Global Langlands for Function Fields - Sophie Morel - Fall 2014
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1 Lecture - 09/11/2014

Fix a finite field $\mathbb{F}_q$ of characteristic p, prime number $\ell \neq p$, let $X/\mathbb{F}_q$ be a smooth projective geometrically connected curve. Let $F = \mathbb{F}_q(X)$ be the function field of the curve. For every closed point $v \in |X|$, i.e. place of F, have completion F_v with ring of integers $\mathcal{O}_v$. Set $\mathbb{O} = \prod_{v\in|X|} \mathcal{O}_v$ and $\mathbb{A} = \prod'_{v\in|X|} F_v$. Fix a split connected reductive group G over $\mathbb{F}_q$ (e.g. $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n},...$ - but not unitary ones because they aren't split).

Cuspidal automorphic forms: Fix a finite closed subscheme N of X (the level). Let $K_N = \ker G(\mathbb{O}) \to G(\mathcal{O}_N)$, an open compact subgroup in $G(\mathbb{A})$. Also, for technical purposes fix a cocompact lattice Ξ in $Z(F)\backslash Z(\mathbb{A})$. Let

$$\mathcal{H}_N = C_c(K_N\backslash G(\mathbb{A})/K_N, \overline{\mathbb{Q}}_\ell),$$

be the global Hecke algebra, with convolution product, for Haar measure with $\mathrm{vol}(K_N) = 1$ and thus 1_{K_N} is the identity of this associative algebra. This acts by right convolution on

$$C_c(G(F)\backslash G(\mathbb{A})/\Xi K_N, \overline{\mathbb{Q}}_\ell),$$

where the Ξ is there to make the space have finite volume (so can take it to be trivial if G is semisimple).

Some elements of this space are our automorphic forms, but we need to add a cuspidality condition. Let F be such a form; say it's cuspidal if for every parabolic subgroup $P \subsetneq G$ with unipotent radical U, then the constant term

$$g \mapsto \int_{U(F)\backslash U(\mathbb{A})} f(ng)dn$$

is zero. So get a subspace of cuspidal automorphic forms

$$C_c^{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/\Xi K_N, \overline{\mathbb{Q}}_\ell).$$

(Remark: Usually there's other conditions for "automorphic forms", but a non-trivial fact that they're implied by the cuspidality condition in this situation). Another nontrivial fact: C_c^{cusp} is a finite-dimensional space, stable under the action of $\mathcal{H}_N$. The representations that appear in this are called the cuspidal automorphic ones. (Remark: $\lim C_c^{\mathrm{cusp}}$ over all levels N has an action of $G(\mathbb{A})$, with its irreducible constituents all of the cuspidal automorphic representations. If N is fixed, irreducible $\mathcal{H}_N$-submodules are the same as irreducible cuspidal automorphic representations with K_N-fixed vectors).

Main theorem (V. Lafforgue): There exists a canonical $\mathcal{H}_N$-equivariant decomposition

$$C_c^{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/\Xi K_N, \overline{\mathbb{Q}}_\ell) = \bigoplus_\sigma h_\sigma$$

where σ runs over Langlands parameters: continuous semisimple homomorphisms $\mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ which are unramified outside of N. This decomposition is compatible with the Satake isomorphism at places $v \nmid |N|$. (Remark this does *not* determine the decomposition in general - for GL_n it does but for other groups strong multiplicity one can fail).

Here, $\widehat{G}$ is the dual group of G (invert the roots). Have $\mathrm{GL}_n, \mathrm{SO}_{2n}$ are self-dual, $\mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$ are dual, $\mathrm{SL}_n, \mathrm{PGL}_n$ are dual, etc. Semisimple means $\overline{\mathrm{Im}\,\sigma}$ (taking the Zariski closure) is reductive. Unramified outside of N means for all $v \notin |N|$, $\sigma|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is trivial on inertia, so $\sigma(\mathrm{Frob}_v)$ is well-defined as a conjugacy class.

What is the Satake isomorphism? Write $K_N = \prod_v K_v$ where $K_v \subseteq G(\mathcal{O}_v0$ is such that if $v \notin |N|$ then $K_v = G(\mathcal{O}_v)$. So $\mathcal{H}_N = \bigotimes' \mathcal{H}_v$ for $\mathcal{H}_v = C_c(K_v\backslash G(F_v)/K_v, \overline{\mathbb{Q}}_\ell)$. Then if $v \notin |N|$ there is a canonical $\overline{\mathbb{Q}}_\ell$-algebra isomorphism (the Satake isomorphism) between $\mathcal{H}_v$ and $K(\mathbf{Rep}_{\widehat{G}}) \otimes \overline{\mathbb{Q}}_\ell$, where $\mathbf{Rep}_{\widehat{G}}$ is the category of algebraic representations of $\widehat{G}$, and K is the Grothendieck group with multiplication coming from tensor product. In particular, $\mathcal{H}_v$ is commutative. We also get characters of $\mathcal{H}_v \to \overline{\mathbb{Q}}_\ell$ is in bijection with characters of $K(\mathbf{Rep}_{\widehat{G}}) \otimes \overline{\mathbb{Q}}_\ell$, which is in bijection with semisimple elements $\widehat{G}(\overline{\mathbb{Q}}_\ell)$ up to conjugacy. Example: If $G = \mathrm{GL}_n$,

$$K(\mathbf{Rep}_{\widehat{G}}) \cong \overline{\mathbb{Q}}_\ell[X_1^{\pm1}, \dots, X_n^{\pm1}]^{S_n}.$$

What do we mean that the isomorphism is compatible with Satake isomorphism outside N? For all σ and all $v \in |N|$, $\mathcal{H}_v$ acts on h_σ via multiplication by the character corresponding to $\sigma(\mathrm{Frob}_v)$.

Very vague idea of proof: For every finite set I and every $W \in \mathbf{Rep}_{\widehat{G}^I}$, we can define a moduli stack $ChtN, I, W \to (X \setminus N)^I$ of G-*shtuka*. A G-shtuka on $S/\mathbb{F}_q$ is a G-bundle $\mathcal{G}$ on $X \times S$ with an isomorphism

$$\varphi : \mathcal{G}_{X\times S \setminus \bigcup_{i\in I} \Gamma_{x_i}} \to \left((\mathrm{id}_X \times \mathrm{Frob}_S)^* \mathcal{G}\right)_{X\times S \setminus \bigcup_{i\in I} \Gamma_{x_i}},$$

where $(x_i)_{i\in I} \in X(S)$ are the "legs of the shtuka", W bounds how far φ is from an isomorphism at the x_i, and we have a level N structure, i.e. a trivialization of $(\mathcal{G}, \varphi)$ on $N \times S$.

Have that $Cht_{N,I,W}$ is a Deligne-Mumford stack, and if $I = \emptyset$ then $Cht_{\varphi,I,1}$ is a discrete stack with points $G(F)\backslash G(\mathbb{A})/K_N$. Lafforgue defines a subspace $H_{I,W}$ of $H^*(Cht_{N,I,W}, \overline{\mathbb{Q}}_\ell)$ (actually intersection cohomology) that admits an action of $\mathcal{H}_N \times \mathrm{Gal}(\overline{F}/F)^I$, with

$$H_{\emptyset,1} = C_c^{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell).$$

Properties: $W \mapsto H_{I,W}$ is functorial in $W \in \mathbf{Rep}_{\widehat{G}^I}$ (comes from geometric Satake). By coalescing and separating the legs, get for all $\xi : I \to J$, an isomorphism $H_{I,W} \cong H_{J,W^\xi}$.

Lafforgue uses these to construct "excursion operators" $H_{\emptyset,1} \to H_{\emptyset,1}$ (depending on $I, W, x \in W, \xi \in W^*, (\gamma_i)_{i\in I}$ with $\gamma_i \in \mathrm{Gal}(\overline{F}/F)$). These operators generate a commutative subalgebra B of $\mathrm{End}(H_{\emptyset,1})$ and get the decomposition of $C_c^{\mathrm{cusp}} \cong H_{\emptyset,1}$ by taking the generalized eigenspace decomposition of B. (Initially this decomposition is indexed by characters of B, but these give parameters σ by Lafforgue's generalization of pseudo-representations). Finally, for $v \notin |N|$ we have a basis consisting of excursion operators.

So that's the outline - lots of steps. The first will be to define the moduli stacks, but we first need to define algebraic stacks.

Fix a base scheme S, let $\mathbf{Aff}_S$ be the category of affine schemes over S. We consider four Grothendieck topologies on this category:

1. Zariski topology, $\mathbf{Aff}_{S,\mathrm{Zar}}$: A covering is a family $(U_i \to U)_{i\in I}$ such that each $U_i \to U$ is an open embedding (of each connected component) and $U = \bigcup \mathrm{img}(U_i)$.

2. Étale topology, $\mathbf{Aff}_{S,\text{ét}}$: A covering family $(f_i : U_i \to U)$ is one such that each $f_i : U_i \to U$ is étale and $U = \bigcup f[U_i]$.

3. Fppf topology, $\mathbf{Aff}_{S,\mathrm{fppf}}$: A covering family $(f_i : U_i \to U)$ is one such that each $f_i : U_i \to U$ is fppf flat of finite presentation and $U = \bigcup f[U_i]$.

4. Fpqc topology, $\mathbf{Aff}_{S,\mathrm{fpqc}}$: A covering family $(f_i : U_i \to U)$ is one such that each $f_i : U_i \to U$ is flat and there exists $J \subseteq I$ with $U = \bigcup_{j\in J} f_j[U_j]$.

Definitions: A presheaf of sets on $\mathbf{Aff}_S$ is a functor $\mathcal{F} : \mathbf{Aff}_S^{op} \to \mathbf{Set}$. If $f : U \to V$ the map $\mathcal{F}(V) \to \mathcal{F}(U)$ is called $s \mapsto f^*s$, or $s \mapsto s|_U$. Let $\mathbf{Psh}(\mathbf{Aff}_S)$ be the category of presheaves, which has all inductive and projective limits (calculated term by term).

For a sheaf, you need a topology *top* (one of the four just defined). Then a presheaf $\mathcal{F}$ on $\mathbf{Aff}_S$ is called a sheaf for *top* (a *top*-sheaf) if, for every covering family $f_i : U_i \to U$, we have that $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective, and the image of this map is the set of tuples (s_i) with $s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$ for all i, j. (A "separating presheaf" is one such that $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective).

The map $\mathbf{Sh}(\mathbf{Aff}_{S,top}) \hookrightarrow \mathbf{Psh}(\mathbf{Aff}_S)$ is full. This embedding has a left adjoint $\mathcal{F} \mapsto \mathcal{F}^{sh}$ called sheafification. Note $\mathbf{Sh}(\mathbf{Aff}_{S,top})$ has all inductive and projective limits. Projective limits and filtered inductive limits are calculated term by term. Other inductive limits are sheafifications of the limit in the category of presheaves.

2 Lecture - 09/16/2014

S a scheme, $\mathbf{Aff}_S$ the category of affine schemes over S (affine scheme with map to S) and $\mathbf{Sch}_S$ the category of schemes over S. Last time defined four sites, $\mathbf{Aff}_{S,\mathrm{Zar}}$, $\mathbf{Aff}_{S,\text{ét}}$, $\mathbf{Aff}_{S,\mathrm{fppf}}$, $\mathbf{Aff}_{S,\mathrm{fpqc}}$. (Remark: for fpqc topology have some set-theoretic issues; need to fix a universe, and things depened on that).

Defined $\mathbf{Psh}(\mathbf{Aff}_S)$ and $\mathbf{Sh}(\mathbf{Aff}_{S,top})$; have two functors, $\mathbf{Sh} \hookrightarrow \mathbf{Psh}$ the fully faithful embedding, and the sheafification functor that's a left adjoint. Yoneda embedding: For $X \in \mathbf{Sch}_S$, get $\underline{X} \in \mathbf{Psh}(\mathbf{Aff}_S)$ defined by $\underline{X}(U) = \mathrm{Hom}(U, X)$. This gives a fully faithful functor $\mathbf{Sch}_S \to \mathbf{Psh}(\mathbf{Aff}_S)$. (Often just write X for $\underline{X}$, identifying $\mathbf{Sch}_S$ as a subcategory of (pre)sheaves).

Theorem (Grothendieck): For any $X \in \mathbf{Sch}_S$, $\underline{X}$ is a fpqc sheaf, so we get a fully faithful embedding $\mathbf{Sch}_S \to \mathbf{Sh}(\mathbf{Aff}_{S,\mathrm{fpqc}})$. (And since our topologies are linearly ordered by coarseness, being an fpqc sheaf is the strongest condition, so get $\mathbf{Sch}_S \to \mathbf{Sh}(\mathbf{Aff}_{S,top})$ for any of the four topologies).

Definition: An *S-space* is a fppf sheaf on $\mathbf{Aff}_S$. An S-space is *representable* by a scheme (or even "is a scheme") if it is isomorphic to some $\underline{X}$.

This point of view is nice if we want a natural way to define a scheme via its functors of points, e.g. moduli problems. For instance:

- Algebraic groups.
- Grassmannians: If $\mathcal{E}$ is a quasicoherent sheaf on S and $r \in \mathbb{N}$, set $\mathrm{Gr}(r, \mathcal{E})$ to be the sheaf taking U to the set of surjective maps $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_U \twoheadrightarrow \mathcal{F}$ with $\mathcal{F}$ a locally free $\mathcal{O}_U$-module with rank r. This is representable.
- In case $S = \operatorname{Spec} \mathbb{Q}$, take sheaf $\mathcal{F}$ mapping U to the set of relative elliptic curves $E \to U$ together with $P \in E(U)$ of order 4 modulo equivalence; this is representable.
- Non-example: $S = \operatorname{Spec} \mathbb{Q}$ and $\mathcal{F}$ like above, but maps U just to the set of relative elliptic curves $E \to U$ modulo equivalence. Unfortunately this isn't even an étale sheaf! There exist nonisomorphic elliptic curves $E, E'/\mathbb{Q}$ that become isomorphic over a number field (so this isn't separated, because a number field is an étale cover of $\mathbb{Q}$). This comes from the fact that $H^1(G_{\mathbb{Q}}, \mathrm{Aut}(E)) \neq 1$.

So the problem in the last case is that some elliptic curves have too many isomorphisms, but we're naively taking equivalence classes anyway. A few ways to fix the problem: we can rigidify the problem (i.e. like in the example above it). Alternatively, and perhaps more natural (at the expense of lots of technical stuff): Get rid of the equivalence, and instead look at a sheaf of categories.

How do we make sense of a "presheaf of categories". Want to say this is a functor $\mathbf{Sch}_S \to \mathbf{Cat}$, but $\mathbf{Cat}$ is really a 2-category so it isn't really reasonable to ask for this. For instance, we could try to define a presheaf by mapping U to the category of vector bundles on U, but we only have $(f \circ g)^* \cong g^* \circ f^*$, rather than equality - so this isn't actually a functor. Instead it's a *pseudofunctor*, which we could use as our definition (but becomes a pain given the compatibility conditions we need to carry around).

Another point of view: filtered categories. Let $\mathbf{C}$ be a category and $p : \mathbf{F} \to \mathbf{C}$ a functor. An arrow $\varphi : E \to F$ in $\mathbf{F}$ is *Cartesian* if, for every $\psi : E' \to F$ in $\mathcal{F}$ and every $h : p(E') \to p(E)$ with $p(\varphi) \circ h = p(\psi)$ there's a unique $\chi : E' \to E$ with $\varphi \circ \chi = \psi$ and $p(\chi) = h$.

Example: Take $\mathbf{C} = \mathbf{Aff}_S$, and $\mathbf{F}$ having objects given by maps $E \to U$ with $U \in \mathbf{Aff}_S$ and $E \to U$ a vector bundle, and morphisms $(E \to U)$ to $(F \to V)$ being commutative diagrams

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[d] & & F \arrow[d] \\ Y \arrow[rr, "f"'] \arrow[dr] & & V \arrow[dl] \\ & S & \end{tikzcd}$$

such that $E \to U \times_V F$ is a map of vector bundles over U. Define $p : \mathbf{F} \to \mathbf{C}$ by mapping $p(E \to U)$ to U and p of a diagram to f. Then a diagram as above is a Cartesian arrow in $\mathbf{F}$ iff it's a Cartesian square.

Remarks: Let $f : U \to V$ be a morphism in $\mathbf{C}$, F an object in $\mathbf{F}$ with $p(F) = V$. If $\varphi : E \to F$ and $\varphi' : E'\ toF$ are Cartesian arrows in $\mathbf{F}$ are such that $p(E) = p(E') = U$ and $p(\varphi) = p(\varphi') = f$ then there exists a unique $\theta : E \to E'$ with $p(\theta) = \mathrm{id}_U$. We call E the *pullback* of F to U. (Pullbacks are unique up to unique isomorphism).

Back to general case: If $\varphi : E \to F$ and $\psi : F \to G$ are arrows in $\mathbf{F}$ with ψ Cartesian, then φ is Cartesian iff $\psi \circ \varphi$ is Cartesian. If $\varphi : E \to F$ is in $\mathbf{F}$ is such that $p(\varphi)$ is an isomorphism, then φ is Cartesian iff it's an isomorphism.

Definition. We say $p : \mathbf{F} \to \mathbf{C}$ is a *fibered category* over $\mathbf{C}$ if pullbacks always exist: For every morphism $f : U \to V$ in $\mathbf{C}$ and every object F in $\mathbf{F}$ with $p(F) = V$, there exists a Cartesian arrow $\varphi : E \to F$ with $p(\varphi) = f$ (which includes $p(E) = U$). (So the example above is a fibered category).

Definition: If $p : \mathbf{F} \to \mathbf{C}$ is fibered, the *fiber* over an object U of $\mathbf{C}$ is the category $\mathbf{F}(U)$ with objects E of $\mathbf{F}$ with $p(E) = U$, and morphisms $E \to E'$ given by morphisms $E \to E'$ from $\mathbf{F}$ such that $p(E) = \mathrm{id}_U$. In our example, $F(\mathbf{U})$ is the category of vector bundles over U.

In general, fibered categories over $\mathbf{C}$ form a 2-category. A morphism of fibered categories (i.e. a 1-morphism) from $p : \mathbf{F} \to \mathbf{C}$ to $p' : \mathbf{F}' \to \mathbf{C}$ is a functor $F : \mathbf{F} \to \mathbf{F}'$ such that $p' \circ F = p$, which sends Cartesian arrows to Cartesian arrows. (Note: This equality of composition of functors is actually a equality, not an isomorphism!). A 2-morphism between $F, G : \mathbf{F} \to \mathbf{F}'$ is a natural transformation $\alpha : F \to G$ such that for every $E \in \mathbf{F}$ the morphism $\alpha_E : F(E) \to G(E)$ satisfies $p'(\alpha_E) = \mathrm{id}_{p(E)}$. (Note $p'(\alpha_E)$ maps from $p'(F(E)) = p(E)$ to $p'(G(E)) = p(E)$ via the equalities $p' \circ F = p = p' \circ G$).

Let $p : \mathbf{F} \to \mathbf{C}$ be a fibered category. A *cleavage* of p is a class K of arrows of $\mathbf{F}$ such that for all $f : U \to V$ in $\mathbf{C}$ and every F in $\mathbf{F}$ with $p(F) = V$, there exists a unique $\varphi : E \to F$ in K with φ Cartesian and $p(\varphi) = f$. (Remark: cleavages always exist by the axiom of choice).

So let K be a cleavage. For $f : U \to V$ in $\mathbf{C}$, get a functor $f^* : \mathbf{F}(V) \to \mathbf{F}(U)$: if F is an object in $\mathbf{F}(V)$, then there exists a unique $\varphi : E \to F$ as before so we can take $f^*(F) = E$; and if $\varphi : F \to F'$ is a map in $\mathbf{F}(V)$ then by Cartesianness there exists a unique $f^*(\psi) : f^*(F) \to f^*(F')$ which makes the appropriate square commute and such that $p(f^*(\psi)) = \mathrm{id}_U$. Also, get isomorphisms of functors $(\mathrm{id}_U)^* \cong \mathrm{id}_{\mathbf{F}(U)}$ and $f \circ g)^* \cong g^* \circ f^*$, plus a bunch of compatibility conditions. This defines a contravariant pseudofunctor $\mathbf{C} \to \mathbf{Cat}$ (by $U \mapsto \mathbf{F}(U)$ and $f \mapsto f^*$). Conversely, a contravariant pseudofunctor $\mathbf{C} \to \mathbf{Cat}$ gives a fibered category.

So, have defined fibered categories, and mentioned that they corresponded to pseudo-functors and thus to "presheaves of categories". Now can move on to stacks, which will be "sheaves of categories". How does the sheaf condition translate? Let $p : \mathbf{F} \to \mathbf{C}$ be a fibered category where $\mathbf{C}$ is a site. (The only examples we really care about are $\mathbf{Aff}_{S,top}$). Fix a cleavage of p. Let $\mathcal{U} = \{f_i : U_i \to U\}$ be a covering family in $\mathbf{C}$. The category of descent data for $\mathcal{U}$, $\mathbf{DD}(\mathcal{U})$, is:

- The objects of $\mathbf{DD}(\mathcal{U})$ is the collection of families $(E_i, \varphi_{ij})_{i,j \in I}$ with $E_i \in \mathbf{F}(U_i)$ and $\varphi_{ij} : \pi_i^* E_i \cong \pi_j^* E_j$ in $\mathcal{F}(U_i \times_U U_j)$, satisfying the cocycle condition (for E_i, E_j, E_k we pull back φ_{ij}, φ_{jk}, φ_{ik} to $U_i \times U_j \times U_k$ and demand compatibility there once we put in all of the canonical isomorphisms).

- Morphisms between (E_i, φ_{ij}) and (E_i', φ_{ij}') are families (ψ_i) with $\psi_i : E_i \to E_i'$ that are compatible with the φ_{ij} and φ_{ij}' in the obvious way.

Given this definition, we have a functor $\mathbf{F}(U) \to \mathbf{DD}(\mathcal{U})$ via $E \mapsto (\pi_i^* E, \mathrm{id})$ for any cover $\mathcal{U}$ of U.

What's a stack? We say $p : \mathbf{F} \to \mathbf{C}$ is a *prestack* (respectively a *stack*) if, for every covering family $\mathcal{U}$ of U, the functor $\mathbf{F}(U) \to \mathbf{DD}(\mathcal{U})$ is fully faithful (respectively an equivalence of categories). Example: U mapping to vector bundles over U is a fppf stack.

3 Lecture - 09/18/2014

Last time: defined filtered category $p : \mathbf{F} \to \mathbf{C}$; talked about how it corresponded to a pseudo-functor $\mathbf{C}^{op} \to \mathbf{Cat}$, which is what we want to call a presheaf of categories in $\mathbf{C}$. If $\mathbf{C}$ is a site can make sense of the sheaf axioms, and call the resulting presheaves a *stack*. Formalize this by defining, for each covering family $\mathcal{U}$ of U, a category of descent data $\mathbf{DD}(\mathcal{U})$, and a functor $\mathbf{F}(U) \to \mathbf{DD}(\mathcal{U})$. (If $\mathbf{F}$ is a presheaf of sets, then $\mathbf{DD}(\mathcal{U})$ is the set of tuples (s_i) with $s_i \in \mathbf{F}(U_i)$ such that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times U_j}$). Then $p : \mathbf{F} \to \mathbf{C}$ is a *prestack* of categories if $\mathcal{F}(U) \to \mathbf{DD}(\mathcal{U})$ is fully faithful for all $\mathcal{U}$, and a *stack* of categories if $\mathbf{F}(U) \to \mathbf{DD}(\mathcal{U})$ is an equivalence of categories for all $\mathcal{U}$. (For presheaves of sets, these conditions are exactly "separated presheaf" and "presheaf").

Definition: A category is called:

- A *set* (or *discrete*) if the only morphisms are identity morphisms.
- A *groupoid* if every morphism is an isomorphism.
- An *equivalence relation* if it's a groupoid, and if $\mathrm{Hom}(A, B) \leq 1$ for all A, B (equivalently, if it's equivalent to a set).

Definition: We say that a fibered category $\mathbf{F} \to \mathbf{C}$ is *fibered in sets/equivalence relations/groupoids* if all of its fibers are of the appropriate type of category. Exercise: $\mathbf{F} \to \mathbf{C}$ is fibered in groupoids iff every map in $\mathbf{F}$ is Cartesian.

Definition: We say that $\mathbf{F} \to \mathbf{C}$ is a *(pre)stack in sets/equivalence relations/groupoids* if it's a (pre)stack and fibered in sets/equivalence relations/groupoids. Convention for the rest of the semester: "(pre)stacks" are always in groupoids. (Remark: from above, prestacks/stacks in sets are just separated presheaves / sheaves).

Remarks: Have 2-categories $\mathbf{Prestack}(\mathbf{C})$ and $\mathbf{Stack}(\mathbf{C})$ of (pre)stacks in groupoids lying over $\mathbf{C}$. Have a fully faithful functor $\mathbf{Sh}(\mathbf{C}) \to \mathbf{Stack}(\mathbf{C})$, by identifying sheaf with the corresponding stack in sets, which has essential image equal to stacks in equivalence relations. Have *2-Yoneda lemma*: for any of our 4 usual topologies *top*, we get a fully faithful embedding

$$\mathbf{Sch}_S \to \mathbf{Sh}(\mathbf{Aff}_{S,top}) \to \mathbf{Stacks}(\mathbf{Aff}_{S,top}).$$

What's the stack corresponding to a scheme X/S? Well, the sheaf corresponding to it was $\underline{X}$ given by $U \mapsto \mathrm{Hom}(U, X) = X(U)$. Then this becomes a stack by seeing it as a pseudofunctor; but what's the corresponding fibered category $\mathbf{F} \to \mathbf{Aff}_S$? Well, the objects of $\mathbf{F}$ are pairs (U, x) where U is an affine scheme and $x \in X(U) = \mathrm{Hom}(U, X)$ is a map of schemes $U \to X$, i.e. the morphisms $f : U \to X$, i.e. affine X-schemes. So the objects of $\mathbf{F}$ are the objects of $\mathbf{Aff}_X$. What are the maps? Given $f : U \to X$ and $g : V \to X$ in this category, an $\mathbf{F}$-morphism between them must be a S-map $h : U \to V$ together with $\varphi : f \to h^*(g)$, which has to be id_f because the stack is fibered in sets. But this means $f = h^*(g)$ so $f = g \circ h$. So an $\mathbf{F}$-morphism between $U \to X$ and $V \to X$ is the same as an X-morphism. Thus we conclude that the stack coming from X is the fibered category $\mathbf{F} = \mathbf{Aff}_X \to \mathbf{Aff}_S$ with the obvious map.

Another way to formulate the stack condition: Let $p : \mathbf{F} \to \mathbf{C}$ be a fibered category, and have objects $U \in \mathbf{C}$, $E, F \in \mathbf{F}(U)$, get a presheaf $\mathrm{Hom}(E, F) : \mathbf{C}_U \to \mathbf{Set}$ (where $\mathbf{C}_U$ is the category of objects of $\mathbf{C}$ over U...), by $f : V \to U$ mapping to $\mathrm{Hom}_{\mathbf{F}(V)}(f^*E, f^*F)$. Fact: $\mathbf{F} \to \mathbf{C}$ is a prestack iff for all U, E, F, $\mathrm{Hom}(E, F)$ is a sheaf. Moreover, $\mathbf{F} \to \mathbf{C}$ is a stack iff every descent datum is effective. (So prestack means "morphisms are defined locally" and stack means "morphisms and objects are defined locally").

Examples: If $\Phi : \mathbf{C} \to \mathbf{Cat}$ is a (pre)stack in categories, then $\Phi^{iso} : \mathbf{C} \to \mathbf{Grpd}$ is a (pre)stack of groupoids, where $\Phi^{iso}(U)$ is the subcategory of $\Phi(U)$ obtained by throwing away all non-isomorphisms. Another example: $\mathbf{C} \to \mathbf{Cat}$ given by $U \mapsto \mathbf{Sh}(\mathbf{C}_U)$ is a stack. Also, can reformulate faithfully flat descent as saying the pseudofunctor $QCoh$ mapping U to quasicoherent sheaves on U is a fpqc stack over $\mathbf{Aff}_S$. Consequence: the map taking U to the category of affine morphisms $V \to U$ is a fpqc stack. Also, U mapping to the category of pairs $(X \to U, \mathcal{L})$ where $X \to U$ is a projective morphism and $\mathcal{L}$ is a relatively ample $\mathcal{O}_X$-module is a fpqc stack.

G-bundles. Fix a field k, all schemes will be over k. Let G be an affine algebraic group over k (which we might as well assume is smooth because it will be in applications). Definition: If X is a scheme, a *vector bundle* over X is a $\mathcal{O}_X$-module that is locally free of finite type (locally could equivalently be for Zariski, etale, fppf, and maybe even fpqc topologies). Then have the category of vector bundles over X, $\mathbf{Vect}(X)$, an exact tensor category. Remember the 4 equivalent definitions of a G-bundle over a scheme X:

1. A sheaf $\mathcal{P}$ on $\mathbf{Aff}_{X,\mathrm{fppf}}$ which is a torsor under $\underline{G}$ (the sheaf $U \mapsto G(U)$). This means you have a right action $\mathcal{P} \times \underline{G} \to \mathcal{P}$ such that $\mathcal{P} \times \underline{G} \to \mathcal{P} \times \mathcal{P}$ (given by $(x, g) \mapsto (x, xg)$) is an isomorphism.

2. A scheme $\widetilde{X} \to X$ with a right action of G compatible with the trivial action on X, such that there exists a morphism $Y \to X$ that's faithfully flat of finite presentation, such that $\widetilde{Y} = Y \times_X \widetilde{X} \cong Y \times G$ as a Y-scheme with axiom of G.

3. $\widetilde{X} \to X$ is fppf and there's a right action of G on $\widetilde{X}$ with $G \times \widetilde{X} \cong \widetilde{X} \times \widetilde{X}$ via $(g, x) \mapsto (x, xg)$.

4. Tannakian description: An exact tensort functor $\mathbf{Rep}_G \to \mathbf{Vect}(X)$.

Remark: (3) says that Y from (2) can be taken to be $\widetilde{X}$, and that $\mathcal{P}$ is representable by $\widetilde{X}$ in (1). Another remark: If G is smooth then every G-bundle is étale-locally trivial. If $G = \mathrm{GL}_n$ then every G-bundle is Zariski-locally trivial.

Construction: IF $\widetilde{G} \to X$ is a G-bundle and if ZS is a scheme with an action of G, then the fiber bundle over X associated to Z and $\mathcal{P}$ is $Z_{\mathfrak{P}} = (\widetilde{X} \times Z)/G$ if this exists as a scheme. (Could introduce this more generally as an algebraic space). Example: this exists as a scheme if Z is affine, or if Z is quasi-projective with a G-equivariant ample line bundle. Applying this to $Z = \mathbb{A}^n$ with the usual action of GL_n gives that the category of GL_n-bundles on X is equivalent to the category of rank-n vector bundles on X, with isomorphisms. (A morphism of G-bundles is automatically an isomorphism because of local triviality, so we need to restrict to only isomorphisms in the category of vector bundles). The inverse functor takes a vector bundle $\mathcal{E}$ to $Isom(\mathcal{O}_X^n, \mathcal{E})$.

More fppf stacks on $\mathbf{Aff}_S$.

- $\mathbf{Vect}_n$ given by V mapping to rank n vector bundles with isomorphisms. (Or even $\mathbf{Vect}$).

- The classifying stack $\mathbf{B}G = [pt/G]$ of G, which maps U to G-bundles on U. (Above, said $\mathbf{B}\mathrm{GL}_n \cong \mathbf{Vect}_n^{iso}$). This is "the quotient in the category of stacks of the point $\operatorname{Spec} k$ by the action of G".

- More generally: If X is a S-scheme with an action of G, define $[X/G]$ as a presheaf in groupoids defined by letting $[X/G](U)$ be the category of diagrams $U \leftarrow \mathcal{P} \to X$ with $\mathcal{P} \to U$ a G-bundle and $\mathcal{P} \to X$ G-equivariant, and homomorphisms are G-equivariant maps $\mathcal{P} \to \mathcal{P}'$ over $U \times X$. This is a stack, and if the action of G on X is free and X/G is a scheme then $[X/G]$ is represented by X/G.

Notation for our several types of quotients: X/G is quotient in category of schemes (which may or may not exist as scheme), $[X/G]$ is quotient in category of stacks.

Proof that $[X/G]$ is a stack: Let $G - \mathbf{Sh}_X$ be the pseudofunctor mapping U to the G-equivariant fppf sheaves on $\mathbf{Aff}_{X \times U}$ with isomorphisms. (i.e. sheaves $\mathcal{P}$ on $\mathbf{Aff}_{X \times U,\mathrm{fppf}}$ with an action of G such that $\mathcal{P} \to X$ is G-equivariant). Then:

1. $G - \mathbf{Sh}_X$ is a stack (by descent...).

2. The map $[X/G] \to G - \mathbf{Sh}_X$ given by taking $U \leftarrow \mathcal{P} \to X$ to the map $\mathcal{P} \to X \times U$ is a 1-morphism of pseudofunctors. This is fully-faithful, so $[X/G]$ is a prestack.

3. Remains to show descent is effective in $[X/G]$. Given a descent datum for $[X/G]$ and $(U_i \to U)$, then it glues to an object $\mathcal{P} \to X \times U$ in G-$\mathbf{Sh}_X$. We just need to check that $\mathcal{P} \to U$ is a G-bundle; but this is an fppf local condition we know on an open cover.

4. Finally, if the action is free and X/G is a scheme, then $X \to X/G$ is a G-bundle. This gives a point of $[X/G](X/G)$, i.e. a map $X/G \to [X/G]$, which is an isomorphism.

4 Lecture - 09/23/2014

Examples of stacks. Last time gave a specific case of a mapping stack: let k be a field, G an affine algebraic group over k, X/k a scheme. Then $Bun_{G,X}$, given by mapping U to G-bundles on $X \times U$, is an fppf stack.

Mapping stacks in general: Fix a base scheme S. Let $\mathcal{Y}$ be a stack on $\mathbf{Aff}_{s,top}$ and X an S-scheme. Define $Maps(X, \mathcal{Y})$ as a presheaf in groupoids, sending $U \in \mathbf{Aff}_S$ to $\mathrm{Hom}_{\mathbf{Stacks}(S,top)}(X \times U, \mathcal{Y})$. If X is affine this equals $\mathcal{Y}(X \times U)$ (but we can't write this in general like this... yet). Then this is a stack. For instance, $Bun_{G,x} = Maps(X, BG)$.

This follows from: Lemma: if $\mathcal{Y}$ is a stack on $\mathbf{Aff}_{S,top}$ then $\mathcal{Y}^{ext} : \mathbf{Sch}_S \to \mathbf{Grpd}$ defined by

$$X \mapsto \mathrm{Hom}_{\mathbf{St}(\mathbf{Aff}_{S,top})}(X, \mathcal{Y})$$

is a stack on $\mathbf{Sch}_{S,top}$. So we get an equivalence of 2-categories

$$\mathbf{St}(\mathbf{Aff}_{S,top}) \cong \mathbf{St}(\mathbf{Sch}_{S,top}).$$

Later on we'll just identify $\mathcal{Y}$ and $\mathcal{Y}^{ext}$ and then identify these two categories. (This justifies writing $\mathcal{Y}(X \times U)$ if X is a non-affine scheme above).

Proof: Step 1 - prove $\mathcal{Y}^{ext}$ is a prestack. Fix an object X in $\mathbf{Sch}_S$ and let $F, G \in \mathcal{Y}^{ext}(X) = \mathrm{Hom}(X, \mathcal{Y})$. We get a presheaf $Isom(F,G)$ on $\mathbf{Sch}_{X,top}$ taking an X-scheme $F : U \to X$ to the set of natural transformations $f^*F \to f^*G$ (since $\mathcal{Y}$ is a groupoid, all natural transformations are actually isomorphisms). As a fibered category over $\mathbf{Aff}_S$, X is just $\mathbf{Aff}_X$. So we have $F, G : \mathbf{Aff}_X \to \mathcal{Y}$ compatible with the natural maps $\mathbf{Aff}_X, \mathcal{Y} \to \mathbf{Aff}_S$, with $f^*F = F|_{\mathbf{Aff}_U}$ and similarly for G.

Then $\mathcal{Y}^{ext}$ is a prestack iff $Isom(F,G)$ is a sheaf for all X, F, G as above. Let $\{f_i : Y_i \to Y\}$ is a covering family on $\mathbf{Sch}_{X,top}$. So suppose $\varphi, \psi \in Isom(F,G)(Y)$ are such that $f_i^*\varphi = f_i^*\psi$ for all i. Fix some $T \in \mathbf{Aff}_Y$. If $T \to Y$ factors through $Y_i \to Y$ then $\varphi(T) = \psi(Y) : F(T) \to G(T)$ in $\mathcal{Y}(T)$. Now, take general T; let $T_i = T \times_Y Y_i$. Then $\{g_i : T_i \to T\}$ is a covering family, and $\varphi(T_i) = \psi(T_i)$ for all i. But $\mathcal{Y}$ is a stack so $\varphi(T) = \psi(T)$.

So have proven $Isom(F,G)$ is a presheaf. Now take a compatible family

$$\{\varphi_i : F|_{\mathbf{Aff}_{Y_i}} \to G|_{\mathbf{Aff}_{Y_i}}\}$$

(i.e. $\varphi_i \in Isom(F,G)(Y_i)$) such that $\varphi_i|_{Y_i \times_Y Y_j} = \varphi_j|_{Y_i \times_Y Y_j}$. Let $T \in \mathbf{Aff}_Y$. We want to define $\varphi(T) : F(T) \to G(T)$, a morphism of $\mathcal{Y}(T)$; let $T_i = T \times_Y Y_i$. Then $T_i \to T$ is a covering, and we have $\varphi(T_i) : F(T_i) \to G(T_i)$ such that the pullbacks agre on $T_i \times_T T_j$. Since $\mathcal{Y}$ is a stack, this gives a $\varphi(T)$.

Step 2 - prove $\mathcal{Y}^{ext}$ is a stack, i.e. descent data are effective. Let $\{X_i \to X\}$ be a covering family in $\mathbf{Sch}_{S,top}$. For all $i \in I$ suppose we have

$$F_i \in \mathcal{Y}^{ext}(X_i) = \mathrm{Hom}_{\mathbf{St}_{top}(S)}(X_i, \mathcal{Y})$$

and for all i, j we have

$$\varphi_{ij} : F_i|_{\mathbf{Aff}_{X_i \times X_j}} \cong F_j|_{\mathbf{Aff}_{X_i \times X_j}}$$

satisfying the cocycle condition. We want $F : \mathbf{Aff}_X \to \mathcal{Y}$, i.e. $F \in \mathcal{Y}^{ext}(X)$, such that $F|_{\mathbf{Aff}_{X_i}} isoF_i$. But if T is an object in $\mathbf{Aff}_X$, get $F(T)$ an object in $\mathcal{Y}(T)$ by gluing the $F_i(T \times_X X_i)$. Same for morphisms. QED

Next: Fiber products of stacks. Let $\mathbf{C}$ be a site, and $\mathbf{St}(\mathbf{C})$ the category of stacks (of groupoids) over $\mathbf{C}$. Then this category has all 2-projective limits and all 2-inductive limits; in principle could do these by calculating these in the category of prestacks and then stackifying (but hopefully not). Will describe fiber products; let $\mathcal{F}, \mathcal{F}', \mathcal{G}$ be categories, and let $f : \mathcal{F} \to \mathcal{G}$ and $f' : \mathcal{F}' \to \mathcal{G}$ be 1-morphisms. Then $\mathcal{F} \times_{\mathcal{G}} \mathcal{F}'$ is defined in the following way. For $U \in \mathbf{C}$, the objects of $(\mathcal{F} \times_{\mathcal{G}} \mathcal{F}')(U)$ are defined as tuples (E, E', φ) where $E \in \mathcal{F}(U)$, $E' \in \mathcal{F}'(U)$, and $\varphi : f(E') \cong f'(E')$ in $\mathcal{G}(U)$. Morphisms $(E, E', \varphi) \to (F, F', \psi)$ are pairs of maps $a : E \to F$ in $\mathcal{F}(U)$ and $a' : E' \to F'$ in $\mathcal{F}'(U)$ such that the obvious diagram commutes:

$$\begin{tikzcd} f(E) \arrow[r, "\varphi"] \arrow[d, "f(a)"'] & f'(E') \arrow[d, "f'(a')"] \\ f(F) \arrow[r, "\psi"'] & f'(F') \end{tikzcd}.$$

Remarks: This is a stack if $\mathcal{F}, \mathcal{F}', \mathcal{G}$ are (exercise: trace through the definition to verify this). Also, if $\mathcal{F}, \mathcal{F}'$ are fibered in sets (i.e. presheaves of sets, not just groupoids) then so is $\mathcal{F} \times_{\mathcal{G}} \mathcal{F}'$. Warning: the square

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{G}} \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{F}' \arrow[r] & \mathcal{G} \end{tikzcd}$$

is only commutative up to natural isomorphism! So to write a UMP we need to have some sort of naturality condition.

Example: Let $p : \mathcal{F} \to \mathbf{C}$ be as before. Then $U, V \in \mathbf{C}$ give $\underline{U}, \underline{V} : \mathbf{C}^{op} \to \mathbf{Set}$; fix $\alpha : \underline{U} \to \mathcal{F}$ and $\beta : \underline{V} \to \mathcal{F}$, i.e. $a\ell \in \mathcal{F}(U)$ and $\beta \neg \mathcal{F}(V)$. We want to calculate $\underline{U} \times_{\mathcal{F}} \underline{V}$. This is a presheaf of sets on $\mathbf{C}$, and sends $T \in \mathbf{C}$ to triples (f, g, φ) where $f : T \to U$, $g : T \to V$, and $\varphi : f^*(a) \cong g^*(b)$ is a morphism in $\mathcal{F}(T)$. If $U = V$ then $\underline{U} \times_{\mathcal{F}} \underline{U}|_{\mathbf{C}_U} = Isom(\alpha, \beta)$. Now look at the 2-fiber product

$$\begin{tikzcd} (\underline{U} \times \underline{V}) \times_{\mathcal{F}^2} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Delta_{\mathcal{F}}"] \\ \underline{U} \times \underline{V} \arrow[r, "(\alpha,\beta)"'] & \mathcal{F}^2 \end{tikzcd}$$

where all unspecified products are over the final objects. Now, $((\underline{U} \times \underline{V}) \times_{\mathcal{F}^2} \mathcal{F})(T)$ turns out to only have identity morphisms, so this stack is a presheaf of sets. In fact we have an isomorphism $(\underline{U} \times \underline{V}) \times_{\mathcal{F}^2} \mathcal{F} \cong \underline{U} \times_{\mathcal{F}} \underline{V}$. If $U = V$ and we restrict to $\mathbf{C}_U$ then we get $Isom(\alpha, \beta) \cong \underline{U} \times_{\mathcal{F}^2} \mathcal{F}|_{\mathbf{C}_U}$.

Schematic maps: Let S be a scheme, $\mathbf{C} = \mathbf{Aff}_{S,top}$. Definition: let $\mathcal{X}, \mathcal{Y}$ be stacks on $\mathbf{Aff}_{S,top}$. A 1-morphism $F : \mathcal{X} \to \mathcal{Y}$ is called schematic if, for all $Y \in \mathbf{Sch}_S$ and all morphisms $Y \to \mathcal{Y}$, the morphism $Y \times_{\mathcal{Y}} \mathcal{X} \to Y$ is a morphism of schemes (since $Y \times_{\mathcal{Y}} \mathcal{X}$ is a scheme!).

Definition: If (P) is a property of morphisms of schemes that is stable by base change and top-local, we say $F : \mathcal{X} \to \mathcal{Y}$ is schematic and has property (P) if

1. F is schematic
2. For all Y in $\mathbf{Sch}_S$, $Y \times_{\mathcal{Y}} \mathcal{X} \to Y$ has (P).

Examples of such (P): smooth, unramified, étale, smooth surjective, closed/open/locally closed immersion, quasi-compact, locally of finite type/presentation, separated, ...

Definition: An *algebraic stack* or *Artin stack* over S is an fppf stack $\mathcal{X}$ on $\mathbf{Aff}_S$ such that:

1. The diagonal $\Delta_X : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is schematic, separated, quasi-compact. (Note: sometimes algebraic stacks are defined without the latter two conditions on the diagonal, but they get added in as hypotheses in most theorems).
2. There exists an S-scheme X and a smooth surjective map $X \to \mathcal{X}$ (called a *presentation* of the stack).

Note: We'll see that (1) implies that every map $X \to \mathcal{X}$, with X a scheme, is schematic.

We say $\mathcal{X}$ is a *Deligne-Mumford stack* if it is algebraic and has an étale presentation. Remark: Let $\mathcal{X} \in \mathbf{St}(\mathbf{Aff}_{S,\text{fppf}})$. The following are equivalent:

1. $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is schematic.
2. For all $U \in \mathbf{Aff}_S$ and all $x, y \in \mathcal{X}(U)$, the fppf sheaf $Isom(x, y)$ on $\mathbf{Aff}_U$ is representable by a scheme.
3. For all $U \in \mathbf{Aff}_S$ and all $x \in \mathcal{X}(U)$, the map $x : U \to \mathcal{X}$ is schematic.
4. For all $X \in \mathbf{Sch}_S$, every $x : X \to \mathcal{X}$ is schematic.

Proof: All rests on $Isom(x, y) \cong U \times_{\mathcal{X}^2} \mathcal{X}|_{\mathbf{Aff}_U}$.

5 Lecture - 09/25/2014

Let S be a scheme; we take the convention that a "stack" on $\mathbf{Aff}_S$ is always with the fppf topology unless otherwise specified. Also, remark that last time we used the convention that un-adorned products were over the final object of $\mathbf{St}(S)$, which is just S itself viewed as a stack.

Definition: A stack $\mathcal{X}$ over S (i.e. over $\mathbf{Aff}_S$) is algebraic stack (Deligne-Mumford stack, respectively) if:

1. $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is schematic, separated, and quasi-compact.

2. There exists an S-scheme X and a map $X \to \mathcal{X}$ that's smooth and surjective (étale surjective, respectively).

Lemma: (a) If $\mathcal{X} \to \mathcal{Y}$ is a schematic (+ some other assumptions?) map of sheaves of groupoids, and $\mathcal{Y}$ is an algebraic (or D-M) stack, then so is $\mathcal{X}$.
(b) If $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{X}' \to \mathcal{Y}$ are maps of stacks with $\mathcal{X}, \mathcal{X}', \mathcal{Y}$ algebraic (or D-M) stacks, then so is $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}'$.

Proof: (a) ??? (b) To prove (2) take presentations of each thing and take lots of base changes - reduce to the case $X = \mathcal{X}$ and $X' = \mathcal{X}'$, etc... For (1) have some base-change diagram with the diagonal map on the top and $Z \to S$ for a scheme Z...

Remark: Condition (2) implies that $\Delta_{\mathcal{X}}$ in (1) is of finite type.

Examples of algebraic stacks. Let k be a field and G a linear algebraic group over k (which includes the condition that G is smooth). First of all, recall we had a stack $BG = [\cdot/G]$ where $BG(S)$ is the set of G-bundles over S, which is $\mathrm{Hom}(S, BG)$. Then BG is an algebraic stack.

Proof: need to show that $BG \to BG \times BG$ is schematic, etc. Long computation........

Now, $[Z/G]$. Let Z be a k-scheme with a right action of G. We let $[Z/G](S)$ be the set of pairs of G-bundles $\widetilde{X} \to X$ with G-equivariant maps $\alpha : \widetilde{X} \to G$. Have morphism $\pi : [Z/G] \to BG$ by $(\widetilde{X}, \alpha) \mapsto \widetilde{X}$.

Given $X \in \mathbf{Sch}_k$ with a morphism $X \to BG$ (which amounts to a G-bundle $\pi_X : \widetilde{X} \to X$), what is $[Z/G] \times_{BG} X$? Well, should be $(Z \times \widetilde{X})/G$; once we expect this we simply need to check that these are equal. In particular, if Z is separated and quasicompact, then $[Z/G]$ is algebraic.

Our main example of algebraic stacks; $Bun_{\mathrm{GL}_n, X} = Vect_{n,X}$, taking S to the category of rank-n vector bundle on $S \times X$, with isomorphisms as morphisms. This is an algebraic stack if X is projective. (Remark: What if X is not projective? Well can look at $X = \mathbb{A}^1$, and find $Bun_{\mathrm{GL}_n, \mathbb{A}^1}(\operatorname{Spec} k)$ is the category of free $k[t]$-modules of rank n, and if x corresponds to $k[t]^n$ then $Isom(x,x)(\operatorname{Spec} A)$ is $\mathrm{GL}_n(A[t])$ which is too big if $n \geq 2$).

6 Lecture - 09/30/2014

Goals: Prove a lemma from last time; give a presentation of $Bun_{\mathrm{GL}_n,X}$. To do this, there's two results we'll be using over and over again which we'll state but not prove:

- Let $S \in \mathbf{Sch}_k$, $p_s : X_S \to X$, $\mathcal{E}$ a coherent $\mathcal{O}_{X_S}$-module. Define a presheaf on $\mathbf{Sch}_S$ by $Quot_{\mathcal{E}/X_S/S}$ by mapping $T \to S$ to the set of isomorphism classes of pairs $(\mathcal{G}, \varphi)$ with $\mathcal{G}$ an $\mathcal{O}_{X_T}$-module that's flat over $\mathcal{O}_T$ and $\varphi : \mathcal{E}_{X_T} \twoheadrightarrow \mathcal{G}$. Theorem (Grothendieck): If $X_S \to S$ is projective then this is representable by a scheme over S.

- Assume $p_S : X_S \to S$ is projective, let $\mathcal{O}(1)$ be a relatively ample line bundle on X_S. Theorem (Serre): Assume S is Noetherian, let $\mathcal{D}$ be a coherent $\mathcal{O}_{X_S}$-module. Set $\mathcal{E}(n) = \mathcal{E} \otimes \mathcal{O}(1)^n$. Then (i) the $R^i p_{S*}\mathcal{E}$ are coherent $\mathcal{O}_S$-modules. (ii) There exists N such that if $n \geq N$ then $R^i p_{S*}\mathcal{E}(n) = 0$ for all $i \geq 1$. (iii) There exists N such that if $n \geq N$ then $p_S^* p_{S*}\mathcal{E}(n) \to \mathcal{E}(n)$ is surjective for $n \geq N$. (Remark: If $R^i p_{S*}\mathcal{E} = 0$ then $p_*\mathcal{E}$ is a vector bundle on S. Why? Well, it's a coherent sheaf and $s \mapsto \dim(\mathcal{E}_S) = \chi(X_S, \mathcal{E})$ is locally constant on S).

Lemma (very important): Let $S \in \mathbf{Sch}_k$, $p_S : X_S \to S$ be flat and projective, and $Y_S \to X_S$ be quasi-projective. Define a presheaf of sets on $\mathbf{Sch}_S$ by

$$Sect(X_S, Y_S)(T) = \mathrm{Hom}_{X_T}(X_T, Y_T) = \mathrm{Hom}_{X_S}(X_T, Y_S)$$

where $X_T = X_S \times_S T$ and $Y_T = Y_S \times_S T$. Then $Sect(X_S, Y_S)$ is a scheme.

Remark: If $S = \operatorname{Spec} k$, $X_S = X$, $Y_S = X \times Y$, then

$$Sect(X_S, Y_S)(T) = \mathrm{Hom}_{X \times T}(X \times T, X \times Y \times T) = \mathrm{Hom}(X \times T, Y) = Maps(X, Y)(T),$$

so $Sect$ is some relative version of the mapping stack.

Proof: (0) If $Y_{1,S}, Y_{2,S}, Y_{3,S}$ are quasiprojective over X_S and we have X_S-morphisms $Y_{1,S} \to Y_{2,S} \leftarrow Y_{3,S}$, then

$$Sect(X_S, Y_{1,S} \times_{Y_{2,S}} Y_{3,S}) = Sect(X_S, Y_{1,S}) \times_{Y_{2,S}} Sect(X_S, Y_{3,S}).$$

This is immediate from the definition.

(1) Case where $Y_S = \mathbb{P}(\mathcal{E})$, $\mathcal{E}$ a vector bundle over X_S. In this case

$$Sect(X_S, \mathcal{P}(\mathcal{E}))(T) = \mathrm{Hom}_{X_T}(X_T, \mathbb{P}(\mathcal{E}_{X_T})) = \{(\mathcal{L}, \alpha)\}/\sim$$

where $\mathcal{L}$ runs over line bundles on X_T and $\alpha : \mathcal{L} \to \mathcal{E}_{X_T}$ is $\mathcal{O}_{X_T}$-linear and condition (*) holds: For all $Z \to X_T$, $\alpha_Z : \mathcal{L}_Z \to \mathcal{E}_Z$ is injective (equivalently, $\mathcal{E}_{X_T}/\mathcal{L}$ is flat over $\mathcal{O}_{X_T}$, or also equivalently $\mathcal{E}_{X_T}/\mathcal{L}$ is a vector bundle over X_T).

Now: Have $Sect(X_S, \mathbb{P}(\mathcal{E}))(T) \to Quot_{\mathcal{E}/X_S/S}(T)$ given by $(\mathcal{L}, \alpha) \mapsto (\mathcal{E}_{X_T} \twoheadrightarrow \mathcal{E}_{X_T}/\mathcal{L})$. Want to prove that this is a schematic open immersion. The image of this is cut out by two conditions on $\varphi : \mathcal{E}_{X_T} \twoheadrightarrow \mathcal{G}$: namely

- (a) $\mathcal{G}$ is $\mathcal{O}_{X_T}$-flat (not just $\mathcal{O}_T$-flat), i.e. $\mathcal{G}$ is a vector bundle.

- (b) $\operatorname{rank} \mathcal{G} = \operatorname{rank} \mathcal{E} - 1$.

Let $Q(a)$ and $Q(a, b)$ be the subsets cut out by these things; want to show both have schematic open embeddings into $Quot_{\mathcal{E}/X_S/S}$. For the first one, want to show that for $T \to S$, $P = Q(a) \times_{Quot} T \to T$ is an open immersion; but $P(T' \to T)$ is $*$ if $\mathcal{G}_{X_{T'}}$ is $\mathcal{O}_{X_{T'}}$-flat, and $\emptyset$ otherwise. Let $U \subseteq X_T$ be the (open) locus of flatness of $\mathcal{G}_{X_T}$; then P is representable by $T \setminus p_T[X_T \setminus U]$. Remains to show $Q(a, b) \hookrightarrow Q(a)$ is a schematic open and closed embedding; this follows because the rank of a vector bundle is locally constant.

(2) If $Y_S \hookrightarrow Z_S$ is an open embedding of X_S-schemes, then $Sect(X_S, Y_S) \to Sect(X_S, Z_S)$ is a schematic open embedding.

Proof: Let $T \to S$, let $\alpha : X_T \to Z_S$ be in $Sect(X_S, Z_S)(T) = \mathrm{Hom}_{X_S}(X_T, Z_S)$. Let $P = Sect(X_S, Y_S) \times_{Sect(X_S,Z_S)} T$ with the product via α. This is a presheaf on $\mathbf{Sch}_T$, and $P(T' \to T)$ is $*$ if $X_{T'} \to X_T \to Z_S$ has image in Y_S and $\emptyset$ otherwise. Let $U_T = \alpha^{-1}[Y_S] \hookrightarrow X_T$ (an open subscheme). Then $X_{T'} \to X_T \to Z_S$ has image in Y_S iff $X_{T'} \to X_T$ has image in Y_T iff $T' \to T$ has image in $T \setminus p_T[X_T \setminus U_T]$, which is open because p_T is proper. So $P \to T$ is representable by $T - p_T[X_T \setminus U_T] \to T$.

(3) If $Y_S \hookrightarrow Z_S$ is a closed embedding of X_S-schemes, then $Sect(X_S, Y_S) \to Sect(X_S, Z_S)$ is a schematic closed immersion. (This will finish the proof).

Proof: Let $T \to S$, let $\alpha : X_T \to Z_S$ be in $Sect(X_S, Z_S)(T) = \mathrm{Hom}_{X_S}(X_T, Z_S)$. Let $P = Sect(X_S, Y_S) \times_{Sect(X_S,Z_S)} T$ with the product via α. This is a presheaf on $\mathbf{Sch}_T$, and $P(T' \to T)$ is $*$ if $X_{T'} \to X_T \to Z_S$ has image in Y_S and $\emptyset$ otherwise. Let $U_T = \alpha^{-1}[Y_S] \hookrightarrow X_T$ (an open subscheme). Set $W_T = \alpha^{-1}[Y_S]$, a closed subscheme of X_T. Then $X_{T'} \to Z_S$ has image in Y_S iff $X_{T'} \to X_T$ has image in W_T. So $P(T' \to T) = Sect(X_T, W_T)(T')$.

All of that is identical as above, but at this point we need to modify the argument; is $Sect(X_T, W_T)(T')$ representable by a closed subscheme of T? If X_T is affine, then $W_T = \{0\} \times_{\mathbb{A}^n} X_T$ where $0 \to \mathbb{A}^n$ is the zero section; want to generalize this. Well, $W_T \hookrightarrow X_T$ is closed so let $\mathcal{J} \subseteq \mathcal{O}_{X_T}$ be ideal of definition. . Let $\mathcal{O}(1)$ be a relatively (for $X_T \to T$) ample line bundle. If $n \gg 0$ then $p_T^* p_{T*}\mathcal{J}(n) \twoheadrightarrow \mathcal{J}(n)$ and $p_T^* p_{T*}\mathcal{O}_{X_T}(n) \twoheadrightarrow \mathcal{O}_{X_T}(n)$ and $p_{T*}\mathcal{O}_{X_T}(n)$ is a vector bundle.

Want a vector bundle $\mathcal{E}$ on X_T and a section α of $\mathcal{E}$ such that $W_T = X_T \times_{\mathcal{E}} X_T$. If $T = \operatorname{Spec} k$ choose $s_1, \ldots, s_m \in \Gamma(X_T, \mathcal{J}(n))$ generating $\mathcal{J}(n)$. Take $\mathcal{E} = \mathcal{O}_X(n)^m$ and $\alpha = (s_1, \ldots, s_N)$. If T is not, exercise (easier version: Assume T quasiprojective, use ample line bundle on X_T). On top of this taking n big enough can assume $R^i p_{T*}\mathcal{E} = 0$ for $i \geq 1$. Now we reduce to step 4.

(4) Let $\mathcal{E} \to X_S$ be a vector bundle. Then $Sect(X_S, \mathcal{E})$ is representable by a scheme, and $Sect(X_S, X_S) \to Sect(X_S, \mathcal{E})$ (coming from the 0-section) is a schematic closed embedding.

Remark: This is sufficient to finish (3) because $Sect(X_T, W_T) \to Sect(X_T, X_T) = T$ is the pullback of $T = Sect(X_T, X_T) \to Sect(X_T, \mathcal{E})$ via $\alpha : X_T \to \mathcal{E}$.

Proof: This is true in general but much easier if we assume $R^i p_{S*}\mathcal{E} = 0$ for all $i \geq 1$ (which we can do by the end of (3)). I want a coherent sheaf $\mathcal{F}$ over S such that for all $T \to S$, $\Gamma(X_T, \mathcal{E}_{X_T}) = \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{F}_T, \mathcal{O}_T)$. (Why? Then $Sect(X_S, \mathcal{E})$ is representable by $\mathrm{Spec}_S(\mathrm{Sym}\,\mathcal{F})$, which has T-points equal to $\mathcal{O}_T$-algebra maps $\mathrm{Sym}\,\mathcal{F} \to \mathcal{O}_T$, which equal to $\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{F}, \mathcal{O}_T)$).

How do we get this $\mathcal{F}$? Take $\mathcal{F} = (p_{S*}\mathcal{E})^\vee$. This $\mathcal{F}$ works by flat base change and Serre duality: if we have $f : T \to S$ then $Lf^* Rp_{S*}\mathcal{E} = Rp_{T*}\mathcal{E}_{X_T}$ but this reduces to $p_{s*}\mathcal{E} = p_{T*}\mathcal{E}_{X_T}$ (in particular flatness gives $Lf^* = f^*$, and our assumption $R^i p_{S*}\mathcal{E} = 0$ for $i > 0$ gives the rest). Without the assumption on R^i still works for

$$\mathcal{F} = H^0(Rp_{S*}(\mathcal{E}^\vee \otimes K_{X_S/S}))$$

and use Grothendieck duality. Note the map $S = Sect(X_S, X_S) \to Sect(X_S, \mathcal{E})$ coming from the zero section $X_S \to \mathcal{E}$ is also the map induced by $\mathrm{Sym}\,\mathcal{F} \twoheadrightarrow \mathrm{Sym}^0\,\mathcal{F} = \mathcal{O}_S$.

7 Lecture - 10/02/2014

Remember: if $\mathcal{Y}$ is a stack and X is a scheme (over a field k) then $Maps(X, \mathcal{Y})$ is the stack

$$S \mapsto \mathrm{Hom}(X \times S, \mathcal{Y}).$$

The reason we care is $Bun_{G,X} = Maps(X, BG)$.

Cor 1: Let $\mathcal{Y}_1 \to \mathcal{Y}_2$ be a schematic quasiprojective morphism of stacks. Then the induced map $Maps(X, \mathcal{Y}_1) \to Maps(X, \mathcal{Y}_2)$ is schematic.

Cor 2: If $G_1 \to G_2$ is an injective map between linear algebraic groups, then for any X, $Bun_{G_1,X} \to Bun_{G_2,X}$.

Cor 3: If $Bun_{\mathrm{GL}_n,X}$ is algebraic for all n, then $Bun_{G,X}$ is algebraic for any linear algebraic group G.

Corollary 2 implies Corollary 3 immediately. To see Corollary 1 implies Corollary 2, need that $BG_1 \to BG_2$ is schematic and quasiprojective. Take $S \in \mathbf{Sch}_k$, $S \to BG_2$ corresponding to G_2-bundle $\widetilde{S} \to S$. Then can calculate

$$BG_1 \times_{BG_2} S = (G_2/G_1 \times \widetilde{S})/G_2,$$

which is a scheme, and the map to S is quasiprojective.

Proof of Corollary 1: Let $\mathcal{Y}_1 \to \mathcal{Y}_2$ be schematic quasiprojective. Let $S \in \mathbf{Sch}_k$, let $X \times S \to \mathcal{Y}_2$ be an S-point of $Maps(X, \mathcal{Y}_2)$. Let

$$Z_S = (X \times S) \times_{\mathcal{Y}_2} \mathcal{Y}_1 \to X \times S$$

Then

$$(Maps(X, \mathcal{Y}_1) \times_{Maps(X,\mathcal{Y}_2)} S)(T)$$

is the set of maps $X \times T \to \mathcal{Y}_1$ such that

$$\begin{tikzcd} X \times T \arrow[r] \arrow[d] & \mathcal{Y}_1 \arrow[d] \\ S \times X \arrow[r] & \mathcal{Y}_2 \end{tikzcd}$$

commutes. But this is $\mathrm{Hom}(X \times T, Z_S) = Sect(X \times S, Z_S)(T)$, and we know this $Sect$ is representable by a scheme.

Goal: to show that if X is projective then $Bun_{GL_n,X}$ is an algebraic stack; missing the most important part, the presentation. By definition

$$Bun_{\mathrm{GL}_n,X}(S) = \{\text{rank } n \text{ vector bundles on } X \times S\}.$$

(Remark: Eventually we'll be using this when X is a smooth projective curve, but don't need that many hypotheses for now).

Let $p_S : X \times S \to S$ be the projection. For all $d \in \mathbb{Z}$ and $N \in \mathbb{N}$ define a stack $\mathcal{U}_{d,N}$ by letting $\mathcal{U}_{d,N}(S)$ be the set of $\mathcal{E} \in Bun_{\mathrm{GL}_n,X}(S)$ satisfying

1. For all $i \geq 1$, $R^i p_{S*}\mathcal{E}(d) = 0$.
2. $p_S^* p_{S*}\mathcal{E}(d) \to \mathcal{E}(d)$ is surjective.
3. The rank of $p_{S*}\mathcal{E}(d)$ is N.

Remark: (1) implies $p_{S*}\mathcal{E}(d)$ is a vector bundle on S, so (3) makes sense.

Claim: the obvious inclusion $\mathcal{U}_{d,N} \to Bun_{\mathrm{GL}_n,X}$ is a schematic open embedding, and $Bun_{\mathrm{GL}_n,X} = \bigcup_{d,N} \mathcal{U}_{d,N}$. The second part follows from Serre's theorem last time. For first part, need to see that (1),(2),(3) are open conditions. Fix S and a map $S \to Bun_{\mathrm{GL}_n,X}$ corresponding to $\mathcal{E}$ on $X \times S$. Then

$$(\mathcal{U}_{d,N} \times_{Bun_{\mathrm{GL}_n,X}} S)(T)$$

is $*$ if $\mathcal{E}_{X\times T} \in U_{d,N}(T)$ and $\emptyset$ otherwise, and need to see this is open. Take

$$U_1 = S \setminus \bigcup_{i\geq 1} \mathrm{supp}(R^i p_{S*}\mathcal{E})$$

which is open in S and cuts out condition (1). Let U_3 be the open closed subscheme of U_1 where $p_{S*}\mathcal{E}(d)$ has rank N, and let

$$U_2 = U_3 \setminus p_S[X \times U_2 \setminus W]$$

for

$$W = X \times U_3 \setminus \mathrm{supp}(Coker(p_S^* p_{S*}\mathcal{E}(d) \to \mathcal{E}(d))).$$

Claim: Fix $T \to S$. Claim $\mathcal{E}_{X\times T} \in \mathcal{U}_{d,N}(T)$ iff $T \to S$ factors through U_2.

Proof: "if" is obvious. Only if: once you see that $T \to S$ has to factor through U_1 it's easy to get that it in fact factors through U_2 (base change, etc.) That it factors through U_1 follows from semicontinuity theorems.

Now, need a presentation of $\mathcal{U}_{d,N}$ (fixed d, N). Let $\mathcal{Y}_{d,N}(S)$ be the set of pairs $(\mathcal{E}, \varphi)$ with $\mathcal{E}$ free of rank n and $R^i p_{S*}\mathcal{E}(d) = 0$ for all $i \geq 1$, and $\varphi : \mathcal{O}^N_{X\times S} \twoheadrightarrow \mathcal{E}(d)$ a map inducing via adjunction an isomorphism $\mathcal{O}_S^N \to p_{S*}\mathcal{E}(d)$. Then map $\mathcal{Y}_{d,N} \to \mathcal{U}_{d,N}$ by $(\mathcal{E}, \varphi) \mapsto \mathcal{E}$.

(a) This is surjective (as a map of stacks): if $\mathcal{E} \in \mathcal{U}_{d,N}(S)$ then fppf locally there's an isomorphism $\mathcal{O}_S^n \to p_{S*}\mathcal{E}(d)$, which gives by pullback $\varphi : \mathcal{O}^N_{X\times S} \to p_S^* p_{S*}\mathcal{E}(d) \twoheadrightarrow \mathcal{E}(d)$. Then $(\mathcal{E}, \varphi) \in \mathcal{Y}_{d,N}(S)$.

(b) This map is schematic smooth and surjective. Let $\mathcal{E} \in \mathcal{U}_{d,N}(S)$. Then $(\mathcal{Y}_{d,N} \times_{\mathcal{U}_{d,N}} S)(T)$ is he set of $\varphi : \mathcal{O}^N_{X\times T} \to \mathcal{E}_{X\times T}(d)$ such that the induced map $\mathcal{O}_T^N \to p_{T*}\mathcal{E}_{X\times T}(d)$ (from adjunction) is an isomorphism. This is equal to the set of isomorphisms $\mathcal{O}_T^N \to p_{T*}\mathcal{E}_{X\times T}(d)$, and thus to $Isom(\mathcal{O}_S^n, p_{S*}\mathcal{E}(d))(T)$, which is a GL_n-bundle on S.

Finally, want to show $\mathcal{Y}_{d,N}$ is a scheme. Let $\mathcal{Z}_N(S)$ be the set of pairs $(\mathcal{G}, \varphi)$ with $\varphi : \mathcal{O}^N_{X\times S} \twoheadrightarrow \mathcal{G}$ and $\mathcal{G}$ locally free of rank N. Then $\mathcal{Y}_{d,N} \to \mathcal{Z}_N$ given by $(\mathcal{E}, \varphi) \mapsto (\mathcal{E}(d), \varphi)$ is a schematic open immersion (just as for $\mathcal{U}_{d,N} \to Bun_{\mathrm{GL}_n, X}$). Then, claim that the map $\mathcal{Z}_N(S) \to Quot_{\mathcal{O}_X^n/X/\operatorname{Spec} k}(S)$ given by $(\mathcal{G}, \varphi) \mapsto (\mathcal{G}, \varphi)$ is a schematic open immersion. Why? If $(\mathcal{G}, \varphi)$ is in $Quot$ then $\varphi : \mathcal{O}^N_{X\times S} \twoheadrightarrow \mathcal{G}$ with $\mathcal{G}$ a coherent $\mathcal{O}_{X\times S}$-module, flat over $\mathcal{O}_S$. The conditions for being in $\mathcal{Z}_N$ is that $\mathcal{G}$ is flat over $\mathcal{O}_{X\times S}$ and of rank n; both open conditions. Since we know $Quot$ is a scheme, we're done.

Properties of stacks and morphisms (over a base scheme S). Let (P) be a property of morphisms of schemes $f : X \to Y$ that's local for the smooth topology, i.e. for every Cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

with $Y' \to Y$ smooth surjective, and $X'' \to X'$ is smooth surjective, and $f'' : X'' \to Y'$ is the map induced by the UMP, then f has (P) iff f' has (P). Examples: surjective, universally open, locally of finite type / presentation, flat, smooth. Non-example: étale.

Definition: Say a map $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks has property (P) if there exists a commutative diagram (iff for every commutative diagram)

$$\begin{tikzcd} X'' \arrow[r] \arrow[dr, "f''"'] & \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ & Y \arrow[r] & \mathcal{Y} \end{tikzcd}$$

with the left square commutative and $Y \to \mathcal{Y}$, $X'' \to \mathcal{X}'$ are presentations, such that f'' has (P). Say $\mathcal{X}'$ has (P) if $\mathcal{X} \to S$ has (P).

Also, let (P) be a property of schemes that's local for the smooth topology, i.e. if $X' \to X$ is smooth surjective then X' has (P) iff X has (P). Examples: locally Noetherian, reduced, regular, of characteristic

p, ... We say an algebraic stack $\mathcal{X}$ has (P) if it has a presentation (iff for all presentations) $X \to \mathcal{X}$ with X having (P).

Definition: We say $\mathcal{X}$ is quasi-compact if there exists a presentation $X \to \mathcal{X}$ with X quasicompact. Define Noetherian as locally Noetherian + quasicompact, and finite type = locally of finite type + quasi-compact.

By looking at points: Theorem: Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then:

(i) f is locally of finite type iff for every filtered projective system $(U_i)_{i \in I}$ in $\mathbf{Aff}_S$, the following square is 2-Cartesian.

$$\begin{tikzcd} \varinjlim \mathcal{X}(U_i) \arrow[r] \arrow[d, "f"'] & \mathcal{X}(\varprojlim U_i) \arrow[d, "f"] \\ \varinjlim \mathcal{Y}(U_i) \arrow[r] & \mathcal{Y}(\varprojlim U_i) \end{tikzcd}$$

In particular if $\mathcal{Y} = S$ this says $\varinjlim \mathcal{X}(U_i) \cong \mathcal{X}(\varprojlim U_i)$.

(ii) If f is locally of finite presentation then f is smooth iff it is formally smooth, i.e. for all $U = \operatorname{Spec} A \in \mathbf{Aff}_S$ with A local strictly Henselian, for all $I \subseteq A$ with $I^2 = 0$ and all commutative diagrams

$$\begin{tikzcd} \operatorname{Spec}(A_0) \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "f"] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

there exists $x : U \to \mathcal{Y}$ making the diagram commute.

8 Lecture - 10/07/2014

Today talk about how to define properties of stacks and which are satisfied by the stacks we've seen. To define dimension: let $\mathcal{X}$ be an algebraic stack over S. We define the *points of* S as the set

$$|\mathcal{X}| = \left(\coprod_{\operatorname{Spec} K \to S} \mathcal{X}(\operatorname{Spec} K) \right) /\!\sim$$

where the disjoint union runs over maps from fields K, and the equivalence relation is defined by $(x', K') \sim (x'', K'')$ if there exists some K with $K', K'' \hookrightarrow K$ and $\operatorname{Spec} K \to S$ through which x', x'' become equivalent.

Let $f : X \to \mathcal{X}$ be a presentation of $\mathcal{X}$. If $x \in |X|$ (so $x : \operatorname{Spec} K \to \mathcal{X}$), then we define the dimension at the point x as

$$\dim_x(\mathcal{X}) = \dim X - \dim(X \times_{\mathcal{X}} \operatorname{Spec} K),$$

i.e. the dimension $\dim X$ minus the relative dimension of f at the point (since this base change of f is a map of schemes). Need to prove this is well-defined, etc. But even then this is difficult to use - need an explicit presentation of the stack (which we were very far from having, e.g. for Bun_G; for that we first reduced to Bun_{GL_n} and then used an open cover...)

In practice, to calculate $\dim_x(\mathcal{X})$ (at least when $\mathcal{X}$ is smooth) we use the *tangent complex*. For this, can consider the lifts of $x : \operatorname{Spec} K \to \mathcal{X}$ to $\operatorname{Spec} K[\varepsilon]$; the set of all such lifts is the fiber of $\mathcal{X}(\operatorname{Spec} K[\varepsilon]) \to \operatorname{Spec} K$ at x, which is a groupoid $\mathbf{X}_{\mathcal{X},x}$ in $\mathcal{X}(\operatorname{Spec} K[\varepsilon])$. This has extra structure:

Definition: A *category in K-vector spaces* is a groupoid $\mathbf{C}$ with a functor $+ : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and functors $\lambda : \mathbf{C} \to \mathbf{C}$ for every $\lambda \in \mathbf{C}$ such that:

- $(\mathbf{C}, +)$ is symmetric monoidal (i.e. addition is commutative)
- For all $\mathbf{C}$, $\mathbf{C} \cdot - : \mathbf{C} \to \mathbf{C}$ is an equivalence of categories
- A bunch of compatibilities for the rest of the vector space axioms for λ.

Example: Let $d : \mathbf{C}^{-1} \to \mathbf{C}^0$ be a 2-step complex of K-vector spaces. Define a category $\mathbf{C}$ with objects $\mathbf{C}^0$ and morphisms $x \to y$ given by $f \in \mathbf{C}^{-1}$ such that $df = x - y$. This is a category in K-vector spaces.

Fact (Deligne): For any $\mathbf{C}$ in the definition, if for every $X \in \mathbf{C}$ the commutativity constraint $X + X \cong X + X$ is id_{X+X}, then $\mathbf{C}$ is equivalent to a category as in the example.

Back to stacks: If $\mathbf{C}_{\mathcal{X},x}$ is what we defined above, it's a category in K-vector spaces, satisfying the condition in the above fact. Thus it corresponds to a complex $\mathbf{C}^{-1} \to \mathbf{C}^0$, and $H^0(\mathbf{C}^{\cdot})$ is the set of isomorphism classes of objects in the category, and H^{-1} is the automorphism group of the trivial lift.

Theorem: If $\mathcal{X}$ is smooth at x, then $\dim_x \mathcal{X} = \dim H^0(\mathbf{C}^{\cdot}) - \dim H^{-1}(\mathbf{C}^{\cdot})$.

Examples: $\operatorname{Spec} k$ is our base scheme, and G is a linear algebraic group.

(1) BG: $\operatorname{Spec} k \to BG$ is a presentation, BG is locally of finite type (check the condition $BG(\operatorname{Spec} \varinjlim A_i) = \varinjlim BG(\operatorname{Spec} A_i)$, follows from something in EGA). In fact it's finite type and smooth, so $\dim BG = \dim(\operatorname{Spec} k) - reldim(f) = -\dim G$. (Remark $|BG|$ is just one point). This implies BG is not Deligne-Mumford, since D-M stacks must have nonnegative dimension.

(2) $Bun_{G,X}$ (for X projective, so $Bun_{G,X}$ is algebraic). It's locally finite type, but not finite type in general (e.g. consider $Bun_{\mathrm{GL}_2,X}$ for X a smooth projective curve, which splits up as an infinite disjoint union indexed by degree). Formally: Proposition 1: For all $n \gg 0$, there's a connected k-variety Y and a map $Y \to Bun_{\mathrm{GL}_2,X}$ and $y_0, y_1 \in Y(k)$ such that $y_0 \mapsto \mathcal{O}_X \oplus \mathcal{O}_X$ and $y_1 \mapsto \mathcal{O}_X(n) \oplus \mathcal{O}_X(-n)$ (in $Bun^{(0)}_{\mathrm{GL}_2,X}(\operatorname{Spec} k)$, where the 0 means degree 0). Proof of this: Y the affine space corresponding to$k$-vector space $\operatorname{Ext}^T 1(\mathcal{O}_X(n), \mathcal{O}_X(-n))$. Then define $Y \to Bun_{\mathrm{GL}_2,X}$ by mapping a SES

$$0 \to \mathcal{O}_X(-n) \to \mathcal{E} \to \mathcal{O}_X n) \to 0$$

to $\mathcal{E}$. Want to see that $\mathcal{F} = \mathcal{O}_X \oplus \mathcal{O}_X$ is an extension of $\mathcal{O}_X(n)$ by $\mathcal{O}_X(-n)$. By Serre's theorem, if $n \gg 0$ then $\mathcal{F}(n)$ is generated by global sections (actually $n \geq 1$ works by ampleness of $\mathcal{O}_X(1)$) so we have an injective map $\mathcal{O}_X \to \mathcal{F}(n)$ (choosing section which is nowhere zero; exercise why "generated by global sections") gives this. Then $\mathcal{L} = \mathcal{F}(n)/\mathcal{O}_X$ is a line bundle, and we have

$$0 \to \mathcal{O}_X(-n) \to \mathcal{F} \to \mathcal{L}(-n) \to 0.$$

Identify $\mathcal{L}(-n)$; note we have $\mathcal{O}_X = \det \mathcal{F} = \mathcal{O}_X(-n) \otimes \mathcal{L}(-n)$ so $\mathcal{L}(-n) \cong \mathcal{O}_X(n)$.
Proposition 2: If $Y \to Bun_{\mathrm{GL}_2,X}$ is a representation such that, for all n there's $y_n \in Y(\operatorname{Spec} k)$ with $y - n \mapsto \mathcal{O}_X(n) \oplus \mathcal{O}_X(-n)$ then Y cannot be of finite type. Proof: Assume we have such a Y. Then $Y \to Bun_{\mathrm{GL}_2,X}$ corresponds to a rank 2 vector bundle $\mathcal{E}$ on $Y \times X$, and $y_n \mapsto \mathcal{O}_X(n) \oplus \mathcal{O}_X(-n)$ means that $\mathcal{E}|_{y_n \times X} \cong \mathcal{O}_X(n) \oplus \mathcal{O}_X(-n)$. If Y were Noetherian then by Serre's theorem there would be $N \in \mathbb{N}$ with $\mathcal{E}(N)|_{y \times X}$ generated by global sections for all points y of Y. But $\mathcal{E}(N)|_{y_{N+1} \times X} \cong \mathcal{O}_X(-1) \oplus \mathcal{O}_X(2N+1)$, contradiction.

Theorem: (i) If X is a curve, then $Bun_{G,X}$ is smooth. (In fact, if $\dim X > 1$ then it's known $Bun_{G,X}$ is not smooth!)
(ii) If X is a smooth curve and G is reductive, then $\dim Bun_{G,X} = (\dim G)(g-1)$.

Idea of proof: (i) We want the infinitesimal lifting criterion. Let A be a strictly Henselian k-algebra, $I \subseteq A$ ideal with $I^2 = 0$. May assume $A/\mathfrak{m}_A = k$ by extending k. We are given $x_0 \in Bun_{G,X}(\operatorname{Spec} A_0)$, i.e. a G-bundle $\mathcal{P}$ on X_{A_0}. Can we lift to a point in $Bun_{G,X}(\operatorname{Spec} A)$, i.e. extend to G-bundle on X_A? Since G is smooth, a G-bundle is étale locally trivial. So G-bundles on X_{A_0} are $\mathcal{G}$-bundles on X, where $\mathcal{G} = \operatorname{Res}_{A_0/k} G_{A_0}$ and G-bundles on X_A are $\mathcal{G}'$-bundles on X, where $\mathcal{G} = \operatorname{Res}_{A/k} G_A$. Now, have

$$1 \to H \to \mathcal{G}' \to \mathcal{G} \to 1$$

where $H = \operatorname{Lie} G \otimes_k I = \operatorname{Lie} \mathcal{G}' \otimes_A I$. Now, $\mathcal{G} \to \operatorname{Out}(H) = \operatorname{Aut}(H)$ as a natural adjoint action on $\operatorname{Lie} G \otimes_k A$, and $\pi_*(\widetilde{X}) = (\widetilde{X} \times \mathcal{G})/\mathcal{G}'$.

We have a fixed $\mathcal{G}$-bundle $\mathcal{P}$ over X. Consider the stack $\mathcal{Y}$ over X, with $\mathcal{U}(U)$ the $\mathcal{G}'$-bundles $\mathcal{P}'$ on U together with $\varphi : \pi_*\mathcal{P}' \cong \mathcal{P}_U$. Does this have a point over X? Well, $\mathcal{Y}$ is a gerbe over X, i.e. has objects locally and any two objects are locally isomorphic. What is the band of the gerbe $\mathcal{Y}$? It should be a sheaf of groups $\mathcal{B}$ on $\mathbf{Aff}_X$; so given $U \to X$ what's $\mathcal{B}(U)$? Take $U' \to U$ covering with $x' \in \mathcal{Y}(U')$. Then $Isom(x', x')$is a sheaf $\mathcal{B}_{U'}$ on $\mathbf{Aff}_{U'}$, and the two pullbacks of x' on $U' \times_U U'$ are locally isomorphic, so this defines gluing data on $\mathcal{B}_{U'}$ for $U' \to U$ and allows us to descend $\mathcal{B}_{U'}$ to $\mathcal{B}_U$ on $\mathbf{Aff}_U$, unique up to unique isomorphism.

9 Lecture - 10/09/2014

Goal: X/k a smooth projective curve, G/k linear algebraic group. Stated two things last time:

1. $Bun_{G,X}$ is smooth.
2. If X is geometrically irreducible and G is reductive, then $\dim(Bun_{G,X}) = \dim(G)(g-1)$ (where g is the genus of X).

Studied this by deforming G-bundles. Let A be a strictly Henselian k-algebra, $I \subseteq A$ an ideal with $I^2 = 0$, $A_0 = A/I$. Assume WLOG (by extending k if necessary) that $A/\mathfrak{m}_A = k$. Want to prove the infinitesimal lifting criterion to prove smoothness.

Last time showed $Bun_{G,X}(\operatorname{Spec} A)$ was the set of $\mathcal{G}'$-bundles on X and $Bun_{G,X}(\operatorname{Spec} A_0)$ was the set of $\mathcal{G}$-bundles on X, where $\mathcal{G}' = \operatorname{Res}_{A/k} G_A$ and $\mathcal{G} = \operatorname{Res}_{A_0/k} G_{A_0}$. Have exact sequence

$$1 \to H \to \mathcal{G}' \to \mathcal{G} \to 1$$

where $\pi : \mathcal{G}' \to \mathcal{G}$ is the obvious thing. Moreover, we identified

$$H = \operatorname{Lie}(\mathcal{G}') \otimes_A I = \operatorname{Lie}(\mathcal{G}) \otimes_{A_0} I = \operatorname{Lie}(G) \otimes_k I.$$

Now, viewing this SES as an extension it gives a map $\mathcal{G} \to \operatorname{Out}(H) = \operatorname{Aut}(H)$ (since H is abelian), which is the adjoint representation.

So: Fix a point $x \in Bun_{G,X}(\operatorname{Spec} A_0)$ corresponding to a $\mathcal{G}$-bundle $\widetilde{X} \to X$. Define stack $\mathcal{Y}/X$ of local lifts of $\widetilde{X}$ to a $\mathcal{G}'$-bundle; namely $\mathcal{Y}(U)$ is the set of $\mathcal{G}'$-bundles $\widetilde{X}' \to U$ together with isomorphisms $(\widetilde{X}' \times \mathcal{G})/\mathcal{G}' \cong \widetilde{X}_U$. Then $\mathcal{Y}$ is a *gerbe* (i.e. lifts always exist locally, and two are locally isomorphic). Let $\mathcal{E} = (H \times \widetilde{X})/\mathcal{G}$ (quotient by the diagonal action), a vector bundle over X; claim this is the band of the gerbe.

Let $x' \in \mathcal{Y}(U)$; so x' corresponds to $\widetilde{X}' \to U$ with $(\widetilde{X}' \times \mathcal{G})/\mathcal{G}' \cong \widetilde{X}_U$. Then $\operatorname{Aut}(x')$ is a fppf sheaf on $\mathbf{Aff}_U$; what is it? Answr is that it's $\mathcal{E}|_U$. To construct the map; let $\varphi \in \operatorname{Aut}(x')(V)$, so $\varphi : \widetilde{X}'_V \to \widetilde{X}'_V$ is a $\mathcal{G}'$-bundle automorphism such that the induced isomorphism

$$(\widetilde{X}' \times \mathcal{G})/\mathcal{G}' \cong (\widetilde{X}' \times \mathcal{G})/\mathcal{G}'$$

is the identity. Fix $W \to V$ and $s \in \widetilde{X}'(W)$. Then $\varphi(s) = s \cdot g$ for some $g \in \mathcal{G}'(W)$. Also $(s,1) \sim (\varphi(s,1)$ mod $\mathcal{G}'(W)$, so there exists $h \in \mathcal{G}'(W)$ such that $(sh, \pi(h)) = (\varphi(s), 1) = (sg, 1)$. So $\varphi(h) = 1$, so $h \in H(W)$, and $g = h^{-1}$ so $g \in H(W)$ as well. If we choose a different section $t \in \widetilde{X}'(W)$ and write $t = sa$ then $\varphi(t) = \varphi(s)a = sga = t(a^{-1}ga)$; so $(\alpha(s,1), h)$ is uniquely determined by φ as an element of $((\widetilde{X} \times H)/\mathcal{G})(W)$. (Note: The action of $\mathcal{G}'$ on H factors through $\mathcal{G}$, as H is commutative).

So: $\widetilde{X}'$ has a point over a cover W of V, which gives some $v \in \mathcal{E}(W)$. By the uniqueness, we can descend to a section of $\mathcal{E}(V)$. So we get a map $\operatorname{Aut}(\varphi) \to \mathcal{E}|_V$, a map of sheaves. Can check it's a group isomorphism locally (since locally $\widetilde{X}'$ is trivial). Hence:

- For every $U \to X$, the set of isomorphism classes of objects of $\mathcal{Y}(U)$ is isomorphic to $H^1_{\text{fppf}}(U, \mathcal{E})$.
- So objects exist locally; need to know we can glue local lifts. The obstruction to lifting is an element of $H^2_{\text{fppf}}(U, \mathcal{E})$.

But $\mathcal{E}$ is a vector bundle, so $H^2_{\text{fppf}}(X, \mathcal{E}) = H^2_{\text{ét}}(X, \mathcal{E}) = H^2(X, \mathcal{E}) = 0$ because $\dim X \leq 1$. So $\mathcal{Y}(X) \neq 0$, proving smoothness.

We now move on to computing the dimension. Specialize to $A = k[\varepsilon]$, $A_0 = k$, and $x \in Bun_{G,X}(\operatorname{Spec} k)$. Then $\mathbf{C} = \mathbf{C}_{Bun_{G,X},x} = \mathcal{Y}(\operatorname{Spec} k)$ is a category in vector spaces over k as defined last time. So this corresponds to a 2-step complex of vector spaces that we need to identify. But H^0 is the set of objects of $\mathbf{C}$ modulo isomorphisms, which is $H^1(X, \mathcal{E})$. For every object x', $\operatorname{Aut}(x') = H^0(X, \mathcal{E})$, so

$$\dim_x(Bun_{G,X}) = \dim H^1(X, \mathcal{E}) - \dim H^0(X, \mathcal{E}) = -\chi(X, \mathcal{E}).$$

So need to calculate this Euler-Poincaré characteristic. If G is reductive, $\mathrm{Lie}(G) \cong \mathrm{Lie}(G)^\vee$ as a G-module, so $\mathcal{E} \cong \mathcal{E}^\vee$, so $\deg(\mathcal{E}) = 0$. Then Riemann-Roch tells us $\chi(X, \mathcal{E}) = \mathrm{rank}(\mathcal{E})(g-1) = \dim(G)(g-1)$.

This finishes the generalities on stacks. Now move onto a new topic, defining Shtuka (reference: Varshavsky's paper on "F-bundles"). The base scheme will be $\mathrm{Spec}\, k$ for k a field. We let G/k be a split connected reductive algebraic group and X/k a smooth projective geometrically irreducible curve. Since we've fixed G, X, write Bun for $Bun_{G,X}$.

Recall the Beauville-Laszlo theorem from last semester. Notation: If A is a k-algebra let $D_A = \mathrm{Spec}\, A[\![t]\!]$ and $D_A^\circ = \mathrm{Spec}\, A(\!(t)\!)$. Then $\mathbf{Vect}(D_A)$ and $\mathbf{Vect}(D_A^\circ)$ are the categories of finite-type projective modules over $A[\![t]\!]$ or $A(\!(t)\!)$, respectively.

Here we only suppose X is a smooth curve: Fix $x \in X(k)$, fix $\widehat{\mathcal{O}}_{X,x} \cong k[\![t]\!]$, get $D_A \to X_A$ and $D_A^\circ \to X_A^\circ$ where $X^\circ = X \setminus \{x\}$. Then:

Theorem (Beauville-Laszlo): For every k-algebra A, define a functor

$$Bun(\mathrm{Spec}\, A) = BG(X_A) \to \mathbf{D}$$

where $\mathbf{D}$ is the gluing data category consisting of triples $(\mathcal{P}_{X^\circ}, \mathcal{P}_{\mathcal{D}}, \beta)$ where $\mathcal{P}_{X^\circ} \in BG(X_A^\circ)$, $\mathcal{P}_D \in BG(D_A)$, and $\beta : \mathcal{P}_{X^\circ}|_{D_A^\circ} \to \mathcal{P}_D|_{D_A^\circ}$ is an isomorphism, and the functor is given by $\mathcal{P} \mapsto (\mathcal{P}|_{X_A^\circ}, \mathcal{P}|_{D_A}, \mathrm{id})$. Then this isomorphism is an equivalence of categories.

Level structures on Bun. Let N be a finite closed subscheme of X. Also, $\mathcal{P}^\circ$ always denotes the trivial G-bundle. Definition: Bun_N maps U to the set of pairs $(\mathcal{P}, \varphi)$ for $\mathcal{P} \in Bun(U)$ and $\varphi : \mathcal{P}|_{U \times N} \to \mathcal{P}^\circ_{U \times N}$. Define a group scheme $G_N : U \mapsto G(U \times N)$, which acts on the right on Bun_N by changing φ (since $G_N(U) = \mathrm{Aut}(\mathcal{P}^\circ_{U\times N})$). If $N \subseteq M$ are two levels, get a map $Bun_M \to Bun_N$; also $Bun = Bun_\emptyset$.

Fact: $Bun_M \to Bun_N$ is a torsor under the linear algebraic group $\ker(G_M \to G_N)$, so it's schematic affine. In particular Bun_N is algebraic.

Proof: Let $(\mathcal{P}, \varphi) \in Bun_N(U)$. Then $(Bun_M \times_{Bun_N} U)(V)$ is the set of $\psi : \mathcal{P}_{M \times V} \cong \mathcal{P}^\circ_{M\times V}$ such that $\psi|_{N \times V} = f^*\varphi$.

Infinite level structure: Fix $x \in X(K)$, and $\widehat{\mathcal{O}}_{X,x} \cong k[\![t]\!]$ (which fixes $D_A \to X_A$). Define $Bun_{\infty x}(\mathrm{Spec}\, A)$ to be the set of pairs $(\mathcal{P}, \varphi)$ with $\mathcal{P} \in Bun(\mathrm{Spec}\, A)$ and $\varphi : \mathcal{P}|_{D_A} \cong \mathcal{P}^\circ_{D_A}$. Also define $G[\![t]\!](\mathrm{Spec}\, A) = G(A[\![t]\!])$ and $G(\!(t)\!)(\mathrm{Spec}\, A) = G(A(\!(t)\!))$. Then $G[\![t]\!]$ acts on $Bun_{\infty x}$ as before (by changing φ: $G(A[\![t]\!]) = \mathrm{Aut}(\mathcal{P}^\circ_{D_A})$). Also, have $Bun_{\infty x} \to Bun$, and this is a $G[\![t]\!]$-torsor; moreover $Bun_{\infty x} = \varprojlim Bun_{nx}$ (i.e. for $N = nx$).

Fact: (1) $Bun_{\infty x}$ is a sheaf in equivalence relations (in fact, a scheme).
(2) The $G[\![t]\!]$-action on $Bun_{\infty x}$ extends to a $G(\!(t)\!)$ action.

Proof: (1) We have to prove that for all $a \in Bun_{\infty x}(U)$, $\mathrm{Aut}(a)(U) = \mathrm{id}_a$. Let $(\mathcal{P}, \varphi) \in Bun_{\infty x}(\mathrm{Spec}\, A)$, and $\varphi : \mathcal{P} \to \mathcal{P}$ in $BG(X_A)$ such that $\psi|_{D_K} = \mathrm{id}$. Let F be the function field of X. Then $BG(X_A) \to BG(\mathrm{Spec}\, A \times \mathrm{Spec}\, K)$ is faithful, and thus $\varphi|_{D_A^\circ} = \mathrm{id}$ (which we know) implies $\varphi|_{\mathrm{Spec}(A \otimes F)} = \mathrm{id}$.

(2) Let $(\mathcal{P}, \varphi) \in Bun_{\infty x}(\mathrm{Spec}\, A)$, $g \in G(A(\!(t)\!))$. Define a G-bundle $\mathcal{P}'$ on X_A by applying B-L theorem to the triple $(\mathcal{P}|_{X_A^\circ}, \mathcal{P}^\circ_{D_A}, g \circ \varphi)$. By construction get $\varphi' : \mathcal{P}'|_{D_A} \to \mathcal{P}^\circ_{D_A}$; take $(\mathcal{P}, \varphi) \cdot g = (\mathcal{P}', \varphi')$.

10 Lecture - 10/14/2014

Notation: k field, X/k smooth projective geometrically irreducible curve, G/k split connected reductive linear algebraic group. Write $Bun = Bun_{G,X}$, and if $N \subseteq X$ a finite closed subscheme, we have $Bun_N \to Bun$.

Study $\mathbb{F}_q$-points, in case $k = \mathbb{F}_q$: Let $F = k(x)$, so for every $v \in |X|$ have $\mathcal{O}_v = \widehat{\mathcal{O}}_{X,v} \subseteq F_v$, and surjective map $\mathcal{O}_v \twoheadrightarrow k(v)$ which is a finite extension of k. Define the adeles $\mathbb{A} = \prod'_v F_v$ with respect to the $\mathcal{O}_v$'s, and $\mathbb{O} = \prod_v \mathcal{O}_v$. Let K_N be the kernel of the projection $G(\mathbb{O}) \to G(O_N)$ where $O_N = \prod_{v \in N} \mathcal{O}_v$.

Claim: The objects of $Bun_N(\mathbb{F}_q)/\sim$ are in bijection with $G(F)\backslash G(\mathbb{A})/K_N$.

Proof: Start by constructing the claimed bijection α. Let $\mathcal{G} \in Bun_N(\mathbb{F}_q)$. Since $H^1(F, G) = 0$, have that $\mathcal{G}$ is generically trivial, so there exists U open dense and $\xi \in \mathcal{G}(U)$. Also, for all $v \in |X|$ have $H^1(k(v), G) = 0$ by Lang's theorem, so $\mathcal{G}$ has a section over $k(v)$ which lifts to a section ξ_v over $\mathcal{O}_v$ by smoothness. If $v \in |N|$ take ξ_v given by the fixed isomorphism $\varphi : \mathcal{G}|_N \to G_k$. Thus for all $v \in |X|$ have two sections $\xi, \xi_v$of $\mathcal{G}$ over $\operatorname{Spec} F_v$, so there exists a unique $g_v \in G(F_v)$ with $\xi = \xi_v \cdot g_v$. If $v \in |U|$, then ξ is defined over $\operatorname{Spec} \mathcal{O}_v$ so $g_v \in G(\mathcal{O}_v)$. So $g = (g_v) \in G(\mathbb{A})$; define $\alpha(\mathcal{G}, \varphi) = g$.

Why is α injective? If $\alpha(\mathcal{G}, \varphi) = \alpha(\mathcal{G}', \varphi')$ choose $U, \xi, \xi', \xi_v, \xi'_v, g_v, g'_v$ as before. Then by assumption there exists $\gamma \in G(F)$ and $(h_v) \in K_N$ such that $\gamma g_v h_v = g'_v$ for all v. We may assume $\gamma \in G(U)$ (just shrink U), so we may assume $\mathcal{G}|_U = \mathcal{G}'|_U$, $\xi = \xi'$ and $\gamma = 1$.

Claim: If $V \supseteq U$ is open, if $\psi : \mathcal{G}|_V \cong \mathcal{G}'|_V$ is such that $\varphi|_U = \mathrm{id}$ and $\varphi(\xi_v) = \xi'_v$ for all $v \in |V| \setminus |U|$ and ψ is compatible with φ, φ' then for all $w \in |X| \setminus |V|$ there exists $\mathcal{G}|_W \cong \mathcal{G}'|_W$ where $W = V \cup \{w\}$ extending ψ and satisfying the same conditions. Proof: Use B-L theorem to glue ψ and the isomorphisms $\mathcal{G}|_{\operatorname{Spec} \mathcal{O}_w} \to \mathcal{G}'|_{\operatorname{Spec} \mathcal{O}_w}$ coming from h_w.

Why is α surjective? Let $g = (g_v) \in G(\mathbb{A})$. Can find $U \subseteq X$ open dense such that $g_v \in G(\mathcal{O}_v)$ for all $v \in |U|$; trying to construct $(\mathcal{G}, \varphi)$ such that $\alpha(\mathcal{G}, \varphi) = g$. Over U take $\mathcal{G}|_U = G_U$ the trivial bundle, and take $\xi = 1$ and $\xi_v = g_v$ for all $v \in |U|$. Extend $\mathcal{G}$ to larger subsets one point at a time by B-L theorem.

Now, talk about truncations. Start with truncations by degree, which appeared when we proved Bun_{GL_n} was an algebraic stack. Notation: if H is an algebraic group and $\mathcal{H} \in BH(S)$, and $\varphi : H \to \mathrm{GL}(V)$ is an algebraic representation of H, let $\mathcal{H}_\varphi$ be the associated vector bundle over S (i.e. $(\mathcal{H} \times V)/H$). (Remember if we let $\varphi : H \to H'$ be a morphism of algebraic groups get a map $\varphi_* : BH \to BH'$ sending $\mathcal{H} \to (\mathcal{H} \times H')/H$). Moreover if φ is a closed immersion then for all $\mathcal{H}' \in bH'(S)$, the isomorphism classes of H-bundles $\mathcal{H}$ together with an isomorphism $\varphi_* \mathcal{H} \cong \mathcal{H}'$ is bijective with the set of sections $\mathcal{H}'/H \to S$, where $\mathcal{H}'/H = (\mathcal{H}' \times H'/H)/H'$ with action $(s, h) \cdot h' = (sh, (h')^{-1} h)$.

Fix $T \subseteq B \subseteq G$, (maximal torus, Borel), let $Z = Z(G)$. Assume that G^{der} is simply connected (mostly to simplify notation). Roots and coroots: have $X^*(T) \supseteq \Phi \subseteq \Phi^+ \supseteq \Delta$ and $X_*(T) \supseteq \Phi^\vee \supseteq \Phi^{\vee +} \supseteq \Delta^\vee$ as usual, and

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^{\vee +}} \alpha \in X_*(T).$$

Have $X_+^*(T)$ and $X_*^+(T)$ the dominant things. On $X_*(T)$ and $X^*(T)$ have the Bruhat order $\lambda_1 \leq \lambda_2$ iff $\lambda_2 - \lambda_1 \in \sum_{\alpha \in \Phi^+} \mathbb{N}\alpha$. Finally, let $\lambda_1, \ldots, \lambda_d$ be the fundamental weights of G (in $X_+^*(T)$).

For all $\lambda \in X_*^+(T)$ let λ be the corresponding irreducible representation of G. If $\mathcal{G}$ is a G-bundle write $\mathcal{G}_\lambda = \mathcal{G}_{V_\lambda}$. Now fix a parabolic subgroup $P \supseteq B$. A P-structure on a G-bundle $\mathcal{G}$ is a P-bundle $\mathcal{P}$ with an isomorphism $(\mathcal{P} \times G)/P \cong \mathcal{G}$ of G-bundles.

Lemma: Let $\mathcal{G}$ be a G-bundle on S. We have a canonical bijection between

1. Isomorphism classes of P-structures on $\mathcal{G}$
2. Families of line subbundles $\mathcal{A}_\lambda \subseteq \mathcal{G}_\lambda$ for $\lambda \in X_*^+(T) \cap X^*(P)$ such that for all λ, μ we have $\mathcal{A}_\lambda \otimes \mathcal{A}_\mu = \mathcal{A}_{\lambda+\mu}$ via the natural embedding $\mathcal{G}_{\lambda+\mu} \hookrightarrow \mathcal{G}_\lambda \otimes \mathcal{G}_\mu$.
3. Collections of line subbundles $\mathcal{A}_{\lambda_i} \subseteq \mathcal{G}_{\lambda_i}$ for all i such that the fundamental weight λ_i lives in $X^*(P)$, such that the Plücker relations are satisfied: for all $k_i \in \mathbb{N}$ then

$$\bigotimes_i (\mathcal{A}_{\lambda_i})^k \subseteq \bigotimes_i (\mathcal{G}_{\lambda_i})^k$$

is contained in $\mathcal{G}_{\sum k_i \lambda_i}$.

Proof: Map from (ii)$\to$(iii) is obvious. Map from (i)$\to$(ii): Let $\mathcal{P}$ be a P-structure on $\mathcal{G}$; if $\lambda \in X^*_+(T) \cap X^*(P)$ the highest weight line $L_\lambda \subseteq V_\lambda$ is stable by P. Take

$$\mathcal{A}_\lambda = (\mathcal{P} \times L_\lambda)/P \hookrightarrow (\mathcal{P} \times V_\lambda)/\mathcal{P} \cong \mathcal{G}.$$

(iii)$\to$(i): Let $\mathcal{A}_{\lambda_i}$ be as in (iii). Want to construct a section s of $\mathcal{G}/P$ that will give back the $\mathcal{A}_{\lambda_i}$ by our map (i)$\to$(ii). Claim such a map exists and is unique. Thanks to uniqueness can check the claim étale locally on S, so we assume $\mathcal{G}$ is trivial. Then $\mathcal{G}/P = (G/P) \times S$. So s is an S-point of G/P. We have $G/P \hookrightarrow \prod_i \mathcal{P}(V_{\lambda_i})$; each $\mathcal{A}_{\lambda_i}$ gives a S-point $x_i \in \mathcal{P}(V_{\lambda_i})$ and the family (x_i) is in the image of G/P iff the Plücker relations are satisfied.

Definition: For all $\mu \in X_*(T) \otimes \mathbb{Q}$, let $Bun_N^{\leq \mu}$ be defined by taking S to the set of $(\mathcal{G}, \varphi) \in Bun_N(S)$ such that for every geometric point s of S, for every B-structure $\mathcal{B}$ on $\mathcal{G}_{X \times \{s\}}$, and for every $\lambda \in X^*_+(T)$, we have $\deg(\mathcal{B}_\lambda) \leq \langle \lambda, \mu \rangle$.

Theorem: $Bun_N^{\leq \mu} \to Bun_N$ is a schematic open immersion. Moreover, for all $\mu \in X_*(T) \otimes \mathbb{Q}$, if $\deg(N)$ is big enough, $Bun_N^{\leq \mu}$ is a countable disjoint union of quasi-projective schemes.

Lemma: $\pi_0(Bun) \cong \pi_1(G) = X^*(Z(\widehat{G})) = X_*(T)/X_*(T \cap G^{der})$.

So, if $\nu \in \pi_1(G)$ get the connected component $Bun^v \subseteq Bun$, and then $Bun_N^{\leq \mu, \nu} = Bun_N^{\leq \mu} \cap Bun^\nu$. The second part of the theorem actually says that if $\deg(N) \gg 0$ (relative to μ) then $Bun_N^{\leq \mu, \nu}$ is a quasi-projective scheme for all ν.

Proofs: May or may not prove lemma; very similar to proof for affine Grassmannian. For the first part of the theorem, only have to consider $Bun^{\leq \mu} \to Bun$ (the level structure doesn't play any role). Let $\mathcal{G} \in Bun(S)$. Then $\mathcal{G} \in Bun^{\leq \mu}(S)$ iff for all geometric points s of S and all $1 \leq i \leq d$ and all B-structures $\mathcal{B}$ on $\mathcal{G}_s$, $\deg(\mathcal{B}_{\lambda_i}) \leq \langle \lambda_i, \mu \rangle$; call this condition (*). Let $U_i \to Bun_N$ be the substack defined by the following: $\mathcal{G} \in U_i(S)$ iff (*) is true for the fixed i. Then $Bun_{\leq \mu} = \bigcap U_i$, so it's sufficient to prove $U_i \to Bun$ is a schematic open embedding.

Fix i, let $P_i \supseteq B$ be the maximal parabolic subgroup corresponding to λ_i. For every s and $\mathcal{B}$ as above, $\mathcal{B}_{\lambda_i}$ only depends on $(\mathcal{B} \times P_i)/B$ (a P_i-structure on $\mathcal{G}_s$). Using the lemma, (*) becomes condition (**), that for every s, $(\mathcal{G}_s)_{\lambda_i}$ has no line subbundle of degree $\leq \langle \lambda_i, \mu \rangle$ satisfying the Plücker relations. This is an open condition on s. If $s_1 \to s_2$ is a specialization, if $\mathcal{L} \subseteq (\mathcal{G}_{s_1})_{\lambda_i}$ is a line subbundle, then it gives a unique line subbundle $\mathcal{L}_2$ of $(\mathcal{G}_{s_2})_{\lambda_i}$ because Grassmannians are projective. The degree conditions and the Plücker relations are closed conditions, so $\mathcal{L}_2$ satisfies them if $\mathcal{L}_1$ does.

For the second statement (about $Bun_N^{\leq \mu}$ being a countable disjoint union of quasiprojective schemes): Case $G = \mathrm{GL}_n$. Fix an ample line bundle $\mathcal{O}(1)$ on X. A B-structure on a G-bundle $\mathcal{G} \in BG(S)$ is a flag $0 = \mathcal{F}_0 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{G}$ with $\mathcal{F}_{i+1}/\mathcal{F}_i$ a line bundle for all i. The $\leq \mu$ condition involves bounding the degrees of these line bundles. Serre's theorem gives that there exists $m \in \mathbb{N}$ such that for all S and all $\mathcal{E} \in Bun^{\leq \mu}(S)$, if $p_S : X \times S \to S$ is the projection, then $R^i p_{S*}\mathcal{E}(m) = 0$ for $i \geq 0$, $p_{S*}\mathcal{E}(m)$ is a vector bundle, and the adjunction map $p_S^* p_{S*}\mathcal{E}(m) \to \mathcal{E}(m)$ is surjective.

11 Lecture - 10/16/2014

From last time: Fixed $T \subseteq B \subseteq G$ and $\mu \in X_*(T) \otimes \mathbb{Q}$. Defined a stack $\mathrm{Bun}^{\leq\mu}$ by letting $\mathrm{Bun}^{\leq\mu}(S)$ be the set of $\mathcal{G} \in \mathrm{Bun}(S)$ such that for every geometric point s of S, every B-structure $\mathcal{B}$ on $\mathcal{G}_s$, and every $\lambda \in X_+^*(T)$ we have $\deg(\mathcal{B}_\lambda) \leq \langle \mu, \lambda \rangle$.

Also, recalled we had an isomorphism $\pi_0(\mathrm{Bun}) \cong \pi_1(G) = X^*(Z(\widehat{G}))$. We let Bun^v be the connected component corresponding to $v \in \pi_1(G)$ and set $\mathrm{Bun}^{\leq\mu,v} = \mathrm{Bun}^{\leq\mu} \cap \mathrm{Bun}^v$ and $\mathrm{Bun}_N^{\leq\mu,v} = \mathrm{Bun}_N \otimes_{\mathrm{Bun}} \mathrm{Bun}^{\leq\mu,v}$.

Result we were working on last time: For all μ there exists d such that if $\deg(N) \geq d$ then $\mathrm{Bun}_N^{\leq\mu,v}$ is a quasi-projective scheme for all v.

Proof: Case $G = \mathrm{GL}_n$ with the standard Borel; then a $\mathcal{B}$-structure on $\mathcal{G} \in \mathrm{Bun}(S)$ is a flag $0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{G}$ with $\mathcal{F}_{i+1}/\mathcal{F}_i$ a line for all i. Take $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Q}^n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Then $\deg(\mathcal{B}_\lambda) \leq \langle \lambda, \mu \rangle$ iff

$$\sum_{i=0}^{k} \deg(\mathcal{F}_{i+1}/\mathcal{F}_i)\lambda_{i+1} \leq \sum_{i=0}^{k} \lambda_{i+1}\mu_{i+1}$$

for all $k \leq n-1$. If we want this to be true for every λ this is equivalent to the condition

$$\sum_{i=0}^{k} \deg(\mathcal{F}_{i+1}/\mathcal{F}_i) \leq \sum_{i=0}^{k} \mu_{i+1}$$

for every $k \leq n-1$ with equality for $k = n-1$.

Fix $\mu \in X_*(T) \otimes \mathbb{Q}$. By Serre's theorem, exists m such that for all S and all $\mathcal{E} \in \mathrm{Bun}^{\leq\mu}(S)$ we have $R^i p_{S*}\mathcal{E}(m) = 0$ for $i \geq 1$, $p_{S*}\mathcal{E}(m)$ is a vector bundle, and $p_S^* p_{S*}\mathcal{E}(m) \to \mathcal{E}(m)$ is surjective. (As usual, $p_S : X \times S \to S$ is the projection and $\mathcal{O}(1)$ is a fixed ample line bundle). (Apparently getting that this is true uniformly for all $\mathcal{E}$ is tricky and uses much more than Serre's theorem ????)

Fix such a m. If $\deg(N)$ is big enough, then for all S and $\mathcal{E}$, $p_{S*}\mathcal{E}(m) \to p_{S*}\mathcal{E}_{N\times S}(m)$ is injective and stays so after base change. So we get $p_{S*}\mathcal{E}(m) \hookrightarrow p_{S*}\mathcal{E}_{N\times S}(m) \cong p_{S*}\mathcal{O}^n_{N\times S}(m)$ (where this isomorphism comes from the level structure φ). Let $\mathcal{Y}_r$ be the stack defined by letting $\mathcal{Y}_r(S)$ be pairs (α, β) such that $\alpha : \mathcal{E}_1 \hookrightarrow p_{S*}\mathcal{O}^n_{N\times S}(m)$ where $\mathcal{E}_1$ is a rank-r vector bundle and $\operatorname{coker}\alpha$ is flat over S, and $\beta : (p_S^*\mathcal{E}_1)(-m) \twoheadrightarrow \mathcal{E}_2$ with $\mathcal{E}_2$ a rank n vector bundle. Thus have a map $\mathrm{Bun}_N^{\leq\mu} \coprod_r \mathcal{Y}_r$ by mapping

$$(\mathcal{E}, \psi) \mapsto \big(\alpha : p_{S*}\mathcal{E}(m) \hookrightarrow p_{S*}\mathcal{O}^n_{N\times S}(m), \beta : (p_S^* p_{S*}\mathcal{E}(m))(-m) \twoheadrightarrow \mathcal{E}\big).$$

Claims:

- This map $\mathrm{Bun}_N^{\leq\mu} \to \coprod \mathcal{Y}_r$ is a schematic open embedding.
- $\mathcal{Y}_r$ is a projective scheme
- If v is fixed, the image of $\mathrm{Bun}_N^{\leq\mu}$ lands in a finite union of $\mathcal{Y}_r$.

General case: Choose an injective morphism $G \to \mathrm{GL}_n$. This gives $\mathrm{Bun}_{G,N} \to \mathrm{Bun}_{\mathrm{GL}_n,N}$ which we have seen is schematic and affine (because GL_n/G is affine). Let $\mu \in X_*(T) \otimes \mathbb{Q}$; then there exists $\mu' \in X_*(T_{\mathrm{GL}_n})$ such that $\mathrm{Bun}_{G,N}^{\leq\mu} \to \mathrm{Bun}_{\mathrm{GL}_n,N}$ factors through $\mathrm{Bun}_{\mathrm{GL}_n,N}^{\leq\mu'}$.

The affine Grassmannian and the Hecke stack. Remember: A fppf sheaf of sets on $\mathbf{Aff}_k$ (what we called a k-space last semester) is an *ind-scheme* if we can write $X = \varinjlim_{i\in\mathbb{N}} X_i$ with the X_i schemes and the transition maps $X_i \to X_j$ closed immersions. We say X is of ind-finite type (or ind-affine, ind-projective) if we can take all of the X_i's to be of finite type (affine, projective); if this is true for one presentation $X = \varinjlim X_i$ then it's true for all.

Notation/examples: Arc group $G[\![t]\!] : \operatorname{Spec} A \mapsto G(A[\![t]\!])$, loop group $G(\!(t)\!) : \operatorname{Spec} A \mapsto G(A(\!(t)\!))$, affine Grassmannian $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$. Have that $G[\![t]\!]$ is a group scheme (not of finite type in general), which embeds in $G(\!(t)\!)$ that's a group ind-scheme (not of ind-finite type in general), and Gr_G is an ind-projective ind-scheme. Then $G[\![t]\!]$ acts on Gr_G on the left in the obvious way, and last year calculated the orbits are indexed by $w \in X_*^+(T)$. If $w \leftrightarrow Orb_w$ and if K/k is algebraically closed extension, then

$$Orb_w(K) = G(K[\![t]\!])t^w G(K[\![t]\!])/G(K[\![t]\!])$$

where $t^w = w(t) \in G(K(\!(t)\!))$. Also, Orb_w is a projective scheme with

$$\overline{Orb}_w = \bigcup_{w' \leq w} Orb_{w'}.$$

How was this connected to G-bundles? Fix $x \in X(k)$, identify $\widehat{\mathcal{O}}_{X,x} \cong k[\![t]\!]$, so get maps $D_A \to X_A$ (where $D_A = \operatorname{Spec} A[\![t]\!]$ is the formal disc). Let $X^\circ = X \setminus \{x\}$ so $D_A^\circ = \operatorname{Spec} A(\!(t)\!)$. We define more stacks

$$\mathrm{Gr}_G^{\mathrm{glob}} : \operatorname{Spec} A \mapsto \{(\mathcal{G}, \varphi) : \mathcal{G} \in BG(X_A), \varphi : \mathcal{G}|_{X_A^\circ} \cong G_{X_A^\circ}\},$$

$$\mathrm{Gr}_G^{\mathrm{loc}} : \operatorname{Spec} A \mapsto \{(\mathcal{G}, \varphi) : \mathcal{G} \in BG(A_A), \varphi : \mathcal{G}|_{D_A^\circ} \cong G_{D_A^\circ}\}.$$

We have maps $\mathrm{Gr}_G^{\mathrm{glob}} \to \mathrm{Gr}_G^{\mathrm{loc}}$ and $\mathrm{Gr}_G \to \mathrm{Gr}_G^{\mathrm{loc}}$ by $(\mathcal{G}, \varphi) \mapsto (\mathcal{G}|_{D_A}, \varphi|_{D_A})$ and $g \mapsto (G_{D_A}, g)$, and last year proved these were isomorphisms by the B-L theorem.

Description of Orb_w in $\mathrm{Gr}_G^{\mathrm{loc}}$. If $g \in G(A((t)))$ then $gG(A[\![t]\!])$ is in $Orb_w(A)$ if $\rho : G \to \mathrm{GL}(V_\lambda)$ then for all $\lambda \in X_+^*(T)$, we have $\rho(g) \in t^{\langle \lambda, w \rangle} \operatorname{End}(A[\![t]\!] \otimes V_\lambda)$ and moreover there exists λ such that $\rho(g) \notin t^{\langle \lambda, w \rangle + 1} \operatorname{End}(A[\![t]\!] \otimes V_\lambda)$.

Now, for all $x \in X(S) = \operatorname{Hom}(S, X)$ let $\Gamma_X \subseteq X \times S$ be the graph of x. For I a finite set and $N \subseteq X$ a closed finite subscheme, let Gr_I be the stack with $\mathrm{Gr}_I(S)$ the set of tuples $(\mathcal{G}, (x_i)_{i \in I}, \varphi)$ where $\mathcal{G} \in BG(X \times S)$, $x_i \in X(S)$, and

$$\varphi : \mathcal{G}_{X \times S \setminus \bigcup_{i \in I} \Gamma_{x_i}} \cong G|_{X \times S \setminus \bigcup_{i \in I} \Gamma_{x_i}}.$$

Also define a stack $Hecke_{N,I}$ by letting $Hecke_{N,I}(S)$ be the set of tuples $(\mathcal{G}, \varphi, \mathcal{G}', \varphi', (x_i), \varphi)$ with $(\mathcal{G}, \varphi), (\mathcal{G}', \varphi') \in \mathrm{Bun}_N(S)$, $x_i \in X(S) \setminus N(S)$, and

$$\varphi : \mathcal{G}_{X \times S \setminus \bigcup_{i \in I} \Gamma_{x_i}} \cong \mathcal{G}'|_{X \times S \setminus \bigcup_{i \in I} \Gamma_{x_i}}.$$

Let $Hecke_I = Hecke_{\emptyset, I}$, where the ψ's can be omitted. Note we could also have defined $\mathrm{Gr}_I(S)$ to be the set of $(\mathcal{G}, \mathcal{G}', (x_i), \varphi, \alpha)$ where $(\mathcal{G}, \mathcal{G}', (x_i), \varphi) \in Hecke_I(S)$ and $\alpha : \mathcal{G}' \cong G_{X \times S}$.

We have lots of maps:

- $\mathrm{Gr}_I \to Hecke_I$ by $(\mathcal{G}, (x_i), \varphi) \mapsto (\mathcal{G}, G_{X \times S}, (x_i), \varphi)$
- Two maps $Hecke_{N,I} \to \mathbf{Bun_N}$ by $(\mathcal{G}, \varphi, \mathcal{G}', \varphi', (x_i), \varphi)$ mapping to $(\mathcal{G}, \psi)$ and $(\mathcal{G}', \psi)$.
- Maps $Hecke_I, \mathrm{Gr}_i \to X^I$ by mapping things to the tuple (x_i) (compatible with the map $\mathrm{Gr}_I \to Hecke_I$).
- $Hecke_{M,I} \to Hecke_{N,I}|_{(X-M)^I}$ if $M \supseteq N$.

Note

$$Hecke_{N,I} = \mathrm{Bun}_N \times_{\mathrm{Bun}} Hecke_I|_{(X-N)^I}$$

so we'll usually work with just $Hecke_I$.

These Gr_I's and $Hecek_{N,I}$'s are not algebraic stacks in general, they're ind-algebraic stacks. But we'll deal with them by truncation just like Gr_G.

Let $\underline{\omega} = (\omega_i) \in X_*^+(T)^I$. We define substacks $\mathrm{Gr}_{I,\underline{\omega}}$ and $Hecke_{N,I,\underline{\omega}}$ of Gr_I and $Hecke_{N,I}$ like so:

- $(\mathcal{G}, \varphi, \mathcal{G}', \varphi', (x_i), \varphi)$ is in $Hecke_{N,I,\underline{\omega}}(S)$ iff for all $\lambda \in X_+^*(T)$, we have $\varphi(\mathcal{G}_\lambda) \subseteq \mathcal{G}_{\lambda'} \cdot (\sum_{i=1}^n \langle \lambda, \omega_i \rangle \Gamma_{x_i})$ - this allows poles at the points x_i. (Recall $\mathcal{G}_\lambda = \mathcal{G}_{V_\lambda}$).

- $(\mathcal{G}, (x_i), \varphi)$ is in $\mathrm{Gr}_{I,\underline{\omega}}(s)$ iff for all $\lambda \in X^*_+(T)$ have $\varphi(\mathcal{G}_\lambda) \subseteq (G_{X\times S})_\lambda \cdot (\sum_{i=1}^n \langle \lambda, \omega_i \rangle \Gamma_{x_i})$.

Note that this has nothing to do with level structure, so

$$Hecke_{N,I,\underline{\omega}} = \mathrm{Bun}_N \times_{\mathrm{Bun}} Hecke_{I,\underline{\omega}}|_{(X-N)^I}$$

12 Lecture - 10/21/2014

Last time we needed this lemma.

Lemma: Suppose that $k = \overline{k}$. Let $n > 0$ be an integer, and $\mu_1, \dots, \mu_n \in \mathbb{Q}$. Then there exists N such that for all rank n vector bundles $\mathcal{E}$ on X, if

- For all complete flags $0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_n$ with $\deg(\mathcal{E}_i) \leq \mu_1 + \cdots + \mu_i$ with equality for $i = n$, then $H^1(X, \mathcal{E}(m)) = 0$ for $m \geq N$.

(Key remark: this condition imposes a bound on the Harder-Norasimhan polynomial of $\mathcal{E}$, which was how Drinfeld and others truncated the shtukas).

Proof for $n = 2$: Suppose $\mathcal{E}$ satisfies this condition so $\deg \mathcal{E} = \mu_1 + \mu_2 = d$. There are 2 cases to consider: either $\mathcal{E}$ is semistable or not. If not then there exists a flag with $\deg(\mathcal{E}_1) > \deg(\mathcal{E}/\mathcal{E}_1)$. Then condition (*) says that if we set $\mathcal{L}_1$ and $\mathcal{L}_2 = \mathcal{E}/\mathcal{E}_1$ we have

$$\deg(\mathcal{L}_1) > \deg(\mathcal{L}_2) \geq d - \mu - 1.$$

So if $m > d - \mu_1 + C$ (for some constant C depending only on the genus of X) then $H^1(X, \mathcal{E}(m)) = 0$. Second case is when $\mathcal{E}$ is semistable; thus every subbundle has slope $\leq$ slope of $\mathcal{E}$, so in particular for every $\mathcal{L} \subseteq \mathcal{E}$ we have $\deg \mathcal{L} \leq (\deg \mathcal{E})/2$. If $H^1(X, \mathcal{E}(m)) \neq 0$ then by Serre duality $H^0(X, \mathcal{E}^\vee(-m) \otimes \omega_X) \neq \emptyset$ so $\mathcal{E}^\vee(-m) \otimes \omega_X$ has a quotient line bundle of degree 0, so $\mathcal{E}(m) \otimes \omega_X^\vee$ has a quotient of degree 0 so $\mathcal{E}$ has a quotient line bundle of degree $-m + \deg(\omega_X)$. Since semistability means every quotient line bundle has degree $\geq \deg(\mathcal{E})/2$, thus $-m + \deg(\omega_X) \geq \deg(\mathcal{E})/2$ so $m \leq \deg(\omega_X) - d/2$. So if $m > \deg(\omega_X) - d/2$ then $H^1(X, \mathcal{E}(m)) = 0$; so can take $N = \max\{\deg(\omega_X) - d/2, d - \mu_1 + C\}$ and this works to prove the $n = 2$ case.

Now, fix a finite set I and $N \subseteq X$ a finite subscheme. Remember $Hecke_{N,I}$ was the stack where an S-point was the collection of data $(\mathcal{G}, \varphi, \mathcal{G}', \varphi', (x_i), \psi)$ with $(\mathcal{G}, \varphi)$ and $(\mathcal{G}', \varphi')$ in $\mathrm{Bun}_N(S)$, the x_i's are points (indexed by I) with $x_i \in (X - N)(S)$, and ψ is an isomorphism

$$\mathcal{G}|_{X \times S \setminus \bigcup \Gamma_{x_i}} \cong \mathcal{G}'|_{X \times S \setminus \bigcup \Gamma_{x_i}}$$

compatible with the level structure in the sense $\varphi' \circ \psi = \varphi$ on $N \times S$. Also defined $\mathrm{Gr}_I(S)$ as the set of tuples $(\mathcal{G}, (x_i), \psi)$ with $\mathcal{G} \in \mathrm{Bun}(S)$, $x_i \in X(S)$, and $\psi : \mathcal{G}|_{X \times S \setminus \bigcup \Gamma_{x_i}} \cong G_{X \times S \setminus \bigcup \Gamma_{x_i}}$.

Remark: There's an involution τ of $Hecke_{N,I}$ sending

$$(\mathcal{G}, \varphi, \mathcal{G}', \varphi', (x_i), \psi) \to (\mathcal{G}', \varphi', \mathcal{G}, \varphi, (x_i), \psi^{-1}).$$

Fix $\underline{\omega} = (\omega_i)$ with $\omega_i \in X_*^+(T)$. Define $Hecke_{N,I,\underline{\omega}}$ and $\mathrm{gr}_{I,\underline{\omega}}$ by requiring the following. Recall that for $\lambda \in X_+^*(T)$ we have the highest-weight representation V_λ and then for any $\mathcal{G}$ can set $\mathcal{G}_\lambda = \mathcal{G}_{V_\lambda}$. Then we say $(\mathcal{G}, \varphi, \mathcal{G}', \varphi', (x_i), \psi) \in Hecke_{N,I,\underline{\omega}}(S)$ iff for all λ,

$$\psi(\mathcal{G}_\lambda) \subseteq \mathcal{G}'_\lambda \left(\sum_{i=1}^n \langle \lambda, \omega_i \rangle \Gamma_{x_i} \right).$$

Similarly $(\mathcal{G}, (x_i), \psi \in \mathrm{Gr}_{I,\underline{\omega}}$ iff for all λ,

$$\psi(\mathcal{G}_\lambda) \subseteq G_\lambda \left(\sum_{i=1}^n \langle \lambda, \omega_i \rangle \Gamma_{x_i} \right).$$

Remark: The Drinfeld case is when $G = \mathrm{GL}_n$, $|I| = 2$, and $\omega_1 = (1, 0, \dots, 0)$, $\omega_2 = (0, \dots, 0, -1)$.

Remark: The involution τ sends $Hecke_{N,I,\underline{\omega}}$ to $Hecke_{N,I,-w_0(\underline{\omega})}$ where w_0 is the longest element of the Weyl group. Why? Assume WLOG $N = \emptyset$, get

$$\psi(\mathcal{G}_\lambda) \subseteq \mathcal{G}'_\lambda(\sum \langle \lambda, \omega_i \rangle \Gamma_{x_i})$$

iff

$$(\mathcal{G}')^\vee_\lambda(-\sum \langle \lambda, \omega_i\rangle \Gamma_{x_i}) \subseteq \psi(\mathcal{G}_\lambda)^\vee$$

iff

$$\psi^{-1}(\mathcal{G}'_{-w_0(\lambda)} \subseteq \mathcal{G}_{-w_0(\lambda)}(-\sum \langle \lambda, \omega_i\rangle \Gamma_{x_i})$$

because $V_\lambda^\vee = V_{-w_0(\lambda)}$, and noting $\langle \lambda, \omega_i\rangle = \langle -w_0(\lambda), -w_0(\omega_i)\rangle$ get the desired relation.

Proposition: For all $\underline{\omega}$, $Hecke_{N,I,\underline{\omega}}$ is an algebraic stack locally of finite type. Moreover, $\mathrm{Gr}_{I,\underline{\omega}}$ is a scheme locally of finite type. In fact, the map $Hecke_{N,I,\underline{\omega}} \to Bun_N \times (X-N)^I$ given by mapping a point of the Hecke stack to $(p', (x_i))$ is schematic projective, and by the remark so is the map $(p, (x_i))$.

Proof: We'll just do the fist statement. Assume $N = \emptyset$. Want to prove $Hecke_{I,\underline{\omega}} \to \mathrm{Bun} \times X^I$ is schematic projective. Let $V = \bigoplus_{i=1}^d V_{\lambda_i}$, where the λ_i's are the fundamental weights, which is a faithful representation of G. Fix N such that for all $i \in I$ and all weights λ of V, $-N \leq \langle \omega_i, \lambda\rangle \leq N$. Let $Hecke'$ be a stack $S \mapsto (\mathcal{G}, \mathcal{G}', (x_i), \psi) \in Hecke_I(S)$ with

$$\mathcal{G}'_V\bigg(-N\sum_i \Gamma_{x_i}\bigg) \subseteq \psi(\mathcal{G}_v) \subseteq \mathcal{G}'_V\bigg(N\sum_i \Gamma_{x_i}\bigg).$$

First note $Hecke_{I,\underline{\omega}} \subseteq Hecke'$ by definition and remarks about τ. Claim that $Hecke_{I,\underline{\omega}} \to Hecke'$ is a schematic closed embedding. Let $(\mathcal{G}, \mathcal{G}', (x_i), \psi) \in Hecke'(S)$. Then $(Hecke_{I,\underline{\omega}} \times_{Hecke'} S)(f : S' \to S)$ is $*$ if $\varphi(f^*\mathcal{G}_\lambda) \subseteq f^*\mathcal{G}'_\lambda(\sum \langle \lambda, w_i\rangle \Gamma_{x_i}$ for all λ, and $\emptyset$ otherwise. But this happens iff

$$\varphi(f^*\mathcal{G}_{\lambda_j}) \subseteq f^*\mathcal{G}'_{\lambda_j}(\sum \langle \lambda_j, w_i\rangle \Gamma_{x_i}$$

for all i,j. Let $\mathcal{F} = \mathcal{G}'_V(N\sum_i \Gamma_{x_i}) = \bigoplus_{j=}^d \mathcal{G}_{\lambda_j}(N\sum \Gamma_{x_i})$; then $\mathcal{E} = \bigoplus \psi(\mathcal{G}_{\lambda_j})$ and $\mathcal{E}' = \bigoplus \mathcal{G}'_{\lambda_j}(\sum \langle \lambda_i, w_i\rangle \Gamma_{x_i})$ embed in $\mathcal{F}$, and the fiber product we're looking at is the closed subscheme where $\mathcal{E} \subseteq \mathcal{E}'$.

So if we know $Hecke'$ is good then this will imply $Hecke$ is good. How do we understand $Hecke'$? Move to $Hecke''$, the stack with $Hecke''(S)$ having points $(\mathcal{E}, \mathcal{G}', (x_i), \psi)$ with $(\mathcal{G}', (x_i)) \in (\mathrm{Bun} \times X^I)(s)$, $\mathcal{E}$ is a rank $\dim V$ vector bundle on $S \times X$, and $\psi : \mathcal{E}|_{X\times S\setminus \bigcup \Gamma_{x_i}} \cong \mathcal{G}'_V|_{X\times S\setminus\bigcup\Gamma_{x_i}}$.

We have $Hecke' \to Hecke''$ given by $(\mathcal{G}, \ldots) \mapsto (\mathcal{G}_V, \ldots)$. Now, claim that $Hecke'' \to \mathrm{Bun} \times X^I$ is schematic projective. This follows because $(Hecke'' \times_{Bun\times X^I} S)(f : S' \to S)$ is the set of pairs $(\mathcal{E}, \psi)$ with $\mathcal{E}$ a rank $\dim V$ vector bundle on $X \times S'$ with $\psi' : \mathcal{E}|_{X\times S'\setminus\bigcup\Gamma_i} \cong f^*\mathcal{G}'|_{X\times S'\setminus\bigcup\Gamma_i}$ satisfying the usual properties. But this is the same as rank $\dim V$ vector bundles $\mathcal{E}$ on $X \times S'$ with

$$f^*\mathcal{G}'_V\left(-N\sum \Gamma_{x_i}\right) \subseteq \mathcal{E} \subseteq f^*\mathcal{G}'_V\left(N\sum\Gamma_{x_i}\right).$$

This is a Grassmannian over S'.

Finally, need to prove $Hecke' \to Hecke''$ is a schematic closed embedding. Proof: Let $(\mathcal{E}, \mathcal{G}', (x_i), \psi) \in Hecke''(S)$. Then if $Z = Hecke' \times_{Hecke''} S$, we have $Z(S')$ is $*$ if there exists $\mathcal{G} \in \mathrm{Bun}(S')$ with $\mathcal{G}_V \cong \mathcal{E}_{X\times S'}$ and $\mathcal{G} = \mathcal{G}'$ on $X \times S' \setminus \bigcup \Gamma_{x_i}$, and $\emptyset$ otherwise. So $Z(S')$ is $*$ iff the section of $\mathcal{E}/\mathcal{G}$ over $X \times S' \setminus \bigcup\Gamma_{x_i}$ given by ψ extends to $X \times S'$ (and this extension is necessarily unique by separatedness). In fact, $\mathcal{E}/\mathcal{G} \to X \times S$ is affine (because G is reductive so $\mathrm{GL}(V)/G$ is affine). So we just need the following:

Lemma. Let S be a scheme, let $x_i \in X(S)$ for $i \in I$, set $U = X \times S \setminus \bigcup \Gamma_{x_i}$, let $s \in \gamma(U, \mathcal{O}_{X\times S})$. Consider the sheaf $\mathcal{F}$ given by sending $f : S' \to S$ to $*$ if f^*s extends to $X \times S'$ and $\emptyset$ otherwise. Then $\mathcal{F} \to S$ is a schematic closed embedding.

Proof: Let $t \in \Gamma(X \times S, j_*\mathcal{O}_U/\mathcal{O}_{X\times S})$ be the image of s, and $t_i = t|_{\Gamma_{x_i}}$. Identify Γ_{x_i} with S by the projection $X \times S \to S$. Then $\mathcal{F} \to S$ is the inclusion of $\bigcap\{t_i = 0\}$.

Corollary: Let $\mu \in X_*(T)$ and $v \in \pi_1(G)$. Then

$$Hecke^{\leq\mu;v}_{N,I,\underline{\omega}} = Hecke_{N,I,\underline{\omega}} \times_{\mathrm{Bun}_N} \mathrm{Bun}_N^{\leq \mu, v}$$

is an algebraic stack of finite type, and even a scheme if $\deg(N) \gg 0$ (relative to μ).

Shtukas. Now take $k = \mathbb{F}_q$. For all $S \in \mathbf{Sch}_{\mathbb{F}_q}$, let $\mathrm{Frob}_S : S \to S$ be the map taking $f \in \mathcal{O}_S$ to f^q. If $\mathcal{G} \in \mathrm{Bun}(S)$ write ${}^\tau\mathcal{G} = (\mathrm{id}_X \times \mathrm{Frob}_S)^*\mathcal{G}$. (If $\mathcal{Y}$ is any stack and $a \in \mathcal{Y}(S)$, let ${}^\tau a = \mathrm{Frob}_S^* a$).

Definition: Let $Cht_{N,I}$ be the stack with S-points $(\mathcal{G}, \varphi, (x_i), \psi)$ with $(\mathcal{G}, \varphi) \in \mathrm{Bun}_N(S)$, $x_i \in X(S)$, and ψ an isomorphism of $\mathcal{G}$ with ${}^\tau\mathcal{G}$ on $X \times S \setminus \bigcup \mathcal{G}_{x_i}$ with ${}^\tau\psi \circ \varphi|_{N \times S} = \psi$. Thus

$$Cht_{N,I} = Hecke_{N,I} \times_{\mathrm{Bun}_N \times \mathrm{Bun}_N} \mathrm{Bun}_N$$

where the map $Hecke_{N,I} \to \mathrm{Bun}_N \times \mathrm{Bun}_N$ is the obvious one and the map $\mathrm{Bun}_N \to \mathrm{Bun}_N \times \mathrm{Bun}_N$ is (id, τ). We have an obvious map $Cht_{N,I} \to \mathrm{Bun}_N \times (X - N)^I$ by dropping the ψ. Can also define

$$Cht^{\leq \mu; v}_{N,I,\underline{\omega}} = Hecke^{\leq \mu, v}_{N,I,\underline{\omega}} \times_{\mathrm{Bun}_N \times \mathrm{Bun}_N} \mathrm{Bun}_N\,.$$

In particular this this is an algebraic stack of finite type. Also have maps $Cht_{M,I} \to Cht_{N,I}$ if $N \subseteq M$.

Proposition: The map

$$Cht_{M,I} \to Cht_{N,I} \times_{X^I} (X \setminus M)^I$$

is schematic, finite étale and Galois of group $G_{M,N}(\mathbb{F}_q)$ with $G_{M,N} = \ker(G_M \to G_N)$. Moreover, $Cht^{\leq \mu; v}_{N,I,\underline{\omega}}$ is Deligne-Mumford of finite type, and a scheme for $\deg(N) \gg 0$ (relative to μ). If $\omega_i = 0$ for all i, then $Cht_{N,I,\underline{\omega}} = \mathrm{Bun}_N(\mathbb{F}_q) \times (X \setminus N)^I$ with the Bun_N a discrete stack.

13 Lecture - 10/23/2014

Recall we defined the stack of shtuka $Cht_{N,I} = Hecke_{N,I} \times_{\mathrm{Bun}_N \times \mathrm{Bun}_N} \mathrm{Bun}_N$, where the first map to $\mathrm{Bun} \times \mathrm{Bun}$ is (p, p') and the second is (id, τ) where if $\mathcal{G}$ is a bundle on $X \times S$ then ${}^\tau\mathcal{G} = (\mathrm{id}_X \times \mathrm{Frob}_S)^*\mathcal{G}$. More concretely, $Cht_{N,I}(S)$ is the set of collections of $(\mathcal{G}, \varphi) \in \mathrm{Bun}_N(S)$ with $(x_i)_{i\in I} \in X(S)^I$ and $\psi : \mathcal{G} \cong {}^\tau\mathcal{G}$ away from $\bigcup \Gamma_{x_i}$ with ${}^\tau\psi \circ \varphi = \psi$. (Recall a level structure is an isomorphism $\prod : \mathcal{G}|N \times S \cong G_{N\times S}$).

Also had various truncations

$$Cht^{\leq \mu, v}_{N,I,\underline{\omega}} = Hecke^{\leq \mu, v}_{N,I,\underline{\omega}} \times_{\mathrm{Bun}_N \times \mathrm{Bun}_N} \mathrm{Bun}_N\,.$$

which is an algebraic stack of finite type. (Here, $\underline{\omega} \in X_*^+(T)^I$, $\mu \in X^*(T) \otimes \mathbb{Q}$, $v \in \pi_1(G)$). If $N \subseteq M$ get $Cht_{M,I} t \circ Cht_{N,I}$; also have maps $Cht_{N,I} \to \mathrm{Bun}_N \times (X \setminus N)^I$.

Proposition: (a)

$$Cht_{M,I} \to Cht_{N,I} \times_{X^I} (X - M)^I$$

is finite étale Galois of group $G_{M,N}(\mathbb{F}_q)$, where $G_{M,N} = \ker(G_M \to G_N)$ with $G_M(A) = G(A \otimes \mathcal{O}_M)$ which we can abuse notation to write as $G_M = \mathrm{Res}_{M/\mathbb{F}_q} G_M$.

(b) $Cht_{N,I,\underline{\omega}}$ is a Deligne-Mumford stack locally of finite type; in fact each $Cht^{\leq \mu, v}_{N,I,\underline{\omega}}$ is the quotient of a quasiprojective scheme by a finite group, and actually is a scheme if $\deg(N) \gg 0$ relative to μ.

(c) If $\omega_i = 0$ then $Cht_{N,I,\underline{\omega}} = \mathrm{Bun}_N(\mathbb{F}_q) \times (X \setminus N)^I$ (with $\mathrm{Bun}_N(\mathbb{F}_q)$ treated as a discrete stack).

(d) $Cht_{N,I,\underline{\omega}} \neq \emptyset$ iff $\sum \omega_i \in X_*(T \cap G^{der})$.

Remark: $\underline{\omega}$ is the highest weight of an irreducible representation of $\widehat{G}^I$, and $\sum \omega_i \in X_*(T \cap G^{der})$ iff the diagonally-embedded $Z(\widehat{G}) \hookrightarrow \widehat{G}^I$ acts trivially on the representation. Drinfeld case: $G = \mathrm{GL}_n$, $\widehat{G} = \mathrm{GL}_n$, $|I| = 2$, with the representation we use is $st \otimes st^\vee$ where st is the standard representation of GL_n.

Lemma: Let $\mathcal{X}/\mathbb{F}_q$ be an algebraic stack locally of finite type, and let $\mathcal{Y}$ be the stack

$$S \mapsto \{(A, \psi) : A \in \mathcal{X}(S), \psi : A \cong {}^\tau A\}$$

(a) $\mathcal{Y}$ is a DM stack, étale over $\mathrm{Spec}\,\mathbb{F}_q$, and $\mathcal{X}(\mathbb{F}_q) \subseteq \mathcal{Y}$ is a schematic open and closed embedding.

(b) If moreover $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ has connected geometric fibers then $\mathcal{Y} = \mathcal{X}(\mathbb{F}_q)$.

Proof: The general case isn't too hard with some stack theory, but we'll just consider the two cases we need (in which case (b) applies): (i) $\mathcal{X}$ is a scheme and (ii) $\mathcal{X} = BG$ with G connected.

Case (i): Obviously $\mathcal{Y} = \mathcal{X}(\mathbb{F}_q)$ (since $\mathcal{Y}(S) = X(S)^{\mathrm{Frob}_S}$). Case (ii): We want to show $\mathcal{Y} = B(G(\mathbb{F}_q))$ for $\mathcal{X} = BG$. Get natural map $B(G(\mathbb{F}_q))(S) \to \mathcal{Y}(S)$; and there's a bijective correspondence between $\mathcal{G} \in BC(S)$ with $c \in H^1_{\text{ét}}(S, \mathcal{G})$, such that ${}^\tau\mathcal{G} = \mathrm{Frob}_G(c)$. (What is c? Choose covering family $\{U_i\}$ of S such that $\mathcal{G}|_{U_i}$ is trivial, $\varphi_i : \mathcal{G}|_{U_i} \cong G_{U_i}$. Take c to come from $c_{ij} = \varphi_i \circ \varphi_j^{-1} \in G(U_i \times_S U_j)$). If ${}^\tau\mathcal{G} \leftrightarrow c'$ then $c'_{ij} = c_{ij} \circ \mathrm{Frob} = \mathrm{Frob}_G \circ c_{ij}$ so $c' = \mathrm{Frob}_G(c)$ as needed.

To finish the proof we want: If $c \in H^1_{\text{ét}}(S, G)$ is such that $c = \mathrm{Frob}_G(c)$ then c is in the image of $H^1_{\text{ét}}(S, G(\mathbb{F}_q))$. Indeed: Choose (U_i) and c_{ij} as before. Then $c = \mathrm{Frob}_G(c)$ iff there exists $h_i \in G(U_i)$ such that $h_i c_{ij} h_j^{-1} = \mathrm{Frob}_G(c_{ij})$. Lang's theorem says $g \mapsto \mathrm{Frob}_G(g)^{-1} g$ is surjective since G is connected. So up to going to another cover, we have $g_i \in G(U_i)$ such that $h_i = \mathrm{Frob}_G(g_i)^{-1} g_i$. Then $g_i c_{ij} g_j^{-1} = \mathrm{Frob}_G(g_i c_{ij} g_j^{-1})$, i.e. $g_i c_{ij} g_j^{-1} \in G(\mathbb{F}_q)$.

Proof of proposition part (a): It is enough to show that $Cht_{N,I} \to Cht_I|_{(X\setminus N)^I}$ is finite étale Galois of group $G_N(\mathbb{F}_q)$. Let $\mathcal{Y}(S)$ be the set of $(\widetilde{\mathcal{G}}, \widetilde{\psi})$ such that $\widetilde{\mathcal{G}} \in BG(N \times S)$, $\widetilde{\psi} : \widetilde{\mathcal{G}} \to {}^\tau\widetilde{\mathcal{G}}$. By the lemma, $\mathcal{Y}_N = B(G_N(\mathbb{F}_q))$. We have a map $Cht_{N,I} \to \mathcal{Y}_N$ by $(\mathcal{G}, \varphi, (x_i), \psi) \mapsto (\mathcal{G}|_{N\times S}, \psi|_{N\times S})$. Now claim that the map

$$Cht_{N,I} \to (Cht_{N,I} \times_{X^I} (X \setminus N)^I) \times_{\mathcal{Y}_N} \mathrm{Spec}\,\mathbb{F}_q$$

given by $((\mathcal{G}, \varphi), (x_i), \psi) \mapsto (\mathcal{G}, (x_i), \psi)$ is well-defined and an isomorphism. So $Cht_{N,I} \to Cht_I \times (X \setminus N)^I$ is a $G_N(\mathbb{F}_q)$-bundle.

(b) Fix $\mu \in X_*(T)$. If $\deg(N) \gg 0$ then $\mathrm{Bun}_N^{\leq \mu, v}$ and $Hecke^{\leq \mu, v}_{N,I,\underline{\omega}}$ are quasi-projective schemes, so $Cht^{\leq \mu, v}_{N,I,\underline{\omega}}$ is also one. In general, $Cht^{\leq \mu, v}_{N,I,\underline{\omega}}$ is a quotient of $Cht^{\leq \mu, v}_{M,I,\underline{\omega}}$ for M big enough, by a finite group $G_{M,N}(\mathbb{F}_q)$, and by (a) the quotient map is étale.

(c) Suppose $\omega_i = 0$ for all i. We want $Cht_{N,I,\underline{\omega}} = \mathrm{Bun}_N(\mathbb{F}_q) \times (X \setminus N)^I$. But note that if $((\mathcal{G}, \varphi), (x_i), \psi) \in Cht_{N,I,\underline{\omega}}$ then for all $\lambda \in X^*_+(T)$ have $\varphi(\mathcal{G}_\lambda) = \tau\mathcal{G}_\lambda$ so ψ extends to $\mathcal{G} \cong {}^\tau\mathcal{G}$. So $Cht_{N,I,\underline{\omega}} \cong Cht_{N,\emptyset} \times (X \setminus N)^I$. We want $Cht_{N,\emptyset} = \mathrm{Bun}_N(\mathbb{F}_q)$.

Fix μ, v. Then if $M \supseteq N$ has big enough degree $Cht^{\leq \mu, v}_{M,\emptyset}$ and $\mathrm{Bun}^{\leq \mu, v}_M$ are schemes, and $Cht^{\leq \mu, v}_{M,\emptyset} = \mathrm{Bun}^{\leq \mu, v}_M(\mathbb{F}_q)$. This implies the result for $Cht^{\leq \mu v}_{N,\emptyset}$, and we use that $Cht_{N,\emptyset}$ is the union of these over all μ, v.

(d) Claim: $Cht_{N,I,\underline{\omega}} \neq \emptyset$ iff $\sum \omega_i \in X_*(T \cap G^{der})$. If this stack is nonempty: Take $(\mathcal{G}, \varphi, (x_i), \psi) \in Cht_{N,I,\underline{\omega}}(S)$. Then if $\lambda \in X^*(G)$, λ, λ^{-1} are dominant so the condition on $\psi(\mathcal{G}_\lambda)$ and ${}^\tau\mathcal{G}_\lambda$ becomes $\psi(\mathcal{G}_\lambda) = {}^\tau\mathcal{G}_\lambda(\sum \langle \lambda, \omega_i \rangle \Gamma_{x_i})$. Let s be a geometric point of S and take degrees:

$$\deg(\mathcal{G}_{\lambda,s}) = \deg(\mathcal{G}_{\lambda,s}) + \sum_{i \in I} \langle \lambda, \omega_i \rangle.$$

So this sum is zero, so $\sum \omega_i \in X_*(T \cap G^{der})$. Conversely, if we assume $\sum \omega_i$ is in this space then $Cht_{N,I,\Omega}$ contains $\mathrm{Bun}(\mathbb{F}_q) \times (X \setminus N)$.

14 Lecture - 11/04/2014

Local models for $Cht_{N,I,\underline{\omega}}$. As always: $X/\mathbb{F}_q$ a smooth projective curve with $X_{\overline{\mathbb{F}}_q}$ connected, and $G/\mathbb{F}_q$ a split connected reductive group. I a finite set.

Local version of $\mathrm{Gr}_{G,I}$: previously we defined this by letting $\mathrm{Gr}_{G,I}(S)$ be the set of tuples $(\mathcal{G},(x_i),\psi)$ with $\mathcal{G} \in BG(X \times S)$, $x_i \in X(S)$, and

$$\psi : \mathcal{G}|_{X\times S\setminus \bigcup \Gamma_{x_i}} \cong G_{X\times S\setminus \bigcup \Gamma_{x_i}}.$$

Notation: Suppose $(x_i)_{i\in I}$ is in $X(S)^I$. If $\mathcal{J}$ is the ideal of $\bigcup \Gamma_{x_i}$ in $\mathcal{O}_{X\times S}$, let

$$\Gamma_{\sum \infty x_i} = \operatorname{Spec} \varprojlim_n \mathcal{O}_{X\times S}/\mathcal{J}^n.$$

If $(n_i) \in \mathbb{N}^I$, let $\Gamma_{\sum n_i x_i}$ be the closed subscheme which is Zariski locally defined by the equation $\prod t_i^{n_i}$ where t_i is a local equation for $\Gamma_{x_i} \subseteq X \times S$.

More notation: Let $G_{\sum n_i x_i} \to X^I$ be the smooth group scheme with S-points consisting of tuple $(x_i; g_i)$ with $(x_i) \in X^I(S)$ and $(g_i) \in G(\Gamma_{\sum n_i x_i})$. Similarly define $G_{\sum \infty x_i} \to X^I$ (relative version of $G[\![t]\!]$).

Other description of $\mathrm{Gr}_{G,I}$: tuples $(\mathcal{G},(x_i),\psi)$ with $(x_i) \in X^I(S)$, $\mathcal{G} \in BG(\Gamma_{\sum\infty x_i})$, and ψ a trivialization of $\mathcal{G}$on $\Gamma_{\sum \infty x_i} \setminus \bigcup \Gamma_{x_i}$. Have obvious map from original $\mathrm{Gr}_{G,I}$ to this, and by B-L theorem these stacks are isomorphic. But this one is easier to work with in many cases: for instance in this description it is obvious that $\mathrm{Gr}_{G,I}$ has a left action of $G_{\sum\infty x_i}$ (by changing ψ). Also, have $\mathrm{Gr}_{I,\underline{\omega}} \hookrightarrow \mathrm{Gr}_I$ with $\mathrm{Gr}_{I,\underline{\omega}}$ closed subscheme stable under the action and Gr_I the inductive limits.

Lemma: Let $\underline{\omega} \in X_*^+(T)^I$. If the n_i's are big enough then the action of $G_{\sum\infty x_i}$ on $\mathrm{Gr}_{I,\underline{\omega}}$ factors through $G_{\sum n_i x_i}$.

Proof: Have $(\mathcal{G},(x_i),\psi)$ is in $\mathrm{Gr}_{I,\underline{\omega}}(S)$ iff for all $\lambda \in X_+^*(T)$, we have $\psi(\mathcal{G}_\lambda) \subseteq G_\lambda(\sum \langle \lambda, \omega_i\rangle \Gamma_{x_i})$.

Next: Fix $\underline{\omega}$, and fix (n_i) as in the lemma. Remember $Cht_I(S)$ consisted of $(\mathcal{G},(x_i),\psi)$ with $\mathcal{G} \in \mathrm{Bun}(S)$ and ψ an isomorphism of $\mathcal{G}$ with ${}^\tau\mathcal{G}$ on $X \times S \setminus \bigcup \Gamma_{x_i}$. Moreover this element is in $Cht_{I,\underline{\omega}}(S)$ iff for all $\lambda \in X_+^*(T)$, we have $\psi(\mathcal{G}_\lambda) \subseteq {}^\tau\mathcal{G}_\lambda(\sum \langle\lambda,\omega_i\rangle\Gamma_{x_i})$. Want to define a map

$$\varepsilon_{I,\underline{\omega}} : Cht_{I,\underline{\omega}} \to G_{\sum_i x_i}\backslash \mathrm{Gr}_{I,\underline{\omega}}.$$

This is defined by descent. If $(\mathcal{G},(x_i),\psi) \in Cht_{I,\underline{\omega}}(S)$, and if I has $\beta : {}^\tau\mathcal{G}|_{\Gamma_{\sum\infty x_i}} \cong G_{\sum\infty x_i}$, then send $(\mathcal{G},(x_i),\psi)$ to

$$(\mathcal{G}|_{\Gamma_{\sum\infty x_i}}, (x_i), \beta\circ\psi) \in \mathrm{Gr}_{I,\underline{\omega}}(S).$$

Moreover the element of x in $G_{\sum n_i x_i}\backslash \mathrm{Gr}_{I,\underline{\omega}}$ does not depend on β. While a β may not exist globally, it does exist locally on S, so can go to a cover, define the image, and descend back.

Proposition: $\varepsilon_{I,\underline{\omega}}$ is smooth of relative dimension $\dim(G_{\sum n_i x_i}/X^I)$.

Proof: Let

$$\mathcal{Y} = Cht_{I,\underline{\omega}} \times_{G_{\sum n_i x_i}\backslash \mathrm{Gr}_{I,\underline{\omega}}} \mathrm{Gr}_{I,\underline{\omega}}.$$

Then $\mathcal{Y} \to Cht_{I,\underline{\omega}}$ is a $G_{\sum n_i x_i}$-torsor (though not really relevant for us); and we want to show that $\pi_2 : \mathcal{Y} \to \mathrm{Gr}_{I,\underline{\omega}}$ is smooth of relative dimension d. Can identify $\mathcal{Y}(S)$ with the set of tuples $(\mathcal{G},(x_i),\psi,\beta)$ with $(\mathcal{G},(x_i),\psi) \in Cht_{I,\underline{\omega}}$ and $\beta : {}^\tau\mathcal{G}|_{\sum n_i x_i} \cong G_{\sum n_i x_i}$.

Let $z = (\mathcal{G},(x_i),\varphi) \in \mathrm{Gr}_{I,\underline{\omega}}(S)$. What is the fiber $\mathcal{Y}' = \mathcal{Y} \times_{\mathrm{Gr}_{I,\underline{\omega}}} S$, with the map $S \to \mathrm{Gr}_{I,\underline{\omega}}$ coming from z? Well, $\mathcal{Y}'(S' \to S)$ is the set of $(\mathcal{F},(x_i),\chi,\psi,\alpha)$ with $\mathcal{F} \in \mathrm{Bun}(S')$, χ an isomorphism of $\mathcal{F}$ with ${}^\tau\mathcal{F}$ on $X \times S \setminus \bigcup \Gamma_{x_i}$, ψ is an isomorphism of $\mathcal{F}$ with G on $\Gamma_{\sum n_i x_i}$, and $\alpha : \mathcal{F}|_{\Gamma_{\sum\infty x_i}} \cong \mathcal{G}$ such that for all

$$\widetilde{\psi} : {}^\tau\mathcal{F}|_{\Gamma_{\sum\infty x_i}} \cong G_{\Gamma_{\sum\infty x_i}}$$

extending ψ (locally defined) we have $\widetilde{\psi}\circ\chi = \varphi\circ\alpha$.

Now, $\mathcal{Y}'$ is also the equalizer of the maps

$$F_1, F_2 : \mathrm{Bun}_{G,\sum n_i x_i} \times_{X^I} S \to \mathrm{Bun}_G \times S,$$

where F_1 is the obvious projection (forgetting level structure) and F_2 is given as follows. If $(\mathcal{F}, \psi) \in \mathrm{Bun}_{G,\sum n_i x_i}(S')$ and if $\widetilde{\psi} : \mathcal{F}|_{\Gamma_{\sum \infty x_i}} \cong G_{\Gamma_{\sum \infty x_i}}$ extending ψ, we call a G-bundle "$\mathcal{F}^\vee$ on $X \times S$" by gluing $\mathcal{F}|_{X \times S' \setminus \bigcup \Gamma_{x_i}}$ and $\mathcal{G}|_{\Gamma_{\sum \infty x_i}}$ via $\varphi^{-1} \circ \widetilde{\psi}$ (by B-L theorem). This does not depend on choice of $\widetilde{\psi}$, so descends to a G-bundle on $X \times S'$ which we call $a_2(\mathcal{F}, \psi)$. Now F_2 is defined by $(\mathcal{F}, \psi, s) \mapsto (a_2({}^\tau\mathcal{F}, \psi), s)$.

Let $\mathcal{E}$ be the equalizer of these maps; then $\mathcal{E}(S')$ consists of $(\mathcal{F}, \psi, (x_i), s)$ with $\mathcal{F} \in \mathrm{Bun}(S')$, $\psi : \mathcal{F} \cong G$ on $\Gamma_{\sum n_i x_i}$, plus $\mathcal{F} \cong a_2({}^\tau(\mathcal{F}, \psi))$ i.e. $\widetilde{\psi}$ extending ${}^\tau\psi$m and an isomorphism of flied bundles... Basically it turns out to be the same thing as $\mathcal{Y}$!

Situation: we have two maps $F_1, F_2 : Y \to X$. We know F_1 is smooth of relative dimension $\dim(G_{\sum n_i x_i}/X^I)$, F_2 has differential 0, and X, Y are smooth. So the equalizer is the pullback of $\Delta : Y \to Y \times Y$ and $Y \times_X Y \to Y \times Y$ (with all of the things smooth), so it's sufficient to check that the intersection is transversal. Check this on tangent spaces of $(y, y) \in Y \times Y$.

15 Lecture - 11/06/2014

The geometric Satake correspondence and fusion. Here: k is a field, X/k a smooth curve, G/k a split connected reductive group, ℓ a prime not the characteristic of k. Fix $G \supseteq B \supseteq T$ and thus $X_*^+(T)$, $X_+ * (T)$, etc., as usual. Will focus on the case of the affine Grassmannian over a point to start.

Reminder of stuff from last year: Let Y/k be a finite type scheme, H a finite type connected affine group schemes, and if $a : H \times Y \to Y$ is a (left) action of H on Y, then a $\overline{\mathbb{Q}}_\ell$-perverse sheaf K on Y is called *H-equivariant* if there exists an isomorphism $a^*K \cong p^*K$ where $p : H \times Y \to Y$ is the projection. (Note that this is not the general definition! This one only works for H connected, and is "morally wrong", but is simpler for our case so we'll use it).

Define $\mathbf{Perv}_H(Y)$ as the full abelian subcategory of H-equivariant perverse sheaves on Y in $\mathbf{Perv}_H(Y)$. (Again, we're "morally wrong" for not requiring H-equivariance of the morphism, but the case of H connected that condition turns out to be automatic. So we're really just treating H-equivariance as an extra property of the perverse sheaf rather than an additional structure). Some other comments:

- If H acts transitively on Y and if the stabilizers of points are connected, then $\mathbf{Perv}_H(Y)$ is equivalent to the category of constant sheaves on Y, given by mapping a constant sheaf $\mathcal{L}$ to $\mathcal{L}[\dim Y]$.
- What if Y is an ind-scheme? If H is an affine group scheme acting on Y, say the action is *good* if we can write $Y = \varinjlim Y_n$ where each Y_n is a finite-type scheme and the action of H on Y_n factors through a finite-type quotient $H \twoheadrightarrow H_n$. Then we define

$$\mathbf{Perv}_H(Y) = \varinjlim \mathbf{Perv}_{H_n}(Y_n).$$

  (This does not depend on the choices).
- There is a relative version we'll need; if $Y \to S$ is a S-scheme (or S-ind-scheme) and $H \to S$ is an S-group scheme (with connected fibers, for simplicity) with a relative action $a : H \times_S Y \to Y$, and if $p : H \times_S Y \to Y$ is the second projection, then we define $\mathbf{Perv}_H(Y)$ similarly (using this a and this p). Need this for dealing with the relative affine Grassmannian.

Main Example: $Y = \mathrm{Gr}_G = G(\!(t)\!)\backslash G[\![t]\!]$, and $H = G[\![t]\!]$ acting by left multiplication. Proved last year that this action was good; this will basically be the only example we care about. Write $\mathbf{Sat}(G) = \mathbf{Perv}_{G[\![t]\!]}(\mathrm{Gr}_G)$, the Satake category. Studied this last semester, and showed:

- The $G[\![t]\!]$-orbits on Gr_G are given by $\mathcal{O}_\mu = G[\![t]\!]t^\mu G[\![t]\!]/G[\![t]\!]$ for $\mu \in X_*^+(T)$ (for $t^\mu = \mu(t) \in G(k(\!(t)\!))$); this is the Cartan decomposition. These orbits are finite-dimensional, smooth, simply connected. Calculated $\dim \mathcal{O}_\mu = \langle 2\rho, \mu\rangle$ for $2\rho = \sum \alpha$ as α runs over all positive roots.
- The closure $\overline{\mathcal{O}}_\mu$ is $\bigcup_{\mu' \leq \mu} \mathcal{O}_{\mu'}$. This is projective but not smooth in general.
- The simple objects of $\mathbf{Sat}(G)$ are the intersection complexes IC_μ for $\mu \in X_*^+(T)$, where

$$IC_\mu = j_{\mu !*}(\overline{\mathbb{Q}}_{\ell,\mathcal{O}_\mu}[\dim \mathcal{O}_\mu]).$$

  (Here $j_\mu : \mathcal{O}_\mu \to \overline{\mathcal{O}}_\mu$ is the inclusion, and $j_{\mu !*}$ is the intermediate extension discussed last year).
- There is a convolution (or fusion) product $*$ on $\mathbf{Sat}(G)$ making it into a Tannakian category (more on it later). The functor $\omega : \mathbf{Sat}(G) \to \overline{\mathbb{Q}}_\ell\text{-}\mathbf{Vect}$ given by $K \mapsto \bigoplus_{i\in\mathbb{Z}} H^i(\mathrm{Gr}_{G,\overline{k}}, k)$ is a fiber functor. So $\mathbf{Sat}(G)$ is actually a neutral Tannakian category.
- If $k = \overline{k}$ then the Tannakian group of $\mathbf{sat}(G)$ is the dual group $\widehat{G}$ of G over $\overline{\mathbb{Q}}_\ell$. That is, we have an equivalence of categories $S : \mathbf{Rep}_{\widehat{G}} \to \mathbf{Sat}(G)$ that is an exact tensor functor (sends tensor product of representations to fusion product) and sends the natural fiber functor $\mathbf{Rep}_{\widehat{G}} \to \overline{\mathbb{Q}}_\ell\text{-}\mathbf{Vect}$ to ω. Finally,

it matches up irreducible objects in the obvious way: if $\mu \in X_*^+(T)$ and V_μ is the corresponding irreducible representation of $\widehat{G}$, then $S(V_\mu) = IC_\mu$. In particular $\mathbf{Sat}(G)$ is a semisimple abelian category.

- For general k, $\mathbf{Sat}(G)$ is not always semisimple, but we do know that the IC_μ are defined over k. So get an exact fully faithful tensor functor $S : \mathbf{Rep}_{\widehat{G}} \to \mathbf{Sat}(G)$, but there can be nontrivial extensions in $\mathbf{Sat}(G)$ (preventing it from being an equivalence). (Note: It's actually possible to identify the Tannakian group in this case; it's basically ${}^L G$).

Goal of this chapter : Let I be a finite set. We have the relative affine Grassmannian $\mathrm{Gr}_I \to X^I$. Let $\mathcal{G}_I = G_{\sum \infty x_i} \to X^I$ be our relative version of $G[\![t]\!]$ (a relative group scheme). This acts on the left on Gr_I (over X^I). Note:

1. If we write $I = I_1 \sqcup I_2$ as a disjoint union of subsets, then define
$$X^{I_1} \mathring{\times} X^{I_2} \subseteq X^{I_1} \times X^{I_2} = X^I$$
as the set of pairs $(x_i; y_j)$ such that $\bigcup_{i \in I_1} \Gamma_{x_i}$ and $\bigcup_{j \in I_2} \Gamma_{x_j}$ are disjoint. Then:
$$\mathrm{Gr}_I |_{X^{I_1} \mathring{\times} X^{I_2}} \cong (\mathrm{Gr}_{I_1} \times \mathrm{Gr}_{I_2})|_{X^{I_1} \mathring{\times} X^{I_2}}.$$
2. If $\zeta : I \to J$ is a map of finite sets, let $\Delta_\zeta : X^J \to X^I$ be the map $(x_j) \mapsto (x_{\zeta(i)})$. Then we have $\mathrm{Gr}_I \times_{X^I} X^J \cong \mathrm{Gr}_J$ (obvious from writing down the definition).
3. We have similar statements for $\mathcal{G}_I$. (All of this is what we call the "factorizable structure", which if we did formally requires compatibility conditions, but those are obvious in our case).

What we want to do is construct a faithful exact $\overline{\mathbb{Q}}_\ell$-linear functor $\mathbf{Rep}_{\widehat{G}^I} \to \mathbf{Perv}_{\mathcal{G}_I}(\mathrm{Gr}_I)$ (denoted $W \mapsto S_{W,I}$) such that:

- If $|I| = 1$ then for all $x \in X(k)$ have $S_{W,I}|_{\mathrm{Gr}_{I,\{x\}}} = S_W$ (since $\mathrm{Gr}_{I,\{x\}} \cong \mathrm{Gr}_G$; this says our functor extends geometric Satake on the fibers).
- For all $W \in \widehat{G}^I$, $S_{I,W}$ is supported on $\bigcup_{\underline{\omega}} \mathrm{Gr}_{I,\underline{\omega}}$ where $\underline{\omega} \in X_*^+(T)^I$ runs over all weights of W.
- $S_{W,I}$ is ULA (universally locally acyclic) with respect to $\mathrm{Gr}_I \to X^I$. (We'll define what this means later, when we actually need to use it).
- If $I = I_1 \sqcup I_2$ and $W = W_1 \boxtimes W_2$ for $W_i \in \mathbf{Rep}_{\widehat{G}^{I_i}}$, then we have canonical isomorphisms
$$S_{W,I}|_{\mathrm{Gr}_I |_{X^{I_1} \mathring{\times} X^{I_2}}} \cong (S_{W_1,I_1} \boxtimes S_{W_2,I_2})|_{(\mathrm{Gr}_{I_1} \times \mathrm{Gr}_{I_2})|_{X^{I_1} \mathring{\times} X^{I_2}}}.$$
- If $\zeta : I \to J$ is a map of finite sets, we have $\Delta_\zeta : X^J \to X^I$ as before, and define $\zeta^* : \widehat{G}^J \to \widehat{G}^I$ given by $(g_j) \mapsto (g_{\zeta(i)})$ and a functor $\mathbf{Rep}_{\widehat{G}^I} \to \mathbf{Rep}_{\widehat{G}^J}$ given by $W \mapsto W^\zeta$ (which is W with the $\widehat{G}^J$-action via ζ^*). Then we have a canonical isomorphism
$$S_{W,I}|_{\mathrm{Gr}_I \times_{X^I} X^J} \cong S_{W^\zeta, J}.$$

The case $|I| = 1$: Write $\mathrm{Gr}_1 = \mathrm{Gr}_{\{1\}}$ and $\mathcal{G}_1 = \mathcal{G}_{\{1\}}$. We want a functor $\mathbf{Rep}_{\widehat{G}} \to \mathbf{Perv}(\mathcal{G}_1)(\mathrm{Gr}_1)$ which we write $W \mapsto S_{W,1}$. Remark: last year we did this for $X = \mathbb{A}^1$; in this case choosing a coordinate on X gives an isomorphism $\mathcal{G}r_1 \cong \mathrm{Gr}_G \times X$, so we could take $S_{W,1} = S_W \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1]$. Want to generalize this construction.

Let Aut be the connected affine group scheme over k given by letting
$$\mathrm{Aut}(R) = \mathrm{Aut}_{R\text{-alg,cont}}(R[\![t]\!]) \cong R[\![t]\!]^\times$$

(with the latter isomorphism given by $\alpha \mapsto \alpha(t)/t$). Then Aut acts on $G((t))$ and $G[\![t]\!]$ (recalling $G((t)) = \mathrm{Hom}(R((t)), G)$ and letting us send $f \in G(R((t)))$ to $f(\alpha(t))$ or maybe $f(\alpha^{-1}(t))$ - which sign works out isn't important for us). So Aut acts on Gr_G. The results of Section 1 imply that every $K \in \mathbf{Sat}(G)$ is Aut-equivariant.

Now let $\mathcal{X} : \mathbf{Aff}_K \to \mathbf{Set}$ be given by

$$\operatorname{Spec} R \mapsto \{(x, s) : x \in X(R), s : \widehat{\mathcal{O}}_{X_R,x} \cong R[\![t]\!] \text{ continuous } R\text{-algebra homomorphism}\}.$$

Obviously have $\mathcal{X} \to X$ given by $(x, s) \mapsto x$, and this is an Aut-torsor. Let $q : \mathrm{Gr}_G \times \mathscr{X} \to \mathrm{Gr}_1$ sends $(\mathcal{G}, \varphi) \times (x, s)$ to $(s^*\mathcal{G}, s^*\varphi)$. Then q is an Aut-torsor, where $\alpha \in \mathrm{Aut}$ sends $(\mathcal{G}, \varphi) \times (x, s)$ to $((\mathcal{G}, \alpha^* \circ \varphi), (x, \alpha^{-1} \circ s))$. (Again, up to a choice of signs that needs to be checked).

So we get two Aut-torsors, $q : \mathrm{Gr}_G \times \mathcal{X} \to \mathrm{Gr}_1$ and $(\mathrm{id}, \pi) : \mathrm{Gr}_G \times \mathcal{X} \to \mathrm{Gr}_G \times X$. If $K \in \mathbf{Sat}(G)$ we have that

$$K \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1] \in \mathbf{Perv}(\mathrm{Gr}_G \times X)$$

is $G[\![t]\!]$ and Aut equivariant, so

$$p^*(K \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1]) \in \mathbf{Perv}(\mathrm{Gr}_G \times \mathcal{X})$$

is equivariant by $G[\![t]\!]$ acting on Gr_G, Aut acting on Gr_G, and Aut acting on $\mathcal{X}$. So there exists a unique $L \in \mathbf{Perv}(\mathrm{Gr}_1)$ such that $q^*L = p^*(K \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1])$, and L is $\mathcal{G}_1$-equivariant. (Remark: $\mathcal{X}$ isn't a finite-type scheme so strictly speaking this doesn't make sense, but we do everything on the level of finite-type quotients like we did with convolution products last year).

So we define: If $W \in \mathbf{Rep}_{\widehat{G}}$ define $S_{W,1} \in \mathbf{Perv}(\mathrm{Gr}_1)$ as the unique perverse sheaf such that

$$q^* S_{W,1} = p^*(S_W \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1]).$$

16 Lecture - 11/11/2014

Remember: $\mathbf{Sat}(G) = \mathbf{Perv}_{G[\![t]\!]}(\mathrm{Gr}_G)$. Geometric Satake says there is a fully faithful exact tensor functor $S : \mathbf{Rep}_{\widehat{G}/\overline{\mathbb{Q}}_\ell} \to \mathbf{Sat}(G)$ (an equivalence if $k = \overline{k}$), such that if $\mu \in X_*^+(T)$ then $S(V_\mu) = IC_{\overline{\mathcal{O}}_\mu}$.

Goal: Relativize this; if I is a finite set have $\mathrm{Gr}_I \to X^I$ the relative affine Grassmannian, and $\mathcal{G}_I \to X^I$ the relative version of $G[\![t]\!]$, which acts on Gr_I. We want an exact functor $\mathbf{Rep}_{\widehat{G}^I} \to \mathbf{Perv}_{\mathcal{G}_I}(\mathrm{Gr}_I)$ that takes W to $S_{W,I}$ in a way that's "compatible with the factorizable structure" and such that if $|I| = 1$ then for all $x \in X(k)$ we have $(S_{W,I})_x = S_W$ on $\mathrm{Gr}_{I,x} \cong \mathrm{Gr}_G$.

Last time for $|I| = 1$: If $X = \mathbb{A}^1$ we recalled that choosing a coordinate on X gives $\mathrm{Gr}_1 \cong \mathrm{Gr}_G \times X$, and then take $S_{W,1} = S_W \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1]$ for all $W \in \mathbf{Rep}_{\widehat{G}}$. Then showed that this construction is canonical and works fppf locally on X and glues.

Remark: The most general statement of geometric Satake (for $k = \overline{k}$ and $|I| = 1$) is that we have an equivalence S_1 of $\widehat{G}$-$\mathbf{Perv}(X)$ (the categories of perverse sheaves on X with a $\widehat{G}$-action; note $\mathbf{Rep}_{\widehat{G}}$ embeds in this) with the category of $K \in \mathbf{Perv}_{\mathcal{G}_1}(\mathrm{Gr}_1)$ such that K is "universally locally acyclic" with respect to $\pi : \mathrm{Gr}_1 \to X$. To construct this, let $R = k[\widehat{G}]$ with its two $\widehat{G}$-actions. As a $\widehat{G}$-module with the first action, $k[\widehat{G}]$ is the union of its finite-dimensional sub-$\widehat{G}$-modules. Set

$$S_{R,1} = \varinjlim_{V \subseteq k[\widehat{G}], V \in \mathbf{Rep}_{\widehat{G}}} S_{V,1},$$

an ind-perverse sheaf on Gr_1. The second action of $\widehat{G}$ on R gives an action of $\widehat{G}$ on $S_{R,1}$. By the algebraic Peter-Weyl theorem, for any $V \in \mathbf{Rep}_{\widehat{G}}$ have $V \cong \mathrm{Hom}_{\widehat{G}}(V^*, k[\widehat{G}]) \cong (V \otimes k[\widehat{G}])^{\widehat{G}}$. So, for all $V \in \mathbf{Rep}_{\widehat{G}}$, have

$$S_{V,1} = (V \otimes S_{R,1})^{\widehat{G}} = (V_{\mathrm{Gr}_1} \otimes S_{R,1})^{\widehat{G}}.$$

So we define $S : \widehat{G}\text{-}\mathbf{Perv} \to \mathbf{Perv}_{\mathcal{G}_1}(\mathrm{Gr}_1)$ by

$$S(K) = (\pi^* K[-1] \otimes S_{R,1})^{\widehat{G}}.$$

However, we won't need this general form, just the easier special case we've been working on.

The case $|I| = 2$: Write $\mathrm{Gr}_2 = \mathrm{Gr}_I$, $\mathcal{G}_2 = \mathcal{G}_I$, $X^2 = X^I$. We have the diagonal map $\Delta : X \hookrightarrow X^2$ and $j : \mathring{X}^2 = X^2 \setminus \Delta(X) \to X^2$. Have a commutative diagram with Cartesian squares:

$$\begin{tikzcd}
\mathrm{Gr}_1 \arrow[r, hook, "\Delta"] \arrow[d] & \mathrm{Gr}_2 \arrow[d] & (\mathrm{Gr}_1 \times \mathrm{Gr}_1)|_{\mathring{X}^2} \arrow[l, hook', "j"'] \arrow[d] \\
X \arrow[r, hook, "\Delta"'] & X^2 & \mathring{X}^2 \arrow[l, hook', "j"]
\end{tikzcd}$$

Last year we saw (at least if $X = \widehat{A}^1$):

Theorem: Let $V, W \in \mathbf{Rep}_{\widehat{G}}$. Then we have canonical isomorphisms

$$\left(\Delta^* j_{!*}(S_{V,1} \boxtimes S_{W,1})|_{\mathring{X}^2}\right)[-1] = \left(\Delta^! j_{!*}(S_{V,1} \boxtimes S_{W,1})|_{\mathring{X}^2}\right)[-1] = S_{V \otimes W,1}.$$

Since X is locally $\mathbb{A}^1$ can reduce the general case to this result from last year.

Remark: More generally $K, L \in \mathbf{Perv}_{\mathcal{G}_1}(\mathrm{Gr}_1)$ are universally locally acyclic with respect to $\mathrm{Gr}_1 \to X$, the isomorphism

$$\left(\Delta^* j_{!*}(K \boxtimes L)|_{\mathring{X}^2}\right)[-1] = \left(\Delta^! j_{!*}(K \boxtimes L)|_{\mathring{X}^2}\right)[-1]$$

and use this to define a convolution product $K * L$.

Remark: If $X = \mathbb{A}^1$ and we fix an isomorphism $\mathrm{Gr}_1 \cong \mathrm{Gr}_G \times X$ and if $K, L \in \mathbf{Sat}(G)$, showed last year that $(K * L) \boxtimes \overline{\mathbb{Q}}_\ell[1]$ is equal to

$$\Delta^* j_{!*}\big((K \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1]) \boxtimes (L \boxtimes \overline{\mathbb{Q}}_{\ell,X}[1])|_{\mathring{X}^2}\big)[-1].$$

In fact we can use this to define $K * L$.

If $W = W_1 \boxtimes W_2 \in \mathbf{Rep}_{\widehat{G}^2}$, we want $S_{W,2} \in \mathbf{Perv}_{\mathcal{G}_2}(\mathrm{Gr}_2)$ to be such that

$$j^* S_{W,2} \cong \big(S_{W_1,1} \boxtimes S_{W_2,2}\big)|_{\mathring{X}^2}.$$

and $\Delta^* S_{W,2} = S_{W,1}[1]$ if we make W into a representation of $\widehat{G}$ via the diagonal map $\widehat{G} \to \widehat{G}^2$. There's an obvious candidate to do this:

$$S_{W,2} = j_{!*}(S_{W_1,1} \boxtimes S_{W_2,1}|_{\mathring{X}^2}).$$

The General Case : Induct on $|I|$. First, assuming $S_{\cdot,J}$ is constructed for all J with $|J| < |I|$, prove the following:

Theorem: Let $\zeta : I \twoheadrightarrow J$, and let $\Delta_\zeta : X^J \to X^I$ be the map $(x_j) \mapsto (x_{\zeta(i)})$. Let $U_J \subseteq \prod_{j \in J} X^{\zeta^{-1}[j]}$ be the set of tuples $((x_i)_{i \in \zeta^{-1}[j]})_{j \in J})$ such that if $j \neq j'$ and $i \in \zeta^{-1}[j]$ and $i' \in \zeta^{-1}[j']$ then $x_i \neq x_{i'}$. Let $j_\zeta : U_J \to X^I$ be the open embedding. We have a diagram of Cartesian squares

$$\begin{tikzcd}
\mathrm{Gr}_1 \arrow[r, hook, "\Delta_\zeta"] \arrow[d] & \mathrm{Gr}_2 \arrow[d] & \big(\prod_{j \in J} \mathrm{Gr}_{\zeta^{-1}[j]}\big)|_{U_J} \arrow[l, hook', "j_\zeta"'] \arrow[d] \\
X^J \arrow[r, hook, "\Delta_\zeta"'] & X^I & U_J \arrow[l, hook', "j_\zeta"]
\end{tikzcd}$$

Let $k = |I| - |J|$, let $W_j \in \mathbf{Rep}_{\widehat{G}\zeta^{-1}[j]}$ for all $j \in J$, and suppose we have canonical isomorphisms (assuming $|\zeta^{-1}[j]| < |J|$):

$$\Delta_\zeta^* j_{\zeta !*} \left(\boxtimes_{j \in J} S_{W_j, \zeta^{-1}[j]}|_{U_j}\right)[-k] = \Delta_\zeta^! j_{\zeta !*} \left(\boxtimes_{j \in J} S_{W_j, \zeta^{-1}[j]}|_{U_j}\right)[-k] \cong S_{\bigotimes_J W_j, J}.$$

Then: If $W = \boxtimes_{i \in I} W_i \in \mathbf{Rep}_{\widehat{G}^I}$, we can define

$$S_{W,I} = j_{\zeta,!*}(\boxtimes_{i \in I} S_{W_i,1})|_{U_J}$$

for ζ the unique map $I \twoheadrightarrow \{1\}$, and extend to $\mathbf{Rep}_{\widehat{G}^I}$ by additivity.

Proof: Omitted; just extends what we talked about before the obvious way.

Perverse sheaves on $Cht_{N,I}$ .If I is a finite set and $W \in \mathbf{Rep}_{\widehat{G}^I}$, let $Cht_{N,I,W} = \bigcup_{\underline{\omega}} Cht_{N,I,\underline{\omega}}$ where $\underline{\omega}$ runs over elements of $X_*^+(T)^I$ that are weights of W, and $\mathrm{Gr}_{I,W} = \bigcup_{\underline{\omega}} \mathrm{Gr}_{I,\underline{\omega}}$ for the same set of $\underline{\omega}$ (so this is the support of $S_{W,I}$). Fix W. Then we have seen that if $(n_i) \in \mathbb{N}^I$ are big enough, we have a smooth morphism

$$\varepsilon_{I,W} : Cht_{N,I,W} \to \mathrm{Gr}_{I,W} / G_{\sum n_i x_i}$$

which has relative dimension $\dim(G_{\sum n_i x_i}/X^I)$. We set

$$\mathcal{F}_{N,I,W} = \varepsilon_{I,W}^*(S')[\cdots]$$

where the twist is by whatever we need to make this perverse on the fibers of $\mathbf{Cht}_{N,I,W} \to X^I$. Here S' is the unique sheaf on $\mathrm{Gr}_{I,W} / G_{\sum n_i x_i}$ that you get by descending $S_{W,I}$ (which we know is equivariant by $\mathcal{G}_I$ and thus by its image $G_{\sum n_i x_i}$).

Technical annoyance to deal with lack of finite type hypotheses: Let $Z = Z(G)$. Then $Z(\mathbb{A})$ acts on $Cht_{N,I,(W)}^{(\leq \mu)}$, and $Z(F)$ acts trivially. Choose a lattice $\Xi \subseteq Z(\mathbb{A})/Z(F)$. Then $Cht_{N,I,W}^{\leq \mu}/\Xi$ is a DM stack of finite type - same as you need to choose auxiliary things like this in automorphic forms to make certain volumes finite. (Or: Take G semisimple, where $\Xi = 1$ works).

Definition: Let $p_{N,I,W} : Cht_{N,I,W} \to X^I$ be the natural map. Then for every $\mu \in X_*(T)$, set

$$\mathcal{H}^{\leq\mu}_{N,I,W} = R^0 p_{N,I,W,!}\big(\mathcal{F}_{N,I,W}|_{Cht^{\leq\mu}_{N,I,W}/\Xi}\big).$$

Right now this is a perverse sheaf on X^I.

Crucial property of these: let $\zeta : I \to J$ be a map, let $W \in \mathbf{Rep}_{\widehat{G}^I}$, let W^ζ be the corresponding object of $\mathbf{Rep}_{\widehat{G}^J}$ (taking the diagonal action via ζ). Remember we have $\Delta_\zeta : X^J \to X^I$. Then there is a canonical (i.e. functorial in W) isomorphism

$$\chi_\zeta : \Delta_\zeta^*(\mathcal{H}^{\leq\mu}_{N,I,W}) \cong \mathcal{H}^{\leq\mu}_{N,J,W}.$$

Why is this? This follows from proper base change, and the canonical isomorphism between $\mathcal{F}_{N,I,W}$ and the pullback of $\mathcal{F}_{N,J,W^\zeta}$ by the map

$$Cht_{N,J} = Cht_{N,I} \times_{(X\setminus N)^I} (X \setminus N)^J \to Cht_{N,I}|_{\mathrm{img}\,\Delta_\zeta}$$

given by the similar isomorphism for $S_{W,I}$ and $S_{W^\zeta,J}$.

Remark: If W is irreducible of highest weight $\underline{\omega} \in X_*^+(T)^I$, then $\mathcal{F}_{N,I,W}$ on $Cht_{N,I,\underline{\omega}}/\Xi$ is just the intersection complex (with trivial coefficients, suitably shifted).

17 Lecture - 11/13/2014

Goal for today: Start proof of main theorem, assuming a bunch of facts that we'll need to prove later on. From last time, we constructed a family of (exact) functors $\mathbf{Rep}_{\widehat{G}^I} \to \mathbf{Perv}(X^I)$ given by $W \mapsto \mathcal{H}^{\leq\mu}_{N,I,W}$ (for parameters I a finite set, N a level, $\mu \in X_*(T)$, and depending on Ξ a lattice in $Z(F)\backslash Z(\mathbb{A})$). These functors satisfy the property that, for all $\zeta : I \to H$ we have a functorial isomorphism (in W)

$$\chi_\zeta : \Delta_\zeta^* \mathcal{H}^{\leq\mu}_{N,I,W} \cong \mathcal{H}^{\leq\mu}_{N,J,W^\zeta}.$$

Where we used Geometric Satake was in making this functorial!

Hecke correspondences : (A) Let $\pi : Z \to Y$ be a finite étale map (for Z, Y irreducible). Then it induces $\pi^* : IC_Y \to \pi_* IC_Z$ and $\mathrm{tr}_\pi : \pi_* IC_Z \to IC_Y$ such that $\mathrm{tr}_\pi \circ \pi^*$ equals multiplication by $\deg(\pi)$ on IC_Y.

Proof of this: Choose $j : V \hookrightarrow Y$ open dense and such that V_{red} is smooth. Write $U = \pi^{-1}[V] \hookrightarrow Z$. Do everything over V, U where the intersection complex is just the constant sheaf. Then have $\pi^* : \mathbb{Q}_V \to \pi_* \mathbb{Q}_U = \pi_* \pi^* \mathbb{Q}_V$ that's an adjunction map, and $\mathrm{tr}_\pi : \pi_* \mathbb{Q}_U = \pi_! \pi^! \mathbb{Q}_V \to \mathbb{Q}_v$ is also an adjunction map. Then the composite is multiplication by $\deg(\pi)$ by SGA4 XVII or XVIII (standard property of trace map). Want to extend this to intersection complexes; use definition $IC_Y = j_{!*}(\mathbb{Q}_V[d])$ for $d = \dim Y = \dim V = \dim Z$, and $IC_Z = j'_{!*}(\mathbb{Q}_U[d])$. Use that $\pi_* = j'_{!*} = j_{!*}$ (recall $j_{!*}$ is the image of $PH^0(j^!) \to PH^0(j^*)$ in the perverse sense), using that $\pi_* j'_! = j_!$ and $\pi_* j'_* = j_*$, and that π_* is exact (in the usual sense and the perverse sense).

(B) If there's two finite étale maps $\pi_1 : Y \to Z$ and $\pi_2 : Y \to Z$, they give a map $\mathrm{tr}_{\pi_2} \circ \pi_1^* : IC_Z \to IC_Z$, hence an endomorphism of the intersection cohomology of Z. The Hecke correspondences will show up in this way.

(C) Come back to the moduli stack of shtuka. Fix $\underline{\omega} \in X_*^+(T)^I$ a dominant weight for $\widehat{G}^I$ and look at $Cht_{N,I,\underline{\omega}}$. Let $K_N = \ker(G(\mathcal{O}) \to G(\mathcal{O}_N))$ as earlier in the semester. Then $K_N = \prod_{v \in |X|} K_{N,v}$ with $K_{N,v} = \mathcal{O}_v$ for $v \notin N$. Recall we have

$$Hecke_N = C_c(K_N \backslash G(\mathbb{A}) / K_N, \overline{\mathbb{Q}}_\ell)$$

which is an algebra by the convoluton product; want to make this act on

$$\varinjlim_{\mu} \mathcal{H}^{\leq\mu}_{N,I,W}$$

where W is the representation with highest weight $\underline{\omega}$. Let $g = (g_v) \in G(\mathbb{A})$, and choose $T \subseteq |X|$ a finite set of places such that for all $v \notin T$ we have $g_v \in K_{N,v}$, and assume that $T \cap N \neq \emptyset$ (to "concreteify" the definition, which will be enough for us).

We'll define the action of the characteristic function $1_{K_n g K_n} \in Hecke_N$. Write

$$K_N g K_N = \prod_{v \notin T} K_{N,v} \times \prod_{v \in T} G(\mathcal{O}_v) \varpi_v^{\lambda_v} G(\mathcal{O}_v)$$

for $\varpi_v \in \mathcal{O}_v$ a uniformizer and $\lambda_v \in X_*^+(T)$. Let $\Gamma_N(g)$ be the stack sending S to the category of pairs of a tuple $(x_i) \in (X \setminus |N| \setminus T)^I(S)$ and a commutative diagram

$$\begin{tikzcd}
(\mathcal{G}', \psi') \arrow[r, "\varphi"] \arrow[d, "\kappa"'] & ({}^\tau\mathcal{G}', {}^\tau\psi') \arrow[d, "\kappa"] \\
(\mathcal{G}, \psi) \arrow[r, "\varphi"'] & ({}^\tau\mathcal{G}, {}^\tau\psi)
\end{tikzcd}$$

where the horizontal arrows are objects of $Cht_{N,I,\underline{\omega}}(S)$, and κ is an isomorphism $\mathcal{G}|_{(X\setminus T)\times S)} \cong \mathcal{G}'|_{(X\setminus T)\times S}$ such that $\psi' \circ \kappa = \psi$ (this is where we use our assumption that $T \cap |N| = \emptyset$) and such that for every place $v \in T$ the relative position of $\mathcal{G}$ and $\mathcal{G}'$ at v is equal to λ_v, i.e.: for all geometric points $s \to S$, if we choose trivializations of $\varphi : \mathcal{G} \to {}^\tau\mathcal{G}$ and $\varphi' : \mathcal{G}' \to {}^\tau\mathcal{G}'$ over $D \times s \cong \operatorname{Spec} k(s)[\![t]\!]$ (where D is the formal disc around v, and these trivializations are unique up to $G(\mathcal{O}_v)$) then

$$\kappa : \mathcal{G}|_{(D\setminus\{v\})\times S} \cong \mathcal{G}'|_{(D\setminus\{v\})\times S}$$

defines an element of $G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v)$ which we require to be $\lambda_v(\varpi_v)$.

We then have two maps $\pi_1, \pi_2 : \Gamma_N(g) \to Cht_{N,I,\underline{\omega}}$ with π_2 taking the top horizontal arrow in the diagram and π_1 the bottom horizontal arrow. Claim: π_1, π_2 are finite étale. This is because there exists $\kappa \in X_*(T)$ (depending on g) such that for all μ, we have

$$\pi_2[\pi_1^{-1}[Cht^{\leq\mu}_{N,I,\underline{\omega}}]] \subseteq Cht^{\mu+\kappa}_{N,I,Uom}.$$

So we get a map $\mathcal{F}^{\leq\mu}_{N,I,\underline{\omega}} \to \mathcal{F}^{\leq\mu}_{N,I,\underline{\omega}}$. Taking perverse direct images, we get

$$\varinjlim_{\mu} \mathcal{H}^{\leq\mu}_{N,I,\underline{\omega}} \to \varinjlim_{\mu} \mathcal{H}^{\leq\mu}_{N,I,\underline{\omega}}$$

which is by definition the action of $1_{K_N g K_N} \in Hecke_N$. We can extend this to $\varinjlim_\mu \mathcal{H}^{\leq\mu}_{N,I,\underline{\omega}}$ for any $W \in \mathbf{Rep}_{\widehat{G}^I}$ by linearity. Claim: This defines an action of $Hecke_N$, i.e. sends convolution to convolution.

A particular case : If $W = 1$ is the trivial representation (i.e. $I = \emptyset$) then we saw that $Cht_{N,I,1}$ is the discrete stack $\operatorname{Bun}_N(\mathbb{F}_q) = G(F)\backslash G(\mathbb{A})/K_N$. So $\varinjlim \mathcal{H}^{\leq\mu}_{N,I,1}$ is the constant sheaf with value

$$C_c^\infty(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell),$$

and the action of $Hecke_N$ is the usual one (by convolution).

The Hecke-finite part : Let $\Delta : X \hookrightarrow X^I$ be the diagonal embedding. Let $\overline{\eta}$ be the generic geometric point of X. Fix N, W as before. Then we say

$$u \in \left(\varinjlim_{\mu} \mathcal{H}^{\leq\mu}_{N,I,W}\right)_{\Delta(\overline{\eta})}$$

is Hecke-finite if there exists a finite-dimensional $\overline{\mathbb{Q}}_\ell$-subspace V such that $u \in V$ and such that v is stable by $Hecke_N$.

Let $H_{W,I}$ be the Hecke-finite vectors in $\varinjlim H^{\leq\mu}_{N,I,W}$. Note that this depends on N but we're suppressing that from the notation. Then:
(a) $W \mapsto H_{I,W}$ is a functor in $\mathbf{Rep}_{\widehat{G}^I}$.
(b) If $\zeta : I \to J$ then we have an isomorphism $\chi_\zeta : H_{I,W} \cong H_{J,W^\zeta}$ functorial in W, and such that $\chi_{\zeta\zeta'} = \chi_\zeta\chi_{\zeta'}$, and χ_ζ is Hecke-invariant.
(c) If $W = 1$ then $H_{I,W} = C_c^{cusp}(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell)$. (This is a purely automorphic calculation - can read it in Lafforgue).

We assume (for now - much of the rest of the semester will be devoted to proving it) the following: For all I, W there's an action of $\operatorname{Gal}(\overline{F}/F)^I$ on $H_{I,W}$ commuting with the maps coming from functoriality, commuting with the Hecke actions, and compatible with χ_ζ via the diagonal map $\operatorname{Gal}(\overline{F}/F)^J \to \operatorname{Gal}(\overline{F}/F)^I$.

How do (a),(b),(c) give the decomposition of

$$H_{\{0\},1} = C_c^{cusp}(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell)?$$

Excursion operators: For I a finite set and $W \in \mathbf{Rep}_{\widehat{G}^I}$, set $\zeta_I : It \circ \{0\}$ (so $W^{\zeta_I} = W$ with the diagonal action) .Let $x : 1 \to W^{\zeta_I}$ and $\zeta : W^{\zeta_I} \to 1$ be $\widehat{G}$-equivariant, and take $(\gamma_i) \in \operatorname{Gal}(\overline{F}/F)^I$. Define $S_{I,W,x,\xi,(\gamma_i)} \in \operatorname{End}(H_{\{0\},1})$ as the composition

$$H_{\{0\},1} \xrightarrow{H(x)} H_{\{0\},W^{\zeta_I}} \xleftarrow[\chi_{\zeta_I}]{\cong} H_{I,W} \xrightarrow{(\gamma_i)} H_{I,W} \xrightarrow{\chi_{\zeta_I}} H_{\{0\},W^{\zeta_I}} \xrightarrow{H(\xi)} H_{\{0\},1} \quad .$$

We will see that $S_{I,W,x,\xi,(\gamma_i)}$ only depends on I, (γ_i), and $f \in \mathcal{O}(\widehat{G}^I)$ defined by $f((g_i)) = \langle \xi, (g_i) \cdot x \rangle$. Note f is left and right invariant by $\widehat{G}$ (because we chose x and ξ to be $\widehat{G}$-invariant). So we'll write $S_{I,f,(\gamma_i)}$.

18 Lecture - 11/18/2014

We have additive functors $W \mapsto H_{I,W}$, where $W \in \mathbf{Rep}_{\widehat{G}^I}$ and $H_{I,W}$ is a $\overline{\mathbb{Q}}_\ell$-vector space with commuting actions of $Hecke_N$ (so this does depend on N but we're suppressing that from the notation) and $\mathrm{Gal}(\overline{F}/F)^I$ compatible with maps $I \to J$. Remark: If W is irreducible it has a highest weight $\underline{\omega} \in X_*^+(T)^I$, and $H_{I,W}$ is the "Hecke-finite part" of

$$\varinjlim_{\mu \in X_*(T)} IH^d(Cht_{N,I,\underline{\omega}}^{\leq \mu}/\Xi)$$

with $d = \dim Cht_{N,I,\underline{\omega}}^{\leq \mu}/\Xi$. If W is the trivial representation then $H_{I,W}$ is $C_c^{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell)$.

Excursion operators: For I finite, $W \in \mathbf{Rep}_{\widehat{G}^I}$, $\zeta_I : I \to \{0\}$, W^{ζ_I} is W with the diagonal action of $\widehat{G}$, fix $x \in (W^{\zeta_I})^{\widehat{G}}$ (which gives $x : 1 \to W^{\zeta_I}$) and $\xi \in (W^{\zeta_I,*})^{\widehat{G}}$ (which gives $\xi : W^{\zeta_I} \to 1$) and $(\gamma_i)_{i\in I} \in \mathrm{Gal}(\overline{F}/F)^I$. The excursion operator $S_{I,W,x,\xi,(\gamma_i)} \in \mathrm{End}(H_{\{0\},1})$ is defined by

$$H_{\{0\},1} \xrightarrow{H(x)} H_{\{0\},W^{\zeta_I}} \xleftarrow[\chi_{\zeta_I}]{\cong} H_{I,W} \xrightarrow{(\gamma_i)} H_{I,W} \xrightarrow{\chi_{\zeta_I}} H_{\{0\},W^{\zeta_I}} \xrightarrow{H(\xi)} H_{\{0\},1} \quad .$$

Remark: Make $\widehat{G}$ act diagonally on $\widehat{G}^I$ on the left and right, and let $\widehat{G}\backslash\backslash\widehat{G}^I//\widehat{G}$ be the coarse quotient, i.e. the one such that

$$\mathcal{O}(\widehat{G}\backslash\backslash\widehat{G}^I//\widehat{G}) = \mathcal{O}(\widehat{G}^I)^{\widehat{G}\times\widehat{G}}$$

over $\overline{\mathbb{Q}}_\ell$. Claim

$$\mathcal{O}(\widehat{G}\backslash\backslash\widehat{G}^I//\widehat{G}) = \{f \in \mathcal{O}(\widehat{G}^I) : \exists W, x, \xi \text{ such that } f((g_i)) = \langle \xi, (g_i)x\rangle.$$

Proof: Obviously any such f is $\widehat{G}\times\widehat{G}$-invariant by $\widehat{G}$-invariance of x, ξ. Conversely, suppose f is in the ring of invariants. Let $W_F \subseteq \mathcal{O}(\widehat{G}^I)$ be the subspace generated by left translates of f under $\widehat{G}^I$ (note that $\dim W_F) < \infty$), let $x_f = f$, let $\xi_f : W_F \to \overline{\mathbb{Q}}_\ell$ map h to $h(1)$. Then we have $f((g_i)) = \langle \xi_f, (g_i)x_f\rangle$.

Note that the W, x, ξ is not unique, but the one we defined for our f in the previous paragraph is the minimal one. If W, x, ξ are any such triple, let $W_x \subseteq W$ be the $\widehat{G}^I$-submodule generated by x. Then we have $\alpha : W_x \hookrightarrow W$ and $\beta : W_x \twoheadrightarrow W_f$ that are $\widehat{G}^I$-invariant (with $\beta(y)(g_i) = \langle \xi, (g_i)y\rangle$). Under this map, $\alpha(x) = x$ and $\beta(x) = x_f$, and $({}^\top\alpha)(\xi) = ({}^\top\beta)(\xi_f) = \xi|_{W_F}$.

Proposition: (0) If W, x, ξ are as before, then $S_{I,W,x,\xi,(\gamma_i)}$ depends only on the function $f \in \mathcal{O}(\widehat{G}\backslash\backslash\widehat{G}^I//\widehat{G})$ defined by W, x, ξ. So we write $S_{I,f,(\gamma_i)}$ instead.
(1) The subalgebra $\mathcal{B}$ of $\mathrm{End}(H_{\{0\},1})$ generated by the $S_{I,f,(\gamma_i)}$ (for $I, f, (\gamma_i)$ varying) is commutative, and for fixed $I, (\gamma_i)$ the map $\mathcal{O}(\widehat{G}\backslash\backslash\widehat{G}^I//\widehat{G}) \to B$ is a $\overline{\mathbb{Q}}_\ell$-algebra map.
(2) For all $\zeta : I \to J$ and all $f \in \mathcal{O}(\widehat{G}\backslash\backslash\widehat{G}^I//\widehat{G})$, if $f^\zeta \in \mathcal{O}(\widehat{G}\backslash\backslash\widehat{G}^J//\widehat{G})$ is defined by the usual formula $f^\zeta(g_j) = f(g_{\zeta(i)})$ then $S_{J,f^\zeta,(\gamma_j)} = S_{I,f,(\gamma_{\zeta(i)})}$
(3) For all I, f, $(\gamma_i), (\gamma_i'), (\gamma_i'')$, if

$$\widetilde{F} \in \mathcal{O}(\widehat{G}\backslash\backslash\widehat{G}^{I\sqcup I\sqcup I}//\widehat{G})$$

is defined by $f(g_i, g_i', g_i'') = f(g_i(g_i')^{-1}g_i'')$ then

$$S_{I\sqcup I\sqcup I,\tilde{f},(\gamma_i,\gamma_i',\gamma_i'')} = S_{I,f,(\gamma_i(\gamma_i')^{-1}\gamma_i'')}.$$

(4) For all I, f the map $\mathrm{Gal}(\overline{F}/F)^I \to \mathcal{B}$ given by $(\gamma_i) \mapsto S_{I,f,(\gamma_i)}$ is a continuous morphism of groups (where $\mathcal{B}$ has the ℓ-adic topology). Remark: $\dim_{\overline{\mathbb{Q}}_\ell} H_{0,1} < \infty$ means $\dim \mathcal{B} < \infty$.
(5) For all $v \in |X \setminus N|$ and all $V \in \mathbf{Rep}_{\widehat{G}}$ irreducible, if $T_{v,v} \in \mathrm{End}(H_{\{0\},1})$ is the action of the element of the unramified Hecke algebra at v corresponding to V, if $f(g_1, g_2) = \chi_v(g_1g_2^{-1})$, if $\mathrm{Frob}_v \in \mathrm{Gal}(\overline{F}/F)$ is a lift of the Frobenius at v, then $T_{V,v} = S_{\{1,2\},f,(\mathrm{Frob}_v,1)}$.

Point (5) is serious and will take us a while to prove. The other points follow from the following easy lemma:

Lemma: (1) If $\mu : W \to W'$ is a map in $\mathbf{Rep}_{\widehat{G}^I}$, if $x \in (W^{\zeta_i})^{\widehat{G}}$, if $\zeta \in ((W')^{\zeta_I,*})^{\widehat{G}}$, then

$$S_{U,W,x,{}^\top\mu(\zeta'),(\gamma_i)} = S_{I,W',\mu(x),\xi',(\gamma_i)}.$$

(2) For all $\zeta : I \to J$ we have $S_{J,W^\zeta,x,\xi,(\gamma_i)} = S_{I,W,x,\xi,(\gamma_i)}$.
(3)

$$S_{I_1 \sqcup I_2, W_1 \boxtimes W_2, x_1 \boxtimes w_2, \xi_1 \boxtimes \xi_2, (\gamma_i^1) \times (\gamma_i^2)} = S_{I_1,w_1,x_1,\zeta_1,(\gamma_i^1)} \circ S_{I_2,w_2,x_2,\zeta_2,(\gamma_i^2)}.$$

(4)

$$S_{I,W,x,\xi,(\gamma_i(\gamma_i')\gamma_i''} = S_{I \sqcup I \sqcup I, W \boxtimes W^* \boxtimes W, \delta_W \boxtimes x, \zeta \boxtimes ev_w, (\gamma_i) \times (\gamma_i') \times (\gamma_i'')}.$$

Now, how do we apply this result? Last year, we proved the following result:
Theorem: Let Γ be a profinite group, let $\overline{H}/\overline{\mathbb{Q}}_\ell$ be connected reductive, let $\Xi_n : \mathcal{O}(H^n//H) \to \mathcal{C}(\Gamma^n, \overline{\mathbb{Q}}_\ell)$ be a family of maps such that:
(a) For all $\zeta : \{1, \ldots, n\} \to \{1, \ldots, m\}$ and all $f \in \mathcal{O}(H^n//H)$ and all $(\gamma_1, \ldots, \gamma_m) \in \Gamma^m$, we have

$$\Xi_m(f^\zeta)(\gamma_j) = \Xi_N(f)(\gamma_{\zeta(i)}).$$

(b) For all $n \geq 1$ and all $f \in \mathcal{O}(H^n//H)$ and all $(\gamma_i) \in \Gamma^{n+1}$, we have

$$\Xi_{n+1}(\widehat{f})(\gamma_1, \ldots, \gamma_{n+1}) = \Xi_n(f)(\gamma_1, \ldots, \gamma_{n-1}, \gamma_n \gamma_{n+1})$$

where $\widehat{f}(g_1, \ldots, g_{n+1}) = f(g_1, \ldots, g_{n-1}, g_n g_{n+1})$.

Then, there exists $\sigma : \Gamma \to H(\overline{\mathbb{Q}}_\ell)$ a continuous morphism of groups, unique up to $\widehat{(\overline{\mathbb{Q}}_\ell)}$-conjugacy, such that $\Xi_n(f)(\gamma_1, \ldots, \gamma_n) = f(\sigma(\gamma_1), \ldots, \sigma(\gamma_n))$. Moreover, if $\Gamma \twoheadrightarrow \overline{\Gamma}$, if for all $n \gg 0$ the image of Ξ_n is in $\mathbb{C}(\overline{\gamma}^n, \overline{\mathbb{Q}}_\ell)$ then σ factors through $\overline{\Gamma}$.

How do we apply this? Remember that $H_{\{0\},1} = C_c^{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K_N \Xi, \overline{\mathbb{Q}}_\ell)$. Write the decomposition of $H_{\{0\},1)}$ into generalized eigenspaces for $\mathcal{B}$, $\bigoplus_{\alpha:\mathcal{B}\to\overline{\mathbb{Q}}_\ell} \underline{h}_\alpha$. Now we change this into something indexed by Langlands parameters: if $\alpha : \mathcal{B} \to \overline{\mathbb{Q}}_\ell$ is a character then for all $\overline{\mathbb{Q}}_\ell$-algebra morphisms and all I, have continuous function

$$\mathcal{O}(\widehat{G}\backslash\backslash \widehat{G}^I // \widehat{G}) \to \mathcal{C}(\mathrm{Gal}(\overline{F}/F)^I, \overline{\mathbb{Q}}_\ell)$$

by mapping f to the function $(\gamma_i) \mapsto \alpha(S_{I,f,(\gamma_i)})$. Using the isomorphisms $\widehat{G}^n//\widehat{G} \cong \widehat{G}$ $\widehat{G}^{n+1}//\widehat{G}$ by $(g_1, \ldots, g_n) \mapsto (1, g_1, \ldots, g_n)$ this defines a family Ξ_n as in the theorem so there exists $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\mathbb{Q}_\ell)$ a continuous group morphism unique up to $\widehat{G}(\mathbb{Q}_\ell)$-conjugacy, semisimple (i.e. the Zariski closure of $\mathrm{Im}\,\sigma$ is reductive) such that for all $I, f, (\gamma_i)$ we have

$$\mathrm{tr}(S_{f,I,(\gamma_i)} | \underline{h}_\alpha) = f(\sigma(\gamma_i)) \dim(\underline{h}_\alpha).$$

Also, σ factors through the Galois group of the maximal extension unramified outside of N. From now on write $\underline{h}_\alpha = \underline{h}_\sigma$.

So, we get

$$C_c^{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K_N \Xi, \overline{\mathbb{Q}}_\ell) = \bigoplus_{\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)} \underline{h}_\sigma,$$

and part (5) of the proposition says that for all $v \in |X \setminus N|$ and $V \in \mathbf{Rep}_{\widehat{G}}$ irreducible, the unique eigenvalue of $T_{V,v}$ on $\underline{h}_\sigma$ is $\chi_v(\sigma(\mathrm{Frob}_v)) = \mathrm{tr}(\sigma(\mathrm{Frob}_v)|V)$, i.e. the decomposition is compatible with the Satake isomorphism at places $v \notin |N|$.

So that's the outline of the argument. Remains to do:

- Define the action of $\mathrm{Gal}(\overline{F}/F)^I$ on $H_{I,W}$ via partial Frobenius maps.
- Study Hecke-finite vectors.
- Prove (5) of the proposition.

The Galois action on the cohomology of $Cht_{N,I}$. Start by discussing "small maps". A map of k-schemes of finite type $f : X \to Y$ is *small* if f is proper surjective birational and for all $r \geq 1$,

$$\operatorname{codim}\{y \in Y : \dim f^{-1}[y] = r\} > 2r$$

Theorem: If f is small, if $V \subseteq Y$ is open dense such that $f : f^{-1}[V] \cong V$, then the canonical isomorphism $\overline{\mathbb{Q}}_{\ell,V} = f_*\overline{\mathbb{Q}}_{\ell,f^{-1}[V]}$ extends to an isomorphism $IC_Y = f_*IC_X$.

Now, more affine Grassmannians. Let I be a finite set and $(I_1, \ldots, I_r)$ a partition of I. Define a stack (actually an ind-scheme) $\mathrm{Gr}_I^{(I_1,\ldots,I_r)}$ by letting $\mathrm{Gr}_I^{(I_1,\ldots,I_r)}(S)$ be the set of tuples of $(x_i) \in X(S)^i$, $\mathcal{G}_j \in \mathrm{Bun}_G(S)$ for $0 \leq j \leq r$,

$$\varphi_j : \mathcal{G}_{j-1}|_{X\times S\setminus\bigcup i\in I_j \Gamma_{x_i}} \cong \mathcal{G}_j|_{X\times S\setminus\bigcup i\in I_j \Gamma_{x_i}}$$

for $1 \leq j \leq r$, and $\varphi : \mathcal{G}_r \cong G \times X \times S$.

Remark:

- We could also require $\mathcal{G}_r = G \times X \times S$ and get rid of ψ.
- $\mathrm{Gr}_I = \mathrm{Gr}_I^{(I)}$
- We have obvious maps $\mathrm{Gr}_I^{(I_1,\ldots,I_r)} \to \mathrm{Gr}_I$ by the full tuple mapping to $((x_i), \mathcal{G}_0, \mathcal{G}_r, \varphi_r \circ \cdots \circ \varphi_1, \varphi)$. More generally if $(I_1, \ldots, I_r)$ refines $(I'_1, \ldots, I'_{r'})$ we have an obvious small map $\mathrm{Gr}_I^{(I_1,\ldots,I_r)} \to \mathrm{Gr}_I^{(I'_1,\ldots,I'_{r'})}$.
- Can also make the G-bundles $\mathcal{G}_j$ defined on $\sum \infty x_i$. This would give the same stack.

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From last time: let I be a finite set and $(I_1,\dots,I_r)$ a partition of I. Define a stack (actually an ind-scheme) $\mathrm{Gr}_I^{(I_1,\dots,I_r)}$ by letting $\mathrm{Gr}_I^{(I_1,\dots,I_r)}(S)$ be the set of tuples of $(x_i)\in X(S)^i$, $\mathcal{G}_j \in \mathrm{Bun}_G(S)$ for $0\le j\le r$,

$$\varphi_j : \mathcal{G}_{j-1}|_{X\times S\setminus\bigcup i\in I_j \Gamma_{x_i}} \cong \mathcal{G}_j|_{X\times S\setminus\bigcup i\in I_j \Gamma_{x_i}}$$

for $1\le j\le r$, and $\varphi:\mathcal{G}_r\cong G\times X\times S$. Can also define $Hecke_I^{(I_1,\dots,I_r)}$ as the same thing but omitting the final trivialization φ. Then have maps $\mathrm{Gr}_I^{(I_1,\dots,I_r)}(S)\to Hecke_I^{(I_1,\dots,I_r)}$ and from both things to $X^I(S)$. If $(I_1,\dots,I_r)$ refines $(I_1',\dots,I_{r'}')$ get a map from the things for the former to the things for the latter.

Truncations: Let $\underline{\omega}=(w_i)\in X_*^+(T)^I$ and let $\underline{\omega}_j=(w_i)_{i\in I_j}$. Say $((x_i),\mathcal{G}_0,\dots,\mathcal{G}_r,\varphi_1,\dots,\varphi_r)\in Hecke_I^{(I_1,\dots,I_r)}$ is in $Hecke_{I,\underline{\omega}}^{(I_1,\dots,I_r)}$ if, for all j and all $\lambda\in X_*^+(T)$ we have

$$\varphi_j(\mathcal{G}_{j-1,\lambda})\subseteq \mathcal{G}_{j,\lambda}\left(\sum_{i\in I_j}\langle\lambda,\omega_i\rangle\Gamma_{x_i}\right).$$

Similarly define $\mathrm{Gr}_{I,\underline{\omega}}^{(I_1,\dots,I_r)}$.

Also, have maps $p,p' : Hecke_I^{(I_1,\dots,I_r)}\to \mathrm{Bun}_G$ by mapping a tuple to $\mathcal{G}_0$ or $\mathcal{G}_r$, respectively. Then

$$Hecke_I^{(I_1,\dots,I_r)}\cong Hecke_{I_1}\times_{\mathrm{Bun}_G}\cdots\times_{\mathrm{Bun}_G} Hecke_{I_r}.$$

Also,

$$\mathrm{Gr}_I^{(I_1,\dots,I_r)} = Hecke_I^{(I_1,\dots,I_r)}\times_{\mathrm{Bun}_G}\operatorname{Spec} k$$

with the maps p' and $\operatorname{Spec} k\to\mathrm{Bun}_G$ giving the trivial bundle.

Proposition: (a) Let $\Delta=\{(x_i)\in X^I : \exists i\ne j, x_i=x_j\}$ be the fat diagonal. Fix $\underline{\omega}$. If $(I_1,\dots,I_r)$ refines $(I_1',\dots,I_r')$ then the morphisms

$$Hecke_{I,\underline{\omega}}^{(I_1,\dots,I_r)}\to Hecke_{I,\underline{\omega}}^{(I_1',\dots,I_{r'}')}$$

and

$$\mathrm{Gr}_{I,\underline{\omega}}^{(I_1,\dots,I_r)}\to \mathrm{Gr}_{I,\underline{\omega}}^{(I_1',\dots,I_{r'}')}$$

are projective and are isomorphisms over $X^I\setminus\Delta$.

(b) If $I=I_1\sqcup I_2$ and we let U_{I_1,I_2} be the open set of tuples (x_i) such that $x_i\ne x_j$ if $i\in I_1$ and $j\in J_2$. Then for all $\underline{\omega}=(\underline{\omega}_1,\underline{\omega}_2)$ the morphism

$$Hecke_{I,\underline{\omega}}^{(I_1,I_2)}\to Hecke_{I,\underline{\omega}}$$

is an isomorphism over U_{I_1,I_2}.

(c) Let $\underline{\omega}\in X_*^+(T)^I$. Over $X^I\setminus\Delta$, $\mathrm{Gr}_{I,\underline{\omega}}$ is canonically isomorphic to $\prod_i \mathrm{Gr}_{\{i\},\omega_i}$ (this is just a restatement of the factorizable structure on the Grassmannian).

(d) There exists $T\to\mathrm{Bun}_G\times X$ smooth surjective with connected fibers such that

$$Hecke_{\{1\},\omega}\times_{\mathrm{Bun}_G\times X} T\cong \mathrm{Gr}_{\{1\},\omega}\times_X T.$$

(e) $\mathrm{Gr}_{1,\omega}\to X$ is a Zariski locally trivial fibration with fiber $\mathrm{Gr}_\omega=\overline{Orb(t^\omega)}\subseteq\mathrm{Gr}_G$.

Proof: (c),(e) we're just restating from earlier results.

(b): Let $((X_i),\mathcal{G},\mathcal{G}',\varphi:\mathcal{G}\to\mathcal{G}')\in Hecke_I(S)$ be such that $(x_i)\in U_{I_1,I_2}(S)$. Use φ to glue $\mathcal{G}|_{X\times S\setminus\bigcup_{i\in I_1}\Gamma_{x_1}}$ to $\mathcal{G}'|_{X\times S\setminus\bigcup_{i\in I_2}\Gamma_{x_1}}$ to get a G-bundle $\mathcal{G}''$, with $\mathrm{id}:\mathcal{G}\to\mathcal{G}'$ an isomorphism over $X\times S\setminus\bigcup_{i\in I_1}\Gamma_{x_i}$ and $\varphi:\mathcal{G}'\to\mathcal{G}''$ an isomorphism over $X\times S\setminus\bigcup_{i\in I_2}\Gamma_{x_i}$. This defines a point of $Hecke_{I,\underline{\omega}}^{(I_1,I_2)}(S)$, and this gives our inverse map.

(a) It's enough to do it for $Hecke$ and for $Hecke'_I = Hecke_I^{(\{i\})} \to Hecke_I$. The last part follows from (b). We know $Hecke_{I,\underline{\omega}} \to \mathrm{Bun}_G \times X^I$ is projective for all $\underline{\omega}$ and all I. And also $Hecke'_{I,\underline{\omega}}$ is a product of $Hecke_{\{i_j\}}$'s, so $Hecke'_{I,\underline{\omega}} \to \mathrm{Bun}_G \times X^I$ is also projective so $Hecke'_I \to Hecke_I$ is projective.

(d) Let $T(S)$ be the set of tuples $(x, \mathcal{G}, \psi)$ with $(x, \mathcal{G}) \in (X \times \mathrm{Bun}_G)(S)$ and $\psi : \mathcal{G}|_{\Gamma_{\infty x}} \cong G \times \Gamma_{\infty x}$, a $G_{\Gamma_{\infty x}}$ torsor. (?????? Mistake in proof need to fix next time).

Definition: $f : Y \to Z$ a morphism of finite-type k-schemes. Said f is small if f is proper surjective birational and for all $r \geq 1$

$$\mathrm{codim}\{y \in Y : \dim f^{-1}[y] = r\} > 2r.$$

Say f is *semismall* if it's proper surjective and for all $r \geq 0$

$$\mathrm{codim}\{y \in Y : \dim f^{-1}[y] = r\} \geq 2r.$$

(The $r = 0$ case implies this is generically finite; note we're taking this instead of assuming birational).

If $f : \mathcal{Y} \to \mathcal{Z}$ is a map of algebraic stacks, call it (semi)mall if for all $Z \to \mathcal{Z}$ smooth with Z finite type scheme, $Z \times_{\mathcal{Y}} \mathcal{Z} \to Z$ is (semi)small.

Theorem: Let $\underline{\omega} \in X_*^+(T)^I$. If $(I_1, \ldots, I_r)$ refines $(I'_1, \ldots, I'_{r'})$ then

$$Hecke_{I,\underline{\omega}}^{(I_1,\ldots,I_r)} \to Hecke_{I,\underline{\omega}}^{(I'_1,\ldots,I'_{r'})}$$

and

$$\mathrm{Gr}_{I,\underline{\omega}}^{(I_1,\ldots,I_r)} \to \mathrm{Gr}_{I,\underline{\omega}}^{(I'_1,\ldots,I'_{r'})}$$

are small.

Proof: By (d) of the proposition (hopefully), these two statements are equivalent for fixed $I, \underline{\omega}$. It's enough to do the extreme map

$$Hecke'_{I,\underline{\omega}} = Hecke_{I,\underline{\omega}}^{(\{i\})} \to Hecke_{I,\underline{\omega}}.$$

If $(I_1, \ldots, I_r$ is a partition of I let $\Delta_{(I_1,\ldots,I_r)}$ be the diagonal such that $x_i = x_{i'}$ whenever $i, i' \in I_j$. Let $X^{\circ(I_1,\ldots,I_r)}$ be

$$\Delta_{(I_1,\ldots,I_r)} \setminus \bigcup \Delta_{(I'_1,\ldots,I'_{r'})}$$

where the union runs over all strict coarsenings; thus it consists of all tuples (x_i) with $x_i = x_{i'}$ iff i, i' are in the same I_j. Then X^I is the disjoint union of all of these $X^{\circ(I_1,\ldots,I_r)}$, and we know π^H and π^G are isomorphisms over the open stratum $X^{\circ(\{i\})} = X^I \setminus \Delta$. The other strata are codimension ≥ 1 and $Hecke_{I,\underline{\omega}}, \mathrm{Gr}_{I,\underline{\omega}} \to X^I$ are equidimensional.

So we just need to show π^H and π^G are small over $X^{\circ(I_1,\ldots,I_r)}$ for everything but the finest partition. But over this set we have that $Hecke_{I,\underline{\omega}}$ is canonically isomorphic to ???

20 Lecture - 11/25/2014

First, fix issue from last time. Defined two stacks, $Hecke_1$ mapping S to the tuples $(x, \mathcal{G}, \mathcal{G}', \varphi)$ with $x \in X(S)$, $\mathcal{G}, \mathcal{G}' \in \mathrm{Bun}_G(S)$, $\varphi : \mathcal{G}|_{X\times S\setminus\Gamma_x} \cong \mathcal{G}'|_{X\times S\setminus\Gamma_x}$, and Gr_1 mapping S to the tuples $(x, \mathcal{G}, \varphi)$ with $x \in X(S)$, $\mathcal{G} \in \mathrm{Bun}_G(S)$, $\varphi : \mathcal{G}|_{X\times S\setminus\Gamma_x} \cong G_{X\times S\setminus\Gamma_x}$. Have natural maps $\mathrm{Gr}_1 \to Hecke_1$ taking $(x, \mathcal{G}, \varphi)$ to $(x, \mathcal{G}, G_{X\times S}, \varphi)$ and $Hecke_1 \to \mathrm{Bun}_G \times X$ by $(x, \mathcal{G}, \mathcal{G}', \varphi) \mapsto (x, \mathcal{G}')$.

Want that there exists $T \to \mathrm{Bun}_G \times X$ smooth surjective with connected fibers such that $Hecke_1 \times_{\mathrm{Bun}_G \times X} T \cong \mathrm{Gr}_1 \times_X T$. For this, take T to be given by mapping S to tuples $(x, \mathcal{G}, \psi)$ with $(x, \mathcal{G}) \in (X \times \mathrm{Bun}_G)(S)$ and $\psi : \mathcal{G}|_{\Gamma_{\infty x}} \cong G_{\Gamma_{\infty x}}$.

To prove this works, recall by B-L theorem $\mathrm{Gr}_1 \cong \mathrm{Gr}_1^{\mathrm{loc}}$ where $\mathrm{Gr}_1^{\mathrm{loc}}$ is defined by mapping S to tuples $(x, \mathcal{G}, \varphi)$ with $x \in X(S)$, $\mathcal{G} \in BG(\Gamma_{\infty x})$, and $\varphi : \mathcal{G}|_{\Gamma_{\infty x}\setminus\Gamma_x} \cong G_{\Gamma_{\infty x}\setminus\Gamma_x}$. Then, explicitly define mutually inverse morphisms between the things we want:

To define $\alpha : \mathrm{Gr}_1^{\mathrm{loc}} \times_X T \to Hecke_1 \times_{X\times \mathrm{Bun}_G} T$, let $(x, \mathcal{G}, \varphi, \mathcal{G}', \psi) \in (\mathrm{Gr}_1 \times_X T)(S)$; this means $x \in X(S)$, $\mathcal{G} \in BG(\Gamma_{\infty x})$, $\varphi : \mathcal{G}|_{\Gamma_{\infty x}\setminus\Gamma_x} \cong G_{\Gamma_{\infty x}\setminus\Gamma_x}$, $\mathcal{G}' \in BG(X \times S)$, and $\psi : \mathcal{G}'|_{\Gamma_{\infty x}} \cong G_{\Gamma_{\infty x}}$. By B-L theorem gluing $\mathcal{G}$ on $\Gamma_{\infty x}$ and $\mathcal{G}'$ on $X \times S \setminus \Gamma_x$ using $\psi^{-1} \circ \varphi$ on $\Gamma_{\infty x} \setminus \Gamma_x$ gives a G-bundle $\mathcal{G}_1$ on $X \times S$ together with $\varphi_1 : \mathcal{G}_1 \cong \mathcal{G}'$ on $X \times S \setminus \Gamma_x$. Take α of the input tuple to map to the pair of tuples $(x, \mathcal{G}_1, \mathcal{G}', \varphi_1)$ and $(x, \mathcal{G}', \psi)$.

To define $\beta : Hecke_1 \times_{X\times \mathrm{Bun}_G} T \to \mathrm{Gr}_1^{\mathrm{loc}} \times_X T$, suppose we have a point $((x, \mathcal{G}, \mathcal{G}', \varphi), (x, \mathcal{G}'', \psi), \chi)$ where $\chi : \mathcal{G}' \cong \mathcal{G}''$. Then on $\Gamma_{\infty x} \setminus \Gamma_x$ by composing φ, χ, ψ we get a chain of isomorphisms $\mathcal{G} \cong \mathcal{G}' \cong \mathcal{G}'' \cong G$, so get $(x, \mathcal{G}_{\Gamma_{\infty x}}, \psi \circ \chi \circ \varphi) \in \mathrm{Gr}_1^{\mathrm{loc}}$. Take $\beta(z)$ to be this point together with the point from $T(S)$.

Remark: If we work with $\mathrm{Gr}_{1,\omega}$ and $Hecke_{1,\omega}$ then we can instead take T_n given by $S \mapsto (x, \mathcal{G}, \varphi : \mathcal{G}|_{\Gamma_{nx}} \cong G_{\Gamma_{nx}})$ for large n (relative to ω).

What we were doing : Last time stopped in the middle of a proof. had $\omega = \omega_1 + \cdots + \omega_r \in X_*^+(T)$. Was trying to show that the map $\mathrm{Gr}''_{1,\omega} \to \mathrm{Gr}_{1,\omega}$ was semi-small where

$$\mathrm{Gr}''_{1,\omega} = \mathrm{Gr}^{\{1\},\ldots,\{r\}}_{\{1,\ldots,r\},(\omega_i)} \times_{X^r} X$$

where $\mathrm{Gr}^{\{1\},\ldots,\{r\}}_{\{1,\ldots,r\},(\omega_i)}(S)$ was tuples $(x_1, \ldots, x_r, \mathcal{G}_0, \ldots, \mathcal{G}_r, \varphi_1, \ldots, \varphi_r)$ with $x_i \in X(S)$, $\mathcal{G}_i \in \mathrm{Bun}_G(S)$, φ_i an isomorphism of $\mathcal{G}_{i-1}$ with $\mathcal{G}_i$ on $X \times S \setminus \Gamma_{x_i}$, with $\mathcal{G}_r = G \times X \times S$, and such that for all $\lambda \in X_+^*(T)$ we have $\varphi_i(\mathcal{G}_{i-1,\lambda}) \subseteq \mathcal{G}_{i,\lambda}(\langle \lambda, \omega_i \rangle \Gamma_{x_i})$. Thus $\mathrm{Gr}''_{1,\omega}(S)$ is the same data but with $x_1 = \cdots = x_r = x$, and the map to $\mathrm{Gr}_{1,\omega}$ maps this tuple to $(x, \mathcal{G}_0, \varphi_r \circ \cdots \circ \varphi_1)$.

Proof: First we may assume $X = \mathbb{A}^1$, so $\mathrm{Gr}''_{1,\omega} \to \mathrm{Gr}_{1,\omega}$ is the product of $|id_X$ and $\mathrm{Gr}_{\omega_1,\ldots,\omega_r} \to \mathrm{Gr}_\omega$ where $\mathrm{Gr}_\omega = \overline{Orb_{t^\omega}}$ and $\mathrm{Gr}_{\omega_1,\ldots,\omega_r}$ is $\mathrm{Gr}_{\omega_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}_{\omega_r}$, where this is defined via a convolution diagram

$$\mathrm{Gr}_{\omega_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}_{\omega_r} = \pi_2[\pi_1^{-1}[\mathrm{Gr}_{\omega_1} \times \cdots \times \mathrm{Gr}_{\omega_r}]] = \pi_1^{-1}[\mathrm{Gr}_{\omega_1} \times \cdots \times \mathrm{Gr}_{\omega_r}]/G[\![t]\!]^{r-1},$$

where $\pi_1, \pi_2 : G(\!(t)\!)^{r-1} \times \mathrm{Gr}_G \to \mathrm{Gr}_G^r$ are such that

$$\pi_1(g_1, \ldots, g_{r-1}, z) = (\overline{g}_1, \ldots, \overline{g}_{r-1}, z)$$

$$\pi_1(g_1, \ldots, g_{r-1}, z) = (\overline{g}_1, \overline{g}_1\overline{g}_2, \ldots, \overline{g}_1 \cdots \overline{g}_{r-1}, g_1 \cdots g_{r-1} z)$$

where $\overline{g}$ is the image of g in Gr_G. Note that π_2 is a $G[\![t]\!]^{r-1}$-torsor. Also, the map $\mathrm{Gr}_{\omega_1,\ldots,\omega_r} \to \mathrm{Gr}_\omega$ is projection to the last factor (viewing them as subsets of Gr_G^r and Gr_G).

This map $\mathrm{Gr}_{\omega_1,\ldots,\omega_r} \to \mathrm{Gr}_\omega$ is a partial Bott-Samelson (or Demazure) resolution. Last year we already used that these were semismall to show that the convolution product preserved perversity. Reference: Ngo-Polo, lemma 9.3 for the case ω_i (quasi)-minuscule, but can be easily adapted to the general case.

Shtuka : Let I be a finte set, $(I_1, \ldots, I_r)$ a partition of I, $N \subseteq X$ a finite closed subscheme. Define $Cht_{N,I}^{(I_1,\ldots,I_r)}(S)$ to be the set of tuples $((x_i)_{i\in I}, (\mathcal{G}_0, \psi_0), \ldots, (\mathcal{G}_r, \psi_r), \varphi_1, \ldots, \varphi_r)$ where $x_i \in X(S) \setminus N(S)$, $(\mathcal{G}_j, \psi_j) \in \mathrm{Bun}_{G,N}(S)$, $(\mathcal{G}_r, \psi_r) = ({}^\tau\mathcal{G}_0, {}^\tau\psi_0)$, and φ_j is an isomorphism of $\mathcal{G}_{j-1}$ and $\mathcal{G}_j$ on $X \times S \setminus \bigcup_{i\in I_j} \Gamma_{x_i}$, compatible with the ψ_{j-1} and ψ_j.

Note $Cht_{N,I}^{(I)} = Cht_{N,I}$, and define $Cht_{N,I}^{(I_1,\dots,I_r),\le\mu}$ by putting the degree condition on $\mathcal{G}_0$. Remark: If $(I_1,\dots,I_r)$ refines $(I'_1,\dots,I'_{r'})$ we get a map $Cht_{N,I}^{(I_1,\dots,I_r)} \to Cht_{N,I}^{(I'_1,\dots,I'_r)}$.

We have a map $\eta : Cht_{N,I}^{(I_1,\dots,I_r)} \to \mathrm{Gr}_I^{(I_1,\dots,I_r)}/G_{\sum\infty x_i}$ defined as follows. If $((x_i),(\mathcal{G}_j,\psi_j),(\varphi_j))$ is in $Cht_{N,I}^{(I_1,\dots,I_r)}(S)$ and ψ a trivialization of $\mathcal{G}_r = {}^\tau\mathcal{G}_0$ on $\Gamma_{\sum\infty x_i}$, send this tuple to $((x_i),\psi\circ\varphi_r\circ\cdots\circ\varphi_1)$. Note ψ exists only locally, but this construction glues to give η.

Definition: set

$$Cht_{N,I,\underline{\omega}}^{(I_1,\dots,I_r)} = \eta^{-1}[\mathrm{Gr}_{I,\underline{\omega}}^{(I_1,\dots,I_r)}/G_{\sum\infty x_i}].$$

If $W \in \mathbf{Rep}_{\widehat{G}^I}$ set

$$Cht_{N,I,W}^{(I_1,\dots,I_r)} = \bigcup_{\underline{\omega}} Cht_{N,I,\underline{\omega}}^{(I_1,\dots,I_r)}$$

where $\underline{\omega}$ runs over highest weights of W. As before, if W is fixed then for n_i big enough then η comes from a natural map

$$\varepsilon_{N,I} : Cht_{N,I,W}^{(I_1,\dots,I_r)} \to \mathrm{Gr}_{I,W}^{(I_1,\dots,I_r)}/G_{\sum n_i x_i}.$$

Proposition: $\varepsilon_{N,I}$ is smooth of relative dimension $\dim(G_{\sum n_i x_i}/X^I)$.

Corollary: The maps $Cht_{N,I,W}^{(I_1,\dots,I_r)} \to Cht_{N,I,W}^{(I'_1,\dots,I'_{r'})}$ are small.

Geometric Satake : Theorem: For all I and all partitions $(I_1,\dots,I_r)$ we have an additive functor $\mathbf{Rep}_{\widehat{G}^I} \to \mathbf{Perv}_{G_{\sum\infty x_i}}(\mathrm{Gr}_I^{(I_1,\dots,I_r)})$. We denote this $W \mapsto S_{I,W}^{(I_1,\dots,I_r)}$. This satisfies some compatibility conditions, and is such that if we have $\pi : \mathrm{Gr}_{N,I,W}^{(I_1,\dots,I_r)} \to \mathrm{Gr}_{N,I,W}^{(I'_1,\dots,I'_{r'})}$ induced by a refinement, then we have a canonical isomorphism

$$S_{I,W}^{(I'_1,\dots,I'_{r'})} \cong \pi_* S_{I,W}^{(I_1,\dots,I_r)}.$$

Remark: If W is irreducible of highest weight $\underline{\omega}$ then $S_{I,W}^{(I_1,\dots,I_r})$ is just the intersection complex of $\mathrm{Gr}_{I,\underline{\omega}}^{(I_1,\dots,I_r)}$, so the last map follows from smallness of π.

Definition: Set $\mathcal{F}_{N,I,W}^{(I_1,\dots,I_r)} = \varepsilon_{N,I}^*(S_{I,W}^{(I_1,\dots,I_r)}[\cdots]$ (with the shift chosen so that this is perverse on fibers of $Cht_{N,I,W}^{(I_1,\dots,I_r}) \to X^I$). Remember: for $p : Cht_{N,I,W} \to X^I$, we defined

$$\mathcal{H}_{N,I,W}^{\le\mu} = {}^pR^0p_!(\mathcal{F}_{N,I,W}|Cht_{N,I,W}^{\le\mu}/\Xi).$$

Then we get: for all partitions of $(I_1,\dots,I_r)$ have

$$p_{(I_1,\dots,I_r)} : Cht_{N,I,W}^{(I_1,\dots,I_r)} \to X^I$$

we have a canonical isomorphism

$$\mathcal{H}_{N,I,W}^{\le\mu} \cong {}^pR^0p_{(I_1,\dots,I_r)!}(\mathcal{F}_{N,I,W}^{(I_1,\dots,I_r)}|Cht_{N,I,W}^{(I_1,\dots,I_r)\le\mu}/\Xi).$$

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Partial Frobenius morphisms . Recall our setup: $(I_1, \ldots, I_r)$ is a partition of I, and defined a stack $Cht_{N,I,W}^{(I_1,\ldots,I_r)}$ with S-points consisting of a tuple $(x_i) \in (X \setminus N)^I(S)$ and a sequence of bundles $(\mathcal{G}_j, \psi_j)$ for $0 \le j \le r$ (with $(\mathcal{G}_r, \psi_r) = {}^\top(\mathcal{G}_0, \psi_0)$) and morphisms $\varphi_j : (\mathcal{G}_{j-1}, \psi_{j-1}) \to (\mathcal{G}_j, \psi_j)$ an isomorphism on $X \times S \setminus \bigcup_{i \in I_j} \Gamma_{x_i}$, plus a truncation condition. Have $p : Cht_{N,I,W}^{(I_1,\ldots,I_r)} \to X^I$, and are interested in

$$\mathcal{H}_{N,I,W}^{\le\mu} = {}^pR^0p_!\Big(\mathcal{F}_{N,I,W}^{(I_1,\ldots,I_r)}|Cht_{N,I,W}^{(I_1,\ldots,I_r),\le\mu}/\Xi\Big)$$

which we showed was independent of the partition.

Now, fix a partition $(I_1, \ldots, I_r)$. Let $\mathrm{Frob}_{I_1} : X^I \to X^I$ be $(\mathrm{Frob}_X)^{I_1} \times (\mathrm{id}_X)^{I \setminus I_1}$. Define a morphism

$$\mathrm{Frob}_{I_1}^{(I_1,\ldots,I_r)} : Cht_{N,I,W}^{(I_1,\ldots,I_r)} \to Cht_{N,I,W}^{(I_2,\ldots,I_r,I_1)}$$

by sending the tuple with (x_i) and

$$(\mathcal{G}_0, \psi_0) \xrightarrow{\varphi_1} (\mathcal{G}_1, \psi_1) \longrightarrow \cdots \longrightarrow (\mathcal{G}_r, \psi_r)$$

to $\mathrm{Frob}_{I_1}(x_i)$ and

$$(\mathcal{G}_1, \psi_1) \longrightarrow \cdots \longrightarrow (\mathcal{G}_r, \psi_r) \xrightarrow{{}^\top\varphi_1} {}^\top(\mathcal{G}_1, \psi_1)$$

recalling that $(\mathcal{G}_r, \psi_r) = {}^\top(\mathcal{G}_1, \psi_1)$.

Note that

$$\mathrm{Frob}_{I_r}^{(I_r,I_1,\ldots,I_{r-1})} \circ \cdots \circ \mathrm{Frob}_{I_1}^{(I_1,\ldots,I_r)}$$

is the absolute Frobenius of $Cht_{N,I,W}^{(I_1,\ldots,I_r)}$, so $\mathrm{Frob}_{I_1}^{(I_1,\ldots,I_r)}$ is a totally radical universal homomorphism and in particular we have a canonical isomorphism

$$(\mathrm{Frob}_{I_1}^{(I_1,\ldots,I_r)})^* \mathcal{F}_{N,I,W}^{(I_2,\ldots,I_r,I_1)} = \mathcal{F}_{N,I,W}^{(I_1,\ldots,I_r)}.$$

Lemma: If $\underline{\omega} = (\omega_i) \in X_*^+(T)^I$ then

$$\mathrm{Frob}_I^{(I_1,\ldots,I_r)} \left[Cht_{N,I,\underline{\omega}}^{(I_1,\ldots,I_r),\le\mu}\right] \subseteq Cht_{N,I,\underline{\omega}}^{(I_1,\ldots,I_r),\le\mu - w_0\sum\omega_i}$$

where w_0 is the longest element of the Weyl group, and

$$(\mathrm{Frob}_I^{(I_1,\ldots,I_r)})^{-1}\left[Cht_{N,I,\underline{\omega}}^{(I_2,\ldots,I_r,I_1),\le\mu}\right] \subseteq Cht_{N,I,\underline{\omega}}^{(I_1,\ldots,I_r),\le\mu+\sum\omega_i}.$$

So for all μ and all W, if $\kappa \in X_*^+(T)$ is big enough then $\mathrm{Frob}_{I_1}^{(I_1,\ldots,I_r)}$ induces

$$F_{I_1} : (\mathrm{Frob}_{I_1})^* \mathcal{H}_{N,I,W}^{\le\mu} \cong \mathcal{H}_{N,I,W}^{\le\mu+\kappa}$$

(for every partition of I).

Drinfeld's Lemma . Let X be as before (smooth projective curve), $F = k(X)$, $\eta = \operatorname{Spec} F$ the generic point, and $\overline{\eta} = \operatorname{Spec} \overline{F}$ a geometric generic point. Let I be finite, $\Delta : X \to X^I$ the diagonal, η^I be the generic point of X^I and $\overline{\eta}^I$ a geometric point over this. Fix a specialization map $sp : \overline{\eta}^I \to \Delta(\overline{\eta})$. Let $E/\mathbb{Q}_p$ be a finite extension (our field of coefficients).

Lemma 0 : (i) If $\mathcal{F}$ is a lisse (constructible) $\mathcal{O}_E$-sheaf (or E-sheaf) on a dense open subset of X^I, admitting "partial Frobenius isomorphisms" (i.e. $F_{\{i\}} : \mathrm{Frob}_{\{i\}}^* \mathcal{F}|_{\eta^I} \to \mathcal{F}|_{\eta^I}$ for every $i \in I$, which commute with each other and compose to the usual Frobenius). Then, here exists an open nonempty subset U such that $\mathcal{F}$ extends to a lisse sheaf on U^I.

(ii) Fix $U \subseteq X$ nonempty. Let $\mathcal{C}(U, I, \mathcal{O}_E)$ be the category of lisse (constructible) $\mathcal{O}_E$-sheaves on U^I admitting partial Frobenius isomorphisms. Then $\mathcal{F} \mapsto \mathcal{F}|_{\Delta(\overline{\eta})}$ is an equivalence between $\mathcal{C}(U, I, \mathcal{O}_E)$ and the category of representations of $\pi_1(U, \overline{\eta})^I$ on $\mathcal{O}_E$-modules of finite type.

Note also that $sp^* : \mathcal{F}|_{\Delta(\overline{\eta})} \to \mathcal{F}|_{\overline{\eta}^I}$ is an isomorphism.

Idea of proof: (i) Let $\Omega \subseteq X^I$ be the biggest open on which $\mathcal{F}$ extends to a lisse sheaf, and let $\Delta = X^I \setminus \Omega$. We want to show that if Δ is finite over each factor X we know Δ is stable by all $\mathrm{Frob}_{\{i\}}$. An easy induction reduces to case $|I| = 2$, $I = \{i_1, i_2\}$. Let Δ_0 be an irreducible component of Δ. If $\pi_{i_j} : \Delta_0 \to X$ are both surjective then ?????. (Proof in class had mistake, exercise).

For proof of (ii), see lemmas below.

Lemma 1: Let $Y_0/\mathbb{F}_q$ be a finite-type scheme and let $k/\mathbb{F}_q$ be algebraically closed. Let $Y = Y_0 \otimes_{\mathbb{F}_q} k$ and $\tau = (\mathrm{id}_{Y_0} \otimes \mathrm{Frob}_k)^*$. Then the obvious functor Φ taking the category of coherent sheaves on Y_0 to coherent sheaves $\mathcal{G}$ on Y with $\psi : {}^\tau\mathcal{G} \cong \mathcal{G}$ is fully faithful, and is an equivalence if Y is projective.

Proof: It's obvious that Φ is fully faithful, so assume Y_0 is projective and choose $\mathcal{O}(1)$ a very ample line bundle. Then $\mathcal{F} \mapsto \bigoplus H^0(Y_0, \mathcal{F}(n))$ is an equivalence of coherent sheaves on Y_0 with finite-type graded modules over $\bigoplus H^0(Y_0, \mathcal{O}(n))$ modulo graded modules that are 0 in degree $\gg 0$, and similarly for Y. So reduce to the case $Y_0 = \operatorname{Spec} \mathbb{F}_q$, and we have to prove the following lemma.

Lemma 2: Let $k/\mathbb{F}_q$ be as before, and V a finite-dimensional k-vector space. Let $\psi : V \to V$ be Frob_k-linear (where Frob_k is $x \mapsto x^q$), and let $V_0 = \ker(\psi - \mathrm{id}_V)$. Then $V_0 \otimes_{\mathbb{F}_q} k \to V$ is injective, and it's an isomorphism if ψ is an isomorphism.

Proof: If $V_0 \otimes_{\mathbb{F}_q} k \to V$ is not injective then get $\alpha_l, \dots, \alpha_n \in k^\times$ and $e_1, \dots, e_n$ linearly independent over $\mathbb{F}_q$ but such that

$$\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$$

in V. Assume n is minimal such that this holds. Applying ψ get

$$\alpha_1^q e_1 + \cdots + \alpha_n^q e_n = 0,$$

and by minimality get there exists c such that $\alpha_i^q = c\alpha_i$. So $(\alpha_i/\alpha_j)^q = \alpha_i/\alpha_j$ for all i, so these are in $\mathbb{F}_q$, so

$$e_1 + \frac{\alpha_2}{\alpha_1} e_2 + \cdots + \frac{\alpha_n}{\alpha_1} e_n = 0$$

is a dependence relation over $\mathbb{F}_q$. Contradiction.

Now assume ψ is an isomorphism, and let $n = \dim_k V$. We want to show that $n = \dim_{\mathbb{F}_q} V_0$, i.e. $|V_0| = q^n$. Consider the closed subscheme Z of $\mathrm{GL}_n \times \mathbb{A}^n$ over k defined by equation

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - g \begin{bmatrix} x_1^q \\ \vdots \\ x_n^q \end{bmatrix} = 0.$$

Then V_0 is isomorphic to the fiber over g_0 the matrix of ψ in some basis. Then $\pi_1 : Z \to \mathrm{GL}_n$ is a finite commutative group scheme, affine and quasi-finite étale. Claim Z is closed in $\mathrm{GL}_n \times \mathbb{P}^n$, where we embed $\mathbb{A}^n$ as $[1 : x_1 : \cdots : x_n]$. Define a homogeneous version of the same equation as

$$x_0^{q-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - g \begin{bmatrix} x_1^q \\ \vdots \\ x_n^q \end{bmatrix} = 0.$$

and find it doesn't have any solutions for $x_0 = 0$, so defines Z in $\mathrm{GL}_n \times \mathbb{P}^1$. So $Z \to \mathrm{GL}_n$ is a finite étale group scheme, hence has constant degree. So $|V_0|$ is degree of the matrix of ψ, which is degree over $1 \in \mathrm{GL}_n <$ which is q^n.

Lemma 3: Let Y_0 be a smooth scheme over $\mathbb{F}_q$, $k/\mathbb{F}_q$ be algebraically closed, $Y = Y_0 \otimes_{\mathbb{F}_q} k$, $F = \mathrm{Frob}_{Y_0} \otimes \mathrm{id}_k$ the relative Frobenius. Then $z_0 \mapsto z_0 \otimes_{\mathbb{F}_q} k$ induces an equivalence of finite étale covers of Y_0 with finite étale covers Z of Y with $\beta : Z \cong F^* Z$.

Proof: First note that giving β is the same as giving an isomorphism $\alpha : {}^\tau Z \cong \mathbb{Z}$. This functor Ψ is fully faithful by Lemma 1, so it's enough to show essential surjectivity locally on Y_0. So we may assume Y_0 is affine. Choose a projective scheme $\widetilde{Y}_0$ and open embedding $Y_0 \hookrightarrow \widetilde{Y}_0$. Let (Z, α) be in the RHS category. Let L/K be ring of fractions of $\mathcal{O}(Z)/\mathcal{O}(Y)$, and let $\widetilde{Z}$ be the normalization of $\widetilde{Y} = \widetilde{Y}_0 \otimes_{\mathbb{F}_q} k$ in L.

Over Y, $\widetilde{Z}$is just Z. As τ does not change the underlying scheme, ${}^\tau \widetilde{Z}$ is the normalization of ${}^\tau \widetilde{Y}$ in ${}^\tau L$, so α extends to ${}^\tau \widetilde{Z} \cong \widetilde{Z}$. Apply Lemma 1 to the coherent $\mathcal{O}_{\widetilde{Y}}$-module $p_* \mathcal{O}_{\widetilde{Z}}$ (for $p : \widetilde{Z} \to \widetilde{Y}$) and its algebra structure, descending it to $\mathcal{O}_{\widetilde{Y}_0}$.

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Lemma 4: Let $Y_{1,0}$ and $Y_{2,0}$ be two smooth schemes over $\mathbb{F}_q$, $Y_i = T_{i,0} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, $Y = Y_1 \times Y_2$. Let F_i be the "partial relative Frobenius" on Y defined by

$$F_i = (\mathrm{Frob}_{Y_{i,0}} \otimes \mathrm{id}_{\overline{\mathbb{F}}_q}) \times \mathrm{id}_{Y_{3-i}}\,.$$

Then we have an equivalence from the categories of maps $\rho : \pi_1(Y_{1,0}) \times \pi(Y_{2,0}) \to \mathrm{Aut}(A)$ for A a finite set, to the category of finite étale maps $Z \to Y$ together with isomorphisms $F_i^* Z \cong Z$. The functor here starts with ρ, moves it to a map $\pi_1(Y_{1,0} \times Y_{2,0}) \to \mathrm{Aut}(A)$ by composing with $\pi_1(Y_{1,0} \times Y_{2,0}) \to \pi_1(Y_{1,0}) \times \pi_2(Y_{2,0})$, using this to get a finite étale map $Z_0 \to Y_{1,0} \times Y_{2,0}$ and then base-changing to $\overline{\mathbb{F}}_q$.

Proof: As usual, the hard part is essential surjectivity. So suppose we've fixed $Z \to Y$ finite étale together with isomorphisms $F_i^* Z \cong Z$ for $i = 1, 2$. Let K_i be the field of fractions of $\mathcal{O}(Y_i)$ and $K = K_1 \times K_2$ (the field of fractions of Y). Then get commutative diagram

$$\begin{tikzcd} & \pi_1(Y_1 \otimes \overline{K}_2) \arrow[r, two heads] \arrow[d, hook] & \pi_1(Y_1) \arrow[r] \arrow[d, hook] & \pi_1(Y_{1,0}) \\ \pi_1(K) \arrow[r] & \pi_1(Y_1 \otimes K_2) \arrow[r, two heads] & \pi_1(Y_1) \times \pi_1(K_2) & \end{tikzcd}.$$

Now, Z corresponds to $\pi_1(Y) \to \mathrm{Aut}(A)$ with A a finite set. By Lemma 3, $\rho|_{\pi_1(Y_1 \otimes \overline{K}_2}$ factors through $\pi_1(Y_{1,0})$ and thus through $\pi_1(Y_1)$. So $\rho|_{\pi_1(Y_1 \otimes K_2)}$ factors through $\pi_1(Y_1) \times \pi_1(K_2)$, hence so does $\rho|_{\pi_1(K)}$. Similarly $\rho|_{\pi_1(K)}$ factors through $\pi_1(K_1) \times \pi_1(Y_2)$. Conclude ρ factors through $\pi_1(Y_1) \times \pi_1(Y_2)$. (?)

Application (Lemma 0 part ii): Let $U \subseteq X$ be open nonempty, $\mathcal{F}$ a lisse ℓ-adic sheaf over U^I with partial Frobenius morphisms. Then the ℓ-adic representation of $\pi_1(U^I)$ corresponding to $\mathcal{F}$ factors through $\pi_1(U)^I$.

Hecke-finite cohomology . We want to apply Drinfeld's lemma to $\mathcal{H}^{\le\mu}_{N,I,W} \in \mathbf{Perv}(U^I)$ (which is lisse). But there's a bit of an issue because these sheaves aren't stable by partial Frobenius morphisms - they increase μ. What these morphisms really act on is $\varinjlim_\mu \mathcal{H}^{\le\mu}_{N,I,W}$, which is not finite type anymore.

Definition: Let $\overline{x}$ be a geometric point of X^I (we only really care about $\overline{x} = \overline{\eta}^I$ or $\overline{x} = \Delta(\overline{\eta})$). We say an element of $\varinjlim_\mu \mathcal{H}^{\le\mu}_{N,I,W}|_{\overline{x}}$ is *Hecke-finite* if it is contained in a finite-type $\mathcal{O}_E$-submodule (for $E/\mathbb{Q}_\ell$ finite the field of coefficients) that is stable by all Hecke operators $T(f)$ for $f \in C_c(K_N\backslash G(\mathbb{A})/K_n, \mathcal{O}_E)$. Let $(\varinjlim \mathcal{H}^{\le\mu}_{N,I,W}|_{\overline{x}})^{HF}$ be the set of Hecke-finite element.

V. Lafforgue conjectures (and hopes to prove, but hasn't so far) conjectures that $(\varinjlim \mathcal{H}^{\le\mu}_{N,I,W}|_{\overline{x}})^{HF}$ is finite-dimensional, which means we could apply Drinfeld's lemma immediately to the corresponding lisse sheaf. Fortunately we only need the following weaker thing to apply Drinfeld's lemma.

Claim: (1) $(\varinjlim \mathcal{H}^{\le\mu}_{N,I,W}|_{\overline{\eta}^I})^{HF}$ is a union of finite-type $\mathcal{O}_E$-submodules that are stable by $C_c(K_N\backslash G(\mathbb{A}_F)/K_N, \mathcal{O}_E)$ and the partial Frobenius.
(2) Remembering we fixed a specialization map $sp : \overline{\eta}^I \to \Delta(\overline{\eta})$. The specialization morphism

$$sp^* : (\varinjlim \mathcal{H}^{\le\mu}_{N,I,W}|_{\Delta(\overline{\eta})})^{HF} \to (\varinjlim \mathcal{H}^{\le\mu}_{N,I,W}|_{\overline{\eta}^I})^{HF}$$

is an isomorphism.

Then, we take $H_{I,W} = (\varinjlim \mathcal{H}^{\le\mu}_{N,I,W}|_{\Delta(\overline{\eta})})^{HF}$, which by Drinfeld's lemma and (2) have a canonical action of $\mathrm{Gal}(\overline{F}/F)^I$. These are the things we used to define the excursion maps $S_{I,f,(\gamma_i)_{i\in I}}$. So what about the rest of the proof (and the proof of this claim):

- We'll have some Eichler-Shimura relations that, together with (1) will imply (2). (Very straightforward argument)
- The Eichler-Shimura relations will also imply (1).

- To get the Eichler-Shimura relations, we need to identify unramified Hecke operators with some excursion operators.

Eichler-Shimura relations : Fix I, N, μ, W as before. Let $f \in C_c(K_N\backslash G(\mathbb{A}_F)/K_N, \mathcal{O}_E)$ (or replace $\mathcal{O}_E$ with E). If M is the set of places where f is not trivial, then $T(f)$ is a map

$$\mathcal{H}^{\leq\mu}_{N,I,W}|_{(X\setminus(N\cup M))^I} \to \mathcal{H}^{\leq\mu+\kappa}_{N,I,W}|_{(X\setminus(N\cup M))^I}$$

for some κ depending on f.

Creation and annihilation operators: Given J finite, $U \in \mathbf{Rep}_{\widehat{G}^J}$, $\zeta_J : J \to \{0\}$, let $U^{\zeta_J} \in \mathbf{Rep}_{\widehat{G}}$ be U with the diagonal action of $\widehat{G}$. Fix $x \in (U^{\zeta_J})^{\widehat{G}}$ and $\xi \in (U^{\zeta_J,*})^{\widehat{G}}$. Then have $\mathrm{id}_W \boxtimes x : W \boxtimes 1 \to W \boxtimes U^{\zeta_J}$ and $\mathrm{id}_W \boxtimes \xi : W \boxtimes U^{\zeta_j} \to W \boxtimes 1$. Let $E_{X\setminus N}$ be the constant sheaf on $X \setminus N$, and $\Delta = \Delta_{\zeta_J} : X \to X^J$ be the diagonal map

Definition: The creation operator $C^{\sharp}_x$ is the composition of the canonical isomorphism

$$\mathcal{H}^{\leq\mu}_{N,I,W} \boxtimes E_{X\setminus N} \cong \mathcal{H}^{\leq\mu}_{N,I\sqcup\{0\},W\boxtimes 1}$$

on $(X \setminus N)^{I\sqcup\{0\}}$, followed by

$$H(x) : \mathcal{H}^{\leq\mu}_{N,I\sqcup\{0\},W\boxtimes 1} \to \mathcal{H}^{\leq\mu}_{N,I\sqcup J,W\boxtimes U^{\zeta_J}}$$

and then the inverse of the fusion isomorphism

$$\mathcal{H}^{\leq\mu}_{N,I\sqcup J,W\boxtimes U}|_{(X\setminus N)^I\times\Delta(X\setminus N)} \to \mathcal{H}^{\leq\mu}_{N,I\sqcup J,W\boxtimes U^{\zeta_J}}.$$

The annihilation operator $C^{\flat}_{\xi}$ is the composition of the fusion isomorphism

$$\mathcal{H}^{\leq\mu}_{N,I\sqcup J,W\boxtimes U}|_{(X\setminus N)^I\times\Delta(X\setminus N)} \to \mathcal{H}^{\leq\mu}_{N,I\sqcup J,W\boxtimes U^{\zeta_J}}.$$

with $H(\xi)$ and then the inverse of the isomorphism

$$\mathcal{H}^{\leq\mu}_{N,I,W} \boxtimes E_{X\setminus N} \cong \mathcal{H}^{\leq\mu}_{N,I\sqcup\{0\},W\boxtimes 1}.$$

Now, fix $v \in |X \setminus N|$, $V \in \mathbf{Rep}_{\widehat{G}}$ irreducible. Get

$$h_{V,v} \in C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \mathcal{O}_E)$$

by classical Satake. Let $\delta_V : 1 \to V \otimes V^*$ and $ev_v : V \otimes V^* \to 1$ the obvious maps from adjunction. Let $S_{V,v}$ be the composition of

$$C^{\sharp}_{\delta_V} : \mathcal{H}^{\leq\mu}_{N,I,W} \boxtimes E_v \to \mathcal{H}^{\leq\mu}_{N,I\sqcup\{1,2\},W\boxtimes V\boxtimes V^*}|_{(X\setminus N)^I\times\Delta(v)}$$

with

$$F^{\deg v}_{\{1\}} : \mathcal{H}^{\leq\mu}_{N,I\sqcup\{1,2\},W\boxtimes V\boxtimes V^*}|_{(X\setminus N)^I\times\Delta(v)} \to \mathcal{H}^{\leq\mu+\kappa}_{N,I\sqcup\{1,2\},W\boxtimes V\boxtimes V^*}|_{(X\setminus N)^I\times\Delta(v)}$$

and then

$$C^{\flat}_{ev_v} : \mathcal{H}^{\leq\mu+\kappa}_{N,I\sqcup\{1,2\},W\boxtimes V\boxtimes V^*}|_{(X\setminus N)^I\times\Delta(v)} \to \mathcal{H}^{\leq\mu+\kappa}_{N,I,W} \boxtimes E_v.$$

Then $S_{V,v}$ descends to a map

$$S_{V,v} : \mathcal{H}^{\leq\mu}_{N,I,W} \to \mathcal{H}^{\leq\mu+\kappa}_{N,I,W}.$$

Crucial Theorem (to be proved later): $S_{V,v}|_{(X\setminus(N\cup\{v\}))^I} = T(h_{V,v})$.

Some propositions following from the theorem:

Proposition 1: For all $f \in C_c(K_N\backslash G(\mathbb{A}_F)/K_N, \mathcal{O}_E)$, $T(f)$ extends to a morphism $T(f) : \mathcal{H}^{\leq\mu}_{N,I,W} \to \mathcal{H}^{\mu+\kappa}_{N,I,W}$ on $(X \setminus N)^I$ (in a way compatible with composition of Hecke operators).

Proof: Assume $f = \otimes f_v$ and extend each $T(f_v)$ defined on $(X \setminus N \cup \{v\})^I$. If $v \in N$, nothing to do. If $v \notin N$,

$$f_v \in C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \mathcal{O}_E),$$

so we may assume $f_v = h_{V,v}$ for some V and then use $S_{V,v}$ to extend $T(f_v)$.

Proposition 2 (Eichler-Shimura relation): Let $V \in \mathbf{Rep}_{\widehat{G}}$ be irreducible and $v \in |X \setminus N|$. Consider $F_{\{0\}}^{\deg v}$ as an endomorphism on

$$\varinjlim_{\mu} \mathcal{H}_{N, I \sqcup \{0\}, W \boxtimes V}^{\leq \mu}|_{(X \setminus N)^I \times v}.$$

Then

$$\sum_{i=0}^{\dim V} (-1)^i (F_{\{0\}}^{\deg v})^i \circ S_{\bigwedge^{\dim V - i} V, v}|_{(X \setminus N)^I \times v} = 0$$

(Strictly speaking this is independent of the crucial theorem, but we use that to interpret the $S_{\bigwedge^{\dim V - i} V,v}$ as an extension of a Hecke operator and thus make this look like a classical Eichler-Shimura relation).

Proof: For all J finite, let

$$\mathcal{A}_J = \frac{1}{|J|!} \sum_{\sigma \in S_J} \operatorname{sgn}(\sigma)\sigma \in \mathbb{Q}[S_J],$$

so $\mathcal{A}_J^2 = \mathcal{A}_J$. If V' is an E-vector space, $\mathcal{A}_J$ acts on $(V')^{\otimes J}$ and its image is $\bigwedge^{|J|} V'$. For all $n \in \mathbb{N}$ and all $U \in \operatorname{End}(V^{\otimes \{0,\ldots,n\}})$ let $C_n(U)$ be the composition of

$$C^{\sharp}_{\delta_v^{\otimes n}} : \mathcal{H}_{N,I,W \boxtimes V}^{\leq \mu}|_{(X \setminus N)^I \times v} \to \mathcal{H}^{\leq \mu}_{N, I \sqcup \{0\} \sqcup \{1,\ldots,2n\} W \boxtimes V \boxtimes V^{\boxtimes n} \boxtimes V^{*\boxtimes n}}|_{(X \setminus N)^I \times \Delta(v)}$$

with $H(\mathrm{id}_W \boxtimes U \boxtimes \mathrm{id}_{(V^*)^{\boxtimes n}})$ and then $\prod_{j=1}^n (F_{\{j\}})^{\deg v}$ and finally

$$C^{\flat}_{ev_v^{\boxtimes n}} : \mathcal{H}^{\leq \mu + n\deg(v)\kappa}_{N, I \sqcup \{0\} \sqcup \{1,\ldots,2n\} W \boxtimes V \boxtimes V^{\boxtimes n} \boxtimes V^{*\boxtimes n}}|_{(X \setminus N)^I \times \Delta(v)} \to \mathcal{H}^{\leq \mu + n\deg(v)\kappa}_{N,I,W \boxtimes V}|_{(X \setminus N)^I \times v}.$$

Claim: For all n,

$$C_n(\mathcal{A}_{\{0,\ldots,n\}}) = \frac{1}{n+1} \sum_{i=0}^{n} (-1)^i (F_{\{0\}}^{\deg v})^i \circ S_{\bigwedge^{\dim V - i} V, v}.$$

If we apply the claim for $n = \dim V$, $\mathcal{A}_{\{0,\ldots,\dim V\}}$ acts by 0 to on $V^{\otimes \{0,\ldots,\dim V\}}$, so $C_{\dim V}(\mathcal{A}_{\{0,\ldots,\dim V\}})$ and so we get the proposition.

Proof of the claim: Finish next time.

23 Lecture - 12/09/2014

Remember from last time: I finite, N the level, $W \in \mathbf{Rep}_{\widehat{G}^I}$. Defined creation morphism, annihilation morphism, and then the operator

$$S_{V,v} : \mathcal{H}^{\leq \mu}_{N,I,W} \boxtimes E_v \to \mathcal{H}^{\leq \mu+\kappa}_{N,I,W} \boxtimes E_v$$

for $v \in |X \setminus N|$ and $V \in \mathbf{Rep}_{\widehat{G}}$. This descends to a morphism $\mathcal{H}^{\leq \mu}_{N,I,W} \to \mathcal{H}^{\leq \mu+\kappa}_{N,I,W}$.

Crucial theorem (still to be proved): If V is irreducible, then $S_{V,v}$ restricted to $(X \setminus (N \cup v))^I$ is equal to the unramified Hecke operator $T(h_{V,v})$ at v corresponding to V by Satake.

Proposition 2 (Eichler-Shimura relation): Consider

$$F^{\deg(v)}_{\{0\}} : \varinjlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I\sqcup\{0\},W\boxtimes V}|_{(X\setminus N)^I \times v} \to \varinjlim_{\mu} \mathcal{H}^{\leq \mu}_{N,I\sqcup\{0\},W\boxtimes V}|_{(X\setminus N)^I \times v}.$$

Then

$$\sum_{i=0}^{\dim V} (-1)^i (F^{\deg v}_{\{0\}})^i \circ S_{\bigwedge^{\dim V - i} V,v}|_{(X\setminus N)^I \times v} = 0$$

Proof: From last time reduce to the claim that for all n

$$C_n(\mathcal{A}_{\{0,\ldots,n\}}) = \frac{1}{n+1}\sum_{i=0}^{n}(-1)^i (F^{\deg v}_{\{0\}})^i \circ S_{\bigwedge^{\dim V - i} V,v}.$$

For all $\sigma \in S_{\{0,\ldots,n\}}$, let $\ell(\sigma, 0)$ be the length of the cycle containing 0. Then we further reduce to the claim that, for each i,

$$C_n\left(\frac{1}{n!}\sum_{\sigma:\ell(\sigma,0)=i+1} \operatorname{sgn}(\sigma)\sigma\right) = (-1)^i (F^{\deg v}_{\{0\}})^i \circ S_{\bigwedge^{\dim V - i} V,v}.$$

Fix $i \in \{0, \ldots, n-1\}$. Note that $C_n(\mu)$ does not change if we compose by σ with $\sigma(0) = 0$. So

$$C_n\left(\frac{1}{n!}\sum_{\sigma:\ell(\sigma,0)=i+1}\right) = C_n\left(\frac{1}{(n-i)!}\sum_{\sigma=(0\ 1\ \cdots\ i)\cdots} \operatorname{sgn}(\sigma)\sigma\right).$$

Now, what's this remaining sum? At this point we've totally separated what our σ's does to $\{0, \ldots, i\}$ and $\{i+1, \ldots, n\}$, so the legs for $\{0, \ldots, i\} \cup \{n+1, \ldots, n+i\}$ and $\{i+1, \ldots, n\} \cup \{n+i+1, \ldots, 2n\}$ play independent roles. So what happens to them?

First, the legs at $\{i+1, \ldots, n\} \cup \{n+i+1, \ldots, 2n\}$:

1. We create the pairs of legs $(i+1, n+i+1), \ldots, (n, 2n)$ by δ_V.
2. We apply
$$\frac{1}{(n-i)!}\sum_{\tau \in S_{\{i+1,\ldots,n\}}} \operatorname{sgn}(\tau)\tau = \mathcal{A}_{\{i+1,\ldots,n\}}$$
to the legs in $\{i+1, \ldots, n\}$.
3. We apply the partial Frobenius to the legs $\{i+1, \ldots, n\}$.
4. We destroy the legs by the pairs $(i+1, n+i+1), \ldots, (n, 2n)$ by ev_v.

The result is $S_{\bigwedge^{\dim V - i} V,v}$ (or $S_{V^{\otimes(\dim V - i)},v}$ if we don't do (2)).

What about the legs in $\{0, \ldots, i\} \cup \{n+1, \ldots, n+i\}$? Renumber the legs by the bijection

$$\{0, n+1, 1, n+2, \ldots, n+i, i\} \leftrightarrow \{0, 1, 2, \ldots, 2i\}$$

(so the first set of legs become even numbers and the second set becomes odd numbers). Then:

1. First we create pairs of legs $(1,2),\ldots,(2i-1,2i)$ using δ_V.
2. We apply the partial Frobenius at $\{2,4,6,\ldots,2i\}$.
3. We kill the pairs of legs $(0,1),\ldots,(2i-2,2i-1)$.

If $i=1$ we get $F_{\{0\}}^{\deg v}$ using that if μ using the fact that for all μ,

$$V \xrightarrow{\mathrm{id}_V \otimes \delta_V} V\otimes V^* \otimes V \xrightarrow{\mathrm{id}_V\otimes \mathrm{id}_V \otimes \mu} V\otimes V^*\otimes V \xrightarrow{ev_V \otimes \mathrm{id}_V} V$$

is equal to μ. By induction on i, the obvious analog of this computation for $i\geq 1$ gives that the answer to the question above is $(F_{\{0\}}^{\deg v})^i$.

Aside: How is this inspired by a proof of Cayley-Hamilton? Well, if we have V and $\mu\in\mathrm{End}(V)$ then we want

$$\sum_{i=1}^{\dim V}(-1)^i \operatorname{tr}(\bigwedge^{\dim V-i}\mu)\mu^i=0.$$

Then, if $n\in\mathbb{N}$ and $U\in\mathrm{End}(V^{\otimes\{0,\ldots,n\}})$, let $C_n(U)$ be the composition of $\mathrm{id}_V\otimes\delta_V\otimes n$ with $U\otimes \mathrm{id}_{(V^*)^{\otimes n}}$, $\mathrm{id}_V\otimes\mathrm{id}_{(V^*)^{\otimes n}}\otimes\mu^{\otimes n}$ and then $(ev_v)^{\otimes n}\otimes\mathrm{id}_V$. Then claim that $C_n(\mathcal{A}_{\{0,\ldots,n\}})=\sum(-1)^i\operatorname{tr}(\bigwedge^{\dim V-i}\mu)\mu^i$, which follows from a very similar proof.

Claim (1) from last time : Now that we have the Eichler-Shimura relation, which we called (2), we want to get statement (1), that

$$\left(\varinjlim_{\mu}\mathcal{H}^{\leq\mu}_{N\cap,W|_{\overline{\eta}^I}}\right)^{HF}$$

is a union of finite-type $\mathcal{O}_E$-submodules stable by the actions of $C_c(K_N\backslash G(\mathbb{A}_F)/K_N,\mathcal{O}_E)$ and the partial Frobenius. (We can then apply Drinfeld's lemma to these finite-type parts and get what we want).

So how do we prove this? We may assume $W=\boxtimes_{i\in I}W_i$. Let

$$\mathcal{N}\subseteq\left(\varinjlim_{\mu}\mathcal{H}^{\leq\mu}_{N\cap,W|_{\overline{\eta}^I}}\right)^{HF}$$

be a finite-type $\mathcal{O}_E$-submodule stable by Hecke operators. We may assume $\mathcal{N}\subseteq\mathcal{H}^{\leq\mu_0}_{N,I,W|_{\overline{\eta}^I}}$ for some μ_0. Let $U\subseteq X^I$ be open dense such that $\mathcal{H}^{\leq\mu_0}_{N,I,W}$ is lisse on U. Then $\mathcal{N}=\mathcal{F}|_{\overline{\eta}^I}$ with $\mathcal{F}$ lisse on U. Let $(v_i)_{i\in I}\in(X\setminus N)^I$ be such that $\times_{i\in I}v_i\in U$. For all $i\in I$, Eichler-Shimura gives that

$$(F_{\{i\}}^{\deg(v_i)})^{\dim W_i}(\mathcal{F}|_{\times v_i})\subseteq\sum_{r=0}^{\dim W_i-1}(F_{\{i\}}^{\deg(v_i)})^r(S_{\bigwedge^{\dim W_i-r}W_i,v_i}\mathcal{F}|_{\times v_i})$$

in $\varinjlim\mathcal{H}^{\leq\mu}_{N,I,W|_{\times v_i}}$. But the LHS here is a lisse sheaf on X^I, so we have a similar inclusion of subschemes of $(\varinjlim_{\mu}\mathcal{H}^{\leq\mu}_{N,I,W})|_{\overline{\eta}^I}$. As $\mathcal{F}|_{\overline{\eta}^I}$ is stable by $S_{\bigwedge^{\dim W_i-r}W_i}$ (by the crucial theorem that this is a Hecke operator) so we get that

$$F_{\{i\}}^{\deg(v_i)\dim(W_i)}(\mathcal{N})\subseteq\sum_{r=0}^{\dim W_i-1}F_{\{i\}}^{\deg(v_i)r}(\mathcal{N})$$

in $\varinjlim_{\mu}\mathcal{H}^{\leq\mu}_{N,I,W|_{\overline{\eta}^I}}$. So $\mathcal{N}'$ is finite type over $\mathcal{O}_E$, stable by Hecke operators and partial Frobenius, where

$$\mathcal{N}=\sum_{(n_i)\in\mathbb{N}^I:0\leq n_i\leq\dim(W_i)\deg(v_i)-1}\prod_{i\in I}F_{\{i\}}^{n_i}(\mathcal{N}).$$

Proof of the crucial theorem : Recall the theorem was if $v \in |X \setminus N|$ and $V \in \mathbf{Rep}_{\widehat{G}}$ is irreducible, then $S_{V,v} = T(h_{V,v})$ as morphisms

$$\mathcal{H}^{\leq\mu}_{N,I,W}|_{(X\setminus(N\cup v))^I} \to \mathcal{H}^{\leq\mu+\kappa}_{N,I,W}|_{(X\setminus(N\cup v))^I}.$$

Simple case to do first: $\deg(v) = 1$, the highest weight ω_V of V is minuscule. (Remark: ω_v minuscule iff ω_V is minimal in $X_*^+(T)$, iff the weights of V are the Weyl group translates of ω_V, which implies $Orb(t^{\omega_v}) = \overline{Orb(t^{\omega_v})}$ is smooth). Also, assume W is irreducible (which is a harmless simplification)

24 Lecture - 12/11/2014

Still studying the map $S_{V,v}$ (for $v \in |X \setminus N|$ and $V \in \mathbf{Rep}_{\widehat{G}}$ irreducible). Main theorem we want to prove: $S_{V,v}|_{(X\setminus(N\cup v))^I} = T(h_{V,v})$. Recall we're first doing the case where $\deg v = 1$ and the highest weight ω_V of V is minuscule.

Correspondences: Let X_1, X_2 be DM stacks of locally of finite type. A *correspondence* from X_1 to X_2 is a morphism $a = (a_1, a_2)$ with a_2 schematic of finite type. (The case where $a_2 = \mathrm{id}$ corresponds to an actual morphism). A *cohomological correspondence* from $\mathcal{F}_1 \in D^b_c(X_1, \overline{\mathbb{Q}}_\ell)$ to $\mathcal{F}_2 \in D^b_c(X_2, \overline{\mathbb{Q}}_\ell)$ with support in a (or A) is a map $\mu : a_1^*\mathcal{F}_1 \to a_2^!\mathcal{F}_2$.

If a_1 is proper, if $f_i : X_i \to S$ are such that $f_1 a_1 = f_2 a_2$, then μ induces a map $H(\mu)$

$$f_{1!}\mathcal{F}_1 \xrightarrow{adj} f_{1!}a_{1*}a_1^*\mathcal{F}_1 \xrightarrow{=} f_{1!}a_{1!}a_1^*\mathcal{F}_1 \xrightarrow{\mu} f_{2!}a_{2!}a_2^!\mathcal{F}_2 \xrightarrow{adj} f_{2!}\mathcal{F}_2 \quad .$$

Sometimes there's a canonical μ with support in a.

Example: if X_1, X_2, A are smooth of dimensions d_1, d_2, d then $a_1^*\overline{\mathbb{Q}}_{\ell,X_1} = \overline{\mathbb{Q}}_{\ell A}$ and $a_2^!\overline{\mathbb{Q}}_{\ell,X_2} = \overline{\mathbb{Q}}_{\ell,A}(d - d_2)[2(d-d_2)]$ so we have $\mu = \mathrm{id}$ is a correspondence from $\overline{\mathbb{Q}}_{\ell,X_1}(d-d_2)[2(d-d_2)]$ to $\overline{\mathbb{Q}}_{\ell,X_2}$.

The Hecke correspondence $T(h_{V,v})$. Write $\mathcal{Z}^{(I)} = Cht_{N,I,W}|_{(X\setminus(N\cup v))^I}$. Then $T(h_{V,v}) = H(\mu)$ where μ is a cohomological correspondence from $\mathcal{F}^{\leq\mu}_{N,I,W}$ to itself, with support in $\Gamma^{(I)}$ where $\Gamma^{(I)}(S)$ is the set of tuples of (x_i) and diagrams

$$\begin{tikzcd}
(\mathcal{G}', \psi') \arrow[r, "\varphi'"] & ({}^\tau\mathcal{G}', {}^\tau\psi') \\
(\mathcal{G}, \psi) \arrow[u, "\kappa"] \arrow[r, "\varphi"'] & ({}^\tau\mathcal{G}, {}^\tau\psi) \arrow[u, "{}^\tau\kappa"']
\end{tikzcd}$$

where κ gives an isomorphism of $\mathcal{G}$ and $\mathcal{G}'$ on $X \times S \setminus \Gamma_v$ (compatible with ψ, ψ') and such that for all λ we have

$$\kappa(\mathcal{G}_\lambda) = \mathcal{G}^\wedge_\lambda(\langle \lambda, \omega_V\rangle \Gamma_v)$$

(here $=$ and $\subseteq$ are equivalent because $\omega - v$ is minuscule). Then, our maps $a_1, a_2 : \Gamma^{(I)} \to \mathcal{Z}^{(I)}$ take the diagram to the lower line and the upper line, respectively; since ω_v is minuscule these are in fact finite étale. Hence

$$a_1^*\mathcal{F}^{\leq\mu}_{N,I,W} = a_2^!\mathcal{F}^{\leq\mu}_{N,I,W} = IC_{\Gamma^{(I)},\leq\mu}.$$

Claim that this first equality is actually our μ; this is because ω_V is minuscule so $h_{V,v} = 1_{G(\mathcal{O}_v)t^{\omega_V}G(\mathcal{O}_v)}$.

Now, we need to write $S_{V,v}$ in the same way. Recall our annihilation operator is from

$$C^\flat : \mathcal{H}_{N,I\sqcup\{1,2\},W\boxtimes V\boxtimes V^*}|_{(X\setminus(N\cup v))^I\times\Delta(v)} \to \mathcal{H}^{\leq\mu}_{N,I,W}|_{(X\setminus(N\cup v))^I\times\Delta(v)}$$

with the domain coming from the sheaf $\mathcal{F}$ (with the same decorations) on

$$Cht^{(1,2,I)}_{N,I\sqcup\{1,2\},W\boxtimes V\boxtimes V^*}|_{(X\setminus(N\cup v))^I\times\Delta(v)}.$$

Let this stack be $\mathcal{Z}^{(1,2,I)}$; so we want a correspondence form $\mathcal{Z}^{(1,2,I)}$ to $\mathcal{Z}^{(I)}$ for the corresponding sheaves $\mathcal{F}^{\leq\mu}$ (with the same lower subscripts).

Let $\iota_1 : \mathcal{Y}_1 \hookrightarrow \mathcal{Z}^{(1,2,I)}$ be the closed substack where $\varphi_2\varphi_1$ extends to an isomorphism on $X \times S$. Similarly let $\alpha_1 : \mathcal{Y}_1 \to \mathcal{Z}^{(I)}$ be given by taking a diagram in $\mathcal{Y}_1$ to $\varphi_3(\varphi_2\varphi_1) : (\mathcal{G}_0, \psi_0) \to ({}^\tau\mathcal{G}_0, {}^\tau\psi_0)$. Then α_1 is smooth of relative dimension $\langle\rho, \omega_V\rangle$. Our geometric correspondence is then ι_1 and α_1.

Lots of details checking this correspondence gives what we want...