Brief history of the vector-field method

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  " The Formation of Shocks in 3-Dimensional Fluids"
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- In *"On the formation of trapped surfaces"* (Acta Math 2011?) Kl-Rodnianski give a significant extension and simplification of Christodoulou's result.

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- Coliander-Keel-Staffilani-Takaoka-Tao *"Global well-posedness and scattering for the energy critical nonl. Schr. eq in $\mathbb{R}^3$."* Prove global existence for the critical, 3D, defocusing, Schr. equation using the **interaction Morawetz estimates.**

MULTIPLIER METHOD

Using Friedrich's a,b,c method **C. Morawetz** discovers how to use the vector-fields,

$$\mathbb{S} = t\partial_t + x^i\partial_i$$

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- 1961. Decay of solutions of the exterior initial-boundary value problem for the wave equation
- 1968. Time decay for the nonlinear Klein-Gordon equations
- 1972. Decay and scattering of solutions of a nonlinear relativistic wave equation.

GENERAL REMARKS

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4. They have all found innumerable applications, most notable in General Relativity

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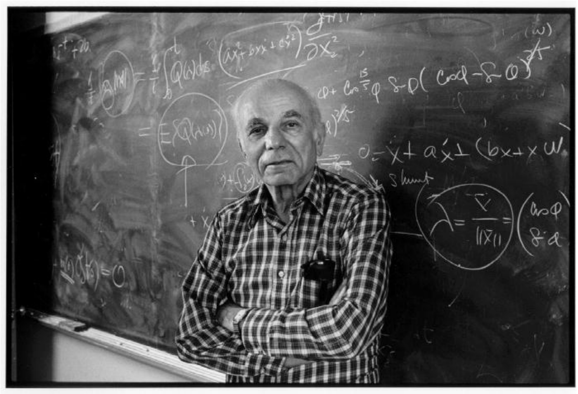
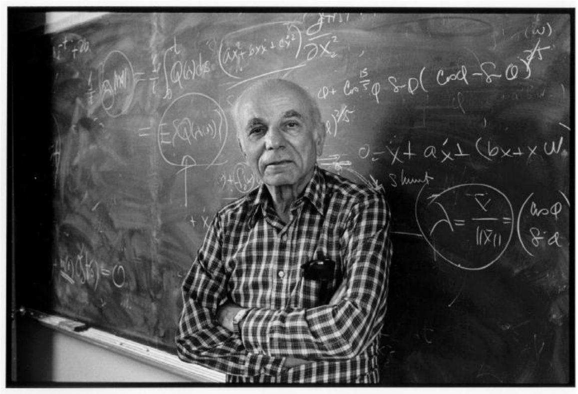
PROOF. Use Kirchoff formula or stationary phase. Intimately tied to the curvature of the light cone, restriction theorems in Harmonic Analysis, Strichartz inequalities.



$$\Box\phi = F(\partial\phi, \partial^2\phi), \quad \phi|_{t=0} = \epsilon f, \ \partial_t\phi|_{t=0} = \epsilon g.$$



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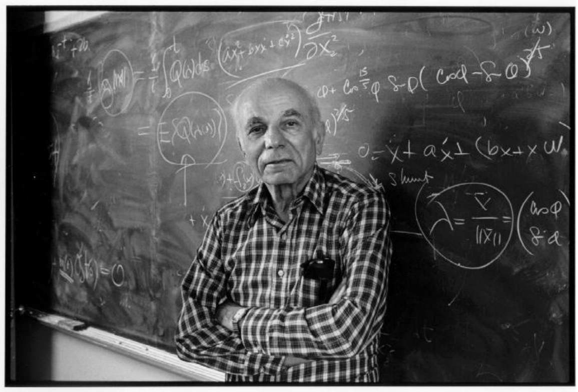

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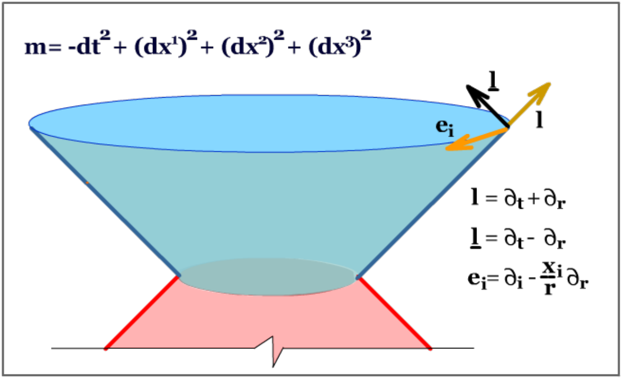
- 1976 "Delayed singularity formation in solutions of nonlinear wave equations in higher dimensions".
- 1983 "Lower bounds for the life span of solutions of nonlinear wave equations in three dimensions".

JOHN'S 1983 PAPER

- Introduces a **null frame** formalism to capture different rates of decay in different directions,



$$\mathbf{m}(l,\underline{l}) = 0,\ \mathbf{m}(l,\underline{l}) = -2,\ \mathbf{m}(l,e_i) = \mathbf{m}(\underline{l},e_i) = 0.$$

- Measures decay using more precise weights.

Proposition. If $\square\phi = 0$, with compactly supported initial data

$$|\partial\phi(t,x)| \lesssim (1+t+|x|)^{-\frac{n-1}{2}}\left(1+\big|t-|x|\big|\right)^{-\frac{1}{2}}$$

and **peeling**

$$\begin{aligned}
|\underline{l}(\partial\phi)(t,x)| &\lesssim (1+t+|x|)^{-\frac{n-1}{2}}\left(1+\big|t-|x|\big|\right)^{-\frac{3}{2}}\\
|l(\partial\phi)(t,x)| &\lesssim (1+t+|x|)^{-\frac{n+1}{2}}\left(1+\big|t-|x|\big|\right)^{-\frac{1}{2}}\\
|e_i(\partial\phi)(t,x)| &\lesssim (1+t+|x|)^{-\frac{n+1}{2}}\left(1+\big|t-|x|\big|\right)^{-\frac{1}{2}}
\end{aligned}$$

Proof. Use the spherical means representation of solutions.

THEOREM[John(1983)] The life-span $T_*(\epsilon)$ of solutions to equation

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Proof. Asymptotic expansion in powers of ϵ,

$$\phi = \epsilon\phi^{(0)} + \epsilon^2\phi^{(1)} + \ldots + \epsilon^N\phi^{(N)} + \psi$$

and estimate each term inductively using, the **weighted decay** estimates. To close need also **energy estimates** for the error ψ.

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FURTHER DEVELOPMENTS

- F. John shows that the result is sharp, in general. Result extended by T. Sideris compressible, ideal, fluids.
- F. John extends the result to nonlinear elasticity (result later simplified in Kl-Sideris).
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Main idea. If $\square\phi = \square\psi = 0$, the **null forms**

$$Q_0(\phi,\psi) = \mathbf{m}^{\mu\nu}\partial_\mu\phi\partial_\nu\psi$$
$$Q_{\alpha\beta}(\phi,\psi) = \partial_\alpha\phi\partial_\beta\psi - \partial_\beta\phi\partial_\alpha\psi$$

decay faster than $\partial\phi\cdot\partial\psi$.

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- Method of space-time resonances of Germain-Masmoudi-Shatah

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- Get peeling estimates by a simple linear algebra calculation

After 2300 years mathematics comes back to Greece ! (P. LAX)

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Christodoulou in 1982

GLOBAL STABILITY OF THE MINKOWSKI SPACE

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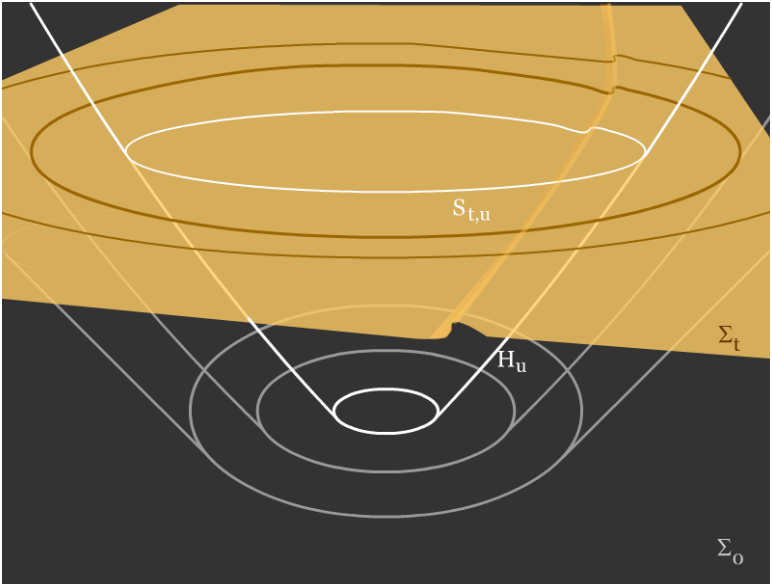
- A family of generalized energy norms of the space-time curvature and its Lie derivatives with respect to these vectorfields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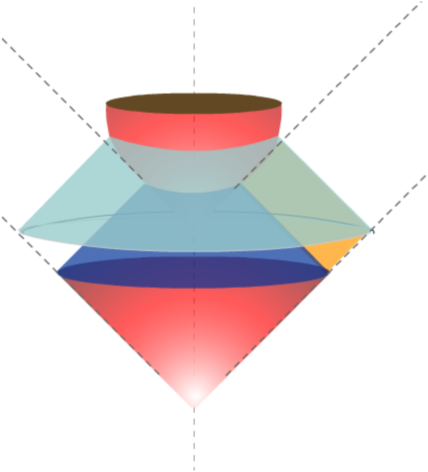
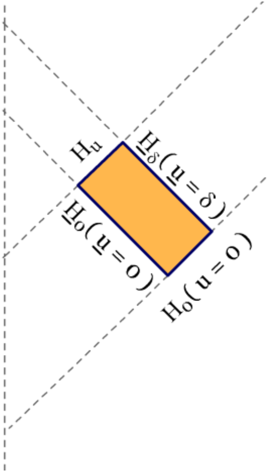
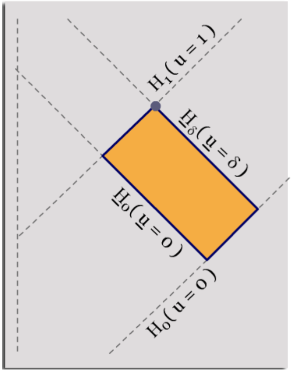
Christodoulou's works *" Formation of shocks..."*, and *" Formation of black holes..."* as well as Kl-Rodnianski's *"Formation of trapped surfaces..."* are based on the same three, steps, even though many important details are different.

Theorem[Chr. 2008, Kl-Rodn. 2010] Specify regular, **characteristic**, initial data, in **vacuum**, and show that its future development must contain a trapped surface





SHORT PULSE. Prescribe non-trivial data on the outgoing null hypersurface H_0,



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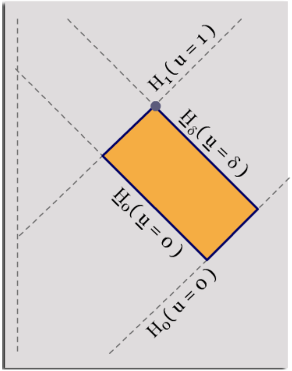

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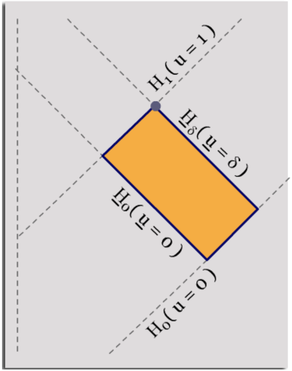

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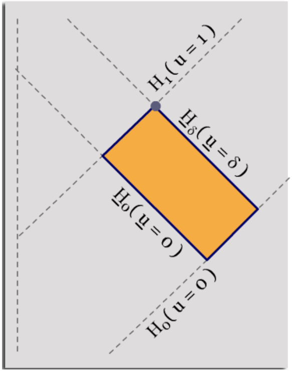

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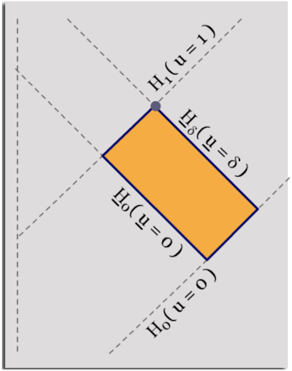

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In Kl-Rodn we prove also formation of scars in prescribed angular sectors.

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Soffer-Blue, Blue-Sterbenz, Dafermos-Rodn., Tataru-Tohaneanu, Anderson-Blue ..., have used an enhanced version of the vectorfield method to study the decay properties of wave equation $\Box_g \phi = 0$ in the exterior region of a black hole (Kerr).

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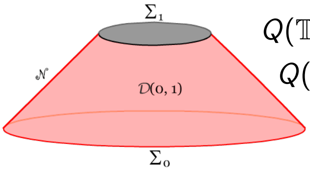
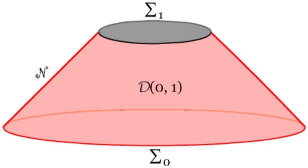


$$Q(\mathbb{T}_0, \mathbb{T}_0) \geq c|D\phi|^2$$

$$Q(\mathbb{T}_0, L) \geq c|L\phi|^2 + |\nabla\!\!\!/ \phi|^2$$

GENERALIZED ENERGY



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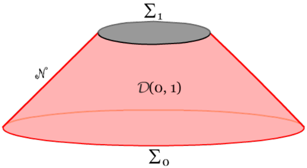

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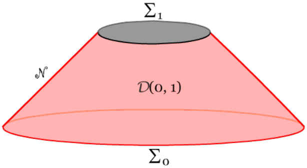
$$\begin{aligned} Q(X,Y) &= X(\phi)Y(\phi) - \frac{1}{2} g(X,Y) L(\phi) \\ Q_w(X,Y) &= Q(X,Y) + \frac{1}{2} w\,\phi\, Y(\phi) - \frac{1}{4} Y(w)\phi^2 \\ \mathrm{Err}(w,X) &= \frac{1}{2}\big(Q\cdot\mathcal{L}_X g + w\,\mathcal{L}(\phi)\big) - \frac{1}{4}\Box(w)\,\phi^2 \end{aligned}$$

Example 1. $\mathcal{L}_X g = 0, \quad g(X,X) < 0, \quad w = 0.$



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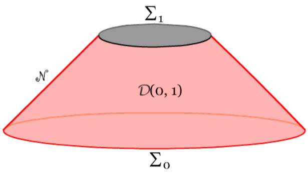

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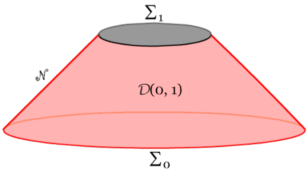

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Example 3. $\mathrm{Err}(w, X) \geq 0$. This is the case of the Morawetz vectorfield $X = \partial_r$ in Minkowski space $\mathbb{R}^{1+3}$.

$$2\pi \int_{t_0}^{t_1} |\phi(t,0)|^2 dt + \int_{\mathcal{D}} \frac{1}{r} |\nabla\!\!\!/ \phi|^2 \quad = \quad \mathrm{RHS}$$

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If $X_1, X_2, \ldots$ are Killing,

$$\Box(\mathcal{L}_{X_1} \mathcal{L}_{X_2} \ldots \mathcal{L}_{X_k} \phi) = 0$$

SYMMETRIES AND DECAY IN MINKOWSKI SPACE $\mathbb{R}^{d+1}$

Theorem. There exists an expression $\mathcal{Q}[\phi](t)$, constructed by the *vectorfield method*, such that $\mathcal{Q}[\phi](t) = \mathcal{Q}[\phi](0)$ if $\Box\phi = 0$ and, with $u = t - |x|, \underline{u} = t + |x|$,

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- Generators of translations : $\mathbb{T}_\mu = \frac{\partial}{\partial x^\mu}$.
- Generators of rotations $\mathbb{L}_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$.
- Generator of scaling: $\mathbb{S} = x^\mu \partial_\mu$.
- Generators of inverted translations $\mathbb{K}_\mu = 2x_\mu x^\rho \frac{\partial}{\partial x^\rho} - (x^\rho x_\rho)\frac{\partial}{\partial x^\mu}$.

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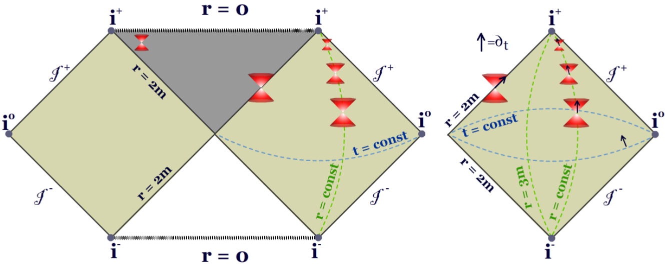

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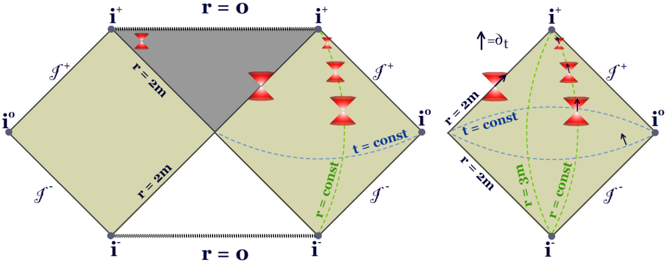

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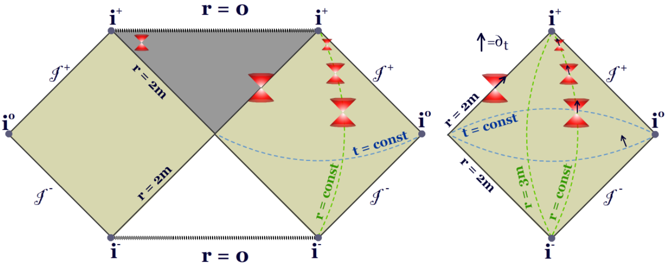

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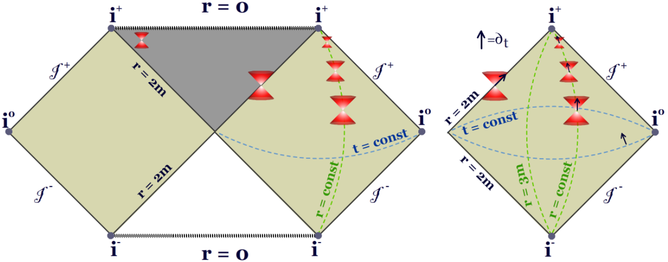

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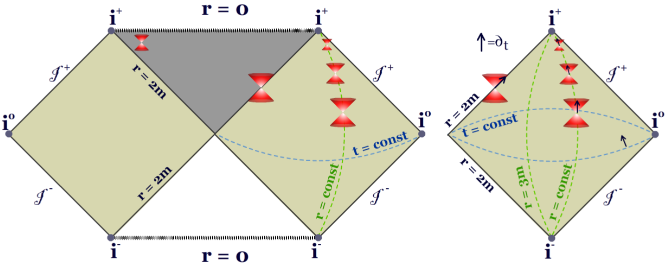

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Can be done near $r = 3m$.

JOHN'S APOLOGY. The science of mathematics depends for its growth on the flow of information between its practitioners. The joy of discovering new results ought to be matched by the joy in studying the achievements of others. Unfortunately this latter enjoyment is made difficult by the overwhelming volume of mathematical output and the work involved in absorbing the context of even a single paper. Every mathematician has to compromise on the amount of energy he can devote to the literature. I myself have been irresistibly attracted to mathematical research almost since my childhood, but always was loath to spend the time needed to keep up with developments. This has severely limited my work. Fortunately there was a compensating factor. I was able to spend most of my mathematical life in the stimulating atmosphere of the Courant Institute of Mathematical Sciences at New York University, where I could draw freely on the knowledge and experience of my colleagues.