Bilinear Estimates and Applications to Nonlinear Wave Equations

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1 Introduction

In this paper we undertake a systematic review of results proved in [21, 22, 25, 26, 27] concerning local well-posedness of the Cauchy problem for certain systems of nonlinear wave equations, with minimal regularity assumptions on the initial data. Moreover we give a vastly simplified and unified treatment of these results and provide also complete proofs for large data. The key ingredient throughout the survey is the use of space-time bilinear estimates; they are intimately tied to the null structure of the equations we consider. The simplest type of bilinear estimates are of L^2 type; they transfom, by Plancherel's identity, to bilinear L^2 convolution estimates in Fourier space. This leads naturally to weighted L^2 spaces; in view of their similarity to the Sobolev H^s spaces we denote them $H^{s,\theta}$, and propose to call them Wave-Sobolev spaces[1]. Though these spaces play a fundamental role, in most applications they need to be refined. In this survey we do this by taking their intersection with suitable weighted $\mathcal{L}_t^q(\mathcal{L}_x^r)$ type spaces. These spaces are described in detail in section 4. The main bilinear estimates are summarized in section 2, the spaces $H^{s,\theta}$ are discussed in section 3. The main nonlinear results are stated below and proved in sections 7–9. In section 10 we discuss some of the main open problems and provide some historical remarks.

On the Minkowski space-time $\mathrm{R} \times \mathrm{R}^n \simeq \mathrm{R}^{1+n}$ we use coordinates $(t,x) = (x^0, \dots, x^n)$, and indices are raised and lowered relative to the metric $m_{\mu\nu} = \mathrm{diag}(-1, 1, \dots, 1)$. The summation convention is used in some sections. We write $\partial_\mu = \partial_{x^\mu}$ and $\partial_t = \partial_0$.

We are interested in the Cauchy problem for systems of the type

$$\Lambda u = \mathcal{N}(u),$$

where $\Lambda = -\partial_t^2 + \Delta$ is the standard wave operator on $\mathrm{R} \times \mathrm{R}^n$, $\Delta = \sum_1^n \partial_j^2$ is the Laplacian on R^n, $u = u(t,x)$ takes values in R^N for some $N \geq 1$ and $\mathcal{N}$ is

[1] These spaces have apppeared before in PDE, in connection with questions of propagation of singularities for nonlinear wave equations. In the context of bilinear estimates and optimal well-posedness of the Cauchy problem they appear first in [4], in the study of periodic solution to KdV and nonlinear Schrödinger equations (see also [14]), and in [18] in connection with semilinear wave equations satisfying the null condition. See section 10 for more complete historical remarks.

an operator which is local with respect to the time variable (i.e., $\mathcal{N}(u)(t,\cdot)$ only depends on $u(t',\cdot)$ for t' in any neighborhood of t).

Cauchy data are prescribed on the initial hypersurface $\{0\} \times \mathrm{R}^n \simeq \mathrm{R}^n$:

$$(u, \partial_t u)\big|_{t=0} = (f, g) \in H^s \times H^{s-1}$$

where $H^s = \{f : (I - \Delta)^{s/2} f \in L^2\}$.

1.1 Statement of Main Results

We shall in fact concentrate on systems of the following types:

(i) *Wave Maps Type:*

$$\text{(WM)} \qquad \Lambda u^I + \sum_{J,K} \Gamma^I_{JK}(u) Q_0(u^J, u^K) = 0.$$

Here, u^I denotes the I-th component function of u, the Γ^I_{JK} are smooth functions from R^N into R and Q_0 is the null form

$$Q_0(\phi, \psi) = \sum_{\mu=0}^{n} \partial_\mu \phi \partial^\mu \psi = -\partial_t \phi \partial_t \psi + \sum_{j=1}^{n} \partial_j \phi \partial_j \psi.$$

(ii) *Yang-Mills Type:*

$$\text{(“YM”)} \qquad \Lambda u = D^{-1} Q(u, u) + Q(D^{-1} u, u),$$

where $D^\alpha = (-\Delta)^{\alpha/2}$ and Q stands for any bilinear operator of the following type: Given vector-valued functions u and v, the I-th component function of $Q(u,v)$ is a linear combination, with constant, real coefficients, of $Q_{ij}(u^J, v^K)$ for all $1 \le i < j \le n$ and all J, K, where Q_{ij} is the null form

$$Q_{ij}(\phi, \psi) = \partial_i \phi \partial_j \psi - \partial_i \phi \partial_j \psi.$$

(The two Q's on the right hand side of (“YM”) may represent two different such operators.)

(iii) *Maxwell-Klein-Gordon Type:*

$$\text{(“MKG”)} \qquad \begin{cases} \Lambda u = D^{-1} Q(v, v), \\ \Lambda v = Q(D^{-1} u, v), \end{cases}$$

where $u = (u^1, \ldots, u^{N_1})$, $v = (v^1, \ldots, v^{N_2})$, $N = N_1 + N_2$ and Q has the same meaning as before. Thus (“MKG”) is a special case of (“YM”).

(iv) *Wave Maps Model Problem:*

$$\text{(WMM)} \qquad \Lambda u^I = \sum_{J,K=1}^{N} a^I_{JK} \widetilde{Q}(u^J, u^K),$$

where the a^I_{JK} are real constants,

$$\widetilde{Q}(\phi,\psi) = \sum_{j=1}^{n} \partial_j \big(R_0 R_j \phi \cdot \psi - \phi \cdot R_0 R_j \psi \big)$$

and $R_\mu = D^{-1}\partial_\mu$.

The following theorem summarizes the main well-posedness results proved[2] in [21, 22, 25, 26, 27].

Main Theorem. *(a) ([21, 26].) If $n \geq 2$ and $s > \frac{n}{2}$, then* (WM) *is locally well-posed for initial data in $H^s \times H^{s-1}$.*

(b) ([25, 27].) If $n \geq 4$ and $s > \frac{n-2}{2}$, then ("MKG") *and* ("YM") *are locally well-posed for initial data in $H^s \times H^{s-1}$.*

(c) ([22].) If $n \geq 3$ and $s > \frac{n-2}{2}$, then (WMM) *is locally well-posed for initial data in $H^s \times H^{s-1}$.*

By *locally well-posed* we mean that for all $(f,g) \in H^s \times H^{s-1}$ there exist $T > 0$ and

$$u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$$

such that u solves the equation on $(0,T) \times \mathrm{R}^n$ in the sense of distributions, and such that the initial condition is satisfied. Moreover, T is bounded below by a strictly positive and continuous function of $\|f\|_{H^s} + \|g\|_{H^{s-1}}$, the map $(f,g) \mapsto u$ is locally Lipschitz[3], and u is unique in some subspace of $C([0,T],H^s) \cap C^1([0,T],H^{s-1})$. Moreover, any additional regularity of the initial data persists in time, but for simplicity we ignore this issue.

1.2 Motivation of the Equations

With the exception of (WM), the equations we work with are model problems derived from the actual Maxwell-Klein-Gordon, Yang-Mills and wave maps equations. Here we review these equations and discuss how our model problems relate to them.

Wave Maps

A *wave map* from the Minkowski space-time into a Riemannian manifold (M,g) is a map $u : \mathrm{R}^{1+n} \to M$ which is a critical point with respect to compactly

[2] Strictly speaking most of these results were proved only for sufficiently small data. Large data recquire some technical considerations discussed in this paper.

[3] In fact, the solution depends smoothly (or even analytically in most of the above examples) on the data, in the sense that if $\varepsilon \mapsto (f_\varepsilon, g_\varepsilon)$ is a smooth map into $H^s \times H^{s-1}$ for $|\varepsilon| < \varepsilon_0$, and if u_ε is the solution corresponding to the initial data $(f_\varepsilon, g_\varepsilon)$, then $\varepsilon \mapsto u_\varepsilon$ is a smooth map into $C([0,T],H^s) \cap C^1([0,T],H^{s-1})$ for some $T > 0$. This is because the solution is obtained by a Picard iteration procedure; see [34].

supported variations of the Lagrangian

$$\mathcal{L}[u] = \frac{1}{2}\int_{\mathbb{R}^{1+n}} \langle\, du, du \,\rangle \; dt\, dx,$$

where $\langle\, du, du \,\rangle = \sum_{\mu=0}^{n} \sum_{a,b} g_{ab} \partial_\mu u^a \partial^\mu u^b$ in local coordinates on M. The Euler-Lagrange equation for this variational problem is exactly of the form (WM), in local coordinates on M, with Γ^I_{JK} the Christoffel symbols of M in the local chart and $N = \dim M$ (see, e.g., Shatah-Struwe [36]).

Maxwell-Klein-Gordon Equations

In the following discussion, the summation convention is in effect. Greek indices are summed from 0 to n, roman indices from 1 to n. Recall that indices are raised and lowered relative to the Minkowski metric $m_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$. For example, $\Lambda = \partial^\mu \partial_\mu$ and $\Delta = \partial^j \partial_j$. We denote by i the imaginary unit.

The unknowns of the equations are a one-form $A_\mu dx^\mu$ (the gauge potential) and a scalar ϕ, both defined on the Minkowski space-time:

$$A_\mu : \mathbb{R}^{1+n} \to \mathbb{R},$$
$$\phi : \mathbb{R}^{1+n} \to \mathbb{C}.$$

The electromagnetic field is the two-form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The covariant derivative relative to the gauge potential is

$$D_\mu \phi = \partial_\mu \phi + i A_\mu \phi.$$

We are looking for critical points of the Lagrangian

$$\mathcal{L}[A_\mu, \phi] = \int_{\mathbb{R}^{1+n}} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \phi \overline{D^\mu \phi} \right) dt\, dx.$$

The corresponding Euler-Lagrange equations are

$$\text{(MKGa)} \qquad \partial^\mu F_{\mu\nu} = -\Im\big(\phi \overline{D_\nu \phi}\big),$$
$$\text{(MKGb)} \qquad D^\mu D_\mu \phi = 0,$$

where $\Im z$ denotes the imaginary part of z.

Let χ be a real-valued function on $\mathbb{R}^{1+n}$, and consider the transformation $(A_\mu, \phi) \to (\widetilde{A}_\mu, \widetilde{\phi})$ given by

$$\widetilde{A}_\mu = A_\mu - \partial_\mu \chi,$$
$$\widetilde{\phi} = e^{i\chi} \phi.$$

Clearly, the electromagnetic field is left unchanged by the gauge transformation $A_\mu \to \widetilde{A}_\mu$, and a simple calculation reveals that if (A_μ, ϕ) verifies (MKG), then so does $(\widetilde{A}_\mu, \widetilde{\phi})$ (keep in mind that D_μ depends on A_μ). This gives an equivalence relation on the set of pairs (A_μ, ϕ) verifying (MKG), and by a *solution* of the latter, we understand an equivalence class of such pairs.

Thus, we have gauge freedom; i.e., we are free to choose any representative of a given solution (equivalence class), and we may stipulate a condition that the gauge potential should satisfy. The traditional gauge conditions are:

- *Lorentz:* $\partial^\mu A_\mu = 0$,
- *Coulomb:* $\partial^j A_j = 0$,
- *Temporal:* $A_0 = 0$.

(MKG) in Lorentz gauge. Coupling the Lorentz condition with (MKG) yields the system

$$\Lambda A_\mu = -\Im\big(\phi\overline{\partial_\mu\phi}\big) + |\phi|^2 A_\mu, \tag{1.1a}$$
$$\Lambda\phi = -2iA^\mu\partial_\mu\phi + A^\mu A_\mu\phi, \tag{1.1b}$$
$$\partial^\mu A_\mu = 0. \tag{1.1c}$$

Now observe that if (A_μ, ϕ) satisfies (1.1a) and (1.1b) with initial data

$$A_\mu\big|_{t=0} = a_\mu, \qquad \partial_t A_\mu\big|_{t=0} = b_\mu, \tag{1.2a}$$
$$\phi\big|_{t=0} = \phi_0, \qquad \partial_t\phi\big|_{t=0} = \phi_1 \tag{1.2b}$$

satisfying the constraints

$$b_0 = \partial^j a_j, \qquad \Delta a_0 - |\phi_0|^2 a_0 = \partial^j b_j - \Im(\phi_0\overline{\phi_1}), \tag{1.3}$$

then (1.1c) is automatically satisfied. For by (1.1a) and (1.1b), $u = \partial^\mu A_\mu$ solves

$$\Lambda u = |\phi|^2 u,$$

and by (1.2a) and (1.3), $u\big|_{t=0} = \partial_t u\big|_{t=0} = 0$. By uniqueness of solutions, $u = 0$.

Thus, (1.1c) is equivalent to the constraint (1.3) on the initial data, so we are left with (1.1a) and (1.1b). Therefore, (MKG) in Lorentz gauge is schematically of the form $\Lambda u = u\partial u + u^3$. Unfortunately[4], generic equations of this type do not have good local regularity properties, so the Lorentz gauge is not very useful for our purposes.

(MKG) in Coulomb gauge. Coupling the Coulomb condition with (MKG) gives

$$\Delta A_0 = -\Im\big(\phi\overline{\partial_t\phi}\big) + |\phi|^2 A_0, \tag{1.4a}$$
$$\Lambda A_j = -\Im\big(\phi\overline{\partial_j\phi}\big) + |\phi|^2 A_j - \partial_j\partial_t A_0, \tag{1.4b}$$
$$\Lambda\phi = -2iA^j\partial_j\phi + 2iA_0\partial_t\phi + i(\partial_t A_0)\phi + A^\mu A_\mu\phi, \tag{1.4c}$$
$$\partial^j A_j = 0. \tag{1.4d}$$

[4] See our discussion concerning the first iterate in section 1.3 below.

Here we have split the gauge potential into its time component A_0 and its spatial component $A = A_j dx^j$. We prescribe initial data at time $t = 0$:

$$A_j\big|_{t=0} = a_j, \qquad \partial_t A_j\big|_{t=0} = b_j, \tag{1.5a}$$

$$\phi\big|_{t=0} = \phi_0, \qquad \partial_t \phi\big|_{t=0} = \phi_1. \tag{1.5b}$$

No initial condition is imposed on A_0; if we set $a_0 = A_0\big|_{t=0}$, then by (1.4a), $\Delta a_0 - |\phi_0|^2 a_0 = -\Im(\phi_0 \overline{\phi_1})$.

Equation (1.4d) is automatically satisfied if the data are divergence-free:

$$\partial^j a_j = \partial^j b_j = 0. \tag{1.6}$$

For if (A_0, A, ϕ) satisfies (1.4a)–(1.4c), then $u = \partial^j A_j$ solves $\Lambda u = |\phi|^2 u$, and if (1.5) and (1.6) are satisfied, then $u\big|_{t=0} = \partial_t u\big|_{t=0} = 0$.

We are then left with the equations (1.4a)–(1.4c). The first of these, being an elliptic equation, is relatively easy to handle, so we leave it out of our model equations. The two remaining equations have terms of three types on the right hand side:

- "Elliptic terms" involving A_0; these are collectively denoted by $\mathcal{E}$.
- Cubic terms in A_j and ϕ; these are collectively denoted by $\mathcal{C}$.
- Quadratic terms with a null-form structure.

The terms falling into the latter category are $-\Im\big(\phi\overline{\partial_j \phi}\big)$ and $-2iA^j\partial_j\phi$. We now uncover the null-form structure inherent in these expressions (due to the Coulomb condition).

Split ϕ into its real and imaginary parts: $\phi = u + iv$. Then

$$-\Im\big(\phi\overline{\partial_j \phi}\big) = u\partial_j v - v\partial_j u,$$

so (1.4b) reads, as an equation of (time-dependent) one-forms on R^n:

$$\Lambda A = udv - vdu + \mathcal{C} - d(\partial_t A_0).$$

Apply d to both sides:

$$\Lambda(dA) = 2du \wedge dv + d\mathcal{C}.$$

But

$$du \wedge dv = \frac{1}{2} Q_{jk}(u,v) dx^j \wedge dx^k,$$

whence

$$\Lambda F_{jk} = Q_{jk}(u,v) + \partial\mathcal{C}.$$

The Coulomb gauge condition implies that $\partial^k F_{jk} = -\Delta A_j$, so we have

$$-\Delta\Lambda A_j = \partial^k Q_{jk}(u,v) + \partial^2\mathcal{C}.$$

Thus, modulo Riesz operators,

$$\Lambda A = D^{-1} Q(\Re\phi, \Im\phi) + \mathcal{C}, \tag{1.7}$$

where Q is some linear combination of the null forms[5] Q_{jk}. Since the cubic term $\mathcal{C}$ is easier to estimate, we leave it out of our model problem.

Now consider equation (1.4c). Separating real and imaginary parts, we have

$$\begin{aligned} \Lambda u &= 2A \cdot \nabla v + \mathcal{C} + \mathcal{E}, \\ \Lambda v &= -2A \cdot \nabla u + \mathcal{C} + \mathcal{E}. \end{aligned}$$

(Here we consider A as a vector field by raising its indices; ∇ denotes the gradient in the space variables.) We claim that the terms $A \cdot \nabla u$ and $A \cdot \nabla v$ have a null-form structure, due to the fact that A is divergence-free (by the Coulomb condition). Let B_{jk} be the unique solution of

$$\Delta B_{jk} = \partial_j A_k - \partial_k A_j \tag{1.8}$$

(with appropriate regularity assumptions). By the Coulomb condition,

$$\Delta \partial^j B_{jk} = \Delta A_k, \quad \text{which implies} \quad \partial^j B_{jk} = A_k. \tag{1.9}$$

Thus,

$$A \cdot \nabla u = \partial^j B_{jk} \partial^k u = \frac{1}{2} Q_{jk}(u, B^{jk}).$$

The above equations for $u = \Re\phi$ and $v = \Im\phi$ can therefore be rewritten

$$\begin{aligned} \Lambda \Re\phi &= Q_{jk}(\Im\phi, B^{jk}) + \mathcal{C} + \mathcal{E}, \\ \Lambda \Im\phi &= Q_{jk}(B^{jk}, \Re\phi) + \mathcal{C} + \mathcal{E}. \end{aligned}$$

But in view of (1.8), B is of the form $D^{-1}A$ modulo Riesz operators. Combining this with (1.7) and discarding the terms $\mathcal{C}$ and $\mathcal{E}$ throughout, we obtain a system of the form ("MKG"), which is our model for (MKG).

Yang-Mills Equations

Let G be one of the classical, compact Lie groups of matrices (such as $\mathrm{SO}(k, \mathbb{R})$ or $\mathrm{SU}(k, \mathbb{C})$), and let $\mathfrak{g}$ be its Lie algebra. The unknown is a $\mathfrak{g}$-valued one-form $A_\mu dx^\mu$ on $\mathbb{R}^{1+n}$. The corresponding covariant derivative is

$$D_\mu H = \partial_\mu H + [A_\mu, H],$$

where H is any $\mathfrak{g}$-valued tensor field on $\mathbb{R}^{1+n}$ and $[\cdot, \cdot]$ is the matrix commutator.

The curvature is the $\mathfrak{g}$-valued two-form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

[5]To be precise, the j-th component of Q is $\sum_k R_k Q_{jk}$, where $R_k = D^{-1}\partial^k$ is the k-th Riesz operator. Since we work with norms which only depend on the size of the Fourier transform, we ignore the Riesz operators.

The Lagrangian is

$$\mathcal{L}[A_\mu] = -\frac{1}{4}\int \langle F_{\mu\nu}, F^{\mu\nu} \rangle \, dt \, dx,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathfrak{g}$ inherited from the ambient space (e.g., $SO(k, \mathrm{R})$ embeds in R^{k^2}, so its Lie algebra can be viewed as a subspace of the latter). The Euler-Lagrange equations are

$$D^\nu F_{\mu\nu} = 0. \tag{YM}$$

Let O be a G-valued function on R^{1+n}. Consider the gauge transformation $A_\mu \to \widetilde{A}_\mu$, given by

$$\widetilde{A}_\mu = O A_\mu O^{-1} - \partial_\mu O O^{-1}.$$

A calculation shows that the curvature then transforms into

$$\widetilde{F}_{\mu\nu} = O F_{\mu\nu} O^{-1}.$$

Denoting by $\widetilde{D}_\mu$ the covariant derivative corresponding to $\widetilde{A}_\mu$, we then have

$$\widetilde{D}^\nu \widetilde{F}_{\mu\nu} = O D^\nu F_{\mu\nu} O^{-1},$$

so (YM) is invariant under gauge transformations. We therefore have gauge freedom, and may impose a gauge condition on A_μ.

(YM) in Coulomb gauge. Relative to the Coulomb condition $\partial^j A_j = 0$, (YM) takes the form (see [20])

$$\Delta A_0 = 2[\partial^j A_0, A_j] + [A^j, \partial_t A_j] + [A^j, [A_0, A_j]], \tag{1.10a}$$

$$\begin{aligned} \Lambda A_j + \partial_t \partial_j A_0 = {} & -2[A^k, \partial_k A_j] + [A^k, \partial_j A_k] + [\partial_t A_0, A_j] + 2[A_0, \partial_t A_j] \\ & - [A_0, \partial_j A_0] - [A^k, [A_k, A_j]] + [A_0, [A_0, A_j]], \end{aligned} \tag{1.10b}$$

$$\partial^j A_j = 0. \tag{1.10c}$$

Unfortunately, assuming the existence of a global Coulomb gauge forces a restrictive smallness assumption on the initial data. In [20] this difficulty was resolved by using local arguments. Following [27], we ignore this complication, and derive our model equation from the system (1.10).

As in the discussion of (MKG), (1.10c) reduces to a constraint on the initial data. The equation for A_0 is elliptic, so we ignore it. As for (1.10b), we only retain the first two terms on the right, since the other terms either involve A_0 (for which we expect to have better estimates than for A_j), or are cubic.

Now write (1.10b) as an equation of time-dependent, $\mathfrak{g}$-valued one-forms on R^n (ignoring all but the first two terms on the right):

$$\Lambda A + d(\partial_t A_0) = S + T,$$

where $A = A_j dx^j$, $S = -2[A^k, \partial_k A_j] dx^j$ and $T = [A^k, \partial_j A_k] dx^j$. Apply the exterior derivative d to both sides:

$$\Lambda dA = dS + dT.$$

Let B be the two-form (in this case $\mathfrak{g}$-valued) determined by equation (1.8). Thus

$$\Lambda\Delta B_{jk} = \partial_j S_k - \partial_k S_j + \partial_j T_k - \partial_k T_j.$$

By (1.9), it follows that

$$-\Lambda\Delta A_j = \partial^k \left(\partial_j S_k - \partial_k S_j + \partial_j T_k - \partial_k T_j\right),$$

so for the purposes of estimates in frequency space, we may replace (1.10b) by

$$\Lambda A = S + D^{-1} dT. \tag{1.11}$$

It remains to identify the null form structure hidden in S and dT. To begin with, we have

$$\begin{aligned} S_j = [\partial_k B^{kl}, \partial_l A_j] &= \frac{1}{2}[\partial_k B^{kl} - \partial_k B^{lk}, \partial_l A_j] \\ &= \frac{1}{2}\left([\partial_k B^{kl}, \partial_l A_j] - [\partial_l B^{kl}, \partial_k A_j]\right) \\ &= \frac{1}{2}\left(\partial_k B^{kl}\partial_l A_j - \partial_l B^{kl}\partial_k A_j + \partial_k A_j \partial_l B^{kl} - \partial_l A_j \partial_k B^{kl}\right), \end{aligned}$$

so each entry of the matrix S_j is a linear combination of terms of the form $Q_{kl}(A, B)$, where A and B stand for any two entries of A_j and B_{kl}. But by (1.9), we may replace B by $D^{-1}A$. Schematically,

$$S \sim Q(A, D^{-1}A). \tag{1.12}$$

Now consider the one-form T. We calculate:

$$\begin{aligned} (dT)_{jk} &= \partial_j [A^l, \partial_k A_l] - \partial_k [A^l, \partial_j A_l] \\ &= [\partial_j A^l, \partial_k A_l] - [\partial_k A^l, \partial_j A_l]. \end{aligned}$$

Thus, each entry of the matrix $(dT)_{jk}$ is a linear combination of terms of the form $Q_{jk}(A, A')$, where A and A' stand for any two entries of A_l, $1 \le l \le n$. Combining this with (1.12) and (1.11), we arrive at the model ("YM") for the Yang-Mills equations.

Wave Maps Model Problem

The (WMM) equation arises from a simple reformulation of Wave Maps whose target manifold has a bi-invariant Lie group structure. Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra, identified with the tangent space $T_e G$, where e is the unit in G. For any $a \in G$, we denote by L_a and R_a the left and right translation operators on G, given by $L_a(g) = ag$ and $R_a(g) = ga$. Their derivatives are denoted by L_{a*} and R_{a*} respectively.

Assume that G is endowed with a Riemannian metric h which is bi-invariant; i.e., $h(L_{a*}X, L_{a*}Y) = h(X, Y)$ and $h(R_{a*}X, R_{a*}Y) = h(X, Y)$ for all $a \in G$ and all tangent vector fields X and Y.

Let $u : \mathrm{R}^{1+n} \to G$. Then for all $0 \le \mu \le n$ and $(t,x) \in \mathrm{R}^{1+n}$, $\partial_\mu u(t,x)$ is a vector in the tangent space $T_{u(t,x)}G$, and we move this vector into the Lie algebra T_eG by left translation. More precisely, we define a $\mathfrak{g}$-valued one-form $A_\mu dx^\mu$ by

$$A_\mu = L_{u^{-1}*}(\partial_\mu u),$$

where u^{-1} denotes the group inverse.

It turns out that u is a wave map if and only if $A_\mu dx^\mu$ satisfies

$$\begin{aligned} \partial^\mu A_\mu &= 0, \\ \partial_\mu A_\nu - \partial_\nu A_\mu &= -[A_\mu, A_\nu], \end{aligned} \tag{1.13}$$

where $[\cdot,\cdot]$ is the Lie bracket. The advantage of this formulation of the wave maps problem is that it avoids the use of local charts in the target manifold. See Christodoulou and Tahvildar-Zadeh [5] for an application of this system to prove global regularity of spherically symmetric wave maps for $n = 2$.

First, let us see how the model equation (WMM) arises from this system. We start by transforming the variables, using the nonlocal operators $R_\mu = D^{-1}\partial_\mu$. We assume that $\mathfrak{g}$ is a Lie algebra of matrices, and that $[\cdot,\cdot]$ is the usual matrix commutator. Set

$$\bar{A}_i = A_i + R_0 R_i A_0.$$

Then it follows from (1.13) that the one-form $A_0 dx^0 + \bar{A}_i dx^i$ satisfies

$$\begin{aligned} \Lambda A_0 &= \partial^i [A_0, \bar{A}_i - R_0 R_i A_0] \\ \partial^i \bar{A}_i &= 0 \\ \partial_i \bar{A}_j - \partial_j \bar{A}_i &= [\bar{A}_j - R_0 R_j A_0, \bar{A}_i - R_0 R_i A_0]. \end{aligned}$$

Since the spatial part $\bar{A}_i$ satisfies an elliptic Hodge system, it is easier to estimate than the temporal part A_0, and therefore we ignore it. In other words, we set $\bar{A}_i = 0$ in the equation for A_0, which gives the model problem (WMM).

We remark that if we set $A_0 = 0$, then the above system describes a time-independent wave map (a *harmonic map*) $u : \mathrm{R}^n \to G$. This formulation of the harmonic map problem was used by F. Hélein [8] to prove regularity of weakly harmonic maps in dimension $n = 2$.

We now outline the derivation of the system (1.13). Following [5, Section 3.1], we first choose an orthonormal basis Ω_I of $\mathfrak{g}$, and we let ω^I be the dual basis of left-invariant one-forms on G. Let e^I_{JK} be the structure constants, defined by

$$[\Omega_J, \Omega_K] = e^I_{JK} \Omega_I.$$

Express A_μ relative to the basis:

$$A_\mu = \psi^I_\mu \Omega_I.$$

Since $\partial_\mu u = L_{u*} A_\mu$, it follows that

$$\psi^I_\mu(t,x) = \omega^I_{u(t,x)}\big(\partial_\mu u(t,x)\big), \tag{1.14}$$

which gives the precise dependence of ψ^I on u and ∂u.

Recall that the wave map Lagrangian is $\mathcal{L}[u] = \int L\,dt\,dx$, where

$$L(u,\partial u) = \frac{1}{2}\langle du, du\rangle = \frac{1}{2}h(\partial_\mu u, \partial^\mu u) = \frac{1}{2}h(A_\mu, A^\mu) = \frac{1}{2}\sum_{\mu,I}(\psi^I_\mu)^2.$$

Here we used the left invariance of h. Using the last expression for $L(u,\partial u)$, together with (1.14) and the Cartan structure equations

$$d\omega^I = -\frac{1}{2}e^I_{JK}\omega^J \wedge \omega^K,$$

a calculation reveals (see [5] for the details) that the Euler-Lagrange equation takes the form

$$\partial^\mu \psi^I_\mu = \sum_{J,K} e^J_{KI}\psi^{J\mu}\psi^K_\mu. \tag{1.15}$$

A direct calculation also gives

$$\partial_\mu \psi^I_\nu - \partial_\nu \psi^I_\mu = -\sum_{J,K} e^I_{JK}\psi^J_\mu \psi^K_\nu.$$

Observe that the last equation is equivalent to the second equation in (1.13). We claim that (1.15) is equivalent to

$$\partial^\mu A_\mu = [A^\mu, A_\mu].$$

Since the right hand side vanishes, we obtain the first equation in (1.13).

To prove the claim, we only have to note that, because of the assumption that the metric on G is bi-invariant, the structure constants satisfy

$$e^I_{JK} = e^J_{KI}.$$

Equivalently,

$$h\big(\Omega_I, [\Omega_J, \Omega_K]\big) + h\big(\Omega_J, [\Omega_I, \Omega_K]\big) = 0.$$

To see this when G is a group of matrices, let e^X denote the exponential map, where $X \in \mathfrak{g}$. Fix $X, Y, Z \in \mathfrak{g}$. By the bi-invariance of h,

$$h\big(e^{tX}Ye^{-tX}, e^{tX}Ze^{-tX}\big) = h(Y,Z).$$

Since

$$\frac{d}{dt}\big(e^{tX}Ye^{-tX}\big)\Big|_{t=0} = XY - YX = [X,Y],$$

it follows that

$$\frac{d}{dt}h\big(e^{tX}Ye^{-tX}, e^{tX}Ze^{-tX}\big)\Big|_{t=0} = h([X,Y],Z) + h(Y,[X,Z]) = 0,$$

which proves the claim.

1.3 Motivation of the Main Theorem

Consider the system

$$\Lambda u = F(u, \partial u), \tag{1.16}$$

where $u : \mathrm{R}^{1+n} \to \mathrm{R}^N$, $\partial u = (\partial_t u, \partial_1 u, \ldots, \partial_n u)$ and F is a smooth R^N-valued function satisfying $F(0) = 0$. For this equation one has the following standard existence and uniqueness result (concerning the proof, see Example 5.2).

Classical Local Existence Theorem. *Equation* (1.16) *is locally well-posed for initial data in* $H^s \times H^{s-1}(\mathrm{R}^n)$ *for all* $s > \frac{n}{2} + 1$.

This result is far from being sharp insofar as the regularity assumption on the initial data is concerned.

To understand better the issue of optimal local well-posedness, in the context of our examples (wave maps, Maxwell-Klein-Gordon and Yang-Mills equations), we need to define the critical well-posedness (henceforth abbreviated WP) exponent s_c. All our equations have a natural scaling associated to them, and s_c is the unique value of s for which the $\dot{H}^s \times \dot{H}^{s-1}$-norm of the initial data is invariant under this scaling. For example, if u solves (WM), then so does

$$u_\lambda(t, x) = u(\lambda t, \lambda x),$$

for any $\lambda > 0$. Since $\|u_\lambda(t)\|_{\dot{H}^s} = \lambda^{\frac{n}{2}-s} \|u(\lambda t)\|_{\dot{H}^s}$, the critical WP exponent for (WM) is $s_c = \frac{n}{2}$.

The same principle works for (MKG), (YM) and (WMM). In fact, they all have critical WP exponent $s_c = \frac{n-2}{2}$.

With this definition we formulate the following, taken from [16]:

General WP Conjecture. *(i) For all basic field theories the initial value problem is locally well posed for initial data in* $H^s \times H^{s-1}$, $s > s_c$.

(ii) The basic field theories are weakly[6] *globally well-posed for all initial data with small* $H^{s_c} \times H^{s_c-1}$*-norm.*

(iii) The basic field theories are ill posed for initial data in $H^s \times H^{s-1}$, $s < s_c$.

Our Main Theorem establishes part (i) of this conjecture for the equations in section 1.1. We prove local existence by Picard iteration in a suitable Banach space, as discussed in section 5. The 0-th iterate u_0 corresponding to a Cauchy problem

$$\Lambda u = \mathcal{N}(u), \qquad (u, \partial_t u)\big|_{t=0} = (f, g)$$

is just the homogeneous part of the solution:

$$\Lambda u_0 = 0, \qquad (u, \partial_t u)\big|_{t=0} = (f, g).$$

[6] The solutions may fail to depend smoothly (analytically) on the data.

The subsequent iterates are given inductively by

$$u_{j+1} = u_0 + \Lambda^{-1}\mathcal{N}(u_j)$$

for $j \geq 0$, where Λ^{-1} is the operator which to any sufficiently regular F assigns the solution v of $\Lambda v = F$ with $(v, \partial_t v)\big|_{t=0} = 0$.

If we are to prove existence of a local solution of $\Lambda u = \mathcal{N}(u)$ with initial data in $H^s \times H^{s-1}$ by iteration, we must be able to prove that the iterates remain in the data space:

$$f \in H^s, \quad g \in H^{s-1} \implies u_j(t) \in H^s, \quad \partial_t u_j(t) \in H^{s-1} \tag{1.17}$$

for all $j \geq 0$ and all t in some interval $(0, T)$. For $j = 0$, (1.17) is trivial, but the case $j = 1$ already offers valuable insights. We will say that the first iterate is *WP for initial data in* H^s if (1.17) holds for $j = 1$ and all $(f, g) \in H^s \times H^{s-1}$.

1.1. Example. Consider the model problem

$$\Lambda u = (\partial_t u)^2,$$

where u is real-valued. This equation has the same scaling properties as (WM), hence the WP-exponent is $s_c = \frac{n}{2}$. We want to find the lower bound for the set of s such that the first iterate u_1 is WP for initial data in H^s. A simple calculation involving Duhamel's principle, done in the Appendix, shows that this reduces to proving an estimate of the type

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} K(\xi, \eta) f(\xi) g(\eta) h(\xi + \eta) \, d\xi \, d\eta \lesssim \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2} \tag{1.18}$$

for all $f, g, h \in L^2(\mathbb{R}^n)$, where

$$K(\xi, \eta) = \frac{\langle \xi + \eta \rangle^{s-1}}{\langle \xi \rangle^{s-1} \langle \eta \rangle^{s-1} \big(1 + \Delta_\pm(\xi, \eta)\big)}, \tag{1.19}$$

$$\Delta_+ = |\xi| + |\eta| - |\xi + \eta|, \qquad \Delta_- = |\xi + \eta| - \big||\xi| - |\eta|\big|. \tag{1.20}$$

Here we use the notation $\langle \cdot \rangle = 1 + |\cdot|$.

In the Appendix we prove the following result concerning integral estimates of the type

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f(\xi) g(\eta) h(\xi + \eta)}{\langle \xi \rangle^a \langle \eta \rangle^b \big(1 + \Delta_\pm(\xi, \eta)\big)^c} \, d\xi \, d\eta \lesssim \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}, \tag{1.21}$$

where $\Delta_\pm$ are given by (1.20).

1.2. Proposition. *Let* $a, b, c \geq 0$. *Then* (1.21) *holds if* $a + b + c > \frac{n}{2}$ *and* $c < \frac{n-1}{4}$.

It should be remarked that the estimate fails if $a+b+c<\frac{n}{2}$ or $c\geq\frac{n-1}{4}$, although we do not prove this here.

The kernel (1.19) clearly satisfies (assuming $s\geq 1$)

$$K\lesssim\left(\langle\xi\rangle^{1-s}+\langle\eta\rangle^{1-s}\right)\left(1+\Delta_\pm(\xi,\eta)\right)^{-\alpha}$$

for any $0\leq\alpha\leq 1$. In view of Proposition 1.2, we must have $\alpha<\frac{n-1}{4}$. In order for the other hypothesis of Proposition 1.2 to be satisfied, we need

$$s-1+\min\left(1,\frac{n-1}{4}\right)>\frac{n}{2},$$

i.e., $s>\max\left(\frac{n}{2},\frac{n+5}{4}\right)$.

Thus, for the model equation $\Lambda u=(\partial_t u)^2$ in dimension $n=3$, the above example shows that the first iterate is WP for initial data in H^s if $s>2=s_c+\frac{1}{2}$; in fact, one can show that this fails to be true if $s\leq 2$. This should be compared to the counterexamples of Lindblad [28] in dimension $n=3$, which show that there are equations of the type $\Lambda u=q(\partial u)$, where q is a quadratic form on R^4, which are ill posed for data in $H^2\times H^1(\mathrm{R}^3)$. However, if the quadratic form q is of null form type, one can go almost all the way to the critical WP-exponent s_c. The next two examples verify this at the level of the first iterate.

1.3. Example. Consider the equation[7]

$$\Lambda u=Q_0(u,u),$$

where u is real-valued. Again the question of WP of the first iterate leads to the problem of proving an estimate of the type (1.18), but because of the special null structure of the operator Q_0, the singular factors $\Delta_\pm$ cancel out completely from the denominator of the kernel. In fact, K is given by

$$K(\xi,\eta)=\langle\xi\rangle^{-s}+\langle\eta\rangle^{-s},$$

so by Proposition 1.2, the first iterate is WP for data in H^s, $s>s_c=\frac{n}{2}$.

1.4. Example. Consider the equation

$$\Lambda u=Q(u,u),$$

where u is vector-valued and $Q(u,u)$ is a vector whose I-th component is a linear combination of $Q_{ij}(u^J,u^K)$ for all i,j,J and K. As in the preceding example, there is a cancellation due to the null strucure of Q_{ij}, but in this case we only get rid of half a power of $\Delta_\pm$. In fact, K is now given by

$$K(\xi,\eta)=\left(\langle\xi\rangle^{-s+\frac{1}{2}}+\langle\eta\rangle^{-s+\frac{1}{2}}\right)\left(1+\Delta_\pm(\xi,\eta)\right)^{-\frac{1}{2}}$$

[7]The equation below can in fact be trivially solved and analyzed, see the first page in the introduction of [18].

so the first iterate is WP for data in H^s, $s > \max\left(\frac{n}{2}, \frac{n+3}{4}\right)$.

By an obvious modification, if we consider instead the equation

$$\Lambda u = Q(D^{-1}u, u),$$

we find that the first iterate is WP for data in H^s, $s > \max\left(\frac{n-2}{2}, \frac{n-1}{4}\right)$.

The preceding examples are worked out in more detail in the Appendix.

1.4 Notation

Throughout the paper, $p \lesssim q$ means that $p \leq Cq$ for some positive constant C. Similarly, $\simeq$ means $=$ modulo a positive constant. The notation $p \sim q$ means $p \lesssim q \lesssim p$.

If $\mathcal{X}$ is a separable Banach space, $L^p(\mathbb{R}^k, \mathcal{X})$ denotes the usual L^p space, relative to Lebesgue measure on $\mathbb{R}^k$, and we write $L^p(\mathbb{R}^k) = L^p(\mathbb{R}^k, \mathbb{C})$. If $(\alpha, \beta) \in \mathbb{R}^k \times \mathbb{R}^l$, we define the mixed norm $\|f(\alpha,\beta)\|_{L^q_\alpha(L^r_\beta)}$ by first taking the $L^r(\mathbb{R}^l)$-norm in β, followed by the $L^q(\mathbb{R}^k)$-norm in α. Thus, $L^q_\alpha(L^r_\beta) = L^q\big(\mathbb{R}^k, L^r(\mathbb{R}^l)\big)$ if $r < \infty$.

The space of Schwartz functions on $\mathbb{R}^k$ is denoted by $\mathcal{S}(\mathbb{R}^k)$, and its dual, the space of tempered distributions, is written $\mathcal{S}'(\mathbb{R}^k)$. If $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ and it makes sense to restrict u to any time-slice $\{t\} \times \mathbb{R}^n$, we write $u(t)$ instead of $u(t, \cdot)$. The Fourier transform of a tempered distribution u is denoted by $\mathcal{F}u$ or $\widehat{u}$, in any space-dimension. In frequency space we use coordinates $(\tau, \xi) = \Xi = (\Xi^0, \dots, \Xi^1)$, where $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ correspond to the time variable t and the space variable x respectively. The Lorentzian inner product on $\mathbb{R}^{1+n}$ is denoted by $\langle \Xi, \widetilde{\Xi} \rangle$. Thus

$$\langle \Xi, \widetilde{\Xi} \rangle = \sum_{\mu=0}^{n} \Xi_\mu \widetilde{\Xi}^\mu = -\Xi^0 \widetilde{\Xi}^0 + \sum_{j=1}^{n} \Xi^j \widetilde{\Xi}^j,$$

and the symbol of the wave operator Λ is $-\langle \Xi, \Xi \rangle = \tau^2 - |\xi|^2$. By $|\Xi|$ we always mean the Euclidean norm.

Let Λ^α, Λ^α_+ and Λ^α_- be the multipliers given by

$$\begin{aligned}
\widehat{\Lambda^\alpha f}(\xi) &= \big(1 + |\xi|^2\big)^{\alpha/2} \widehat{f}(\xi), \\
\widehat{\Lambda^\alpha_+ u}(\Xi) &= \big(1 + |\Xi|^2\big)^{\alpha/2} \widehat{u}(\Xi), \\
\widehat{\Lambda^\alpha_- u}(\Xi) &= \left(1 + \frac{\langle \Xi, \Xi \rangle^2}{1 + |\Xi|^2}\right)^{\alpha/2} \widehat{u}(\Xi).
\end{aligned}$$

Observe that these operators are isomorphisms of $\mathcal{S}(\mathbb{R}^{1+n})$ as well as $\mathcal{S}'(\mathbb{R}^{1+n})$. Moreover, Λ^α may also be regarded as an isomorphism of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, since it only acts in the space variable.

We also need homogeneous versions of these operators: Let D^α, D_+^α and D_-^α be the multipliers with symbols

$$|\xi|^\alpha, \quad (|\tau| + |\xi|)^\alpha, \quad \big||\tau| - |\xi|\big|^\alpha$$

respectively.

If $u, v \in \mathcal{S}'$ and $\widehat{u}, \widehat{v}$ are tempered functions, we write $u \preceq v$ iff $|\widehat{u}| \le \widehat{v}$, and - means $\preceq$ up to a constant. If $u = (u^1, \ldots, u^N)$ and $v = (v^1, \ldots, v^N)$, then $u \preceq v$ (resp. u - v) means $u^I \preceq v^I$ (resp. u^I - v^I) for $I = 1, \ldots, N$.

If $\mathcal{X}$ is a normed vector space of tempered distributions such that $\widehat{u}$ is a tempered function whenever $u \in \mathcal{X}$, then we say that the norm on $\mathcal{X}$ depends only on the size of the Fourier transform if $\|u\| = \|v\|$ whenever $|\widehat{u}| = |\widehat{v}|$, and we say that the norm is compatible with the relation $\preceq$ if $\|u\| \le \|v\|$ whenever $u \preceq v$.

The solution of the homogeneous wave equation $\Lambda u = 0$ with initial data $(u, \partial_t u)\big|_{t=0} = (f, g)$ can be decomposed into *half waves*: $u = u_+ + u_-$, where $u_\pm(t) = e^{\pm itD} \frac{1}{2}(f \pm i^{-1} D^{-1} g)$. We shall often restrict ourselves to the reduced initial value problem $\Lambda u = 0$ with data $(u, \partial_t u)\big|_{t=0} = (f, 0)$; the general case can easily be reduced to this.

The symbol $\hookrightarrow$ means continuous inclusion. For example, we have the Sobolev embeddings

$$\dot{H}^{\frac{n}{2} - \frac{n}{p}}(\mathrm{R}^n) \hookrightarrow L^p(\mathrm{R}^n) \quad \text{iff} \quad 2 \le p < \infty, \tag{1.22}$$

$$H^s \hookrightarrow L^\infty(\mathrm{R}^n) \quad \text{iff} \quad s > \frac{n}{2}. \tag{1.23}$$

If $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are normed function spaces, then $\mathcal{X} \cdot \mathcal{Y} \hookrightarrow \mathcal{Z}$ means that $\|uv\|_{\mathcal{Z}} \cdot \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}$ for all $(u, v) \in \mathcal{X} \times \mathcal{Y}$. More generally, if $B(u, v)$ is some bilinear operator on $\mathcal{X} \times \mathcal{Y}$, we shall write $B(\mathcal{X}, \mathcal{Y}) \hookrightarrow \mathcal{Z}$ to mean that B is bounded from $\mathcal{X} \times \mathcal{Y}$ into $\mathcal{Z}$.

2 Estimates for the Wave Equation

Here we review some of the well known estimates for solutions of the homogeneous wave equation $\Lambda u = 0$ which will be needed throughout the paper.

Without loss of generality, we restrict ourselves to the reduced initial value problem

$$\Lambda u = 0, \qquad (u, \partial_t u)\big|_{t=0} = (f, 0). \tag{2.1}$$

Estimates for the general case $(u, \partial_t u)\big|_{t=0} = (f, g)$ can easily be deduced from this.

We start by recalling the Strichartz type estimates

$$\|u\|_{L_t^q(L_x^r)} \cdot \|f\|_{\dot{H}^s}, \tag{2.2}$$

where u solves (2.1) and $\|f\|_{\dot H^s} = \|D^s f\|_{L^2}$. Scaling considerations impose the condition

$$s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q}. \tag{2.3}$$

The pair (q, r) is said to be *wave admissible* if

$$2 \le q \le \infty, \quad 2 \le r < \infty, \quad \frac{2}{q} \le (n-1)\left(\frac{1}{2} - \frac{1}{r}\right). \tag{2.4}$$

For the proof of the next result, and further references, see [10]. The case $q, r = 4$, $n = 3$ corresponds to the original inequality of Strichartz [39].

Theorem A. *The estimate* (2.2) *is satisfied by the solution of* (2.1) *for all* $f \in \dot H^s$ *iff* (2.3) *holds and* (q, r) *is wave admissible.*

The next result is a generalization of Theorem A to bilinear estimates of the type

$$\left\| D^{-\sigma}(uv) \right\|_{L_t^{q/2}(L_x^{r/2})} \cdot \|f\|_{\dot H^{s_1}} \, \|g\|_{\dot H^{s_2}}\,, \tag{2.5}$$

where u and v solve

$$\Lambda u = \Lambda v = 0, \qquad (u, \partial_t u)\big|_{t=0} = (f, 0), \qquad (v, \partial_t v)\big|_{t=0} = (g, 0). \tag{2.6}$$

Note that if $\sigma = 0$ and $s_1 = s_2 = s$, then (2.5) reduces to (2.2) by Hölder's inequality.

The following theorem was first proved by Klainerman-Machedon [23] in the case $q, r = 4$; the general statement was proved by Klainerman-Tataru [27].

Theorem B. *Assume that* $n \ge 2$, *the pair* (q, r) *is wave admissible, i.e.,*

$$2 \le q \le \infty, \quad 2 \le r < \infty, \quad \frac{2}{q} \le (n-1)\left(\frac{1}{2} - \frac{1}{r}\right),$$

and that the last inequality is strict if $q = 2$. *Assume also that*

$$\begin{aligned} 0 < \sigma < n - \frac{2n}{r} - \frac{4}{q}, \\ 0 < s_1, s_2 < \frac{n}{2} - \frac{n}{r} - \frac{1}{q}, \\ s_1 + s_2 + \sigma = n - \frac{2n}{r} - \frac{2}{q}. \end{aligned}$$

Then (2.5) *holds for all solutions of* (2.6).

In practically all our applications of the above theorem, $s_1 = s_2$. It should be remarked that in the asymmetric case $s_1 \neq s_2$, the above conditions on s_1, s_2 are not optimal (cf. the proof of (8.29) in section 8.1).

Now consider more general bilinear estimates, of the form

$$\left\| D^{\gamma} D_+^{\gamma_+} D_-^{\gamma_-} (uv) \right\|_{L_t^{q/2}(L_x^{r/2})} \lesssim \|f\|_{\dot H^{s_1}} \|g\|_{\dot H^{s_2}} , \tag{2.7}$$

where u and v solve (2.6). In the case $q, r = 4$ all such estimates are known. Special cases of the following theorem have appeared first in [18] and later in [23, 26, 25]. The complete solution was carried out recently by Foschi-Klainerman [7], see also [41].

Theorem C. *Let $n \geq 2$ and $\gamma, \gamma_-, \gamma_+, s_1, s_2 \in \mathrm{R}$. The estimate*

$$\left\| D^{\gamma} D_+^{\gamma_+} D_-^{\gamma_-} (uv) \right\|_{L^2(\mathrm{R}^{1+n})} \lesssim \|f\|_{\dot H^{s_1}} \|g\|_{\dot H^{s_2}}$$

is satisfied by the solutions of (2.6) for all f, g iff the following conditions hold:

$$\begin{aligned}
\gamma + \gamma_+ + \gamma_- &= s_1 + s_2 - \frac{n-1}{2}, \\
\gamma_- &\geq -\frac{n-3}{4}, \\
\gamma &> -\frac{n-1}{2}, \\
s_i &\leq \gamma_- + \frac{n-1}{2}, \quad i = 1, 2, \\
s_1 + s_2 &\geq \frac{1}{2}, \\
(s_i, \gamma_-) &\neq \left(\frac{n+1}{4}, -\frac{n-3}{4} \right), \quad i = 1, 2, \\
(s_1 + s_2, \gamma_-) &\neq \left(\frac{1}{2}, -\frac{n-3}{4} \right).
\end{aligned}$$

3 Wave-Sobolev Spaces

We define the space $H^{s,\theta}$, which is adapted to the wave operator on R^{1+n} in the same way that H^s is adapted to the Laplacian on R^n, and we show that the estimates in Theorems A, B and C for solutions of the homogeneous wave equation imply corresponding estimates for elements of $H^{s,\theta}$.

3.1. Definition. *For $s, \theta \in \mathrm{R}$, define*

$$H^{s,\theta} = \left\{ u \in \mathcal{S}' : \Lambda^s \Lambda_-^\theta u \in L^2 \right\}$$

with norm $\|u\|_{s,\theta} = \left\| \Lambda^s \Lambda_-^\theta u \right\|_{L^2}$ (see section 1.4 for the definition of the operators Λ^s and Λ_-^θ).

Since $\Lambda^s \Lambda_-^\theta (\mathcal{S}) = \mathcal{S}$ and $\mathcal{S}$ is dense in L^2, it is immediate from the definition that $\mathcal{S}$ is dense in $H^{s,\theta}$.

There is a remarkably simple connection between $H^{s,\theta}$ and the space of solutions of the homogeneous wave equation with data in H^s. In effect, every $u \in H^{s,\theta}$ is of the form

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it\lambda} u_\lambda(t)}{(1+|\lambda|)^\theta} d\lambda \qquad (H^s\text{-valued integral}) \tag{3.1}$$

where $\{u_\lambda\}_{\lambda \in \mathrm{R}}$ is a one-parameter family of solutions of (2.1) with data in H^s; i.e., $\Lambda u_\lambda = 0$ and $(u_\lambda, \partial_t u_\lambda)\big|_{t=0} = (f_\lambda, 0)$, where $\lambda \mapsto f_\lambda$ belongs to $L^2(\mathrm{R}, H^s)$. Moreover, $\|u\|_{s,\theta}^2 = \int \|f_\lambda\|_{H^s}^2 \, d\lambda$. This is a slight simplification (a precise description is given below), but for most practical purposes it will suffice.

An important consequence of (3.1) is the following:

3.2. Principle. *A linear or multilinear space-time estimate for solutions of the homogeneous wave equation with data in H^s implies a corresponding estimate for elements of $H^{s,\theta}$.*

This is made precise in Proposition 3.5 below. To illustrate this principle, let us interpret Theorem A in terms of $H^{s,\theta}$. Assume that the hypotheses of Theorem A are satisfied, and take the $L_t^q(L_x^r)$-norm in (3.1). By Minkowski's integral inequality,

$$\|u\|_{L_t^q(L_x^r)} \cdot \int_{-\infty}^{\infty} \frac{\|u_\lambda\|_{L_t^q(L_x^r)}}{(1+|\lambda|)^\theta} d\lambda,$$

and by Theorem A, $\|u_\lambda\|_{L_t^q(L_x^r)} \cdot \|f_\lambda\|_{H^s}$. Thus, if $\theta > 1/2$,

$$\int_{-\infty}^{\infty} \frac{\|f_\lambda\|_{H^s}}{(1+|\lambda|)^\theta} d\lambda \le C_\theta \left(\int \|f_\lambda\|_{H^s}^2 \, d\lambda \right)^{\frac{1}{2}} = C_\theta \|u\|_{s,\theta} \,,$$

whence $\|u\|_{L_t^q(L_x^r)} \cdot \|u\|_{s,\theta}$. We summarize:

Theorem D. *The embedding*

$$H^{\frac{n}{2} - \frac{n}{r} - \frac{1}{q}, \theta} \hookrightarrow L_t^q(L_x^r)$$

holds whenever (q,r) is wave admissible and $\theta > 1/2$.

Theorem D may be viewed as an analog for $H^{s,\theta}$ of the Sobolev embedding (1.22) for the standard Sobolev spaces.

Just as in the linear case, via (3.1) we can interpret the bilinear estimates of Theorem D in $H^{s,\theta}$.

Theorem E. *If $n \ge 2$ and q, r, s_1, s_2 and σ satisfy the hypotheses of Theorem D, then*

$$\left\| D^{-\sigma}(uv) \right\|_{L_t^{q/2}(L_x^{r/2})} \cdot \|u\|_{s_1,\theta} \|v\|_{s_2,\theta}$$

provided $\theta > 1/2$.

The crucial observation is that since $D^{-\sigma}$ does not involve the time variable, the integral formula (3.1) implies

$$D^{-\sigma}(u^2)(t) = \frac{1}{4\pi^2} \iint \frac{e^{it(\lambda_1+\lambda_2)} D^{-\sigma}(u_{\lambda_1} u_{\lambda_2})(t)}{(1+|\lambda_1|)^\theta (1+|\lambda_2|)^\theta} d\lambda_1 \, d\lambda_2.$$

Take the $L_t^{q/2}(L_x^{r/2})$-norm, use Minkowski's integral inequality, Theorem D and finally the Cauchy-Schwarz inequality to obtain the estimate in Theorem E (see Proposition 3.5 and Remark 3.6 for the details).

Theorem C, in contrast to Theorem B, does not have an obvious interpretation in terms of $H^{s,\theta}$ via the integral representation (3.1), since the operator $(u,v) \mapsto D^\gamma D_+^{\gamma_+} D_-^{\gamma_-}(uv)$ acts in both space and time. Nevertheless, if we set $\gamma_+ = 0$, Theorem C does have an $H^{s,\theta}$-analog, but with $D_-^{\gamma_-}$ replaced by the operator R^{γ_-} appearing in the following lemma.

3.3. Lemma. *If $\alpha > 0$, then*

$$D_-^\alpha(uv) \text{ - } (D_-^\alpha u)v + uD_-^\alpha v + R^\alpha(u,v),$$

for all u and v with nonnegative Fourier transforms, where R^α is the symmetric bilinear operator given by

$$\mathcal{F}R^\alpha(u,v)(\Xi) = \int_{\mathbb{R}^{1+n}} \left[r(\Xi - \Xi'; \Xi') \right]^\alpha \widehat{u}(\Xi - \Xi') \widehat{v}(\Xi') \, d\Xi',$$

$$r(\tau,\xi;\lambda,\eta) = \begin{cases} |\xi| + |\eta| - |\xi+\eta| & \textit{if} \quad \tau\lambda \geq 0, \\ |\xi+\eta| - \big||\xi| - |\eta|\big| & \textit{if} \quad \tau\lambda < 0. \end{cases}$$

Moreover, the same estimate holds with D_-^α replaced by Λ_-^α.

Proof. It is enough to show that

$$\big||\tau+\lambda| - |\xi+\eta|\big| \leq \big||\tau| - |\xi|\big| + \big||\lambda| - |\eta|\big| + r(\tau,\xi,\lambda,\eta).$$

The proof splits into four cases, corresponding to the four quadrants of the (τ,λ)-plane. For example, if $\tau, \lambda \geq 0$, then

$$\begin{aligned} \big||\tau+\lambda| - |\xi+\eta|\big| &= \big|\tau - |\xi| + \lambda - |\eta| + |\xi| + |\eta| - |\xi+\eta|\big| \\ &\leq \big|\tau - |\xi|\big| + \big|\lambda - |\eta|\big| + |\xi| + |\eta| - |\xi+\eta|, \end{aligned}$$

and the remaining cases are similar. Finally, note that the symbol of Λ_- is comparable to $1 + \big||\tau| - |\xi|\big|$. □

We now state the $H^{s,\theta}$-version of Theorem C.

Theorem F. *If $n \geq 2$ and γ, γ_-, s_1 and s_2 satisfy the hypotheses of Theorem C with $\gamma_+ = 0$, then*

$$\|D^\gamma R^{\gamma_-}(u,v)\|_{L^2(\mathbb{R}^{1+n})} \cdot \|D^{s_1}u\|_{0,\theta} \|D^{s_2}v\|_{0,\theta}$$

provided $\theta > 1/2$.

To prove this result we need the precise version of the integral representation (3.1). First note that any $u \in H^{s,\theta}$ has a unique decomposition

$$u = u_+ + u_- \tag{3.2}$$

where u_+ and u_- belong to $H^{s,\theta}$ and have Fourier transforms supported in $[0, \infty) \times \mathrm{R}^n$ and $(-\infty, 0] \times \mathrm{R}^n$ respectively. Moreover,

$$\|u\|_{s,\theta}^2 = \|u_+\|_{s,\theta}^2 + \|u_-\|_{s,\theta}^2 \,.$$

The notation in the decomposition (3.2) is intentionally the same as the one used in the decomposition of a solution of the homogeneous wave equation into half-waves (see section 1.4).

3.4. Proposition. *If $u \in H^{s,\theta}$, there exist $f_+, f_- \in L^2(\mathrm{R}, H^s)$ such that*

$$u_\pm(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it(\lambda \pm D)} f_\pm(\lambda)}{(1 + |\lambda|)^\theta} \, d\lambda \qquad (H^s\text{-valued}) \tag{3.3}$$

and $\|u_\pm\|_{s,\theta} = \|f_\pm\|_{L^2(\mathrm{R}, H^s)}$.

Thus, elements of $H^{s,\theta}$ may be thought of as superpositions of half-waves with data in H^s. In fact, f_+ is given by

$$\mathcal{F}\{f_+(\lambda)\}(\xi) = \begin{cases} (1 + |\lambda|)^\theta \widehat{u}(\lambda + |\xi|\,, \xi) & \text{if } \lambda + |\xi| \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and f_- has a similar definition. It is then easy to verify (3.3) by applying the spacetime Fourier transform (see [33]).

The following is the precise statement of Principle 3.2.

3.5. Proposition. *Assume that $T : H^{s_1}(\mathrm{R}^n) \times \cdots \times H^{s_k}(\mathrm{R}^n) \longrightarrow H^\sigma(\mathrm{R}^n)$ is k-linear, and let $\theta > 1/2$.*

(a) Fix a k-tuple $\varepsilon \in \{-1, 1\}^k$. If

$$\left\| T(e^{\varepsilon_1 itD} f_1, \dots, e^{\varepsilon_k itD} f_k) \right\|_{L_t^q(L_x^r)} \cdot \|f_1\|_{H^{s_1}} \cdots \|f_k\|_{H^{s_k}} \,, \tag{3.4}$$

for all $(f_1, \dots, f_k) \in H^{s_1} \times \cdots \times H^{s_k}$, then

$$\left\| T\big(u_1(t), \dots, u_k(t)\big)(x) \right\|_{L_t^q(L_x^r)} \leq C_\theta \|u_1\|_{s_1,\theta} \cdots \|u_k\|_{s_k,\theta} \tag{3.5}$$

for all $(u_1, \dots, u_k) \in H^{s_1,\theta} \times \cdots \times H^{s_k,\theta}$ such that

$$\operatorname{supp} \widehat{u_j} \subseteq \begin{cases} [0, \infty) \times \mathrm{R}^n & \textit{if} \quad \varepsilon_j = 1, \\ (-\infty, 0] \times \mathrm{R}^n & \textit{if} \quad \varepsilon_j = -1. \end{cases} \tag{3.6}$$

(b) If (3.4) holds for all $\varepsilon \in \{-1, 1\}^k$ and all $(f_1, \dots, f_k) \in H^{s_1} \times \cdots \times H^{s_k}$, then (3.5) holds for all $(u_1, \dots, u_k) \in H^{s_1,\theta} \times \cdots \times H^{s_k,\theta}$.

Proof. By proposition 3.4 and the condition (3.6), which is equivalent to

$$u_j = \begin{cases} u_{j+} & \text{if} \quad \varepsilon_j = 1, \\ u_{j-} & \text{if} \quad \varepsilon_j = -1, \end{cases}$$

there exist $f_j \in L^2(\mathrm{R}, H^{s_j})$ for $j = 1, \dots, k$ such that

$$u_j = \int_{-\infty}^{\infty} \frac{e^{it\lambda} e^{\varepsilon_j itD} f_j(\lambda)}{(1 + |\lambda|)^\theta} \, d\lambda$$

and $\|u_j\|_{s_j,\theta} = \|f_j\|_{L^2(\mathrm{R}, H^{s_j})}$. By linearity,

$$\begin{aligned} &T\big(u_1(t), \dots, u_k(t)\big) \\ &\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e^{it(\lambda_1 + \dots + \lambda_k)} T\big(e^{\varepsilon_1 itD} f_1(\lambda_1), \dots, e^{\varepsilon_k itD} f_k(\lambda_k)\big)}{(1 + |\lambda_1|)^\theta \cdots (1 + |\lambda_k|)^\theta} d\lambda_1 \cdots d\lambda_k, \end{aligned}$$

so by Minkowski's integral inequality, (3.4) and the Cauchy-Schwarz inequality,

$$\begin{aligned} &\|T(u_1, \dots, u_k)\|_{L_t^q(L_x^r)} \\ &\quad \le \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{C \|f_1(\lambda_1)\|_{H^{s_1}} \cdots \|f_k(\lambda_k)\|_{H^{s_k}}}{(1 + |\lambda_1|)^\theta \cdots (1 + |\lambda_k|)^\theta} d\lambda_1 \cdots d\lambda_k \\ &\quad \le C \|f_1\|_{L^2(\mathrm{R}, H^{s_1})} \cdots \|f_k\|_{L^2(\mathrm{R}, H^{s_k})} . \end{aligned}$$

This concludes the proof of part (a), and to prove part (b) we simply write $u_j = u_{j+} + u_{j-}$, use the multilinearity of T, and apply part (a). □

3.6. Remark. Theorems A,B and C remain true with u and v replaced by any of their half-waves (in fact, this is how the estimates are proved). Thus, part (b) of Proposition 3.5, applied to Theorems A and B, proves Theorems D and E respectively. Notice also that Theorems D and E remain true when $H^{s,\theta}$ is replaced by the space with norm $\left\|D^s \Lambda_-^\theta u\right\|_{L^2}$. The reason is that any estimate of the form (2.7) may be rewritten as follows:

$$\left\|D^\gamma D_+^{\gamma_+} D_-^{\gamma_-} (D^{-s_1} u D^{-s_2} v)\right\|_{L_t^{q/2}(L_x^{r/2})} \cdot \|f\|_{L^2} \|g\|_{L^2} .$$

This fact will be used freely in the rest of the paper.

3.7. Definition. *Let S_+^α and S_-^α be the symmetric bilinear operators given by*

$$\mathcal{F} S_+^\alpha(f, g)(\xi) = \int_{\mathrm{R}^n} \big(|\xi - \eta| + |\eta| - |\xi|\big)^\alpha \widehat{f}(\xi - \eta) \widehat{g}(\eta) \, d\eta,$$

$$\mathcal{F} S_-^\alpha(f, g)(\xi) = \int_{\mathrm{R}^n} \big(|\xi| - \big||\xi - \eta| - |\eta|\big|\big)^\alpha \widehat{f}(\xi - \eta) \widehat{g}(\eta) \, d\eta.$$

The relation between the above operators and D_-^α and R^α is given in the following lemma. Keep in mind that the latter two operators act on functions defined on the space-time. If u, v are two such functions, then $S(u, v)$ denotes the function $(t, x) \mapsto S\big(u(t), v(t)\big)(x)$, where as usual we write $u(t) = u(t, \cdot)$.

3.8. Lemma. *Let f, g be defined on R^n and u, v on R^{1+n}. Then*

(a) $D_-^\alpha(e^{itD}f \cdot e^{\pm itD}g) = S_\pm^\alpha(e^{itD}f, e^{\pm itD}g)$,

(b) $R^\alpha(u,v) = S_+^\alpha(u_+,v_+) + S_-^\alpha(u_+,v_-) + S_-^\alpha(u_-,v_+) + S_+^\alpha(u_-,v_-)$.

Proof. We prove (a) for S_+; the proof for S_- is similar. We have

$$\mathcal{F}(e^{itD}f \cdot e^{\pm itD}g)(\tau,\xi) = \int_{\mathrm{R}^n} \delta(\tau - |\xi-\eta| - |\eta|)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta,$$

whence

$$\begin{aligned}\mathcal{F}\big\{D_-^\alpha(e^{itD}f \cdot e^{\pm itD}g)\big\}(\tau,\xi) & \\ &= \big||\tau| - |\xi|\big|^\alpha \int_{\mathrm{R}^n} \delta(\tau - |\xi-\eta| - |\eta|)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta \\ &= \int_{\mathrm{R}^n} \delta(\tau - |\xi-\eta| - |\eta|)\big(|\xi-\eta| + |\eta| - |\xi|\big)^\alpha \widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta.\end{aligned}$$

But the last expression equals

$$\mathcal{F}\big\{S_\pm^\alpha(e^{itD}f, e^{\pm itD}g)\big\}(\tau,\xi).$$

As for (b), using the decomposition (3.2) and the bilinearity of R^α, we have

$$R^\alpha(u,v) = R^\alpha(u_+,v_+) + R^\alpha(u_+,v_-) + R^\alpha(u_-,v_+) + R^\alpha(u_-,v_-),$$

so it suffices to prove $R^\alpha(u_+,v_\pm) = S_\pm^\alpha(u_+,v_\pm)$ and $R^\alpha(u_-,v_\mp) = S_\pm^\alpha(u_-,v_\mp)$. We will only prove the case

$$R^\alpha(u_+,v_+) = S_+^\alpha(u_+,v_+). \tag{3.7}$$

The Fourier transform of the right hand side, at fixed t, is

$$\int_{\mathrm{R}^n} \big(|\xi-\eta| + |\eta| - |\xi|\big)^\alpha \widehat{u}_+(t)(\xi-\eta)\widehat{v}_+(t)(\eta)\,d\eta,$$

and applying the Fourier transform in t yields the following expression for the space-time Fourier transform at (τ,ξ) of the right hand side of (3.7):

$$\int_{\mathrm{R}^n}\int_{\mathrm{R}} \big(|\xi-\eta| + |\eta| - |\xi|\big)^\alpha \widehat{u_+}(\tau-\lambda,\xi-\eta)\widehat{v_+}(\lambda,\eta)\,d\lambda\,d\eta.$$

But the latter is, by the definition of R^α, equal to the space-time Fourier transform of the left hand side of (3.7). The remaining cases are proved in a similar manner. □

We can now prove Theorem F. If the hypotheses of the theorem are satisfied, then by Lemma 3.8(a) and Theorem C, we have (cf. remark 3.6)

$$\big\|D^\gamma S_\pm^{\gamma_-}(e^{itD}D^{-s_1}f, e^{\pm itD}D^{-s_2}g)\big\|_{L^2} \cdot \|f\|_{L^2}\|g\|_{L^2},$$

for all $f, g \in L^2(\mathrm{R}^n)$, so by Proposition 3.5(a),

$$\left\| D^\gamma S_\pm^{\gamma_-}(D^{-s_1}u_+, D^{-s_2}v_\pm) \right\|_{L^2} \lesssim \|u_+\|_{0,\theta} \|v_\pm\|_{0,\theta} \tag{3.8}$$

for all $u, v \in H^{0,\theta}$, where $\theta > 1/2$. A similar argument gives

$$\left\| D^\gamma S_\pm^{\gamma_-}(D^{-s_1}u_-, D^{-s_2}v_\mp) \right\|_{L^2} \lesssim \|u_-\|_{0,\theta} \|v_\mp\|_{0,\theta} . \tag{3.9}$$

By (3.8), (3.9) and Lemma 3.8(b), we conclude that

$$\left\| D^\gamma R^{\gamma_-}(u, v) \right\|_{L^2} \lesssim \|D^{s_1}u\|_{0,\theta} \|D^{s_2}v\|_{0,\theta} ,$$

so Theorem F is proved.

The following embedding is an easy consequence of the integral formula (3.3) and the dominated convergence theorem for H^s-valued integrals. We omit the details. Here $C_b(\mathrm{R}, H^s)$ denotes the space of bounded, continuous maps from R to H^s, with the supremum norm.

3.9. Proposition. *$H^{s,\theta} \hookrightarrow C_b(\mathrm{R}, H^s)$ if $\theta > 1/2$.*

Associated to $H^{s,\theta}$ we have the following space.

3.10. Definition. *For $s, \theta \in \mathrm{R}$, define*

$$\mathcal{H}^{s,\theta} = \left\{ u : u \in H^{s,\theta} \text{ and } \partial_t u \in H^{s-1,\theta} \right\}$$

with norm $|u|_{s,\theta} = \|u\|_{s,\theta} + \|\partial_t u\|_{s-1,\theta}$.

3.11. Remark. An equivalent, but less intuitive definition is

$$\mathcal{H}^{s,\theta} = \left\{ u \in \mathcal{S}' : \Lambda^{s-1}\Lambda_+\Lambda_-^\theta u \in L^2 \right\}$$

with norm $|u|_{s,\theta} = \left\| \Lambda^{s-1}\Lambda_+\Lambda_-^\theta u \right\|_{L^2}$. We will use these two definitions of $\mathcal{H}^{s,\theta}$ and its norm interchangeably.

The following embedding is a corollary to Proposition 3.9.

3.12. Proposition. *$\mathcal{H}^{s,\theta} \hookrightarrow C_b(\mathrm{R}, H^s) \cap C_b^1(\mathrm{R}, H^{s-1})$ if $\theta > \frac{1}{2}$.*

4 The Space $\mathcal{L}_t^q(\mathcal{L}_x^r)$

Optimal local well-posedness for (WM) (part (a) of the Main Theorem) will be proved by iteration in the space $\mathcal{H}^{s,\theta}$, defined in the previous section. An attempt to prove the corresponding results for ("MKG")/("YM") (part (b) of the Main Theorem) and (WMM) (part (c) of the Main Theorem) by iteration in the same space, leads to estimates which are false. It turns out that the iteration works out if we replace $\mathcal{H}^{s,\theta}$ with the subspace defined by a norm

$$\|u\| = |u|_{s,\theta} + \left\| \Lambda^\gamma \Lambda_-^{\gamma_-} u \right\|_{L_t^q(L_x^r)} ,$$

where the choice of exponents γ, γ_-, q and r is dictated by the specific equation under consideration.

However, since we want a space whose norm only depends on the size of the Fourier transform, the space $L_t^q(L_x^r)$ must be modified. To motivate the following definition, recall that if $u \in \mathcal{S}'$ and $1 \leq q, r \leq \infty$, then

$$\|u\|_{L_t^q(L_x^r)} = \sup_v |\langle u, v \rangle| = \sup_v |\langle \widehat{u}, \widehat{v} \rangle| ,$$

where the supremum is over all $v \in \mathcal{S}$ such that $\|v\|_{L_t^{q'}(L_x^{r'})} = 1$ (q' and r' being the dual exponents of q and r respectively). Of course, if $\widehat{u}$ is a tempered function, then $\langle \widehat{u}, \widehat{v} \rangle = \int_{\mathrm{R}^{1+n}} \widehat{u}\widehat{v}$.

4.1. Definition. *If $1 \leq q, r \leq \infty$, $u \in \mathcal{S}'$ and $\widehat{u}$ is a tempered function, set*

$$\|u\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} = \sup \left\{ \int_{\mathrm{R}^{1+n}} |\widehat{u}(\Xi)| \, \widehat{v}(\Xi) \, d\Xi : v \in \mathcal{S}, \, \widehat{v} \geq 0 \text{ and } \|v\|_{L_t^{q'}(L_x^{r'})} = 1 \right\} ,$$

where q' and r' are the conjugate exponents of q and r respectively; i.e., $1 = \frac{1}{q} + \frac{1}{q'}$ and $1 = \frac{1}{r} + \frac{1}{r'}$. Let $\mathcal{L}_t^q(\mathcal{L}_x^r)$ be the corresponding subspace of $\mathcal{S}'$.

Clearly, $\|\cdot\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)}$ is a translation invariant norm on $\mathcal{L}_t^q(\mathcal{L}_x^r)$, it is compatible with the relation $\preceq$, and it only depends on the size of the Fourier transform. Note that $\mathcal{L}_t^2(\mathcal{L}_x^2) = L^2(\mathrm{R}^{1+n})$. Observe also that

$$\|u\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} \leq \|u\|_{L_t^q(L_x^r)} \quad \text{whenever} \quad \widehat{u} \geq 0. \tag{4.1}$$

The above definition is inspired by the norms introduced in [24, 25]. Another way of modifying the norm on $L_t^q(L_x^r)$ so that it only depends on the size of the Fourier transform can be found in [27].

4.2. Proposition. *Let $s, \gamma, \gamma_+, \gamma_- \in \mathrm{R}$, $\theta > \frac{1}{2}$ and $1 \leq q, r \leq \infty$. Define*

$$\mathcal{X}^s = \{u : \|u\| < \infty\},$$

where

$$\|u\| = |u|_{s,\theta} + \left\| \Lambda^\gamma \Lambda_+^{\gamma_+} \Lambda_-^{\gamma_-} u \right\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} .$$

Then $\mathcal{X}^s$ is a Banach space.

Proof. Assume that (u_j) is a Cauchy sequence in $\mathcal{X}^s$. Then (u_j) is Cauchy in $\mathcal{H}^{s,\theta}$, so it converges in the latter space to some limit u. It remains to prove that $\|u_j - u\| \to 0$ as $j \to \infty$. Fix $\varepsilon > 0$. There exists $M \in \mathrm{N}$ such that

$$\left\| \Lambda^\gamma \Lambda_+^{\gamma_+} \Lambda_-^{\gamma_-} (u_j - u_k) \right\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} \leq \varepsilon$$

for all $j, k \geq M$. We claim that

$$\left\| \Lambda^\gamma \Lambda_+^{\gamma_+} \Lambda_-^{\gamma_-} (u_j - u) \right\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} \leq \varepsilon$$

for all $j \geq M$. To see this, fix $v \in \mathcal{S}$ such that $\widehat{v} \geq 0$ and $\|v\|_{L_t^{q'}(L_x^{r'})} = 1$, where q', r' are conjugate to q, r. Then

$$\int \left|\mathcal{F}\left\{\Lambda^\gamma \Lambda_+^{\gamma_+} \Lambda_-^{\gamma_-}(u_j - u_k)\right\}(\Xi)\right| \widehat{v}(\Xi)\, d\Xi \leq \varepsilon$$

for all $j, k \geq M$, so it suffices to prove that

$$\lim_{k\to\infty} \int \left|\mathcal{F}\left\{\Lambda^\gamma \Lambda_+^{\gamma_+} \Lambda_-^{\gamma_-}(u_j - u_k)\right\}(\Xi)\right| \widehat{v}(\Xi)\, d\Xi = \int \left|\mathcal{F}\left\{\Lambda^\gamma \Lambda_+^{\gamma_+} \Lambda_-^{\gamma_-}(u_j - u)\right\}(\Xi)\right| \widehat{v}(\Xi)\, d\Xi \tag{4.2}$$

for fixed j. To prove this, we write

$$\int \left|\mathcal{F}\left\{\Lambda^\gamma \Lambda_+^{\gamma_+} \Lambda_-^{\gamma_-}(u_j - u_k)\right\}(\Xi)\right| \widehat{v}(\Xi)\, d\Xi = \int \left|\mathcal{F}\left\{\Lambda^{s-1} \Lambda_+ \Lambda_-^{\theta}(u_j - u_k)\right\}(\Xi)\right| \widehat{v'}(\Xi)\, d\Xi,$$

where $v' = \Lambda^{\gamma+1-s}\Lambda_+^{\gamma_+-1}\Lambda_-^{\gamma_--\theta} v \in \mathcal{S}$. Since $\Lambda^{s-1}\Lambda_+\Lambda_-^{\theta}(u_j - u_k)$ converges to $\Lambda^{s-1}\Lambda_+\Lambda_-^{\theta}(u_j - u)$ in L^2, we conclude that (4.2) holds. □

For later use, we mention some basic properties of $\mathcal{L}_t^q(\mathcal{L}_x^r)$.

First, a version of Hölder's inequality holds.

4.3. Proposition. *Suppose* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ *and* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, *where the* q*'s and* r*'s all belong to* $[1, \infty]$. *Then*

$$\|uv\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} \leq \|u\|_{\mathcal{L}_t^{q_1}(\mathcal{L}_x^{r_1})} \|v\|_{L_t^{q_2}(L_x^{r_2})}$$

for all $u \in \mathcal{L}_t^{q_1}(\mathcal{L}_x^{r_1})$ *and* $v \in \mathcal{S}$ *with* $\widehat{v} \geq 0$.

Proof. Since the norm $\|\cdot\|_{\mathcal{L}_t^{\cdot}(\mathcal{L}_x^{\cdot})}$ is compatible with the relation $\preceq$ and only depends on the size of the Fourier transform, it suffices to prove the inequality when $\widehat{u} \geq 0$. Thus, we fix u and v such that $v \in \mathcal{S}$ and $\widehat{u}, \widehat{v} \geq 0$. Let q', r', q_1' etc. denote the dual exponents. Then

$$\|uv\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} = \sup_w \int uvw\, dt\, dx \leq \|u\|_{\mathcal{L}_t^{q_1}(\mathcal{L}_x^{r_1})} \|vw\|_{L_t^{q_1'}(L_x^{r_1'})},$$

where the supremum is over all $w \in \mathcal{S}$ such that $\widehat{w} \geq 0$ and $\|w\|_{L_t^{q'}(L_x^{r'})} = 1$. But by Hölder's inequality,

$$\|vw\|_{L_t^{q_1'}(L_x^{r_1'})} \leq \|v\|_{L_t^{q_2}(L_x^{r_2})} \|w\|_{L_t^{q'}(L_x^{r'})},$$

finishing the proof. □

When applying the previous proposition, the following is useful:

4.4. Lemma. *$\{u \in \mathcal{S} : \widehat{u} \geq 0\}$ is dense in $\{u \in H^{a,\alpha} : \widehat{u} \geq 0\}$ for all $a, \alpha \in \mathrm{R}$.*

Proof. Since $\Lambda^a \Lambda_-^\alpha$ is an isomorphism of $H^{a,\alpha}$ onto L^2, which preserves the Schwartz class and positivity of the Fourier transform, we may take $a = \alpha = 0$. Since the Fourier transform is an isomorphism of both $\mathcal{S}$ and L^2, it then suffices to prove that the set $\{v \in \mathcal{S} : v \geq 0\}$ is dense in $\{v \in L^2 : v \geq 0\}$. But the standard proof that $\mathcal{S}$ is dense in L^2 shows this to be true. $\square$

The following duality argument is fundamental to our approach.

4.5. Proposition. *Let $1 \leq a, b, q, r \leq \infty$.*

(a) If

$$\|G\|_{L_t^{a'}(L_x^{b'})} \lesssim \left\|\Lambda^\alpha \Lambda_-^\beta G\right\|_{L_t^{q'}(L_x^{r'})} \tag{4.3}$$

for all $G \in \mathcal{S}$, then

$$\|F\|_{L_t^q(L_x^r)} \lesssim \left\|\Lambda^\alpha \Lambda_-^\beta F\right\|_{L_t^a(L_x^b)}$$

for all F.

(b) If (4.3) holds for all $G \in \mathcal{S}$ with $\widehat{G} \geq 0$, then

$$\|F\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} \lesssim \left\|\Lambda^\alpha \Lambda_-^\beta F\right\|_{\mathcal{L}_t^a(\mathcal{L}_x^b)}$$

for all F.

Proof. We have

$$\begin{aligned} \|F\|_{L_t^q(L_x^r)} &= \sup_G \left| \int FG \right| \\ &= \sup_G \left| \int \Lambda^\alpha \Lambda_-^\beta F \Lambda^{-\alpha} \Lambda_-^{-\beta} G \right| \\ &\leq \left\|\Lambda^\alpha \Lambda_-^\beta F\right\|_{L_t^a(L_x^b)} \sup_G \left\|\Lambda^{-\alpha} \Lambda_-^{-\beta} G\right\|_{L_t^{a'}(L_x^{b'})}, \end{aligned}$$

where the supremum is over all $G \in \mathcal{S}$ with $\|G\|_{L_t^{q'}(L_x^{r'})} = 1$. Part (a) follows.

For part (b), we have

$$\begin{aligned} \|F\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} &= \sup_G \int |\widehat{F}| \widehat{G} \\ &= \sup_G \int \left|\mathcal{F}\left(\Lambda^\alpha \Lambda_-^\beta F\right)\right| \mathcal{F}\left(\Lambda^{-\alpha} \Lambda_-^{-\beta} G\right) \\ &\leq \left\|\Lambda^\alpha \Lambda_-^\beta F\right\|_{\mathcal{L}_t^a(\mathcal{L}_x^b)} \sup_G \left\|\Lambda^{-\alpha} \Lambda_-^{-\beta} G\right\|_{L_t^{a'}(L_x^{b'})}, \end{aligned}$$

where the supremum is over all $G \in \mathcal{S}$ such that $\widehat{G} \geq 0$ and $\|G\|_{L_t^{q'}(L_x^{r'})} = 1$. $\square$

4.6. Corollary. *Let* $1 \le a, b, q, r \le \infty$. *If*

$$\Lambda^{-\alpha}\Lambda_-^{-\beta} L_t^a(L_x^b) \hookrightarrow L_t^q(L_x^r),$$

then

$$\Lambda^{-\alpha}\Lambda_-^{-\beta} \mathcal{L}_t^a(\mathcal{L}_x^b) \hookrightarrow \mathcal{L}_t^q(\mathcal{L}_x^r).$$

Proof. We apply Proposition 4.5. By part (a), we get

$$\Lambda^{-\alpha}\Lambda_-^{-\beta} L_t^{q'}(L_x^{r'}) \hookrightarrow L_t^{a'}(L_x^{b'}).$$

Then, by part (b),

$$\Lambda^{-\alpha}\Lambda_-^{-\beta} \mathcal{L}_t^a(\mathcal{L}_x^b) \hookrightarrow \mathcal{L}_t^q(\mathcal{L}_x^r).$$

□

The next estimate is used in the proof of part (b) of the Main Theorem. The proof is based on an idea from [25].

4.7. Proposition. *If* $n \ge 4$, $1 < p \le \frac{2(n-1)}{n+1}$, $s = \frac{n}{p} - \frac{n}{2} - \frac{1}{2}$ *and* $\theta > \frac{1}{2}$, *then*

$$\left\| D^{-s}\Lambda_-^{-\theta} u \right\|_{L_t^\infty(L_x^2)} \cdot \|u\|_{L_t^\infty(L_x^p)}$$

for all $u \in \mathcal{S}$ *with nonnegative Fourier transform.*

Proof. Set $U = D^{-s}\Lambda_-^{-\theta} u$. Since $\|U\|_{L_t^\infty(L_x^2)} \le \left\| \int \widehat{U}(\tau,\xi)\, d\tau \right\|_{L_\xi^2}$, it suffices to show that

$$\int_{\mathrm{R}^{1+n}} |\xi|^{-s} \left(1 + \big||\tau| - |\xi|\big|\right)^{-\theta} \widehat{u}(\tau,\xi) \check{f}(\xi)\, d\tau\, d\xi \; . \; \|u\|_{L_t^\infty(L_x^p)} \|f\|_{L^2}$$

for all $f \in \mathcal{S}(\mathrm{R}^n)$ whose inverse Fourier transform $\check{f}$ is nonnegative. The integral on the left hand side is dominated by

$$\int_{\mathrm{R}^{1+n}} u(v_+ + v_-)\, dt\, dx, \tag{4.4}$$

where

$$\mathcal{F}^{-1} v_\pm(\tau,\xi) = |\xi|^{-s} \left(1 + \big|\tau \mp |\xi|\big|\right)^{-\theta} \check{f}(\xi).$$

By Hölder's inequality, (4.4) is bounded by $\left\|u\right\|_{L_t^\infty(L_x^p)}$ times $\|v_\pm\|_{L_t^1(L_x^r)}$, where $1 = \frac{1}{r} + \frac{1}{p}$, so it suffices to show that

$$\|v_\pm\|_{L_t^1(L_x^r)} \cdot \|f\|_{L^2}\,.$$

But $v_\pm(t,\cdot) = c(t) D^{-s} e^{\pm itD} f$, where $\widehat{c}(\tau) = (1 + |\tau|)^{-\theta}$, and since $c \in L^2(\mathrm{R})$, it follows that

$$\|v_\pm\|_{L_t^1(L_x^r)} \cdot \left\| D^{-s} e^{\pm itD} f \right\|_{L_t^2(L_x^r)}\,.$$

By Theorem A (see also Remark 3.6), the right hand side is dominated by $\|f\|_{L^2}$. □

The dual statement is as follows:

4.8. Proposition. *If* $\frac{2(n-1)}{n-3} \le r < \infty$, $s = \frac{n}{2} - \frac{n}{r} - \frac{1}{2}$ *and* $\theta > \frac{1}{2}$, *then*

$$\|u\|_{\mathcal{L}_t^1(\mathcal{L}_x^r)} \lesssim \left\|\Lambda^s \Lambda_-^\theta u\right\|_{\mathcal{L}_t^1(\mathcal{L}_x^2)}$$

for all u.

Proof. If $1 = 1/p + 1/r$, the hypotheses of Proposition 4.7 are satisfied, so the result follows by Proposition 4.5(b). ☐

5 The Iteration Space

The main point we want to make here is that proving local well-posedness for a system $\square u = \mathcal{N}(u)$ with initial data in $H^s \times H^{s-1}$ by iteration in some functional Banach space $\mathcal{X}^s$ (which should satisfy certain conditions), reduces to proving estimates of the type

$$\left\|\Lambda_+^{-1}\Lambda_-^{-1}\mathcal{N}(u)\right\|_{\mathcal{X}^s} \le A\left(\|u\|_{\mathcal{X}^s}\right), \tag{5.1}$$

$$\left\|\Lambda_+^{-1}\Lambda_-^{-1}\left(\mathcal{N}(u) - \mathcal{N}(v)\right)\right\|_{\mathcal{X}^s} \le A'\left(\max\{\|u\|_{\mathcal{X}^s}, \|v\|_{\mathcal{X}^s}\}\right)\|u - v\|_{\mathcal{X}^s}, \tag{5.2}$$

where A and A' are continuous functions and $A(0) = 0$.

Let us briefly describe the conditions that $\mathcal{X}^s$ should satisfy.

Firstly, we require that $\mathcal{X}^s$ embed in the continuation of the data space $H^s \times H^{s-1}$, namely

$$C_b(\mathbb{R}, H^s) \cap C_b^1(\mathbb{R}, H^{s-1}) \tag{5.3}$$

with norm $\sup_{t\in\mathbb{R}}\left(\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}}\right)$. In particular, this ensures that restriction to any time-slab $S_T = [0,T] \times \mathbb{R}^n$ is well-defined for elements of $\mathcal{X}^s$, and we denote by $\mathcal{X}_T^s$ the corresponding restriction space.

Secondly, we limit our attention to spaces in which we have estimates for the Cauchy problem for the *linear* wave equation with data in $H^s \times H^{s-1}$. Thus, we require that the solution of $\square u = F$ with initial data $(u, \partial_t u)\big|_{t=0} = (f, g)$ satisfy[8]

$$\|u\|_{\mathcal{X}_T^s} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}} + \left\|\Lambda_+^{-1}\Lambda_-^{-1}F\right\|_{\mathcal{X}^s} \tag{5.4}$$

for all $0 < T < 1$, say. Let us write

$$u = u_0 + \square^{-1}F,$$

[8]Here $\|u\|_{\mathcal{X}_T^s}$ denotes the norm on the restriction space $\mathcal{X}_T^s$. For example, if $\mathcal{X}^s$ is the data continuation space (5.3), it is obvious which norm to use on $\mathcal{X}_T^s$. It is less obvious if $\mathcal{X}^s$ is the space $\mathcal{H}^{s,\theta}$, since the norm is then nonlocal in time. However, there is an abstract way of defining a norm on $\mathcal{X}_T^s$ such that it becomes a Banach space; see the statement of Theorem 5.3.

where u_0 is the homogeneous part of the solution, i.e., $\Lambda u_0 = 0$ with initial data (f, g), and where Λ^{-1} is the operator which to any sufficiently regular F assigns the solution v of $\Lambda v = F$ with $(v, \partial_t v)\big|_{t=0} = 0$.

Thus, (5.4) splits into two estimates:

$$\|u_0\|_{X_T^s} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}} ,$$
$$\left\|\Lambda^{-1} F\right\|_{\mathcal{X}_T^s} \lesssim \left\|\Lambda_+^{-1}\Lambda_-^{-1} F\right\|_{\mathcal{X}^s} .$$

The latter says that for the purpose of local-in-time estimates, we may replace Λ^{-1} by the much nicer operator $\Lambda_+^{-1}\Lambda_-^{-1}$.

The essential point is then the following: If we have a space $\mathcal{X}^s$ with the above properties (i.e., $\mathcal{X}^s$ embeds in the data continuation space and (5.4) holds), and if the estimates (5.1) and (5.2) for the nonlinearity $\mathcal{N}$ are true, then the equation $\Lambda u = \mathcal{N}(u)$ is locally well-posed for initial data in $H^s \times H^{s-1}$. A precise statement is given in Theorem 5.4 below.

5.1. Remark. If the norm on $\mathcal{X}^s$ only depends on the size of the Fourier transform and is compatible with the relation $\preceq$ (this terminology is defined in section 1.4), and if there are operators $\mathcal{N}_1, \dots, \mathcal{N}_m$ such that

$$u \preceq v \implies \mathcal{N}(u) \preceq \mathcal{N}_1(v) + \cdots + \mathcal{N}_m(v),$$

then in order to prove (5.1), it suffices to prove

$$\left\|\Lambda_+^{-1}\Lambda_-^{-1}\mathcal{N}_j(v)\right\|_{\mathcal{X}^s} \lesssim A\big(\|v\|_{\mathcal{X}^s}\big) \qquad (1 \le j \le m)$$

for all v with $\widehat{v} \ge 0$.

We start by discussing in fairly general terms how estimates imply local well-posedness.

5.1 Well-Posedness: A General Point of View

Consider again the generic Cauchy problem

$$\Lambda u = \mathcal{N}(u), \qquad (u, \partial_t u)\big|_{t=0} = (f, g). \tag{5.5}$$

Assume that $\mathcal{N}(0) = 0$ and $\mathcal{N}$ is local in time.

Associated to the Cauchy problem (5.5) is the sequence (u_j) of iterates, defined inductively by setting $u_{-1} \equiv 0$ and

$$\Lambda u_j = \mathcal{N}(u_{j-1}), \qquad (u, \partial_t u)\big|_{t=0} = (f, g)$$

for $j \ge 0$. Thus u_0 is the homogeneous part of the solution, and the subsequent iterates are given by

$$u_{j+1} = u_0 + \Lambda^{-1}\mathcal{N}(u_j) \tag{5.6}$$

for $j \ge 0$.

The strategy for proving local existence for (5.5) in a time slab $S_T = [0,T] \times \mathrm{R}^n$, for some $T > 0$ and for given data $(f,g) \in H^s \times H^{s-1}$, is to find a Banach space

$$\mathcal{X}_T^s \hookrightarrow C([0,T], H^s) \cap C^1([0,T], H^{s-1}) \tag{5.7}$$

in which $(u_j | S_T)$ is Cauchy. The limit u will then be a solution of (5.5) on S_T, provided that $\mathcal{N}(u_j) \to \mathcal{N}(u)$ in the sense of distributions on $(0,T) \times \mathrm{R}^n$ (this always follows from the estimates involving $\mathcal{N}$).

To prove that (u_j) is Cauchy, we need estimates. Firstly, the inductive step (5.6) must be well-defined, so we need (for $0 < T < 1$, say)

$$\|u_0\|_{\mathcal{X}_T^s} \le C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}}\big), \tag{5.8}$$

$$\big\|\Lambda^{-1}\mathcal{N}(u)\big\|_{\mathcal{X}_T^s} \le C_T A\big(\|u\|_{\mathcal{X}_T^s}\big), \tag{5.9}$$

where A is a continuous function vanishing at 0. We may always assume that A is increasing. Secondly, we need estimates for the difference of two iterates; i.e., we need

$$\big\|\Lambda^{-1}\big(\mathcal{N}(u) - \mathcal{N}(v)\big)\big\|_{\mathcal{X}_T^s} \le C_T A'\big(\max\{\|u\|_{\mathcal{X}_T^s}, \|v\|_{\mathcal{X}_T^s}\}\big)\, \|u - v\|_{\mathcal{X}_T^s}, \tag{5.10}$$

where A' is continuous.

5.2. Example. To prove the Classical Local Existence Theorem (section 1.3) for a system $\Lambda u = F(u, \partial u)$, where F is smooth and vanishes at the origin, we set

$$\mathcal{X}_T^s = C([0,T], H^s) \cap C^1([0,T], H^{s-1}).$$

Then (5.8), (5.9) and (5.10) hold for any $s > \frac{n}{2} + 1$, with $C_T = O(T)$ as $T \to 0$. Indeed, by the energy inequality,

$$\|u_0\|_{\mathcal{X}_T^s} \le C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}}\big)$$

and

$$\big\|\Lambda^{-1}F(u, \partial u)\big\|_{\mathcal{X}_T^s} \le C \int_0^T \big\|F\big(u(t), \partial u(t)\big)\big\|_{H^{s-1}}\, dt \tag{5.11}$$

for all $0 \le T \le 1$, say. Recall the Moser inequality, which says that if Γ is smooth and vanishes at the origin, and if $\sigma \ge 0$, then there exists a continuous function $g : [0,\infty) \to [0,\infty)$ such that

$$\|\Gamma(f)\|_{H^\sigma} \le g(\|f\|_{L^\infty})\, \|f\|_{H^\sigma} \tag{5.12}$$

for all $f \in H^\sigma \cap L^\infty$ (f may be R^N-valued). See, e.g., Meyer [31]. By applying this, we get

$$\begin{aligned} \big\|F\big(u(t), \partial u(t)\big)\big\|_{H^{s-1}} & \\ \le g\big(\|u(t)\|_{L^\infty} + \|\partial u(t)\|_{L^\infty}\big)&\big(\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}}\big). \end{aligned} \tag{5.13}$$

Since $s - 1 > \frac{n}{2}$, the L^∞ Sobolev embedding (1.23) gives

$$\|u(t)\|_{L^\infty} + \|\partial u(t)\|_{L^\infty} \, . \, \|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}} . \tag{5.14}$$

By combining (5.11),(5.13) and (5.14), we obtain (5.9). The proof of the difference estimate (5.10) is similar.

If (5.8) and (5.9) hold, and we let R be twice the right hand side of (5.8), then it follows by induction that $\|u_j\|_{\mathcal{X}^s_T} \le R$ provided $2C_T A(R) \le R$ (keep in mind that A is increasing). There are two ways to ensure that the latter inequality holds: Take T small (if $\lim_{T \to 0^+} C_T = 0$), or (since $A(0) = 0$) require that R be small, i.e., the data have small norm.

Then, by the difference estimate (5.10), we have

$$\|u_{j+1} - u_j\|_{\mathcal{X}^s_T} \le \frac{1}{2} \|u_j - u_{j-1}\|_{\mathcal{X}^s_T}$$

provided $2C_T A'(R) \le R$ (so we need either $\lim_{T \to 0^+} C_T = 0$ or $A'(0) = 0$).

It follows that (u_j) is Cauchy, establishing local existence. With a little more work one can then prove uniqueness of local solutions in $\mathcal{X}^s_T$ for any $T > 0$, and local Lipschitz continuity. For a fuller discussion we refer to Selberg [33, 34], where the following is proved.

5.3. Theorem. *Let $\mathcal{X}^s$ be a Banach space which embeds in* (5.3) *and is time-translation invariant:*

$$\|u(\cdot + T, \cdot)\|_{\mathcal{X}^s} = \|u\|_{\mathcal{X}^s} \quad (\forall T).$$

Also assume that for all $\phi \in C^\infty_c(\mathrm{R})$, the multiplication map $u \mapsto \phi(t)u(t,x)$ is bounded from $\mathcal{X}^s$ into itself.

For any $T > 0$, let $\mathcal{X}^s_T$ be the restriction of $\mathcal{X}^s$ to $[0,T] \times \mathrm{R}^n$. (That is, we define an equivalence relation $\sim_T$ on $\mathcal{X}^s$ by

$$u \sim_T v \iff u(t) = v(t) \quad \forall 0 \le t \le T.$$

Since $\mathcal{X}^s$ embeds in (5.3)*, the equivalence classes are closed sets in $\mathcal{X}^s$, so the quotient $\mathcal{X}^s_T = \mathcal{X}^s/{\sim_T}$, with norm*

$$\|u\|_{\mathcal{X}^s_T} = \inf_{v \sim_T u} \|v\|_{\mathcal{X}^s},$$

is a Banach space.)

Consider the system (5.5)*, where $\mathcal{N}$ is local in time and satisfies $\mathcal{N}(0) = 0$. Assume that the estimates* (5.8)*,* (5.9) *and* (5.10) *hold for all $0 < T < 1$, $(f,g) \in H^s \times H^{s-1}$ and $u, v \in \mathcal{X}^s_T$. Moreover, assume that*

$$\lim_{T \to 0^+} C_T = 0.$$

Then (5.5) *is locally well-posed for initial data in $H^s \times H^{s-1}$, in the following precise sense:*

(a) ***(Existence)*** *For all* $(f,g) \in H^s \times H^{s-1}$ *there is a time*

$$0 < T = T_s(\|f\|_{H^s} + \|g\|_{H^{s-1}})$$

which depends continuously on the norm of the data, and there is a $u \in \mathcal{X}^s_T$ *which solves* (5.5) *on* $S_T = [0,T] \times \mathrm{R}^n$.

(b) ***(Uniqueness)*** *If* $T > 0$ *and* $u, u' \in \mathcal{X}^s_T$ *are two solutions of* (5.5) *on* S_T *with the same data* (f,g), *then* $u = u'$ *in* $\mathcal{X}^s_T$.

(c) ***(Continuous dependence on initial data)*** *If* $u \in \mathcal{X}^s_T$ *solves* (5.5) *on* S_T *for some* $T > 0$, *then for all* $(f', g') \in H^s \times H^{s-1}$ *sufficiently close to* (f,g) *there exists a* $u' \in \mathcal{X}^s_T$ *which solves* (5.5) *on* S_T *with data* (f', g'), *and*

$$\|u - u'\|_{\mathcal{X}^s_T} . \|f - f'\|_{H^s} + \|g - g'\|_{H^{s-1}} .$$

5.2 Specialization to the Relevant Spaces

Here we specialize the preceding discussion to suit our present needs. The following theorem gives a precise form to the idea outlined at the beginning of this section.

5.4. Theorem. *Let* $\mathcal{X}^s$ *be a Banach space with the following properties:*

(a) $\mathcal{X}^s$ *embeds in* (5.3).

(b) *The norm on* $\mathcal{X}^s$ *is invariant under time-translation.*

(c) *The estimate*

$$\|u_0\|_{\mathcal{X}^s_T} \le C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}}\big)$$

holds for all $(f,g) \in H^s \times H^{s-1}$ *and* $0 < T < 1$, *where* u_0 *is the solution of the homogeneous wave equation with initial data* (f,g) *and* $\mathcal{X}^s_T$ *is the restriction space.*

(d) *For the purposes of local-in-time estimates,* Λ^{-1} *may be replaced with* $\Lambda_+^{-1}\Lambda_-^{-1}$. *More precisely, assume that for* $\varepsilon > 0$ *sufficiently small,*

$$\left\|\Lambda^{-1}F\right\|_{\mathcal{X}^s_T} \le C_{T,\varepsilon} \left\|\Lambda_+^{-1}\Lambda_-^{\varepsilon-1}F\right\|_{\mathcal{X}^s} \tag{5.15}$$

for all $F \in \Lambda_+\Lambda_-^{1-\varepsilon}\mathcal{X}^s$ *and* $0 < T < 1$. *Furthermore, suppose*

$$\lim_{T \to 0^+} C_{T,\varepsilon} = 0 \tag{5.16}$$

for $\varepsilon > 0$.

(e) *For all* $\phi \in C_c^\infty(\mathrm{R})$, *the multiplication map* $u \mapsto \phi(t)u(t,x)$ *is bounded from* $\mathcal{X}^s$ *into itself.*

Consider the Cauchy problem (5.5)*. Assume that $\mathcal{N}$ is local in time and $\mathcal{N}(0) = 0$. Furthermore, suppose*

$$\left\| \Lambda_+^{-1} \Lambda_-^{\varepsilon-1} \mathcal{N}(u) \right\|_{\mathcal{X}^s} \le A\big(\|u\|_{\mathcal{X}^s}\big), \tag{5.17}$$

$$\left\| \Lambda_+^{-1} \Lambda_-^{\varepsilon-1} \big(\mathcal{N}(u) - \mathcal{N}(v)\big) \right\|_{\mathcal{X}^s} \le A'\Big(\frac{1}{3}\max\{\|u\|_{\mathcal{X}^s}, \|v\|_{\mathcal{X}^s}\}\Big) \|u - v\|_{\mathcal{X}^s} \tag{5.18}$$

for all $u, v \in \mathcal{X}^s$, where A and A' are continuous and $A(0) = 0$.

Then (5.5) *is locally well-posed for initial data in $H^s \times H^{s-1}$, with uniqueness of solutions in $\mathcal{X}_T^s$ for any $T > 0$.*

Remark. In our applications of this theorem, ε must be strictly positive to ensure that (5.16) holds. The latter is not needed, however, if one imposes instead a smallness assumption on the norms of the initial data. More precisely, if we take $\varepsilon = 0$ in the above theorem, and if we replace (5.16) with the assumption that

$$\limsup_{t \to 0^+} \|u\|_{\mathcal{X}_t^s} \le C(\|u(0)\|_{H^s} + \|\partial_t u(0)\|_{H^{s-1}}) \tag{5.19}$$

for all $u \in \mathcal{X}^s$, then the conclusion of the theorem still holds, but we must require that the initial data satisfy $\|f\|_{H^s} + \|g\|_{H^{s-1}} < \delta$ for some sufficiently small $\delta > 0$.

Proof of Theorem 5.4. By Theorem 5.3, it suffices to prove the estimates (5.9) and (5.10) for $0 < T < 1$.

Fix T and $u \in \mathcal{X}_T^s$. Let $u' \in \mathcal{X}^s$ be any extension of u (meaning $u' \sim_T u$). Since $\mathcal{N}$ is local in time, we have $\Lambda^{-1}\mathcal{N}(u) = \Lambda^{-1}\mathcal{N}(u')$ on $[0, T] \times \mathrm{R}^n$, so by (5.15) and (5.17),

$$\left\| \Lambda^{-1} \mathcal{N}(u) \right\|_{\mathcal{X}_T^s} \le C_{T,\varepsilon} A\big(\|u'\|_{\mathcal{X}^s}\big).$$

Let $\|u'\|_{\mathcal{X}^s} \to \|u\|_{\mathcal{X}_T^s}$. Since A is continuous, (5.9) follows.

To prove (5.10), fix $u, v \in \mathcal{X}_T^s$, and let $u', v' \in \mathcal{X}^s$ be any two extensions of u and v. By (5.15) and (5.18),

$$\left\| \Lambda^{-1} \big(\mathcal{N}(u) - \mathcal{N}(v)\big) \right\|_{\mathcal{X}_T^s} \le C_{T,\varepsilon} A'\Big(\frac{1}{3}\max\{\|u'\|_{\mathcal{X}^s}, \|v'\|_{\mathcal{X}^s}\}\Big) \|u' - v'\|_{\mathcal{X}^s}.$$

Let $w = u' - v'$, and write

$$\max\{\|u'\|_{\mathcal{X}^s}, \|v'\|_{\mathcal{X}^s}\} \le \max\{\|u'\|_{\mathcal{X}^s}, \|w\|_{\mathcal{X}^s} + \|u'\|_{\mathcal{X}^s}\} = \|w\|_{\mathcal{X}^s} + \|u'\|_{\mathcal{X}^s}.$$

We may assume that A' is increasing. Thus

$$\left\| \Lambda^{-1} \big(\mathcal{N}(u) - \mathcal{N}(v)\big) \right\|_{\mathcal{X}_T^s} \le C_{T,\varepsilon} A'\Big(\frac{1}{3}\{\|w\|_{\mathcal{X}^s} + \|u'\|_{\mathcal{X}^s}\}\Big) \|w\|_{\mathcal{X}^s}.$$

Let $\|u'\|_{\mathcal{X}^s} \to \|u\|_{\mathcal{X}_T^s}$ and $\|w\|_{\mathcal{X}^s} \to \|u - v\|_{\mathcal{X}_T^s}$. Since

$$\|u - v\|_{\mathcal{X}_T^s} + \|u\|_{\mathcal{X}_T^s} \le 2\|u\|_{\mathcal{X}_T^s} + \|v\|_{\mathcal{X}_T^s} \le 3\max\{\|u\|_{\mathcal{X}_T^s}, \|v\|_{\mathcal{X}_T^s}\},$$

we conclude that (5.10) holds. □

The next theorem gives sufficient conditions for $\mathcal{X}^s$ to satisfy properties (c) and (d) of Theorem 5.4.

5.5. Theorem. *Let $\mathcal{X}^s$ be a Banach space satisfying:*

(a) $\mathcal{X}^s \hookrightarrow \mathcal{H}^{s,\theta}$ *for some* $\frac{1}{2} < \theta < 1$*;*

(b) $\|u\|_{\mathcal{X}^s} \le \|v\|_{\mathcal{X}^s}$ *whenever* $u \preceq v$*;*

(c) There exists $\alpha \le \frac{5}{4} + \frac{\theta}{2}$ *such that*

$$\|u\|_{\mathcal{X}^s} \;.\; \left\|\mathcal{F}\Lambda^{s-1}\Lambda_+\Lambda_-^{\alpha}u(\tau,\xi)\right\|_{L^2_\xi(L^\infty_\tau)}$$

for all $u \in \mathcal{X}^s$.

Let $\mathcal{X}^s_T$ *be the restriction, defined as in Theorem 5.3. Fix* $0 \le \varepsilon < 1-\theta$*. Then the solution of the linear Cauchy problem*

$$\Lambda u = F, \qquad (u, \partial_t u)\big|_{t=0} = (f,g)$$

satisfies

$$\|u\|_{\mathcal{X}^s_T} \;.\; \|f\|_{H^s} + \|g\|_{H^{s-1}} + T^{\varepsilon/2}\left\|\Lambda_+^{-1}\Lambda_-^{\varepsilon-1}F\right\|_{\mathcal{X}^s}$$

for all $0 < T \le 1$, $(f,g) \in H^s \times H^{s-1}$ *and* $F \in \Lambda_+\Lambda_-^{1-\varepsilon}\mathcal{X}^s$.

The proof can be found in [34].

Next, we verify that the iteration spaces used in the proof of the Main Theorem satisfy the hypotheses of the previous theorem. The spaces in parts (a)–(c) below are the iteration spaces used to prove parts (a)–(c), respectively, of the Main Theorem.

5.6. Proposition. *The hypotheses of Theorem 5.5 are satisfied by the following spaces:*

(a) $\mathcal{X}^s = \mathcal{H}^{s,\theta}$*, provided* $\frac{1}{2} < \theta < \frac{3}{2}$.

(b) The space $\mathcal{X}^s$ *given by the norm*

$$\|u\| = |u|_{s,\theta} + \left\|\Lambda^{\gamma}\Lambda_-^{\frac{1}{2}}u\right\|_{\mathcal{L}^1_t(\mathcal{L}^{2n}_x)},$$

where $n \ge 4$*,* $s > \frac{n-2}{2}$*,* $\frac{1}{2} < \theta < \frac{3}{2}$ *and* $0 < \gamma \le s - \frac{n-2}{2}$.

(c) The space $\mathcal{X}^s$ *given by the norm*

$$\|u\| = |u|_{s,\theta} + \left\|\Lambda^{-1}\Lambda_- u\right\|_{\mathcal{L}^q_t(\mathcal{L}^\infty_x)},$$

where $n \ge 3$*,* $s > \frac{n-2}{2}$*,* $\frac{1}{2} < \theta < \frac{3}{2}$ *and* $1 \le q \le 2$.

Proof. For $\mathcal{H}^{s,\theta}$, we only have to note that

$$|u|_{s,\theta} = \left\|\mathcal{F}\Lambda^{s-1}\Lambda_+\Lambda_-^{\theta} u(\tau,\xi)\right\|_{L^2_{\tau,\xi}} \le C_\delta \left\|\mathcal{F}\Lambda^{s-1}\Lambda_+\Lambda_-^{\theta+\frac{1}{2}+\delta} u(\tau,\xi)\right\|_{L^2_\xi(L^\infty_\tau)},$$

where

$$C_\delta^2 \simeq \int_{\mathrm{R}} (1+|\lambda|^2)^{-\frac{1}{2}-\delta}\, d\lambda < \infty$$

for any $\delta > 0$. This proves part (a).

Let $\mathcal{X}^s$ be the space in part (b). Since $n \ge 4$, Proposition 4.8 gives

$$\left\|\Lambda^\gamma \Lambda_-^{\frac{1}{2}} u\right\|_{\mathcal{L}^1_t(\mathcal{L}^{2n}_x)} \cdot \left\|\Lambda^{\gamma+\frac{n-2}{2}} \Lambda_-^{1+\delta} u\right\|_{\mathcal{L}^1_t(\mathcal{L}^2_x)}$$

for any $\delta > 0$. We claim that

$$\|u\|_{\mathcal{L}^1_t(\mathcal{L}^2_x)} \cdot \|\widehat{u}(\tau,\xi)\|_{L^2_\xi(L^\infty_\tau)} \tag{5.20}$$

for all u; this finishes the proof of part (b), if we choose δ sufficiently small.

To prove the claim, notice that

$$\|u\|_{\mathcal{L}^1_t(\mathcal{L}^2_x)} = \sup_v \int |\widehat{u}(X)|\, \widehat{v}(X)\, dX \le \|\widehat{u}(\tau,\xi)\|_{L^2_\xi(L^\infty_\tau)} \sup_v \|\widehat{v}(\tau,\xi)\|_{L^2_\xi(L^1_\tau)},$$

where the supremum is over all $v \in \mathcal{S}$ such that $\widehat{v} \ge 0$ and $\|v\|_{L^\infty_t(L^2_x)} = 1$. But for such v, we have $\|\widehat{v}(\tau,\xi)\|_{L^2_\xi(L^1_\tau)} = (2\pi)^{-1}\, \|v(0,\cdot)\|_{L^2(\mathrm{R}^n)} \le (2\pi)^{-1}$ by Fourier inversion.

Finally, we prove part (c). Since the norm only depends on the size of the Fourier transform and is compatible with the relation $\preceq$, it suffices, by (4.1), to prove that

$$\left\|\Lambda^{-1}\Lambda_- u\right\|_{L^q_t(L^\infty_x)} \cdot \left\|\mathcal{F}\Lambda^s \Lambda_-^\alpha u(\tau,\xi)\right\|_{L^2_\xi(L^\infty_\tau)}$$

for some $\alpha \le \frac{5}{4} + \frac{\theta}{2}$. Let $\frac{1}{q'} = 1 - \frac{1}{q}$ and choose β satisfying $\frac{1}{q'} < \beta \le \frac{1}{4} + \frac{\theta}{2}$. By the inequalities of Hausdorff-Young, Minkowski and Hölder,

$$\begin{aligned} \|v\|_{L^q_t(L^\infty_x)} &\le \left\|\widehat{v(t)}(\xi)\right\|_{L^q_t(L^1_\xi)} \le \left\|\widehat{v(t)}(\xi)\right\|_{L^1_\xi(L^q_t)} \le \left\|\widehat{v}(\tau,\xi)\right\|_{L^1_\xi(L^{q'}_\tau)} \\ &\cdot \left\|(1+|\xi|)^{-\frac{n}{2}-\varepsilon}\big(1+\big||\tau|-|\xi|\big|\big)^{-\beta} \mathcal{F}\Lambda^{\frac{n}{2}+\varepsilon}\Lambda_-^\beta v(\tau,\xi)\right\|_{L^1_\xi(L^{q'}_\tau)} \\ &\cdot \left\|\mathcal{F}\Lambda^{\frac{n}{2}+\varepsilon}\Lambda_-^\beta v(\tau,\xi)\right\|_{L^2_\xi(L^\infty_\tau)} \end{aligned}$$

for any $\varepsilon > 0$. This finishes the proof. □

Finally, we check that condition (e) of Theorem 5.4 holds for the spaces that we use.

5.7. Proposition. *Let $\gamma, \gamma_+, \gamma_- \in \mathrm{R}$ and $1 \le q, r \le \infty$. Let $\phi \in \mathcal{S}(\mathrm{R})$, $M_\phi u(t,x) = \phi(t)u(t,x)$. Then*

$$\left\|\Lambda^\gamma \Lambda_+^{\gamma_+}\Lambda_-^{\gamma_-} M_\phi u\right\|_{\mathcal{L}^q_t(\mathcal{L}^r_x)} \le C_{\gamma_+,\gamma_-,\phi} \left\|\Lambda^\gamma \Lambda_+^{\gamma_+}\Lambda_-^{\gamma_-} u\right\|_{\mathcal{L}^q_t(\mathcal{L}^r_x)}.$$

Proof. Since $\Lambda^\gamma M_\phi u = M_\phi(\Lambda^\gamma u)$, we may assume $\gamma = 0$.

Let χ and v be defined by $\widehat{\chi}(\tau) = (1 + |\tau|^2)^{\frac{\gamma_+ + \gamma_-}{2}} \bigl|\widehat{\phi}\bigr|$ and $\widehat{v} = |\widehat{u}|$. Since

$$\bigl||\tau| - |\xi|\bigr| \leq \bigl||\tau| - |\lambda|\bigr| + \bigl||\lambda| - |\xi|\bigr| \leq |\tau - \lambda| + \bigl||\lambda| - |\xi|\bigr|,$$
$$|\tau| + |\xi| \leq |\tau - \lambda| + |\lambda| + |\xi|\,,$$

we have

$$\Lambda_+^{\gamma_+} \Lambda_-^{\gamma_-} M_\phi u \text{ - } M_\chi(\Lambda_+^{\gamma_+} \Lambda_-^{\gamma_-} v).$$

Thus, it suffices to prove

$$\|M_\chi v\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} \cdot \|v\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)}$$

for all v such that $\widehat{v} \geq 0$.

Since $\chi \in H^\alpha(\mathrm{R})$ for all $\alpha > 0$, we have $\chi \in C^\infty$ and $\chi^{(j)} \in L^\infty$ for all $k \geq 0$, so M_χ maps the set $\{w \in \mathcal{S}(\mathrm{R}^{1+n}) : \widehat{w} \geq 0\}$ into itself. Thus, if $\widehat{v} \geq 0$,

$$\begin{aligned}\|M_\chi v\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} &= \sup_w \int (M_\chi v) w \, dt \, dx \\ &= \sup_w \int v M_\chi w \, dt \, dx \leq \|v\|_{\mathcal{L}_t^q(\mathcal{L}_x^r)} \sup_w \|M_\chi w\|_{L_t^{q'}(L_x^{r'})}\,,\end{aligned}$$

where q', r' denote the dual exponents of q, r and the supremum is over all $w \in \mathcal{S}$ such that $\widehat{w} \geq 0$ and $\|w\|_{L_t^{q'}(L_x^{r'})} = 1$. But by Hölder's inequality and Sobolev embedding,

$$\|M_\chi w\|_{L_t^{q'}(L_x^{r'})} \cdot \|\chi\|_{L^\infty(\mathrm{R})} \|w\|_{L_t^{q'}(L_x^{r'})} \cdot \|\chi\|_{H^{\frac{1}{2}+\varepsilon}(\mathrm{R})} \|w\|_{L_t^{q'}(L_x^{r'})}$$

for any $\varepsilon > 0$, and we have $\|\chi\|_{H^{\frac{1}{2}+\varepsilon}(\mathrm{R})} = \|\phi\|_{H^{\gamma_+ + \gamma_- + \frac{1}{2}+\varepsilon}(\mathrm{R})}$. □

6 Some Special Embeddings

We collect here some embeddings that are used repeatedly in the proof of the Main Theorem.

(6.1) $H^{0,\theta} \hookrightarrow L_t^\infty(L_x^2), \qquad \theta > 1/2,$

(6.2) $H^{s,\theta} \hookrightarrow L^\infty(\mathrm{R}^{1+n}), \qquad s > n/2,\ \theta > 1/2,$

(6.3) $H^{s,0} \hookrightarrow L_t^2(L_x^\infty), \qquad s > n/2,$

(6.4) $H^{1,0} \hookrightarrow L_t^2(L_x^{\frac{2n}{n-2}}), \qquad n \geq 3,$

(6.5) $H^{1,\theta} \hookrightarrow L_t^2(L_x^{\frac{2n}{n-3}}), \qquad n \geq 4,\ \theta > \frac{1}{2},$

(6.6) $H^{\frac{n-1}{2},0} \hookrightarrow L_t^2(L_x^{2n}),$

(6.7) $H^{\frac{n-1}{2}+\varepsilon,\theta} \hookrightarrow L_t^2(L_x^\infty), \qquad n \geq 4,\ \varepsilon > 0,\ \theta > \frac{1}{2},$

(6.8) $H^{\frac{n-2}{2},\theta} \hookrightarrow L_t^2(L_x^{2n}), \qquad n \geq 4,\ \theta > \frac{1}{2},$

(6.9) $H^{\frac{n-3}{2},0} \hookrightarrow L_t^2(L_x^{\frac{2n}{3}}), \qquad n \geq 3$

(6.10) $H^{\frac{n-3}{2},\theta} \hookrightarrow L_t^2(L_x^n), \qquad n \geq 5,\ \theta > \frac{1}{2},$

(6.11) $H^{\frac{n-3}{2},\theta} \hookrightarrow L_t^\infty(L_x^{\frac{2n}{3}}), \qquad n \geq 3,\ \theta > \frac{1}{2},$

(6.12) $\Lambda^{-\frac{1}{2}-\varepsilon} L_t^1(L_x^{2n}) \hookrightarrow L_t^1(L_x^\infty), \qquad \varepsilon > 0,$

(6.13) $\Lambda^{-\frac{1}{2}-\varepsilon} \mathcal{L}_t^1(\mathcal{L}_x^{2n}) \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^\infty), \qquad \varepsilon > 0,$

(6.14) $\mathcal{L}_t^q(\mathcal{L}_x^\infty) \cdot H^{0,\theta} \hookrightarrow H^{0,-\alpha}, \qquad 1 \leq q \leq 2,\ \alpha > \frac{1}{q} - \frac{1}{2},\ \theta > \frac{1}{2},$

(6.15) $\Lambda^{-\frac{1}{2}-\varepsilon} \mathcal{L}_t^1(\mathcal{L}_x^{2n}) \cdot H^{0,\theta} \hookrightarrow H^{0,-\theta}, \qquad \varepsilon > 0,\ \theta > \frac{1}{2}.$

6.1. Remark. In view of Corollary 4.6, the estimates (6.1)–(6.11) remain valid if we replace the $L_t^q(L_x^r)$-space on the right by the corresponding $\mathcal{L}_t^q(\mathcal{L}_x^r)$-space.

Proofs:

(i) (6.1) follows from Proposition 3.9; then, since

$$\Lambda^{-s} L^2(\mathrm{R}^n) \hookrightarrow L^\infty(\mathrm{R}^n)$$

for $s > \frac{n}{2}$ by Sobolev embedding, and since $H^{s,\theta} = \Lambda^{-s} H^{0,\theta}$ and $H^{s,0} = \Lambda^{-s} L^2$, (6.2) and (6.3) follow.

(ii) (6.5), (6.8) and (6.10) are special cases of Theorem D.

(iii) (6.4), (6.6) and (6.9) hold by Sobolev embedding.

(iv) (6.7) follows from (6.8) and

$$L_t^2(L_x^{2n}) \hookrightarrow \Lambda^{\frac{1}{2}+\varepsilon} L_t^2(L_x^\infty).$$

The latter holds by Sobolev embedding.

(v) (6.11) follows from (6.1) and the Sobolev embedding $H^{\frac{n-3}{2}} \hookrightarrow L^{\frac{2n}{3}}(\mathrm{R}^n)$, since these imply

$$H^{\frac{n-3}{2},\theta} = \Lambda^{-\frac{n-3}{2}} H^{0,\theta} \hookrightarrow \Lambda^{-\frac{n-3}{2}} L_t^\infty(L_x^2) \hookrightarrow L_t^\infty(L_x^{\frac{2n}{3}}).$$

(vi) (6.12) holds by Sobolev embedding; (6.13) follows by Corollary 4.6.

(vii) (6.14) is proved as follows: By interpolation between (6.1) and $L^2 \hookrightarrow L^2$,

$$L_t^q(L_x^2) \hookrightarrow H^{0,-\alpha}$$

for $1 \le q \le 2$ and $\alpha > \frac{1}{q} - \frac{1}{2}$, and it follows that

$$\mathcal{L}_t^q(\mathcal{L}_x^2) \hookrightarrow H^{0,-\alpha}$$

for such q, α. Combining this with Proposition 4.3, Lemma 4.4 and (6.1), we get (6.14).

(viii) (6.15) follows from (6.14) and (6.13).

7 Main Estimates for (WM)

Our aim here is to prove part (a) of the Main Theorem: (WM) is locally well-posed for initial data in $H^s \times H^{s-1}$ for $s > \frac{n}{2}$ and $n \ge 2$.

It suffices to verify the hypotheses of Theorem 5.4. Take $\mathcal{X}^s = \mathcal{H}^{s,\theta}$, where $\theta = \theta(s,n) > \frac{1}{2}$ is to be determined. The nonlinearity is

$$\mathcal{N}^I(u) = -\sum_{J,K=1}^{N} \Gamma_{JK}^I(u) Q_0(u^J, u^K), \qquad 1 \le I \le N.$$

Let us first check that conditions (a)–(e) of Theorem 5.4 are satisfied. Condition (a) holds by Proposition 3.12; condition (b) is obviously satisfied; conditions (c) and (d) follow from Theorem 5.5, in view of Proposition 5.6; finally, condition (e) holds by Proposition 5.7.

It remains to prove (5.17) and (5.18). Let us first prove (5.17) in the case where the Γ_{JK}^I's are constants, and let us set $\varepsilon = 0$. The proof of the general case is quite similar; it appears in section 7.2.

7.1 The Simplified Case

Since $\Lambda_+\Lambda_-\mathcal{H}^{s,\theta} = H^{s-1,\theta-1}$, what we want to prove is the following:

7.1. Theorem. *Suppose $n \ge 2$, $s > \frac{n}{2}$ and $\frac{1}{2} < \theta \le s - \frac{n-1}{2}$. Then*

$$Q_0\big(\mathcal{H}^{s,\theta}, \mathcal{H}^{s,\theta}\big) \hookrightarrow H^{s-1,\theta-1}.$$

By estimating the Fourier symbol of the null form Q_0 in absolute value, it is easy to prove (see Lemma 7.6 below) that

$$Q_0(\phi,\psi) \preceq D_+D_-(\phi'\psi') + (D_+D_-\phi')\psi' + \phi' D_+D_-\psi'$$

whenever $\phi \preceq \phi'$ and $\psi \preceq \psi'$. Therefore, in view of Remark 5.1, it suffices to prove

$$\mathcal{H}^{s,\theta} \cdot \mathcal{H}^{s,\theta} \hookrightarrow \mathcal{H}^{s,\theta},$$
$$H^{s-1,\theta-1} \cdot \mathcal{H}^{s,\theta} \hookrightarrow H^{s-1,\theta-1}.$$

To keep the discussion as simple as possible (the complete details appear in section 7.2), let us for the moment ignore the difference between $\mathcal{H}^{s,\theta}$ and $H^{s,\theta}$. Thus, we want to prove

$$H^{s,\theta} \cdot H^{s,\theta} \hookrightarrow H^{s,\theta}, \tag{7.1}$$
$$H^{s-1,\theta-1} \cdot H^{s,\theta} \hookrightarrow H^{s-1,\theta-1}. \tag{7.2}$$

These are special cases of the following.

7.2. Theorem. *Let $n \ge 2$, $s > \frac{n}{2}$ and $\frac{1}{2} < \theta \le s - \frac{n-1}{2}$. Then*

$$H^{a,\alpha} \cdot H^{s,\theta} \hookrightarrow H^{a,\alpha}$$

for all a, α satisfying

$$0 \le \alpha \le \theta,$$
$$-s + \alpha < a \le s.$$

(Hence, by duality, for all $-\theta \le \alpha \le 0$ and $-s \le a < s + \alpha$.)

The proof is achieved by interpolating between four different points in the (a,α)-plane. One of these points is (s,θ), which corresponds to the estimate (7.1). We give the proof of the latter here. The proofs of the remaining estimates are similar, and can be found in the Appendix.

We may restate (7.1) as follows:

7.3. Theorem. *$H^{s,\theta}$ is an algebra if $n \ge 2$, $s > \frac{n}{2}$ and $\frac{1}{2} < \theta \le s - \frac{n-1}{2}$.*

For the proof we need the following "Leibniz rule", which is an immediate consequence of the triangle inequality.

7.4. Lemma. *If* $\alpha > 0$*, then*

$$\Lambda^\alpha(uv) \text{ - } (\Lambda^\alpha u)v + u\Lambda^\alpha v$$

for all u *and* v *with* $\widehat{u}, \widehat{v} \geq 0$*. Moreover, the same estimate holds with* Λ^α *replaced by either of the operators* D^α, D_+^α *or* Λ_+^α*.*

By Lemma 7.4, the proof of Theorem 7.3 reduces to showing

$$H^{0,\theta} \cdot H^{s,\theta} \hookrightarrow H^{0,\theta}.$$

But by Lemma 3.3, the latter reduces to three estimates:

$$\begin{aligned} H^{0,\theta} \cdot H^{s,0} &\hookrightarrow L^2, \\ L^2 \cdot H^{s,\theta} &\hookrightarrow L^2, \\ R^\theta(H^{0,\theta}, H^{s,\theta}) &\hookrightarrow L^2. \end{aligned}$$

The first one follows from Hölder's inequality, the energy embedding (6.1) and the Sobolev embedding (6.3); the second one holds by Hölder's inequality and (6.2); the third one is a special case of Theorem F.

7.2 The General Case

Here we prove the following.

7.5. Theorem. *Let* $n \geq 2$*,* $s > \frac{n}{2}$*. Suppose*

$$\begin{aligned} \frac{1}{2} < \theta &\leq \min\Big(1, s - \frac{n-1}{2}\Big), \\ 0 \leq \varepsilon &\leq \min\Big(1 - \theta, s - \frac{n-1}{2} - \theta\Big). \end{aligned}$$

Let $\Gamma : \mathrm{R}^N \to \mathrm{R}$ *be smooth. Then there exist continuous functions* $g, h : [0, \infty) \to [0, \infty)$ *such that*

$$\big\|\Gamma(u)Q_0(u^J, u^K)\big\|_{s-1,\theta+\varepsilon-1} \leq g\big(\|u\|_{s,\theta}\big)\, |u|_{s,\theta}^2 \tag{7.3}$$

and

$$\begin{aligned} &\big\|\Gamma(u)Q_0(u^J, u^K) - \Gamma(U)Q_0(U^J, U^K)\big\|_{s-1,\theta+\varepsilon-1} \\ &\qquad\qquad \leq h\big(\|u\|_{s,\theta} + \|U\|_{s,\theta}\big)\, |u - U|_{s,\theta} \end{aligned} \tag{7.4}$$

for all R^N*-valued* $u, U \in \mathcal{H}^{s,\theta}$ *and* $1 \leq J, K \leq N$*.*

As a consequence, we obtain part (a) of the Main Theorem.
We shall need the following.

7.6. Lemma. *If* $0 \leq \alpha \leq 1$*, then*

$$Q_0(\phi, \psi) \text{ - } D_+^{1-\alpha} D_-^{1-\alpha}(D_+^\alpha \phi' D_+^\alpha \psi') + D_+ D_-^{1-\alpha} \phi' D_+^\alpha \psi' + D_+^\alpha \phi' D_+ D_-^{1-\alpha} \psi'$$

whenever $\phi \preceq \phi'$ *and* $\psi \preceq \psi'$*.*

Proof. The symbol of Q_0 is $q_0(\Xi, \Theta) \simeq \langle \Xi, \Theta \rangle$. Recall that $\langle \cdot, \cdot \rangle$ denotes the Minkowskian inner product on R^{1+n}, while $|\cdot|$ always denotes the Euclidean norm. Since

$$\langle \Xi + \Theta, \Xi + \Theta \rangle = \langle \Xi, \Xi \rangle + \langle \Theta, \Theta \rangle + 2 \langle \Xi, \Theta \rangle,$$

we have

$$|q_0(\Xi, \Theta)| \;.\; |\langle \Xi, \Xi \rangle| + |\langle \Theta, \Theta \rangle| + |\langle \Xi + \Theta, \Xi + \Theta \rangle| \,.$$

Take this to the power $1 - \alpha$, and take the trivial estimate

$$|q_0(\Xi, \Theta)| \;.\; |\Xi| \, |\Theta|$$

to the power α. The product of the left hand sides of the resulting inequalities is then bounded by the product of the right hand sides, and keeping in mind that the symbol of $D_+ D_-$ is $|\langle \Xi, \Xi \rangle|$, we get the desired estimate. □

We first prove Theorem 7.5 in the case of constant Γ; the general case is then reduced to this, by virtue of Theorem 7.2 and the following result, which is an analogue of the Moser inequality (5.12).

7.7. Theorem. *Assume that* $\Gamma \in C^\infty(\mathrm{R}^N)$ *and* $\Gamma(0) = 0$*. If* n, s, θ *are as in Theorem 7.3 and* $\theta \leq 1$*, there exists a continuous function* $g = g_{s,\theta} : [0, \infty) \to [0, \infty)$ *such that*

$$\|\Gamma(u)\|_{s,\theta} \leq g\big(\|u\|_{n/2+\varepsilon,\theta}\big) \, \|u\|_{s,\theta}$$

for all R^N*-valued* $u \in H^{s,\theta}$*, where* $\varepsilon = \theta - 1/2$*.*

This was proved in [33].

Let us now prove Theorem 7.5. Throughout the rest of this section we assume that n, s, θ and ε satsify the hypotheses of Theorem 7.5.

Step 1. We assume $\Gamma = 1$; i.e., we prove

$$Q_0\big(\mathcal{H}^{s,\theta}, \mathcal{H}^{s,\theta}\big) \hookrightarrow H^{s-1,\theta+\varepsilon-1}. \tag{7.5}$$

If we apply Lemma 7.6 with $\alpha = \varepsilon$, and then apply Lemma 7.4 to the first term on the right hand side, we get

$$Q_0(\phi, \psi) \text{ - } D_-^{1-\alpha}(D_+ \phi' D_+^\alpha \psi') + D_+ D_-^{1-\alpha} \phi' D_+^\alpha \psi' + \text{symmetric terms},$$

where $\phi \preceq \phi'$ and $\psi \preceq \psi'$. Thus (7.5) reduces to two estimates:

$$H^{s-1,\theta} \cdot H^{s-\varepsilon,\theta} \hookrightarrow H^{s-1,\theta},$$
$$H^{s-1,\theta+\varepsilon-1} \cdot H^{s-\varepsilon,\theta} \hookrightarrow H^{s-1,\theta+\varepsilon-1},$$

both of which are special cases of Theorem 7.2.

Step 2. We prove the theorem for general Γ. First, we write

$$\Gamma(u)Q_0(u^J,u^K) = \big\{\Gamma(u)-\Gamma(0)\big\}Q_0(u^J,u^K) + \Gamma(0)Q_0(u^J,u^K).$$

Since

$$H^{s,\theta}\cdot H^{s-1,\theta+\varepsilon-1} \hookrightarrow H^{s-1,\theta+\varepsilon-1} \tag{7.6}$$

by Theorem 7.2, (7.3) reduces to

$$\|\Gamma(u)-\Gamma(0)\|_{s,\theta} \le g\big(\|u\|_{s,\theta}\big), \tag{7.3a}$$

$$\big\|Q_0(u^J,u^K)\big\|_{s-1,\theta+\varepsilon-1} \cdot \big|u^J\big|_{s,\theta}\,\big|u^K\big|_{s,\theta}\,. \tag{7.3b}$$

The former holds by Theorem 7.7, the latter by Step 1.

To prove (7.4), write

$$\begin{aligned}&\Gamma(u)Q_0(u^J,u^K)-\Gamma(U)Q_0(U^J,U^K)\\ &= \big\{\Gamma(u)-\Gamma(U)\big\}Q_0(u^J,u^K)+\Gamma(U)\big\{Q_0(u^J-U^J,u^K)+Q_0(U^J,u^K-U^K)\big\}.\end{aligned}$$

The second term on the right hand side is covered by the proof of (7.3), while the first term reduces, in view of (7.6) and (7.3b), to the estimate

$$\|\Gamma(u)-\Gamma(U)\|_{s,\theta} \le h\big(\|u\|_{s,\theta}+\|U\|_{s,\theta}\big)\,|u-U|_{s,\theta}\,. \tag{7.7}$$

But

$$\begin{aligned}\Gamma(u)-\Gamma(U) &= \int_0^1 d\Gamma\big((1-\lambda)U+\lambda u\big)\cdot(u-U)\,d\lambda\\ &= \int_0^1 \big\{d\Gamma\big((1-\lambda)U+\lambda u\big)-d\Gamma(0)\big\}\cdot(u-U)\,d\lambda + d\Gamma(0)\cdot(u-U),\end{aligned}$$

and since $H^{s,\theta}$ is an algebra (Theorem 7.3), it follows that

$$\begin{aligned}&\|\Gamma(u)-\Gamma(U)\|_{s,\theta}\\ &\cdot \int_0^1 \big\|d\Gamma\big((1-\lambda)U+\lambda u\big)-d\Gamma(0)\big\|_{s,\theta}\,\|u-U\|_{s,\theta}\;d\lambda + |d\Gamma(0)|\,\|u-U\|_{s,\theta}\,.\end{aligned}$$

Thus, (7.7) follows after another application of Theorem 7.7.

8 Main Estimates for (“MKG”)/(“YM”)

Here we prove part (b) of the Main Theorem: (“MKG”)/(“YM”) are locally well-posed for initial data in $H^s\times H^{s-1}(\mathrm{R}^n)$ for $s>\frac{n-2}{2}$ and $n\ge 4$.

The full details of the proof appear in section 8.2. The reader who wants to get the gist of the argument, without getting bogged down in technicalities, is advised to read first the informal discussion in section 8.1.

8.1 Informal Proof

In order to prove part (b) of the Main Theorem by iteration in the space $\mathcal{H}^{s,\theta}$, we would need two types of estimates:

$$D^{-1}Q_{ij}\big(\mathcal{H}^{s,\theta},\mathcal{H}^{s,\theta}\big) \hookrightarrow H^{s-1,\theta-1}, \tag{8.1}$$

$$Q_{ij}\big(D^{-1}\mathcal{H}^{s,\theta},\mathcal{H}^{s,\theta}\big) \hookrightarrow H^{s-1,\theta-1}, \tag{8.2}$$

for all $s > \frac{n-2}{2}$, $n \geq 4$ and some $\theta = \theta(s,n) > \frac{1}{2}$.

We shall use the following.

8.1. Lemma. *The estimate*

$$Q_{ij}(\phi,\psi) \preceq D^{\frac{1}{2}}D_-^{\frac{1}{2}}(D^{\frac{1}{2}}\phi' D^{\frac{1}{2}}\psi') + D^{\frac{1}{2}}(D^{\frac{1}{2}}D_-^{\frac{1}{2}}\phi' D^{\frac{1}{2}}\psi') + D^{\frac{1}{2}}(D^{\frac{1}{2}}\phi' D^{\frac{1}{2}}D_-^{\frac{1}{2}}\psi')$$

holds whenever $\phi \preceq \phi'$ and $\psi \preceq \psi'$.

For the proof, see [24].

In view of Lemma 8.1 and Remark 5.1, proving (8.1) reduces to proving, if we replace D^{-1} by Λ^{-1} and ignore the difference between $\mathcal{H}^{s,\theta}$ and $H^{s,\theta}$,

$$H^{s-\frac{1}{2},\theta} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow H^{s-\frac{3}{2},\theta-\frac{1}{2}}, \tag{8.3}$$

$$H^{s-\frac{1}{2},\theta-\frac{1}{2}} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow H^{s-\frac{3}{2},\theta-1}, \tag{8.4}$$

Similarly, (8.2) can be reduced to

$$H^{s+\frac{1}{2},\theta} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow H^{s-\frac{1}{2},\theta-\frac{1}{2}}, \tag{8.5}$$

$$H^{s+\frac{1}{2},\theta-\frac{1}{2}} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow H^{s-\frac{1}{2},\theta-1}, \tag{8.6}$$

$$H^{s+\frac{1}{2},\theta} \cdot H^{s-\frac{1}{2},\theta-\frac{1}{2}} \hookrightarrow H^{s-\frac{1}{2},\theta-1}. \tag{8.7}$$

8.2. Theorem. *The estimates* (8.3)–(8.5) *and* (8.7) *hold for all $s > \frac{n-2}{2}$, $n \geq 4$ and $\frac{1}{2} < \theta \leq s - \frac{n-3}{2}$. However,* (8.6) *fails if $s < \frac{n}{2} - \theta$.*

Remark. The condition $\theta - \frac{1}{2} \leq s - \frac{n-2}{2}$ is necessary by scaling. Therefore, in view of the above theorem, (8.6) cannot hold unless $s \geq \frac{n}{2} - \frac{3}{4}$. In fact, we expect that (8.3)–(8.7) are all true for $s > \frac{n}{2} - \frac{3}{4}$, $n \geq 4$ with $\theta = s - \frac{n-3}{2}$. This has been verified in dimension $n = 3$; see Cuccagna [6] and Keel-Tao [12]. We do not pursue this question.

Remark. The failure of our attempt to iterate in $\mathcal{H}^{s,\theta}$ when $s < \frac{n}{2} - \theta$ is not due to any loss of information through the use of Lemma 8.1: The proof of the last statement in Theorem 8.2 (see the Appendix) shows that (8.2) also fails for such s.

The first (positive) statement in Theorem 8.2 is proved in section 8.2, and the second (negative) statement is proved in the Appendix. The following heuristic arguments should convince the reader that the result is reasonable.

Observe that if we consider the idealized case $\theta = \frac{1}{2}$, then the estimates (8.3)–(8.7) are all of the form (using duality if necessary)

$$H^{a,\theta} \cdot H^{b,\theta} \hookrightarrow H^{-c,0}.$$

The latter is morally equivalent to a product estimate for two solutions of the homogeneous wave equation:

$$\left\| D^{-c}(uv) \right\|_{L^2} \lesssim \|f\|_{H^a} \|g\|_{H^b},$$

where $\Lambda u = \Lambda v = 0$, $(u, \partial_t u)\big|_{t=0} = (f, 0)$ and $(v, \partial_t v)\big|_{t=0} = (g, 0)$. By Theorem C, a necessary condition for this estimate to hold is

$$a + b \geq \frac{1}{2}. \tag{8.8}$$

Note that if $a + b + c = \frac{n-1}{2}$, then (8.8) is equivalent to $c \leq \frac{n-2}{2}$.

Let us reexamine our estimates in the light of condition (8.8). Taking $s = \frac{n-2}{2}$ and $\theta = \frac{1}{2}$, and using duality where necessary, (8.3)–(8.7) reduce to

$$H^{\frac{n-3}{2},\theta} \cdot H^{\frac{n-3}{2},\theta} \hookrightarrow H^{\frac{n-5}{2},0}, \tag{8.3'}$$

$$H^{\frac{5-n}{2},\theta} \cdot H^{\frac{n-3}{2},\theta} \hookrightarrow H^{\frac{3-n}{2},0}, \tag{8.4'}$$

$$H^{\frac{n-1}{2},\theta} \cdot H^{\frac{n-3}{2},\theta} \hookrightarrow H^{\frac{n-3}{2},0}, \tag{8.5'}$$

$$H^{\frac{3-n}{2},\theta} \cdot H^{\frac{n-3}{2},\theta} \hookrightarrow H^{\frac{1-n}{2},0}, \tag{8.6'}$$

$$H^{\frac{n-1}{2},\theta} \cdot H^{\frac{3-n}{2},\theta} \hookrightarrow H^{\frac{3-n}{2},0}. \tag{8.7'}$$

Condition (8.8) is satisfied in all of the above except (8.6′), where $a + b = 0$ (and $c = \frac{n-1}{2}$). The latter estimate is therefore far from being true (it is half a derivative off the mark). On the other hand, since $n \geq 4$, it is easily checked that the other four estimates above are in fact true by Theorem F, if we take $\theta > \frac{1}{2}$. Let us now take a closer look at the estimate which fails, namely (8.6). By Lemma 7.4, this reduces to

$$\begin{aligned} &H^{1,\theta-\frac{1}{2}} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow H^{0,\theta-1}, \\ &H^{s+\frac{1}{2},\theta-\frac{1}{2}} \cdot H^{0,\theta} \hookrightarrow H^{0,\theta-1}. \end{aligned} \tag{8.9}$$

The former is true for $s > \frac{n-2}{2}$, $n \geq 4$ (see section 8.2 for the proof), while the latter fails for $s < \frac{n}{2} - \theta$ (see the Appendix).

For simplicity, throughout the remainder of section 8.1 we will only consider the idealized case where $s = \frac{n-2}{2}$ and $\theta = \frac{1}{2}$. *Thus, when we say that an estimate holds, we mean up to a logarithmic divergence.* The informal arguments in this section are easily made rigorous (see section 8.2).

Since we are assuming $s = \frac{n-2}{2}$ and $\theta = \frac{1}{2}$, the problematic estimate (8.9) reads

$$H^{\frac{n-1}{2},0} \cdot H^{0,\theta} \hookrightarrow H^{0,-\theta}.$$

An easy way to fix the problem with this estimate is to replace $H^{\frac{n-1}{2},0}$ on the left hand side with

$$H^{\frac{n-1}{2},0} \cap L_t^1(L_x^\infty).$$

In other words, we claim that

$$\big(H^{\frac{n-1}{2},0} \cap L_t^1(L_x^\infty)\big) \cdot H^{0,\theta} \hookrightarrow H^{0,-\theta}. \tag{8.10}$$

This is a trivial consequence of energy estimates and Hölder's inequality. Indeed, by (6.4),

$$H^{0,\theta} \hookrightarrow L_t^\infty(L_x^2).$$

The dual of this embedding is

$$L_t^1(L_x^2) \hookrightarrow H^{0,-\theta}.$$

Since

$$L_t^1(L_x^\infty) \cdot L_t^\infty(L_x^2) \hookrightarrow L_t^1(L_x^2)$$

by Hölder's inequality, we obtain (8.10).

This suggests taking

$$\mathcal{H}^{s,\theta} \cap \Lambda^{\frac{1}{2}}\Lambda_-^{-\frac{1}{2}}\big(\mathcal{L}_t^1(\mathcal{L}_x^\infty)\big) \tag{8.11}$$

as our iteration space. This works for systems of the type ("MKG"), but leads to problems for ("YM") (cf. Remarks 8.4 and 8.5 below). A better choice turns out to be

$$\mathcal{X}^s = \mathcal{H}^{s,\theta} \cap \Lambda^{-\gamma}\Lambda_-^{-\frac{1}{2}}\big(\mathcal{L}_t^1(\mathcal{L}_x^{2n})\big), \tag{8.12}$$

where $\gamma > 0$ is sufficiently small. Since we are assuming $s = \frac{n-2}{2}$ and $\theta = \frac{1}{2}$, we will take $\gamma = 0$ here.

We are now faced with the task of proving

$$D^{-1}Q_{ij}(\mathcal{X}^s, \mathcal{X}^s) \hookrightarrow \Lambda_+\Lambda_-\mathcal{X}^s,$$
$$Q_{ij}(D^{-1}\mathcal{X}^s, \mathcal{X}^s) \hookrightarrow \Lambda_+\Lambda_-\mathcal{X}^s,$$

where $\mathcal{X}^s$ is given by (8.12). In fact, we can prove

$$D^{-1}Q_{ij}(\mathcal{H}^{s,\theta}, \mathcal{H}^{s,\theta}) \hookrightarrow \Lambda_+\Lambda_-\mathcal{X}^s, \tag{8.13}$$

$$Q_{ij}(D^{-1}\mathcal{X}^s, \mathcal{H}^{s,\theta}) \hookrightarrow \Lambda_+\Lambda_-\mathcal{X}^s. \tag{8.14}$$

In view of the definition of $\mathcal{X}^s$, (8.13) is equivalent to

$$D^{-1}Q_{ij}(\mathcal{H}^{s,\theta}, \mathcal{H}^{s,\theta}) \hookrightarrow H^{s-1,\theta-1}, \tag{8.15}$$

$$D^{-1}Q_{ij}(\mathcal{H}^{s,\theta}, \mathcal{H}^{s,\theta}) \hookrightarrow \Lambda_+\Lambda_-^{\frac{1}{2}}\big(\mathcal{L}_t^1(\mathcal{L}_x^{2n})\big). \tag{8.16}$$

Similarly, (8.14) is equivalent to

$$Q_{ij}(D^{-1}\mathcal{X}^s, \mathcal{H}^{s,\theta}) \hookrightarrow H^{s-1,\theta-1} \tag{8.17}$$

$$Q_{ij}(D^{-1}\mathcal{X}^s, \mathcal{H}^{s,\theta}) \hookrightarrow \Lambda_+\Lambda_-^{\frac{1}{2}}\big(\mathcal{L}_t^1(\mathcal{L}_x^{2n})\big). \tag{8.18}$$

Observe that (8.15) follows from Theorem 8.2. Also, (8.17) is the estimate that motivated the introduction of the new space $\mathcal{X}^s$. It therefore remains to prove (8.16) and (8.18).

8.3. Remark. For the system ("MKG") the estimate (8.18) is not needed. In fact, it is clear from the special structure of ("MKG") that if (8.13) and (8.17) are true, then we can iterate in the space

$$\big\{(u,v) : u \in \mathcal{X}^s, v \in \mathcal{H}^{s,\theta}\big\}.$$

See also Remark 8.4 below.

Informal Proof of (8.16)

By Lemma 8.1, (8.16) can be reduced to proving (again we ignore the difference between D^{-1} and Λ^{-1})

$$H^{s-\frac{1}{2},\theta} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{\frac{1}{2}}\Lambda_+\mathcal{L}_t^1(\mathcal{L}_x^{2n}), \tag{8.19}$$

$$H^{s-\frac{1}{2},\theta-\frac{1}{2}} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{\frac{1}{2}}\Lambda_+\Lambda_-^{\frac{1}{2}}\mathcal{L}_t^1(\mathcal{L}_x^{2n}). \tag{8.20}$$

Clearly, $\Lambda^{\frac{1}{2}}\Lambda_+$ may be replaced by $\Lambda^{\frac{3}{2}}$ on the right hand side of both estimates.

First, (8.19) holds by Theorem E (again up to logarithmic divergence).

To prove (8.20), we use the following special case of Proposition 4.8 (valid since $n \geq 4$):

$$\|u\|_{\mathcal{L}_t^1(\mathcal{L}_x^{2n})} \;.\; \big\|\Lambda^{\frac{n-2}{2}}\Lambda_-^{\theta}u\big\|_{\mathcal{L}_t^1(\mathcal{L}_x^2)}. \tag{8.21}$$

Thus, since we are assuming $s = \frac{n-2}{2}$ and $\theta = \frac{1}{2}$, (8.20) reduces to

$$H^{\frac{n-3}{2},0} \cdot H^{\frac{n-3}{2},\theta} \hookrightarrow \Lambda^{\frac{5-n}{2}}\mathcal{L}_t^1(\mathcal{L}_x^2).$$

For simplicity, we consider only the case $n = 4$; the latter estimate then reads

$$H^{\frac{1}{2},0} \cdot H^{\frac{1}{2},\theta} \hookrightarrow \Lambda^{\frac{1}{2}}\mathcal{L}_t^1(\mathcal{L}_x^2). \tag{8.22}$$

Set $I = \big\|\Lambda^{-\frac{1}{2}}(uv)\big\|_{\mathcal{L}_t^1(\mathcal{L}_x^2)}$. We have to prove $I \;.\; \|u\|_{\frac{1}{2},0}\|v\|_{\frac{1}{2},\theta}$. Since the norms involved only depend on the size of the Fourier transform and are compatible with the relation $\preceq$, we may assume that $\widehat{u}, \widehat{v} \geq 0$.

Now write

$$\Lambda^{-\frac{1}{2}}(uv) = \Lambda^{-\frac{1}{2}}\big(u\Lambda^{\frac{1}{2}}\Lambda^{-\frac{1}{2}}v\big), \tag{8.23}$$

and apply the estimate (valid for any $\alpha > 0$)

$$u\Lambda^{\alpha} v \;.\; \Lambda^{\alpha}(uv) + \Lambda^{\alpha} u \cdot v. \tag{8.24}$$

(This holds by the triangle inequality.) Thus, $I \;.\; I_1 + I_2$, where

$$\begin{aligned} I_1 &= \left\| u\Lambda^{-\frac{1}{2}} v \right\|_{\mathcal{L}_t^1(\mathcal{L}_x^2)}, \\ I_2 &= \left\| \Lambda^{-\frac{1}{2}} \left(\Lambda^{\frac{1}{2}} u \cdot \Lambda^{-\frac{1}{2}} v \right) \right\|_{\mathcal{L}_t^1(\mathcal{L}_x^2)}. \end{aligned}$$

For I_1, (4.1) and Hölder's inequality give

$$I_1 \leq \|u\|_{L_t^2(L_x^{\frac{8}{3}})} \left\| \Lambda^{-\frac{1}{2}} v \right\|_{L_t^2(L_x^8)}.$$

Since $n = 4$, we have

$$\|u\|_{L_t^2(L_x^{\frac{8}{3}})} \;.\; \|u\|_{\frac{1}{2},0}$$

by Sobolev embedding, and

$$\left\| \Lambda^{-\frac{1}{2}} v \right\|_{L_t^2(L_x^8)} \;.\; \|v\|_{1,\theta}$$

by the following special case of Theorem D:

$$H^{1,\theta} \hookrightarrow L_t^2(L_x^8) \qquad (n = 4). \tag{8.25}$$

For I_1, (4.1) and Sobolev embedding, followed by Hölder's inequality, gives

$$I_2 \;.\; \left\| \Lambda^{\frac{1}{2}} u \cdot \Lambda^{-\frac{1}{2}} v \right\|_{L_t^1(L_x^{\frac{8}{5}})} \leq \left\| \Lambda^{\frac{1}{2}} u \right\|_{L^2} \left\| \Lambda^{-\frac{1}{2}} v \right\|_{L_t^2(L_x^8)}.$$

And by (8.25) again, the right hand side is $.\; \|u\|_{\frac{1}{2},0} \|v\|_{\frac{1}{2},\theta}$.

This concludes the discussion of (8.16).

8.4. Remark. An inspection of the above arguments reveals that up to this point we could just as well have been working in the space (8.11). For (8.13) still holds if we let $\mathcal{X}^s$ be defined by (8.11), and with essentially the same proof as above (only a few obvious modifications are needed). It is only when we try to prove (8.14) that we run into problems if we choose the space (8.11).

Informal Proof of (8.18)

By Lemma 8.1, (8.18) reduces to

$$\Lambda^{-\frac{1}{2}} \mathcal{X}^s \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{\frac{1}{2}} \mathcal{L}_t^1(\mathcal{L}_x^{2n}), \tag{8.26}$$

$$\Lambda^{-\frac{1}{2}} \Lambda_-^{\frac{1}{2}} \mathcal{X}^s \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{\frac{1}{2}} \Lambda_-^{\frac{1}{2}} \mathcal{L}_t^1(\mathcal{L}_x^{2n}), \tag{8.27}$$

$$\Lambda^{-\frac{1}{2}} \mathcal{X}^s \cdot \mathcal{H}^{s-\frac{1}{2},\theta-\frac{1}{2}} \hookrightarrow \Lambda^{\frac{1}{2}} \Lambda_-^{\frac{1}{2}} \mathcal{L}_t^1(\mathcal{L}_x^{2n}). \tag{8.28}$$

8.5. Remark. It is the estimate (8.27) which necessitates the use of the space (8.12) rather than (8.11) (more precisely, the problem comes up with the estimate (8.30) below, which derives from (8.27)). The other two estimates, (8.26) and (8.28), are easily seen to be true also with $\mathcal{X}^s$ given by (8.11), and with essentially the same proof as below.

For (8.26) it suffices to prove

$$H^{s+\frac{1}{2},\theta} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{\frac{1}{2}} \mathcal{L}_t^1(\mathcal{L}_x^{2n}).$$

Equivalently,

$$\left\| \Lambda^{-\frac{1}{2}} \left(\Lambda^{-\frac{1}{2}} u \cdot \Lambda^{\frac{1}{2}} v \right) \right\|_{\mathcal{L}_t^1(\mathcal{L}_x^{2n})} \cdot \left\| u \right\|_{s,\theta} \left\| v \right\|_{s,\theta} . \tag{8.29}$$

By (8.24),

$$\left\| \Lambda^{-\frac{1}{2}} \left(\Lambda^{-\frac{1}{2}} u \cdot \Lambda^{\frac{1}{2}} v \right) \right\|_{\mathcal{L}_t^1(\mathcal{L}_x^{2n})} \cdot \left\| \Lambda^{-\frac{1}{2}} u \cdot v \right\|_{L_t^1(L_x^{2n})} + \left\| \Lambda^{-\frac{1}{2}} (uv) \right\|_{L_t^1(L_x^{2n})},$$

where we also used (4.1). By Theorem E,

$$\left\| \Lambda^{-\frac{1}{2}} (uv) \right\|_{L_t^1(L_x^{2n})} \cdot \left\| u \right\|_{s,\theta} \left\| v \right\|_{s,\theta} .$$

Take $n = 4$ for simplicity. Then by Hölder's inequality and Sobolev embedding,

$$\left\| \Lambda^{-\frac{1}{2}} u \cdot v \right\|_{L_t^1(L_x^8)} \le \left\| \Lambda^{-\frac{1}{2}} u \right\|_{L_t^2(L_x^\infty)} \left\| v \right\|_{L_t^2(L_x^8)} \cdot \left\| u \right\|_{L_t^2(L_x^8)} \left\| v \right\|_{L_t^2(L_x^8)} .$$

Now apply (8.25). This finishes the proof of (8.26).

By applying (8.21), we reduce (8.27) to

$$\Lambda^{-\frac{1}{2}} \Lambda_-^{\frac{1}{2}} \mathcal{X}^s \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{\frac{3-n}{2}} \mathcal{L}_t^1(\mathcal{L}_x^2).$$

By Lemma 7.4, it suffices to prove (recall that $s = \frac{n-2}{2}$):

$$\Lambda^{\frac{n-4}{2}} \Lambda_-^{\frac{1}{2}} \mathcal{X}^s \cdot H^{\frac{n-3}{2},\theta} \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2), \tag{8.30}$$

$$\Lambda^{-\frac{1}{2}} \Lambda_-^{\frac{1}{2}} \mathcal{X}^s \cdot H^{0,\theta} \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2). \tag{8.31}$$

For simplicity, we take $n = 4$ again. Then (8.30) becomes

$$\Lambda_-^{\frac{1}{2}} \mathcal{X}^s \cdot H^{\frac{1}{2},\theta} \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2),$$

and in view of the definition of $\mathcal{X}^s$, it is enough to prove

$$\mathcal{L}_t^1(\mathcal{L}_x^8) \cdot H^{\frac{1}{2},\theta} \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2).$$

Equivalently,

$$\left\| uv \right\|_{\mathcal{L}_t^1(\mathcal{L}_x^2)} \cdot \left\| u \right\|_{\mathcal{L}_t^1(\mathcal{L}_x^8)} \left\| v \right\|_{\frac{1}{2},\theta} .$$

In view of Lemma 4.4, we may assume that $v \in \mathcal{S}$ and $\widehat{v} \geq 0$. Therefore, by Proposition 4.3,

$$\|uv\|_{\mathcal{L}_t^1(\mathcal{L}_x^2)} \leq \|u\|_{\mathcal{L}_t^1(\mathcal{L}_x^8)} \|v\|_{L_t^\infty(L_x^{\frac{8}{3}})},$$

and by (6.11) (since $n = 4$),

$$\|v\|_{L_t^\infty(L_x^{\frac{8}{3}})} \lesssim \|v\|_{\frac{1}{2},\theta}.$$

This proves (8.30). As for (8.31), it is enough to prove

$$\Lambda^{-\frac{1}{2}}\mathcal{L}_t^1(\mathcal{L}_x^8) \cdot H^{0,\theta} \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2).$$

Reasoning as above, we have

$$\|uv\|_{\mathcal{L}_t^1(\mathcal{L}_x^2)} \leq \|u\|_{\mathcal{L}_t^1(\mathcal{L}_x^\infty)} \|v\|_{L_t^\infty(L_x^2)}.$$

By Sobolev embedding, or more accurately by (6.13),

$$\|u\|_{\mathcal{L}_t^1(\mathcal{L}_x^\infty)} \lesssim \left\|\Lambda^{\frac{1}{2}}u\right\|_{\mathcal{L}_t^1(\mathcal{L}_x^8)}.$$

By the energy embedding (6.1),

$$\|v\|_{L_t^\infty(L_x^2)} \lesssim \|v\|_{0,\theta}.$$

This finishes the proof of (8.27).

As in the proof of (8.27), by applying (8.21) followed by Lemma 7.4, (8.28) reduces to proving (recall that $s = \frac{n-2}{2}$ and $\theta = \frac{1}{2}$)

$$\Lambda^{\frac{n-4}{2}}\mathcal{X}^s \cdot H^{\frac{n-3}{2},0} \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2), \tag{8.32}$$

$$\Lambda^{-\frac{1}{2}}\mathcal{X}^s \cdot L^2 \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2). \tag{8.33}$$

Again we take $n = 4$. For (8.32) it is then enough to prove

$$H^{1,\theta} \cdot H^{\frac{1}{2},0} \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2).$$

In view of (4.1), we may replace $\mathcal{L}_t^1(\mathcal{L}_x^2)$ on the right by $L_t^1(L_x^2)$, and by Hölder's inequality,

$$\|uv\|_{L_t^1(L_x^2)} \leq \|u\|_{L_t^2(L_x^8)} \|v\|_{L_t^2(L_x^{\frac{8}{3}})}$$

To the first factor on the right we apply (8.25), to the second factor we apply Sobolev embedding.

For (8.33) it is enough to prove

$$H^{\frac{3}{2},\theta} \cdot L^2 \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2).$$

As above, we simply note that by Hölder,

$$\|uv\|_{L_t^1(L_x^2)} \leq \|u\|_{L_t^2(L_x^\infty)} \|v\|_{L^2}.$$

Apply (6.7) (with $n = 4$) to the first factor on the right. This finishes the proof of (8.28).

8.2 Proof of Main Theorem, part (b)

We shall prove the following.

8.6. Theorem. *Let $n \geq 4$, $s > \frac{n-2}{2}$. Assume that θ and ε satisfy*

$$\frac{1}{2} < \theta \leq \min\left(\frac{3}{4}, \frac{1}{2} + s - \frac{n-2}{2}\right),$$

$$0 \leq \varepsilon \leq \frac{1}{8}\min\left(\frac{3}{2} - 2\theta, s - \frac{n-2}{2} + 1 - 2\theta\right),$$

and let $\gamma = \theta - \frac{1}{2} + 3\varepsilon$. Let $\mathcal{X}^s$ be the Banach space given by the norm

$$\|u\| = |u|_{s,\theta} + \left\|\Lambda^\gamma \Lambda_-^{\frac{1}{2}} u\right\|_{\mathcal{L}_t^1(\mathcal{L}_x^{2n})}.$$

Then

$$D^{-1}Q_{ij}\big(\mathcal{H}^{s,\theta}, \mathcal{H}^{s,\theta}\big) \hookrightarrow \Lambda_+\Lambda_-^{1-\varepsilon}\mathcal{X}^s, \tag{8.34}$$

$$Q_{ij}\big(D^{-1}\mathcal{X}^s, \mathcal{H}^{s,\theta}\big) \hookrightarrow \Lambda_+\Lambda_-^{1-\varepsilon}\mathcal{X}^s. \tag{8.35}$$

This implies part (b) of the Main Theorem, in view of Theorem 5.4, since conditions (a)–(e) of the latter are satisfied by the space $\mathcal{X}^s$ (condition (a) holds by Proposition 3.12; condition (b) is obviously satisfied; conditions (c) and (d) follow from Theorem 5.5, in view of Proposition 5.6; finally, condition (e) holds by Proposition 5.7).

By the definition of $\mathcal{X}^s$, (8.34) is equivalent to two estimates:

$$D^{-1}Q_{ij}\big(\mathcal{H}^{s,\theta}, \mathcal{H}^{s,\theta}\big) \hookrightarrow H^{s-1,\theta+\varepsilon-1},$$

$$D^{-1}Q_{ij}\big(\mathcal{H}^{s,\theta}, \mathcal{H}^{s,\theta}\big) \hookrightarrow \Lambda^{-\gamma}\Lambda_+\Lambda_-^{\frac{1}{2}-\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^{2n}),$$

and (8.35) is equivalent to

$$Q_{ij}\big(D^{-1}\mathcal{X}^s, \mathcal{H}^{s,\theta}\big) \hookrightarrow H^{s-1,\theta+\varepsilon-1},$$

$$Q_{ij}\big(D^{-1}\mathcal{X}^s, \mathcal{H}^{s,\theta}\big) \hookrightarrow \Lambda^{-\gamma}\Lambda_+\Lambda_-^{\frac{1}{2}-\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^{2n}).$$

We split these four estimates into what we call high and low frequency cases. The high frequency estimates are the ones obtained by replacing D^{-1} by Λ^{-1}:

$$\Lambda^{-1}Q_{ij}\big(\mathcal{H}^{s,\theta}, \mathcal{H}^{s,\theta}\big) \hookrightarrow H^{s-1,\theta+\varepsilon-1}, \tag{8.36}$$

$$\Lambda^{-1}Q_{ij}\big(\mathcal{H}^{s,\theta}, \mathcal{H}^{s,\theta}\big) \hookrightarrow \Lambda^{-\gamma}\Lambda_+\Lambda_-^{\frac{1}{2}-\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^{2n}), \tag{8.37}$$

$$Q_{ij}\big(\Lambda^{-1}\mathcal{X}^s, \mathcal{H}^{s,\theta}\big) \hookrightarrow H^{s-1,\theta+\varepsilon-1}, \tag{8.38}$$

$$Q_{ij}\big(\Lambda^{-1}\mathcal{X}^s, \mathcal{H}^{s,\theta}\big) \hookrightarrow \Lambda^{-\gamma}\Lambda_+\Lambda_-^{\frac{1}{2}-\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^{2n}). \tag{8.39}$$

In the low frequency estimates, D^{-1} is replaced by $\Lambda^{-M} D^{-1}$, where $M > 0$ can be chosen arbitrarily large. In view of the trivial estimates[9]

$$Q_{ij}(\phi, \psi) \precsim D(D^{\frac{1}{2}} \phi' D^{\frac{1}{2}} \psi'),$$
$$Q_{ij}(\phi, \psi) \precsim D^{\frac{1}{2}}(D\phi' D^{\frac{1}{2}} \psi'),$$

where $\phi \preceq \phi'$ and $\psi \preceq \psi'$, the low frequency estimates reduce to

$$H^{s-\frac{1}{2},\theta} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow H^{-M,0}, \tag{8.40}$$
$$H^{s-\frac{1}{2},\theta} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{-M} \mathcal{L}_t^1(\mathcal{L}_x^{2n}), \tag{8.41}$$
$$H^{M,\theta} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow H^{s-\frac{1}{2},0}, \tag{8.42}$$
$$H^{M,\theta} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{1-\gamma} \mathcal{L}_t^1(\mathcal{L}_x^{2n}), \tag{8.43}$$

where $M > 0$ can be taken arbitrarily large.

The estimates (8.41) and (8.43) hold by Theorem E, while (8.40) and (8.42) are special cases of the following theorem, which is essentially a corollary of Theorem F (see the Appendix for the proof).

8.7. Theorem. *Let $n \geq 4$ and $\theta > \frac{1}{2}$. Then*

$$H^{a,\theta} \cdot H^{b,\theta} \hookrightarrow H^{-c,0}.$$

for all a, b, c satisfying

$$a, b \geq -c,$$
$$a + b \geq \frac{1}{2},$$
$$a + b + c \geq \frac{n-1}{2}.$$

Let us now turn to the proofs of (8.36)–(8.39). By Lemma 8.1, (8.36) reduces to

$$\mathcal{H}^{s-\frac{1}{2},\theta} \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow H^{s-\frac{3}{2},\theta+\varepsilon-\frac{1}{2}}, \tag{8.44}$$
$$\mathcal{H}^{s-\frac{1}{2},\theta-\frac{1}{2}} \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow H^{s-\frac{3}{2},\theta+\varepsilon-1}; \tag{8.45}$$

(8.37) reduces to

$$\mathcal{H}^{s-\frac{1}{2},\theta} \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{\frac{1}{2}-\gamma} \Lambda_+ \Lambda_-^{-\varepsilon} \mathcal{L}_t^1(\mathcal{L}_x^{2n}), \tag{8.46}$$
$$\mathcal{H}^{s-\frac{1}{2},\theta-\frac{1}{2}} \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{\frac{1}{2}-\gamma} \Lambda_+ \Lambda_-^{\frac{1}{2}-\varepsilon} \mathcal{L}_t^1(\mathcal{L}_x^{2n}); \tag{8.47}$$

[9] These follow from the fact that the symbol of Q_{ij} is bounded in absolute value by $|\xi \wedge \eta|$, where $\xi \wedge \eta$ is the exterior product of vectors in $\mathbb{R}^n$. We have $|\xi \wedge \eta| \leq |\xi| \, |\eta|$, and since $\xi \wedge \eta = \xi \wedge (\xi + \eta) = (\xi + \eta) \wedge \eta$, we also have $|\xi \wedge \eta| \leq |\xi| \, |\xi + \eta| \, , |\xi + \eta| \, |\eta|$. By combining these we get the desired estimates.

(8.38) reduces to

$$\mathcal{H}^{s+\frac{1}{2},\theta} \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow H^{s-\frac{1}{2},\theta+\varepsilon-\frac{1}{2}}, \tag{8.48}$$

$$\Lambda^{-\frac{1}{2}}\Lambda_-^{\frac{1}{2}}\mathcal{X}^s \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow H^{s-\frac{1}{2},\theta+\varepsilon-1}, \tag{8.49}$$

$$\mathcal{H}^{s+\frac{1}{2},\theta} \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow H^{s-\frac{1}{2},\theta+\varepsilon-1}; \tag{8.50}$$

and (8.39) reduces to

$$\mathcal{H}^{s+\frac{1}{2},\theta} \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{-\frac{1}{2}-\gamma}\Lambda_+\Lambda_-^{-\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^{2n}), \tag{8.51}$$

$$\Lambda^{-\frac{1}{2}}\Lambda_-^{\frac{1}{2}}\mathcal{X}^s \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{-\frac{1}{2}-\gamma}\Lambda_+\Lambda_-^{\frac{1}{2}-\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^{2n}), \tag{8.52}$$

$$\mathcal{H}^{s+\frac{1}{2},\theta} \cdot \mathcal{H}^{s-\frac{1}{2},\theta-\frac{1}{2}} \hookrightarrow \Lambda^{-\frac{1}{2}-\gamma}\Lambda_+\Lambda_-^{\frac{1}{2}-\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^{2n}). \tag{8.53}$$

For the proofs, we need a few technical lemmas.

8.8. Lemma. *Let $\alpha > 0$. Then*

$$\Lambda_-^\alpha(uv) \text{ - } \Lambda_+^\alpha u \Lambda_+^\alpha v$$

for all u and v with $\widehat{u}, \widehat{v} \geq 0$.

The trivial proof is omitted.

8.9. Lemma. *Let $\alpha, \beta \geq 0$. Then*

$$\Lambda_-^\alpha(uv) \text{ - } (\Lambda_+^\alpha u)v + (\Lambda^{-\beta}u)\Lambda_-^{\alpha+\beta}v$$

for all u and v with $\widehat{u}, \widehat{v} \geq 0$.

Proof. Since $R^\alpha(u,v)$ - $(D^\alpha u)v$ by the triangle inequality, Lemma 3.3 implies

$$\Lambda_-^\alpha(uv) \text{ - } (\Lambda_-^\alpha u)v + u\Lambda_-^\alpha v + (D^\alpha u)v.$$

To finish the proof, combine this with

$$u\Lambda_-^\alpha v \text{ - } (\Lambda^\alpha u)v + (\Lambda^{-\beta}u)\Lambda_-^{\alpha+\beta}v.$$

The latter is proved by considering two cases: $\big||\lambda|-|\eta|\big| \leq |\xi|$ and $\big||\lambda|-|\eta|\big| > |\xi|$, where (τ, ξ) and (λ, η) are the frequencies of u and v respectively. □

8.10. Lemma. *Let $\alpha, \beta \geq 0$. Then*

(a) $\Lambda_-^{-\beta}(uv)$ - $\Lambda_-^{-\alpha-\beta}(\Lambda_+^\alpha u \Lambda_+^\alpha v)$,

(b) $\Lambda_-^{-\beta}(uv)$ - $\Lambda_-^{-\alpha-\beta}(u\Lambda^\alpha v) + u\Lambda^{-\beta}v$,

for all u and v with $\widehat{u}, \widehat{v} \geq 0$.

Proof. Part (a) follows from Lemma 8.8. Part (b) is proved by considering two cases: $\big||\tau+\lambda| - |\xi+\eta|\big| \leq |\eta|$ and $\big||\tau+\lambda| - |\xi+\eta|\big| > |\eta|$, where (τ, ξ) and (λ, η) are the frequencies of u and v respectively. □

Proof of (8.44). Set $\delta = \theta + \varepsilon - 1/2$. In view of the hypotheses of Theorem 8.6,

$$\delta \le \frac{1}{2} \min\left(\frac{1}{2}, s - \frac{n-2}{2}\right).$$

By Lemma 8.8, it suffices to prove

$$H^{s-\frac{1}{2}-\delta,\theta} \cdot H^{s-\frac{1}{2}-\delta,\theta} \hookrightarrow H^{s-\frac{3}{2},0}.$$

This estimate holds by Theorem 8.7.

Proof of (8.45). Set $\zeta = \theta + 2\varepsilon - 1/2$. In view of the hypotheses of Theorem 8.6,

$$\zeta \le \frac{1}{2} \min\left(\frac{1}{2}, s - \frac{n-2}{2}\right).$$

By Lemma 8.10(a), it suffices to prove

$$H^{s-\frac{1}{2}-\zeta,\theta-\frac{1}{2}} \cdot H^{s-\frac{1}{2}-\zeta,\theta} \hookrightarrow H^{s-\frac{3}{2},-\frac{1}{2}-\varepsilon}.$$

Since $\theta > \frac{1}{2}$, this estimate is weaker than

$$H^{s-\frac{1}{2}-\zeta,0} \cdot H^{s-\frac{1}{2}-\zeta,\theta} \hookrightarrow H^{s-\frac{3}{2},-\frac{1}{2}-\varepsilon},$$

which by duality is equivalent to

$$H^{\frac{3}{2}-s,\frac{1}{2}+\varepsilon} \cdot H^{s-\frac{1}{2}-\zeta,\theta} \hookrightarrow H^{-s+\frac{1}{2}+\zeta,0}.$$

The latter holds by Theorem 8.7.

Proof of (8.48). Again we let $\delta = \theta + \varepsilon - 1/2$. By Lemma 7.4, it suffices to prove:

$$\mathcal{H}^{1,\theta} \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow H^{0,\theta+\varepsilon-\frac{1}{2}}, \tag{8.54}$$

$$\mathcal{H}^{s+\frac{1}{2},\theta} \cdot \mathcal{H}^{0,\theta} \hookrightarrow H^{0,\theta+\varepsilon-\frac{1}{2}}. \tag{8.55}$$

By Lemma 8.8, (8.54) reduces to

$$H^{1-\delta,\theta} \cdot H^{s-\frac{1}{2}-\delta,\theta} \hookrightarrow L^2,$$

which holds by Theorem 8.7. By Lemma 8.9, (8.55) reduces to

$$H^{s+\frac{1}{2}-\delta,\theta} \cdot H^{0,\theta} \hookrightarrow L^2,$$
$$H^{s+\frac{1}{2}+\theta-\delta,\theta} \cdot L^2 \hookrightarrow L^2.$$

The former holds by Theorem 8.7, the latter by the embedding (6.2).

Proof of (8.50). Set $\zeta = \theta + 2\varepsilon - 1/2$. By Lemma 7.4, it suffices to prove:

$$\mathcal{H}^{1,\theta} \cdot \mathcal{H}^{s-\frac{1}{2},\theta-\frac{1}{2}} \hookrightarrow H^{0,\theta+\varepsilon-1}, \tag{8.56}$$

$$\mathcal{H}^{s+\frac{1}{2},\theta} \cdot \mathcal{H}^{0,\theta-\frac{1}{2}} \hookrightarrow H^{0,\theta+\varepsilon-1}. \tag{8.57}$$

By Lemma 8.10(a), (8.56) reduces to

$$H^{1-\zeta,\theta} \cdot H^{s-\frac{1}{2}-\zeta,0} \hookrightarrow H^{0,-\frac{1}{2}-\varepsilon}$$

which by duality is equivalent to

$$H^{1-\zeta,\theta} \cdot H^{0,\frac{1}{2}+\varepsilon} \hookrightarrow H^{-s+\frac{1}{2}+\zeta,0}.$$

This estimate holds by Theorem 8.7. By Lemma 8.10(b), (8.57) reduces to

$$H^{s+\frac{1}{2}-\zeta,\theta} \cdot L^2 \hookrightarrow H^{0,-\frac{1}{2}-\varepsilon},$$
$$H^{s+\frac{3}{2}-\theta-\varepsilon,\theta} \cdot L^2 \hookrightarrow L^2.$$

The former holds by Theorem 8.7, the latter by the embedding (6.2).

Proof of (8.46). This reduces to

$$H^{s-\frac{1}{2},\theta} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow D^{\frac{3}{2}-\gamma-\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^{2n}),$$

and in view of (4.1) it suffices to prove

$$H^{s-\frac{1}{2},\theta} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow D^{\frac{3}{2}-\gamma-\varepsilon}L_t^1(L_x^{2n}).$$

The last estimate holds by Theorem E (since $n \geq 4$).

Proof of (8.47). Since $n \geq 4$, we may apply Proposition 4.8. Thus, it suffices to prove

$$\mathcal{H}^{s-\frac{1}{2},\theta-\frac{1}{2}} \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{\frac{3}{2}-\frac{n}{2}-\gamma}\Lambda_+\Lambda_-^{-2\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^2).$$

Replace $\Lambda_+\Lambda_-^{-2\varepsilon}$ on the right hand side by $\Lambda^{1-2\varepsilon}$ and apply Lemma 7.4, thereby reducing to (since $s > \frac{n-2}{2} + \gamma + 2\varepsilon$)

$$H^{\frac{n-3}{2},0} \cdot H^{\frac{n-3}{2},\theta} \hookrightarrow \Lambda^{\frac{5}{2}-\frac{n}{2}}\mathcal{L}_t^1(\mathcal{L}_x^2). \tag{8.58}$$

We consider the cases $n = 4$ and $n \geq 5$ separately.

If $n = 4$, (8.58) is just (8.22), and in view of (8.23) and (8.24) (with $\alpha = \frac{1}{2}$), it suffices to prove:

$$H^{\frac{1}{2},0} \cdot H^{1,\theta} \hookrightarrow L_t^1(L_x^2), \tag{8.59}$$

$$L^2 \cdot H^{1,\theta} \hookrightarrow \Lambda^{\frac{1}{2}}L_t^1(L_x^2). \tag{8.60}$$

Here we also used (4.1). By Hölder's inequality, (8.59) reduces to

$$H^{\frac{1}{2},0} \hookrightarrow L_t^2(L_x^{\frac{8}{3}}),$$
$$H^{1,\theta} \hookrightarrow L_t^2(L_x^8).$$

These are just (6.9) and (6.5) in dimension $n = 4$. By Sobolev embedding we reduce (8.60) to

$$L^2 \cdot H^{1,\theta} \hookrightarrow L_t^1(L_x^{\frac{8}{5}}).$$

But this holds by (6.5).

Now assume $n \geq 5$. By Lemma 7.4, (8.66) reduces to

$$H^{1,0} \cdot H^{\frac{n-3}{2},\theta} \hookrightarrow L_t^1(L_x^2), \tag{8.61}$$
$$H^{\frac{n-3}{2},0} \cdot H^{1,\theta} \hookrightarrow L_t^1(L_x^2).$$

Using Hölder's inequality, these follow from (6.4), (6.10), (6.9) and (6.5).

Proof of (8.49). By Lemma 7.4, it suffices to prove

$$\Lambda^{s-1}\Lambda_-^{\frac{1}{2}}\mathcal{X}^s \cdot \mathcal{H}^{s-\frac{1}{2},\theta} \hookrightarrow H^{0,\theta+\varepsilon-1}, \tag{8.62}$$

$$\Lambda^{-\frac{1}{2}}\Lambda_-^{\frac{1}{2}}\mathcal{X}^s \cdot \mathcal{H}^{0,\theta} \hookrightarrow H^{0,\theta+\varepsilon-1}. \tag{8.63}$$

Set $\zeta = \theta + 2\varepsilon - 1/2$. By Lemma 8.10(a), (8.62) reduces to

$$H^{1-\zeta,0} \cdot H^{s-\frac{1}{2}-\zeta,\theta} \hookrightarrow H^{0,-\frac{1}{2}-\varepsilon},$$

which by duality is equivalent to

$$H^{0,\frac{1}{2}+\varepsilon} \cdot H^{s-\frac{1}{2}-\zeta,\theta} \hookrightarrow H^{-1+\zeta,0}.$$

The latter holds by Theorem 8.7.

By Lemma 8.10(b), (8.63) reduces to two estimates:

$$\Lambda^{\theta+\varepsilon-\frac{3}{2}}\Lambda_-^{\frac{1}{2}}\mathcal{X}^s \cdot H^{0,\theta} \hookrightarrow L^2, \tag{8.64}$$

$$\Lambda^{\zeta-\frac{1}{2}}\Lambda_-^{\frac{1}{2}}\mathcal{X}^s \cdot H^{0,\theta} \hookrightarrow H^{0,-\frac{1}{2}-\varepsilon}. \tag{8.65}$$

(Recall that $\zeta = \theta + 2\varepsilon - 1/2$.) In view of (6.3) and (6.1),

$$H^{s+\frac{3}{2}-\theta-\varepsilon,0} \cdot H^{0,\theta} \hookrightarrow L^2,$$

which implies (8.64). Since $\zeta + \varepsilon = \gamma$, (8.65) follows from (6.15).

Proof of (8.51). This reduces to

$$H^{s+\frac{1}{2},\theta} \cdot H^{s-\frac{1}{2},\theta} \hookrightarrow \Lambda^{\frac{1}{2}-\gamma-\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^{2n}).$$

In view of (8.23) and (8.24) (with $\alpha = \frac{1}{2} - \gamma - \varepsilon$), this reduces to

$$H^{s+\frac{1}{2},\theta} \cdot H^{s-\gamma-\varepsilon,\theta} \hookrightarrow L_t^1(L_x^{2n}),$$
$$H^{s+\gamma+\varepsilon,\theta} \cdot H^{s-\gamma-\varepsilon,\theta} \hookrightarrow \Lambda^{\frac{1}{2}-\gamma-\varepsilon}L_t^1(L_x^{2n}).$$

The former holds by (6.7) and (6.8), the latter holds by Theorem E.

Proof of (8.52). By Lemma 8.10(b), this reduces to

$$\Lambda^{-\frac{1}{2}}\Lambda_-^{\frac{1}{2}}\mathcal{X}^s \cdot H^{s-\frac{1}{2}-2\varepsilon,\theta} \hookrightarrow \Lambda^{-\frac{1}{2}-\gamma}\Lambda_+\Lambda_-^{\frac{1}{2}+\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^{2n}), \tag{8.66}$$
$$\Lambda^{-\frac{1}{2}}\Lambda_-^{\frac{1}{2}}\mathcal{X}^s \cdot H^{s-\varepsilon,\theta} \hookrightarrow \Lambda^{-\frac{1}{2}-\gamma}\Lambda_+\mathcal{L}_t^1(\mathcal{L}_x^{2n}). \tag{8.67}$$

By Proposition 4.8, (8.66) reduces to

$$\Lambda^{-\frac{1}{2}}\Lambda_-^{\frac{1}{2}}\mathcal{X}^s \cdot H^{s-\frac{1}{2}-2\varepsilon,\theta} \hookrightarrow \Lambda^{\frac{3-n}{2}-\gamma}\mathcal{L}_t^1(\mathcal{L}_x^2).$$

By Lemma 7.4, the latter reduces to two estimates:

$$\Lambda^{\frac{n-4}{2}+\gamma}\Lambda_-^{\frac{1}{2}}\mathcal{X}^s \cdot H^{\frac{n-3}{2},\theta} \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2). \tag{8.68}$$
$$\Lambda^{-\frac{1}{2}}\Lambda_-^{\frac{1}{2}}\mathcal{X}^s \cdot H^{0,\theta} \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2). \tag{8.69}$$

If $n = 4$, then (8.68) follows from

$$\mathcal{L}_t^1(\mathcal{L}_x^8) \cdot H^{\frac{1}{2},\theta} \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2),$$

which holds by Proposition 4.3, Lemma 4.4 and (6.11). If $n \geq 5$, then (8.68) reduces to (8.61).

In view of (6.1), (8.69) reduces to

$$\Lambda^{-\frac{1}{2}-\gamma}\mathcal{L}_t^1(\mathcal{L}_x^{2n}) \cdot L_t^\infty(L_x^2) \hookrightarrow \mathcal{L}_t^1(\mathcal{L}_x^2),$$

which holds by Proposition 4.3, Lemma 4.4 and (6.13) (since $\gamma > 0$).

By Lemma 7.4, (8.67) follows from

$$H^{\frac{n-1}{2},0} \cdot H^{\frac{n-2}{2},\theta} \hookrightarrow D^{\frac{1}{2}}L_t^1(L_x^{2n}).$$

By Sobolev embedding, $L_t^1(L_x^n) \hookrightarrow D^{\frac{1}{2}}L_t^1(L_x^{2n})$, so it suffices to have

$$H^{\frac{n-1}{2},0} \cdot H^{\frac{n-2}{2},\theta} \hookrightarrow L_t^1(L_x^n).$$

This follows from (6.6) and (6.8).

Proof of (8.53). By Proposition 4.8, this reduces to

$$\Lambda^{-\frac{1}{2}}\mathcal{X}^s \cdot \mathcal{H}^{s-\frac{1}{2},\theta-\frac{1}{2}} \hookrightarrow \Lambda^{\frac{3-n}{2}-\gamma-2\varepsilon}\mathcal{L}_t^1(\mathcal{L}_x^2).$$

By Lemma 7.4, it suffices to prove

$$H^{1,\theta} \cdot H^{\frac{n-3}{2},0} \hookrightarrow L_t^1(L_x^2).$$
$$H^{\frac{n-1}{2}+\varepsilon,\theta} \cdot L^2 \hookrightarrow L_t^1(L_x^2).$$

The first of these holds by (6.5) and (6.9). The second follows from (6.7).

9 Main Estimates for (WMM)

Here we prove the following.

9.1. Theorem. *Let $n \geq 3$, $s > \frac{n-2}{2}$. Assume that θ, ε and q satisfy*

$$\frac{1}{2} < \theta < \min\left(1, \frac{1}{2} + \frac{1}{2}\left[s - \frac{n-2}{2}\right]\right),$$
$$0 \leq \varepsilon < \min\left(1-\theta, \theta - \frac{1}{2}, \frac{1}{2}\left[s - \frac{n-2}{2} + 1 - 2\theta\right]\right),$$
$$\frac{1}{2} \leq \frac{1}{q} < \frac{3}{2} - \theta - \varepsilon.$$

Let $\mathcal{X}^s$ be the Banach space given by the norm

$$\|u\| = |u|_{s,\theta} + \left\|\Lambda^{-1}\Lambda_- u\right\|_{\mathcal{L}_t^q(\mathcal{L}_x^\infty)},$$

then

$$\widetilde{Q}(\mathcal{X}^s, \mathcal{X}^s) \hookrightarrow \Lambda_+\Lambda_-^{1-\varepsilon}\mathcal{X}^s, \tag{9.1}$$

where $\widetilde{Q}$ is the null form appearing in (WMM).

This implies part (c) of the Main Theorem, in view of Theorem 5.4, since conditions (a)–(e) of the latter are satisfied by the space $\mathcal{X}^s$ (condition (a) holds by Proposition 3.12; condition (b) is obviously satisfied; conditions (c) and (d) follow from Theorem 5.5, in view of Proposition 5.6; finally, condition (e) holds by Proposition 5.7).

By the definition of $\mathcal{X}^s$, (9.1) is equivalent to two estimates:

$$\Lambda^{-1}\widetilde{Q}(\mathcal{X}^s, \mathcal{X}^s) \hookrightarrow H^{s,\theta+\varepsilon-1}, \tag{9.2}$$

$$\Lambda^{-1}\widetilde{Q}(\mathcal{X}^s, \mathcal{X}^s) \hookrightarrow \Lambda_+\Lambda_-^{-\varepsilon}\mathcal{L}_t^q(\mathcal{L}_x^\infty). \tag{9.3}$$

The latter can be proved without using the null structure of $\widetilde{Q}$. In fact, we shall rely on the following crude estimate:

$$\Lambda^{-1}\widetilde{Q}(\phi,\psi) \precsim D^{-1}\Lambda_+\phi' D^{-1}\Lambda_+\psi' \quad \text{whenever} \quad \phi \preceq \phi', \, \psi \preceq \psi'.$$

(The trivial proof of this is omitted.) Thus, (9.3) reduces to

$$D^{-1}H^{s-1,\theta} \cdot D^{-1}H^{s-1,\theta} \hookrightarrow \Lambda^{1-\varepsilon}\mathcal{L}_t^q(\mathcal{L}_x^\infty).$$

By Sobolev embedding, this reduces to

$$D^{-1}H^{s-1,\theta} \cdot D^{-1}H^{s-1,\theta} \hookrightarrow \Lambda^{1-\frac{n}{r}-2\varepsilon}\mathcal{L}_t^q(\mathcal{L}_x^r)$$

for any $2 \le r < \infty$. The latter holds by Theorem E, if we take r so large (to ensure that $(2q, 2r)$ is wave admissible) that

$$\frac{1}{q} \le \frac{n-1}{2} - \frac{n-1}{2r}.$$

(We can do this since $q > 1$ and $n \ge 3$.)

To prove (9.2) we need to take into account the null structure. In fact, proving (9.2) can be reduced to proving four estimates:

$$D^{-1}H^{M,\theta} \cdot H^{0,\theta} \hookrightarrow H^{0,\theta+\varepsilon-1}, \tag{9.4}$$

$$\mathcal{H}^{1,\theta-1} \cdot H^{s,\theta} \hookrightarrow H^{0,\theta+\varepsilon-1}, \tag{9.5}$$

$$\mathcal{L}_t^q(\mathcal{L}_x^\infty) \cdot H^{0,\theta} \hookrightarrow H^{0,\theta+\varepsilon-1}, \tag{9.6}$$

$$R(H^{s+1,\theta}, H^{0,\theta}) \hookrightarrow H^{0,\theta+\varepsilon-1}. \tag{9.7}$$

Here $M > 0$ can be taken arbitrarily large ((9.4) is a low frequency estimate which comes up because we want to replace D^{-1} by Λ^{-1} in certain places). The important estimates are (9.5)–(9.7).

We will need the following theorem (essentially a corollary of Theorem F; see the Appendix for the proof).

9.2. Theorem. *Let $n \ge 3$ and $\theta > \frac{1}{2}$. Then*

$$H^{a,\theta} \cdot H^{b,\theta} \hookrightarrow H^{-c,\theta}$$

holds for all a, b, c satisfying

$$\begin{aligned} a, b, c &\ge 0, \\ c &< \frac{n-1}{2}, \\ a + b + c &\ge \frac{n-1}{2} + \theta. \end{aligned}$$

The basic estimate for the null form $\widetilde{Q}$ is as follows.

9.3. Lemma. *The estimate*

$$D^{-1}\widetilde{Q}(\phi, \psi) \precsim D^{-1}D_-\phi' \cdot \psi' + R(D^{-1}\phi', \psi') + \textit{symmetric terms}$$

holds whenever $\phi \preceq \phi'$ and $\psi \preceq \psi'$.

The proof can be found in [22, Lemmas 2.3 and 2.4]. We shall also make use of the following:

9.4. Lemma. *Let* $\gamma, M \geq 0$. *Then*

$$\Lambda^{\gamma-1}\widetilde{Q}(\phi,\psi) \text{ - } \Lambda^{-M}D^{-1}\Lambda_+\phi' \cdot \Lambda^\gamma\psi' + \Lambda^{\gamma-1}\Lambda_-\phi' \cdot \psi' \\ + \Lambda^{-1}\Lambda_-\phi' \cdot \Lambda^\gamma\psi' + R(\Lambda^{-1}\phi', \Lambda^\gamma\psi') + \textit{symmetric terms}$$

whenever $\phi \preceq \phi'$ *and* $\psi \preceq \psi'$.

Proof. By Lemma 9.3,

$$\Lambda^{-1}\widetilde{Q}(\phi,\psi) \text{ - } D^{-1}\Lambda_-\phi' \cdot \psi' + R(D^{-1}\phi', \psi') + \text{symmetric terms.}$$

Since $D^{-1}u \mathbin{.} \Lambda^{-1}u + D^{-1}\Lambda^{-M}u$ whenever $\widehat{u} \geq 0$, we conclude that

$$\Lambda^{-1}\widetilde{Q}(\phi,\psi) \text{ - } \Lambda^{-M}D^{-1}\Lambda_+\phi' \cdot \psi' + R(\Lambda^{-M}D^{-1}\phi', \psi') \\ + \Lambda^{-1}\Lambda_-\phi' \cdot \psi' + R(\Lambda^{-1}\phi', \psi') + \text{symmetric terms.}$$

Since $R(u,v) \text{ - } (Du)v$ whenever $\widehat{u}, \widehat{v} \geq 0$, the second term on the right hand side is subsumed in the first. It is easy to see that

$$\Lambda^\gamma R(\Lambda^{-1}u, v) \text{ - } R(\Lambda^\gamma u, \Lambda^{-1}v) + R(\Lambda^{-1}u, \Lambda^\gamma v)$$

provided $\widehat{u}, \widehat{v} \geq 0$. Combining this with Lemma 7.4 yields the desired estimate. □

We shall also need the following estimate for the operator R.

9.5. Lemma. *Let* $\alpha \in [0,1]$, $\delta \geq 0$. *Then*

$$R(\phi,\psi) \text{ - } \Lambda_-^{1-\alpha}R^\alpha(\phi', \psi') + \Lambda_-\phi' \cdot \psi' + \Lambda_-^{-\delta}(\Lambda^\delta\phi' \cdot \Lambda_-\psi')$$

whenever $\phi \preceq \phi'$ *and* $\psi \preceq \psi'$.

Proof. It is readily verified that the symbol r of R satisfies

$$r(\tau,\xi;\lambda,\eta) \leq A + B + C,$$

where $A = \big||\tau+\lambda| - |\xi+\eta|\big|$, $B = \big||\tau| - |\xi|\big|$ and $C = \big||\lambda| - |\eta|\big|$. We consider three cases, corresponding to A, B or C being the maximum of the three.

If A is the maximum, then $r \mathbin{.} A^{1-\alpha}r^\alpha$.

If B is the maximum, then $r \mathbin{.} B$.

If C is the maximum, we consider two subcases: (i) $|\xi| \geq A$; and (ii) $|\xi| < A$. In case (i), $r \mathbin{.} A^{-\delta}A^\delta C \mathbin{.} A^{-\delta}|\xi|^\delta C$. In case (ii), $r \mathbin{.} |\xi|^{1-\alpha} r^\alpha \mathbin{.} A^{1-\alpha}r^\alpha$. □

Applying Lemma 9.4 to (9.2) (with $\gamma = s$), we see that (9.2) reduces to (9.4)–(9.7).

Proof of (9.4). This is weaker than

$$D^{-1}H^{M,0}\cdot H^{0,\theta}\hookrightarrow L^2.$$

By Hölder's inequality and (6.1), the latter reduces to

$$D^{-1}H^{M,0}\hookrightarrow L_t^2(L_x^\infty),$$

which holds by Sobolev embedding for $M>\frac{n-2}{2}$, since $n\geq 3$.

Proof of (9.5). By Lemma 8.10(b), this reduces to

$$\mathcal{H}^{2-\theta-\varepsilon,\theta-1}\cdot H^{s,\theta}\hookrightarrow L^2, \tag{9.8}$$

$$H^{1-\delta,\theta-1}\cdot H^{s,\theta}\hookrightarrow H^{0,-\frac{1}{2}-\varepsilon}, \tag{9.9}$$

where $\delta=\theta+\varepsilon-\frac{1}{2}$. Observe that (9.8) is weaker than

$$H^{1-\theta-\varepsilon,\theta}\cdot H^{s,\theta}\hookrightarrow L^2,$$

which holds by Theorem F. (9.9) is equivalent to

$$H^{0,\frac{1}{2}+\varepsilon}\cdot H^{s,\theta}\hookrightarrow H^{\delta-1,1-\theta},$$

and the latter holds by Theorem 9.2.

Proof of (9.6). This holds by (6.14).

Proof of (9.7). By Lemma 9.5 (with $\alpha=\theta+\varepsilon$ and $\delta=\theta+\varepsilon-\frac{1}{2}$), this reduces to three estimates, one of which is (9.6); the other two are

$$R^{\theta+\varepsilon}(H^{s+1,\theta},H^{0,\theta})\hookrightarrow L^2,$$

$$H^{s+1-\delta,\theta}\cdot H^{0,\theta-1}\hookrightarrow H^{0,-\frac{1}{2}-\varepsilon}.$$

The former holds by Theorem F, and the latter is equivalent to

$$H^{s+1-\delta,\theta}\cdot H^{0,\frac{1}{2}+\varepsilon}\hookrightarrow H^{0,1-\theta},$$

which holds by Theorem 9.2.

10 Further Results, Open Problems and Historical Remarks

(1) The results discussed in the Main Theorem (see section 1.1) confirm part (i) of the General WP Conjecture (see section 1.3) for the equations that we consider, with two notable exceptions. The first concerns (MKG) and (YM),

as well as the simplified model problems ("MKG") and ("YM"), in dimension $n = 3$. In both cases the critical WP exponent is $s_c = \frac{1}{2}$, and one can prove local well-posedness for $s > s_c + \frac{1}{4}$ using the $H^{s,\theta}$ spaces with $\theta = s$; see [6] and [12]. It is easy to see that while the first iterate, in all the above-mentioned cases, is well-posed for $s > \frac{1}{2}$, it fails to belong to the corresponding $H^{s,\theta}$ space for $\frac{1}{2} < s \leq \frac{3}{4}$, $\theta > \frac{1}{2}$. One can show that any strategy based on norms which depend only on the size of the Fourier transform, such as those used in this survey, is bound to fail to prove well-posedness for s in the range $\frac{1}{2} < s \leq \frac{3}{4}$. Is it possible that well-posedness fails in that range?

The second case in which our present techniques do not allow us to go all the way to the critical exponent is the (WMM) equation in dimension $n = 2$. Even though the system (1.13) is a equivalent, through a simple transformation, to the standard wave maps equation (WM), for which we can prove well-posedness for all $s > s_c(\mathrm{WM}) = 1$ $(n = 2)$, we cannot treat it for s close to the corresponding critical exponent $s_c(\mathrm{WMM}) = 0$ $(n = 2)$. In fact, the best result proved so far for (WMM) in dimension $n = 2$, is that it is well-posed for $s > \frac{1}{4}$; see [33]. In comparison with the higher-dimensional $(n \geq 3)$ case, discussed in section 9, we remark that the estimate (9.2) is true for all $s > 0$ in dimension $n = 2$, whereas the "$L^1_t(L^\infty_x)$" estimate (9.3) fails for $0 < s < \frac{1}{8}$, by a counterexample given in [33]. For a more in-depth discussion of the (WMM) equation, see [33, 35].

(2) Close to nothing is known concerning part (ii) of the General WP Conjecture, except for semilinear scalar wave equations of the type $\Lambda\phi + V'(\phi) = 0$; see [9], [37]. An important advance was made recently by D. Tataru [42, 43] who was able to prove, in the case of (WM) equations, global well-posedness for data in the Besov space $B^{2,1}_{\frac{n}{2}}$, in any dimension $n \geq 2$. It would be interesting to extend Tataru's result to the other cases covered by the Main Theorem. We expect that all classical field theories are globally well-posed (in the strong sense of our Main Theorem) for small data in $B^{2,1}_{s_c}$ with s_c the critical WP exponent. The fundamental problem of well-posedness in H^{s_c} is far more difficult (see the relevant discussion on well-posedness and its connection to the issue of global regularity in [16]). Ultimately the issue of optimal well-posedness must be tied to that of global regularity for all finite energy data, in the case of critical nonlinearities, or that of spontaneous formation of singularities[10] in the case of supercritical equations.

(3) Recently M. Keel and T. Tao were able to prove global existence for the full (MKG) system for arbitrarily large H^s initial data with $\frac{3}{4} < s < 1$. Local well-posedness in the same range was dealt with in [6]. Global H^1 well-posedness for the harder case of the (YM) equations, corresponding to the energy norm, was treated in [20].

(4) The issue of optimal well-posedness for quasilinear wave equations has only very recently started to be investigated. We refer the interested reader to the

[10]See [38] for an up to date survey concerning weak solutions and formation of singularities, as well as known results in the case of equivariant or spherically symmetric wave maps.

works of Chemin-Bahouri [1, 2], Tataru [44, 45] and Klainerman [17]. See also the relevant discussion in [16].

(5) The first improved[11] space-time regularity results for null quadratic forms appear in [18]. Those estimates were used, by virtue of Duhamel's principle, to set up an iteration procedure with respect to a space $\mathcal{X}_T^s$ (see the discussion in section 5) defined by the space-time norm

$$\|u\|_{L_t^\infty([0,T],H^s)} + \|\partial_t u\|_{L_t^\infty([0,T],H^{s-1})} + T^{\frac{1}{2}} \left\|D^{s-1}\Lambda u\right\|_{L^2([0,T]\times\mathbb{R}^n)} \tag{10.1}$$

and derive improved well-posedness results for a general class of nonlinear wave equations verifying the null condition (see [15]), including (WM). The same type of estimates and a similar version of the $\mathcal{X}_T^s$ iteration space were used in [19, 20] to derive global well-posedness results in the energy norm for the full (MKG) and (YM) systems. Observe that the norm (10.1) is essentialy the same as that of the spaces $\mathcal{H}^{s,\theta}$ for $\theta = 1$.

It is noteworthy that in [18] there appear also sharper bilinear estimates corresponding to the homogeneous $H^{s,\theta}$, $\theta = \frac{1}{2}$ spaces (see section 3). These better estimates could not, however, be used in an iterative procedure; $\theta = \frac{1}{2}$ leads to an obvious logarithmic divergence. The use of the $H^{s,\theta}$ spaces, for $\theta > \frac{1}{2}$, was initiated in [21] under the influence of the works of Bourgain [4] and Kenig-Ponce-Vega [14] for dispersive equations. The new idea, provided by these works, was to introduce a time cut-off function which allows one to replace Λ by $\Lambda_+\Lambda_-$. The inhomogeneous $H^{s,\theta}$ for $s > s_c$, $\theta > \frac{1}{2}$, avoids the above logarithmic divergences and allowed one to prove a well-posedness result for $s > s_c$, in the case of the (WM) system, for $n \geq 3$. The case $n = 2$ was treated later in [26] with the help of the new bilinear estimates proved in [23]. The (WM) system is the only one for which the $\mathcal{H}^{s,\theta}$ spaces alone suffice to prove optimal WP results for $s > s_c$. More precisely, for (MKG), (YM) $n \geq 4$ and (WMM) $n \geq 3$, some of the product properties of the $\mathcal{H}^{s,\theta}$ spaces, necessary to carry the step by step iteration, fail by a lot. The starting point in [22],[24] and [25] was the observation that, despite this failure, one can check nevertheless that the second iterates belong to $\mathcal{H}^{s,\theta}$. Moreover the second iterates satisfy additional trilinear properties which, when taken into account, allow one to prove inductively that all iterates belong both to $\mathcal{H}^{s,\theta}$ and satisfy the same trilinear conditions. In the wake of the bilinear estimates of Theorem B, proved in [27], it became clear that the additional trilinear conditions can be more conveniently rephrased in terms of the $\mathcal{L}_t^q(\mathcal{L}_x^r)$ spaces discussed in this paper. See also [30] for a review of this circle of ideas.

(6) The spaces $H^{s,\theta}$ are by no means new in PDE. Before [18] and the systematic use of such spaces by Bourgain [4] in the study of optimal well-posedness for periodic initial conditions for KdV and nonlinear Schrödinger equations (see also [14]), such spaces were used in microlocal analysis, in particular in the the study of propagation of singularities for nonlinear wave equations; see [3]. The novel idea, in both [18] and [4], was to estimate directly, in a space-time L^2

[11] By comparison to what can be derived by Strichartz-type estimates.

norm, the principal quadratic part of the nonlinear term[12]. These new types of estimates[13], which we now refer to as bilinear, provide additional regularity information in connection with the issue of optimal well-posedness. The L^2 set-up of the $H^{s,\theta}$ spaces is the most convenient[14] way to take into account possible cancellations between the symbol of the special quadratic part of the nonlinear equation and the symbol of the corresponding linear operator. Aside from simplicity there is in fact no reason to stop at L^2; as we have seen above, additional information can be provided by combining the $H^{s,\theta}$ norm with a suitable $\mathcal{L}_t^q(\mathcal{L}_x^r)$ norm. Further progress in this respect may be expected from the $L_t^q(L_x^r)$ bilinear estimates conjectured in [7] and proved partially in [46] and [40]. Clever modifications of the $H^{s,\theta}$ spaces appear also in [42] and [43].

A Appendix

A.1 Counterexamples

Here we prove the negative statement in Theorem 8.2. The argument below is a slight modification of the counterexample used in [23]. We construct, for all sufficiently large $L > 0$, functions u_L and v_L such that for any s and θ,

$$|u_L|_{s,\theta} \sim L^{s+\theta+\frac{n}{2}}, \qquad |v_L|_{s,\theta} \sim L^{2s+\frac{n+1}{2}}, \tag{A.1}$$

$$\mathcal{F}\left\{Q_{ij}(D^{-1}u_L, v_L)\right\} \sim L^2\mathcal{F}(u_L v_L) \sim \mathcal{F}\left\{\Lambda^{\frac{1}{2}}(\Lambda^{-\frac{1}{2}}\Lambda_-^{\frac{1}{2}}u_L \cdot \Lambda^{\frac{1}{2}}v_L)\right\}. \tag{A.2}$$

Moreover, for all (τ, ξ) in a certain set C with measure $\sim L^{n+1}$,

$$\mathcal{F}\left\{\Lambda^{s-1}\Lambda_-^{\theta-1}(u_L v_L)\right\}(\tau,\xi) \sim L^{2(s-1)+n}. \tag{A.3}$$

It follows from (A.2), (A.3) and Plancherel's theorem that

$$\begin{aligned}\left\|Q_{ij}(D^{-1}u_L, v_L)\right\|_{s-1,\theta-1} &\sim \left\|\Lambda^{\frac{1}{2}}(\Lambda^{-\frac{1}{2}}\Lambda_-^{\frac{1}{2}}u_L \cdot \Lambda^{\frac{1}{2}}v_L)\right\|_{s-1,\theta-1} \\ &\gtrsim L^{2(s-1)+n+2}\sqrt{|C|} \sim L^{2s+n+\frac{n+1}{2}}.\end{aligned}$$

But by (A.1),

$$|u_L|_{s,\theta}\,|v_L|_{s,\theta} \sim L^{3s+\theta+n+\frac{1}{2}}.$$

[12] In the case of the KdV equation $u_t + u_{xxx} + uu_x = 0$, treated in [4], this was uu_x. In [18] one relies on space-time L^2 estimates for the null quadratic forms Q_0 and $Q_{\alpha\beta}$.

[13] Previous attempts to prove optimal well-posedness results relied on the idea of treating the nonlinear part of the equation as a source term and using the best available estimates, such as Strichartz, for the corresponding linear inhomogeneous equation (see e.g. [13], [32]). In some situations, such as nonlinear wave equations of the type $\Lambda\phi = \pm\phi^p$, this procedure is in fact optimal, see [29].

[14] Indeed, in view of the Plancherel identity it suffices to estimate bilinear weighted convolutions.

We conclude that the estimates

$$\left\| Q_{ij}(D^{-1}u, v) \right\|_{s-1,\theta-1} \lesssim |u|_{s,\theta} \, |v|_{s,\theta} \, ,$$
$$\left\| \Lambda^{\frac{1}{2}} (\Lambda^{-\frac{1}{2}} \Lambda_-^{\frac{1}{2}} u \cdot \Lambda^{\frac{1}{2}} v) \right\|_{s-1,\theta-1} \lesssim |u|_{s,\theta} \, |v|_{s,\theta}$$

must fail if $s < \frac{n}{2} - \theta$.

We may of course take $i = 1 < j$. Let A be the set of $(\lambda, \eta) \in \mathrm{R}^{1+n}$ such that

$$|\lambda - \eta_1| \leq 1, \qquad \frac{L}{2} \leq \eta_1 \leq L, \qquad \frac{L}{2} \leq |\eta'| \leq L,$$

where we write $\eta = (\eta_1, \ldots, \eta_n)$ and $\eta' = (\eta_2, \ldots, \eta_n)$. With the same notation, let B be the set of (τ, ξ) such that

$$\big|\tau - |\xi|\big| \leq 8, \qquad \frac{L^2}{2} \leq \xi_1 \leq 4L^2, \qquad |\xi'| \leq 2L,$$

and let C be the set determined by

$$\big|\tau - |\xi|\big| \leq 1, \qquad L^2 \leq \xi_1 \leq 2L^2, \qquad |\xi'| \leq L.$$

Let $\widehat{u_L}$ and $\widehat{v_L}$ be the characteristic functions of A and B respectively.

Clearly, (A.1) is satisfied. Also,

$$(\lambda, \eta) \in A, (\tau, \xi) \in C \implies (\tau - \lambda, \xi - \eta) \in B, \tag{A.4}$$

since

$$\begin{aligned} \big|\tau - \lambda - |\xi - \eta|\big| &\leq \big|\tau - |\xi|\big| + |\lambda - \eta_1| + |\xi| - \xi_1 + |\xi - \eta| - (\xi_1 - \eta_1) \\ &\leq 2 + \frac{|\xi'|^2}{|\xi| + \xi_1} + \frac{|\xi' - \eta'|^2}{|\xi - \eta| + \xi_1 - \eta_1} \leq 2 + 1 + 5 = 8. \end{aligned}$$

Observe that (A.4) implies

$$\mathcal{F}(u_L v_L)(\tau, \xi) = |A| \sim L^n \quad \text{for all} \quad (\tau, \xi) \in C,$$

and (A.3) follows.

To prove (A.2), write

$$\begin{aligned} &\mathcal{F}\left\{Q_{1j}(D^{-1}u_L, v_L)\right\}(\tau, \xi) \\ &\qquad = \int_{\mathrm{R}^{1+n}} \left(\frac{\eta_j}{|\eta|}(\xi_1 - \eta_1) - \frac{\eta_1}{|\eta|}(\xi_j - \eta_j) \right) \widehat{u_L}(\lambda, \eta) \widehat{v_L}(\tau - \lambda, \xi - \eta) \, d\lambda \, d\eta. \end{aligned}$$

Obviously,

$$(\lambda, \eta) \in A, (\tau, \xi) \in B \implies \frac{\eta_j}{|\eta|}\xi_1 - \frac{\eta_1}{|\eta|}\xi_j \sim L^2,$$

whence

$$\mathcal{F}\left\{Q_{1j}(D^{-1}u_L, v_L)\right\} \sim L^2 \mathcal{F}(u_L v_L).$$

It is also easy to see that

$$\mathcal{F}\left\{\Lambda^{\frac{1}{2}} (\Lambda^{-\frac{1}{2}} \Lambda_-^{\frac{1}{2}} u_L \cdot \Lambda^{\frac{1}{2}} v_L)\right\} \sim L^2 \mathcal{F}(u_L v_L),$$

so we have established (A.2). This concludes the proof.

A.2 Proof of Theorem 7.2

We first prove the following.

A.1. Proposition. *Let $n \geq 1$ and $a, b, c, \alpha, \beta, \gamma \geq 0$. Then*

$$H^{a,\alpha} \cdot H^{b,\beta} \hookrightarrow H^{-c,-\gamma},$$

provided $a + b + c > \frac{n}{2}$ and $\alpha + \beta + \gamma > \frac{1}{2}$.

Proof. Assume that $s = a+b+c > \frac{n}{2}$ and $\theta = \alpha+\beta+\gamma > \frac{1}{2}$ $(a, b, c, \alpha, \beta, \gamma \geq 0)$.
By Hölder's inequality,

$$L^2 \cdot L^\infty \hookrightarrow L^2,$$
$$L_t^\infty(L_x^2) \cdot L_t^2(L_x^\infty) \hookrightarrow L^2,$$

so by (6.2),(6.1) and (6.3) , we have

$$L^2 \cdot H^{s,\theta} \hookrightarrow L^2, \tag{A.5}$$
$$H^{0,\theta} \cdot H^{s,0} \hookrightarrow L^2. \tag{A.6}$$

Once we have these estimates, the others follow by interpolation:

Step 1. Assume $\gamma = 0$ (so $\theta = \alpha + \beta > \frac{1}{2}$). Interpolation between (A.5) and (A.6) gives

$$H^{0,\alpha} \cdot H^{s,\beta} \hookrightarrow L^2.$$

Step 2. By Step 1, we have

$$H^{0,\alpha+\gamma} \cdot H^{s,\beta} \hookrightarrow L^2,$$
$$L^2 \cdot H^{s,\beta} \hookrightarrow H^{0,-\alpha-\gamma}.$$

Interpolation between these yields

$$H^{0,\alpha} \cdot H^{s,\beta} \hookrightarrow H^{0,-\gamma}.$$

Step 3. Assume $c = 0$ (so $s = a + b > \frac{n}{2}$). By Step 2,

$$H^{0,\alpha} \cdot H^{s,\beta} \hookrightarrow H^{0,-\gamma},$$
$$H^{s,\alpha} \cdot H^{0,\beta} \hookrightarrow H^{0,-\gamma},$$

and by interpolation,

$$H^{a,\alpha} \cdot H^{b,\beta} \hookrightarrow H^{0,-\gamma}.$$

Step 4. By Step 3,

$$H^{a+c,\alpha} \cdot H^{b,\beta} \hookrightarrow H^{0,-\gamma},$$
$$H^{0,\alpha} \cdot H^{b,\beta} \hookrightarrow H^{-a-c,-\gamma},$$

so interpolation gives

$$H^{a,\alpha} \cdot H^{b,\beta} \hookrightarrow H^{-c,-\gamma}.$$

This concludes the proof. □

We now turn to the proof of Theorem 7.2, which we restate here for convenience.

Theorem. *Let $n \geq 2$, $s > \frac{n}{2}$ and $\frac{1}{2} < \theta \leq s - \frac{n-1}{2}$. Then*

$$H^{a,\alpha} \cdot H^{s,\theta} \hookrightarrow H^{a,\alpha}$$

for all a, α satisfying

$$0 \leq \alpha \leq \theta,$$
$$-s + \alpha < a \leq s.$$

(Hence, by duality, for all $-\theta \leq \alpha \leq 0$ and $-s \leq a < s + \alpha$.)

By interpolation, it suffices to prove:

(A) $\quad H^{s,\theta} \cdot H^{s,\theta} \hookrightarrow H^{s,\theta},$

(B) $\quad H^{s,0} \cdot H^{s,\theta} \hookrightarrow H^{s,0},$

(C) $\quad H^{-s,0} \cdot H^{s,\theta} \hookrightarrow H^{-s,0},$

(D) $\quad H^{a,\theta} \cdot H^{s,\theta} \hookrightarrow H^{a,\theta}, \qquad -s + \theta < a < 0.$

The estimates A,B,C and D correspond to the vertices of a trapezoid in the (a, α)-plane.

Estimate A. By Lemma 7.4 it suffices to prove

$$H^{0,\theta} \cdot H^{s,\theta} \hookrightarrow H^{0,\theta}.$$

By Lemma 3.3, the last estimate reduces to three estimates:

$$H^{0,\theta} \cdot H^{s,0} \hookrightarrow L^2,$$
$$L^2 \cdot H^{s,\theta} \hookrightarrow L^2,$$
$$R^\theta(H^{0,\theta}, H^{s,\theta}) \hookrightarrow L^2.$$

The first two are covered by Proposition A.1; the third follows from Theorem F.

Estimates B and C. These are equivalent by duality, so it suffices to prove B, which by lemma 7.4 reduces to:

$$
\begin{aligned}
H^{s,0} \cdot H^{0,\theta} &\hookrightarrow L^2, \\
L^2 \cdot H^{s,\theta} &\hookrightarrow L^2.
\end{aligned}
$$

Both of these are covered by Proposition A.1.

Estimate D. Since D is equivalent to

$$
H^{-a,-\theta} \cdot H^{s,\theta} \hookrightarrow H^{-a,-\theta}
$$

by duality, and since $a < 0$, by Lemma 7.4 it suffices to prove

$$
\begin{aligned}
H^{-a,-\theta} \cdot H^{s+a,\theta} &\hookrightarrow H^{0,-\theta}, \\
H^{0,-\theta} \cdot H^{s,\theta} &\hookrightarrow H^{0,-\theta}.
\end{aligned}
$$

By duality, the last two estimates are equivalent to

$$
\begin{aligned}
H^{0,\theta} \cdot H^{s+a,\theta} &\hookrightarrow H^{a,\theta}, \\
H^{0,\theta} \cdot H^{s,\theta} &\hookrightarrow H^{0,\theta}.
\end{aligned}
$$

The last estimate was proved above (estimate for A), and the second to last reduces, by Lemma 3.3, to three estimates:

$$
H^{0,\theta} \cdot H^{s+a,0} \hookrightarrow H^{a,0}, \tag{A.7}
$$

$$
L^2 \cdot H^{s+a,\theta} \hookrightarrow H^{a,0}, \tag{A.8}
$$

$$
R^{\theta}(H^{0,\theta}, H^{s+a,\theta}) \hookrightarrow H^{a,0}. \tag{A.9}
$$

By interpolation between the estimates

$$
\begin{aligned}
H^{0,\theta} \cdot H^{s,0} &\hookrightarrow L^2, \\
H^{0,\theta} \cdot L^2 &\hookrightarrow H^{-s,0},
\end{aligned}
$$

which are dual to each other and hold by Proposition A.1, we get (A.7). Proposition A.1 also covers (A.8) (via duality). Finally, for (A.9) we consider two cases:

(i) If $s = \frac{n-1}{2} + \theta$, then $a > -\frac{n-1}{2}$, and (A.9) holds by Theorem F.

(ii) If $s > \frac{n-1}{2} + \theta$, then $-s + \theta < -\frac{n-1}{2}$, so we may assume that $-s + \theta < a < -\frac{n-1}{2}$ (then the estimate for $-\frac{n-1}{2} \le a < 0$ follows by interpolation with estimate A). Choose $\varepsilon > 0$ so small that $\theta + \varepsilon < s + a$ and $\varepsilon \le \frac{n-1}{2}$. Then by Theorem F,

$$
R^{\theta}(H^{0,\theta}, H^{\theta+\varepsilon,\theta}) \hookrightarrow H^{-\frac{n-1}{2}+\varepsilon,0},
$$

which implies (A.9).

This concludes the proof of Theorem 7.2.

A.3 Proof of Theorem 8.7

Let $n \geq 4$, $\theta > \frac{1}{2}$. Assume

$$a, b \geq -c,$$
$$a + b \geq \frac{1}{2},$$
$$a + b + c \geq \frac{n-1}{2}.$$

We want to prove

$$H^{a,\theta} \cdot H^{b,\theta} \hookrightarrow H^{-c,0}. \tag{A.10}$$

Step 1. Assume $c \leq 0$. Then by Lemma 7.4, (A.10) reduces to

$$H^{a+c,\theta} \cdot H^{b,\theta} \hookrightarrow L^2.$$

This can be reduced to the extreme case

$$H^{0,\theta} \cdot H^{\frac{n-1}{2},\theta} \hookrightarrow L^2, \tag{A.11}$$

which holds by Theorem F.

Step 2. Assume $-c < 0 \leq a, b$. If $a + b \geq \frac{n-1}{2}$, then (A.10) follows from

$$H^{a,\theta} \cdot H^{b,\theta} \hookrightarrow L^2,$$

which again reduces to (A.11). If $a + b < \frac{n-1}{2}$, set $\gamma = a + b - \frac{n-1}{2}$. Then (A.10) follows from

$$H^{a,\theta} \cdot H^{b,\theta} \hookrightarrow H^{\gamma,0},$$

and the latter holds by Theorem F.

Step 3. Assume $-c \leq a < 0$. By Lemma 7.4, (A.10) reduces to two estimates:

$$H^{0,\theta} \cdot H^{a+b,\theta} \hookrightarrow H^{-c,0},$$
$$H^{0,\theta} \cdot H^{b,\theta} \hookrightarrow H^{-a-c,0}.$$

These estimates hold by Steps 1 and 2.

A.4 Proof of Theorem 9.2

Let $n \geq 3$, $\theta > \frac{1}{2}$. Assume

$$a, b, c \geq 0,$$
$$c < \frac{n-1}{2},$$
$$a + b + c \geq \frac{n-1}{2} + \theta.$$

We want to prove

$$H^{a,\theta} \cdot H^{b,\theta} \hookrightarrow H^{-c,\theta}.$$

By Lemma 3.3, this reduces to

$$H^{a,0} \cdot H^{b,\theta} \hookrightarrow H^{-c,0},$$
$$H^{a,\theta} \cdot H^{b,0} \hookrightarrow H^{-c,0},$$
$$R^{\theta}(H^{a,\theta}, H^{b,\theta}) \hookrightarrow H^{-c,0}.$$

The first two hold by Proposition A.1, the last one by Theorem F (take $\gamma = -c$, $\gamma_- = \theta$ and choose $0 \le s_1 \le a$, $0 \le s_2 \le b$ such that $c + s_1 + s_2 = \frac{n-1}{2} + \theta$).

A.5 Analysis of the First Iterate

Here we work out in more detail the examples considered in section 1.3.

Step 1. If u solves $\Lambda u = F$ with vanishing initial data at time $t = 0$, then

$$\left|\widehat{u(t)}(\xi)\right| \le \frac{C_t}{|\xi|} \int_{\mathbb{R}} \frac{\left|\widehat{F}(\tau,\xi)\right|}{1 + \left||\tau| - |\xi|\right|} \, d\tau,$$
$$\left|\widehat{u(t)}(\xi)\right| \le t^2 \int_{\mathbb{R}} \left|\widehat{F}(\tau,\xi)\right| d\tau,$$

for all $t > 0$. The first estimate is an immediate consequence of the formula

$$\widehat{u(t)}(\xi) = \int_{\mathbb{R}} \frac{\widehat{F}(\tau,\xi)}{4\pi\,|\xi|} \left(\frac{e^{it\tau} - e^{it|\xi|}}{\tau - |\xi|} - \frac{e^{it\tau} - e^{-it|\xi|}}{\tau + |\xi|} \right) d\tau,$$

which is easily derived from Duhamel's principle (see, e.g., [33, Section 3.6.3]). As for the second estimate, Duhamel's principle implies

$$\left|\widehat{u(t)}(\xi)\right| \le t \int_0^t \left|\widehat{F}(t')(\xi)\right| dt',$$

and clearly, $\left|\widehat{F}(t')(\xi)\right| \le \int_{\mathbb{R}} \left|\widehat{F}(\tau,\xi)\right| d\tau$.

Step 2. Let B be a bilinear operator of the form

$$\mathcal{F}\left(B(u,v)\right)(\tau,\xi) = \int_{\mathbb{R}^{1+n}} b(\tau-\lambda, \xi-\eta; \lambda, \eta) \widehat{u}(\tau-\lambda, \xi-\eta) \widehat{v}(\lambda,\eta) \, d\lambda \, d\eta.$$

Assume that

$$\Lambda v = \Lambda w = 0, \qquad (v, \partial_t v)\big|_{t=0} = (v_0, 0), \qquad (w, \partial_t w)\big|_{t=0} = (w_0, 0).$$

As in section 1.4, we decompose v and w into half-waves. Thus,

$$B(v,w) \simeq B(v_+,w_+) + B(v_+,w_-) + B(v_-,w_+) + B(v_-,w_-),$$

where $v_\pm = e^{\pm itD}v_0$ and $w_\pm = e^{\pm itD}w_0$. It suffices to consider the first two terms on the right hand side. Since $\widehat{v_+}(\tau,\xi) \simeq \delta(\tau - |\xi|)\widehat{v_0}(\xi)$ and $\widehat{w_\pm}(\tau,\xi) \simeq \delta(\tau \mp |\xi|)\widehat{w_0}(\xi)$, we have

$$\begin{aligned}
&\mathcal{F}B(v_+,w_\pm)(\tau,\xi)\\
&= \int b(\tau-\lambda,\xi-\eta;\lambda,\eta)\delta(\tau-\lambda-|\xi-\eta|)\delta(\lambda\mp|\eta|)\widehat{v_0}(\xi-\eta)\widehat{w_0}(\eta)\,d\lambda\,d\eta\\
&= \int k_\pm(\xi-\eta,\eta)\delta(\tau\mp|\eta|-|\xi-\eta|)\widehat{v_0}(\xi-\eta)\widehat{w_0}(\eta)\,d\lambda\,d\eta.
\end{aligned}$$

where $k_\pm(\xi,\eta) = b(|\xi|,\xi;\pm|\eta|,\eta)$.

Step 3. Let $u_\pm$ be the solution of $\Lambda u_\pm = B(v_+,w_\pm)$ with vanishing initial data. Set $f(\xi) = \langle\xi\rangle^s|\widehat{v_0}(\xi)|$ and $g(\xi) = \langle\xi\rangle^s|\widehat{w_0}(\xi)|$. By Steps 1 and 2,

$$\langle\xi\rangle^s\left|\widehat{u}_\pm(t)(\xi)\right| \le C_t\int K(\xi-\eta,\eta)f(\xi-\eta)g(\eta)\,d\eta,$$

where

$$K(\xi,\eta) = \frac{\langle\xi+\eta\rangle^s|k_\pm|(\xi,\eta)}{\langle\xi\rangle^s\langle\eta\rangle^s}\min\left(1,\frac{1}{|\xi+\eta|\,\big(1+\Delta_\pm(\xi,\eta)\big)}\right)$$

and $\Delta_\pm$ is defined by (1.20).

Step 4. By Step 3, proving $\|u_\pm(t)\|_{H^s} \le C_t\|v_0\|_{H^s}\|w_0\|_{H^s}$ reduces to proving

(A.12) $$\int_{\mathrm{R}^n\times\mathrm{R}^n} K(\xi,\eta)f(\xi)g(\eta)h(\xi+\eta)\,d\xi\,d\eta \lesssim \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}$$

for all $f,g,h \in L^2(\mathrm{R}^n)$, where K is as in Step 3. Write $K = K_1 + K_2 + K_3$, where K_1, K_2 and K_3 are supported in the mutually disjoint regions

$$\begin{aligned}
\Omega_1 &= \{(\xi,\eta) : |\xi+\eta| < 1\},\\
\Omega_2 &= \{(\xi,\eta) : |\xi+\eta| \ge 1, |\eta| < |\xi|\},\\
\Omega_3 &= \{(\xi,\eta) : |\xi+\eta| \ge 1, |\eta| \ge |\xi|\}
\end{aligned}$$

respectively.

Obviously, $K_1 \lesssim |k_\pm|\langle\xi\rangle^{-s}\langle\eta\rangle^{-s}$. Assuming that

(A.13) $$|k_\pm| \lesssim \langle\xi\rangle^s\langle\eta\rangle^s,$$

it follows that K_1 is bounded, whence

$$\sup_\eta\int K_1^2(\xi-\eta,\eta)\,d\xi < \infty.$$

Therefore (A.12) holds by Lemma A.3 below (after a linear change of variables). Observe that (A.13) is satisfied in the examples we consider (we always have $s \geq 1$ and $|k_\pm| \lesssim |\xi|\,|\eta|$).

Next, for K_2 we have

$$K_2(\xi,\eta) \lesssim \frac{|k_\pm|\,(\xi,\eta)}{\langle\xi\rangle\,\langle\eta\rangle^s\,\big(1+\Delta_\pm(\xi,\eta)\big)}$$

Let us now consider this expression for the operators $B(v,w)$ appearing in our examples (the estimates for K_3 are the same, since the operators are symmetric).

(i) In Example 1.1, $B(v,w) = \partial_t v \cdot \partial_t w$, so $b(\tau,\xi;\lambda,\eta) \simeq \tau\lambda$. Therefore,

$$|k_\pm| \leq |\xi|\,|\eta|$$

and

$$K_2(\xi,\eta) \lesssim \frac{1}{\langle\eta\rangle^{s-1}\,\big(1+\Delta_\pm(\xi,\eta)\big)}.$$

(ii) In Example 1.3, $B(v,w) = Q_0(v,w)$, so $b(\tau,\xi;\lambda,\eta) \simeq \tau\lambda - \xi\cdot\eta$. Therefore,

$$k_\pm(\xi,\eta) \simeq \pm|\xi|\,|\eta| - \xi\cdot\eta = \pm|\xi|\,|\eta|\,(1 \mp \cos\angle(\xi,\eta))\ .$$

And in view of Lemma A.2 below, this implies

$$|k_\pm| \leq \max(|\xi|\,,|\eta|)\Delta_\pm(\xi,\eta).$$

Hence,

$$K_2(\xi,\eta) \lesssim \langle\eta\rangle^{-s}\,.$$

(iii) In Example 1.4, $B(v,w) = Q_{ij}(v,w)$, so $b(\tau,\xi;\lambda,\eta) \simeq \xi_i\eta_j - \xi_j\eta_i$. Therefore,

$$|k_\pm(\xi,\eta)| \lesssim |\xi\wedge\eta| = |\xi|\,|\eta|\,\sqrt{1-\cos^2\theta} \lesssim |\xi|\,|\eta|\,\sqrt{1\mp\cos\theta}.$$

In view of Lemma A.2, this implies

$$K_2(\xi,\eta) \lesssim \frac{1}{\langle\eta\rangle^{s-\frac12}\,\big(1+\Delta_\pm(\xi,\eta)\big)^{\frac12}}.$$

Proof of Proposition 1.2

Instead of proving Proposition 1.2 as stated, we prove a homogeneous version. The proof is easily modified to give the inhomogeneous statement in Proposition 1.2. We first prove two lemmas.

A.2. Lemma. *Let $\Delta_\pm$ be defined by* (1.20). *Then*

$$\min(|\xi|\,,|\eta|)(1\pm\cos\theta) \leq 2\Delta_\mp(\xi,\eta),$$

where θ is the angle between ξ and η.

Proof. We have

$$\Delta_+(\xi,\eta) = \frac{\Delta_+(\xi,\eta)(|\xi|+|\eta|+|\xi+\eta|)}{|\xi|+|\eta|+|\xi+\eta|} \geq \frac{|\xi|\,|\eta|\,(1-\cos\theta)}{|\xi|+|\eta|},$$

and a similar computation gives the proof for Δ_-. □

A.3. Lemma. *If K is a measurable function on $\mathrm{R}^n \times \mathrm{R}^n$ such that at least one of the numbers*

$$\sup_\xi \int K^2(\xi,\eta)\,d\eta, \quad \sup_\eta \int K^2(\xi,\eta)\,d\xi$$

is finite, then

$$\int_{\mathrm{R}^n\times\mathrm{R}^n} K(\xi,\eta) f(\xi) g(\eta) h(\xi+\eta)\,d\xi\,d\eta \leq C\,\|f\|_{L^2}\,\|g\|_{L^2}\,\|h\|_{L^2}$$

for all $f, g, h \in L^2$.

To prove this, simply apply the Cauchy-Schwarz inequality twice.

Proposition. *If $a, b, c \geq 0$ and $\Delta(\xi,\eta)$ is either of the expressions defined in* (1.20)*, then*

$$\int_{\mathrm{R}^n\times\mathrm{R}^n} \frac{f(\xi)g(\eta)h(\xi+\eta)}{|\xi|^a\,|\eta|^b\,\Delta^c(\xi,\eta)}\,d\xi d\eta \leq C\,\|f\|_{L^2}\,\|g\|_{L^2}\,\|h\|_{L^2}$$

for all $f, g, h \in L^2(\mathrm{R}^n)$, provided

$$a+b+c = \frac{n}{2}, \quad a, b < \frac{n}{2} - c, \quad c < \frac{n-1}{4}.$$

Proof. Set $K(\xi,\eta) = |\xi|^{-a}\,|\eta|^{-b}\,\Delta^{-c}(\xi,\eta)$ and write $K = K_1 + K_2$, where K_1 is supported in the region $|\eta| \leq |\xi|$ and K_2 is supported in $|\eta| > |\xi|$. By lemma A.2,

$$K_1(\xi,\eta) \leq \frac{2}{|\xi|^a\,|\eta|^{b+c}\,(1\pm\cos\theta)^c},$$

where θ is the angle between ξ and η. Thus, for all ξ, integration in polar coordinates $(r,\omega) = (|\eta|, \eta/|\eta|)$ yields

$$\begin{aligned}\int K_1^2(\xi,\eta)\,d\eta &\leq \frac{4}{|\xi|^{2a}} \int_0^{|\xi|} r^{n-1-2(b+c)}\,dr \int_{S^{n-1}} \frac{d\sigma(\omega)}{(1\pm\cos\theta)^{2c}} \\ &= \frac{4}{n-2(b+c)} \int_{S^{n-1}} \frac{d\sigma(\omega)}{(1\pm\cos\theta)^{2c}},\end{aligned}$$

and the last integral is finite iff $2c < (n-1)/2$. By symmetry, this implies that $\sup_\eta \int K_2^2(\xi,\eta)\,d\xi$ is also finite, so we may apply lemma A.3. □

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