MAT 540 : Problem Set 1

Due Thursday, September 19

1.

(a). (2 points) In the category **Set**, show that a morphism is a monomorphism (resp. an epimorphism) if and only it is injective (resp. surjective).

(b). (2 points) Let $\mathscr{C}$ be a category and $F : \mathscr{C} \to \mathbf{Set}$ be a *faithful* functor, show that any morphism f of $\mathscr{C}$ whose such that $F(f)$ is injective (resp. surjective) is a monomorphism (resp. an epimorphism).

(c). (2 points) What are the monomorphisms and epimorphisms in ${}_R\mathbf{Mod}$?

(d). (2 points) What are the monomorphisms in **Top** ? Give an example of a continuous morphism with dense image that is not an epimorphism in **Top**. [1]

(e). (2 points) Find a category $\mathscr{C}$, a faithful $F : \mathscr{C} \to \mathbf{Set}$ and a monomorphism f in $\mathscr{C}$ such that $F(f)$ is not injective.

(f). (1 point) Find an epimorphism in **Ring** that is not surjective.

(g). The goal of this question is to show that any epimorphism in **Grp** is a surjective map. Let $\phi : G \to H$ be a morphism of groups, and suppose that it is an epimorphism in **Grp**. Let $A = \mathrm{Im}(\phi)$. Let $S = \{*\} \sqcup (H/A)$, where $\{*\}$ is a singleton, and let $\mathfrak{S}$ be the group of permutations of S. We denote by σ the element of $\mathfrak{S}$ that switches $*$ and A and leaves the other elements of H/A fixed. For every $h \in H$, we denote by $\psi_1(h)$ the element of $\mathfrak{S}$ that leaves $*$ fixed and acts on H/A by left translation by H; this defines a morphism of groups $\psi_1 : H \to \mathfrak{S}$. We denote by $\psi_2 : H \to \mathfrak{S}$ the morphism $\sigma\psi_1\sigma^{-1}$.

   (i) (2 points) Show that $\psi_1 = \psi_2$.

   (ii) (1 point) Show that $A = H$.

2. Let $F : \mathscr{C} \to \mathscr{C}'$ be a functor.

(a). (3 points) If F has a quasi-inverse, show that it is fully faithful and essentially surjective.

(b). (4 points) If F is fully faithful and essentially surjective, construct a functor $G : \mathscr{C}' \to \mathscr{C}$ and isomorphisms of functors $F \circ G \simeq \mathrm{id}_{\mathscr{C}}$ and $G \circ F \simeq \mathrm{id}_{\mathscr{C}'}$.

3. Let $\mathscr{C}$ be the full subcategory of **Ab** whose objects are finitely generated abelian groups.

(a). (2 points) Show that every natural endomorphism of $\mathrm{id}_{\mathscr{C}}$ is multiplication by some $n \in \mathbb{Z}$.

[1] In fact, the epimorphisms in **Top** are the surjective continuous maps.

(b). (3 points) Consider the functor $F : \mathscr{C} \to \mathscr{C}$ that sends an abelian group A to $A_{\mathrm{tor}} \oplus (A/A_{\mathrm{tor}})$ (and acts in the obvious way on morphisms), where A_{tor} is the torsion subgroup of A. Show that there is no natural isomorphism $F \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}}$.

4. (2 points, extra credit) Let k be a field, and let $F : \mathbf{Mod}_k \to \mathbf{Mod}_k$ be the functor sending a k-vector space V to $V \otimes_k V$ and a k-linear transformation f to $f \otimes f$. Show that the only morphism of functors from $\mathrm{id}_{\mathbf{Mod}_k}$ to F is the zero one, i.e. the morphism $u : \mathrm{id}_{\mathbf{Mod}_k} \to F$ such that $u(V) = 0$ for every k-vector space V.

5. (4 points) Let $\mathscr{C}$ be a category. Remember that the category $\mathrm{PSh}(\mathscr{C})$ of presheaves on $\mathscr{C}$ is the category $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set})$.

Let F be a presheaf on $\mathscr{C}$ and X be an object of $\mathscr{C}$. Let $\Phi : \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, F) \to F(X)$ be the map defined by $\Phi(u) = u(X)(\mathrm{id}_X)$. Let $\Psi : F(X) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, F)$ be the map sending $x \in F(X)$ to the morphism of functors $\Psi(x) : h_X \to F$ such that $\Psi(x)(Y) : h_X(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X) \to F(Y)$ sends $f : Y \to X$ to $F(f)(x) \in F(Y)$. Show that Φ and Ψ are bijections that are inverses of each other.

6.

(a). (2 points) Show that the categories $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$ are not equivalent. (Hint : If $F : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is an equivalence of categories, show that $F(\varnothing)$ is a singleton and that $F(X) = \varnothing$ for X a singleton.)

(b). (1 point) Let $\mathscr{C}$ be the full subcategory of $\mathbf{Set}$ whose objects are finite sets. Show that $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$ are not equivalent.

(c). (1 point) Show that $\mathbf{Rel}$ and $\mathbf{Rel}^{\mathrm{op}}$ are equivalent.

(d). (2 points) Let $\mathscr{D}$ be the full subcategory of $\mathbf{Ab}$ whose objects are finite abelian groups. Show that $\mathscr{D}$ and $\mathscr{D}^{\mathrm{op}}$ are equivalent.

7. (4 points) Let $\mathscr{C}$ and $\mathscr{C}'$ and $F : \mathscr{C} \to \mathscr{C}'$, $G : \mathscr{C}' \to \mathscr{C}$ be two functors. We consider the two bifunctions $H_1, H_2 : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ defined by $H_1 = \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), \cdot)$ and $H_2 = \mathrm{Hom}_{\mathscr{C}}(\cdot, G(\cdot))$. Suppose that we are given, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $Y \in \mathrm{Ob}(\mathscr{C}')$, a bijection $\alpha(X, Y) : H_1(X, Y) \xrightarrow{\sim} H_2(X, Y)$. Show that the two following statements are equivalent :

(i) The family of bijections $(\alpha(X, Y))_{X \in \mathrm{Ob}(\mathscr{C}), Y \in \mathrm{Ob}(\mathscr{C}')}$ defines an isomorphism of functors $H_1 \xrightarrow{\sim} H_2$.

(ii) For every morphism $f : X_1 \to X_2$ in $\mathscr{C}$, every morphism $g : Y_1 \to Y_2$ in $\mathscr{C}'$, and for all $u \in \mathrm{Hom}_{\mathscr{C}'}(F(X_1), Y_1)$ and $v \in \mathrm{Hom}_{\mathscr{C}'}(F(X_2), Y_2)$, the square

$$\begin{tikzcd} F(X_1) \arrow[r, "u"] \arrow[d, "F(f)"'] & Y_1 \arrow[d, "g"] \\ F(X_2) \arrow[r, "v"'] & Y_2 \end{tikzcd}$$

is commutative if and only if the square

$$\begin{tikzcd} X_1 \arrow[r, "\alpha(X_1,Y_1)(u)"] \arrow[d, "f"'] & G(Y_1) \arrow[d, "G(g)"] \\ X_2 \arrow[r, "\alpha(X_2,Y_2)(v)"'] & G(Y_2) \end{tikzcd}$$

is commutative.

8. Remember that a functor $F : \mathscr{C} \to \mathbf{Set}$ is called representable if there exists an object X of $\mathscr{C}$ and an element x of $F(X)$ such that the morphism of functors $u : \mathrm{Hom}_{\mathscr{C}}(X, \cdot) \to F$ defined by $u(Y) : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to F(Y)$, $(f : X \to Y) \longmapsto F(f)(x)$ is an isomorphism. The couple (X, x) is then said to represent F.

The following functors are representable. For each of them, give a couple representing the functor. (If the functor is only defined on objects, it is assumed to act on morphisms in the obvious way.) (1 point per functor)

(a). The identity endofunctor of **Set**.

(b). The functor $F : \mathbf{Grp} \to \mathbf{Set}$, $G \longmapsto G^n$, where $n \in \mathbb{N}$.

(c). The forgetful functor $\mathbf{Mod}_R \to \mathbf{Set}$, where R is a ring.

(d). The forgetgul functor $\mathbf{Ring} \to \mathbf{Set}$.

(e). The functor $\mathbf{Ring} \to \mathbf{Set}$, $R \longmapsto R^\times$.

(f). The functor $F : \mathbf{Cat} \to \mathbf{Set}$ that takes a category to its set of objects.

(g). The functor $F : \mathbf{Cat} \to \mathbf{Set}$ that takes a category to its set of morphisms (i.e. $\bigcup_{X,Y \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X, Y)$).

(h). The functor $F : \mathbf{Cat} \to \mathbf{Set}$ that takes a category to its set of isomorphisms.

(i). The functor $F : \mathbf{Top}_* \to \mathbf{Set}$ that takes a pointed topological space (X, x) to the set of continuous loops on X with base point x.

(j). The functor $F : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ such that $F(X) = \mathfrak{P}(X)$ and, for every map $f : X \to Y$, $F(f) : \mathfrak{P}(Y) \to \mathfrak{P}(X)$ is the map $A \longmapsto f^{-1}(A)$.

(k). The functor $F : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$ that sends a topological space to its set of open subsets. (If $f : X \to Y$ is a continuous map, $F(f) : F(Y) \to F(X)$ is the map $U \longmapsto f^{-1}(U)$.)

(l). If k is a field, the functor $F : \mathbf{Mod}_k^{\mathrm{op}} \to \mathbf{Set}$ that sends a k-vector space to the underlying set of V^* (so F is the composition of the duality functor $\mathbf{Mod}_k^{\mathrm{op}} \to \mathbf{Mod}_k$ and of the forgetful functor from $\mathbf{Mod}_k$ to **Set**.)

9. (extra credit) The simplicial category Δ is defined in Example I.2.1.8(5) of the notes. It is the category whose objects are the finite sets $[n] = \{0, 1, \dots, n\}$ with their usual order and whose morphisms are the nondecreasing maps between these sets.

The category **sSet** of *simplicial sets* if $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Set})$. So a simplicial set is by definition a functor $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$; in that case, we write X_n for $X([n])$ and, if $f : [n] \to [m]$, we often write $f^* : X_m \to X_n$ for $X(f)$. For example, for each $n \in \mathbb{N}$, the *standard simplex* of dimension n is the simplicial set $\mathrm{Hom}_\Delta(\cdot, [n])$.

If X is a simplicial set, a simplicial subset Y of X is the data of a subset Y_n of X_n, for every $n \in \mathbb{N}$, such that $\alpha^*(Y_m) \subset Y_n$ for every morphism $\alpha : [n] \to [m]$ in Δ. We can form images of morphisms of simplicial sets, and unions and intersections of simplicial subsets, in the obvious way.

If we see each poset $[n]$ as a category in the usual way, then the morphisms of Δ become functors, so this allows us to see Δ as a subcategory of **Cat**.

Let $\mathscr{C}$ be a category. Its *nerve* $N(\mathscr{C})$ is the restriction to Δ^{op} of the functor $\mathrm{Hom}_{\mathbf{Cat}}(\cdot, \mathscr{C})$ on $\mathbf{Cat}^{\mathrm{op}}$; it is a functor from Δ^{op} to **Set**, i.e. a simplicial set. As $\mathrm{Hom}_{\mathbf{Cat}}$ is a bifunctor, this construction is functorial in $\mathscr{C}$, and we get a nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$.

(a). (3 points) If $\mathscr{C}$ is a category, show that $N(\mathscr{C})_0 \simeq \mathrm{Ob}(\mathscr{C})$ and $N(\mathscr{C})_1 \simeq \coprod_{X,Y \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X, Y)$. Can you give a similar description of $N(\mathscr{C})_n$ for $n \geq 2$?

(b). (1 point) Let $n \in \mathbb{N}$. Show that the nerve of $[n]$ is isomorphic to Δ_n.

(c). (1 point) Let $n \in \mathbb{N}$. Show that there exists $e_n \in \Delta_n([n])$ such that, for every simplicial set X, the map $\mathrm{Hom}_{\mathbf{sSet}}(\Delta_n, X) \overset{\sim}{\to} X_n$ sending u to $u_n(e_n)$ is bijective.

(d). (1 point) For every category $\mathscr{C}$ and every simplicial set X, if $u, v : X \to N(\mathscr{C})$ are two morphisms of simplicial sets such that $u_i, v_i : X_i \to N(\mathscr{C})_i$ are equal for $i \in \{0, 1\}$, show that $u = v$.

(e). (1 point) We denote by $\Delta_{\leq 2}$ the full subcategory of Δ whose objects are [0], [1] and [2]; if X is a simplicial set, we denote by $X_{\leq 2}$ its restriction to $\Delta_{\leq 2}$ (which is a functor $\Delta_{\leq 2}^{\mathrm{op}} \to \mathbf{Set}$).

Let X be a simplicial set and $\mathscr{C}$ be a category. Show that every morphism $X_{\leq 2} \to N(\mathscr{C})_{\leq 2}$ extends to a morphism $X \to N(\mathscr{C})$.

(f). (2 points) Show that the functor $N : \mathbf{Cat} \to \mathbf{sSet}$ is fully faithful.

Let $n \in \mathbb{N}$ For every $k \in [n]$, we denote by δ_k the unique injective increasing map $[n-1] \to [n]$ such that $k \notin \mathrm{Im}(\delta_k)$. This induces a map $\Delta_{n-1} \to \Delta_n$, that we also denote by δ_k; the image of this map is called the kth facet of Δ_n.

If $k \in [n]$, the *horn* Λ_k^n is the union of all the facets of Δ_n except for the kth one; in other words, it is the simplicial subset of Δ_n defined by

$$\Lambda_k^n([m]) = \{f \in \mathrm{Hom}_\Delta([m], [n]) \mid \exists l \in [n] - \{k\} \text{ and } g \in \mathrm{Hom}_\Delta([m], [n-1]) \text{ with } f = \delta_l \circ g\}.$$

(g). (1 point) Let $\mathscr{C}$ be a category. If $n \geq 3$ and $k \in [n]$, show that every morphism of simplicial sets $\Lambda_k^n \to X$ extends uniquely to a morphism $\Delta_n \to X$.

(h). (1 point) Let $\mathscr{C}$ be a category. Show that every morphism of simplicial sets $\Lambda_1^2 \to X$ extends uniquely to a morphism $\Delta_2 \to X$.

(i). (2 points) Show that a simplicial set X is the nerve of a category if and only if, for every $n \in \mathbb{N}$, every $0 < k < n$ and every morphism of simplicial sets $u : \Lambda_k^n \to X$, the morphism u extends uniquely to a morphism $\Delta_n \to X$.