({}_R\mathbf{Mod})}$, and it is easy to see that $\iota(\varepsilon)$ is the inverse of $\eta(\iota) : \iota \to \iota \circ \kappa \circ \iota$. We want to apply Proposition I.4.6 of the notes to show that (κ, ι) is a pair of adjoint functors. It remains to prove that the composition

$$\kappa \xrightarrow{\kappa(\eta)} \kappa \circ \iota \circ \kappa \xrightarrow{\varepsilon(\kappa)} \kappa$$

is the identity, but this also follows immediately from the definitions.

(d). Let $F \in \mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod})$. Suppose that we have found $(M, \mathrm{Fil}_* M) \in \mathrm{Ob}(\mathrm{Fil}({}_R\mathbf{Mod}))$ and $\alpha : \iota(M, \mathrm{Fil}_* M) \to F$ such that $\alpha(n) : \mathrm{Fil}_n M \to F(n)$ is surjective for every $n \in \mathbb{Z}$. We define $(N, \mathrm{Fil}_* N)$ by $N = \mathrm{Ker}(M \to \varinjlim F)$ and $\mathrm{Fil}_n N = \mathrm{Ker}(\mathrm{Fil}_n M \to F(n))$. This is clearly a filtered R-module. If $x \in N$, there exists $n \in \mathbb{Z}$ such that $x \in \mathrm{Fil}_n M$; as the image of x in $\varinjlim F$ is 0, there exists $m \geq n$ such that the image of x in $F(m)$ is 0, and then $x \in \mathrm{Fil}_m N$. So $N = \bigcup_{n \in \mathbb{Z}} \mathrm{Fil}_n N$, and it si clear from the way cokernels are calculated in $\mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod})$ that F is the cokernel of the morphism $\iota(N, \mathrm{Fil}_* N) \to \iota(M, \mathrm{Fil}_* M)$.

So, to answer the question, it suffices to find $(M, \mathrm{Fil}_* M)$ satufying the conditions of the previous paragraph. Let $M = \bigoplus_{n \in \mathbb{Z}} F(n)$ and, for every $m \in \mathbb{Z}$, let $\mathrm{Fil}_m M = \bigoplus_{n \leq m} F(n)$. Then $(M, \mathrm{Fil}_* M)$ is an object of $\mathrm{Fil}({}_R\mathbf{Mod})$. Let $\alpha : \iota(M, \mathrm{Fil}_* M) \to F$ be the morphism of functors such that $\alpha(m) : \bigoplus_{n \leq m} F(n) \to F(m)$ is given on the factor $F(n)$ by $F(u_{nm}) : F(n) \to F(m)$, where u_{nm} is the unique morphism from n to m in $\mathbb{Z}$; this clearly defines a morphism of functors.

□

5. Admissible "topology" on $\mathbb{Q}$ If you have not seen sheaves (on a topological space) in a while, you might want to go read about them a bit, otherwise (b) will be very hard, and (f) won't be as shocking as it should be. Also, if the construction of sheafification that you learned used stalks, you should go and read a construction that uses open covers instead; see for example Section III.1 of the notes.

If $a, b \in \mathbb{R}$, we write

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

and

$$]a, b[= \{x \in \mathbb{R} \mid a < x < b\}.$$

Consider the space $\mathbb{Q}$ with its usual topology. An *open rational interval* is an open subset of $\mathbb{Q}$ of the form $\mathbb{Q}\cap]a,b[$ with $a,b\in\mathbb{Q}$. An *closed rational interval* is a closed subset of $\mathbb{Q}$ of the form $\mathbb{Q}\cap[a,b]$, with $a,b\in\mathbb{Q}$.

We say that an open subset U of $\mathbb{Q}$ is *admissible* if we can write U as a union $\bigcup_{i\in I} A_i$ of open rational intervals such that, for every closed rational interval $B=\mathbb{Q}\cap[a,b]\subset U$, there exists a finite subset J of I and closed rational intervals $B_j\subset A_j$, for $j\in J$, such that $B\subset\bigcup_{j\in J}B_j$.

If U is an admissible open subset of $\mathbb{Q}$ and $U=\bigcup_{i\in I}U_i$ is an open cover of U, we say that this cover is *admissible* if, for every closed rational interval $B=\mathbb{Q}\cap[a,b]\subset U$, there exist a finite subset J of I and closed rational intervals $B_j\subset U_j$, for $j\in J$, such that $B\subset\bigcup_{j\in J}B_j$.

Let Open_a be the poset of admissible open subsets of $\mathbb{Q}$ (ordered by inclusion), and let $\mathrm{PSh}_a=\mathrm{Func}(\mathrm{Open}_a,\mathbf{Ab})$. This is called the category of presheaves of abelian groups on the admissible topology of $\mathbb{Q}$. If $F:\mathrm{Open}_a^{\mathrm{op}}\to\mathbf{Set}$ is a presheaf and $U\subset V$ are admissible open subsets of $\mathbb{Q}$, we denote the map $F(V)\to F(U)$ by $s\longmapsto s_{|U}$.

We say that a presheaf $F:\mathrm{Open}_a^{\mathrm{op}}\to\mathbf{Ab}$ is a *sheaf* if, for every admissible open subset U of $\mathbb{Q}$ and for every admissible cover $(U_i)_{i\in I}$ of U, the following two conditions hold :

(1) the map $F(U)\to\prod_{i\in I}F(U_i)$, $s\longmapsto(s_{|U_i})$ is injective;

(2) if $(s_i)\in\prod_{i\in I}F(U_i)$ is such that $s_{i|U_i\cap U_j}=s_{j|U_i\cap U_j}$ for all $i,j\in I$, then there exists $s\in F(U)$ such that $s_i=s_{|U_i}$ for every $i\in I$.

The full subcategory Sh_a of PSh_a whose objects are sheaves is called the category of sheaves of abelian groups on the admissible topology of $\mathbb{Q}$. [2]

(a). Let U be an open subset of $\mathbb{Q}$, and let $V(U)$ be the union of all the open subsets V of $\mathbb{R}$ such that $V\cap\mathbb{Q}=U$. Show that $V(U)$ is the union of all the intervals $[a,b]$, for $a,b\in\mathbb{Q}$ such that $\mathbb{Q}\cap[a,b]\subset U$.

(b). (2 points) Show that every open set in $\mathbb{Q}$ is admissible. [3]

(c). (1 point) Give an open cover of an open subset of $\mathbb{Q}$ that is not an admissible open cover.

(d). (3 points) Show that the inclusion functor $\mathrm{Sh}_a\to\mathrm{PSh}_a$ has a left adjoint $F\longmapsto F^{sh}$. (The sheafification functor.)

(e). (4 points) Show that Sh_a is an abelian category.

(f). (3 points) Show that the inclusion $\mathrm{Sh}_a\to\mathrm{PSh}_a$ is left exact but not exact, and that the sheafification functor $\mathrm{PSh}_a\to\mathrm{Sh}_a$ is exact.

For every $x\in\mathbb{Q}$ and every presheaf $F\in\mathrm{Ob}(\mathrm{PSh}_a)$, we define the *stalk* of F at x to be $F_x=\varinjlim_{U\ni x}F(U)$, that is, the colimit of the functor $\phi:\mathrm{Open}_a(\mathbb{Q},x)^{\mathrm{op}}\to\mathbf{Ab}$, where $\mathrm{Open}_a(\mathbb{Q},x)$ is the full subcategory of Open_a of admissible open subsets containing x and ϕ is the restriction of F.

(g). (2 points) For every $x\in\mathbb{Q}$, show that the functor $\mathrm{Sh}_a\to\mathbf{Ab}$, $F\longmapsto F_x$ is exact.

(h). (4 points) Let PSh (resp. Sh) be the usual category of presheaves (resp. sheaves) of abelian groups on $\mathbb{R}$. Show that the functor $\mathrm{Sh}\to\mathrm{PSh}_a$ sending a sheaf F on $\mathbb{R}$ to the presheaf $U\longmapsto F(V(U))$ on $\mathbb{Q}$ is fully faithful, that its essential image is Sh_a, and that it is exact as a functor from Sh_a to Sh.

[2] Of course, we could also define presheaves and sheaves with values in **Set**.

[3] There is a similar notion of admissible open subset in $\mathbb{Q}^n$, where intervals are replaced by products of intervals, and this result does not hold for $n\geq 2$.

(i). (2 points) Find a nonzero object F of Sh_a such that $F_x = 0$ for every $x \in \mathbb{Q}$.

Solution.

(a). The set $V(U)$ is obviously an open subset of $\mathbb{R}$, we have $V(U) \cap \mathbb{Q} = U$, and $V(U)$ is maximal among open subsets of $\mathbb{R}$ satisfying this condition.

Let $a, b \in \mathbb{Q}$ such that $\mathbb{Q} \cap [a, b] \subset U$. Then $(]a, b[\cup V(U)) \cap \mathbb{Q} = U$, so $]a, b[\subset V(U)$ by the maximality of $V(U)$; as $a, b \in U \subset V$, we get that $[a, b] \subset V(U)$.

Conversely, let $x \in V(U)$. As $V(U)$ is open in $\mathbb{R}$ and $\mathbb{Q}$ is dense, there exist $a, b \in \mathbb{Q}$ such that $a < x < b$ and $[a, b] \subset V(U)$. Then $\mathbb{Q} \cap [a, b] \subset \mathbb{Q} \cap V(U) \subset U$.

(b). Let U be an open subset of $\mathbb{Q}$, and let $V = V(U)$. Let $((a_i, b_i))_{i \in I}$ be the family of all couples $(a_i, b_i) \in \mathbb{Q}$ such that $a_i < b_i$ and that the open interval $]a_i, b_i[$ is contained in $V(U)$. For every $i \in I$, let $A_i =]a_i, b_i[\cap \mathbb{Q}$; this is an open rational interval. We have $U = \bigcup_{i \in I} A_i$, and we claim that this is an admissible cover of U, which implies that U is admissible. Let B be a closed rational interval, and let $a, b \in \mathbb{Q}$ such that $B = \mathbb{Q} \cap [a, b]$. Then $[a, b] \subset V$ by question (a). As $[a, b]$ is compact, there exists a finite subset J of I such that $[a, b] \subset \bigcup_{j \in J}]a_j, b_j[$. By the shrinking lemma (and the finiteness of J), there exists $\varepsilon > 0$ such that $[a, b] \subset \bigcup_{j \in J} [a_j + \varepsilon, b_j - \varepsilon]$, so $B \subset \bigcup_{j \in J} B_j$, where $B_j = \mathbb{Q} \cap [a_j + \varepsilon, b_j - \varepsilon]$.

(c). Let $U =]0, 2[\cap \mathbb{Q}$. Let $(x_n)_{n \in \mathbb{N}}$ be an increasing sequence of rational numbers that converges to $\sqrt{2}$. For every $n \in \mathbb{N}$, let $U_n = \mathbb{Q} \cap (]0, x_n[\cup]2/x_n, 2[)$; note that $x_n < \sqrt{2} < 2/x_n$. Then $U = \bigcup_{n \in \mathbb{N}} U_n$. We claim that this is not an admissible cover. Let $a, b \in \mathbb{Q}$ such that $0 < a < 1/\sqrt{2} < b < 1$, and let $B = \mathbb{Q} \cap [a, b]$. If there existed a finite subset M of $\mathbb{N}$ such that $B \subset \bigcup_{n \in M} U_n$, then, as the family U_n is increasing, there would exist $N \in \mathbb{N}$ such that $B \subset U_N$, which is absurd because $x_{N+1} \in B \setminus U_{N+1}$.

(d). The same construction as in Section III.1 of the notes gives an additive functor $\mathscr{F} \longmapsto F^{\mathrm{sh}} = \mathscr{F}^{++}$ from PSh_a to Sh_a and a morphism of functors $\iota : \mathrm{id}_{\mathrm{PSh}_a} \to (\cdot)^{\mathrm{sh}}$ such that, if $\mathscr{F}$ is a sheaf, then $\iota(\mathscr{F})$ is an isomorphism. Indeed, we never used the fact that we have a topology in this construction. We only used the fact that we have a notion of open subsets and a notion of covers of open subsets, such that :

(1) any two covers of an open subset admit a common refinement;

(2) if we have a cover $(U_i)_{i \in I}$ of U and we take a cover of each U_i, then the union of these gives a cover of U;

(3) if we have a cover of U and we intersect it with an open subset V of U, then we get a cover of V.

Let's check these properties.

(1) Let $(U_i)_{i \in I}$ and $(V_j)_{j \in J}$ be two admissible covers of U. We claim that $(U_i \cap V_j)_{i \in I, j \in J}$ is an admissible cover of U. Let $B \subset$ a rational closed interval. There exist finite subsets $I' \subset I$ and $J' \subset J$ and closed rational intervals $A_i \subset U_i$ and $B_j \subset V_j$, for $i \in I'$ and $j \in J'$, such that $B \subset \bigcup_{i \in I'} A_i$ and $B \subset \bigcup_{j \in J'} B_j$. For every $(i, j) \in I' \times J'$, $C_{ij} = A_i \cap B_j \subset U_i \cap V_j$ is a closed rational interval, and we have $B \subset \bigcup_{(i,j) \in I' \times J'} C_{ij}$.

(2) Let $(U_i)_{i \in I}$ be an admissible open cover of U. For every $i \in I$, let $(U_{ij})_{j \in J_i}$ be an admissible open cover of U_i. Let $B = \mathbb{Q} \cap [a, b] \subset U$ be a closed rational interval. Let $I' \subset I$ be a finite subset and $B_i \subset U_i$ be closed rational intervals, for $i \in I'$, such that $B \subset \bigcup_{i \in I'} B_i$. For every $i \in I'$, let $J'_i \subset J_i$ be a finite subset and $B_{ij} \subset U_{ij}$ be closed rational intervals, for $j \in J'_i$, such that $B_i \subset \bigcup_{j \in J'_i} B_{ij}$. Then $B \subset \bigcup_{i \in I'} \bigcup_{j \in J'_i} B_{ij}$, and the set $\bigcup_{i \in I'} J'_i$ is still finite.

(3) Let $(U_i)_{i\in I}$ be an admissible cover of U, let $V \subset U$ be another open set of $\mathbb{Q}$, and let $(V_i)_{i\in I} = (V \cap U_i)_{i\in I}$. Let $B = \mathbb{Q} \cap [a,b]$ be a closed rational interval such that $B \subset V$. Then there exist a finite subset J of I and closed rational intervals $B_j = \mathbb{Q} \cap [a_j, b_j] \subset U_j$, for $j \in J$, such that $B \subset \bigcup_{j\in J} B_j$. After replacing each B_j by its intersection with $[a,b]$, we may assume that $a_j \geq a$ and $b_j \leq b$ for every $j \in J$. Then $B_j \subset B \subset V$, so $B_j \subset V_j$, and we still have $B \subset \bigcup_{j\in J} B_j$.

The fact that $\iota(\mathscr{F})$ is an isomorphism for $\mathscr{F}$ a sheaf means that $\iota(G) : G \to G \circ F \circ G$ is an isomorphism of functors; as G is fully faithful, $\iota(G)^{-1} : G \circ F \circ G \to G$ comes from a unique isomorphism of functors $\varepsilon : F \circ G \to \mathrm{id}_{\mathrm{Sh}_a}$. By Lemma I.4.5, this ε induces a functorial morphism

$$\alpha : \mathrm{Hom}_{\mathrm{PSh}_a}(\cdot, G(\cdot)) \to \mathrm{Hom}_{\mathrm{Sh}_a}(F(\cdot), \cdot)$$

sending $f : \mathscr{F} \to \mathscr{F}'$ (with $\mathscr{F}$ a presheaf and $\mathscr{F}'$ a sheaf) to $\alpha(\mathscr{F}, \mathscr{F}')(f) = \iota(\mathscr{F}')^{-1} \circ f^{\mathrm{sh}} : \mathscr{F}^{\mathrm{sh}} \to \mathscr{F}'$. As ι is a morphism of functors, we have a commutative square

$$\begin{tikzcd}
\mathscr{F}^{\mathrm{sh}} \arrow[r, "f^{\mathrm{sh}}"] & \mathscr{F}'^{\mathrm{sh}} \\
\mathscr{F} \arrow[u, "\iota(\mathscr{F})"] \arrow[r, "f"'] & \mathscr{F}' \arrow[u, "\iota(\mathscr{F}')"']
\end{tikzcd}$$

hence $\alpha(\mathscr{F}, \mathscr{F}')(f) \circ \iota(\mathscr{F}) = f$, and, by the analogue of the uniqueness statement of Proposition III.1.10(vi) of the notes, $\alpha(\mathscr{F}, \mathscr{F}')(f)$ is the unique morphism from $\mathscr{F}^{\mathrm{sh}} \to \mathscr{F}'$ having that property. This implies that $\alpha(\mathscr{F}, \mathscr{F}')(f)$ determines f (so that $\alpha(\mathscr{F}, \mathscr{F}')$ is injective), but also that, if $g : \mathscr{F}^{\mathrm{sh}} \to \mathscr{F}'$ is any morphism of sheaves such that $f = g \circ \iota(\mathscr{F})$, then $g = \alpha(\mathscr{F}, \mathscr{F}')(f)$; the last part gives a construction of a map $\beta : \mathrm{Hom}_{\mathrm{Sh}_a}(\mathscr{F}^{\mathrm{sh}}, \mathscr{F}') \to \mathrm{Hom}_{\mathrm{PSh}_a}(\mathscr{F}, \mathscr{F}')$ such that $\alpha(\mathscr{F}, \mathscr{F}') \circ \beta$ is the identity, so $\alpha(\mathscr{F}, \mathscr{F}')$ is surjective.

(e). We will actually show that the category of sheaves has all small limits and colimits, and that the inclusion functor $\mathrm{Sh}_a \to \mathrm{PSh}_a$ commutes with limits.

If $\mathscr{F}$ is a presheaf, then $\mathscr{F}$ is a sheaf if and only if the sequence

$$0 \to \mathscr{F}(U) \overset{u(\mathscr{F},\mathscr{U})}{\to} \prod_{i\in I} \mathscr{F}(U_i) \overset{v(\mathscr{F},\mathscr{U})}{\to} \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j)$$

is exact, for every open subset U of $\mathbb{Q}$ and every admissible open cover $\mathscr{U} = (U_i)_{i\in I}$ of U, where $u(\mathscr{F}, \mathscr{U})$ sends $s \in \mathscr{F}(U)$ to $(s_{|U_i})_{i\in I}$ and $v(\mathscr{F}, \mathscr{U})$ sends $(s_i) \in \prod_{i\in I} \mathscr{F}(U_i)$ to $(s_{i|U_i\cap U_j} - s_{j|U_i\cap U_j})_{i,j\in I}$. This sequence is functorial in $\mathscr{F}$, and functorial and contravariant in $\mathscr{U}$. Also, the functors appearing in the sequence commute with limits in $\mathscr{F}$, because of the way limits are formed in categories of presheaves and because direct products (being limits) commute with limits. So, if we have a functor $\alpha : \mathscr{I} \to \mathrm{Sh}_a$ with $\mathscr{I}$ a small category, the presheaf $\varprojlim(G \circ \alpha)$ is also a sheaf; as sheaves form a full subcategory of PSh_a, this limit satisfies the universal property of the limit in the category Sh_a, so it is (the image by G of) the limit of α. Or, in other terms : to form a limit in the category of sheaves, it suffices to take the limit in the category of sheaves.

Now we show that α also has a colimit. In fact, we show that the sheafification of $\varinjlim(G \circ \alpha)$

is the colimit of α. Indeeed, for every sheaf $\mathscr{G}$, we have isomorphisms, functorial in $\mathscr{G}$:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sh}_a}(F(\varinjlim(G\circ\alpha)),\mathscr{G}) &\simeq \mathrm{Hom}_{\mathrm{PSh}_a}(\varinjlim(G\circ\alpha),G(\mathscr{G}))\\ &\simeq \varprojlim_{i\in\mathrm{Ob}(\mathscr{I}^{\mathrm{op}})}\mathrm{Hom}_{\mathrm{PSh}_a}(G(\alpha(i)),G(\mathscr{G}))\\ &= \varprojlim_{i\in\mathrm{Ob}(\mathscr{I}^{\mathrm{op}})}\mathrm{Hom}_{\mathrm{Sh}_a}(\alpha(i),\mathscr{G}),\end{aligned}$$

which is the universal property of the colimit.

As Sh_a is an additive subcatgeory of PSh_a, the fact that it has all limits and colimits shows that every morphism of Sh_a has a kernel and a cokernel; we also showed that the kernel of a morphism of Sh_a is its kernel in PSh_a, and that its cokernel is the sheafification of its cokernel in PSh_a.

Now let $f:\mathscr{F}\to\mathscr{G}$ be a morphism in Sh_a. We want to show that the canonical morphism $\mathrm{Coim}(f)\to\mathrm{Im}(f)$ is an isomorphism. By definition, $\mathrm{Im}(f)$ is the kernel of $p:\mathscr{G}\to\mathrm{Coker}(f)$; so, for every open subset U of $\mathbb{Q}$, an element $s\in\mathscr{G}(U)$ is in $(\mathrm{Im}\,f)(U)$ if and only if there exists an admissible open cover $(U_i)_{i\in I}$ of U such that, for every $i\in I$, we have $p(s_{|U_i})=0$, that is, $s_{|U_i}\in\mathrm{Im}(\mathscr{F}(U_i)\to\mathscr{G}(U_i))$. In other words, $\mathrm{Im}(f)$ is the sheafification of the separated presheaf $\mathscr{C}:U\longmapsto\mathrm{Im}(\mathscr{F}(U)\to\mathscr{G}(U))$. On the other hand, $\mathrm{Coim}(f)$ is the sheafification of the presheaf $\mathscr{I}:U\longmapsto\mathscr{F}(U)/(\ker f)(U)$. The canonical morphism $\mathrm{Coim}(f)\to\mathrm{Im}(f)$ is induced by the morphism $\mathscr{I}\to\mathscr{C}$ sending an element s of $\mathscr{F}(U)/(\mathrm{Im}\,f)(U)$ to $f(s)\in\mathscr{G}(U)$, which is an isomorphism; so $\mathrm{Coim}(f)\to\mathrm{Im}(f)$ is also an isomorphism.

(f). We saw that $G:\mathrm{Sh}_a\to\mathrm{PSh}_a$ commutes with limits, so it is left exact. To show that G is not exact, it suffices to find a surjective morphism $u:\mathscr{F}\to\mathscr{G}$ in Sh_a such that $\mathscr{F}(\mathbb{Q})\to\mathscr{G}(\mathbb{Q})$ is not surjective. Let $\mathscr{F}$ be the constant sheaf with value $\mathbb{Z}$ on $\mathbb{Q}$, that is, the sheafification of the constant presheaf $\mathscr{F}_0:U\longmapsto\mathbb{Z}$. Let $\mathscr{G}$ be the sheaf sending an open subset U of $\mathbb{Q}$ to $\mathbb{Z}^{U\cap\{0,1\}}$ (with the obvious restriction maps), with the (usual) convention that $\mathbb{Z}^{\varnothing}=0$. For each open sbuset U of $\mathbb{Q}$, we have the diagonal map $\mathscr{F}_0(U)\to\mathscr{G}(U)$. This gives a morphism of presheaves $\mathscr{F}_0\to\mathscr{G}$, so we get a morphism of sheaves $\mathscr{F}\to\mathscr{G}$. This morphism is surjective in Sh_a because, if U is an open subset of $\mathbb{Q}$ such that $\{0,1\}\subset U$ and $s\in\mathscr{G}(U)$, then we can find an admissible open cover (U_0,U_1) of U such that $U_i\cap\{0,1\}=\{i\}$ for $i=0,1$, and then $s_{|U_i}\in\mathrm{Im}(\mathscr{F}(U_i)\to\mathscr{G}(U_i))=\mathscr{G}(U_i)$ for $i=0,1$. However, the morphism $\mathscr{F}(\mathbb{Q})=\mathbb{Z}\to\mathscr{G}(\mathbb{Q})=\mathbb{Z}^2$ is the diagonal morphism, which is not surjective. (To calculate $\mathscr{F}(\mathbb{Q})$, it is easiest to use question (i); the sheaf $\mathscr{F}$ is then identitied to the constant sheaf with values $\mathbb{Z}$ on $\mathbb{R}$, and its global sections are $\mathbb{Z}$ because $\mathbb{R}$ is connected.)

It remains to show that the sheafification functor is exact. By the construction of colimits in Sh_a, we already know that it is right exact, so it suffices to show that it preserves injective morphisms. Let $\mathscr{F}\to\mathscr{F}'$ is an injective morphism of presheaves. If U is an open subset of $\mathbb{Q}$, then $\check{\mathrm{H}}^0(\mathscr{U},\mathscr{F})\to\check{\mathrm{H}}^0(\mathscr{U},\mathscr{F}')$ is injective for every admissible open cover $\mathscr{U}$ of U (by definitions of these groups), so $\mathscr{F}^+(U)\to\mathscr{F}'^+(U)$ is injective because filtrant colimits are exact in **Ab**. Applying this reasoning twice, we see that $\mathscr{F}^{\mathrm{sh}}(U)\to\mathscr{F}'^{\mathrm{sh}}(U)$ is injective for every U. As kernels in Sh_a are calculated by taking kernels in PSh_a, this means that $\mathscr{F}^{\mathrm{sh}}\to\mathscr{F}'^{\mathrm{sh}}$ is injective.

(g). We can define the stalks of a presheaf (with the same formula). If $0\to\mathscr{F}_1\to\mathscr{F}_2\to\mathscr{F}_3\to 0$ is a short exact sequence of presheaves, then the complex $0\to\mathscr{F}_1(U)\to\mathscr{F}_2(U)\to\mathscr{F}_3(U)\to 0$ is exact for every open subset U of $\mathbb{Q}$, so, as stalks are defined by filtrant colimits and filtrant colimits are exact in **Ab**, the complex $0\to\mathscr{F}_{1,x}\to\mathscr{F}_{2,x}\to\mathscr{F}_{3,x}\to 0$ is exact for every $x\in\mathbb{Q}$. As the inclusion $\mathrm{Sh}_a\subset\mathrm{PSh}_a$

is left exact, we conclude that the functor $\mathrm{Sh}_a \to \mathbf{Ab}$, $\mathscr{F} \longmapsto \mathscr{F}_x$ is left exact for every $x \in \mathbb{Q}$. To show that it is exact, it is therefore enough to show that it sends surjections to surjections. Let $f : \mathscr{F} \to \mathscr{G}$ be a surjective morphism, let $x \in \mathbb{Q}$, and let $s_x \in \mathscr{G}_x$. Choose an open suset $U \ni x$ of $\mathbb{Q}$ and a section $s \in \mathscr{G}(U)$ representing s_x. As we saw in the solution of (d), the surjectivity of f means that there exists an admissible open cover $(U_i)_{i \in I}$ of U and sections $t_i \in \mathscr{F}(U_i)$ such that $f(t_i) = s_{|U_i}$ for every $i \in I$. Let $i_0 \in I$ such that $x \in U_{i_0}$, and let $t_x \in \mathscr{F}_x$ be the image of t_{i_0}. Then $f_x : \mathscr{F}_x \to \mathscr{G}_x$ sends t_x to s_x.

(h). We prove the following facts :

(A) Let U and U' be open subsets of $\mathbb{Q}$. We claim that $V(U \cap U') = V(U) \cap V(U')$. Indeed, the set $V(U) \cap V(U')$ is an open subset of $\mathbb{R}$ such that $\mathbb{Q} \cap V(U) \cap V(U') = U \cap U'$, so $V(U) \cap V(U') \subset V(U \cap U')$ by maximality of $V(U \cap U')$. Conversely, if $a, b \in \mathbb{Q}$ are such that $B := \mathbb{Q} \cap [a, b] \subset U \cap U'$, then $[a, b] \subset V(U)$ and $[a, b] \subset V(U')$, so $[a, b] \subset V(U) \cap V(U')$; this shows that $V(U \cap U') \subset V(U) \cap V(U')$.

(B) Let U be an open subset of $\mathbb{Q}$ and let $(U_i)_{i \in I}$ be an admissible open cover of U. We claim that $V(U) = \bigcup_{i \in I} V(U_i)$. Indeed, $V' := \bigcup_{i \in I} V(U_i)$ is an open subset of $\mathbb{R}$ such that $\mathbb{Q} \cap V' = U$, so $V' \subset V(U)$. Conversely, let $a, b \in \mathbb{Q}$ such that $B := \mathbb{Q} \cap [a, b] \subset U$; by the admissibility conditions, there exists a finite subset J of I and rational closed intervals $B_j = \mathbb{Q} \cap [a_j, b_j] \subset U_j$, for $j \in J$, such that $B \subset \bigcup_{j \in J} B_j$. This implies that $[a, b] = \bigcup_{j \in J} [a_j, b_j]$; moreover, for every $j \in J$, the fact that $B_j \subset U_j$ implies that $[a_j, b_j] \subset V(U_j)$; so we finally get that $[a, b] \subset V'$, as desired.

(C) Let U be an open subset of $\mathbb{Q}$, and let $(V_i)_{i \in I}$ be an open cover of $V(U)$. We claim that, after replacing $(V_i)_{i \in I}$ by a refinement, the open cover $(U \cap V_i)_{i \in I}$ of U is admissible; also, if all the V_i are open intervals, then no refinement is necessary. Indeed, after replacing $(V_i)_{i \in I}$ by a refinement, we can assume that all the V_i are open intervals in $\mathbb{R}$. Let $B = \mathbb{Q} \cap [a, b]$ be a closed rational interval contained in U. Then $[a, b] \subset V(U)$. As $[a, b]$ is compact, there exists a finite subset J of I such that $[a, b] \subset \bigcup_{j \in J} V_j$. By the shrinking lemma, there exist closed intervals with rational end points $[a_j, b_j] \subset V_j$ such that $[a, b] \subset \bigcup_{j \in J}]a_j, b_j[$. If $B_j = \mathbb{Q} \cap [a_j, b_j]$ for every $j \in J$, we have $B_j \subset U \cap V_j$ and $B \subset \bigcup_{j \in J} B_j$. So the open cover $(U \cap V_i)_{i \in I}$ of U is admissible.

(D) If A is an open inteval of $\mathbb{R}$, then $V(A \cap Q) = A$. Indeed, it is clear that $A \subset V(A \cap \mathbb{Q})$. Conversely, write $A =]x, y[$, and let $a, b \in \mathbb{Q}$ such that $\mathbb{Q} \cap [a, b] \subset A \cap Q$; then $x < a$ and $b < y$, so $[a, b] \subset A$. Hence $A \supset V(A \cap \mathbb{Q})$.

Now let $\mathscr{F}$ be a sheaf on $\mathbb{R}$ for the usual topology, and let $\Phi(\mathscr{F})$ be the presheaf $U \longmapsto \mathscr{F}(V(U))$ on $\mathbb{Q}$. For an admissible open cover $(U_i)_{i \in I}$ of an open subset U of $\mathbb{Q}$, the sequence

$$0 \to \Phi(\mathscr{F})(U) \to \prod_{i \in I} \Phi(\mathscr{F})(U_i) \to \prod_{i,j \in I} \Phi(\mathscr{F})(U_i \cap U_j)$$

(where the first map sends $s \in \Phi(\mathscr{F})(U)$ to $(s_{|U_i})_{i \in I}$ and the second map sends $(s_i) \in \prod_{i \in I} \Phi(\mathscr{F})(U_i)$ to the family $(s_{i|U_i \cap U_j} - s_{j|U_i \cap U_j})_{i,j \in I})$ is exact, because it is equal to the sequence

$$0 \to \mathscr{F}(V(U)) \to \prod_{i \in I} \mathscr{F}(V(U_i)) \to \prod_{i,j \in I} V(U_i) \cap V(U_j)$$

by (A), and because $(V(U_i))_{i \in I}$ is an open cover of $V(U)$ by (B). So $\Phi(\mathscr{F})$ is a sheaf for the admissible topology on $\mathbb{Q}$.

The functor $\Phi : \mathrm{Sh} \to \mathrm{Sh}_a$ is clearly additive and left exact. We show that it is faithful. Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of Sh such that $\Phi(f) = 0$. Then, for every open interval A of $\mathbb{R}$, we have $\mathscr{F}(A) = \Phi(\mathscr{F})(A \cap \mathbb{Q})$ and $\mathscr{G}(A) = \Phi(\mathscr{G})(A \cap \mathbb{Q})$ by (D), so the morphism $f(A) : \mathscr{F}(A) \to \mathscr{G}(A)$ is zero. As open intervals form a basis of the topology of $\mathbb{R}$, this implies that $f = 0$.

We show that Φ is full. Let $\mathscr{F}$, $\mathscr{G}$ be sheaves on $\mathbb{R}$, and let $g : \Phi(\mathscr{F}) \to \Phi(\mathscr{G})$ be a morphism of sheaves on $\mathbb{Q}$. For every open interval A of $\mathbb{R}$, we define $f(A) : \mathscr{F}(A) = \Phi(\mathscr{F})(A \cap \mathbb{Q}) \to \mathscr{G}(A) = \Phi(\mathscr{G})(A \cap \mathbb{Q})$ to be $g(A \cap \mathbb{Q})$ (we are using (D) again). If $A \subset A'$ are open intervals of $\mathbb{R}$, the diagram

$$\begin{tikzcd} \mathscr{F}(A') \arrow[r, "F(A')"] \arrow[d] & \mathscr{G}(A') \arrow[d] \\ \mathscr{F}(A) \arrow[r, "f(A)"'] & \mathscr{G}(A) \end{tikzcd}$$

(where the vertical arrows are restriction maps) is commutative because g is a morphism of presheaves. As open intervals form a basis of the topology of $\mathbb{R}$, there is a unique morphism of sheaves $f : \mathscr{F} \to \mathscr{G}$ that is equal to $f(A)$ on sections over any open interval A. It is clear that $\Phi(f) = g$.

We show that the essential image of Φ is Sh_a. Let $\mathscr{F}_0$ be a sheaf on $\mathbb{Q}$ for the admissible topology. We want to define a sheaf $\mathscr{F}$ on $\mathbb{R}$ such that $\Phi(\mathscr{F}) \simeq \mathscr{F}_0$. As open intervals form a base of the topology of $\mathbb{R}$, it suffices to define $\mathscr{F}$ on open intervals (and to check the sheaf condition for covers of an open interval by open intervals). If A is an open interval of $\mathbb{R}$, we set $\mathscr{F}(A) = \mathscr{F}_0(A \cap \mathbb{Q})$; by (D), we then have $\mathscr{F}_0(A \cap \mathbb{Q}) = \mathscr{F}(V(A \cap \mathbb{Q}))$. The sheaf condition for $\mathscr{F}$ follows from the sheaf condition from $\mathscr{F}_0$ and from (C), and the fact that $\Phi(\mathscr{F}) = \mathscr{F}_0$ is obvious.

Finally, we have shown that Φ is an equivalence of categories from Sh to Sh_a. In particular, it commutes with all limits and colimits that exist in these categories, so it is exact.

(i). Let $\mathscr{F}$ be the skryscaper sheaf on $\mathbb{R}$ supported at $\sqrt{2}$ and with value $\mathbb{Z}$. In other words, if V is an open subset of $\mathbb{R}$, we have $\mathscr{F}(V) = 0$ if $\sqrt{2} \notin V$ and $\mathscr{F}(V) = \mathbb{Z}$ if $\sqrt{2} \in V$; the restriction morphisms are either 0 or $\mathrm{id}_{\mathbb{Z}}$. Let $\mathscr{F}_0 = \Phi(\mathscr{F})$; then $\mathscr{F}_0$ is the sheaf on $\mathbb{Q}$ given by $\mathscr{F}(U) = 0$ if $\sqrt{2} \notin V(U)$, and $\mathscr{F}(U) = \mathbb{Z}$ if $\sqrt{2} \in V(U)$. If $x \in \mathbb{Q}$, then there exists an open neighborhood U of x in $\mathbb{Q}$ such that $\sqrt{2} \notin V(U)$ (for example an open rational interval), so $\mathscr{F}_x = 0$.

□

6. Canonical topology on an abelian category Let $\mathscr{A}$ be an abelian category. Let $\mathrm{PSh} = \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Ab})$ be the category of presheaves of abelian groups on $\mathscr{A}$. We say that a presheaf $F : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ is a *sheaf* (in the canonical topology) if, for every epimorphism $f : X \to Y$ in $\mathscr{A}$, the following sequence of abelian groups is exact :

$$0 \longrightarrow F(Y) \xrightarrow{F(g)} F(X) \xrightarrow{F(p_1)-F(p_2)} F(X \times_Y X)\ ,$$

where $p_1, p_2 : X \times_Y X \to X$ are the two projections. We denote by Sh the full subcategory of PSh whose objects are the sheaves.[4]

[4] Again, we could define sheaves of sets.

(a). (2 points) If $f : X \to Y$ is an epimorphism in $\mathscr{A}$, show that it is the cokernel of the morphism $p_1 - p_2 : X \times_Y X \to X$, where $p_1, p_2 : X \times_Y X \to X$ are the two projections as before.

(b). (2 points) Let $f : X \to Y$ be an epimorphism in $\mathscr{A}$ and $g : Z \to Y$ be a morphism. Consider the second projection $p_Z : X \times_Y Z \to Z$. Show that p_Z is an epimorphism.

(c). (1 point) Show that every representable presheaf on $\mathscr{A}$ is a sheaf.

(d). (3 points) Show that the inclusion functor Sh $\to$ PSh has a left adjoint $F \longmapsto F^{sh}$. (The sheafification functor.)

(e). (4 points) Show that Sh is an abelian category.

(f). (3 points) Show that the inclusion Sh $\to$ PSh is left exact but not exact, and that the sheafification functor PSh $\to$ Sh is exact.

Solution.

(a). We have a cartesian square

$$\begin{tikzcd}
X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"'] \arrow[ddr, "g"] & \\
X \arrow[r, "f"] \arrow[drr, "g"'] & Y \arrow[dr, dashed] & \\
& & Z
\end{tikzcd}$$

As f is surjective, Proposition II.2.1.15 of the notes implies that this square is also cocartesian, that is, Y is the coproduct $X \sqcup_{X \times_Y X} X$. We now show that $f = \mathrm{Coker}(p_1, p_2)$. We have $f \circ p_1 = f \circ p_2$ by definition of the fiber product. Let $g : X \to Z$ be a morphism such that $g \circ p_1 = g \circ p_2$. By the universal property of the coproduct, there exists a unique morphism $h : Y \to Z$ such that $g = h \circ f$. This is also the universal property of $\mathrm{Coker}(p_1, p_2)$.

(b). This is Corollary II.2.1.16(ii) of the notes.

(c). If $f : X \to Y$ is surjective, then, by (a), the sequence $X \times_Y X \overset{p_1 - p_2}{\to} X \overset{f}{\to} Y \to 0$ is exact. So every left exact functor $\mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ is a sheaf, and in particular every representable functor.

In fact, every sheaf $\mathscr{F} : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ that is an additive functor is automatically a left exact functor $\mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$. Indeed, let $0 \to Z \overset{g}{\to} X \overset{f}{\to} Y \to 0$ be an exact sequence. As the sequence $X \times_Y X \overset{p_1 - p_2}{\to} X \overset{f}{\to} Y \to 0$ is exact, we have $Z \overset{\sim}{\to} \mathrm{Im}(g) = \mathrm{Ker}(f) = \mathrm{Im}(p_1 - p_2)$, so there exists a unique morphism $h : X \times_Y X \to Z$ such that $g \circ h = p_1 - p_2$. Applying $\mathscr{F}$, we get a commutative diagram where the top row is exact

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}(Y) \arrow[r, "\mathscr{F}(f)"] & \mathscr{F}(X) \arrow[r, "\mathscr{F}(p_1 - p_2)"] \arrow[dr, "\mathscr{F}(g)"'] & \mathscr{F}(X \times_Y X) \\
& & & \mathscr{F}(Z) \arrow[u, "\mathscr{F}(h)"']
\end{tikzcd}$$

We have $\mathscr{F}(g) \circ \mathscr{F}(f) = 0$ because $f \circ = 0$, so $\mathrm{Ker}\,\mathscr{F}(g) \supset \mathrm{Im}\,\mathscr{F}(f)$. On the other hand, $\mathrm{Im}\,\mathscr{F}(f) = \mathrm{Ker}\,\mathscr{F}(p_1 - p_2)$ because $\mathscr{F}$ is a sheaf, and $\mathrm{Ker}\,\mathscr{F}(g) \subset \mathrm{Ker}\,\mathscr{F}(p_1 - p_2)$ because $\mathscr{F}(p_1 - p_2) = \mathscr{F}(h) \circ \mathscr{F}(g)$. So $\mathrm{Ker}\,\mathscr{F}(g) = \mathrm{Im}\,\mathscr{F}(f)$, and the sequence $\mathscr{F}(Z) \to \mathscr{F}(X) \to \mathscr{F}(Y) \to 0$ is exact.

(d). As in problem 5, it suffices to construct a functor PSh $\to$ Sh, $\mathscr{F} \longmapsto \mathscr{F}^{\mathrm{sh}}$ and a morphism of functors $\iota(\mathscr{F}) : \mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$ such that $\iota(\mathscr{F})$ is an isomorphism for $\mathscr{F}$ a sheaf and that, if $f : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves and $\mathscr{G}$ is a sheaf, then there exists a unique morphism of sheaves $f' : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$ such that the following diagram commutes

$$\begin{tikzcd} \mathscr{F} \arrow[r, "f"] \arrow[d, "\iota(\mathscr{F})"'] & \mathscr{G} \arrow[d, "\iota(\mathscr{G})"] \\ \mathscr{F}^{\mathrm{sh}} \arrow[ur, "f'"'] \arrow[r, "f^{\mathrm{sh}}"'] & \mathscr{G}^{\mathrm{sh}} \end{tikzcd}$$

The rest of the proof is the same as is question 5(c).

The construction of the sheafification functor follows the same lines as the construction of Section III.1 of the notes, except that we have to use the correct notion of cover. Let X be an object of $\mathscr{A}$. The category of covering families of X is the category $\mathscr{I}_X$ whose objects are surjective morphisms $Y \to X$ and whose morphisms are commutative diagrams $\begin{tikzcd} Y \arrow[r] \arrow[d] & Y' \arrow[dl] \\ X & \end{tikzcd}$. Let $\mathscr{F}$ be a presheaf on $\mathscr{A}$. If X is an object of $\mathscr{A}$ and $f : Y \to X$ is a surjective morphism, we set

$$\check{\mathrm{H}}^0(Y \to X, \mathscr{F}) = \mathrm{Ker}(\mathscr{F}(p_1 - p_2) : \mathscr{F}(Y) \to \mathscr{F}(Y \times_X Y)),$$

where $p_1, p_2 : Y \times_X Y \to Y$ are as before the two projections. As $\mathscr{F}(p_1 - p_2) \circ \mathscr{F}(f) = 0$, we have a morphism $\mathscr{F}(X) \to \check{\mathrm{H}}^0(Y \to X, \mathscr{F})$ induced by $\mathscr{F}(f)$. Let $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ be two surjective morphisms, and suppose that there exists $g : Y_2 \to Y_1$ such that $f_1 \circ g = f_2$ (that is, g is a morphism in $\mathscr{I}_X$). If $q_1, q_2 : Y_2 \times_X Y_2 \to Y_2$ are the two projections, then $f_1 \circ g \circ q_1 = f_2 \circ q_1 = f_2 \circ q_2 = f_1 \circ g \circ q_2$, so $(g \circ q_1, g \circ q_2)$ defines a morphism $g' : Y_2 \times_X Y_2 \to Y_1 \times_X Y_1$ such that the compositions of g' with the projections $p_1, p_2 : Y_1 \times_X Y_1 \to Y_1$ are equal to $g \circ q_1$ and $g \circ q_2$. In particular, we have $g \circ (q_1 - q_2) = (p_1 - p_2) \circ g'$, hence $\mathscr{F}(q_1 - a_2) \circ \mathscr{F}(g) = \mathscr{F}(g') \circ \mathscr{F}(p_1 - p_2)$. So morphism $\mathscr{F}(g) : \mathscr{F}(Y_1) \to \mathscr{F}(Y_2)$ sends $\check{\mathrm{H}}^0(Y_1 \to X, \mathscr{F})$ to $\check{\mathrm{H}}^0(Y_2 \to X, \mathscr{F})$. As in Section III.1 of the notes, we can show that this morphism does not depend on g. Indeed, let $h : Y_2 \to Y_1$ be another morphism such that $f_1 \circ h = f_2$. Then (g, h) defines a morphism $k : Y_2 \to Y_1 \times_X Y_1$ such that $p_1 \circ k = g$ and $p_2 \circ k = h$. If $s \in \check{\mathrm{H}}^0(Y_1 \to X, \mathscr{F})$, then

$$\mathscr{F}(g)(s) = \mathscr{F}(k)(\mathscr{F}(p_1)(s)) = \mathscr{F}(k)(\mathscr{F}(p_2)(s)) = \mathscr{F}(h)(s),$$

because $\mathscr{F}(p_1)(s) = \mathscr{F}(p_2)(s)$ by definition of $\check{\mathrm{H}}^0(Y_1 \to X, \mathscr{F})$.

In summary, we have made $\check{\mathrm{H}}^0(\cdot, \mathscr{F})$ into a functor $(\mathscr{I}_X^0)^{\mathrm{op}} \to \mathbf{Ab}$, where $\mathscr{I}_X^0$ is the category that we get from $\mathscr{I}_X$ by contracting all the nonempty Hom sets to singletons. We denote by $\mathscr{F}^+(X)$ the colimit of this functor. We have a canonical morphism $\mathscr{F}(X) \to \mathscr{F}^+(X)$, given by the morphisms $\mathscr{F}(X) \to \check{\mathrm{H}}^0(Y \to X, \mathscr{F})$.

If X' is another object of $\mathscr{A}$ and $u : X' \to X$ is a morphism, then we get a functor $\mathscr{I}_X \to \mathscr{I}_{X'}$ by sending a surjection $Y \to X$ to $Y \times_X X' \to X'$ (which is a surjection by (b)). This allows us to define a morphism $\mathscr{F}^+(X) \to \mathscr{F}(X')$ as in the notes, and so $\mathscr{F}^+$ is a presheaf. It is easy to see that the morphisms $\mathscr{F}(X) \to \mathscr{F}^+(X)$ define a morphism of presheaves $\iota_0(\mathscr{F}) : \mathscr{F} \to \mathscr{F}^+$. It is also easy to see that $\mathscr{F} \longmapsto \mathscr{F}^+$ is a functor, and that the $\iota_0(\mathscr{F})$ define a morphism of functors.

We set $\mathscr{F}^{\mathrm{sh}} = \mathscr{F}^{++}$, with the morphism $\iota(F) : \mathscr{F} \to \mathscr{F}^{++}$ given by $\iota(F) = \iota_0(\mathscr{F}^+) \circ \iota_0(\mathscr{F}^+)$. If $\mathscr{F}$ is a sheaf, then $\mathscr{F}(X) \xrightarrow{\sim} \check{\mathrm{H}}^0(Y \to X, \mathscr{F})$ for every surjective morphism $Y \to X$, so $\iota(\mathscr{F})$ is an isomorphism.

The proof of Proposition III.1.10 of the notes now goes through, provided that we can prove that $(\mathscr{I}_X^0)^{\mathrm{op}}$ is filtrant. If $f : Y \to X$ and $f' : Y' \to X$ are surjective maps, then the two projections $p_1 : Y \times_X Y' \to Y$ and $p_2 : Y \times_X Y' \to Y'$ are surjective by (b), and $f \circ p_1 = f' \circ p_2$, so we have morphisms $(Y \to X) \to (Y \times_X Y' \to X)$ and $(Y' \to X) \to (Y \times_X Y' \to X)$ in $(\mathscr{I}_X^0)^{\mathrm{op}}$, which suffices because the Homs of $(\mathscr{I}_X^0)^{\mathrm{op}}$ are empty or singletons.

(e) and (f) The proof of questions (d) and (e) of problem 5 applies (provided we replace admissible open covers by surjective morphisms), except for the counterexample showing that the inclusion $\mathrm{Sh} \subset \mathrm{PSh}$ does not preserve surjections. Anticipating a bit on problem 3 of problem set 4, we can make the following counterexample : Let $A \to B$ be a surjective morphism in $\mathscr{A}$. Then the induced morphism $\mathrm{Hom}_{\mathscr{A}}(\cdot, A) \to \mathrm{Hom}_{\mathscr{A}}(\cdot, B)$ is surjective in Sh. (See the solution of that problem.) But it is not true in general that $\mathrm{Hom}_{\mathscr{A}}(C, A) \to \mathrm{Hom}_{\mathscr{A}}(C, B)$ is surjective for every object C of $\mathscr{A}$ and every choice of surjective $A \to B$, unless $\mathscr{A}$ is a semisimple abelian category. Indeed, if $\mathscr{A}$ is not semisimple, then we can find an exact sequence $0 \to A' \to B \to A \to 0$ that is not split, and than $\mathrm{id}_A \in \mathrm{Hom}_{\mathscr{A}}(A, A)$ does not come from an element of $\mathrm{Hom}_{\mathscr{A}}(A, B)$.

□