MAT 540 : Problem Set 11

Due Thursday, December 19

1 The model structure on complexes (continued)

(a). Let R be a ring, let $\mathscr{C} = \mathscr{C}^*({}_R\mathbf{Mod})$ with $* \in \{-, \varnothing\}$ and consider the sets of morphisms

$$I = \{S^n \to D^{n-1},\ n \in \mathbb{Z}\}$$

and

$$J = \{0 \to D_n,\ n \in \mathbb{Z}\}$$

in $\mathscr{C}$. We use the notation S^n and D^n of problem 2 of problem set 10, and we denote by W the set of quasi-isomorphisms in $\mathscr{C}$.

(i) (1 point) Show that S^n is small in $\mathscr{C}$ for every $n \in \mathbb{Z}$.

(ii) (1 point) Show that $I - \mathrm{inj}$ is the set of surjective quasi-isomorphisms.

(iii) (1 point) Show that $J - \mathrm{inj}$ is the set of surjective morphisms.

(iv) (1 point) Show that I and J are the sets of generating cofibrations and generating acyclic cofibrations of the model structure of problem 2 of problem set 10 on $\mathscr{C}$.

(v) (3 points) Show that, if $f \in I - \mathrm{cell}$, then f is injective and $\mathrm{Coker}(f_n)$ is a free R-module for every $n \in \mathbb{Z}$.

(b). Let $\mathscr{C}' = \mathscr{C}^{\leq 0}({}_R\mathbf{Mod})$, and consider the following sets of morphisms in $\mathscr{C}'$:

$$I' = \{S^n \to D^{n-1},\ n \leq 0\} \cup \{0 \to S^0\}$$

and

$$J' = \{0 \to D_n,\ n \leq -1\}.$$

We still denote by W the set of quasi-isomorphisms in $\mathscr{C}'$.

(i) (1 points) Show that $J' - \mathrm{inj}$ is the set of morphisms f such that f_n is surjective for $n \leq -1$.

(ii) (2 points) Show that $I' - \mathrm{inj} = W \cap J' - \mathrm{inj}$.

(iii) (2 points) Show that I' and J' satisfy the conditions of Theorem VI.5.4.5 of the notes.

2 The model structure on simplicial R-modules

Remember the simplicial category Δ and the category of simplicial sets **sSet** from problem 9 of problem set 1 and problem 2 of problem set 2. As in problem 9 of problem set 1 we define morphisms $\delta_0, \delta_1, \dots, \delta_n : [n-1] \to [n]$ in Δ by the condition that δ_i is the unique increasing map $[n-1] \to [n]$ such that $i \notin \mathrm{Im}(\delta_i)$.

(a). Let $\mathscr{A}$ be an additive category. Let $X_\bullet$ be an object of $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$. For $n \in \mathbb{N}$ and $i \in \{0, 1, \dots, n\}$, we denote the morphism $X_\bullet(\delta_i^n)$ by $d_i^n : X_n \to X_{n-1}$. The *unnormalized chain complex* of $X_\bullet$ is the complex $C(X_\bullet)$ in $\mathscr{C}^{\leq 0}(\mathscr{A})$ given by: for every $n \geq 0$,

$$C(X_\bullet)^{-n} = X_n$$

and

$$d_{C(X_\bullet)}^{-n} = \sum_{i=0}^{n} (-1)^i d_i^n.$$

(i) (1 point) Show that $C(X_\bullet)$ is a complex.

From now on, we assume that $\mathscr{A}$ is also pseudo-abelian. We use the notation of problem 1 of problem set 10. In particular, we denote by $\mathscr{C}$ the category $\mathrm{kar}((\mathbb{Z}[\Delta])^\oplus)$, identified to full subcategory of $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Ab})$, and we extend object of $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$ to additive functors from $\mathscr{C}$ to $\mathbf{Ab}$. Let $X_\bullet$ be an object of $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$.

(ii) (1 point) For every $r \in \mathbb{N}$ and every $n \geq 0$, we consider the following direct summand of $C(X_\bullet)^{-n}$:

$$C_{\leq r}(X_\bullet)^{-n} = \begin{cases} X_\bullet(\mathbb{Z}^{(\Delta_n^{\leq r})}) & \text{if } r \leq n-1 \\ X_\bullet(\mathbb{Z}^{(\Delta_n^{\leq n-1})}) & \text{otherwise.} \end{cases}$$

(With the convention that $\Delta_0^{\leq -1} = \varnothing$.)

Show that this defines a subcomplex $C_{\leq r}(X^\bullet)$ of $C(X^\bullet)$.

(iii) (2 points) Let $i_r : C_{\leq r}(X^\bullet) \to C_{\leq r+1}(X^\bullet)$ be the obvious inclusion. Show that there exists a morphism $f_r : C_{\leq r+1}(X^\bullet) \to C_{\leq r}(X^\bullet)$ such that $f_r \circ i_r$ is the identity morphism.

(iv) (2 points, probably hard) Show that i_r is a homotopy equivalence.

(v) (2 points) If $\mathscr{A}$ is an abelian category, show that the inclusion $N(X^\bullet) \subset C(X^\bullet)$ is a quasi-isomorphism.

(b). (2 points) Let R be a ring, and let $\mathscr{C} = \mathrm{Func}(\Delta^{\mathrm{op}}, {}_R\mathbf{Mod})$. Show that there is a model structure on $\mathscr{C}$ for which the weak equivalences are the morphisms $f : X_\bullet \to Y_\bullet$ such that $C(f) : C(X_\bullet) \to C(Y_\bullet)$ is a quasi-isomorphism, and the cofibrations are the morphisms $f : X_\bullet \to Y_\bullet$ such that $N(f)^n : N(X_\bullet)^{-n} \to N(X_\bullet)^{-n}$ is injective with projective cokernel for every $n \geq 0$.

3 A Quillen adjunction

Let k be a commutative ring, let G be a group, and let $R = k[G]$. Consider the categories $\mathscr{C} = \mathscr{C}^-({}_R\mathbf{Mod})$ and $\mathscr{D} = \mathscr{C}^-({}_k\mathbf{Mod})$ with the projective model structures, and the functor $F : \mathscr{C} \to \mathscr{D}$ sending a complex X to the complex $\mathrm{H}_0(G, X)$.

(a). (1 point) Show that F has a rigth adjoint G.

(b). (2 points) Show that (F, G) is a Quillen adjunction.

4 Kähler differentials

Let R be a commutative ring. If B is a commutative R-algebra and M is a B-module, a R-linear *derivation* from B to M is a R-linear map $d : B \to M$ such that, for all $b, b' \in B$, we have

$$d(bb') = bd(b') + b'd(b).$$

We denote by $\mathrm{Der}_R(B, M)$ the abelian group of derivations from B to M.

We fix a commutative R-algebra B.

(a). (2 points) Show that the functor ${}_B\mathbf{Mod} \to \mathbf{Ab}$, $M \longmapsto \mathrm{Hom}_R(B, M)$ is representable and give a pair representing it.

(b). (2 points) Show that the functor ${}_B\mathbf{Mod} \to \mathbf{Ab}$, $M \longmapsto \mathrm{Der}_R(B, M)$ is representable by a pair $(\Omega^1_{B/R}, d_{\mathrm{univ}})$, where $\Omega^1_{B/R}$ is a B-module (called the module of *Kähler differentials*) and $d_{\mathrm{univ}} : B \to \Omega^1_{B/R}$ is a R-linear derivation. (Hint: The functor $M \longmapsto \mathrm{Der}_R(B, M)$ is a subfunctor of $M \longmapsto \mathrm{Hom}_R(B, M)$, so $\Omega^1_{B/R}$ should be a quotient of the B-module representing the functor of (a).)

(c). (2 points) If B is the polynomial ring $R[X_i, i \in I]$ (where I is a set), show that $\Omega^1_{B/R}$ is a free B-module on the set I.

5 Abelianization and Kähler differentials

Let $\mathscr{C}$ is a category that has finite products, and denote a final object of $\mathscr{C}$ by $*$. An *abelian group in* $\mathscr{C}$ is a triple (X, m, e), where X is an object of $\mathscr{C}$, and $m : X \times X \to X$ and $e : * \to X$ are morphisms such that, for every object Y of $\mathscr{C}$, the morphisms

$$m_* : \mathrm{Hom}_{\mathscr{C}}(Y, X \times X) \simeq \mathrm{Hom}_{\mathscr{C}}(Y, X) \times \mathrm{Hom}_{\mathscr{C}}(Y, X) \to \mathrm{Hom}_{\mathscr{C}}(Y, X)$$

and

$$e_* : \mathrm{Hom}_{\mathscr{C}}(Y, *) = * \to \mathrm{Hom}_{\mathscr{C}}(Y, X)$$

(where we also denote by $*$ a final object of $\mathbf{Set}$) define the structure of an abelian group on the set $\mathrm{Hom}_{\mathscr{C}}(Y, X)$. The morphism m is called the *multiplication morphism* of the group, and the morphism e is called the *unit*.

If $G = (X, m, e)$ and $G' = (X', m', e')$ are two abelian groups in $\mathscr{C}$, a *morphism from G to G'* is a morphism $f : X \to X'$ in $\mathscr{C}$ such that $f \circ e = e'$ and that the following diagram commutes:

$$\begin{tikzcd} X \times X \arrow[r, "m"] \arrow[d, "f \times f"'] & X \arrow[d, "f"] \\ X' \times X' \arrow[r, "m'"'] & X' \end{tikzcd}$$

We denote by $\mathscr{C}_{\mathrm{ab}}$ the category of abelian groups in $\mathscr{C}$.

An *abelianization functor* on $\mathscr{C}$ is a left adjoint to the forgetful functor $\mathscr{C}_{\mathrm{ab}} \to \mathscr{C}$.

(a). (4 points) Show that $\mathbf{Set}_{\mathrm{ab}} \simeq \mathbf{Ab}$, that $\mathbf{Grp}_{\mathrm{ab}} \simeq \mathbf{Ab}$, that $\mathbf{Top}_{\mathrm{ab}}$ is equivalent to the category of commutative topological groups and that $\mathbf{sSet}_{\mathrm{ab}} \simeq \mathbf{sAb}$.

(b). (3 points) Show that $\mathbf{Set}$, $\mathbf{Grp}$ and $\mathbf{sSet}$ have abelianization functors, and give formulas for these functors.

Let R be a commutative ring and A be a commutative R-algebra. We denote by $\mathscr{C}$ the slice category $R-\mathbf{CAlg}/A$ (see Definition I.2.2.6 of the notes).

If M is a A-module, we define an A-algebra structure on $A\oplus M$ by taking the mutliplication given by the formula

$$(a,m)(a',m') = (aa', am' + a'm),$$

for $a, a' \in A$ and $m, m' \in M$. We have a morphism of A-algebras $A \oplus M \to A$ sending (a,m) to a. This gives a functor ${}_A\mathbf{Mod} \to \mathscr{C}$.

If $B \to A$ is an object of $\mathscr{C}$ and M is a A-module, we denote by $\mathrm{Der}_R(B,M)$ the abelian group of R-linear derivations from B to M (where M is seen as a B-module using the morphism $B \to A$).

(c). (2 points) Show that we have an isomorphism of functors $\mathscr{C} \times {}_A\mathbf{Mod} \to \mathbf{Ab}$:

$$\mathrm{Hom}_{\mathscr{C}}(B, A\oplus M) \simeq \mathrm{Der}_R(B,M).$$

(d). (3 points) If M is an A-module, show that $A \oplus M$ is an abelian group in $\mathscr{C}$, and give formulas for its multiplication and unit.

(e). (4 points) Show that the functor ${}_A\mathbf{Mod} \to \mathscr{C}$ sending M to $A \oplus M$ factors through the subcategory $\mathscr{C}_{\mathrm{ab}}$, and that it induces an equivalence of categories ${}_A\mathbf{Mod} \to \mathscr{C}_{\mathrm{ab}}$.

(f). (1 point) Show that the functor $\mathscr{C} \to {}_A\mathbf{Mod}$ sending $B \to A$ to $A \otimes_B \Omega^1_{B/R}$ is an abelianization functor for $\mathscr{C}$.