MAT 540 : Problem Set 10

Due Saturday, December 7

1 The Dold-Kan correspondence

You need to look at the results of problems 1 and 2 of problem set 3 to do this problem.

Remember the simplicial category Δ and the category of simplicial sets **sSet** from problem 9 of problem set 1 and problem 2 of problem set 2. Let $\mathscr{C} = \mathrm{kar}((\mathbb{Z}[\Delta])^{\oplus})$ (see problems 1 and 2 of problem set 3), so that $\mathscr{C}$ is an additive pseudo-abelian category.

The category $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Ab})$ is called the category of *simplicial abelian groups* and denoted by **sAb**; it is an abelian category, where kernel, cokernels and images are calculated in the obvious way (that is, $\mathrm{Ker}(X \to Y) = (\mathrm{Ker}(X_n \to Y_n))_{n\in\mathbb{N}}$ etc).

By the Yoneda lemma, the functor $h_{\mathscr{C}} : \mathscr{C} \to \mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab})$ is fully faithful; by problems 1 and 2 of problem set 3, we have an equivalence $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab}) \simeq \mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Ab}) = \mathbf{sAb}$. So we get a fully faithful functor $\mathscr{C} \to \mathbf{sAb}$, and we identify $\mathscr{C}$ with the essential image of this functor.

If X is a simplical set, we denote by $\mathbb{Z}^{(X)}$ the "free simplicial abelian group on X" : it is the simplicial abelian group sending $[n]$ to the free abelian group $\mathbb{Z}^{(X_n)}$ and $\alpha : [n] \to [m]$ to the unique group morphism from $\mathbb{Z}^{(X_m)}$ to $\mathbb{Z}^{(X_n)}$ extending $\alpha^* : X_m \to X_n$. If $u : X \to Y$ is a morphism of simplicial sets, we simply write $u : \mathbb{Z}^{(X)} \to \mathbb{Z}^{(Y)}$ for the morphism of simplicial abelian groups induced by u. If $\alpha : [n] \to [m]$, we also use α to denote the morphism $\Delta_n \to \Delta_m$ that is the image of α by the Yoneda embedding $h_\Delta : \Delta \to \mathbf{sSet}$.

(a). (1 point) For every $n \in \mathbb{N}$, show that the simplicial abelian group $\mathbb{Z}^{(\Delta_n)}$ is in $\mathscr{C}$. (Hint : It's the image of the object $[n]$ of Δ. Follow the identifications !)

Let $n \geq 1$. Remember from problem 9 of problem set 1 that we have defined morphisms $\delta_0, \delta_1, \dots, \delta_n : [n-1] \to [n]$ in Δ by the condition that δ_i is the unique increasing map $[n-1] \to [n]$ such that $i \notin \mathrm{Im}(\delta_i)$. According to our previous conventions, we get morphisms $\delta_i : \Delta_{n-1} \to \Delta_n$ in **sSet** and $\delta_i : \mathbb{Z}^{(\Delta_{n-1})} \to \mathbb{Z}^{(\Delta_n)}$ in **sAb**. Remember also that, for $k \in [n]$, the *horn* Λ^n_k is the union of the images of the δ_i, for $i \in [n] - \{k\}$.

(b). (1 point) Show that $\mathbb{Z}^{(\Lambda^n_k)} = \sum_{i\in[n]-\{k\}} \mathrm{Im}(\delta_i)$, where the sum is by definition the image of the canonical morphism $\bigoplus_{i\in[n]-\{k\}} \mathrm{Im}(\delta_i) \to \mathbb{Z}^{(\Delta_n)}$ and we have identified $\mathbb{Z}^{(\Lambda^n_k)}$ to its image in $\mathbb{Z}^{(\Delta_n)}$.

If $f : [n] \to X$ is a map from $[n]$ to a set X, we also use the notation $(f(0) \to f(1) \to \dots \to f(n))$ to represent f. Let $n \in \mathbb{N}$, and let S_n be the set of sequences $(a_1, \dots, a_n) \in [n]$ such that $a_i \in \{i-1, i\}$ for every $i \in \{1, \dots, n\}$; if $\underline{a} = (a_1, \dots, a_n)$, we write $f_{\underline{a}} = (0 \to a_1 \to \dots \to a_n) \in \mathrm{Hom}_{\mathbf{Set}}([n], [n])$ and $\varepsilon(\underline{a}) = (-1)^{\mathrm{card}(\{i | a_i \neq i\})}$.

(c). (1 point) For every $\underline{a} \in S_n$, show that $f_{\underline{a}} \in \mathrm{Hom}_\Delta([n], [n])$.

(d). (2 points) Let $p_n = \sum_{\underline{a}\in S_n} \varepsilon(\underline{a}) f_{\underline{a}} \in \mathrm{End}_{\mathscr{C}}(\mathbb{Z}^{(\Delta_n)})$. Show that p_n is a projector.

(e). (3 points) Show that $\mathbb{Z}^{(\Lambda_0^n)} = \mathrm{Im}(\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n) = \mathrm{Ker}(p_n)$. In particular, $\mathbb{Z}^{(\Lambda_0^n)}$ is an object of $\mathscr{C}$.

(f). (1 point) Let $I_n = \mathrm{Im}(p_n)$. This is also an object of $\mathscr{C}$. Show that we have an isomorphism $\mathbb{Z}^{(\Delta_n)} \simeq \mathbb{Z}^{(\Lambda_0^n)} \oplus I_n$ in $\mathscr{C}$.

(g). (2 points) If X is an object of **sAb** and $f : X \to I_n$ is a surjective morphism (that is, such that f_r is surjective for every $r \geq 0$), show that there exists a morphism $g : I_n \to X$ such that $f \circ g = \mathrm{id}_{I_n}$.

For every $k \in [n]$, define a simplicial subset $\Delta_n^{\leq k}$ of Δ_n by taking $\Delta_n^{\leq k}([m])$ equal to the set of nondecreasing $\alpha : [m] \to [n]$ such that either $\mathrm{card}(\mathrm{Im}(\alpha)) \leq k$, or $\mathrm{card}(\mathrm{Im}(\alpha)) = k+1$ and $0 \in \mathrm{Im}(\alpha)$. In particular, question (h)(i) says that $\Delta_n^{\leq n-1} = \Lambda_0^n$. (On the geometric realizations, $|\Delta_n|$ is a simplex of dimension n with vertices numbered by $0, 1, \ldots, n$, and $|\Delta_n^{\leq k}|$ is the union of its faces of dimension $\leq k$ that contain the vertex 0.)

(h). (i) (1 point) For every $k \in [n]$ and every $m \in \mathbb{N}$, show that

$$\Lambda_k^n([m]) = \{\alpha : [m] \to [n] \mid \text{either } \mathrm{card}(\mathrm{Im}(\alpha)) \leq n-1, \text{ or } \mathrm{card}(\mathrm{Im}(\alpha)) = n \text{ and } k \in \mathrm{Im}(\alpha)\}.$$

(ii) (1 point) For every $m \in \mathbb{N}$, show that the set

$$\{\alpha : [m] \to [n] \mid \mathrm{Im}(\alpha) \supset [n] - \{0\}\}$$

is a basis of the $\mathbb{Z}$-module $I_n([m])$.

(iii) (1 point) For every $k \in \{1, \ldots, n\}$, show that

$$\mathbb{Z}^{(\Delta_n^{\leq k})} / \mathbb{Z}^{(\Delta_n^{\leq k-1})} \simeq I_k^{\binom{n}{k}}.$$

(iv) (1 point) For every $k \in \{1, \ldots, n\}$, show that

$$\mathbb{Z}^{(\Delta_n^{\leq k})} \simeq \mathbb{Z}^{(\Delta_n^{\leq k-1})} \oplus I_k^{\binom{n}{k}}.$$

(i). (1 point) Show that there is an isomorphism $\mathbb{Z}^{(\Delta_n)} \simeq \bigoplus_{k=0}^n I_k^{\binom{n}{k}}$ in $\mathscr{C}$.

(j). (2 points) For all $n, m \in \mathbb{N}$, show that $\mathrm{Hom}_{\mathscr{C}}(I_n, I_m)$ is a free $\mathbb{Z}$-module of finite type. We denote its rank by $a_{n,m}$.

(k). (2 points) Show that $a_{n,n} \geq 1$ and $a_{n,n+1} \geq 1$ for every $n \in \mathbb{N}$. (Hint for the second: $\delta_0 : [n] \to [n+1]$.)

(l). (2 points) Show that, for all $n, m \in \mathbb{N}$, we have

$$\binom{n+m+1}{m} = \sum_{k=0}^{n} \sum_{l=0}^{m} a_{k,l} \binom{n}{k} \binom{m}{l}.$$

(m). (2 points) Show that, for all $n, m \in \mathbb{N}$, we have

$$\binom{n+m+1}{m} = \sum_{k=0}^{m} \binom{n+1}{k} \binom{m}{k}.$$

(n). (2 points) Show that $a_{n,n} = a_{n,n+1} = 1$ for every $n \in \mathbb{N}$ and $a_{n,m} = 0$ if $m \notin \{n, n+1\}$.

(o). (2 points) Let $\mathscr{I}$ be the full subcategory of $\mathscr{C}$ whose objects are the I_n for $n \in \mathbb{N}$. If $\mathscr{A}$ is an additive category, we consider the category $\mathscr{C}^{\leq 0}(\mathscr{A})$ of complexes of objects of $\mathscr{A}$ that are concentrated in degree ≤ 0 (that is, complexes $X \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$ such that $X^n = 0$ for $n \geq 1$).

Give an equivalence of categories from $\mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$ to $\mathscr{C}^{\leq 0}(\mathscr{A})$.

(p). (2 points) Deduce an equivalence of categories from $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$ to $\mathscr{C}^{\leq 0}(\mathscr{A})$, if $\mathscr{A}$ is a pseudo-abelian additive category. This is called the *Dold-Kan equivalence*.

(q). (2 points) Suppose that $\mathscr{A}$ is an abelian category, and let $X_\bullet$ be an object of $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$. For $n \in \mathbb{N}$ and $i \in \{0, 1, \ldots, n\}$, we denote the morphism $X_\bullet(\delta_i^n)$ by $d_i^n : X_n \to X_{n-1}$. The *normalized chain complex* of $X_\bullet$ is the complex $N(X_\bullet)$ in $\mathscr{C}^{\leq 0}(\mathscr{A})$ given by: for every $n \geq 0$,

$$N(X_\bullet)^{-n} = \bigcap_{1 \leq i \leq n} \mathrm{Ker}(d_i^n)$$

and $d_{N(X_\bullet)}^{-n}$ is the restriction of d_0^n. This defines a functor $N : \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A}) \to \mathscr{C}^{\leq 0}(\mathscr{A})$. Show that this functor is isomorphic to the equivalence of categories of the previous question.

2 The model structure on complexes

Let R be a ring, and let $\mathscr{A} = {}_R\mathbf{Mod}$. [1]

We denote by W the set of quasi-isomorphisms of $\mathscr{C}(\mathscr{A})$, by Fib the set of morphisms $f : X \to Y$ in $\mathscr{C}(\mathscr{A})$ such that $f^n : X^n \to Y^n$ is surjective for every $n \in \mathbb{Z}$ and by Cof the set of morphisms of $\mathscr{C}(\mathscr{A})$ that have the left lifting property relatively to every morphism of $W \cap \mathrm{Fib}$. We say that $X \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$ is fibrant (resp. cofibrant) if the unique morphism $X \to 0$ (resp. $0 \to X$) is in Fib (resp. in Cof). The goal of this problem is to show that $(W, \mathrm{Fib}, \mathrm{Cof})$ is a model structure on $\mathscr{C}(\mathscr{A})$.

For every $M \in \mathrm{Ob}(\mathscr{A})$, let $K(M, n) = M[-n] \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$, and let $D^n(M)$ be the complex X such that $X^n = X^{n+1} = M$, $d_X^n = \mathrm{id}_M$ and $X^i = 0$ if $i \not\in \{n, n+1\}$. We also write $S^n = K(R, n)$ and $D^n = D^n(R)$. For every $M \in \mathrm{Ob}(\mathscr{A})$, the identity of M induces a morphism of complexes $K(M, n) \to D^{n-1}(M)$ (which is clearly functorial in M).

(a). (2 points) Show that the functor $D^n : {}_R\mathbf{Mod} \to \mathscr{C}(\mathscr{A})$ is left adjoint to the functor $\mathscr{C}(\mathscr{A}) \to \mathscr{A}$, $X \longmapsto X^n$, and that the functor $K(\cdot, n) : \mathscr{A} \to \mathscr{C}(\mathscr{A})$ is left adjoint to the functor Z^n.

(b). (1 point) Show that a morphism of $\mathscr{C}(\mathscr{A})$ is in Fib is and only if it has the right lifting property relatively to $0 \to D^n$ for every $n \in \mathbb{Z}$.

(c). (1 point) Show that D^n is cofibrant for every $n \in \mathbb{Z}$.

(d). (2 points) Show that S^n is cofibrant for every $n \in \mathbb{Z}$.

(e). Let $p : X \to Y$ be a morphism of $\mathscr{C}(\mathscr{A})$.

  (i) (2 points) If p is in $W \cap \mathrm{Fib}$, show that it has the right lifting property relatively to the canonical morphism $S^n = K(R, n) \to D^{n-1}$ for every $n \in \mathbb{Z}$.

  (ii) (3 points) If p has the right lifting property relatively to the canonical morphism $S^n \to D^{n-1}$ for every $n \in \mathbb{Z}$, show that it is in $W \cap \mathrm{Fib}$.

[1] We only need $\mathscr{A}$ to have all small limits and colimits and a nice enough projective generator, but we take $\mathscr{A} = {}_R\mathbf{Mod}$ to simplify the notation.

(f). (1 point) Show that the canonical morphism $S^n \to D^{n-1}$ is in Cof.

(g). Let $f : X \to Y$ be a morphism of $\mathscr{C}(\mathscr{A})$. Let $E = X \oplus \bigoplus_{n\in\mathbb{Z},\ y\in Y^n} D^n$, let $i : X \to E$ be the obvious inclusion and let $p : E \to Y$ be the morphism that is equal to f on the summand X and that, for every $n \in \mathbb{Z}$ and $y \in Y^n$, is equal on the corresponding summand D^n to the morphism $D^n \to Y$ corresponding to $y \in Y^n = \mathrm{Hom}_R(R, Y^n)$ by the adjunction of question (a). We clearly have $p \circ i = f$.

(i) (1 point) Show that i is in W.

(ii) (1 point) Show that i has the left lifting property relatively to any morphism of Fib.

(iii) (1 point) Show that p is in Fib.

(h). Let $f : X \to Y$ be a morphism of $\mathscr{C}(\mathscr{A})$. Let $X_0 = X$ and $f_0 = f$. For every $i \in \mathbb{N}$, we construct morphisms of complexes $j_i : X_i \to X_{i+1}$ and $f_{i+1} : X_{i+1} \to Y$ such that j_i is a monomorphism and in Cof and $f_{i+1} \circ j_i = f_i$ in the following way: Suppose that we already have $f_i : X_i \to Y$. Consider the set $\mathscr{D}_i$ of commutative squares

$$(D) \qquad \begin{tikzcd} S^{n_D} \arrow[r, "f_D"] \arrow[d] & X_i \arrow[d, "f_i"] \\ D^{n_D-1} \arrow[r, "g_D"'] & Y \end{tikzcd}$$

(for some $n_D \in \mathbb{Z}$). Let $j_i : X_i \to X_{i+1}$ be defined by the cocartesian square

$$\begin{tikzcd} \bigoplus_{D\in\mathscr{D}_i} S^{n_D} \arrow[r, "\sum f_D"] \arrow[d] & X_i \arrow[d, "j_i"] \\ \bigoplus_{D\in\mathscr{D}_i} D^{n_D-1} \arrow[r] & X_{i+1} \end{tikzcd}$$

The morphisms $f_i : X_i \to Y$ and $\sum g_D : \bigoplus_{D\in\mathscr{D}_i} D^{n_D-1} \to Y$ induce a morphism $f_{i+1} : X_{i+1} \to Y$, and we clearly have $f_{i+1} \circ j_i = f_i$.

Finally, let $F = \varinjlim_{i\in\mathbb{N}} X_i$ (where the transition morphisms are the j_i), let $j : X \to F$ be the morphism induced by j_0 and let $q : F \to Y$ be the morphism induced by the f_i.

(i) (1 points) Show that $q \circ j = f$.

(ii) (1 point) Show that j is a monomorphism.

(iii) (2 points) Show that j is in Cof.

(iv) (2 points) Show that q is in $W \cap \mathrm{Fib}$.

(i). (1 point) Show that every element of Cof is a monomorphism.

(j). (2 points) Show that every element of $W \cap \mathrm{Cof}$ has the left lifting property relatively to elements of Fib. (Hint: Use question (g).)

(k). (3 points) Show that $(W, \mathrm{Fib}, \mathrm{Cof})$ is a model structure on $\mathscr{C}(\mathscr{A})$.

(l). (2 points) Show that the intersections of $(W, \mathrm{Fib}, \mathrm{Cof})$ with $\mathscr{C}^-(\mathscr{A})$ also give a model structure on this category.

(m). (2 points) Let $f : A \to B$ be a morphism of $\mathscr{A}$. Show that f has the left lifting property relatively to epimorphisms of $\mathscr{A}$ if and only if it is injective with projective cokernel.

(n). (3 points) Let $i : X \to Y$ be a morphism of $\mathscr{C}^-(\mathscr{A})$. Show that i is in Cof if and only if, for every $n \in \mathbb{Z}$, the morphism i^n is injective with projective cokernel.