SHARP L^1 ESTIMATES FOR SINGULAR TRANSPORT EQUATIONS

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Abstract. We provide L^1 estimates for a transport equation which contains singular integral operators. The form of the equation was motivated by the study of Kirchhoff-Sobolev parametrices in a Lorentzian space-time verifying the Einstein equations. While our main application is for a specific problem in General Relativity we believe that the phenomenon which our result illustrates is of a more general interest.

1. Introduction

The goal of this paper is to prove an L^1 type estimate for solutions of the following transport equation,

$$\partial_t u(t,x) - a(t,x) M u(t,x) = g(t,x), \qquad u(0,x) = 0. \tag{1}$$

Here $a = a(t,x)$ and $g = g(t,x)$ are assumed to be smooth, compactly supported functions defined[1] on $[0,1]\times\mathbb{R}^2$ and M is a classical, translation invariant, Calderon-Zygmund operator in $\mathbb{R}^2$, given by a smooth[2] multiplier. Though, for simplicity, we shall proceed as if the equation (1) is scalar, all our results extend easily to systems, i.e. u and g take values in $\mathbb{R}^N$ and aM is a $N\times N$ matrix valued operator.

Ideally, the desired estimate would take the form

$$\sup_{t\in[0,1]} \|u(t)\|_{L^1(\mathbb{R}^2)} \le C(\|a\|_{L^\infty([0,1]\times\mathbb{R}^2)})\, \|g\|_{L^1([0,1]\times\mathbb{R}^2)}$$

As it is well known however such L^1-type estimates cannot possibly hold due to the failure of L^1 boundedness of Calderon-Zygmund operators. To illustrate this consider first the case of a constant coefficient transport equation with $a \equiv 1$. In this case we may write

$$u(t,x) = \int_0^t e^{(t-s)M} g(s)\,ds \tag{2}$$

where,

$$e^{tM} = I + tM + \frac{1}{2}(tM)^2 + \cdots + \frac{1}{n!}(tM)^n + \cdots$$

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[1] Similar results can be easily extended to higher dimensions.

[2] The smoothness assumption is only imposed to eliminate logarithmic divergences at infinity in $\mathbb{R}^2$, is irrelevant to our main concerns.

The problem of L^1 estimates for (1) is then reduced to the corresponding question for the operators M^n. Each of M^n is a Calderon-Zygmund operator and as such does not map L^1 to L^1. A well known way to resolve this problem is to consider instead mapping properties of the Hardy space[3] $\mathcal{H}_1$ to L^1. Since translation invariant Calderon-Zygmund operators M map $\mathcal{H}_1$ into $\mathcal{H}_1$ (see [Ste2]) we easily infer that a solution u of the transport equation

$$\partial_t u - Mu = g, \qquad u(0,x) = 0$$

belongs to the space $L^\infty([0,1];\mathcal{H}_1)$. Indeed,

$$\begin{aligned}\|u(t)\|_{\mathcal{H}_1} &\le \sum_{n=0}^{\infty}\int_0^t \frac{(t-s)^n}{n!}\|M^n g(s)\|_{\mathcal{H}_1} \le \sum_{n=0}^{\infty}\int_0^t \frac{C^n(t-s)^n}{n!}\|g(s)\|_{\mathcal{H}_1}\,ds\\ &\le e^{Ct}\int_0^t \|g(s)\|_{\mathcal{H}_1}\,ds\end{aligned}$$

While this may be considered a satisfactory solution of the problem for the transport equation (1) with constant coefficients, the situation changes drastically in the variable coefficient case. Consider the transport equation

$$\partial_t u - a(x)Mu = g, \qquad u(0,x) = 0 \tag{3}$$

with a time-independent coefficient $a(x)$. As before we may write

$$u(t,x) = \int_0^t e^{(t-s)aM} g(s)ds \tag{4}$$

where,

$$e^{t\,aM} = I + t\,aM + \frac{1}{2}(t\,aM)^2 + \cdots + \frac{1}{n!}(t\,aM)^n + \cdots$$

The multiplication operator a and Calderon-Zygmund operator M do not commute[4]. We need instead that the operator aM has the same mapping properties as M, i.e. it maps $\mathcal{H}_1$ to itself, in which case we would easily conclude that solutions of the transport equation (3) belong to the space $L^\infty([0,1];\mathcal{H}^1)$. To insure this condition we are led to the requirement that multiplication by the function $a = a(x)$ maps Hardy space into itself. It is well known however that a multiplication by a bounded function does not preserve $\mathcal{H}_1$. Instead, such a function a should satisfy the Dini condition

$$\int_0^\infty \sup_{|x-y|\le\lambda} |a(x)-a(y)|\,\frac{d\lambda}{\lambda} < \infty,$$

see [Steg]. Functions satisfying the Dini condition can not be sharply characterized in terms of the standard Lebesgue type spaces. Specifically, one can easily see that even if a is a single atom in the Besov space $B^0_{\infty,1}(\mathbb{R}^2)$ or even in $B^1_{2,1}(\mathbb{R}^2)$, both sharp Besov refinements of the $L^\infty(\mathbb{R}^2)$ space, does not guarantee that the Dini condition is satisfied. Yet, in view of the specific applications we have in mind, we need to consider precisely the situation when a belongs to the space $B^1_{2,1}$, and

[3]The classical Hardy space $\mathcal{H}_1$, defined by the norm $\|f\|_{\mathcal{H}_1} = \|f\|_{L^1(\mathbb{R}^2)} + \sup_{j=1,2}\|R_j f\|_{L^1(\mathbb{R}^2)}$, is a can be viewed as a logarithmic improvement of L^1. Here $R_j = (-\Delta)^{1/2}\partial_j$ are the standard Riesz operators in $\mathcal{R}^2$.

[4]If they did we could write $(aM)^n$ as a^nM^n and derive $\|u(t)\|_{L^1(\mathbb{R}^2)} \le C\int_0^t \frac{\|a\|^n_{L^\infty(\mathbb{R}^2)}(t-s)^n}{n!}\|M^n g(s)\|_{L^1(\mathbb{R}^2)}\,ds \le e^{Ct\|a\|_{L^\infty(\mathbb{R}^2)}}\int_0^t\|g(s)\|_{\mathcal{H}_1}\,ds$.

allow even more general functions in the time-dependent case. As a consequence to accomplish our goal we need to give up on the Hardy space $\mathcal{H}_1$ and consider in fact estimates[5] for solutions u of transport equation (3) of the form,

$$\sup_{t\in[0,1]} \|u(t)\|_{L^1(\mathbb{R}^2)} \le C(\|a\|_{B^1_{2,1}(\mathbb{R}^2)})N(g), \tag{5}$$

where the expression $N(g)$ reflects a logarithmic loss[6] relative to the L^1 norm of g. The proper definition of $N(g)$ is given below in (14). In the particular case of g with compact support $N(g)$ becomes simply $\|g\|_{L^1(\mathbb{R}^2)} \log^+ \|g\|_{L^\infty(\mathbb{R}^2)} + 1$.

The key feature of estimate (5) is that only one logarithmic loss is present. This means that we are not able to attack the problem by merely considering the mapping properties of the operator aM. Indeed the best we can prove is the estimate,

$$\sup_{t\in[0,1]} \|aMg(t)\|_{L^1(\mathbb{R}^2)} \le C(\|a\|_{B^1_{2,1}(\mathbb{R}^2)})N(g),$$

which leads, by iteration, to a loss of $\big(\log^+ \|g\|_{L^\infty(\mathbb{R}^2)}\big)^n$ for $(aM)^n$. Instead we analyze directly the mapping properties of the multilinear expressions

$$(a(x)M)^n = a(x)\, M\, a(x)\, M \,....\, a(x)M \tag{6}$$

and their sums. Using commutator estimates and appropriate interpolations between the weak L^1 and L^2 mapping properties of the operators M we are able to show that in fact we lose only one logarithm for $\|(aM)^n g\|_{L^1}$, regardless of the exponent n. Note however that under our assumptions on $a(x)$ the commutator $[a(x), M]$ is not a bounded operator[7] on $L^1(\mathbb{R}^2)$ and thus the problem can not be simply reduced to the weak-L^1 estimate for the Calderon-Zygmund operator M^n. Instead using the assumption that $a \in B^1_{2,1}$ we first reduce the problem to the case where in the multilinear expression (6) the function a is replaced by its atoms

$$M a_{k_1} M ... a_{k_{n-1}} M,$$

with $a_k = P_k a$ and the Littewood-Paley projection P_k associated with the dyadic band of frequencies of size 2^k. We then decompose

$$M = M_{\ge k_1} + M_{<k_1} = P_{<k_1} M + P_{\ge k_1} M$$

and observe that $[M_{\ge k_1}, a_{k_1}]$ is a bounded operator on L^1. It follows that

$$\begin{aligned} M a_{k_1} M ... a_{k_{n-1}} M &= a_{k_1} M_{\ge k_1} M ... a_{k_{n-1}} M + [M_{\ge k_1}, a_{k_1}] M ... a_{k_{n-1}} M \\ &\quad + M_{<k_1} a_{k_1} M ... a_{k_{n-1}} M. \end{aligned}$$

We now proceed inductively. The first two terms can be reduced to the problem of L^1 estimates for the multilinear expressions $M^2 a_{k_2} ... a_{k_{n-1}} M$ and $M ... a_{k_{n-1}} M$,

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[5] To prove such estimates we need the the symbol $m(\xi)$ of M is smooth at the origin, i.e., $|\partial^\alpha m(\xi)| \le c(1+|\xi)^{-|\alpha|}, \qquad \forall \xi \in \mathbb{R}^2$.

[6] Recall that according to the result of Stein [Ste1] the Hardy space $\mathcal{H}_1$ contains precisely such logarithmic loss, as the finiteness of the local, i.e. the norm $\|f\|_{L^1} + \|R_j f\|_{L^1}$ computed over balls B, $\mathcal{H}_1$ norm of g is equivalent to bounds on $\int_B |f(x)| \log^+ f(x)\, dx$.

[7] The classical result of Coifman-Rochberg-Weiss [CRW] requires only that $a \in$ BMO for the commutator to be bounded on L^p with $p \in (1,\infty)$. Extensions of this result from L^p to the Hardy space $\mathcal{H}_1$ however impose once again a Dini type condition on a.

each containing only $(n-1)$ Calderon-Zygmund operators and $(n-2)$ atoms a_{k_i}. The remaining term $M_{<k_1} a_{k_1} M ... a_{k_{n-1}} M$ can be written in the form

$$M_{<k_1} a_{k_1} M a_{k_2} ... a_{k_{n-1}} M = \sum_{\ell_2,...,\ell_{n-1}} M_{<k_1} a_{k_1} M_{k_1} a_{k_2} M ... a_{k_{n-1}} M_{\ell_{n-1}}.$$

The operator $M_{<k_1}$ is handled with the help of the weak-L^1 estimate, which comes on one hand with a logarithmic loss but on the other hand has a certain important redeeming property in the choice of the constants, which in particular made dependent on the multi-index $\ell_1, .., \ell_n$. The remaining argument consists in showing that the operator $M_{k_1} a_{k_2} M_{\ell_2} ... a_{k_{n-1}} M_{\ell_{n-1}}$ is bounded on L^1 with the bound reflecting exponential gains in the differences of either of the adjacent frequencies $|\ell_m - \ell_{m-1}|$ or $|k_m - k_{m-1}|$.

The problem of L^1 estimates for the transport equation (1) with variable time-dependent coefficient $a(t, x)$ exemplifies even more the need for such multilinear estimates. In this case a solution u does not quite have an exponential map representation similar to (4). Instead it can be written in the form

$$u(t) = \int_0^t T\Big\{ e^{\int_s^t a(\tau) M \, d\tau} \Big\} g(s) \, ds.$$

Here T is the Quantum Field Theory (QFT) notation for the time ordered product. Thus, we have

$$\begin{aligned} u(t) &= \int_0^t \sum_{n=0}^{\infty} \frac{1}{n!} T\Big\{ \int_s^t \int_s^t ... \int_s^t a(t_1) M a(t_2) M ... a(t_n) M \, dt_1 ... dt_n \Big\} g(s) \, ds \\ &= \int_0^t \sum_{n=0}^{\infty} \int_0^t a(t_1) M dt_1 \int_0^{t_1} a(t_2) M \, dt_2 .... \int_0^{t_{n-1}} a(t_n) M \int_0^{t_n} g(s) \, ds \qquad (7) \end{aligned}$$

The time ordering T arranges variables $t_1, ..., t_n$ in the decreasing order $t_1 \ge t_2 \ge ... \ge t_n$. Our method for deriving L^1 estimates for solutions of the transport equation (1) involves analyzing each of the multilinear expressions in the above expansion. As in the case of the time-independent coefficient a we will be able to derive an L^1 estimate with a logarithmic loss under the assumption that a is a $B^1_{2,1}$ valued function with an appropriate (in fact L^1) time dependence. The infinite series representation (7) will also help us to uncover another phenomenon. In the case when the time-dependent coefficient a can be written as a time derivative of a function b, i.e., $a = \partial_t b$, the L^1 estimate for solutions of the transport equation (1) does not require Besov regularity of the coefficient a and instead needs $L^2([0, 1]; H^1)$ regularity of a together with $L^2([0, 1]; H^2)$ regularity of b. Our main result is the L^1 estimate for solutions of the transport equation (1) with the coefficient $a = \partial_t b + c$ with $c \in L^1([0, 1]; B^1_{2,1})$ and b satisfying the above conditions.

To treat this general case we consider multilinear expressions appearing in (7) and decompose each of the $a(t_i)$ into its Littlewood-Paley components to form a term

$$J_{n.\mathbf{k}}(t) = \int_0^t \int_0^{t_1} ... \int_0^{t_n} a_{k_1}(t_1) M a_{k_2}(t_2) M ... a_{k_n}(t_n) M g(s) \, dt_1 ... dt_n \, ds$$

with $\mathbf{k} = (k_1, ..., k_n)$. For each $\mathbf{k}$ will be able to show the desired estimate

$$\sup_{t\in[0,1]} \|J_{n,\mathbf{k}}(t)\|_{L^1(\mathbb{R}^2)} \le CN(g).$$

The constant C above depends on the $L^1([0,1];H^1)$ norms of a_{k_i} and grows with n. As a consequence we face two major summation problems: first with respect to a given multi-index $\mathbf{k}$ followed by summation in n. Difficulties with summation over $\mathbf{k}$ are connected with the fact that a no longer has Besov regularity $B^1_{2,1}$. This lack of regularity is due to the term $\partial_t b$ in the decomposition of a. We notice however that upon substitution into $J_n(t)$ the term $\partial_t b_{k_j}$ can be integrated by parts which results in a gain of $1/2$ derivative[8] or, alternatively, a factor of $2^{-k_j/2}$. The problem however is that this gain needs to be spread across all remaining $(n-1)$ terms in $J_n(t)$, which leads us to choose k_j to be the highest frequency among all k_i. If the highest frequency is occupied by a Besov term c_{k_j}, appearing the decomposition of a we select the second highest frequency and continue the process, which in the end ensures summability with respect to $\mathbf{k}$. This analysis may potentially lead to violent growth of the constant C with respect to n and extreme care is needed. We ensure that C decays exponentially in n by imposing smallness conditions on the space-time norms of the coefficients b and c.

We now state our result precisely. Consider the transport equation

$$\partial_t u - a(t,x)Mu = g(t,x), \qquad u(0,x) = 0.$$

We assume that for the coefficient a

$$\|a\|_1 := \|a\|_{L^2_t H^1} = (\int_0^1 \|a(t)\|^2_{H^1(\mathbb{R}^2)})^{1/2} \le \Delta_0. \tag{8}$$

In addition a can be decomposed as follows,

$$a = \partial_t b + c \tag{9}$$

where,

$$\|b\|_2 := (\int_0^1 \|b(t)\|^2_{H^2(\mathbb{R}^2)} + \int_0^1 \|\partial_t b(t)\|^2_{H^1(\mathbb{R}^d)})^{1/2} \le \Delta_0 \tag{10}$$

$$\|c\|_3 := \int_0^1 \|c(t)\|_{B^1_{2,1}(\mathbb{R}^2)} dt \le \Delta_0 \tag{11}$$

with $B^1_{2,1}(\mathbb{R}^2)$ the classical inhomogeneous Besov space defined by the norm,

$$\|v\|_{B^1_{2,1}(\mathbb{R}^2)} = \|P_{\le 0} v\|_{L^2} + \sum_{k\in\mathbb{Z}_+} 2^k \|P_k v\|_{L^2(\mathbb{R}^2)}.$$

The operator M is the classical translation invariant Calderon-Zygmund operator on $\mathbb{R}^2$, given by the symbol $m(\xi)$ verifying

$$|\partial^\alpha m(\xi)| \le c(1+|\xi)^{-|\alpha|}, \qquad \forall \xi \in \mathbb{R}^2. \tag{12}$$

We prove the following theorem,

[8]The fact that the gain is only 1/2 derivative rather than the whole derivative is due to the L^2 in time integrability assumption on b.

Theorem 1.1 (Main Theorem)**.** *Under the above assumptions , if Δ_0 is sufficiently small, we have the estimate,*

$$\sup_{t\in[0,1]} \|u(t)\|_{L^1(\mathbb{R}^2)} \lesssim CN(g) \tag{13}$$

where,

$$N(g) = \|g\|_{L^1([0,1]\times\mathbb{R}^2)} \ \log^+ \big\{ \| < x >^3 g\|_{L^\infty([0,1]\times\mathbb{R}^2)} \big\} + 1. \tag{14}$$

Remark 1.2. For a function g of compact support the expression $N(g)$ can be controlled as follows

$$N(g) \lesssim \|g\|_{L^1([0,1]\times\mathbb{R}^2)} \ \log^+ \|g\|_{L^\infty([0,1]\times\mathbb{R}^2)} + 1 \tag{15}$$

Remark 1.3. Condition (12) implies that the symbol of the operator M is smooth at the origin, which in principle eliminates a large class of Calderon-Zygmund operators from our consideration. We argue however that this condition is not particularly restrictive and can be replaced with assumptions of additional spatial decay on the coefficients $a(t, x)$. Moreover, in our application (see the paragraph below) we consider the corresponding transport equation on a compact manifold (2-sphere) instead of $\mathbb{R}^2$, where the existence of a spectral gap ensures that condition (12) holds. In that context a prototype for M is the operator $(-\Delta)^{-1}\nabla^2$. Moreover, in that case $N(g)$ can be replaced by the $L \log L$ type expression (15).

The above theorem is a vastly simplified model case for the type of result we need in [Kl-Ro6] to prove a conditional regularity result for the Einstein vacuum equations. The main assumption in [Kl-Ro6], concerning the pointwise boundedness of the deformation tensor of the unit, future, normal vectorfield to a space-like foliation, allows us to bound the flux of the space -time curvature through the boundary $\mathcal{N}^-(p)$ of the causal past of any point p of the space-time under consideration. In [Kl-Ro1]–[Kl-Ro4], see also [Q], we were able to show that the boundedness of the flux of curvature through $\mathcal{N}^-(p)$ suffices to control the radius of injectivity of $\mathcal{N}^-(p)$. This result, together with the construction of a first order parametrix in [Kl-Ro5], is used in [Kl-Ro6] to derive pointwise bounds for the curvature tensor of the corresponding spacetime. To control the main error term generated by the parametrix one needs however to bound the L^1 norm of the first two tangential derivatives of $\mathrm{tr}\chi$ along $\mathcal{N}^-(p)$, with $\mathrm{tr}\chi$ the trace of the null second fundamental form of $\mathcal{N}^-(p)$. One can show that the second tangential derivatives of $\mathrm{tr}\chi$ verifies a transport equation along the null geodesic generators of $\mathcal{N}^-(p)$ which can be modeled, very roughly, by (1), with g a term whose L^1 norm along $\mathcal{N}^-(p)$ is bounded by the flux of curvature . In fact a more realistic model would be to consider a transport, similar to (1), along the null geodesics of a past null cone $\mathcal{N}^-(p)$ in Minkowski space $\mathbb{R}^{3+1}$ with t denoting the value of the standard afine parameter along null geodesics and $x = (x^1, x^2)$ denoting the standard sperical coordinates on the 2-spheres S_t, corresponding to constant value of t along $\mathcal{N}^-(p)$. Thus the singular integral operator M would act on S_t.

Finally we believe that our result, or rather our proof of the result, can be applied to other situations where one needs to make L^1 or L^∞ estimates for singular transport equations, where a simple logarithmic loss is unavoidable.

2. Preliminary results

We recall briefly the classical Littlewood-Paley decomposition of functions defined on $\mathbb{R}^d$,

$$f = f_0 + \sum_{k\in\mathbb{Z}_+} f_k$$

with frequency localized components f_k, i.e. $\widehat{f_k}(\xi) = 0$ for all values of ξ outside the annulus $2^{k-1} \le |\xi| \le 2^{k+1}$ and a function f_0 with frequency localized in the ball $|\xi| \le 1$. Such a decomposition can be easily achieved by choosing a test function $\chi = \chi(|\xi|)$ in Fourier space, supported in $\frac{1}{2} \le |\xi| \le 2$, and such that, for all $\xi \ne 0$, $\sum_{k\in\mathbb{Z}} \chi(2^{-k}\xi) = 1$. Then for $k > 0$ set $\widehat{f_k}(\xi) = \chi(2^k\xi)\hat{f}(\xi)$ or, in physical space,

$$P_k f = f_k = p_k * f$$

where $p_k(x) = 2^{nk}p(2^k x)$ and $p(x)$ the inverse Fourier transform of χ, while

$$\hat{f}_0(\xi) = \left(1 - \sum_{k\in\mathbb{Z}_+} \chi(2^{-k}\xi)\right)\hat{f}(\xi)$$

and $f_0 = P_0 f$. The operators P_k are called cut-off operators or, somewhat improperly, Littlewood-Paley projections.

Let M be a Calderon-Zygmund operator with multiplier m, i.e.,

$$\widehat{Mf}(\xi) = m(\xi)\hat{f}(\xi) \tag{16}$$

Here m is a smooth function satisfying

$$|\partial_\xi^\alpha m(\xi)| \le c(1+|\xi|)^{-|\alpha|}, \qquad \forall \xi \in \mathbb{R}^d \tag{17}$$

for all multiindices α with $|\alpha| \le d+6$ and a fixed constant $c > 0$. According to Michlin-Hörmander theorem we have,

$$|m(x)| \le c|x|^{-d}, \qquad |\partial_x m(x)| \le c|x|^{-d-1} \tag{18}$$

Due to the smoothness of the symbol of M at the origin we can also add the estimate

$$|m(x)| \le c(1+|x|)^{-d-6} \tag{19}$$

We shall make use of the standard Calderon-Zygmund estimates in L^p, $1 < p < \infty$,

$$\|Mf\|_{L^p} \le C_p \|f\|_{L^p}$$

as well as the weak-L^1 estimate

$$|\{x : |Mf(x)| > \lambda\} \le C\lambda^{-1}\|f\|_{L^1}$$

Our first result is a global version of the standard local L^1 estimate for a multiplier M. The local estimate in a ball B_R does not require the condition (19) and takes the form

$$\|Mf\|_{L^1(B_R)} \le C_R(\|f\|_{L^1}\log^+\|f\|_{L^\infty} + 1).$$

We have the following

Lemma 2.1. *Let M be a multiplier satisfying* (19)*. Fix an $L^1(\mathbb{R}^d)$ positive function β and a constant $\mu > 0$. Then for any smooth function f of compact support*

$$\|Mf\|_{L^1} \le C N_{\mu,\beta}(f),$$

where

$$N_{\mu,\beta}(f) = \mu\|\beta\|_{L^1} + \|f\|_{L^1} \log^+\{\sup_{\mathbf{a}\in\mathbb{Z}^d} \frac{\sum_{|\mathbf{b}-\mathbf{a}|\le 3} \|\chi_{\mathbf{b}} f\|_{L^\infty}}{\mu\|\chi_{\mathbf{a}}\beta\|_{L^1}}\},$$

$\chi_{\mathbf{a}}$ is a partition of unity adapted to the balls of radius one with centers at integer lattice points $\mathbf{a}$ and $\log^+ x = \log(2 + |x|)$.

Proof We first note that the problem can be reduced to the case when the kernel of M, given by the function $m(x)$, has compact support. This follows since

$$Mf(x) = M_0 f(x) + M_1 f(x), \qquad M_1 f(x) = \int \chi(x-y) m(x-y) f(y)\, dy,$$

where χ is a smooth cut-off function vanishing on the ball of radius one. Assumption (19) guarantees that $\chi(x)m(x)$ is integrable. As a consequence,

$$\|M_1 f\|_{L^1} \le C\|f\|_{L^1}.$$

To deal with M_0 we proceed in the usual fashion by writing

$$\|M_0 f\|_{L^1} = \int_0^\infty |\{x : |M_0 f(x)| > \lambda\}|\, d\lambda \le \int_0^\infty |\{x : |M_0 f_{<\lambda}(x)| > \lambda\}|\, d\lambda + \int_0^\infty |\{x : |M_0 f_{\ge\lambda}(x)| > \lambda\}|\, d\lambda,$$

where $f_{<\lambda}(x)$ is the function coinciding with $f(x)$ on the set where $|f(x)| < \lambda$ and vanishing on its complement, and $f_{\ge\lambda} = f(x) - f_{<\lambda}$. To estimate the term with $f_{<\lambda}$ we use the weak-L^2 estimate

$$\int_0^\infty |\{x : |M_0 f_{<\lambda}(x)| > \lambda\}|\, d\lambda \le C \int_0^\infty \frac{\|f_{<\lambda}\|_{L^2}^2}{\lambda^2} = C \int \int_{|f(x)|}^\infty \lambda^{-2} |f(x)|^2\, d\lambda\, dx = C \int |f(x)|\, dx$$

To estimate the term with $f_{\ge\lambda}$ we decompose $f_{\ge\lambda}$ into the sum of functions $f^{\mathbf{a}}_{\ge\lambda} = \chi_{\mathbf{a}} f_{\ge\lambda}$

$$f_{\ge\lambda} = \sum_{\mathbf{a}\in\mathbb{Z}^d} \chi_{\mathbf{a}} f_{\ge\lambda},$$

where $\chi_{\mathbf{a}}$ is a partition of unity, parametrized by integer lattice points in $\mathbb{R}^d$ with the property that the support of $\chi_{\mathbf{a}}$ is contained in the ball of radius two around the point $\mathbf{a} \in \mathbb{R}^d$. Since the kernel of M_0 is supported in a ball of radius one, the support of $M_0 f^{\mathbf{a}}_{\ge\lambda}$ is contained in the ball of radius three around k. As a consequence, there are at most $3^d C$ functions $M_0 f^{\mathbf{a}}_{\ge\lambda}$ containing any given point x in their support. Therefore,

$$|\{x : |M_0 f_{\ge\lambda}(x)| > \lambda\}| \le \sum_{\mathbf{a}\in\mathbb{Z}^d} |\{x : |M_0 f^{\mathbf{a}}_{\ge\lambda}(x)| > \lambda(3^d C)^{-1}\}|.$$

We also have the trivial estimate, with another constant still denoted C,

$$|\{x : |M_0 f^{\mathbf{a}}_{\ge\lambda}(x)| > \lambda(3^d C)^{-1}\}| \le 3^d C.$$

Thus, using a weak-L^1 estimate we obtain

$$\begin{aligned}
J_{\mathbf{a}} : &= \int_0^\infty |\{x : |M_0 f^{\mathbf{a}}_{\geq\lambda}(x)| > \lambda(3^d C)^{-1}\}|\, d\lambda \\
&\leq \int_0^{\lambda_0} 3^d C + 3^d C \int \int_{\lambda_0}^\infty \lambda^{-1} \|\chi_\alpha f_{\geq\lambda}\|_{L^1}\, d\lambda \\
&\leq 3^d C\lambda_0 + 3^d C \int_{\lambda_0}^\infty \int_{|f(x)|\geq\lambda} \lambda^{-1} |\chi_{\mathbf{a}} f(x)|\, dx\, d\lambda \\
&\leq 3^d C\lambda_0 + 3^d C \int \chi_{\mathbf{a}}(x)|f(x)|\, \Big|\log \frac{|f(x)|}{\lambda_0}\Big|\, dx \\
&\lesssim 3^d C\lambda_0 + 3^d C \int_{|f(x)|\geq\lambda_0} \chi_{\mathbf{a}}(x)|f(x)|\, \log \frac{|f(x)|}{\lambda_0}\, dx \\
&\lesssim 3^d C\lambda_0 + 3^d C \int \chi_{\mathbf{a}}(x)|f(x)|\, \log^+ \frac{|f(x)|}{\lambda_0}\, dx
\end{aligned}$$

for some $\lambda_0 > 0$. We now choose $\lambda_0 = \mu \int \chi_{\mathbf{a}}(x)\beta(x)\, dx$. The above estimate then becomes

$$\begin{aligned}
J_{\mathbf{a}} &\leq 3^d C \left(\mu\|\chi_{\mathbf{a}}\beta\|_{L^1} + \int \chi_{\mathbf{a}}(x)|f(x)|\, \log^+ \frac{|f(x)|}{\mu\|\chi_{\mathbf{a}}\beta\|_{L^1}} \right). \\
&\lesssim 3^d C \left(\mu\|\chi_{\mathbf{a}}\beta\|_{L^1} + \int |f(x)|\chi_{\mathbf{a}}(x)\, |\log^+ \sum_{\mathbf{b}} \frac{\chi_{\mathbf{b}}(x)|f(x)|}{\mu\|\chi_a\beta\|_{L^1}} \right) \\
&\lesssim 3^d C \left(\mu\|\chi_{\mathbf{a}}\beta\|_{L^1} + \int |f(x)|\chi_{\mathbf{a}}(x)\, \log^+ \sum_{|\mathbf{b}-\mathbf{a}|\leq 3} \frac{\chi_{\mathbf{b}}(x)|f(x)|}{\mu\|\chi_a\beta\|_{L^1}} \right) \\
&\lesssim 3^d C \left(\mu\|\chi_{\mathbf{a}}\beta\|_{L^1} + \int |f(x)|\chi_{\mathbf{a}}(x)\, \log^+ \sum_{|\mathbf{b}-\mathbf{a}|\leq 3} \frac{\|\chi_{\mathbf{b}}(x)|f(x)|}{\mu\|\chi_a\beta\|_{L^1}} \right) \\
&\lesssim 3^d C \left(\mu\|\chi_{\mathbf{a}}\beta\|_{L^1} + \|f\chi_{\mathbf{a}}\|_{L^1}\, \log^+ \sup_{\mathbf{a}\in\mathbb{Z}^d} \sum_{|\mathbf{b}-\mathbf{a}|\leq 3} \frac{\|\chi_{\mathbf{b}} f\|_{L^\infty}}{\mu\|\chi_a\beta\|_{L^1}} \right)
\end{aligned}$$

Now,

$$\begin{aligned}
\|M_0 f\|_{L^1} &\lesssim \int_0^\infty |\{x : |M_0 f_{<\lambda}(x)| > \lambda\}|\, d\lambda + \int_0^\infty |\{x : |M_0 f_{\geq\lambda}(x)| > \lambda\}|\, d\lambda \\
&\lesssim C\|f\|_{L^1} + \sum_{\mathbf{a}\in\mathbb{Z}^{\mathbf{d}}} J_{\mathbf{a}} \\
&\lesssim C\|f\|_{L^1} + 3^d C \left(\mu\|\beta\|_{L^1} + \|f\|_{L^1} \log^+ \sup_{\mathbf{a}\in\mathbb{Z}^d} \sum_{|\mathbf{b}-\mathbf{a}|\leq 3} \frac{\|\chi_{\mathbf{b}} f\|_{L^\infty}}{\mu\|\chi_a\beta\|_{L^1}} \right)
\end{aligned}$$

as desired.

$\blacksquare$

We also need to consider powers of M^n of M with multipliers $m^{(n)}(\xi) = m(\xi)^n$. Clearly, there exists a constant $C > 0$ depending only on c and d such that,

$$|m^{(n)}(x)| \le C^n |x|^{-d}, \quad |\partial_x m^{(n)}(x)| \le C^n |x|^{-d-1}, \quad |m^{(n)}(x)| \le C^n (1+|x|)^{-d-6} \tag{20}$$

Thus, for a similar $C > 0$,

$$\|M^n f\|_{L^1} \le C^n N_{\mu,\beta}(f) \tag{21}$$

Let $m_k(\xi) = \chi(2^k\xi)m(\xi)$ and denote by M_k the operator defined by the multiplier m_k. Clearly $M_k f = P_k(Mf)$. We shall also denote by M_J the operator $P_J M$ with multiplier $m_J = \sum_{k\in J} m_k$ for any interval $J \subset \mathbb{Z}$. In physical space,

$$M_k f(x) = \int_{\mathbb{R}^d} m_k(x-y) f(y) dy, \quad M_{\ge k} f = \int_{\mathbb{R}^d} m_{\ge k}(x-y) f(y) dy$$

We have the following,

Lemma 2.2. *Let $k \in \mathbb{Z}_+ \cup \{0\}$ and assume that a_k is a function whose frequency is supported in the band $2^{k-1} \le |\xi| \le 2^{k+1}$, or in the case $k = 0$ in the ball $|\xi| \le 1$. Then, there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\|[(M^n)_{\ge k}\, ,\, a_k] f\|_{L^1} \le C^n \|a_k\|_{L^\infty} \|f\|_{L^1}$$

Proof : We have,

$$\begin{aligned} C(a_k)f: \;&=\; (M^n)_{\ge k}(a_k f)(x) - a_k(x)(M^n)_{\ge k} f(x \\ &=\; \int m^{(n)}{}_{\ge k}(x-y)\big(a_k(y) - a_k(x)\big) f(y) dy \end{aligned}$$

To show that the the integral operator $C(a_k)$ maps L^1 into L^1 it suffices to show that,

$$\begin{aligned} I &= \sup_y I(y) \\ I(y) &= \int |m^{(n)}{}_{\ge k}(x-y)||a_k(y) - a_k(x)| dx \le C^n \|\alpha_k\|_{L^\infty} \end{aligned}$$

We write,

$$\begin{aligned} I(y) &\le I_1(y) + I_2(y) \\ I_1(y) &= \int_{|x-y|\ge 2^{-k}} |m^{(n)}{}_{\ge k}(x-y)||a_k(y) - a_k(x)| dx \\ I_2(y) &= \int_{|x-y|\le 2^{-k}} |m^{(n)}{}_{\ge k}(x-y)||a_k(y) - a_k(x)| dx \end{aligned}$$

We have,

$$|a_k(y) - a_k(x)| \le |x-y| \sup_{z\in[x,y]} |\partial a_k(z)| \lesssim 2^k |x-y|\, \|a_k\|_{L^\infty}$$

We also have,

$$|m^{(n)}{}_{\ge k}(x)| \le C^n |x|^{-d}$$

Thus,

$$I_2(y) \le C^n \|a_k\|_{L^\infty} \int_{|x-y|\le 2^{-k}} |x-y|^{-d} 2^k |x-y| dx \lesssim C^n \|a_k\|_{L^\infty}$$

Also, since,

$$|m^{(n)}{}_{\geq k}(x)| \leq C^n 2^{-k}|x|^{-d-1}$$

$$I_1(y) \quad \leq \quad C^n \|a_k\|_{L^\infty} \int_{|x-y|\geq 2^{-k}} 2^{-k}|x-y|^{-d-1}dx \lesssim C^n \|a_k\|_{L^\infty}$$

as desired. ■

We shall now prove the following,

Proposition 2.3. *Let M be a Calderon-Zygmund operator on $\mathbb{R}^2$ with the symbol satisfying* (17) *and $a = a(x)$ a smooth function verifying the bound,*

$$\|a\|_{B^1_{2,1}(\mathbb{R}^2)} \leq A \tag{22}$$

Then, for every positive integer n we have,

$$\|(aM)^n f\|_{L^1} \leq C^n A^n N(f) \tag{23}$$

with $N(f)$ defined by (13)*.*

Remark 2.4. Observe that the proposition remains valid if we replace $(aM)^n$ by $a_{(1)}M_{(1)}a_{(2)}M_{(2)}\dots a_{(n)}M_{(n)}$ with

$$\|a_{(i)}\|_{B^1_{2,1}(\mathbb{R}^d)} \leq A, \qquad i = 1, \dots n$$

and $M_1, M_2, \dots M_n$ translation invariant Calderon-Zygmund operators with symbols which are uniformly bounded by the same constant c, see (17).

The proof follows immediately from the following lemma.

Lemma 2.5. *Let $(k_1, ..., k_n)$ be an n-tuple of non-negative integers and assume that the functions a_{k_i} with $0 \leq i \leq n$ have frequencies supported in the dyadic shells $[2^{k_{i-1}}, 2^{k_{i+1}}]$, or in the case $k_i = 0$ in the ball $|\xi| \leq 1$. Then for some positive constant B,*

$$\|Ma_{k_1}M \dots a_{k_n}Mf\|_{L^1} \quad \lesssim \quad B^n A_{k_1\dots k_n} N(f) \tag{24}$$

where

$$A_{k_1\dots k_n} = \|a_{k_1}\|_{H^1} \cdots \|a_{k_n}\|_{H^1} \tag{25}$$

Proof : We prove by induction on n the following stronger version of estimate (24),

$$\|M^l a_{k_1}M \dots a_{k_n}Mf\|_{L^1} \lesssim B_1^{n+l} B_2^n A_{k_1\dots k_n} N(f) \tag{26}$$

with appropriately chosen constants constants B_1, B_2. Assume that the estimate has been proved for $(n-1)$ and any $l \in \mathbb{N}$. Splitting $\bar{M} := M^l = \bar{M}_{<k_1} + \bar{M}_{\geq k_1}$ we need to prove,

$$\|\bar{M}_{\geq k_1}(a_{k_1}Ma_{k_2}\dots a_{k_n}M)f\|_{L^1} \lesssim B_1^{n+l}B_2^n A_{k_1\dots k_n}N(f) \tag{27}$$

$$\|\bar{M}_{<k_1}(a_{k_1}Ma_{k_2}\dots a_{k_n}M)f\|_{L^1} \lesssim B_1^{n+l}B_2^n A_{k_1\dots k_n}N(f) \tag{28}$$

To deal with the first inequality we write,

$$\begin{aligned}\bar{M}_{\geq k_1}a_{k_1}Ma_{k_2}\dots a_{k_n}M \quad &= \quad a_{k_1}\bar{M}_{\geq k_1}Ma_{k_2}\dots a_{k_n}M \\ &+ \quad [\bar{M}_{\geq k_1}, a_{k_1}]Ma_{k_2}\dots a_{k_n}M\end{aligned}$$

According to Lemma 2.2 and the Bernstein inequality $\|a_k\|_{L^\infty} \lesssim \|a_k\|_{H^1}$, we have,

$$\|[\bar{M}_{\ge k_1}, a_{k_1}]Ma_{k_2}\dots a_{k_n}Mf\|_{L^1} \lesssim C^l\|a_{k_1}\|_{H^1}\|Ma_{k_2}\dots a_{k_n}Mf\|_{L^1}$$

Also,

$$\|a_{k_1}\bar{M}_{\ge k_1}Ma_{k_2}\dots a_{k_n}Mf\|_{L^1} \quad \lesssim \quad \|a_{k_1}\|_{L^\infty}\|M^{l+1}a_{k_2}\dots a_{k_n}Mf\|_{L^1} \tag{29}$$

Thus, taking into account our induction hypothesis,

$$\begin{aligned} \|M_{\ge k_1}(a_{k_1}Ma_{k_2}\dots a_{k_n}M)f\|_{L^1} &\lesssim C^l\|a_{k_1}\|_{H^1}\cdot\|Ma_{k_2}M\dots a_{k_n}Mf\|_{L^1} \\ &+ \|a_{k_1}\|_{H^1}\|M^{l+1}a_{k_2}\dots a_{k_n}Mf\|_{L^1} \\ &\lesssim (C^lB_1^nB_2^{n-1} + B_1^{n+l}B_2^{n-1})A_{k_1\dots k_n}N(f) \\ &\lesssim B_1^{n+l}B_2^nA_{k_1\dots k_n}N(f) \end{aligned}$$

as desired, provided that the constants B_1, B_2 are sufficiently large, in fact we need $B_1 \ge C$ and $B_2 \ge 1$.

We now consider the more difficult term

$$\bar{M}_{<k_1}(a_{k_1}Ma_{k_2}\dots a_{k_n}M)f = \bar{M}_{<k_1}\big(a_{k_1}M(g)\big) = \bar{M}_{<k_1}\big(a_{k_1}M_{k_1}(g)\big)$$

with $g = (a_{k_2}Ma_{k_3}\dots a_{k_n}M)f$. Note that if $k_1 = 0$ the operator $\bar{M}_{<k_1}$ is a multiplier with a smooth symbol of compact support. As a consequence it is bounded on L^1 and, with $a_0 = a_{k_1}$,

$$\begin{aligned} \|\bar{M}_{<0}(a_0Ma_{k_2}\dots a_{k_n}M)f\|_{L^1} &\le C^l\|a_{k_1}\|_{H^1}\|Ma_{k_2}\dots a_{k_n}M)f\|_{L^1} \\ &\lesssim C^lB_1^nB_2^{n-1}A_{k_1\dots k_n}N(f). \end{aligned}$$

Therefore to prove (28) we need to consider the case $k_1 > 0$ and estimate,

$$\|\bar{M}_{<k_1}(a_{k_1}Ma_{k_2}\dots a_{k_n}Mf)\|_{L^1}$$

We further decompose as follows,

$$\begin{aligned} \bar{M}_{<k_1}(a_{k_1}Ma_{k_2}\dots a_{k_n}Mf) &= \sum_{[l]_n}\bar{M}_{<k_1}M_{[k]_n,[l]_n}(f) \\ M_{[k]_n,[l]_n}(f) &= a_{k_1}M_{l_1}a_{k_2}\dots M_{l_{n-1}}a_{k_n}M_{l_n}f \end{aligned} \tag{30}$$

with $[l]_n$ denoting an arbitrary integer n-tuple $(l_1, ..., l_n) \in (\mathbb{Z}_+ \cup \{0\})^n$ and $[k]_n = (k_1, \dots, k_n)$. Whenever there is no possibility of confusion we shall drop the index n and write simply simply write $[k], [l]$. By the triangle inequality

$$\|\bar{M}_{<k_1}(a_{k_1}Ma_{k_2}\dots a_{k_n}Mf)\|_{L^1} \le \sum_{[l]_n}\|\bar{M}_{<k_1}M_{[k]_n,[l]_n}(f)\|_{L^1}$$

We note that in the expression $\bar{M}_{<k_1}a_{k_1}M_{l_1}(a_{k_2}\dots a_{k_n}M_{l_n}f)$ the frequency l_1 is forced to be of the order of k_1. This allows us to insert a factor of $2^{-|k_1-l_1|}$ in the above expression. Using (21) we then derive,

$$\|\bar{M}_{<k_1}M_{[k],[l]}(f)\|_{L^1} \lesssim 2^{-|k_1-l_1|}B_1^lB_2N_{\mu(\,[l]\,),\beta}\big(\,M_{[k],[l]}(f)\,\big) \tag{31}$$

Here, the notation $\mu(\,[l]\,)$ indicates that the scalar μ will be chosen dependent on the multi-index $[l]=[l]_n$. Recall that[9],

$$N_{\mu,\beta}(g)=\mu\|\beta\|_{L^1}+\|g\|_{L^1}\log^+\{\sup_{\mathbf{a}\in\mathbb{Z}^d}\frac{\|\chi_{\mathbf{a}}g\|_{L^\infty}}{\mu\|\chi_{\mathbf{a}}\beta\|_{L^1}}\}$$

We now make the following choice for the scalar μ. The choice will be justified in the lemmas below.

$$\begin{aligned}\mu([l]) &= A_{k_1\dots k_n}2^{-\alpha([l]_n)},\\ \alpha([l] &= \frac{1}{2}\sum_{m=2}^{n}\min\Big(|l_m-l_{m-1}|,|l_m-k_m|\Big)\end{aligned}$$

We also choose the function

$$\beta=(1+|x|)^{-3}.$$

Observe that the following holds true,

$$\left(\frac{<\mathbf{b}>}{<\mathbf{a}>}\right)^{-3}\|\chi_{\mathbf{b}}\beta\|_{L^1}\le\|\chi_{\mathbf{a}}\beta\|_{L^1}\le\left(\frac{<\mathbf{b}>}{<\mathbf{a}>}\right)^{3}\|\chi_{\mathbf{b}}\beta\|_{L^1} \tag{32}$$

We will need to make use of the following,

Lemma 2.6. *The following estimates hold true for the expression,*

$$M_{[k],[l]}(f)=a_{k_1}M_{l_1}a_{k_2}\dots.a_{k_n}M_{l_n}f,$$

$$\|M_{[k],[l]}(f)\|_{L^1} \lesssim C^n2^{-2\alpha([l]_n)}A_{k_1..k_n}\|f\|_{L^1} \tag{33}$$

$$\|\chi_{\mathbf{a}}M_{[k],[l]}(f)\|_{L^\infty} \lesssim C^nA_{k_1..k_n}\sum_{\mathbf{b}\in\mathbb{Z}^2}<|\mathbf{b}-\mathbf{a}|>^{-3}\|\chi_{\mathbf{b}}f\|_{L^\infty} \tag{34}$$

We postpone the proof of the lemma to the end of this section.

Now, using (31)

$$\begin{aligned}&\|\bar M_{<k_1}(a_{k_1}Ma_{k_2}\dots a_{k_n}Mf)\|_{L^1}\le\sum_{[l]_n}\|\bar M_{<k_1}M_{[k],[l]}(f)\|_{L^1}\\ &\lesssim\sum_{[l]}2^{-|k_1-l_1|}\left(\mu(\,[l])\,\|\beta\|_{L^1}+\|M_{[k],[l]}(f)\|_{L^1}\log^+\{\sup_{\mathbf{a}\in\mathbb{Z}^d}\frac{\|\chi_{\mathbf{a}}M_{[k],[l]}(f)\|_{L^\infty}}{\mu(\,[l]\,)\|\chi_{\mathbf{a}}\beta\|_{L^1}}\}\right)\end{aligned}$$

Given our choice of $\mu(\,[l]\,)$ we have,

$$\begin{aligned}\sum_{[l]}2^{-|k_1-l_1|}\mu(\,[l]\,) &= A_{k_1\dots k_n}\sum_{[\,l\,]}2^{-|k_1-l_1|}2^{-\alpha(\,[l])}\\ &= A_{k_1\dots k_n}\sum_{[l]}\Big(2^{-|k_1-l_1|}\cdot2^{-\frac{1}{2}\min(|l_2-l_1|,|l_2-k_2|)}\cdot\ldots\cdot2^{-\frac{1}{2}\min(|l_n-l_{n-1}|,|l_n-k_n|)}\Big)\\ &\lesssim A_{k_1\dots k_n}\end{aligned}$$

[9]For simplicity of notation we drop the summation $\sum_{|\mathbf{b}-\mathbf{a}|\le3}$ which will only adds a finite number of terms of the same type.

Thus, in order to end he proof of (28) it suffices to show that

$$\sum_{[l]} 2^{-|k_1-l_1|}\|M_{[k],[l]}(f)\|_{L^1}\log^+\{\sup_{\mathbf{a}\in\mathbb{Z}^d}\frac{\|\chi_{\mathbf{a}}M_{[k],[l]}(f)\|_{L^\infty}}{\mu(\,[l]\,)\|\chi_{\mathbf{a}}\beta\|_{L^1}}\}\lesssim C^n A_{k_1\dots k_n}N(f) \quad (35)$$

Using (33) and (34) and recalling the definition of $\mu[l]$, $\beta(x)$, we obtain

$$\begin{aligned}
&\sum_{[l]} 2^{-|k_1-l_1|}\|M_{[k],[l]}(f)\|_{L^1}\log^+\{\sup_{\mathbf{a}\in\mathbb{Z}^d}\frac{\|\chi_{\mathbf{a}}M_{[k],[l]}(f)\|_{L^\infty}}{\mu(\,[l]\,)\|\chi_{\mathbf{a}}\beta\|_{L^1}}\}\\
&\lesssim C^n A_{k_1\dots k_n}\sum_{[l]}2^{-|k_1-l_1|}2^{-2\alpha([l])}\|f\|_{L^1}\ \log^+\Big\{C^{n-1}\sup_{\mathbf{a}\in\mathbb{Z}^d}\sum_{\mathbf{b}\neq\mathbf{a}}<|\mathbf{b}-\mathbf{a}|>^{-3}\ \frac{2^{\alpha([l])}\|\chi_{\mathbf{b}}f\|_{L^\infty}}{\|\chi_{\mathbf{a}}\beta\|_{L^1}}\Big\}\\
&\lesssim C^{2n}A_{k_1\dots k_n}\sum_{[l]}2^{-|k_1-l_1|}2^{-\alpha([l])}\|f\|_{L^1}\ \log^+\Big\{\sup_{\mathbf{a}\in\mathbb{Z}^d}\frac{\|\chi_{\mathbf{a}}f\|_{L^\infty}}{\|\chi_{\mathbf{a}}\beta\|_{L^1}}\Big\}\\
&\lesssim C^{2n}A_{k_1\dots k_n}\|f\|_{L^1}\ \log^+\Big\{\sup_{\mathbf{a}\in\mathbb{Z}^d}<|\mathbf{a}|>^3\ \|\chi_{\mathbf{a}}f\|_{L^\infty}\Big\}\\
&\lesssim C^{2n}A_{k_1\dots k_n}N(f),
\end{aligned}$$

as desired. Here we have used,

$$(1+|\mathbf{a}|)^3\lesssim(1+|\mathbf{b}-\mathbf{a}|)^3(1+|\mathbf{b}|)^3$$

and the finiteness of the sum

$$\sum_{[l]}2^{-|k_1-l_1|}2^{-\alpha(\,[l]\,)}=\sum_{[l]}\Big(2^{-|k_1-l_1|}2^{-\frac12\min(|l_2-l_1|,|l_2-k_2|)}\cdot\ldots\cdot2^{-\frac12\min(|l_n-l_{n-1}|,|l_n-k_n|)}\Big)$$

It remains to prove Lemma 2.6. Estimate (33) follows recursively provided that we can establish the following

$$\|M_{l_{m-1}}a_{k_m}P_{l_m}h\|_{L^1}\lesssim\|a_{k_m}\|_{H^1}2^{-\min(|l_m-l_{m-1}|,|l_m-k_m|)}\|h\|_{L^1} \quad (36)$$

In fact, since $M_{l_{m-1}}$ is bounded in L^1, it suffices to prove,

$$\|P_{l_{m-1}}a_{k_m}P_{l_m}h\|_{L^1}\lesssim\|a_{k_m}\|_{H^1}2^{-\min(|l_m-l_{m-1}|,|l_m-k_m|)}\|h\|_{L^1} \quad (37)$$

On the other hand, estimate (34) is a localized version of the trivial estimate

$$\|a_{k_1}M_{l_1}a_{k_2}\ldots.a_{k_n}M_{l_n}f\|_{L^\infty}\lesssim C^nA_{k_1..k_n}\|f\|_{L^\infty},$$

which holds since each of the frequency localized Calderon-Zygmund operators M_l are bounded on L^p including $p=1,\infty$. Its localized version follows inductively from the estimate,

$$\|\chi_{\mathbf{a}}M_l\chi_{\mathbf{b}}g\|_{L^\infty}\le C(1+|\mathbf{b}-\mathbf{a}|)^{-3}\|g\|_{L^\infty},\qquad l\ge0 \quad (38)$$

which holds true on account of the sharp localization of the kernel of M_l, in physical space, due to the smoothness of the symbol of M at zero. Indeed the kernel of $m(x-y)$ of the operator $\chi_{\mathbf{a}}M_l\chi_{\mathbf{b}}$ verifies,

$$|m(x-y)|\le C\chi_{\mathbf{a}}(x)(1+|x-y|)^{-6}\chi_{\mathbf{b}}(y)\le C(1+|\mathbf{b}-\mathbf{a}|)^{-3}m_1(x-y)$$

with $m_1(x-y)=(1+|x-y|)^{-3}$ in L^1.

To prove (37) we distinguish the following cases.

(1) Assume $l_{m-1} < k_m$. Observe that $P_{l_{m-1}}(a_{k_m} P_{l_m} h) = 0$ unless $|l_m - k_m| \leq 2$. Therefore, since

$$\min\big(|l_m - l_{m-1}|, |l_m - k_m|)\big) \approx 1$$

we have ,

$$\begin{aligned} \|P_{l_{m-1}}(a_{k_m} P_{k_m} h)\|_{L^1} &\lesssim \|a_{k_m}\|_{H^1} \|h\|_{L^1} \\ &\lesssim 2^{-\min(|l_m - l_{m-1}|, |l_m - k_m|)} \|a_{k_m}\|_{H^1} \|h\|_{L^1} \end{aligned}$$

as desired.

(2) Assume $l_{m-1} > k_m$. In this case $P_{l_{m-1}}(a_{k_m} P_{l_m} h) = 0$ unless $|l_{m-1} - l_m| \leq 2$. Therefore we have again,

$$\min\big(|l_m - l_{m-1}|, |l_m - k_m|)\big) \approx 1$$

and

$$\begin{aligned} \|P_{l_{m-1}}(a_{k_m} P_{l_{m-1}} h)\|_{L^1} &\lesssim \|a_{k_m}\|_{H^1} \|h\|_{L^1} \\ &\lesssim 2^{-\min(|l_m - l_{m-1}|, |l_m - k_m|)} \|a_{k_m}\|_{H^1} \|h\|_{L^1} \end{aligned}$$

(3) If $l_{m-1} = k_m$, then $P_{l_{m-1}}(a_{k_m} P_{l_m} h) = 0$ unless $l_m \leq k_m$. Then, using the Bernstein inequality $\|P_{l_m} h\|_{L^2} \lesssim 2^{l_m} \|h\|_{L^2}$ we derive,

$$\begin{aligned} \|P_{l_{m-1}}(a_{k_m} P_{l_m} h)\|_{L^1} &\lesssim \|(a_{k_m} P_{l_m} h)\|_{L^1} \lesssim \|a_{k_m}\|_{L^2} \|P_{l_m} h)\|_{L^2} \\ &\lesssim 2^{-k_m} \|a_{k_m}\|_{H^1} \|P_{l_m} h\|_{L^2} \\ &\lesssim 2^{-k_m + l_m} \|a_k\|_{H^1} \|h\|_{L^1} \end{aligned}$$

Since in this case $l_m \leq k_m = l_{m-1}$ we have,

$$\min\big(|l_m - l_{m-1}|, |l_m - k_m|\big) = k_m - l_m$$

Therefore,

$$\|P_{l_{m-1}}(a_{k_m} P_{l_m} h)\|_{L^1} \lesssim 2^{-\min(|l_m - l_{m-1}|, |l_m - k_m|)} \|a_{k_m}\|_{H^1} \|h\|_{L^1}$$

as desired.

Thus in all cases inequality (37) is verified. ■

3. Proof of the main theorem

We need to prove the estimate

$$\sup_{t \in [0,1]} \|u(t)\|_{L^1(\mathbb{R}^d)} \lesssim C N(g)$$

where $d = 2$ and

$$N(g) = \|g\|_{L^1([0,1] \times \mathbb{R}^2)} \ \log^+ \big\{ \sup_{\mathbf{a} \in \mathbb{Z}^2} |\mathbf{a}|^2 \|\chi_{\mathbf{a}} g\|_{L^\infty([0,1] \times \mathbb{R}^2)} \big\} + 1$$

for a solution to (1)

$$\partial_t u - a(t,x) M u = g, \qquad u(0,x) = 0,$$

where the coefficient a admits the decomposition

$$a = \partial_t b + c \tag{39}$$

with a, b and c satisfying the conditions (8), (10) and (11).

We define the iterates $u^0 = 0, u^1, \dots u^n, u^{n+1}$ according to the recursive formula,

$$\partial_t u^{(n+1)}(t,x) = a(t_0,x) M u^{(n)}(t,x) + g(t,x), \quad u^{(n+1)}(0) = 0. \tag{40}$$

3.1. **First iterates.** To illustrate our method consider first the case of the iterate,

$$u^{(2)}(t_0) = \int_0^{t_0} g(t_1)dt_1 + \int_0^{t_0} a(t_1)dt_1 M \int_0^{t_1} g(t_2)dt_2$$

Thus,

$$\begin{aligned} \| \sup_{t_0\in[0,1]} u^{(2)}(t_0)\|_{L^1(\mathbb{R}^d)} &\lesssim \| \sup_{t_0\in[0,1]} \int_0^{t_0} g(t_1)dt_1\|_{L^1} + \| \sup_{t_0\in[0,1]} I(t_0)\|_{L^1} \\ I(t_0) &= \int_0^{t_0} a(t_1)dt_1 M \int_0^{t_1} g(t_2)dt_2 \end{aligned}$$

The first term is trivial. To estimate the second term we need to make use of the decomposition (39). Thus,

$$\begin{aligned} I(t_0) &= I_b(t_0) + I_c(t_0) \\ I_c(t_0) &= \int_0^{t_0} c(t_1)dt_1 \int_0^{t_1} Mg(t_2)dt_2 \\ I_b(t_0) &= \int_0^{t_0} \partial_{t_1} b(t_1)dt_1 \int_0^{t_1} Mg(t_2)dt_2 \\ &= b(t_0) \int_0^{t_0} Mg(t_2)dt_2 - \int_0^{t_0} b(t_1)Mg(t_1)dt_1 \\ &:= I_{b,1}(t_0) + I_{b,2}(t_0) \end{aligned}$$

To estimate I_c we use the fact that, for $d = 2$, the Besove space $B^1_{2,1}(\mathbb{R}^d)$ embedds in $L^\infty(\mathbb{R}^d)$ and the estimate,

$$\|Mg(t)\|_{L^1(\mathbb{R}^d)} \lesssim \|g(t)\|_{L^1(\mathbb{R}^d)} \log^+ \|g(t)\|_{L^\infty(\mathbb{R}^d)} + 1 \lesssim N(g(t))$$

Thus,

$$\begin{aligned} \| \sup_{t_0\in[0,1]} I_c(t_0)\|_{L^1} &\lesssim \int_0^1 \|c(t_1)\|_{L^\infty} dt_1 \int_0^{t_1} \|Mg(t_2)\|_{L^1(\mathbb{R}^d)} dt_2 \\ &\lesssim \int_0^1 \|c(t_1)\|_{B^1_{2,1}(\mathbb{R}^d)} dt_1 \int_0^{t_1} N(g)(t_2)dt_2 \\ &\lesssim \|c\|_3 N(g) \end{aligned}$$

On the other hand, decomposing $b = b_0 + \sum_{k\in\mathbb{Z}_+} b_k$,

$$\begin{aligned}\|\sup_{t_0\in[0,1]} I_{b,1}(t_0)\|_{L^1(\mathbb{R}^d)} &\lesssim \|\sup_{t_0\in[0,1]} b(t_0)\|_{L^\infty(\mathbb{R}^d)}\int_0^{t_0}\|Mg(t_2)\|_{L^1(\mathbb{R}^d)}dt_2\\ &\lesssim N(g)\|\sup_{t_0\in[0,1]} b(t_0)\|_{L^\infty(\mathbb{R}^d)}\\ &\lesssim N(g)\sum_{k\in\mathbb{Z}_+\cup\{0\}}\|\sup_{t_0\in[0,1]} b_k(t_0)\|_{L^\infty(\mathbb{R}^d)}\end{aligned}$$

We now appeal to the following straightforward lemma,

Lemma 3.2. *The following calculaus inequality holds true (see* (10)*) for $k \geq 0$,*

$$\sup_{t\in[0,1]}\|b_k(t)\|_{H^1(\mathbb{R}^d)} \lesssim \|\partial_t b_k\|^{1/2}_{L^2_tH^1}\|b_k\|^{1/2}_{L^2_tH^1} \lesssim 2^{-k/2}\|b_k\|_2$$

Also,

$$\|\sup_{t\in[0,1]} b_k(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\partial_t b_k\|^{1/2}_{L^2_tH^1}\|b_k\|^{1/2}_{L^2_tH^1} \lesssim 2^{-k/2}\|b_k\|_2$$

In view of the Lemma we deduce,

$$\begin{aligned}\|\sup_{t_0\in[0,1]} I_{b,1}(t_0)\|_{L^1(\mathbb{R}^d)} &\lesssim N(g)\sum_{k\in\mathbb{Z}_+\cup\{0\}}\|b_k\|_{L^2_tH^1}\\ &\lesssim N(g)\sum_{k\in\mathbb{Z}_+\cup\{0\}}2^{-k/2}\|b_k\|_2 \lesssim N(g)\|b\|_2\end{aligned}$$

Similarly,

$$\begin{aligned}\|\sup_{t_0\in[0,1]} I_{b,2}(t_0)\|_{L^1(\mathbb{R}^d)} &\lesssim \|\int_0^1 b(t_1)Mg(t_1)dt_1\|_{L^1(\mathbb{R}^d)}\\ &\lesssim N(g)\sup_{t_1\in[0,1]}\|b(t_1)\|_{L^\infty}\\ &\lesssim N(g)\|b\|_2\end{aligned}$$

Therefore,

$$\|\sup_{t_0\in[0,1]} u^{(2)}(t_0)\|_{L^1(\mathbb{R}^d)} \lesssim N(g)\big(\|b\|_2 + \|c\|_3\big)$$

Remark 3.3. Observe that there is room of a $1/2$ derivative in the estimates for I_b. This room will play an important role for treating the general iterates $u^{(n+1)}$.

Consider now the more dificult case of the iterate $u^{(3)}$,

$$\begin{aligned}u^{(3)} &= \int_0^{t_0} g(t_1)dt_1 + \int_0^{t_0} a(t_1)Mu^{(2)}(t_1)dt_1\\ &= \int_0^{t_0} g(t_1)dt_1 + \int_0^{t_0} a(t_1)dt_1 M\big(\int_0^{t_1} g(t_2)dt_2\big)\\ &+ \int_0^{t_0}\int_0^{t_1}\int_0^{t_2} a(t_1)Ma(t_2)Mg(t_3)dt_1dt_2dt_3\end{aligned}$$

We concentrate our attention on the last term,

$$I(t_0)=\int_0^{t_0}\int_0^{t_1}\int_0^{t_2} a(t_1)Ma(t_2)Mg(t_3)dt_1dt_2dt_3$$

As we decompose each $a(t_i)=\partial_t b(t_i)+c(t_i)$ with $i=1,2$ we notice that we can only integrate by parts only one of the potentially two terms containing $\partial_t b(t_i)$. We need to make that choice judiciously, based on the relative strength of the terms. We begin by decomposing $a(t_1), a(t_2)$ into their Littlewood-Paley pieces and write,

$$\begin{aligned} I(t_0) &= \int_0^{t_0}\int_0^{t_1}\int_0^{t_2}\sum_{k_1,k_2\in\mathbb{Z}_+\cup\{0\}} a_{k_1}(t_1)Ma_{k_2}(t_2)Mg(t_3)dt_1dt_2dt_3 \\ &= \int_0^{t_0}\int_0^{t_1}\int_0^{t_2}\sum_{0\le k_1<k_2} + \int_0^{t_0}\int_0^{t_1}\int_0^{t_2}\sum_{0\le k_1=k_2} + \int_0^{t_0}\int_0^{t_1}\int_0^{t_2}\sum_{k_1>k_2\ge 0} \end{aligned}$$

In what follows we will tacitly assume that all the integer indices k_i take values in the set of non-negative integers and will not write this constraint explicitly. Consider the last term,

$$J(t_0)=\int_0^{t_0}\int_0^{t_1}\int_0^{t_2}\sum_{k_1>k_2} a_{k_1}(t_1)Ma_{k_2}(t_2)Mg(t_3)dt_1dt_2dt_3$$

We further decompose,

$$a_{k_1}(t_1)=\partial_t b_{k_1}(t_1)+c_{k_1}(t_1)$$

and concentrate on the term,

$$\begin{aligned} J_b(t_0) &= \int_0^{t_0}\int_0^{t_1}\int_0^{t_2}\sum_{k_1>k_2} \partial_{t_1} b_{k_1}(t_1)Ma_{k_2}(t_2)Mg(t_3)dt_1dt_2dt_3 \\ &= \sum_{k_1>k_2} b_{k_1}(t_0)\int_0^{t_0}\int_0^{t_2} Ma_{k_2}(t_2)Mg(t_3)dt_2dt_3 \\ &- \sum_{k_1>k_2}\int_0^{t_0}\int_0^{t_1} b_{k_1}(t_1)Ma_{k_2}(t_1)Mg(t_3)dt_1dt_3 \end{aligned}$$

Let,

$$J_{b1}(t_0)=\sum_{k_1>k_2} b_{k_1}(t_0)\int_0^{t_0}\int_0^{t_2} Ma_{k_2}(t_2)Mg(t_3)dt_2dt_3$$

and estimate

$$\|J_{b1}(t_0)\|_{L^1} \quad\lesssim\quad \sum_{k_1>k_2}\|b_{k_1}(t_0)\|_{L^\infty}\int_0^{t_0}\int_0^{t_2}\|Ma_{k_2}(t_2)Mg(t_3)\|_{L^1}dt_2dt_3$$

Using Lemma 2.6 we have,

$$\|Ma_{k_2}(t_2)Mg(t_3)\|_{L^1}\lesssim\|a_{k_2}(t_2)\|_{H^1}N(g)(t_3)$$

Also, according to Lemma 3.2, using the norm $\|\,\|_2$ introduced in (11),

$$\|b_{k_1}(t_0)\|_{L^\infty}\lesssim 2^{-k_1/2}\|b_{k_1}\|_2$$

Hence,

$$\begin{aligned}\|J_{b1}(t_0)\|_{L^1} &\lesssim \sum_{k_1>k_2\geq 0} 2^{-k_1/2}\|b_{k_1}\|_2 \int_0^{t_0} \|a_{k_2}(t_2)\|_{H^1} dt_2 \int_0^{t_2} N(g)(t_3)dt_3dt_3 \\ &\lesssim N(g)\sum_{k_1>k_2\geq 0} 2^{-k_1/2}\|b_{k_1}\|_2\|a_{k_2}\|_1 \lesssim N(g)\|b\|_2\|a\|_1\end{aligned}$$

The term $J_{b2} = \sum_{k_1>k_2}\int_0^{t_0}\int_0^{t_2} b_{k_1}(t_1)Ma_{k_2}(t_1)Mg(t_3)dt_1dt_3$ can be treated in exactly the same fashion. Thus,

$$\|J_b(t_0)\|_{L^1} \quad\lesssim\quad N(g)\|b\|_2\|a\|_1 \tag{41}$$

Consider now the term,

$$J_c(t_0) \quad=\quad \int_0^{t_0}\int_0^{t_1}\int_0^{t_2}\sum_{k_1>k_2} c_{k_1}(t_1)Ma_{k_2}(t_2)Mg(t_3)dt_1dt_2dt_3$$

We further decompose

$$a_{k_2}(t_2) = \partial_t b_{k_2}(t_2) + c_{k_2}(t_2)$$

We show how to treat the term,

$$\begin{aligned}J_c(t_0) &= \int_0^{t_0}\int_0^{t_1}\int_0^{t_2}\sum_{k_1>k_2} c_{k_1}(t_1)M\partial_t b_{k_2}(t_2)Mg(t_3)dt_1dt_2dt_3 \\ &= \sum_{k_1>k_2}\int_0^{t_0}\int_0^{t_1} c_{k_1}(t_1)Mb_{k_2}(t_1)Mg(t_3)dt_1dt_3 \\ &- \sum_{k_1>k_2}\int_0^{t_0}\int_0^{t_1} c_{k_1}(t_1)Mb_{k_2}(t_2)Mg(t_2)dt_1dt_2\end{aligned}$$

Hence, using first Lemma 2.6 followed by Lemma 3.2,

$$\begin{aligned}\|J_c(t_0)\|_{L^1} &\lesssim \sum_{k_1>k_2}\int_0^{t_0}\int_0^{t_1} \|c_{k_1}(t_1)Mb_{k_2}(t_1)Mg(t_3)\|_{L^1}dt_1dt_3 \\ &+ \sum_{k_1>k_2}\int_0^{t_0}\int_0^{t_1} \|c_{k_1}(t_1)Mb_{k_2}(t_2)Mg(t_2)\|_{L^1}dt_1dt_2 \\ &\lesssim \sum_{k_1>k_2}\int_0^{t_0}\int_0^{t_1} \|c_{k_1}(t_1)\|_{H^1}\|b_{k_2}(t_1)\|_{H^1}N(g)(t_3)dt_1dt_3 \\ &+ \sum_{k_1>k_2}\int_0^{t_0}\int_0^{t_1} \|c_{k_1}(t_1)\|_{H^1}\|b_{k_2}(t_2)\|_{H^1}N(g)(t_2)dt_1dt_2 \\ &\lesssim \sum_{k_1>k_2}\sup_{t\in[0,1]}\|b_{k_2}(t)\|_{H^1}\int_0^{t_0}\int_0^{t_1}\|c_{k_1}(t_1)\|_{H^1}N(g)(t_2)dt_1dt_2 \\ &\lesssim N(g)\sum_{k_1>k_2\geq 0} 2^{-k_2/2}\|b_{k_2}\|_2\|c_{k_1}\|_{L^1H^1} \lesssim N(g)\|b\|_2\sum_{k_1}\|c_{k_1}\|_{L^1H^1} \\ &\lesssim N(g)\|b\|_2\|c\|_3\end{aligned}$$

3.4. **General case.** Treatment of the general case will follow the scheme laid down for the third iterate $u^{(3)}$. Additional challenge however is presented in controlling constants in the estimates, which may grow uncontrollably with respect to the order of the iterates. Recalling (40) we write,

$$\begin{aligned} u^{(n+1)}(t) &= \int_0^t g(t_1)dt_1 + \int_0^t a(t_1)dt_1 \int_0^{t_1} Mg(t_2)dt_2 + \dots \\ + &\int_0^t \int_0^{t_1} \dots \int_0^{t_n} a(t_1)Ma(t_2)M \dots a(t_n)Mg(t_{n+1})dt_1dt_2 \dots dt_{n+1} \end{aligned}$$

To simplify notations introduce the simplex $\Delta_n(t)$ defined by,

$$t \geq t_1 \geq t_2 \dots \geq t_n \geq t_{n+1} \geq 0$$

and write,

$$u^{(n+1)}(t) \quad = \quad u^{(n)}(t) + J_n(t) \tag{42}$$

where,

$$\begin{aligned} J_n(t) \quad &= \quad \int_{\Delta_n(t_0)} a(t_1)Ma(t_2)M \dots a(t_m)Mg(t_{n+1}) \\ &:= \quad \int \dots \int_{\Delta_n(t_0)} dt_1 \dots dt_{n+1}\ a(t_1)Ma(t_2)M \dots a(t_m)Mg(t_{n+1}) \end{aligned}$$

To prove (13) it will suffice to show that

$$\sup_{t\in[0,1]} \|J_n(t)\|_{L^1(\mathbb{R}^d)} \lesssim C^n \Delta^n N(g) \tag{43}$$

We decompose each $a(t_i)$ in the expression for J_n into its Littlewood-Paley components according to,

$$a(t_i) = \sum_{k\in\mathbb{Z}_+\cup\{0\}} P_k a(t_i) = a_0(t_i) + \sum_{k_i\in\mathbb{Z}_+} a_{k_i}(t_i)$$

Thus, writing $\mathbf{k} = (k_1, \dots k_n) \in (\mathbb{Z}_+ \cup \{0\})^n$

$$J_n(t) = J(t) \quad = \quad \sum_{\mathbf{k}\in(\mathbb{Z}_+\cup\{0\})^n} \int_{\Delta_n(t)} a(t_1)_{k_1} M \dots a_{k_n}(t_n)Mg(t_{n+1}) \tag{44}$$

For each $1 \leq j \leq n$ we define,

$$[k_j] = \{(k_1, k_2, \dots k_n) \in (\mathbb{Z}_+ \cup \{0\})^n \mid k_i \leq k_j \quad \forall i\} \tag{45}$$

to be the set on n-tuples $(k_1, ..., k_n)$ with the property that for each $i = 1, .., n$ $k_i \leq k_j$. In what follows we will tacitly assume that all indices k_i take values in the set of non-negative integers and will not write this constraint explicitly. Let,

$$J^j_n(t) = J^j(t) \quad = \quad \sum_{\mathbf{k}\in[k_j]} \int_{\Delta_n(t)} a_{k_1}(t_1)M \dots a_{k_n}(t_n)Mg(t_{n+1}) \tag{46}$$

Clearly,

$$\|J_n(t)\|_{L^1(\mathbb{R}^d)} \lesssim \sum_{j=1}^n \|J^j_n(t)\|_{L^1(\mathbb{R}^d)}$$

We now fix j and decompose in view of (39),

$$a_{k_j}(t_j) = \partial_t b_{k_j}(t_j) + c_{k_j}(t_j) \tag{47}$$

Thus,

$$\begin{aligned} J^j(t) &= J^j_b(t) + J^j_c(t) = \sum_{\mathbf{k}\in[k_j]} J^j_{b,\mathbf{k}}(t) + \sum_{\mathbf{k}\in[k_j]} J^j_{c,\mathbf{k}}(t) \qquad (48)\\ J^j_{b,\mathbf{k}}(t) &= \int_{\Delta_n(t)} a_{k_1}(t_1)M \dots \partial_t b_{k_j}(t_j)M \dots a_{k_n}(t_n)Mg(t_{n+1})dt_1 \dots dt_{n+1}\\ J^j_{c,\mathbf{k}}(t) &= \int_{\Delta_n(t)} a_{k_1}(t_1)M \dots c_{k_j}(t_j)M \dots a_{k_n}(t_n)Mg(t_{n+1})dt_1 \dots dt_{n+1} \end{aligned}$$

with the summation convention,

$$\sum_{\mathbf{k}\in[k_j]} = \sum_{k_j\in\mathbb{Z}} \sum_{\mathbf{k}'\le k_j}, \qquad \mathbf{k}' = (k_1, \dots \widehat{k_j} \dots k_n).$$

We first estimate[10] $J_b = J^j_b$. Integrating by parts,

$$\begin{aligned} J_{b,\mathbf{k}}(t) &= \int_{\Delta_{n-1}(t)} \dots a_{k_{j-1}}(t_{j-1})Mb_{k_j}(t_{j-1})Ma_{k_{j+1}}(t_{j+1}) \dots Mg(t_{n+1})dt_1 \dots \widehat{dt_j} \dots dt_{n+1}\\ &- \int_{\Delta_{n-1}(t)} \dots a_{k_{j-1}}(t_{j-1})Mb_{k_j}(t_{j+1})Ma_{k_{j+1}}(t_{j+1}) \dots Mg(t_{n+1})dt_1 \dots \widehat{dt_j} \dots dt_{n+1}\\ &= J^-_{b,\mathbf{k}}(t) + J^+_{b,\mathbf{k}}(t) \end{aligned}$$

Now, with the help of Lemma 2.6, we proceed as in the previous subsection,

$$\|J^-_{b,\mathbf{k}}(t)\|_{L^1} \lesssim C^n \sup_t \|b_{k_j}(t)\|_{H^1} \int_{\Delta_{n-1}(t)} A_{\mathbf{k}}(t_1, \dots \widehat{t_j} \dots t_n) N(g)(t_{n+1}) dt_1 \dots \widehat{dt_j} \dots dt_{n+1}$$

where,

$$A_{\mathbf{k},j}(\dots \widehat{t_j} \dots) = \|a_{k_1}(t_1)\|_{H^1} \dots \widehat{\|a_{k_j}(t_j)\|}_{H^1} \dots \|a_{k_n}(t_n)\|_{H^1}$$

Henceforth, with the help of Lemma 3.2,

$$\|J^-_{b,\mathbf{k}}(t)\|_{L^1} \lesssim C^n N(g) 2^{-k_j/2} \|b_{k_j}\|_2 \, |\Delta_{n-2}(t)|^{1/2} \Big(\int_{\Delta_{n-2}(t)} A_{\mathbf{k}}(\dots \widehat{t_j} \dots)^2 dt_1 \dots \widehat{dt_j} \dots dt_n \Big)^{1/2}$$

where $|\Delta_{n-2}(t)|$ is the volume of the $n-2$ dimensional simplex[11]. Consequently,

$$\|J^-_{b,\mathbf{k}}(t)\|_{L^1} \lesssim C^n ((n-1)!)^{-1/2} N(g) 2^{-k_j/2} \|b_{k_j}\|_2 \|a_{k_1}\|_1 \dots \widehat{\|a_{k_j}\|_1} \dots \|a_{k_n}\|_1$$

and, by triangle inequality and then Cauchy-Schwartz,

$$\begin{aligned} \|\sum_{\mathbf{k}\in[k_j]} J^-_{b,\mathbf{k}}(t)\|_{L^1} &\lesssim C^n ((n-1)!)^{-1/2} N(g) \sum_{\mathbf{k}\in[k_j]} 2^{-k_j/2} \|b_{k_j}\|_2 \|a_{k_1}\|_1 \dots \widehat{\|a_{k_j}\|_1} \dots \|a_{k_n}\|_1\\ &\lesssim C^n ((n-1)!)^{-1/2} N(g) \Big(\sum_{\mathbf{k}\in[k_j]} 2^{-k_j}\Big)^{1/2} \Big(\sum_{\mathbf{k}\in[k_j]} \|b_{k_j}\|_2^2 \|a_{k_1}\|_1^2 \dots \|a_{k_n}\|_1^2 \Big)^{1/2}\\ &\lesssim C^n \Big(\frac{n!}{(n-1)!}\Big)^{1/2} N(g) \|b\|_2 \|a\|_1^{n-1}\\ &\lesssim n^{\frac12} C^n N(g) \|b\|_2 \|a\|_1^{n-1} \end{aligned}$$

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[10] For simplicity, since j is kept fix we drop the j upper index below

[11] In our notations it corresponds to an actual $(n-1)$-dimensional simplex.

Proceeding exactly in the same way we derive,

$$\|\sum_{\mathbf{k}\in[k_j]} J^+_{b,\mathbf{k}}(t)\|_{L^1} \quad \lesssim \quad nC^n N(g)\|b\|_2\|a\|_1^{n-1}$$

Therefore, recalling that $J_b(t)=\sum_{\mathbf{k}\in[k_j]} J_{b,\mathbf{k}}(t)$,

$$\|J^j_b(t)\|_{L^1(\mathbb{R}^d)} \quad \lesssim \quad nC^n N(g)\|b\|_2\|a\|_1^{n-1} \tag{49}$$

To estimate $J^j_c(t)=\sum_{\mathbf{k}\in[k_j]} J_{c,\mathbf{k}}(t)$ we have to do a further decomposition. We define,

$$[k_j,k_l]=\{(k_1,k_2,\dots k_n)\in(\mathbb{Z}_+\cup\{0\})^n \mid k_i\le k_l\le k_j \quad \forall i\neq l,j\} \tag{50}$$

For fixed j we have precisely $n-1$ such regions covering $[k_j]$. Fix $l\neq j$ and consider,

$$J^{jl}_c(t)=\sum_{\mathbf{k}\in[k_j,k_l]} J^{jl}_{c,\mathbf{k}}(t) \tag{51}$$

Clearly,

$$\|J^j_c(t)\|_{L^1(\mathbb{R}^d)} \quad \lesssim \quad \sum_{l\neq j}\|J^{jl}_{c,\mathbf{k}}(t)\|_{L^1(\mathbb{R}^d)} \tag{52}$$

In view of (39) we decompose,

$$a_{k_l}(t_l)=\partial_t b_{k_l}(t_l)+c_{k_l}(t_l) \tag{53}$$

Thus, dropping the upper indices j,l,

$$\begin{aligned}
J_c(t) &= J_{cb}(t)+J_{cc}(t)=\sum_{\mathbf{k}\in[k_j,k_l]} J_{cb,\mathbf{k}}(t)+\sum_{\mathbf{k}\in[k_j,k_l]} J_{cc,\mathbf{k}}(t) \qquad (54)\\
J_{cb,\mathbf{k}}(t) &= \int_{\Delta_n(t)} a_{k_1}(t_1)M\dots c_{k_j}(t_j)M\dots\partial_t b_{k_l}(t_l)M\dots a_{k_n}(t_n)Mg(t_{n+1})dt_1\dots dt_{n+1}\\
J_{cc,\mathbf{k}}(t) &= \int_{\Delta_n(t)} a(t_1)_{k_1}M\dots c_{k_j}(t_j)M\dots c_{k_l}(t_l)\dots a_{k_n}(t_n)Mg(t_{n+1})dt_1\dots dt_{n+1}
\end{aligned}$$

Integrating by parts, and droping the operators M for a moment,

$$\begin{aligned}
J_{cb,\mathbf{k}}(t) &= \int_{\Delta_{n-1}(t)} \dots c_{k_j}(t_j)\dots a_{k_{l-1}}(t_{l-1})b_{k_l}(t_{l-1})a_{k_{l+1}}(t_{l+1})\dots g(t_{n+1})dt_1\dots\widehat{dt_l}\dots dt_{n+1}\\
&- \int_{\Delta_{n-1}(t)} \dots c_{k_j}(t_j)\dots a_{k_{l-1}}(t_{l-1})b_{k_l}(t_{l+1})a_{k_{l+1}}(t_{l+1})a_{k_{l+2}}(t_{l+2})\dots g(t_{n+1})dt_1\dots\widehat{dt_l}\dots dt_{n+1}\\
&= J^-_{cb,\mathbf{k}}(t)+J^+_{cb,\mathbf{k}}(t)
\end{aligned}$$

Using Lemma 2.6 as before,

$$\|J^\pm_{cb,\mathbf{k}}(t)\|_{L^1} \quad \lesssim \quad C^n\sup_t\|b_{k_l}(t)\|_{H^1}\int_{\Delta_{n-1}(t)} B_{\mathbf{k}}(t_1,\dots\widehat{t_l}\dots t_n)N(g)(t_{n+1})dt_1\dots\widehat{dt_l}\dots dt_{n+1}$$

where,

$$B_{\mathbf{k}}(\dots\widehat{t_l}\dots) \quad = \quad \|a_{k_1}(t_1)\|_{H^1}\dots\|c_{k_j}(t_j)\|_{H^1}\dots\|\widehat{a_{k_l}(t_l)\|}_{H^1}\dots\|a_{k_n}(t_n)\|_{H^1}$$

Therefore, exactly as before with the help of Lemma 3.2,

$$\begin{aligned}\|J^{\pm}_{cb,\mathbf{k}}(t)\|_{L^1} &\lesssim C^n N(g)2^{-k_l/2}\|b_{k_l}\|_2\; P_{\mathbf{k},n-2}(t)\\ P_{\mathbf{k},n-2}(t) &= \int_{\Delta_{n-2}(t)} B_{\mathbf{k}}(\dots\widehat{t_l}\dots)dt_1\dots\widehat{dt_l}\dots dt_n\end{aligned}$$

Observe that,

$$P_{\mathbf{k},n-2}(t) \leq \int_{\Delta_{n-2}(t)} \|a_{k_1}(t_1)\|_{H^1}\dots\|c_{k_j}(t_j)\|_{H^1}\dots\|\widehat{a_{k_l}(t_l)}\|_{H^1}\dots\|a_{k_n}(t_n)\|_{H^1}dt_1\dots\widehat{dt_l}\dots dt_n$$

Thus,

$$\|\sum_{\mathbf{k}\in[k_j,k_l]} J^{\pm}_{cb,\mathbf{k}}(t)\|_{L^1} \lesssim C^n((n-2)!)^{-1/2}N(g)Q$$

with,

$$Q = \sum_{k_l\leq k_j} 2^{-k_l/2}\|b_{k_l}\|_2\|c_{k_j}\|_3\sum_{\mathbf{k''}\leq k_l}\|a_{k_1}\|_1\dots\widehat{\|a_{k_j}\|_1}\dots\widehat{\|a_{k_l}\|_1}\dots\|a_{k_n}\|_1$$

with $k'' = (k_1,k_2\dots,\widehat{k_j},\dots,\widehat{k_l}\dots k_n)$. Therefore, by Cauchy-Schwartz,

$$\begin{aligned}Q &\lesssim \sum_{k_l\leq k_j} 2^{-k_l/2}k_l^{(n-2)/2}\|b_{k_l}\|_2\|c_{k_j}\|_3\big(\sum_{\mathbf{k''}\leq k_l}\|a_{k_1}\|_1^2\dots\|a_{k_n}\|_1^2\big)^{1/2}\\ &\lesssim \|a\|_1^{n-2}\sum_{k_j\in\mathbb{Z}}\|c_{k_j}\|_3\sum_{k_l\leq k_j}2^{-k_l/2}k_l^{(n-2)/2}\|b_{k_l}\|_2\\ &\lesssim \|a\|_1^{n-2}\|b\|_2\sum_{k_j\in\mathbb{Z}}\|c_{k_j}\|_3\big(\sum_{k_l=0}^{k_j}2^{-k_l}k_l^{(n-2)}\big)^{1/2}\\ &\lesssim ((n-1)!)^{1/2}\|a\|_1^{n-2}\|b\|_2\|c\|_3\end{aligned}$$

Consequently,

$$\begin{aligned}\|\sum_{\mathbf{k}\leq k_l\leq k_j} J^{\pm}_{cb,\mathbf{k}}(t)\|_{L^1} &\lesssim C^n\big(\frac{(n-1)!}{(n-2)!}\big)^{1/2}N(g)\|a\|_1^{n-2}\|b\|_2\|c\|_3\\ &\lesssim n^{\frac{1}{2}}C^nN(g)\|a\|_1^{n-2}\|b\|_2\|c\|_3\end{aligned}$$

Therefore,

$$\sup_{t\in[0,1]}\|J^{jl}_{cb}(t)\|_{L^1(\mathbb{R}^d)} \lesssim n^{\frac{1}{2}}C^nN(g)\|a\|_1^{n-2}\|b\|_2\|c\|_3 \tag{55}$$

To treat the term $J_{cc,\mathbf{k}}(t)$ we decompose once more. Continuing in the same manner after m steps we arrive at the integral,

$$J^{j_1j_2\dots j_{m-1}}_{c_1\dots c_{m-1}}(t) = \sum_{[k_{j_1}\dots k_{j_{m-1}}]}\int_{\Delta_n(t)}\dots \tag{56}$$

with the integrand containing $c_1 = c_{k_{j_1}}, c_2 = c_{k_{j_2}}\dots c_{m-1} = c_{k_{j_{m-1}}}$ and

$[k_{j_1},\dots k_{j_{m-1}}] = \{(k_1\dots k_n)\in\mathbb{Z}^n \mid k_i\leq k_{j_m}\leq\dots\leq k_{j_1} \quad \forall i\neq j_1,j_2\dots j_{m-1}\}$

Clearly $[k_{j_1}, \dots k_{j_{m-1}}]$ can be covered by precisely $n-m+1$ regions of the form $[k_{j_1}, \dots k_{j_m}]$. We have,

$$J^{j_1 j_2 \dots j_{m-1}}_{c_1 \dots c_{m-1}}(t) = \sum_{j_m} J^{j_1 j_2 \dots j_m}_{c_1 \dots c_{m-1}}(t), \qquad k_{j_m} \le k_{j_{m-1}} \tag{57}$$

$$J^{j_1 j_2 \dots j_m}_{c_1 \dots c_{m-1}}(t) = \sum_{[k_{j_1} \dots k_{j_m}]} \int_{\Delta_n(t)} \dots \tag{58}$$

In view of (39) we decompose,

$$a_{k_{j_m}}(t_{j_m}) = \partial_t b_{k_{j_m}}(t_{j_m}) + c_{k_{j_m}}(t_{j_m}) \tag{59}$$

and, respectively,

$$J^{j_1 j_2 \dots j_m}_{c_1 \dots c_{m-1}}(t) = \sum_{\mathbf{k} \in [k_{j_1} \dots k_{j_m}]} J^{j_1 j_2 \dots j_m}_{c_1 \dots c_{m-1} b_m, \mathbf{k}}(t) + \sum_{\mathbf{k} \in [k_{j_1} \dots k_{j_m}]} J^{j_1 j_2 \dots j_m}_{c_1 \dots c_m, \mathbf{k}}(t)$$

where $b_m = b_{k_{j_m}}$, $c_m = c_{k_{j_m}}$ Proceeding exactly as before, integrating by parts and using Lemma 2.6, we write,

$$\|J^{j_1 j_2 \dots j_m}_{c_1 \dots c_{m-1} b_m, \mathbf{k}}(t)\|_{L^1} \lesssim C^n \sup_t \|b_{k_{j_m}}(t)\|_{H^1} \int_{\Delta_{n-1}(t)} B_{\mathbf{k}}(t_1, \dots \widehat{t}_{j_m} \dots t_n) N(g)(t_{n+1})$$

where,

$$\begin{aligned} B_{\mathbf{k}}(\dots \widehat{t}_{j_m} \dots) &= \|c_{k_{j_1}}(t_{j_1})\|_{H^1} \dots \|c_{k_{j_{m-1}}}(t_{j_{m-1}})\|_{H^1} \\ &\cdot \ \|a_{k_{j_{m+1}}}(t_{j_{m+1}})\|_{H^1} \dots \|a_{k_{j_n}}(t_{j_n})\|_{H^1} \end{aligned}$$

Therefore,

$$\begin{aligned} \|J^{j_1 j_2 \dots j_m}_{c \dots cb, \mathbf{k}}(t)\|_{L^1} &\lesssim C^n N(g) 2^{-k_{j_m}/2} \|b_{k_{j_m}}\|_2 \ P_{\mathbf{k}, n-2}(t) \\ P_{\mathbf{k}, n-2}(t) &= \int_{\Delta_{n-2}(t)} B_{\mathbf{k}}(\dots \widehat{t}_{j_m} \dots) \end{aligned}$$

where $k_{j_{m+1}}, \dots k_{j_n}$ are the labels for all other frequencies different from $k_{j_1}, \dots k_{j_{m-1}}$.

To estimate $P_{\mathbf{k}, n-2}(t)$ we make use of the following obvious lemma.

Lemma 3.5. *Let $f_1, f_2, \dots f_n$ be an ordered sequence of n positive, integrable, functions defined on the interval $[0,1] \subset \mathbb{R}$ among which m, say $f_{i_1}, i = 1, \dots m$ are in L^1 and $n-m$, say $f_{j_1}, \dots f_{j_{n-m}}$ are in L^2. Then,*

$$\begin{aligned} \int_{\Delta_{n-2}(t)} f_1(t_1) \dots f_n(t_n) dt_1 \dots dt_n &\lesssim \Big(\frac{1}{(n-m)!}\Big)^{1/2} \|f_{i_1}\|_{L^1} \dots \|f_{i_m}\|_{L^1} \\ &\cdot \ \|f_{j_1}\|_{L^1} \dots \|f_{j_{n-m}}\|_{L^1} \end{aligned}$$

According to Lemma 3.5 we have,

$$P_{\mathbf{k}, n-2}(t) \lesssim \Big(\frac{1}{(n-m-1)!}\Big)^{1/2} \|c_{k_{j_1}}\|_{L^1 H^1} \dots \|c_{k_{j_{m-1}}}\|_{L^1 H^1} \cdot \|a_{k_{j_{m+1}}}\|_1 \dots \|a_{k_{j_n}}\|_1$$

Observe that,

$$\begin{aligned} \sum_{k'' \le k_{j_m}} \|a_{k_{j_{m+1}}}\|_1 \dots \|a_{k_{j_n}}\|_1 &\lesssim \big(k_{j_m}\big)^{(n-1-m)/2} \Big(\sum_{k'' \le k_{j_m}} \|a_{k_{j_{m+1}}}\|_1^2 \dots \|a_{k_{j_n}}\|_1^2 \Big)^{1/2} \\ &\lesssim \big(k_{j_m}\big)^{(n-1-m)/2} \|a\|_1^{m-n} \end{aligned}$$

where $k'' = (k_{j_{m+1}}, \dots k_{j_n})$. Observe also that,

$$\sum_{k_{j_1} \le k_{j_2} \dots \le k_{j_{m-1}}} \|c_{k_{j_1}}\|_{L^1 H^1} \dots \|c_{k_{j_{m-1}}}\|_{L^1 H^1} \lesssim \frac{1}{(m-1)!} \|c\|_3^{m-1} \tag{60}$$

Indeed this follows by symmetry in view of the fact that,

$$\sum_{k_{j_1}, \dots, k_{j_m}} \|c_{k_{j_1}}\|_{L^1 H^1} \dots \|c_{k_{j_{m-1}}}\|_{L^1 H^1} \lesssim \|c\|_3^{m-1}$$

Finally, by Cauchy-Schwartz,

$$\sum_{k_{j_m} \in \mathbb{Z}} 2^{-k_{j_m}/2} \big(k_{j_m}\big)^{(n-1-m)/2} \|b_{k_{j_m}}\|_2 \lesssim ((n-m)!)^{1/2} \|b\|_2$$

Hence,

$$\sum_{[k_{j_1} \dots k_{j_m}]} \|J^{j_1 j_2 \dots j_m}_{c \dots cb, \mathbf{k}}(t)\|_{L^1} \quad \lesssim \quad C^n \frac{1}{(m-1)!} \Big(\frac{(n-m)!}{(n-m-1)!} \Big)^{1/2} N(g) \|b\|_2 \|a\|_1^{n-m} \|c\|_3^{m-1}$$

In other words,

$$\sum_{[k_{j_1} \dots k_{j_m}]} \|J^{j_1 j_2 \dots j_m}_{c \dots cb, \mathbf{k}}(t)\|_{L^1} \quad \lesssim \quad n^{\frac{1}{2}} C^n \frac{1}{(m-1)!} \Delta_0^n \tag{61}$$

We are ready to estimate $J_n(t) = J(t)$ in formula (46). We have,

$$\|J(t)\|_{L^1)} \lesssim \sum_{j_1=1}^{n} \|J^{j_1}(t)\|_{L^1}$$

and,

$$\begin{aligned} \|J^{j_1}(t)\|_{L^1} \quad &\lesssim \quad \|J^{j_1}_{b_1}(t)\|_{L^1} + \|J^{j_1}_{c_1}(t)\|_{L^1} \\ &\lesssim \quad n^{\frac{1}{2}} C^n \Delta_0^n + \|J^{j_1}_{c_1}(t)\|_{L^1} \end{aligned}$$

Hence,

$$\|J(t)\|_{L^1} \quad \lesssim \quad n^{\frac{3}{2}} C^n \Delta_0^n + \sum_{j_1=1}^{n} \|J^{j_1}_{c_1}(t)\|_{L^1}$$

On the other hand, for each j_1,

$$\|J^{j_1}_{c_1}(t)\|_{L^1} \lesssim \sum_{j_2 \ne j_1}^{n} \|J^{j_1 j_2}_{c_1}(t)\|_{L^1}$$

and,

$$\begin{aligned} \|J^{j_1 j_2}_{c_1}(t)\|_{L^1} \quad &\lesssim \quad \|J^{j_1 j_2}_{c_1 b_2}(t)\|_{L^1} + \|J^{j_1 j_2}_{c_1 c_2}(t)\|_{L^1} \\ &\lesssim \quad n^{\frac{1}{2}} \frac{C^n \Delta_0^n}{1!} + \|J^{j_1 j_2}_{c_1 c_2}(t)\|_{L^1} \end{aligned}$$

Therefore,

$$\|J(t)\|_{L^1(\mathbb{R}^d)} \quad \lesssim \quad n^{\frac{1}{2}} n C^n \Delta_0^n + n^{\frac{1}{2}} \frac{n(n-1)}{1!} C^n \Delta_0^n + \sum_{j_1 \ne j_2} \|J^{j_1 j_2}_{c_1 c_2}(t)\|_{L^1}$$

Continuing in this way we derive,

$$\begin{aligned}
\|J_n(t)\|_{L^1} &\lesssim N(g) n^{\frac{3}{2}} \Delta_0^n C^n \big(1 + \frac{(n-1)}{1!} + \frac{(n-1)(n-2)}{2!} \ldots + \frac{(n-1)\ldots(n-m)}{(m-1)!} + \ldots 1\big) \\
&\lesssim n^{\frac{3}{2}} \Delta_0^n C^n (1+1)^{n-1} N(g) \lesssim n^{\frac{3}{2}} \Delta_0^n (2C)^n N(g),
\end{aligned}$$

as claimed in (43).

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