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Chapter 1

Introduction

The main goal of this book is to revisit the proof of the global stability of the Minkowski space by D.Christodoulou and S.Klainerman, [Ch-Kl]. We provide a new self contained proof of the main part of that result, concerning the full solution of the radiation problem in vacuum [1], for arbitrary asymptotically flat initial data sets. The proof, which is a significant modification of the arguments in [Ch-Kl], is based on a "double null foliation" of the spacetime instead of the mixed "null-maximal foliation" used in [Ch-Kl]. This approach is more naturally adapted with the radiation features of the Einstein equations and leads to important technical simplifications. For the convenience of the reader we supplement the proof with two introductory chapters concerning the Cauchy problem in General Relativity.

In this first chapter we review some basic notions of differential geometry which will be systematically used in all the remaining chapters. We then introduce the Einstein equations, initial data sets and discuss some of the basic features of the Initial Value Problem in General Relativity. We shall review, without proofs, well established results concerning local and global existence and uniqueness and formulate our main result.

The second chapter provides the technical motivation for the proof of our main theorem. We start by reviewing the standard proof of local existence and uniqueness for systems of nonlinear wave equations. We then discuss methods for proving global existence results stressing the importance of symmetries. We also emphasize the importance of a structural condition, called the null condition, in establishing global results in $3+1$ dimensions.

[1] Following the terminology introduced in [Ch-Mu]. As explained in Chapter 8 this can also be interpreted as a proof of the global stability of the external region of the Schwarzschild spacetime.

The cancellation encompassed in this formal condition illustrates the need to work with null frames. An essential ingredient in these results is the derivation of uniform decay estimates for the linearized equations using only energy inequalities and the symmetries of the Minkowski spacetime. We proceed to show how the same method can be used to get full decay estimates for the electromagnetic and Weyl fields verifying the linear Maxwell and Bianchi equations in flat spacetime. The latter provides a crucial stepping stone to the Einstein equations. Finally we provide the reader with a detailed discussion of the main ideas in the proof of the *Main Theorem*. We also compare it to the proof of the D.Christodoulou and S.Klainerman theorem, hereafter called *C-K Theorem*, in [Ch-Kl]. All the remaining chapters, with the exception of the last, are dedicated to the proof of our main theorem. The proof is essentially self contained, except for a few topics which are treated in [Ch-Kl] and to which we provide ample reference. In the last Chapter we derive the most important consequences of our *Main Theorem*, in particular we give a rigorous derivation of the Bondi mass law and discuss the asymptotic properties of our spacetime. Due to our approach, based on the double null foliation, we are able to provide a straightforward definition of the outgoing null infinity. This makes the derivation of our asymptotic results simpler and more intuitive than the corresponding ones in the last chapter of [Ch-Kl]. In particular we are able to give a simple derivation of the connection between the Bondi mass and the ADM mass.

Acknowledgments: *We want to thank D. Christodoulou for discussing with us many important ideas concerning our work. While we regret his decision to discontinue the original collaboration we would like to acknoweledge his essential role in the original setup of our proof of the main theorem. This is particularly true in connection with Chapter 7 in which we follow his suggestions concerning the formulation of the last slice problem and the proof of local existence. We are also happy to acknowledge a set of personal notes regarding the setup of the double null foliation. Their content is reflected in section 3.1 of our book.*

1.1 Generalities about Lorentz manifolds

1.1.1 Lorentz metric, vector and tensor fields, covariant derivative, Lie derivative

A Lorentzian manifold [2], or simply a spacetime, consist of a pair $(\mathcal{M}, \mathbf{g})$ where $\mathcal{M}$ is an orientable $n+1$-dimensional manifold, whose points correspond to physical events, and $\mathbf{g}$ is a Lorentzian metric defined on it, that is a smooth, non degenerate, 2-covariant symmetric tensor field of signature $(n, 1)$. This means that at each point $p \in \mathcal{M}$ one can choose a basis of $n+1$ vectors, $\{e_{(\alpha)}\}$, belonging to the tangent space $T\mathcal{M}_p$, such that

$$\mathbf{g}(e_{(\alpha)}, e_{(\beta)}) = \eta_{\alpha\beta} \tag{1.1.1}$$

for all $\alpha, \beta = 0, 1, ..., n$, where η is the diagonal matrix with entries $-1, 1, ..., 1$. If X is an arbitrary vector at p expressed, in terms of the basis $\{e_{(\alpha)}\}$, as $X = X^\alpha e_{(\alpha)}$, we have

$$\mathbf{g}(X, X) = \eta_{\alpha\beta} X^\alpha X^\beta = -(X^0)^2 + (X^1)^2 + (X^2)^2 + ... + (X^n)^2 \tag{1.1.2}$$

The primary example of a spacetime is the Minkowski spacetime, the spacetime of Special Relativity. It plays the same role, in Lorentzian geometry, as the Euclidean space in Riemannian geometry. In this case the manifold $\mathcal{M}$ is diffeomorphic to R^{n+1} and there exists globally defined systems of coordinates, x^α, relative to which the metric takes the diagonal form $-1, 1, ..., 1$. All such systems are related through Lorentz transformations and are called inertial. We shall denote the Minkowski spacetime of dimension $n+1$ by (M^{n+1}, η).

In view of 1.1.2 we see that the Lorentzian metric divides the vectors in the tangent space $T\mathcal{M}_p$ at each p, into timelike, null or spacelike according to whether the quadratic form

$$(X, X) = g_{\mu\nu} X^\mu X^\nu \tag{1.1.3}$$

is, respectively, negative zero or positive. The set of null vectors N_p form a double cone, called the null cone of the corresponding point p. The set of timelike vectors I_p form the interior of this cone. The vectors in the union of I_p and N_p are called causal. The set S_p of spacelike vectors is the complement of $I_p \cup N_p$.

[2] We assume that our reader is already familiar with the basics concepts of differential geometry such as manifolds, tensor fields, covariant, Lie and exterior differentiation. For a short introduction to thes concepts see Chapter 1 of [Haw-El].

Together with the orthonormal fames we will use in the following the null frames[3], $\{e_3, e_4, e_a\}$, satisfying

$$\mathbf{g}(e_3,e_3)=\mathbf{g}(e_4,e_4)=0\ ,\ \mathbf{g}(e_3,e_4)=-2$$
$$\mathbf{g}(e_3,e_a)=\mathbf{g}(e_4,e_a)=0\ ,\ \mathbf{g}(e_a,e_b)=\delta_{ab}$$

where e_a are orthonormal spacelike vectors with $a=1,...,n-2$

Notations: We will use systematically throughout the book the following notational conventions:
We shall use boldface characthers to denote the spacetime metric $\mathbf{g}$, the Riemann curvature tensor $\mathbf{R}$, its conformal part $\mathbf{C}$, as well as the connection $\mathbf{D}$.
Their components relative to arbitrary frames will also be denoted by boldface characthers. Thus given a frame $\{e_{(\alpha)}\}$ we write $\mathbf{g}_{\alpha\beta} = \mathbf{g}(e_\alpha, e_\beta)$, $\mathbf{R}_{\alpha\beta\gamma\delta} = \mathbf{R}(e_\alpha, e_\beta, e_\gamma, e_\delta)$ and, for an arbitrary tensor T,

$$T_{\alpha\beta\gamma\delta...} \equiv T(e_\alpha, e_\beta, e_\gamma, e_\delta, ...)$$
$$\mathbf{D}_\alpha\mathbf{D}_\beta...\mathbf{D}_\delta T_{\epsilon...\lambda} \equiv (\mathbf{D}.\mathbf{D}.\ ...\mathbf{D}.T)(e_\alpha, e_\beta, ..., e_\delta, e_\epsilon, ..., e_\lambda)\ .$$

On the other hand we do not use boldface characthers for the components of all tensors, relative to a generic system of coordinates. Thus, for instance, in 1.1.3 $g_{\mu\nu} = \mathbf{g}(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu})$.
We use the first greek letters $\alpha, \beta, \gamma, \delta, ...$ to denote the indices associated to arbitrary frames and the greek letters $\mu, \nu, \rho, \sigma, ...$ whenever we refer to spacetime coordinates.
When we refer to tensor quantities defined on a spacelike three dimensional hypersurface, Σ, we use the latin letters $i, j, l, k, ...$ In this case it will be clear from the text which kind of components we are using.
When we consider tensors restricted to two dimensional surfaces S, diffeomorphic to S^2, we use the latin letters $a, b, c, d, ...$ only to indicate their components with respect the an adapted orthonormal frame $\{e_a\}$. We will point out explicitly the cases when the components are written with respect to a generic frame or to a set of coordinates of S.
We will, however, in the sequel, restrict ourselves mainly to orthonormal or null frames and, of course, to $n=3$.

We conclude this part stating, without proof, a proposition which shows already at this level, the fundamental role played by the null cones in the Lorentz geometry, see for its proof [Haw-El], Chapter 1.

[3] We often write e_α instead of $e_{(\alpha)}$ to simplify the notations.

Proposition 1.1.1 *The specification of the null cones N_p uniquely determines the metric up to a factor of proportionality. In other words any two Lorentzian metrics on $\mathcal{M}$ which have the same structure are conformally equivalent.*

We recall the three fundamental operators of the differential geometry on a Riemann or Lorentz manifold, the exterior derivative, the Lie derivative and the connection with its associated covariant derivative.

The exterior derivative

Given a scalar function f its differential df is the 1-form defined by

$$df(X) = X(f)$$

for any vector field X. This definition can be extended for all differential forms on $\mathcal{M}$ in the following way:

Definition 1.1.1

i) d is a linear operator defined from the space of all k-forms to that of $k+1$-forms on $\mathcal{M}$. Thus for all k-forms A,B and real numbers λ, μ

$$d(\lambda A + \mu B) = \lambda dA + \mu dB$$

ii) For any k-form A and arbitrary form B

$$d(A \wedge B) = dA \wedge B + (-1)^k A \wedge dB$$

iii) For any form A

$$d^2 A = 0 \ .$$

We recall that, if Φ is a smooth map defined from $\mathcal{M}$ to another manifold $\mathcal{M}'$, then

$$d(\Phi^* A) = \Phi^*(dA) \ .$$

Finally if A is a one form and X, Y arbitrary vector fields, we have the equation

$$dA(X,Y) = \frac{1}{2}\Big(X(A(Y)) - Y(A(X)) - A([X,Y])\Big)$$

which can be easily generalised to arbitrary k forms, see [Sp], Vol.I, Chapter 7, Theorem 13.

The Lie derivative

Consider an arbitrary vector field X. In the local coordinates x^μ, the flow of X is given by the system of differential equations

$$\frac{dx^\mu}{dt} = X^\mu(x^0(t), ..., x^n(t)) \ .$$

The corresponding curves, $x^\mu(t)$, are the integral curves of X. For each point $p \in \mathcal{M}$ there exists an open neighborood $\mathcal{U}$, a small $\epsilon > 0$ and a family of diffeomorphism $\Phi_t : \mathcal{U} \to \mathcal{M}$, $|t| \leq \epsilon$, obtained by taking each point in $\mathcal{U}$ to a parameter distance t, along the integral curves of X. We use these diffeomorphisms to construct, for any given tensor T at p, the family of tensors $(\Phi_t)_* T$ at $\Phi_t(p)$.

Definition 1.1.2 *The Lie derivative $\mathcal{L}_X T$ of a tensor field T, with respect to X, is:*

$$\mathcal{L}_X T|_p \equiv \lim_{t\to 0} \frac{1}{t} \left(T|_p - (\Phi_t)_* T|_p\right) \ .$$

It has the following properties:

i) $\mathcal{L}_X$ maps linearly (p,q)-tensor fields into tensor fields of the same type.

ii) $\mathcal{L}_X$ commutes with contractions.

iii) For any tensor fields S, T,

$$\mathcal{L}_X(S \otimes T) = \mathcal{L}_X S \otimes T + S \otimes \mathcal{L}_X T \ .$$

If X is a vector field we easily check that

$$\mathcal{L}_X Y = [X, Y] \ .$$

If A is a k-form we have, as a consequence of the commutation formula of the exterior derivative with the pull-back Φ^*,

$$d(\mathcal{L}_X A) = \mathcal{L}_X(dA) \ .$$

We remark that the Lie bracket of two coordinate vector fields vanishes, $[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}] = 0$. The converse is also true, namely, see [Sp], Vol.I, Chapter 5,

Proposition 1.1.2 *If $X_{(0)}, ..., X_{(k)}$ are linearly independent vector fields in a neighbourhood of a point p and the Lie bracket of any two of them is zero then there exists a coordinate system x^μ, around p such that $X_{(\rho)} = \frac{\partial}{\partial x^\rho}$ for each $\rho = 0, ..., k$.*

The above proposition is the main step in the proof of Frobenius Theorem. To state the theorem we recall the definition of a k-distribution in $\mathcal{M}$. This is an arbitrary smooth assignement of a k-dimensional plane π_p at every point in a domain $\mathcal{U}$ of $\mathcal{M}$. The distribution is said to be involute if, for any vector fields X, Y on $\mathcal{U}$ with $X|_p, Y|_p \in \pi_p$, for any $p \in \mathcal{U}$, we have $[X,Y]|_p \in \pi_p$. This is clearly the case for integrable distributions[4]. Indeed if $X|_p, Y|_p \in T\mathcal{N}_p$ for all $p \in \mathcal{N}$, then X, Y are tangent to $\mathcal{N}$ and so is also their commutator $[X,Y]$. The Frobenius Theorem establishes that the converse is also true[5], that is being in involution is also a sufficient condition for the distribution to be integrable,

Theorem 1.1.1 *(Frobenius Theorem) A necessary and sufficient condition for a distribution $(\pi_p)_{p\in\mathcal{U}}$ to be integrable is that it is involute.*

The connection and the covariant derivative

Definition 1.1.3 *A connection[6] $\mathbf{D}$ at a point $p \in \mathcal{M}$, is a rule which assignes to each $X \in T\mathcal{M}_p$ a differential operator $\mathbf{D}_X$. This operator maps vector fields Y into vector fields $\mathbf{D}_X Y$ in such a way that, with $\alpha, \beta \in C$ and f, g scalar functions on $\mathcal{M}$,*

$$\begin{aligned} a)\ \ & \mathbf{D}_{fX+gY} Z = f\mathbf{D}_X Z + g\mathbf{D}_Y Z \\ b)\ \ & \mathbf{D}_X(\alpha Y + \beta Z) = \alpha\mathbf{D}_X Y + \beta\mathbf{D}_X Z \\ c)\ \ & \mathbf{D}_X fY = X(f)Y + f\mathbf{D}_X Y \end{aligned} \tag{1.1.4}$$

Therefore, at a generic point p,

$$\mathbf{D}Y \equiv Y^{\alpha}_{;\beta}\,\theta^{(\beta)} \otimes e_{(\alpha)} \tag{1.1.5}$$

where the $\theta^{(\beta)}$ are the one forms of the dual basis respect to the orthonormal frame $e_{(\beta)}$ [7]. On the other side, from c),

$$\mathbf{D}fY = df \otimes Y + f\mathbf{D}Y$$

so that

$$\mathbf{D}Y = \mathbf{D}(Y^{\alpha}e_{(\alpha)}) = dY^{\alpha} \otimes e_{(\alpha)} + Y^{\alpha}\mathbf{D}e_{(\alpha)}$$

[4]Recall that a distribution π on $\mathcal{U}$ is said to be integrable if through every point $p \in \mathcal{U}$ there passes a unique submanifold $\mathcal{N}$, of dimension k, such that $\pi_p = T\mathcal{N}_p$.

[5]For a proof see [Sp], Vol.I, Chapter 6.

[6]Recall that the notion of affine connection does not depend on the metric of $\mathcal{M}$.

[7]Immediately $Y^{\alpha}_{;\beta} = \theta^{(\alpha)}(\mathbf{D}_{e_{(\beta)}}Y)$.

and finally [8]

$$\mathbf{D}Y = \left(e_{(\beta)}(Y^\alpha) + Y^\gamma \theta^{(\alpha)}(\mathbf{D}_{e_{(\beta)}} e_{(\gamma)})\right) \theta^{(\beta)} \otimes e_{(\alpha)} \tag{1.1.6}$$

Therefore

$$Y^\alpha_{;\beta} = \left(e_{(\beta)}(Y^\alpha) + \mathbf{\Gamma}^\alpha_{\beta\gamma} Y^\gamma\right)$$

and the connection is, therefore, determined by its connection coefficients,

$$\mathbf{\Gamma}^\alpha_{\beta\gamma} = \theta^{(\alpha)}(\mathbf{D}_{e_{(\beta)}} e_{(\gamma)}) \tag{1.1.7}$$

which, in a coordinate basis, are the usual Christoffel symbols and have the expression

$$\Gamma^\mu_{\nu\rho} = dx^\mu(\mathbf{D}_{\frac{\partial}{\partial x^\nu}} \frac{\partial}{\partial x^\rho})$$

Finally

$$\mathbf{D}_X Y = \left(X(Y^\alpha) + \mathbf{\Gamma}^\alpha_{\beta\gamma} X^\beta Y^\gamma\right) e_{(\alpha)} \tag{1.1.8}$$

In the particular case of a coordinate frame we have

$$\mathbf{D}_X Y = \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} + \Gamma^\nu_{\rho\sigma} X^\rho Y^\sigma\right) \frac{\partial}{\partial x^\nu}$$

Definition 1.1.4 *The Levi-Civita connection on $\mathcal{M}$ is the unique connection on $\mathcal{M}$ which satisfies $\mathbf{Dg} = 0$.*

Thus for any three vector fields X, Y, Z

$$Z(\mathbf{g}(X,Y)) = \mathbf{g}(\mathbf{D}_Z X, Y) + \mathbf{g}(X, \mathbf{D}_Z Y)$$

and relative to a system of coordinates, x^μ, the Christoffel symbol of the connection is given by the standard formula

$$\Gamma^\mu_{\rho\nu} = \frac{1}{2} g^{\mu\tau} \left(\partial_\rho g_{\nu\tau} + \partial_\nu g_{\tau\rho} - \partial_\tau g_{\nu\rho}\right) \ .$$

The Levi-Civita connection is torsion free namely

$$\mathbf{D}_X Y - \mathbf{D}_Y X = [X,Y] \ .$$

This allows to connect it to the Lie derivative. Thus if T is a k-covariant tensor we have, in a coordinate basis,

$$(\mathcal{L}_X T)_{\sigma_1 \dots \sigma_k} = X^\mu T_{\sigma_1 \dots \sigma_k ; \mu} + X^\mu_{\ ;\sigma_1} T_{\mu \sigma_2 \dots \sigma_k} + \dots . + X^\mu_{\ ;\sigma_k} T_{\sigma_1 \dots \sigma_{k-1} \mu} \ .$$

The covariant derivative is also connected to the exterior derivative according to the following simple formula. If A is a k-form, we have [9] $A_{[\sigma_1 \dots \sigma_k ; \mu]} =$

[8] Using that, from the previous definitions, $df(\cdot) = e_{(\alpha)}(f)\theta^{(\alpha)}(\cdot)$.

[9] $[\sigma_1 ... \sigma_k ; \mu]$ indicates the antisimmetrisation with respect to all indices and $,\mu$ indicates the ordinary derivative with respect to x^μ.

$A_{[\sigma_1 \dots \sigma_k, \mu]}$ and

$$dA = \sum A_{\sigma_1 \dots \sigma_k ; \mu} dx^\mu \wedge dx^{\sigma_1} \wedge dx^{\sigma_2} \wedge \dots \wedge dx^{\sigma_k} \ .$$

Definition 1.1.5 *Given a smooth curve $\mathbf{x} : [0,1] \to \mathcal{M}$, parametrized by t, let $T = \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}}$ be the corresponding tangent vector field. A vector field X, defined on the curve, is said to be parallel transported along it if $\mathbf{D}_T X = 0$. Let the curve have the parametric equations $x^\nu = x^\nu(t)$, then $T^\mu = \frac{dx^\mu}{dt}$ and the components $X^\mu = X^\mu(\mathbf{x}(t))$ satisfy the ordinary differential system of equations*

$$\frac{\mathbf{D}}{dt} X^\mu \equiv \frac{dX^\mu}{dt} + \Gamma^\mu_{\rho\sigma}(\mathbf{x}(t)) \frac{dx^\rho}{dt} X^\sigma = 0 \ .$$

The curve is said to be geodesic if, at every point of the curve, $\mathbf{D}_T T$ is tangent to the curve, $\mathbf{D}_T T = \lambda T$. In this case one can reparametrize the curve such that, relatively to the new parameter s, the tangent vector $S = \left(\frac{\partial}{\partial s}\right)_{\mathbf{x}}$ satisfies $\mathbf{D}_S S = 0$. Such a parameter is called an "affine parameter". The affine parameter is defined up to a transformation $s = as' + b$ for a, b constants. Relative to an affine parameter s and arbitrary coordinates x^μ the geodesic curves satisfy the equations

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0 \ .$$

A geodesic curve parametrized by an affine parameter is simply called a geodesic. Timelike geodesics correspond to histories of particles freely falling in the gravitational field represented by the connection coefficients. In this case the affine parameter s is called the proper time of the particle.

Given a point $p \in \mathcal{M}$ and a vector X in the tangent space $T_p\mathcal{M}$, let $\mathbf{x}(t)$ be the unique geodesic starting at p with "velocity" X. We define the exponential map:

$$\exp_p : T_p\mathcal{M} \to \mathcal{M} \ .$$

This map may not be defined for all $X \in T_p\mathcal{M}$. The theorem of existence for systems of ordinary differential equations implies that the exponential map is defined in a neighbourhood of the origin in $T_p\mathcal{M}$. If the exponential map is defined for all $T_p\mathcal{M}$, for every point p the manifold $\mathcal{M}$ is said geodesically complete. In general if the connection is a C^r connection[10] there exists an open neighbourhood $\mathcal{U}_0$ of the origin in $T_p\mathcal{M}$ and an open neighbourhood of the point p in $\mathcal{M}$, $\mathcal{V}_p$, such that the map $\exp_p$ is a C^r diffeomorphism of $\mathcal{U}_0$ onto $\mathcal{V}_p$. The neighbourhood $\mathcal{V}_p$ is called a normal neighbourhood of p[11].

[10] A C^r connection is such that if Y is a C^{r+1} vector field then $\mathbf{D}Y$ is a C^r vector field.

[11] For a more general discussion of the exponential map see [Sp], Vol.I and [Haw-El].

1.1.2 Riemann curvature tensor, Ricci tensor, Bianchi identities

In the flat spacetime if we parallel transport a vector along any closed curve we obtain the vector we have started with. This fails in general because the second covariant derivatives of a vector field do not commute. This lack of commutation is measured by the Riemann curvature tensor,

$$\mathbf{R}(X,Y)Z = \mathbf{D}_X(\mathbf{D}_Y Z) - \mathbf{D}_Y(\mathbf{D}_X Z) - \mathbf{D}_{[X,Y]}Z \tag{1.1.9}$$

or written in components relative to an arbitrary frame,

$$\mathbf{R}^{\alpha}{}_{\beta\gamma\delta} = \theta^{(\alpha)}\Big((\mathbf{D}_\gamma\mathbf{D}_\delta - \mathbf{D}_\delta\mathbf{D}_\gamma)e_{(\beta)}\Big) \tag{1.1.10}$$

Relatively to a coordinate system x^μ and written in terms of the $g_{\mu\nu}$ components, the Riemann components have the expression

$$R^{\mu}{}_{\nu\rho\sigma} = \frac{\partial\Gamma^{\mu}_{\sigma\nu}}{\partial x^\rho} - \frac{\partial\Gamma^{\mu}_{\rho\nu}}{\partial x^\sigma} + \Gamma^{\mu}_{\rho\tau}\Gamma^{\tau}_{\sigma\nu} - \Gamma^{\mu}_{\sigma\tau}\Gamma^{\tau}_{\rho\nu} \tag{1.1.11}$$

The fundamental property of the curvature tensor, first proved by Riemann, states that if $\mathbf{R}$ vanishes identically in a neighbourhood of a point p one can find families of local coordinates such that, in a neighbourhood of p, $g_{\mu\nu} = \eta_{\mu\nu}$ [12].

The trace of the curvature tensor, relative to the metric $\mathbf{g}$, is a symmetric tensor called the Ricci tensor [13],

$$\mathbf{R}_{\alpha\beta} = \mathbf{g}^{\gamma\delta}\mathbf{R}_{\alpha\gamma\beta\delta}$$

The scalar curvature is the trace of the Ricci tensor

$$\mathbf{R} = \mathbf{g}^{\alpha\beta}\mathbf{R}_{\alpha\beta}\ .$$

The Riemann curvature tensor of an arbitrary spacetime $(\mathcal{M},\mathbf{g})$ has the following symmetry properties [14],

$$\begin{aligned}\mathbf{R}_{\alpha\beta\gamma\delta} &= -\mathbf{R}_{\beta\alpha\gamma\delta} = -\mathbf{R}_{\alpha\beta\delta\gamma} = \mathbf{R}_{\gamma\delta\alpha\beta}\\ \mathbf{R}_{\alpha\beta\gamma\delta} &+ \mathbf{R}_{\alpha\gamma\delta\beta} + \mathbf{R}_{\alpha\delta\beta\gamma} = 0\end{aligned} \tag{1.1.12}$$

[12] For a thorough discussion and proof of this fact we refer to the book of Spivak, [Sp], Vol.II.

[13] In a generic frame $\mathbf{R}_{\alpha\beta} \equiv \mathrm{Ricci}(e_{(\alpha)}, e_{(\beta)})$, $\mathbf{g}^{\alpha\beta} \equiv \mathbf{g}^{-1}_{\alpha\beta}$.

[14] The second ones are called: first Bianchi identities.

It also satisfies the second Bianchi identities, which we refer here as *Bianchi equations* and, in a generic frame, have the form:

$$\mathbf{D}_{[\epsilon}\mathbf{R}_{\gamma\delta]\alpha\beta}=0 \tag{1.1.13}$$

The traceless part of the curvature tensor, $\mathbf{C}$, has the following expression, in a generic frame,

$$\begin{aligned}\mathbf{C}_{\alpha\beta\gamma\delta} &= \mathbf{R}_{\alpha\beta\gamma\delta}-\frac{1}{n-1}\Big(\mathbf{g}_{\alpha\gamma}\mathbf{R}_{\beta\delta}+\mathbf{g}_{\beta\delta}\mathbf{R}_{\alpha\gamma}-\mathbf{g}_{\beta\gamma}\mathbf{R}_{\alpha\delta}-\mathbf{g}_{\alpha\delta}\mathbf{R}_{\beta\gamma}\Big)\\ &+ \frac{1}{n(n-1)}(\mathbf{g}_{\alpha\gamma}\mathbf{g}_{\beta\delta}-\mathbf{g}_{\alpha\delta}\mathbf{g}_{\beta\gamma})\mathbf{R}\end{aligned} \tag{1.1.14}$$

Observe that $\mathbf{C}$ verifies all the symmetry properties of the Riemann tensor:

$$\begin{aligned}&\mathbf{C}_{\alpha\beta\gamma\delta}=-\mathbf{C}_{\beta\alpha\gamma\delta}=-\mathbf{C}_{\alpha\beta\delta\gamma}=\mathbf{C}_{\gamma\delta\alpha\beta}\\ &\mathbf{C}_{\alpha\beta\gamma\delta}+\mathbf{C}_{\alpha\gamma\delta\beta}+\mathbf{C}_{\alpha\delta\beta\gamma}=0\end{aligned} \tag{1.1.15}$$

and, in addition, $\mathbf{g}^{\alpha\gamma}\mathbf{C}_{\alpha\beta\gamma\delta}=0$.

We say that two metrics $\mathbf{g}$ and $\hat{\mathbf{g}}$ are conformal if $\hat{\mathbf{g}}=\lambda^2\mathbf{g}$ for some non zero differentiable function λ. Then the following theorem holds, see [Haw-El], chapter 1,

Theorem 1.1.2 *Let* $\hat{\mathbf{g}}=\lambda^2\mathbf{g}$, $\hat{\mathbf{C}}$ *the Weyl tensor relative to* $\hat{\mathbf{g}}$ *and* $\mathbf{C}$ *the Weyl tensor relative to* $\mathbf{g}$. *Then*

$$\hat{\mathbf{C}}^{\alpha}_{\beta\gamma\delta}=\mathbf{C}^{\alpha}_{\beta\gamma\delta}\ ,$$

showing that $\mathbf{C}$ *is conformally invariant.*

1.1.3 Isometries and conformal isometries, Killing and conformal Killing vector fields

Definition 1.1.6 *A diffeomorphism* $\Phi:\mathcal{U}\subset\mathcal{M}\to\mathcal{M}$ *is said to be a conformal isometry if, at every point* p, $\Phi_*\mathbf{g}=\Lambda^2\mathbf{g}$, *that is,*

$$(\Phi^*\mathbf{g})(X,Y)|_p=\mathbf{g}(\Phi_*X,\Phi_*Y)|_{\Phi(p)}=\Lambda^2\mathbf{g}(X,Y)|_p$$

with $\Lambda\neq 0$. *If* $\Lambda=1$, Φ *is called an isometry of* $\mathcal{M}$.

Definition 1.1.7 *A vector field K which generates a one parameter group of isometries, respectively, conformal isometries is called a Killing, respectively, conformal Killing vector field.*

Let K be such a vector field and Φ_t the corresponding one parameter group. Since the $(\Phi_t)_*$ are conformal isometries, we infer that $\mathcal{L}_K\mathbf{g}$ must be proportional to the metric $\mathbf{g}$. Moreover $\mathcal{L}_K\mathbf{g} = 0$ if K is a Killing vector field.

Definition 1.1.8 *Given an arbitrary vector field X we denote ${}^{(X)}\pi$ the deformation tensor of X defined by the formula*

$$ {}^{(X)}\pi_{\alpha\beta} = (\mathcal{L}_X g)_{\alpha\beta} = \mathbf{D}_\alpha X_\beta + \mathbf{D}_\beta X_\alpha \ . $$

${}^{(X)}\pi$ measures, in a precise sense, how much the diffeomorphism generated by X differs from an isometry or a conformal isometry. The following Proposition holds, see [Haw-El], chapter 1, page 43,

Proposition 1.1.3 *The vector field X is Killing if and only if ${}^{(X)}\pi = 0$. It is conformal Killing if and only if ${}^{(X)}\pi$ is proportional to $\mathbf{g}$.*

Remark: One can choose local coordinates $x^0, x^1, ..., x^n$ such that $X = \frac{\partial}{\partial x^0}$. It then immediately follows that, relative to these coordinates the metric $\mathbf{g}$ is independent of x^0.

Proposition 1.1.4 *On any spacetime $\mathcal{M}$, of dimension $n+1$, there can be no more than $\frac{1}{2}(n+1)(n+2)$ linearly independent Killing vector fields.*

Proof: Proposition 1.1.4 is an easy consequence of the following relation, valid for an arbitrary vector field X, obtained by a straightforward computation and the use of the Bianchi identities.

$$ \mathbf{D}_\beta\mathbf{D}_\alpha X_\lambda = \mathbf{R}_{\lambda\alpha\beta\delta}X^\delta + {}^{(X)}\Gamma_{\alpha\beta\lambda} \tag{1.1.16} $$

where

$$ {}^{(X)}\Gamma_{\alpha\beta\lambda} = \frac{1}{2}\left(\mathbf{D}_\beta\pi_{\alpha\lambda} + \mathbf{D}_\alpha\pi_{\beta\lambda} - \mathbf{D}_\lambda\pi_{\alpha\beta}\right) \tag{1.1.17} $$

and $\pi \equiv {}^{(X)}\pi$ is the X deformation tensor. In fact, if X is a Killing vector field the previous equation 1.1.16 becomes

$$ \mathbf{D}_\beta(\mathbf{D}_\alpha X_\lambda) = \mathbf{R}_{\lambda\alpha\beta\delta}X^\delta \tag{1.1.18} $$

and this implies that any Killing vector field is completely determined by the $\frac{1}{2}(n+1)(n+2)$ values of X and $\mathbf{D}X$ at a given point. Then the argument goes in this way: let p, q be two points connected by a curve $x(t)$ with tangent vector T. Let $L_{\alpha\beta} \equiv \mathbf{D}_\alpha X_\beta$; along $x(t)$, X, L verify the system of differential equations

$$\frac{\mathbf{D}}{dt}X = T \cdot L \quad , \quad \frac{\mathbf{D}}{dt}L = \mathbf{R}(\cdot, \cdot, X, T)$$

therefore the values of X, L along the curve are uniquely determined by their values at p.

The spacetime which possesses the maximum number of Killing and conformal Killing vector fields is the Minkowski spacetime $\mathbf{M}^{n+1}$. Let us review their associated isometries and conformal isometries.

Let x^μ be an inertial coordinate system, positively oriented, we have:

1. Translations: for any given vector $a = (a^0, a^1, ..., a^n) \in \mathbf{M}^{n+1}$,

$$x^\mu \to x^\mu + a^\mu$$

2. Lorentz rotations: Given any $\Lambda = \Lambda^\rho_\sigma \in \mathbf{O}(1, n)$,

$$x^\mu \to \Lambda^\mu_\nu x^\nu$$

3. Scalings: Given any real number $\lambda \neq 0$,

$$x^\mu \to \lambda x^\mu$$

4. Inversion: Consider the transformation $x^\mu \to I(x^\mu)$, where

$$I(x^\mu) = \frac{x^\mu}{(x, x)}$$

defined for all points $x \in \mathbf{M}^{n+1}$ such that $(x, x) \neq 0$.

The first two sets of transformations are isometries of $\mathbf{M}^{n+1}$, the group generated by them is called the Poincarè group. The last two type of transformations are conformal isometries. the group generated by all the above transformations is called the Conformal group. In fact the Liouville theorem, whose infinitesimal version will be proved later on, states that it is the group of all the conformal isometries of $\mathbf{M}^{n+1}$.
Let us list the Killing and conformal Killing vector fields which generate the above transformations.

i. The generators of translations in the x^μ directions, $\mu = 0, 1, ..., n$:

$$T_\mu = \frac{\partial}{\partial x^\mu}$$

ii. The generators of the Lorentz rotations in the (μ,ν) plane:

$$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$

iii. The generators of the scaling transformations:

$$S = x^\mu \partial_\mu$$

iv. The generators of the inverted translations [15]:

$$K_\mu = 2x_\mu x^\rho \frac{\partial}{\partial x^\rho} - (x^\rho x_\rho)\frac{\partial}{\partial x^\mu}$$

We also list below the commutator relations between these vector fields,

$$\begin{cases} [L_{\alpha\beta}, L_{\gamma\delta}] = \eta_{\alpha\gamma} L_{\beta\delta} - \eta_{\beta\gamma} L_{\alpha\delta} + \eta_{\beta\delta} L_{\alpha\gamma} - \eta_{\alpha\delta} L_{\beta\gamma} \\ [L_{\alpha\beta}, T_\gamma] = \eta_{\alpha\gamma} T_\beta - \eta_{\beta\gamma} T_\alpha \\ [T_\alpha, T_\beta] = 0 \\ [T_\alpha, S] = T_\alpha \\ [T_\alpha, K_\beta] = 2(\eta_{\alpha\beta} S + L_{\alpha\beta}) \\ [L_{\alpha\beta}, S] = [K_\alpha, K_\beta] = 0 \\ [L_{\alpha\beta}, K_\gamma] = \eta_{\alpha\gamma} K_\beta - \eta_{\beta\gamma} K_\alpha \end{cases} \qquad (1.1.19)$$

Denoting $\mathcal{P}(1,n)$ the Lie algebra generated by the vector fields $T_\alpha, L_{\beta\gamma}$ and $\underline{\mathcal{K}}(1,n)$ the Lie algebra generated by all the vector fields $T_\alpha, L_{\beta\gamma}, S, K_\delta$ we state the following version of the Liouville theorem,

Theorem 1.1.3

1) $\mathcal{P}(1,n)$ is the Lie algebra of all Killing vector fields in $\mathbf{M}^{n+1}$.
2) If $n > 1$, $\underline{\mathcal{K}}(1,n)$ is the Lie algebra of all conformal Killing vector fields in $\mathbf{M}^{n+1}$.
3) If $n = 1$, the set of all conformal Killing vector fields in $\mathbf{M}^{1+1}$ is given by the following expression

$$f(x^0 + x^1)(\partial_0 + \partial_1) + g(x^0 - x^1)(\partial_0 - \partial_1)$$

where f, g are arbitrary smooth functions of one variable.

[15] Observe that the vector fields K_μ can be obtained applying I_* to the vector fields T_μ.

Proof: The proof for part 1 of the theorem follows immediately, as a particular case, from Proposition 1.1.4. From 1.1.16 as $\mathbf{R} = 0$ and X is Killing we have

$$D_\mu D_\nu X_\lambda = 0 \ .$$

Therefore, there exist constants $a_{\mu\nu}, b_\mu$ such that $X^\mu = a_{\mu\nu}x^\nu + b_\mu$. Since X is Killing $D_\mu X_\nu = -D_\nu X_\mu$ which implies $a_{\mu\nu} = -a_{\nu\mu}$. Consequently X can be written as a linear combination, with real coefficients, of the vector fields $T_\alpha, L_{\beta\gamma}$.
Let now X be a conformal Killing vector field. There exists a function Ω such that

$$^{(X)}\pi_{\rho\sigma} = \Omega\eta_{\rho\sigma} \tag{1.1.20}$$

From 1.1.16 and 1.1.17 it follows that

$$D_\mu D_\nu X_\lambda = \frac{1}{2}\left(\Omega_{,\mu}\eta_{\nu\lambda} + \Omega_{,\nu}\eta_{\mu\lambda} - \Omega_{,\lambda}\eta_{\nu\mu}\right) \tag{1.1.21}$$

Taking the trace with respect to μ, ν, on both sides of 1.1.21 we infer that

$$\begin{aligned} \Box\, X_\lambda &= -\frac{n-1}{2}\Omega_\lambda \\ D^\mu X_\mu &= \frac{n+1}{2}\Omega \end{aligned} \tag{1.1.22}$$

and applying D^λ to the first equation, $\Box$ to the second one and subtracting we obtain

$$\Box\, \Omega = 0 \tag{1.1.23}$$

Applying D_μ to the first equation of 1.1.22 and using 1.1.23 we obtain

$$\begin{aligned} (n-1)D_\mu D_\lambda \Omega &= \frac{n-1}{2}(D_\mu D_\lambda \Omega + D_\lambda D_\mu \Omega) = -\Box\,(D_\mu X_\lambda + D_\lambda X_\mu) \\ &= -(\Box\,\Omega)\eta_{\mu\lambda} = 0 \end{aligned} \tag{1.1.24}$$

Hence for $n \neq 1$, $D_\mu D_\lambda \Omega = 0$. This implies that Ω must be a linear function of x^μ. We can therefore find a linear combination, with constant coefficients, $cS + d^\alpha K_\alpha$ such that the deformation tensor of $X - (cS + d^\alpha K_\alpha)$ must be zero. This is the case because $^{(S)}\pi = 2\eta$ and $^{(K_\mu)}\pi = 4x_\mu\eta$. Therefore $X - (cS + d^\alpha K_\alpha)$ is Killing which, in view of the first part of the theorem, proves the result.

Part 3 can be easily derived by solving 1.1.20. Indeed posing $X = a\partial_0 + b\partial_1$, we obtain $2D_0X_0 = -\Omega$, $2D_1X_1 = \Omega$ and $D_0X_1 + D_1X_0 = 0$. Hence a, b verify the system

$$\frac{\partial a}{\partial x^0} = \frac{\partial b}{\partial x^1} \ , \ \frac{\partial b}{\partial x^0} = \frac{\partial a}{\partial x^1} \ .$$

Hence the one form $adx^0 + bdx^1$ is exact, $adx^0 + bdx^1 = d\phi$, and $\frac{\partial^2 a}{\partial x^{0\,2}} = \frac{\partial^2 b}{\partial x^{1\,2}}$, that is $\Box\, \phi = 0$. In conclusion

$$X = \frac{1}{2}\left(\frac{\partial \phi}{\partial x^0} + \frac{\partial \phi}{\partial x^1}\right)(\partial_0 + \partial_1) + \frac{1}{2}\left(\frac{\partial \phi}{\partial x^0} - \frac{\partial \phi}{\partial x^1}\right)(\partial_0 - \partial_1)$$

which proves the result.

1.2 The Einstein equations

The Einstein equations links the metric $g_{\mu\nu}$ to the matter fields ψ, with energy-momentum tensor $T(\psi)_{\mu\nu}$, by

$$\begin{aligned} G_{\mu\nu} &= 8\pi T_{\mu\nu} \\ F(\psi) &= 0 \end{aligned} \tag{1.2.1}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor, R, the scalar curvature, is the trace of the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$; the second line of 1.2.1 summarizes the dynamical equations of the matter fields. As a consequence of the twice contracted Bianchi identities the energy momentum tensor $T_{\mu\nu}$ satisfies the local conservation laws,

$$D^\mu T_{\mu\nu} = 0 \ .$$

It is also important to emphasize that a solution of the coupled Einstein-matter field equations is, in fact, a class of equivalence of solutions. More precisely if Φ is a diffeomorphism of $\mathcal{M}$ then $\{\mathcal{M}, \mathbf{g}, \psi\}$ and $\{\mathcal{M}, \Phi^*\mathbf{g}, \Phi^*\psi\}$ describe the "same" solution of the Einstein equations.

1.2.1 The initial value problem, initial data sets, constraint equations

The general formulation of the initial value problem is given in the definitions below:

Definition 1.2.1 *An initial data set is given by a set* $\{\Sigma, \overline{g}, \overline{k}, \overline{\psi}\}$ *where* Σ *is a three-dimensional manifold,* $\overline{\psi}$ *the prescribed matter fields on* Σ*,* $\overline{g}$ *a Riemannian metric* [16] *and* $\overline{k}$ *a covariant symmetric tensor field satisfying the constraint equations* [17]*:*

$$\begin{aligned} &\nabla^j \overline{k}_{ij} - \nabla_i tr\overline{k} = 8\pi j_i \\ &\overline{R} - |\overline{k}|^2 + (tr\overline{k})^2 = 16\pi\rho \end{aligned} \tag{1.2.2}$$

Two initial data sets $\{\Sigma, \overline{g}_1, \overline{k}_1, \overline{\psi}_1\}$ *and* $\{\Sigma, \overline{g}_2, \overline{k}_2, \overline{\psi}_2\}$ *are said to be equivalent if there exists a diffeomorphism* χ *of* Σ *such that* $\bar{g}_2 = \chi^* \bar{g}_1$*,* $\bar{k}_2 = \chi^* \bar{k}_1$*,* $\bar{\psi}_2 = \chi^* \bar{\psi}_1$*.*

Definition 1.2.2 (The Cauchy problem) *To solve the Einstein-matter field equations with a given initial data set means to find a four dimensional manifold* $\mathcal{M}$*, a Lorentz metric* g *and fields* ψ *satisfying the coupled Einstein-matter equations as well as an imbedding*

$$i : \Sigma \to \mathcal{M}$$

such that $i^*(g) = \overline{g}$*,* $i^*(k) = \overline{k}$*,* $i^*(\psi) = \overline{\psi}$ *where* g *is the induced metric and* k *is the second fundamental form (the extrinsic curvature) of the submanifold* $i(\Sigma) \subset \mathcal{M}$*. The constraint equations for* $\overline{g}$ *and* $\overline{k}$ *are thus the pull back of the Codazzi and Gauss equations induced on* $i(\Sigma)$*. Two equivalent initial data sets are supposed to lead to equivalent solutions.*

Definition 1.2.3 *The spacetime manifold* $\mathcal{M}$ *defined above is called a development of the initial data set* $\{\Sigma, \overline{g}, \overline{k}, \overline{\psi}\}$*.*

Definition 1.2.4 *If the spacetime* $\mathcal{M}$ *is globally hyperbolic* [18]*, it is called a Cauchy development of the initial data set* $\{\Sigma, \overline{g}, \overline{k}, \overline{\psi}\}$*.*

From now on we will restrict ourselves to the Einstein equations in the vacuum case. In other words we assume everywhere $T(\psi) = 0$. Therefore the Einstein equations take the form

$$R_{\mu\nu} = 0\ .$$

[16] The differentiability class of g and k will be discussed later on.

[17] Here $j_i = T_{0i}(\overline{\psi})$, $\rho = T_{00}(\overline{\psi})$. Also $\overline{R}_{ij}$ denotes the Ricci curvature of Σ with the metric $\overline{g}$, $\overline{R}$ the scalar curvature.

[18] A spacetime $\mathcal{M}$ is globally hyperbolic if it contains a Cauchy surface, see [Haw-El], chapter 6.

1.3 Local existence for the Einstein's vacuum equations

1.3.1 Reduction to the non linear wave equations

The first successfull approach to the general initial value problem in General Relativity is due to Y.Choquet-Bruhat, [Br1], [Br2]. She made use of wave-like coordinates in order to get round the diffeomorphism invariance and to write the Einstein equations as an hyperbolic system in the sense of Leray, [Le], so to obtain local in time existence and uniqueness. Later the result was revisited and improved by many authors, see in particular the result of Hughes, Kato and Marsden, [Hu-Ka-Ms] and of A.Fisher and J.Mardsen [F-Ms1], [F-Ms2], who expressed the reduced equations, 1.3.1, in the form of a symmetric hyperbolic system to which they could apply the general theory developed by T.Kato, see also [Ch-Mu].
In what follows we shall give a short review of the wave-like coordinates and the derivation of the reduced Einstein equations.

Let $\hat{g}$ be a given Lorentz metric on $\mathcal{M}$. For a metric g on $\mathcal{M}$, we introduce[19]

$$V^\alpha = g^{\mu\nu}(\Gamma^\alpha_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\nu})$$

and define, with $\hat{D}$ the covariant deivative associated to the metric $\hat{g}$,

$$R^{(h)}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}(\hat{D}_\alpha V_\beta + \hat{D}_\beta V_\alpha) \ .$$

A simple calculation shows,

$$R^{(h)}_{\alpha\beta} = -\frac{1}{2}\Box_g g_{\alpha\beta} + H_{\alpha\beta}(g, \partial g)$$

with H a quadratic expression depending on g and its first derivatives[20] and

$$\Box_g g_{\alpha\beta} = g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} \ .$$

The condition $V^\alpha = 0$ is satisfied if and only if the identity map

$$\mathrm{Id} : (\mathcal{M}, g) \to (\mathcal{M}, \hat{g})$$

[19] $\Gamma^\alpha_{\mu\nu}$ and $\hat{\Gamma}^\alpha_{\mu\nu}$ are not tensors, but their difference is.
[20] Of course, depending also on $\hat{g}$, $\partial\hat{g}$ and $\partial^2\hat{g}$.

is a wave map. This means that x has to satisfy the following equation

$$\Box_g x^\alpha + \hat{\Gamma}^\alpha_{\beta\gamma}\partial_\lambda x^\beta \partial^\lambda x^\gamma = -(\hat{\Gamma}^\alpha - \Gamma^\alpha) = 0$$

In that case, the vacuum equations are "reduced" to $R^{(h)}_{\alpha\beta} = 0$, that is to

$$\Box\,_g g_{\alpha\beta} = 2H_{\alpha\beta}(g, \partial g) \tag{1.3.1}$$

which is a "weakly coupled" system[21] of non linear wave equations. The general case described above was used in the work of Friedrich, see [Fr3] and also [Fr4].

In the case the background metric is the Minkowski one, $\hat{g}_{\alpha\beta} = \eta_{\alpha\beta}$, we have $\hat{\Gamma}^\alpha_{\mu\nu} = 0$, and $V^\alpha = g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha$. The condition (1.3.1) reduces to the more familiar wave-like coordinates condition

$$\Box\,_g x^\alpha = 0, \qquad \alpha = 0, 1, 2, 3. \tag{1.3.2}$$

In what follows we restrict ourselves to the choice $\hat{g}_{\alpha\beta} = \eta_{\alpha\beta}$. To construct solutions of the Einstein equations one solves the reduced equations 1.3.1, subject to initial conditions satisfying the constraint equations ${G^0}_\mu = 0$, where $G_{\mu\nu}$ is the Einstein tensor. Observe that in the constraint equations the second derivatives with respect to t of the metric are absent. Moreover if we choose Σ as the hyperplane $t = 0$ and define $k_{ij} = -\frac{1}{2}\frac{\partial g_{ij}}{\partial t}$ they coincide with equations 1.2.2 with $\rho = j_i = 0$.

The main goal of the approach described above is, therefore, to reduce the general system of the Einstein equations $R(g)_{\mu\nu} = 0$ to the hyperbolic system

$$R^{(h)}(g)_{\mu\nu} = 0 \tag{1.3.3}$$

This has to be connected with the initial value formulation. In this respect the crucial observation is that if the constraints and the condition $\Gamma^\alpha = 0$ are satisfied by the initial data then they are automatically propagated by the solutions of the reduced equation, 1.3.1. The precise statement is given by the following Proposition whose proof is in [Ch-Mu] and [F-Ms1], see also [Br-Fo].

Proposition 1.3.1 *Let $g_{\mu\nu}$ be the components of a metric tensor g written in a specific set of coordinates, x^μ, with $x^0 = t$, $x = (x^1, x^2, x^3)$, such that:*

[21] The system has a diagonal structure with respect to the highest order terms.

i) On $\mathcal{M}$ [22] *it satisfies the "reduced" Einstein equations* $R^{(h)}_{\mu\nu}(g) = 0$.
ii) On $\Sigma \equiv \Sigma_{t=0}$ *it satisfies the initial conditions*

$$(g_{\mu\nu}(0,x), \frac{\partial g_{\mu\nu}}{\partial t}(0,x)) = (\phi_{\mu\nu}(x), \psi_{\mu\nu}(x))$$

where $\{\phi_{\mu\nu}(x), \psi_{\mu\nu}(x)\}$ *satisfy the conditions* $\overline{\Gamma}^{\alpha}(x^{\lambda}) = 0$ *and the constraints* $\overline{g}^{0\nu}\overline{G}_{\nu\mu}(x) = 0$. *Then* $\Gamma^{\alpha}(x^{\lambda}) = 0$ *on the whole* $\mathcal{M}$ *and, therefore,* $g_{\mu\nu}$ *is also a solution of the Einstein equations* $R_{\mu\nu}(g) = 0$.

The proof of the proposition is achieved in two steps which we sketch below, see also [Hu-Ka-Ms], [Wa2],

Step 1: Let $\{\Sigma, \bar{g}, \bar{k}\}$ be an initial data set. Let us require that the coordinate system is also Gaussian on Σ, adding therefore to $\bar{g}_{ij} \equiv g_{ij}(0,x)$, $g_{00}(0,x) = -1$, $g_{0i}(0,x) = 0$. There exists a coordinate transformation ξ, $x^{\lambda} \to x'^{\lambda} = \xi^{\lambda}(x^{\alpha})$, such that on Σ:

$$\begin{aligned} &(\xi^0(0,x), \xi^j(0,x)) = (0, x^j) \ , \ (\frac{\partial \xi^0}{\partial t}(0,x), \frac{\partial \xi^j}{\partial t}(0,x)) = (1,0) \\ &\frac{\partial^2 \xi^{\lambda}}{\partial t^2}(0,x) = \frac{1}{g^{00}(0,x)}\Gamma^{\lambda}(0,x) \end{aligned} \qquad (1.3.4)$$

In the new coordinates x' we have

$$\begin{aligned} &{g'}_{\mu\nu}(0,x) = g_{\mu\nu}(0,x) \ , \ {k'}_{ij}(0,x) = k_{ij}(0,x) \\ &{k'}_{0i}(0,x) = \frac{1}{2}\frac{\partial {g'}_{0i}}{\partial t'}(0,x) = k_{0i}(0,x) + \frac{1}{2}g_{ij}(0,x)\Gamma^j(0,x) \\ &{k'}_{00}(0,x) = \frac{1}{2}\frac{\partial {g'}_{00}}{\partial t'}(0,x) = k_{00}(0,x) - \Gamma^0(0,x) \end{aligned} \qquad (1.3.5)$$

and, from the transformation rule of Γ^{α},

$$\Gamma'^{\alpha} = \Gamma^{\rho}\frac{\partial \xi^{\alpha}}{\partial x^{\rho}} - g^{\rho\sigma}\frac{\partial^2 \xi^{\alpha}}{\partial x^{\rho}\partial x^{\sigma}} = -\Box_g \xi^{\alpha} \qquad (1.3.6)$$

it follows that $\Gamma'^{\alpha}(0,x) = 0$. Moreover the conditions $\frac{\partial \Gamma'^{\mu}}{\partial t} = 0$ on Σ are automatically fulfilled from the constraint equations $G^0_{\mu} = 0$ when g satisfies the reduced equations.

Step 2: $\Gamma^{\alpha}(0,x) = 0$ and $\frac{\partial \Gamma^{\mu}}{\partial t}(0,x) = 0$ imply $\Gamma^{\alpha} = 0$ on the whole $\mathcal{M}$. This is achieved by observing that the twice contracted Bianchi equations lead to the following system of linear equations satisfied by the Γ^{α},

$$\frac{1}{2}g^{\beta\gamma}\frac{\partial^2 \Gamma^{\mu}}{\partial x^{\beta}\partial x^{\gamma}} + A^{\beta\mu}_{\nu}(g, \partial g)\frac{\partial \Gamma^{\nu}}{\partial x^{\beta}} = 0$$

[22] Here with $\mathcal{M}$ we indicate the region of R^{3+1} where the reduced equations are satisfied.

with $g_{\rho\sigma}$ solution of the reduced equations 1.3.3. The uniqueness properties of the initial value problem for such system proves the result.

Therefore, given Proposition 1.3.1, to reduce the solution of the Einstein vacuum equations to the solution of the "reduced system" 1.3.1, we need initial data which satisfy the constraint equations and also the conditions $\Gamma^\alpha = 0$.
Let $\{\Sigma, \bar{g}, \bar{k}\}$ be an initial data set and let the initial conditions for the reduced system 1.3.1 given by

$$g_{\mu\nu}(0,x) = \phi_{\mu\nu}\ ,\ \frac{\partial g_{\mu\nu}}{\partial t}(0,x) = \psi_{\mu\nu}\ , \tag{1.3.7}$$

we have to connect the latter, $\{\phi_{\mu\nu}, \psi_{\mu\nu}\}$, to the former, $\{\bar{g}_{ij}, \bar{k}_{ij}\}$. To achieve that we restrict to a Gaussian coordinate system requiring that $\phi_{00} = -1, \phi_{0i} = 0$. Then from the first line of 1.3.5 we obtain immediately

$$\phi_{ij} = \overline{g}_{ij},\ \psi_{ij} = 2\overline{k}_{ij}\ .$$

The remaining data $\psi_{00},\ \psi_{0i}$ are determined from the next two lines of 1.3.5 $\Gamma^\mu|_\Sigma = 0$. The result is:

$$\psi_{00} = -4tr_g\overline{k}_{ij}\ ,\ \psi_{0i} = \triangle_g x_i \tag{1.3.8}$$

Proof: The third line of 1.3.5 can be rewritten as

$$\psi_{00} = \frac{\partial g_{00}}{\partial t}(0,x) - 2\Gamma^0(0,x) = -4\mathrm{tr}_{\bar{g}}\bar{k}$$

where the last equality comes from the explicit computation of $\Gamma^0(0,x)$: $\Gamma^0(0,x) = \frac{1}{2}\frac{\partial g_{00}}{\partial t}(0,x) + 2\mathrm{tr}_{\bar{g}}\bar{k}$.
The second relation follows from the explicit computation of the second line of 1.3.5

$$\psi_{0i} = g_{ij}(0,x)^{(3)}\Gamma^j(0,x) = -\triangle_{\bar{g}}\xi^i$$

where ${}^{(3)}\Gamma^j$ is the contracted Christoffel symbol relative to Σ. The last equality arises from the definition $\triangle_{\bar{g}}\xi^i \equiv \bar{g}^{ls}\frac{\partial^2}{\partial x^l\partial x^s}\xi^i - {}^{(3)}\Gamma^j\frac{\partial}{\partial x^l}\xi^i$ and equation 1.3.4.

In view of the fact that $G^0{}_\mu$ does not depend on $\partial_t g_{0\mu}$ it follows immediately that the constraint equations are also satisfied for this choice of initial data.

1.3.2 Local existence for the Einstein vacuum equations using wave coordinates

Before stating the local existence and uniqueness theorem[23] for the Einstein equations we briefly recall the definition of local Sobolev spaces H^s_{loc}.

Definition 1.3.1 *Given a three dimensional Riemannian manifold* Σ *and an integer*[24] $s \geq 0$ *we say that* $f \in H^s_{loc}$ *if, for any compact subset* $K \subset \Sigma$, *we have* $\int_K |D^s f|^2 < \infty$.

Theorem 1.3.1 (local existence) *Let* $\{\Sigma, \bar{g}, \bar{k}\}$ *an initial data set and assume that* Σ *admits a locally finite* C^1 *covering*[25] *by open coordinate charts* $\{U_\alpha\}$ *such that* $(\bar{g}, \bar{k}) \in H^s \times H^{s-1}$ *with some* $s > \frac{5}{2}$. *There exists a globally hyperbolic development* $(\mathcal{M}, \mathbf{g})$ *of* $(\Sigma, \bar{g}, \bar{k})$ *for which* Σ *is a Cauchy hypersurface. Moreover, if* $s > \frac{5}{2} + 1$, *the above development is unique in the sense that any other* H^s *development must be diffeomorphic to it.*

Sketch of the proof: According to the discussion of the previous section, it suffices to prove existence of solutions to the reduced Einstein equations 1.3.1. By a simple domain of dependance argument it suffices to consider that the initial data are supported in a fixed coordinate patch. One is thus reduced to study the initial value problem for systems of non linear wave equations. We shall discuss this issue in more detail in section 1 of the next chapter. For a detailed account of the proof of Theorem 1.3.1 we refer the reader to [Hu-Ka-Ms], previous proofs of the local existence theorem are in [Br1] and [F-Ms1], for a survey see [Fr-Re]. The result applies, in particular, to asymptotically flat initial data sets[26]. More precise information concerning the behavior at spacelike infinity, for asymptotically flat initial data sets, can be derived by using weighted Sobolev spaces, see [Ch-Mu].

Concerning the uniqueness we observe that the Cauchy development $(\mathcal{M}, \mathbf{g})$ described above is not unique. In fact a coordinate transformation $z = \sigma(x)$, which on Σ takes the form

$$\sigma^\mu(x) = x^\mu \ , \ \frac{\partial \sigma^\mu}{\partial x^0}(x) = \delta^\mu_0 \ , \ \frac{\partial^2 \sigma^\mu}{\partial {x^0}^2}(x) = 0 \tag{1.3.9}$$

[23]This version is due to Hughes, Kato and Mardsen, see [Hu-Ka-Ms].

[24]The definition can be extended to nonintegers with the help of the Fourier transform.

[25]This means that any point in Σ has a neighborhood which intersects only a finite number of the open sets U_α. The sets U_α are related by C^1 coordinate transformations.

[26]For these initial data sets one can derive a uniform existence time. The uniformity of time can be made precise by using the geodesic distance function from a point of the spacetime to the initial hypersurface.

does not change Γ^α on Σ_0, nor the other initial conditions. Therefore, if ξ is the transformation to the wavelike coordinates which on Σ connects $\{\Sigma, \overline{g}, \overline{k}\}$ with the initial conditions $(g_{\mu\nu}(0,x) = \phi_{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial t}(0,x) = \psi_{\mu\nu})$, see 1.3.7, then also $\{\sigma \circ \xi\}$ does.
Let g_{re} be the solution of the reduced equations 1.3.3 with initial data $(\phi_{\mu\nu}, \psi_{\mu\nu})$, then $g(x) = \xi^* g_{re}(\xi(x))$ and $\tilde{g}(x) = (\sigma \circ \xi)^* g_{re}(\sigma \circ \xi(x))$ are different solutions of the vacuum Einstein equations, $R_{\mu\nu} = 0$, corresponding to the same initial data set.
To prove the uniqueness in the sense stated in the theorem, we have also to show that any two developments of the Einstein equations, corresponding to the same initial conditions [27], are connected by a coordinate transformation, a diffeomorphism. The idea of the proof is very simple: If $\tilde{g}$ and g are two spacetime metrics corresponding to the same initial data set, then on Σ they share the same Γ^α. We then define, according to 1.3.7, the coordinate transformations $\tilde{\xi}$ and ξ, respectively, and check that the two sets of initial conditions for the reduced equations 1.3.1 coincide. Therefore the spacetime metrics $g, \tilde{g}$ produce two solutions, $\xi^{*-1}g$, $\tilde{\xi}^{*-1}\tilde{g}$, of the reduced Einstein equations 1.3.1, with the same initial conditions and satisfying $\Gamma^\alpha = 0$. In view of the uniqueness results for hyperbolic systems, the two solutions coincide. Then the composition of the transformations $\tilde{\xi}^{-1} \circ \xi$ gives the diffeomorphism we are looking for.
We remark however, see [Hu-Ka-Ms] , that the proof of uniqueness, outlined above, requires one degree more of differentiabilty than needed in the proof of existence.

1.3.3 General foliations of the Einstein spacetime

We first recall the following result due to R.Geroch, [Ge], see also [Haw-El] chapter 6.

Theorem 1.3.2 *Assume that the spacetime* $(\mathcal{M}, \mathbf{g})$ *is globally hyperbolic, then* $(\mathcal{M}, \mathbf{g})$ *can be foliated along a timelike direction and is diffeomorphic to* $R \times \mathcal{S}$ *where* $\mathcal{S}$ *is a three dimensional Riemannian manifold.*

Sketch of the proof: To construct the diffeomorphism $\mathcal{T} : R \times \mathcal{S} \to \mathcal{M}$ one proceeds in the following steps:

1) One shows first that there exists a continuous function $t(\cdot)$ on $\mathcal{M}$ such that

$$\Sigma_a = \{p \in \mathcal{M} | t(p) = a\}$$

[27] The same applies to equivalent initial data sets.

is a Cauchy hypersurface. We identify $\mathcal{S}$ with Σ_0. Then one proves, using a smoothing procedure in the definition of the time function, see [Se], that there exists a global C^2 time function t whose level sets are Cauchy hypersurfaces.

2) One defines on $\mathcal{M}$ a timelike vector field V such that, in view of the properties of the Cauchy hypersurfaces, its integral curves $\Psi(s;p)$ solution of

$$\frac{d\Psi^\mu}{ds} = V^\mu(\Psi(s))$$

define a map $\beta: \ q \in \mathcal{M} \rightarrow \beta(q) \in \mathcal{S}$ through the relation

$$q = \Psi(\bar{s}; \beta(q)) \ .$$

3) We define the diffeomorphism $\mathcal{T}$ through the relation

$$\mathcal{T}^{-1}(q) = (t(q), \beta(q)) \ .$$

Therefore the diffeomorphism $\mathcal{T}$ is specified once are defined on $\mathcal{M}$ the function $t(q)$ and the vector field

$$V|_q = \left.\frac{\partial \Psi^\mu}{\partial s}\right|_q \frac{\partial}{\partial x^\mu} \ .$$

In general the vector field V is not orthogonal to the hypersurface Σ_t. To decompose it into its orthogonal and tangent components, let us introduce the timelike vector field orthogonal to Σ_t,

$$F = \left(g^{\mu\nu}\frac{\partial t}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu} \tag{1.3.10}$$

and define the future directed unit normal vector field to Σ_t,

$$N = \frac{1}{(-\mathbf{g}(F,F))^{\frac{1}{2}}}F \tag{1.3.11}$$

Then we decompose V in a component parallel to Σ and in a orthogonal one, $V = V_{||} + V_\perp$ where

$$\begin{aligned} {V_\perp}^\mu &= -(V^\nu N_\nu)N^\mu = \frac{\mathbf{g}(V,F)}{\mathbf{g}(F,F)}F^\mu \\ {V_{||}}^\mu &= V^\mu - \frac{\mathbf{g}(V,F)}{\mathbf{g}(F,F)}F^\mu = g^{\mu\nu}(g_{\nu\rho} + N_\nu N_\rho)V^\rho = h^\mu_\rho V^\rho \end{aligned} \tag{1.3.12}$$

where $h^\mu_\nu \equiv (g^\mu_\nu + N^\mu N_\nu)$ is the projection tensor on Σ_t. The function

$$\Phi(q) \equiv (-\mathbf{g}(q)(F,F))^{-\frac{1}{2}} \tag{1.3.13}$$

is called the "lapse function" and the vector field, tangent to Σ_t,

$$X^\nu(q) \equiv \frac{{V_{||}}^\nu(q)}{\mathbf{g}(q)(V,F)} \tag{1.3.14}$$

is called the "shift vector".

Lemma 1.3.1 *The following relation holds*

$$\frac{\partial}{\partial t} = \Phi N + X \ .$$

Proof: Observe that from

$$t = t(q) = t(x^\mu(q)) = t(\Psi^\mu(s(t,p);p)$$

we have

$$1 = \frac{\partial t}{\partial t} = \frac{\partial t}{\partial x^\mu}\frac{\partial \psi^\mu}{\partial s}\frac{\partial s}{\partial t} = \mathbf{g}(F,V)\frac{\partial s}{\partial t} \tag{1.3.15}$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial x^\mu} = \frac{\partial s}{\partial t}\frac{\partial \Psi^\mu}{\partial s}\frac{\partial}{\partial x^\mu} = \frac{1}{\mathbf{g}(F,V)}V_\perp + \frac{1}{\mathbf{g}(F,V)}V_{||} \\ &= \frac{F}{\mathbf{g}(F,F)} + X = \Phi N + X \ .\end{aligned}$$

The next lemma, whose proof is in the appendix to this chapter, gives the explicit form of the metric components in the coordinates $(t(q), \beta(q))$.

Lemma 1.3.2 *Choosing as coordinates of the generic point* $q \in \mathcal{M}$

$$x^0(q) = t(q) \quad \textit{and} \quad \tilde{x}^i(q) = \beta^i(q) \ ,$$

the metric tensor $\mathbf{g}$ *has the following expression:*

$$\mathbf{g}(q)(\cdot,\cdot) = -\Phi^2(q)dt^2 + g_{ij}(q)(d\tilde{x}^i + X^i dt)(d\tilde{x}^j + X^j dt)$$

where

$$-\Phi^2(q) = \mathbf{g}(q)(F,F)^{-1} \quad , \quad g_{0i}(q) = X_i \quad .$$

In what follows we assume our spacelike foliation to be given by the level hypersurfaces of the time function t. Let (g_{ij}, k_{ij}) be the induced metric and the second fundamental form on Σ, with k given by $k_{ij} = -\frac{1}{2}(\mathcal{L}_N g)_{ij}$. Consider a frame $\{e_0 = N, e_1, e_2, e_3\}$ satisfying $[\partial_t, e_i] = 0$, and split the Ricci tensor $R_{\alpha\beta}$ into its various components. We obtain the following evolution and constraint equations [28]

Evolution equations:

$$\begin{aligned} \partial_t g_{ij} &= -2\Phi k_{ij} + \mathcal{L}_X g_{ij} \\ \partial_t k_{ij} &= -\nabla_i \nabla_j \Phi + \Phi(-R_{ij} + {}^{(3)}R_{ij} + \mathrm{tr}k k_{ij} - 2k_{im}k^m_j) + \mathcal{L}_X k_{ij} \end{aligned} \tag{1.3.16}$$

Constraint equations:

$$\begin{aligned} {}^{(3)}R - |k|^2 + (\mathrm{tr}k)^2 &= 2R_{TT} + R \\ \nabla_i \mathrm{tr}k - \nabla^j k_{ij} &= R_{Ti} \end{aligned} \tag{1.3.17}$$

with ${}^{(3)}R_{ij}$ denoting [29] the Ricci curvature of the induced metric.

1.3.4 Maximal foliations of the Einstein spacetime

Let us recall that in a Lorentz manifold a maximal hypersurface is one that is spacelike and maximimizes the volume among all possible compact perturbations of it. It satisfies the equation $\mathrm{tr}k = 0$. The constraint equations for the level hypersurfaces of a maximal foliation take, in this case, the form

$$\begin{aligned} {}^{(3)}R - |k|^2 &= 2R_{TT} + R \\ \nabla^j k_{ij} &= R_{Ti} \\ \mathrm{tr}k &= 0 \end{aligned} \tag{1.3.18}$$

1.3.5 A proof of the local existence using the maximal foliation

We review the proof of local existence and uniqueness for the Einstein vacuum equations in the maximal foliation, [Ch-Kl]. The specific gauge conditions are $X = 0$, $\mathrm{tr}k = 0$. Thus the equations, 1.3.16, 1.3.17, in vacuum, take the form [30]

[28] See [An-Mon].

[29] Whenever there is no danger of confusion we will omit the upper left index for ${}^{(3)}R$.

[30] Everywhere here $R_{ij} = {}^{(3)}R_{ij}$.

Evolution equations:

$$\frac{\partial g_{ij}}{\partial t} = -2\Phi k_{ij}$$
$$\frac{\partial k_{ij}}{\partial t} = -\nabla_j\nabla_i\Phi + \Phi(R_{ij} - 2k_{il}k^l_j) \tag{1.3.19}$$

Constraint equations:

$$R - |k|^2 = 0$$
$$\text{div}k = 0\ ,\ \text{tr}k = 0 \tag{1.3.20}$$

Lapse equation [31]*:*

$$\triangle\Phi = |k|^2\Phi \tag{1.3.21}$$

It is easy to check that the evolution-lapse equations preserve the constraint equations, see Proposition 1.3.1, in other words it suffices to assume the constraints satisfied by the initial data set $\{\Sigma, g_0, k_0\}$[32]. We can then try to solve the evolution equations for g, k coupled with the elliptic equation satisfied by Φ. This system is however not in standard hyperbolic form. This is due not only to the fact that the lapse equation is elliptic but also, ignoring Φ, that the evolution system for g, k is not hyperbolic. Indeed, the principal part of the Ricci curvature R_{ij}, expressed relative to the metric g_{ij}, is not elliptic. This problem can be overcome by differentiating the evolution equation for k_{ij} with respect to t.
The detailed proof is given in [Ch-Kl], Chapter 10. The final result is as follows:

Theorem 1.3.3 *Let $\{\Sigma, g_0, k_0\}$ be an initial data set verifying the following conditions:*

1. *$\{\Sigma, g_0\}$ is a complete Riemannian manifold diffeomorphic to R^3.*

2. *The isoperimetric constant $\mathcal{I}(\Sigma, g_0)$ is finite, where $\mathcal{I}$ is defined to be*

$$\sup_S \frac{V(S)}{A(S)^{3/2}}$$

with S an arbitrary surface in Σ, $A(S)$ its area and $V(S)$ the enclosed volume.

---

[31] In the asymptotically flat case one has to normalize Φ by the condition $\Phi \to 1$ at spacelike infinity.

[32] $\Sigma \equiv i(\Sigma)$, $i^*(g_0) = \overline{g}$, $i^*(k_0) = \overline{k}$.

3. *The Ricci curvature* $Ric(g_0)$ *verifies, relative to the distance function* d_0 *from a given point* O,

$$Ric(g_0) \in H_{2,1}(\Sigma, g_0)$$

4. k *is a 2-covariant symmetric trace-free tensorfield on* Σ *verifying*

$$k \in H_{3,1}(\Sigma, g_0)$$

where for a given tensorfield h, $\|h\|_{H_{s,\tau}(\Sigma,g_0)}$ *denotes* [33] *the norm*

$$\|h\|_{H_{s,\tau}(\Sigma,g_0)} = \left(\sum_{i=0}^{s} \int \sigma_0^{2\tau+2i} |\nabla_0^i h|^2 d\mu_{g_0}\right)^{1/2}$$

and $\sigma_0 = \sqrt{1 + d_0^2}$.

5. (g_0, k_0) *verify the constraint equations on* Σ.

Then there exists a unique, local in time smooth development, foliated by a normal, maximal time foliation t *with range in some interval* $[0, t_*]$ *and with* $t = 0$ *corresponding to the initial slice* Σ. *Moreover*

$$\begin{aligned} g(t) - g_0 &\in \mathcal{C}^1([0, t_*]\,;\, H_{3,1}(\Sigma, g_0)) \\ k(t) &\in \mathcal{C}^0([0, t_*]\,;\, H_{3,1}(\Sigma, g_0)) \end{aligned}$$

1.3.6 Maximal Cauchy developments

We recall the general result of Y.Choquet-Bruhat and R.Geroch, [Br-Ge], concerning the existence and uniqueness of a maximal Cauchy development of an initial data set. Without going into details[34] it is intuitively clear what it means for one Cauchy development to be an extension of another. An extension is called proper if it strictly larger than the other development. A Cauchy development which has no proper extensions is called maximal. **Check if the proof with $s \geq 4$ is the one in [Br-Ge], or vicevers is the one of Chruschel**

[33] These weighted Sobolev norms, see [Ch-Mu] and [Br-Ch2], give more control on the behavior of solutions at spacelike infinity. In particular they prove an H^s version of the propagation of asymptotically flatness condition.

[34] For a precise statement of the Bruhat-Geroch result see also [Br-Y].

Theorem 1.3.4 *Let $\{\Sigma, \overline{g}, \overline{k}\}$ be an arbitrary $\mathcal{H}^s$ initial data set with $s \geq 4$. There exists a unique, future, maximal globally hyperbolic vacuum extension, MGHVE, $(\mathcal{M}^*, g^*)$.*
Moreover the development can be represented by $\mathcal{M}^* = [0,1) \times \Sigma$ and $g^*(t,\cdot) \in C^0\Big([0,1); \mathcal{H}^s(\Sigma)\Big) \cap C^1\Big([0,1); \mathcal{H}^{s-1}(\Sigma)\Big)$.

Though the result of Bruhat and Geroch can be deduced quite easily from Theorem 1.3.1, it is conceptually very important as it allows us to associate, *to any initial data set*, a unique maximal global hyperbolic spacetime. Therefore any construction, obtained by an evolutionary approach from initial data, must necessarily be included in the corresponding MGHVE spacetime which should, therefore, be viewed as our main object of study.

1.3.7 The Hawking-Penrose singularities, the cosmic censorship

Soon after the formulation of the General Relativity theory it was realized that the Einstein equations could lead to formation of singularities[35]. A standard example is given by the Friedman-Robertson-Walker spacetime with positive curvature, which evolves from the "big bang singularity" to the "big crunch" singularity. Therefore the question if singularities generally occurr in vacuum Einstein spacetimes has been an important and open question for years. This problem is considered, basically, satisfactorily settled by Hawking and Penrose in their famous singularity teorems, see[Haw-El].

Rephrased in the langauge of the initial value problem the question is that of timelike and null geodesic completeness of the maximal future Cauchy vacuum development. The singularity theorems answer, therefore, to this question in the negative. In particular we recall the Penrose theorem [Pe3] which, in the vacuum Einstein case, can be stated in the following way

Theorem: *If the initial data set $\{\Sigma, \overline{g}, \overline{k}\}$ have Σ non compact and, moreover, Σ contains a closed trapped surface [36] $\mathcal{S}$, then the corresponding maximal future development is incomplete.*

The singularity theorems motivated some efforts in trying to formulate some precise statements about the predictivity power of the Einstein equa-

[35] See [Chr] for a review of the problem.

[36] A closed trapped surface is a C^2 compact, without boundary, spacelike two dimensional surface such that a displacement of $\mathcal{S}$ in $\mathcal{M}$ along the congruence of the future outgoing null directions decreases, pointwise, the area element.

tions and the nature of the singularities. In this line of thought Penrose proposed the following two "cosmic censorships conjectures" [37]:

The first Penrose conjecture, called the "weak cosmic censorship", can be formulated in many ways. The version we state here [38] makes use of the following result, a direct corollary of the result proved in this book [39].

Corollary 1.3.5 *For any asymptotically flat initial data set* $\{\Sigma, \overline{g}, \overline{k}\}$ *with maximal future development* $(\mathcal{M}, \mathbf{g})$*, one can find a suitable domain* Ω_0 *with compact closure in* Σ *such that the boundary of its domain of influence* $I^+(\Omega_0)$ *in* $\mathcal{M}$ *has complete null generating geodesics.*

The above corollary can be used to introduce the concept of complete future null infinity [40]

Definition 1.3.2 *The maximal future Cauchy development* $(\mathcal{M}, \mathbf{g})$ *of an asymptotically flat initial data set possesses a complete future null infinity if, for any positive real number* A*, we can find a domain* Ω *containing the set* Ω_0 *of the previous corollary, such that the boundary* $\mathcal{D}^-$ *of the domain of dependance of* Ω *in* $\mathcal{M}$ *has the property that each of its null generating geodesic has a total affine length, in* $\mathcal{D}^- \cap I^+(\Omega_0)$*, greater or equal to* A*.*

The weak cosmic censorship (WCC): *Generic asymptotically flat initial data have maximal future developments possessing a complete future null infinity.*

Remark: So far the only satisfactory rigorous proof of the conjecture [41], due to Christodoulou, was obtained for the special case of spherically symmetric solutions of the Einstein equations coupled with a scalar field, see [Ch5]. Christodoulou had previously proved the existence of naked singularities for his model, [Ch4], and thus had to show that the WCC conjecture holds true only in a generic sense.

The weak cosmic censorship conjecture does not preclude, however, the possibilities that singularities may be visible by local observers. This could

[37] There are many references on this subject, see for instance [An-Mon] and [Chr].

[38] Due to D.Christodoulou [Ch6].

[39] A proof of Corollary 1.3.5 can be also derived indirectly from [Ch-Kl]. The result proven in this book avoids however a great deal

[40] This concept is usually defined in the General relativity literature through the concept of a regular conformal compactification of a spacetime, by attaching a boundary at infinity. (The notion of conformal compactification, due to Penrose, is discussed in [Haw-El] and [Wa2].) The definition given here, due to [Ch6], avoids the technical issue of the specific degree of smoothness of the compactification

[41] Nevertheless see also [Chr], section 1.4.

lead to the paradoxical situation of lack of unique predictability of outcomes of observations made by such observers. Since predictability is a fundamental requirement of all classical physics it seems reasonable to want it valid throughout the whole spacetime. Predictability is known to fail, however, within the black hole of a Kerr solution in which case the maximal development of any complete spacelike hypersurface has a future boundary, called a Cauchy horizon, where the Kerr solution is perfectly smooth and yet beyond which there are many possible smooth extensions. This failure of predictability is due to a global pathology of the geometry of characteristics and not to a loss of local regularity. It is to avoid this pathology and ensure uniqueness that we want the maximal development of generic initial data to be inextendible. Motivated by these considerations Penrose introduced the strong cosmic censorship which forbids such undesirable feature of singularities.

The strong cosmic censorship (SCC): *Generic initial data sets have maximal Cauchy future developments which are locally inextendible, in a continuous manner, as Lorentz manifolds. In other words every maximal Hasdorff development of a generic initial data set, compact or asymptotically flat, is a Cauchy development.*

Remarks: In more technical terms this means that, disregarding some possible exceptional initial conditions, the maximal future development of an initial data set is such that along any future, inextendible, timelike geodesics of finite length [42], the spacetime curvature components expressed relative to a parallel transported orthonormal frame along the geodesic, must become infinity as the value of the arclength tends to its limiting value.
The formulation above leads open the sense in which the maximal future developments are inextendible. The precise notion of extendibility, which is to be avoided by SCC, is a subtle issue which, probably, can only be settled together with a complete solution of the conjecture.

Moreover if the strong cosmic censorship does not hold this implies the existence of Cauchy horizons, which suggests that the uniqueness of $(\mathcal{M}, \mathbf{g})$ is lost beyond them.

1.3.8 The statements of the *C-K Theorem* and of the *K-N Main Theorem*

We conclude this first chapter stating the two theorems we are discussing. The rest of the book is devoted to the proof of the second one, but at the

[42] That is of bounded proper time.

end of Chapter 3 a survey of the first result is also given.
We preliminary introduce the definitions [43] of the "asymptotically flat initial data sets" and of the "strong asymptotically flat initial data sets" which enter in the statement of both theorems and introduce also a functional associated to these definitions which collects the initial data norms.

Definition 1.3.3 *We say that a data set, $\{\Sigma, \bar{g}, \bar{k}\}$, is asymptotically flat if there exists a coordinate system (x^1, x^2, x^3) defined outside a sufficiently large compact set such that, relative to this coordinate system*

$$\begin{aligned} \bar{g}_{ij} &= (1+\frac{2M}{r})\delta_{ij} + o(r^{-1}) \\ \bar{k}_{ij} &= o(r^{-2}) \end{aligned} \tag{1.3.22}$$

Definition 1.3.4 *An initial data set $\{\Sigma, \bar{g}, \bar{k}\}$ is "strongly asymptotically flat", if there exists a coordinate system (x^1, x^2, x^3) defined outside a sufficiently large compact set such that, relative to this coordinate system*

$$\begin{aligned} \bar{g}_{ij} &= (1+2M/r)\delta_{ij} + o_4(r^{-\frac{3}{2}}) \\ \bar{k}_{ij} &= +o_3(r^{-\frac{5}{2}}) \end{aligned} \tag{1.3.23}$$

We also introduce the following functional associated to any asymptotically flat initial data set,

$$\begin{aligned} J_0(\Sigma_0, g, k) &= \sup_{\Sigma_0}\left((d_0^2+1)^3|Ric|^2\right) \\ &+ \int_{\Sigma_0}\sum_{l=0}^{3}(d_0^2+1)^{l+1}|\nabla^l k|^2 + \int_{\Sigma_0}\sum_{l=0}^{1}(d_0^2+1)^{l+3}|\nabla^l B|^2 \end{aligned} \tag{1.3.24}$$

where d_0 is the geodesic distance fom a fixed point O on Σ and B is the Bach tensor[44].

C-K Theorem *(Global stability of the Minkowski space using a maximal foliation)*

There exists an ϵ sufficiently small such that if $J_0(\Sigma_0, g, k) \leq \epsilon$ then the initial data set $\{\Sigma_0, g, k\}$, strongly asymptotically flat and maximal, has a

[43] More precise definitions are given in Chapter 3, where the *Main Theorem* is stated in every detail.

[44] See [Ch-Kl] for the definition of B and discussions about the quantity J_0.

unique, globally hyperbolic, smooth, geodesically complete solution [45]*. This development is globally asymptotically flat which means that the Riemann curvature tensor tends to zero along any causal or spacelike geodesic. Moreover there exists a global maximal time function t and an optical function u, defined everywhere outside* [46] *an "internal region". The outgoing null foliation defined by u corresponds to the propagation properties of the spacetime.*

Main Theorem *(Global stability using a double null foliation)*

Consider an initial data set $\{\Sigma_0, g, k\}$, strongly asymptotically flat and maximal, and assume $J_0(\Sigma_0, g, k)$ bounded. Then, given a sufficiently large compact set $K \subset \Sigma_0$ such that $\Sigma_0 \backslash K$ is diffeomorphic to R^3/B_1 and under additional smallness assumptions which are made precise in section 3.7, there exists a unique development $(\mathcal{M}, \mathbf{g})$ [47] *with the following properties*

1) $(\mathcal{M}, \mathbf{g})$ can be foliated by a double null foliation $\{C(\lambda)\}$ and $\{\underline{C}(\nu)\}$ whose outgoing leaves $C(\lambda)$ are complete [48]*.*

2) We have a detailed control of all the quantities associated to the double null foliations of the spacetime and of the asymptotic behavior of the Riemann curvature tensor along the null outgoing and the spacelike geodesics.

3)If $J(\Sigma_0, g, k)$ is small we can extend $(\mathcal{M}, \mathbf{g})$ to a smooth, complete solution compatible with the global stability of the Minkowski space.

In this work we only provide complete proofs for 1) and 2), see section 3.7 for a complete discussion of our result.

1.4 Appendix

Proof of Lemma 1.3.2

The proof goes along the following three steps:

Step 1: As the coordinates $(t, \tilde{x}^i)$ are the adapted coordinates of $R \times \mathcal{S}$, we have to compute $\tilde{\mathbf{g}}(t, p) \equiv \mathcal{T}^* \mathbf{g}(q)$, the metric induced on $R \times \mathcal{S}$. It is

[45] Which thus coincide with the maximally hyperbolic development of Choquet-Bruhat and Geroch, [Br-Ge].

[46] See details in [Ch-Kl].

[47] Which coincides, roughly speaking, with the complement of the domain of influence of the compact set K. This means, in particular, that for any point $p \in (\mathcal{M}, g)$ any causal curve passing through it intersects $\Sigma_0 \backslash K$ once and only once.

[48] This definiton means that the null geodesics generating $C(\lambda)$ can be indefinitely extended toward the future.

defined through $\mathcal{T}$ in the following way,

$$\tilde{\mathbf{g}}(t,p)((a,\tilde{Y}),(b,\tilde{Y})) = \mathbf{g}(q)(\mathcal{T}_*(a,\tilde{Y}),\mathcal{T}_*(b,\tilde{Y})) \ .$$

Step 2: Let us consider the tangent space of $R \times \Sigma_0$:

$$\begin{aligned} &T(R\times\Sigma_0) = TR \times T\Sigma_0 \\ &T(R\times\Sigma_0)_{(t,p)} = R_t \times (T\Sigma_0)_p \quad , \ p \in \Sigma_0 \end{aligned}$$

The generic vector of $T(R \times \Sigma_0)$ is

$$(a,\tilde{X})_{(t,p)} \equiv \left(a\frac{\partial}{\partial t}\bigg|_t , \tilde{X}^i_p \frac{\partial}{\partial \tilde{x}^i}\bigg|_p \right)$$

and[49]

$$\mathcal{T}_*(a,\tilde{X})_{(t,p)} \in (T\mathcal{M})_q \ , \ \mathcal{T}_*(0,\tilde{X})_{(t,p)} \in (T\Sigma_t)_q$$

where

$$t(q) = t \ , \ \Psi(\bar{s},p) = q \ .$$

We derive

$$\mathcal{T}_*(1,0)_{(t,p)} = \frac{V^\mu_q}{\mathbf{g}(F,V)}\frac{\partial}{\partial x^\mu}\bigg|_q \ , \ \mathcal{T}_*(0,\tilde{Y})_{(t,p)} = \tilde{Y}^i_p \frac{\partial \Psi^\mu}{\partial \tilde{x}^i}\bigg|_{(\bar{s},p)} \tag{1.4.1}$$

Step 3: Putting together Step 1 and Step 2 we obtain

$$\begin{aligned} \tilde{g}_{00}(t,p) &= -(-\mathbf{g}(q)(F,F))^{-1} + \mathbf{g}(q)(X,X) \\ \tilde{g}_{0i}(t,p) &= \mathbf{g}(q)(X, \frac{\partial \Psi}{\partial \tilde{x}^i}) = X_i \\ g_{ij}(t,p) &= \mathbf{g}(q)\left(\frac{\partial \Psi}{\partial \tilde{x}^i}\frac{\partial \Psi}{\partial \tilde{x}^j}\right) \end{aligned} \tag{1.4.2}$$

completing the proof of the lemma.

The more delicate part to prove is equation 1.4.1 of Step 2. Let f be a function on $\mathcal{M}$, then the vector $Y_q \equiv \mathcal{T}_*(a,\tilde{Y})_{(t,p)}$, applied to f gives

$$\begin{aligned} Y_q(f) &= Y^\mu_q \frac{\partial f}{\partial x^\mu}\bigg|_q = (a,\tilde{Y})_{(t,p)}(f\circ\mathcal{T}) \\ &= a\frac{\partial}{\partial t}(f\circ\mathcal{T})|_{(t,p)} + \tilde{Y}^i_p \frac{\partial}{\partial \tilde{x}^i}(f\circ\mathcal{T})|_{(t,p)} \end{aligned} \tag{1.4.3}$$

[49]Observe that, viceversa, $\mathcal{T}_*(a,\tilde{0})_{(t,p)}$ does not belong to $(T\Sigma_t)^\perp_q$.

where

$$a\frac{\partial}{\partial t}(f\circ\mathcal{T})|_{(t,p)} = a\frac{\partial f}{\partial x^\mu}\bigg|_q \frac{\partial \mathcal{T}^\mu}{\partial t}\bigg|_{(t,p)}$$

and, using 1.3.15,

$$\frac{\partial \mathcal{T}^\mu}{\partial t}\bigg|_{(t,p)} = \frac{\partial \Psi^\mu}{\partial s}\bigg|_{\bar{s}}\frac{\partial s}{\partial t}\bigg|_{(t,p)} = \frac{1}{\mathbf{g}(F,V)}V_q^\mu \ .$$

Moreover

$$\frac{\partial}{\partial \tilde{x}^i}(f\circ\mathcal{T})\bigg|_{(t,p)} = \frac{\partial f}{\partial x^\mu}\bigg|_q \frac{\partial \Psi^\mu}{\partial \tilde{x}^i}|_{(\bar{s},p)}$$

so that finally

$$\mathcal{T}_*(1,\tilde{0})_{(t,p)} = \frac{V_q}{\mathbf{g}(F,V)} = \frac{V_q^\mu}{\mathbf{g}(F,V)}\frac{\partial}{\partial x^\mu}\bigg|_q = \frac{\partial \Psi^\mu}{\partial s}\bigg|_{(\bar{s},p)}\frac{\partial}{\partial x^\mu}\bigg|_q$$

$$\mathcal{T}_*(0,\frac{\partial}{\partial \tilde{x}^i})_{(t,p)} = \frac{\partial \Psi}{\partial \tilde{x}^i}\bigg|_q = \frac{\partial \Psi^\mu}{\partial \tilde{x}^i}\bigg|_{(\bar{s},p)}\frac{\partial}{\partial x^\mu}\bigg|_q \ .$$

Therefore,

$$Y = \left(a\frac{V_q^\mu}{\mathbf{g}(F,V)} + \tilde{Y}_p^i\frac{\partial \Psi^\mu}{\partial \tilde{x}^i}\bigg|_{(\bar{s},p)}\right)\frac{\partial}{\partial x^\mu}\bigg|_q$$

Step 3 is then simply achieved with the following substitutions,

$$\begin{aligned}
\tilde{g}_{00}(t,p) &= \tilde{\mathbf{g}}(t,p)\left((1,\tilde{0}),(1,\tilde{0})\right) = \frac{\mathbf{g}(q)(V,V)}{\mathbf{g}(q)(V,F)^2} \\
&= \frac{1}{\mathbf{g}(q)(V,F)^2}\left(\mathbf{g}(q)(V_\perp,V_\perp) + \mathbf{g}(q)(V_{||},V_{||})\right) \\
&= -(-\mathbf{g}(q)(F,F))^{-1} + \mathbf{g}(q)(X,X)
\end{aligned}$$

$$g_{ij}(t,p) = \tilde{\mathbf{g}}(t,p)\left((0,\frac{\partial}{\partial \tilde{x}^i}),(0,\frac{\partial}{\partial \tilde{x}^j})\right) = \mathbf{g}(q)\left(\frac{\partial \Psi}{\partial \tilde{x}^i}\frac{\partial \Psi}{\partial \tilde{x}^j}\right)$$

Finally [50]

$$\begin{aligned}
\tilde{g}_{0i}(t,p) &= \tilde{\mathbf{g}}(t,p)((1,\tilde{0}),(0,\frac{\partial}{\partial \tilde{x}^j})) = \mathbf{g}(q)\left(\frac{V}{\mathbf{g}(V,F)},\frac{\partial \Psi}{\partial \tilde{x}^i}\right) \\
&= \frac{1}{\mathbf{g}(V,F)}\mathbf{g}(q)(V_{||},\frac{\partial \Psi}{\partial \tilde{x}^i}) = \mathbf{g}(q)(X,\frac{\partial \Psi}{\partial \tilde{x}^i}) = X_i \ .
\end{aligned}$$

[50] The vector components X_i are relative to the $(t,\tilde{x}^i)$ coordinates.

Chapter 2

Analytic methods in the study of the initial value problem

The goal of this chapter is to introduce the reader to the global analytic methods which play a fundamental role in the remaining chapters of the book. We start with a discussion of local and global existence results for systems of non linear wave equations. As we have pointed out in the previous sections, the Einstein vacuum equations can be reduced to such systems of partial differential equations with the help of wavelike coordinates. Thus the general framework of systems of nonlinear wave equations provides a very convenient first introduction to some of the basic analytic tools in the study of the evolution problem in General Relativity.

2.1 Local and global existence for systems of non-linear wave equations

2.1.1 Local existence for the non linear wave equations

Recall that, written relative to a system of wavelike coordinates, the Einstein equations take the reduced form:

$$\frac{1}{2}g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} = H_{\mu\nu}(g, \partial g)$$

where H is a quadratic expression relative to the first derivatives of g. Writing $g_{\alpha\beta} = \eta_{\alpha\beta} + u_{\alpha\beta}$, with η the Minkowski metric and u a small perturbation we derive a system of equations of the form

$$\Box u = N(u, \partial u, \partial^2 u) \tag{2.1.1}$$

where $u = (u^{(1)}, ..., u^{(k)})$ is a vector $\in R^k$. We shall denote by ∂ the space-time gradient $\partial = (\partial_0, \partial_1, ..., \partial_n)$, by D the space gradient $D = (\partial_1, ..., \partial_n)$ and by $\Box = \Box_\eta = \eta^{\alpha\beta}\partial_\alpha\partial_b$ the D'Alembertian with respect to the Minkowski metric of R^{n+1}. The nonlinear part N of the Einstein equations consists of a large number of terms which can be organized in two categories:

1. Terms wich can be written as a product of a real analytic function of u, a component of u and a second partial derivative of a component of u. Schematically,
$$F(u) \cdot u \cdot \partial^2 u.$$

2. Terms wich can be written as a product between a real analytic function of u and a product of first derivatives of two components of u. Schematically,
$$F(u) \cdot \partial u \cdot \partial u.$$

From the point of view of proving local and global existence results the terms of the first type are considerably more difficult to treat. It makes sense, therefore, to start with a treatment of equations which contain only terms of the second type. In doing this we shall make, for the sake of clarity, two more simplifications.
We will assume that the nonlinearity is quadratic in the first derivatives of u, that is $F(u)$ constant and u a scalar function. Both simplifications are irrelevant in so far as the main ideas of the proof are concerned. Indeed it will be clear from our discussion how to extend the proof to the general case. In fact we shall see that an appropriate modification of the argument presented below will be used also in the global theory. We are therefore reduced to an equation of the form:

$$\Box u = N = \partial u \cdot \partial u \tag{2.1.2}$$

We have to solve 2.1.2 subject to the initial conditions at $t = 0$,

$$u(0, x) = f(x), \;\; \partial_t u(0, x) = g(x). \tag{2.1.3}$$

The solution of equations 2.1.2, 2.1.3 can be expressed in the following form:

$$u = u^0 + \Box^{-1} N. \tag{2.1.4}$$

Here u^0 is a solution of the homogeneous equation

$$\square\, u^0 = 0 \tag{2.1.5}$$

subject to the initial conditions 2.1.3 and, for an arbitrary spacetime function F, $\square^{-1}F$ is defined to be the unique solution v of

$$\square\, v = F \tag{2.1.6}$$

subject to zero initial data, that is $v = \partial_t v = 0$ at $t = 0$.

In view of the classical contraction argument to find a "local in time" solution u of 2.1.2 amounts to find a $T > 0$ and a space of functions $X = X(T)$, defined in the time slab $[0, T] \times R^n$, in which we can apply a contraction mapping. In other words one has to find a space X verifying the following properties:

I. The homogeneous solution u^0 belongs to X.

II. If $u \in X$ then $\mathcal{R}[u] = \square^{-1}N(u) \in X$.

III. The mapping $\mathcal{R}:\ u \longrightarrow \square^{-1}N(u)$ is a contraction.

To achieve II and III we need "good estimates" for the inhomogeneous problem 2.1.6. More precisely, since the nonlinear term F depends on the derivatives of u, we need estimates which gain a derivative. This means that we need estimates for the first derivatives of u, in an appropriate norm, in terms of estimates for F itself. The energy estimate is precisely such an estimate [1]

Lemma 2.1.1 *Let u be a general solution of the inhomogeneous equation $\square\, u = F$. Define*

$$Q[u](t) = \left(\frac{1}{2}\int_{\Sigma_t} |\partial u|^2 dx\right)^{\frac{1}{2}} \tag{2.1.7}$$

where $|\partial u|^2 = |\partial_0 u|^2 + |\partial_1 u|^2 + ... + |\partial_n u|^2$. Then

$$Q[u](t) \le Q[u](0) + \int_0^t \|F(s)\|_{L^2} ds. \tag{2.1.8}$$

In the particular case of the homogeneous wave equation $\square\, u = 0$ we have the "energy identity"

[1] It is very important to remark that, for dimension n larger or equal to 2, the energy estimate is, in fact, the only L^p-type estimate with this property. This is easily seen in the case of norms which are L^p in space and uniform in time. The case of general local spacetime L^p norms is harder, see T.Wolff, [W].

$$Q[u](t) = Q[u](0) \tag{2.1.9}$$

Both inequalities follow easily by multiplying the wave equation by $\partial_t u$ and then integrating on the spacetime slab $[0,T] \times R^n$ where we perform a simple integration by parts argument.

It thus makes sense to ask whether the space of functions u endowed with the norm $\sup_{t\in[0,T]} Q[u](t)$ satisfies the right properties. The answer is clearly negative; property II fails due to the lack of sufficient differentiability. The problem is that we cannot bound $\|(\partial u)^2\|_{L^2}$ in terms of the energy norm $\|\partial u\|^2_{L^2}$. However the following modification works: consider the operators $D^I = \partial_1^{i_1} \ldots . \partial_n^{i_n}$ with $I = (i_1, \ldots, i_n)$ and $|I| = i_1 + \cdots i_n$. Let's define, for $i \geq 0$

$$Q_i[u](t) = \Big(\sum_{|I|\leq i} Q^2[D^I u](t)\Big)^{\frac{1}{2}} \tag{2.1.10}$$

In view of the fact that D^I commutes with $\Box$ we have, for the solutions of $\Box\, u = F$,

$$Q_i[u](t) \leq Q_i[u](0) + \int_0^t \|F(s)\|_{H^i} \tag{2.1.11}$$

where $H^i = H^i(R^n)$ denotes the Sobolev space [2] of functions f in R^n endowed with the norm

$$\|f\|_{H^i} = \Big(\sum_{|I|\leq i} \int |D^I f(x)|^2 dx\Big)^{\frac{1}{2}} \tag{2.1.12}$$

Moreover, for the solutions of the homogeneous problem 2.1.5,

$$Q_i[u^0](t) = Q_i[u^0](0) \tag{2.1.13}$$

Motivated by this we define, in the slab $[0,T] \times R^n$, the function space $X = X(T;s)$ of functions $u \in C^1\Big([0,T]; H^{s-1}(R^n)\Big) \cap C^0\Big([0,T]; H^s(R^n)\Big)$ endowed with the norm

$$\|u\|_X = \sup_{[0,T]} Q_{s-1}[u] \tag{2.1.14}$$

We claim that, for $s > \frac{n}{2} + 1$, the space X verifies both properties I, II. The first property is obviously true, the second one follows from

[2] Sobolev type spaces play an important role in the subject because of the energy type inequalities of Lemma 3.1. For a useful monograph on the subject see [Ad].

Proposition 2.1.1 *For $s > \frac{n}{2}$ the Sobolev space $H^s = H^s(R^n)$ forms an algebra, that is*

$$\|f \cdot g\|_{H^s} \leq c\|f\|_{H^s} \cdot \|g\|_{H^s}.$$

In fact let $v = \square^{-1} N(u)$. In view of 2.1.11 and 2.1.13 and using Proposition 2.1.1 we derive, for $s \geq \frac{n}{2} + 1$,

$$\|v\|_X \leq Q_{s-1}[u^0] + cT\|u\|_X^2. \tag{2.1.15}$$

To prove the contraction property III we restrict ourselves to the ball

$$\|u\|_X \leq \Delta,$$

with Δ sufficiently large so that $Q_{s-1}[u^0] \leq \frac{1}{2}\Delta$ and then choose T sufficiently small, proportional to Δ^{-1}, such that $\Delta \leq Q_{s-1}[u^0] + cT\Delta^2$. With this choice of T and Δ the operator $\mathcal{R}$ maps the ball $\|u\|_X \leq \Delta$ into itself. Finally, using the same argument as in the derivation of 2.1.15 we show that

$$\|\mathcal{R}[u_1] - \mathcal{R}[u_2]\|_X \leq cT\Delta\|u_1 - u_2\|_X \tag{2.1.16}$$

Therefore, for a small choice of $T > 0$, we infer that the map $\mathcal{R}$ is a contraction proving, therefore, the following theorem:

Theorem 2.1.1 *Assume that $f \in H^s(R^n)$, $g \in H^{s-1}(R^n)$, with (f, g) the initial data 2.1.3. Then, if $s \geq s_0$ for a fixed $s_0 > \frac{n}{2} + 1$, there exists a time $T > 0$, depending only on the size of $\|f\|_{H^{s_0}(R^n)} + \|g\|_{H^{s_0-1}(R^n)}$ and a unique solution $u \in C^1\Big([0,T]; H^{s-1}(R^n)\Big) \cap C^0\Big([0,T]; H^s(R^n)\Big)$ verifying 2.1.2 and the initial conditions 2.1.3.*

The proof of Proposition 2.1.1 is standard, it can for example be easily derived by Fourier transform methods; it can also be derived from the following more general Moser type estimates, see e.g. [Ho],

Proposition 2.1.2 *For every $s \geq 0$ the space $H^s(R^n) \cap L^\infty(R^n)$ forms an algebra. Moreover we have the estimate*

$$||f \cdot g||_{H^s} \leq c\,(||f||_{L^\infty}||g||_{H^s} + ||g||_{L^\infty}||f||_{H^s})\,, \tag{2.1.17}$$

together with the following classical version of the Sobolev inequality,

Proposition 2.1.3 *The Sobolev space $H^s(R^n)$, for $s > \frac{n}{2}$, is contained in the space of bounded continuous functions in R^n and we have the estimate*

$$||f||_{L^\infty} \le c||f||_{H^s} \tag{2.1.18}$$

Using Proposition 2.1.2 to estimate the term $\|\partial u \cdot \partial u\|_{H^i}$ on the right hand side of the inequality 2.1.11 and applying the standard Gronwall inequality[3], we derive the following a priori estimates for solutions u of 2.1.2

$$Q_s[u](t) \le c_s Q_s[u](0) \exp\left(\int_0^t \|\partial u(t)\|_{L^\infty} dt\right) \tag{2.1.19}$$

These estimates can be used to prove the following characterization of the maximal time of existence in Theorem 2.1.1.

Theorem 2.1.2 *Under the same assumptions as in the previous theorem, the unique solution u can be extended in any slab $[0,T] \times R^n$ as long as*

$$\int_0^T ||\partial u(s)||_{L^\infty} ds < \infty.$$

Both Theorems 2.1.1 and 2.1.2 are valid for more general equations. In fact the argument presented above extends easily over equations of the type $\square u = F(u)\partial u \cdot \partial u$. To treat the general quasilinear case, the previous approach has to be somewhat modified. The idea is to appropriately modify the energy norm 2.1.7 so that we can still rely on "energy type estimates" as we have done before. Consider, first, scalar equations of the form

$$A^{\alpha\beta}(u)\partial_\alpha\partial_\beta u = N(u, \partial u) \tag{2.1.20}$$

with $A^{\alpha\beta}$ a Lorentz metric depending on u. We define the mapping

$$u \to \mathcal{R}(u) = v \tag{2.1.21}$$

where v is the unique solution of the linear wave equation

$$A^{\alpha\beta}(u)\partial_i\partial_j v = N(u, \partial u). \tag{2.1.22}$$

subject to the given initial conditions. We are therefore reduced to prove that the mapping $\mathcal{R}$ is a contraction. This can be done by following precisely the same steps as before. The only modification is in the definition

[3]The Gronwall inequality is a basic estimate used to study the dependence on initial conditions which will be used repeatedly in this book, see e.g. [Ho].

of the energy integral norm which can be defined now with the help of the energy-momentum tensor $T_{\alpha\beta} = \partial_\alpha v \partial_\beta v - A_{\alpha\beta}(u) A^{\mu\nu}(u) \partial_\mu v \partial_\nu v$ associated to solutions v of the linear equation 2.1.22. The integral norm $Q[v](t)$ can now be defined on spacelike hypersurfaces Σ_t by integrating the energy density $T(n, n)$, with n the future unit normal to Σ_t. One then proceeds precisely as in the case of the simple model equation 2.1.2 described above and show that the results of both Theorems 2.1.1 and 2.1.2 hold true for general equations of the type 2.1.1.

Discussion 2.1.1 *The basic building blocks in the proof of the local existence theorem 2.1.1 were:*

1. *Basic energy estimate, see 2.1.8*
2. *Higher energy estimates, see 2.1.11*
3. *Sobolev inequality, see 2.1.18*
4. *Bootstrap estimate, see 2.1.15*
5. *Contraction estimate, see 2.1.16*

These elements are typical to all local existence results, and we shall also encounter them, in a modified form, in the global theory.

2.1.2 The global existence for the non linear wave equations

In trying to prove a global result for the Einstein equations it makes sense to start with wavelike coordinates and therefore study the question of existence of global smooth solutions for the reduced system 1.3.1. In the spirit of our discussion in subsection 2.1.1 we first look at the scalar model equation,

$$\square u = \partial u \cdot \partial u, \tag{2.1.23}$$

subject to the initial conditions $u = f$, $\partial_t u = g$ at $t = 0$, and ask whether the local solutions can be continued for infinite time. In Theorem 2.1.2 we have shown that the solution given by the local existence theorem can be extended in any interval of time $[0, T]$ for which

$$\int_0^T ||\partial u(s)||_{L^\infty} ds < \infty.$$

In fact in the case of equations of the type 2.1.23 we had the precise estimate, see 2.1.19,

$$Q_s[u](t) \leq c_s Q_s[u](0) \exp\left(\int_0^T ||\partial u(t)||_{L^\infty} dt\right) \tag{2.1.24}$$

with Q_s the energy type norms introduced in 2.1.10. This suggests that, to obtain a global solution, we have to control the asymptotic behaviour of the L^∞ norm of ∂u. If u is a solution of the linear wave equation

$$\begin{aligned} \Box\, u &= 0 \\ u(0) = f,\ \partial_t u(0) &= g \end{aligned} \tag{2.1.25}$$

it is possible to show from the explicit form of the fundamental solution that, as $|t|$ goes to infinity

$$||u(t)||_{L^\infty} \leq C|t|^{-\frac{n-1}{2}} \tag{2.1.26}$$

where C depends in a specific way on data f and g. This method of deriving the asymptotic behaviour of u, based on the explicit form of the fundamental solution, is very cumbersome in applications to non linear problems. It would be particularly difficult to implement it for quasilinear wave equations such as 1.3.1.

Another method for deriving the asymptotic behavior of solutions to 2.1.25 is the *conformal method* introduced by Penrose, [Pe2], to obtain the asymptotic behaviour of linear, massless, field equations. This technique was later developed by Christodoulou [Ch1], [Ch2] and Friedrich [Fr1], [Fr2], [Fr3]. The problem with the conformal method is that it requires a lot of decay of the initial data f, g at spacelike infinity, incompatible with long range properties of asymptotically flat initial data sets.

In what follows we give a short outline of a different method, see[4] [Kl4], [Kl3] and [Ho], of deriving not only the uniform asymptotic behavior but also the propagation properties of solutions to the linear wave equation based on the conformal symmetries [5] of the Minkowski spacetime. This method can be easily generalized to non linear situations and its main ideas will turn out to be central in our discussion of the Einstein vacuum equations.

[4]See also [Kl5] for new applications of this technique to Strichartz type inequalities and improved regularity results for quasilinear wave equations.

[5]As in Penrose's method the conformal structure is essential, however one has the flexibilty to use it in a way which is best adapted to the problem at hand.

The Minkowski spacetime is equipped with a family of Killing and conformal Killing vector fields

$$\begin{cases} T_\mu = D_\mu \\ O_{\mu\nu} = x_\mu D_\nu - x_\nu D_\mu \\ S = t\partial_t + x^i\partial_i \\ K_0 = (t^2+r^2)\frac{\partial}{\partial t} + 2tx^i\frac{\partial}{\partial x^i} \\ K_\mu = -2x_\mu S + < x, x > \partial_\mu \end{cases} \tag{2.1.27}$$

The Killing vector fields T_μ and $O_{\mu\nu}$ commute with $\Box$ while S preserves the space of solutions in the sense that $\Box u = 0$ implies $\Box L_S u = 0$ as $[\Box\,, S] = 2\Box$. Based on this observation we define the following "generalized Sobolev norms"

$$\begin{aligned} E_0[u](t) &= ||u(t,\cdot)||_{L^2(R^3)} \\ E_k[u](t) &= \sum_{X_{i_1},..,X_{i_j}} E_0[L_{X_{i_1}}L_{X_{i_2}}...L_{X_{i_j}}u](t) \end{aligned} \tag{2.1.28}$$

with the sum taken over $0 \leq j \leq k$ and over all Killing vector fields T, Ω as well as the scaling vector field S.
The crucial point of this method is that the generalized "energy type norms"

$$Q_k[u](t) \equiv E_k[Du](t) \tag{2.1.29}$$

are conserved by solutions to equation 2.1.25. The desired decay estimates of solutions to 2.1.25 can now be derived from the following global version[6] of the Sobolev inequalities (compare it with Proposition 2.1.3),

Proposition 2.1.4 *Let u be an arbitrary function in R^{n+1} such that $E_s[u]$ is finite for some $s > \frac{n}{2}$. Then for $t > 0$*

$$|u(t,x)| \leq c\frac{1}{(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}}E_s[u] \tag{2.1.30}$$

Therefore if the data f, g in 2.1.25 are such that the quantity $Q_s[u] < \infty$, it follows that, for $t > 0$,

$$|Du(t,x)| \leq c\frac{1}{(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}} \tag{2.1.31}$$

[6]For details see [Kl3].

an estimate which fits very well the expected propagation properties of the linear equation $\square u = 0$.

We can now proceed as in the derivation of the estimate 2.1.24 and prove an estimate of the same type expressed in terms of the new generalized energy norms defined by 2.1.29. Combining that estimate with the global Sobolev inequality 2.1.31 one derives,

$$\mathcal{Q}(T) \leq \mathcal{Q}(0) \exp c \left(\int_0^T (1+t)^{-\frac{n-1}{2}} \right) \mathcal{Q}(T) \tag{2.1.32}$$

where

$$\mathcal{Q}(T) = \sup_{[0,T]} Q_s[u](t) \tag{2.1.33}$$

for some $s > \frac{n}{2} + 1$. This leads to a global bound for $\mathcal{Q}$ provided that $\mathcal{Q}(0)$ is small and $n > 3$. Therefore, for $n > 3$ and sufficiently small data, the local solution provided by Theorem 2.1.1, can be extended for all time, see [Kl4], [Ho].

For $n = 3$, the case of interest for General Relativity, the estimate in 2.1.32 leads to a logarithmic divergence[7]. Nevertheless there are still interesting situations, in space dimension $n = 3$, where one can prove the existence of small global solutions. One favorable situation is, for instance, the case where the nonlinear part consists only of terms of order higher than quadratic, such as $Du \cdot Du \cdot Du$. A much more interesting situation, which turns out to be of great relevance in our discussion below, is when we allow quadratic terms, but require that they satisfy the *null condition*, see [Kl1], [Kl2], [Ch2] and also [Ho]. Roughly speaking, see [Kl2] and [Ho] for details, this means that the quadratic terms of the equation appear only through the intermediary of the "null quadratic forms"

$$\begin{aligned} Q_0(u,v) &= \eta^{\alpha\beta}\partial_\alpha u \partial_\beta v \\ Q_{\alpha\beta}(u,v) &= \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v \end{aligned} \tag{2.1.34}$$

with η, the Minkowski metric. If the *null condition* is satisfied one can prove a small data global existence result[8] even for $n = 3$. The basic observation

[7]This logarithmic divergence is not an artifact of the proof. There are examples, see [John1], [John2], of nonlinear wave equations in $n = 3$ for which all perturbations of the trivial solution form singularity in finite time. Moreover this situation is generic.

[8]The result is proved for a general class of quasilinear systems of wave equations in [Kl2], see also [Ho], based on the ideas sketched here and in [Ch2] with the help of the conformal method. As we have remarked above the conformal method requires more regularity for the data at spacelike infinity.

at the origin of this result has to do with the propagation properties of waves expressed relative to null frames.

Consider again the linear equation 2.1.25 and the estimate 2.1.31. The derivatives of Du , expressed relative to the standard cartesian frame, do not behave any better, along the null directions, than $|t|^{-\frac{n-1}{2}}$. We get, however, a more detailed picture of the behavior of derivatives of u by considering a null frame $\{e_3, e_4, e_a\}$ [9] , with null vectors $e_3 = \frac{\partial}{\partial t} - \frac{\partial}{\partial r}$, $e_4 = \frac{\partial}{\partial t} + \frac{\partial}{\partial r}$ and e_a an orthonormal frame spanning the orthogonal complement of $\{e_3, e_4\}$. It is in fact easy to prove, from 2.1.30, the following estimates, for $t > 0$ and $s > \frac{n}{2} + 1$

$$
\begin{aligned}
|D_{e_4} u(t,x)| &\leq c \frac{1}{(1+t+|x|)^{(\frac{n-1}{2}+1)}(1+|t-|x||)^{\frac{1}{2}}} E_s[u](t) \\
|D_{e_a} u(t,x)| &\leq c \frac{1}{(1+t+|x|)^{(\frac{n-1}{2}+1)}(1+|t-|x||)^{\frac{1}{2}}} E_s[u](t) \\
|D_{e_3} u(t,x)| &\leq c \frac{1}{(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{3}{2}}} E_s[u](t) \qquad (2.1.35)
\end{aligned}
$$

Thus, for $t > 0$, only D_{e_3} fails to improve. By symmetry D_{e_4} fails to improve for $t < 0$.

The null condition, for systems of wave equations of type 2.1.23, simply prevents the presence of terms such as $(D_{e_3} u)^2$ and $(D_{e_4} u)^2$. This allows us to overcome the logarithmic divergence in 2.1.32 and thus prove a small data global existence result.

Discussion 2.1.2 *The main ingredients in the proof of global existence discussed above were*

a) Generalized energy type norms.

b) Killing and conformal Killing vector fields.

c) Null frames.

d) Some appropriate version of the null condition.

The last point d) is crucial in $3+1$ dimensions, without which there is no global existence. It turns out, however, that the reduced Einstein equations 1.3.1 do not satisfy such a condition. This was first pointed out by Y.Choquet-Bruhat,[Br3] and later substantiated by L.Blanchet and T.Damour

[9] An explicit expression of a null frame in the Minkowski spacetime R^{3+1} is given by e_3, e_4 as well as

$$
e_\theta = \frac{1}{r}\frac{\partial}{\partial \theta} \ , \ e_\phi = \frac{1}{r sin\theta}\frac{\partial}{\partial \phi}
$$

[Bl-D]. The problem is connected with the wavelike gauge itself which behaves badly in the large. We have thus to abandon the wavelike coordinates altogether. As we return to the Einstein equations we realize that the main difficulties to face are:

1) The problem of coordinates.

2) The strong non linear features of the Einstein equations.

3) The long range terms in the initial data.

4) The non trivial propagation properties of the expected solutions.

The first two problems have already been discussed. The strongly non linear character of the equations requires one to rely on a quite rigid analytic approach based on energy estimates, background symmetries and some subtle cancellation properties manifest in the nonlinear structure of the equations. This last point, in particular, calls for an invariant approach. But this is not all. We can certainly not expect that the spacetime we plan to construct admits any Killing and conformal Killing vector fields. The best we can hope is that it admits some approximate ones, namely vector fields whose deformation tensors are small in an appropriate way [10]*. In trying to do this we encounter the difficulties 3) and 4). The* $1/r$ *decay of the metric, due to the presence of the mass term, has the long range effect of changing the asymptotic behavior of the null geodesics. Thus the causal structure of the spacetime we construct is not asymptotic to that of the Minkowski spacetime.*

To deal with these problems one has to devise a strategy which is independent, as much as possible, of a specific choice of coordinates. From this point of wiew it makes sense to try to derive the main propagation properties of our solutions in terms of the Riemann curvature tensor. As it will turn out the propagation properties of the Riemann curvature tensor are least sensitive to problems 3 and 4 and best suited, as a starting point, to exhibit the "null structure" properties of the Einstein equations. The key to doing that are the Bianchi equations. In the next section we shall analyze the main properties of that system of equations in the Minkowski spacetime.

2.2 Electromagnetic and Weyl fields, Maxwell and Bianchi equations in the Minkowski spacetime

We start defining the Weyl tensor field as a tensor field with all the algebraic properties of the conformal part of the Riemann tensor field, see 1.1.14,

[10] More precisely, for some of these vector fields X, the property that the null components of their deformation tensors, as defined in 1.1.8, be asymptotically small.

1.1.15. In this section we restrict ourselves to $3+1$ dimensions. A more detailed discussion about Weyl tensor fields appears in Chapter 3, section 3.2. We intend this section as an introduction for it.

Definition 2.2.1 *Given a spacetime $(\mathcal{M}, \mathbf{g})$, we call Weyl field a tensor field W which satisfies the properties*

$$\begin{aligned}
&W_{\alpha\beta\gamma\delta} = -W_{\beta\alpha\gamma\delta} = -W_{\alpha\beta\delta\gamma} = W_{\gamma\delta\alpha\beta} \\
&W_{\alpha\beta\gamma\delta} + W_{\alpha\gamma\delta\beta} + W_{\alpha\delta\beta\gamma} = 0 \\
&g^{\alpha\gamma} W_{\alpha\beta\gamma\delta} = 0
\end{aligned} \tag{2.2.1}$$

We say that a Weyl tensor field is a solution of the Bianchi equation in $(\mathcal{M}, \mathbf{g})$ if, relative to the Levi-Civita connection of $\mathbf{g}$, it verifies the equation

$$\mathbf{D}_{[\lambda} W_{\gamma\delta]\alpha\beta} = 0 \tag{2.2.2}$$

When the spacetime $(\mathcal{M}, \mathbf{g})$ is a solution of the Einstein vacuum equations $\mathbf{R}_{\alpha\beta} = 0$, the curvature tensor coincides with its conformal part $\mathbf{C}$ and it is, therefore, a Weyl tensor field, which satisfies the Bianchi equations 2.2.2

In this section we review the main properties of Weyl tensor fields and of the Bianchi equations 2.2.2 in a fixed background space $(\mathcal{M}, \mathbf{g})$, see [Ch-Kl1]. We start recalling the following definition and properties of the Hodge duals of a given Weyl field

$$^\star W_{\alpha\beta\gamma\delta} \equiv \frac{1}{2} \in_{\alpha\beta\mu\nu} W^{\mu\nu}{}_{\gamma\delta} \ , \ W^\star_{\alpha\beta\gamma\delta} = W_{\alpha\beta}{}^{\mu\nu} \frac{1}{2} \in_{\mu\nu\gamma\delta}$$

Proposition 2.2.1

I. If W is a Weyl field then $W^\star = {}^\star W$ and $^\star({}^\star W) = -W$.

II. The following four sets of equations are equivalent

$$\begin{aligned}
&D_{[\sigma} W_{\gamma\delta]\alpha\beta} = 0 \ , \ D^\lambda W_{\lambda\gamma\alpha\beta} = 0 \\
&D_{[\sigma} {}^\star W_{\gamma\delta]\alpha\beta} = 0 \ , \ D^{\lambda\star} W_{\lambda\gamma\alpha\beta} = 0 \ .
\end{aligned}$$

III. The Bianchi equations 3.2.1, are conformally invariant[11] *, see [Pe1], [Pe2] and also [Ch-Kl1], [Ch-Kl].*

[11] This means that whenever we perform a conformal transformation Φ of the spacetime $(\mathcal{M}, \mathbf{g})$ with $\tilde{\mathbf{g}} = \Phi_* \mathbf{g} = \Lambda^2 \mathbf{g}$, then $\tilde{W} = \Lambda^{-1} \Phi_* W$ is a solution of the Bianchi equations for the spacetime $(\mathcal{M}, \tilde{\mathbf{g}})$.

IV. If W satisfies the Bianchi equations and X is a conformal Killing vector field then the modified Lie derivative [12]

$$\hat{\mathcal{L}}_X W = \mathcal{L}_X W - \frac{1}{2}{}^{(X)}[W] + \frac{3}{8} tr^{(X)}\pi W \tag{2.2.3}$$

is also a solution of these equations.

These equations look complicated, nevertheless they are quite similar to the more familiar Maxwell equations. This becomes apparent if we decompose W into its "electric and magnetic" parts. Given vector fields X, Y we introduce $i_{(X,Y)}$ through the relation $(i_{(X,Y)}W)_{\mu\nu} = W_{\mu\rho\nu\sigma}X^\rho Y^\sigma$, then, with $X = Y = T_0$, define

$$E = i_{(T_0,T_0)}W \ , \ H = i_{(T_0,T_0)}{}^\star W. \tag{2.2.4}$$

These two covariant symmetric and traceless tensor fields E and H, tangent to the hyperplanes $\Sigma_t \equiv \{p \in \mathcal{M} | t(p) = t\}$, determine completely the Weyl tensor field. It is easy to write the Bianchi equations for this decomposition and obtain the following "Maxwell-type equations",

$$\begin{aligned}
&\Phi^{-1}\partial_t E + \mathrm{curl} H = \rho(E, H)\\
&\Phi^{-1}\partial_t H - \mathrm{curl} E = \sigma(E, H)\\
&\mathrm{div} E = k \wedge H\\
&\mathrm{div} H = -k \wedge E \ .
\end{aligned}$$

where ∇ is the covariant derivative with respect to Σ_t, $(\mathrm{div}E)_i \equiv \nabla^j E_{ij}$, $(\mathrm{curl}E)_{ij} \equiv \epsilon_i^{lk}\nabla_l E_{kj}$ and the analogous expressions hold for H. Moreover $(k \wedge E)_i \equiv \epsilon_i{}^{mn} k_m{}^l E_{ln}$ and analogous expression for H.

This strong formal analogy with the Maxwell equations goes even further. In fact, just like the Maxwell equations, the Bianchi equations possess an analogue of the electromagnetic tensor, the Bel-Robinson tensor, see [Bel], which allows to derive, in the case of the Minkowski spacetime, conserved quantities.

Definition 2.2.2 *The Bel-Robinson tensor of the Weyl field W is the four covariant tensor field:*

$$Q_{\alpha\beta\gamma\delta} = W_{\alpha\rho\gamma\sigma}W_\beta{}^\rho{}_\delta{}^\sigma + {}^\star W_{\alpha\rho\gamma\sigma}{}^\star W_\beta{}^\rho{}_\delta{}^\sigma. \tag{2.2.5}$$

[12] ${}^{(X)}[W]_{\alpha\beta\gamma\delta} = {}^{(X)}\pi_\alpha^\lambda W_{\lambda\beta\gamma\delta} + {}^{(X)}\pi_\beta^\lambda W_{\alpha\lambda\gamma\delta} + {}^{(X)}\pi_\gamma^\lambda W_{\alpha\beta\lambda\delta} + {}^{(X)}\pi_\delta^\lambda W_{\alpha\beta\gamma\lambda}$, where ${}^{(X)}\pi$ is the deformation tensor relative to the vector field X.

The Bel-Robinson tensor has the following important properties, which recall those of the energy momentum tensor of the Maxwell equations, see [Ch-Kl], [Ch-Kl1]:

Proposition 2.2.2
a) Q is symmetric and traceless relative to all pairs of indices.
b) Q satisfies the following positivity condition: $Q(X_1, X_2, X_3, X_4)$ is non negative [13] *for any non spacelike future directed vector fields X_1, X_2, X_3, X_4.*
c) If W is a solution of the Bianchi equations then

$$D^\alpha Q_{\alpha\beta\gamma\delta} = 0 \tag{2.2.6}$$

Proposition 2.2.3 *Let $Q(W)$ be the Bel Robinson tensor of a Weyl field W and X, Y, Z a triplet of vector fields. We define the covariant vector field P associated to the triplet as*

$$P_\alpha = Q_{\alpha\beta\gamma\delta} X^\beta Y^\gamma Z^\delta. \tag{2.2.7}$$

Using all the symmetry properties of Q we have

$$\begin{aligned} Div P &= Div Q_{\beta\gamma\delta} X^\beta Y^\gamma Z^\delta \\ &+ \frac{1}{2} Q_{\alpha\beta\gamma\delta} \left({}^{(X)}\pi^{\alpha\beta} Y^\gamma Z^\delta + {}^{(Y)}\pi^{\alpha\gamma} X^\beta Z^\delta + {}^{(Z)}\pi^{\alpha\delta} X^\beta Y^\gamma \right) \end{aligned} \tag{2.2.8}$$

Thus, to any X, Y, Z Killing or conformal Killing vectorfields we can associate a conserved quantity. More precisely,

Theorem 2.2.1 *Let W be a solution of Bianchi equations and $X, Y, Z, V_1, \dots V_k$ be Killing or conformal Killing vector fields, then*

a) $Div P = 0$ where P is defined by 2.2.7

b) The integral $\int_{\Sigma_t} Q[W](X, Y, Z, T_0) d^3x$ is finite and constant for all t provided that it is finite at $t = 0$.

c) The integrals

$$\int_{\Sigma_t} Q[\hat{\mathcal{L}}_{V_1} \hat{\mathcal{L}}_{V_2} \dots \hat{\mathcal{L}}_{V_k} W](X, Y, Z, T_0) d^3x$$

are finite and constant for all t provided that they are finite at $t = 0$.

[13] If we restrict to timelike vector fields $Q(X_1, X_2, X_3, X_4)$ is positive.

2.2.1 Asymptotic behaviour of the Weyl fields in the Minkowski spacetime

The Minkowski spacetime is equipped with the following geometric structures

1) *Hyperplanes:* Consists of the level hypersurfaces of the time function t, $\Sigma_t \equiv \{p \in \mathcal{M} | t(p) = t\}$.

2) *Canonical Null Foliations:* consists of the double family null cones $\{C(u), \underline{C}(\underline{u})\}$ defined as the level hypersurfaces of the functions $u = t - r$ and $t + r$.

$$\begin{aligned} C(u) &\equiv \{p \in \mathcal{M} | u(p) = u = t - r\} \ , \ \text{(outgoing)} \\ \underline{C}(\underline{u}) &\equiv \{p \in \mathcal{M} | \underline{u}(p) = \underline{u} = t + r\} \ , \ \text{(incoming)} \end{aligned} \qquad (2.2.9)$$

3) *Canonical Sphere Foliation:* Consists of the family of 2-spheres $S(t, u) = \Sigma_t \cap C(u)$, or $S(u, \underline{u}) = C(u) \cap C(\underline{u})$. For each fixed t the family $\{S(t, u)\}$ produces an S^2-foliation of the hyperplane Σ_t. This coincides, of course, with the standard foliation by the surfaces $S_{t,r} = \{(t, x) \in \Sigma_t || x| = r\}$

4) *Canonical Null Pair:* given by the vector fields

$$e_3 = \partial_t + \partial_r, \quad e_4 = \partial_t - \partial_r$$

We can complete the pair e_3, e_4 to a null frame $\{e_1, e_2, e_3, e_4\}$, at a generic point p, by chosing an orthonormal frame $\{e_a\}$, $a \in (1, 2)$ on the tangent space to the sphere $S(t, u)$ passing through p.

5) *Conformal Structure:* The Minkowski spacetime has a family of Killing and conformal Killing vector fields, see subsection 2.1.2, among which we note

$$T_0 = \frac{1}{2}(e_3 + e_4) \ , \ S = \frac{1}{2}(ue_3 + \underline{u}e_4) \ , \ K_0 = \frac{1}{2}(u^2 e_3 + \underline{u}^2 e_4) \qquad (2.2.10)$$

T_0 corresponds to time translations, S to scaling transformations and K_0 to inverted time translations. In addition to these we shall also make use of the rotation vector fields:

$$^{(i)}O = \epsilon_{ijk}(x_j \partial_k - x_k \partial_j) \ . \qquad (2.2.11)$$

We next define the null components [14] of the Weyl tensor.

[14] This null decomposition of W originates in the work of E.T.Newman, R.Penrose, [Ne-Pe2].

Definition 2.2.3 *Let e_3, e_4 be a null pair and W a Weyl field. At a given point p we introduce the following tensors defined on the tangent space to the sphere $S(t,u)$ passing through the point p,*

$$\begin{aligned}
&\alpha(W)(X,Y) = W(X,e_4,Y,e_4)\ ,\ \underline{\alpha}(W)(X,Y) = W(X,e_3,Y,e_3)\\
&\beta(W)(X) = \frac{1}{2}W(X,e_4,e_3,e_4)\ ,\ \ \underline{\beta}(W)(X) = \frac{1}{2}W(X,e_3,e_3,e_4)\\
&\rho(W) = \frac{1}{4}W(e_3,e_4,e_3,e_4)\ ,\ \sigma(W) = \frac{1}{4}\rho({}^\star W) = \frac{1}{4}{}^\star W(e_3,e_4,e_3,e_4)
\end{aligned} \tag{2.2.12}$$

where X, Y are arbitrary vector fields tangent to $S(t,u)$.

It is easy to verify that α and $\underline{\alpha}$ are symmetric traceless tensors, β and $\underline{\beta}$ are vectors field and ρ, σ are scalar fields. The total number of independent components is, as expected, ten and they completely describe the Weyl tensor field. The Bianchi equations satisfied by W, see for instance [Ch-Kl1], expressed in terms of these components, are:

Bianchi equations:

$$\begin{aligned}
&\not{D}_4\underline{\alpha} + \tfrac{1}{2}\mathrm{tr}\chi\underline{\alpha} = -\not{\nabla}\widehat{\otimes}\underline{\beta} &&,\ \not{D}_3\underline{\beta} + 2\mathrm{tr}\underline{\chi}\ \underline{\beta} = -\not{\mathrm{d}}\mathrm{iv}\,\underline{\alpha}\\
&\not{D}_4\underline{\beta} + \mathrm{tr}\chi\underline{\beta} = -\not{\nabla}\rho + {}^\star\not{\nabla}\sigma &&,\ \mathbf{D}_3\rho + \tfrac{3}{2}\mathrm{tr}\underline{\chi}\rho = -\not{\mathrm{d}}\mathrm{iv}\,\underline{\beta}\\
&\mathbf{D}_4\rho + \tfrac{3}{2}\mathrm{tr}\chi\rho = \not{\mathrm{d}}\mathrm{iv}\,\beta &&,\ \mathbf{D}_3\sigma + \tfrac{3}{2}\mathrm{tr}\underline{\chi}\sigma = -\not{\mathrm{d}}\mathrm{iv}\,{}^\star\underline{\beta}\\
&\mathbf{D}_4\sigma + \tfrac{3}{2}\mathrm{tr}\chi\sigma = -\not{\mathrm{d}}\mathrm{iv}\,{}^\star\beta &&,\ \not{D}_3\beta + \mathrm{tr}\underline{\chi}\beta = \not{\nabla}\rho + {}^\star\not{\nabla}\sigma\\
&\not{D}_4\beta + 2\mathrm{tr}\chi\beta = \not{\mathrm{d}}\mathrm{iv}\,\alpha &&,\ \not{D}_3\alpha + \tfrac{1}{2}\mathrm{tr}\underline{\chi}\alpha = \not{\nabla}\widehat{\otimes}\beta
\end{aligned} \tag{2.2.13}$$

where, here, $tr\chi = -tr\underline{\chi} = \frac{2}{r}$, $\not{D}_4$ and $\not{D}_3$ are the projection on the tangent space to $S(t,u)$, of the covariant derivatives along the null directions, $\not{\mathrm{d}}\mathrm{iv}$ and $\not{\nabla}$ are the projection on the tangent space to $S(t,u)$, of the divergence and the covariant derivatives relative to Σ_t and $\widehat{\otimes}$ denotes twice the traceless part of the symmetric tensor product. The Hodge operator * indicates the dual of the tensor fields relative to the tangent space of $S(t,u)$. [correction of the definition of $\widehat{\otimes}$, a factor 2 was missing]

Our first goal is to show how to derive the asymptotic properties of a solution to the Bianchi equations in Minkowski spacetime for initial data at $t = 0$, compatible with the assumptions we will use, later on, to study the Einstein equations. From this perspective we expect that, for a given spacetime which satisfies the Einstein vacuum equations, the curvature tensor $\mathbf{R}$ behaves, on the initial hypersurface, like r^{-3} as $r \to \infty$. This is due to the presence of an $\mathcal{ADM}$ mass term different from zero, see [Ar-De-Mi], in the definition of the asymptotic flatness for the initial data of the Einstein vacuum equations. Moreover, since the $\mathcal{ADM}$ mass is a time independent constant, we expect that the time and the angular derivatives of $\mathbf{R}$ behave better.

With this in mind it makes perfect sense to assume initial data for our Bianchi equations in Minkowski spacetime such that all the terms in the following sum are bounded at $t = 0$.

$$\begin{aligned}\mathcal{Q}(t) &= \int_{\Sigma_t} Q(\hat{\mathcal{L}}_O W)(K_0, K_0, T_0, T_0)\\ &+ \int_{\Sigma_t} Q(\hat{\mathcal{L}}_O^2 W)(K_0, K_0, T_0, T_0)\\ &+ \int_{\Sigma_t} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_O W)(K_0, K_0, T_0, T_0) \qquad (2.2.14)\\ &+ \int_{\Sigma_t} Q(\hat{\mathcal{L}}_{T_0} W)(K_0, K_0, K_0, T_0)\\ &+ \int_{\Sigma_t} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_{T_0} W)(K_0, K_0, K_0, T_0)\end{aligned}$$

with $\hat{\mathcal{L}}_X$ defined by 2.2.3. Note also that $|\hat{\mathcal{L}}_O f|^2 \equiv \sum_{i=1,2,3} |\hat{\mathcal{L}}_{{}^{(i)}O} f|^2$.

To understand the meaning of this quantity one should observe that $K_0 = \frac{1}{2}(u^2 e_3 + \underline{u}^2 e_4)$ and $T_0 = \frac{1}{2}(e_3 + e_4)$ are the only[15] future directed, causal conformal Killing vector fields in Minkowski spacetime. Thus the only choices for the vector fields X, Y, Z in 3.2.7, such that $Q(X, Y, Z, T_0)$ be a positive quantity consistent with our above discussion, are between T_0 and K_0. In view of Theorem 2.2.1, we can conclude that $\mathcal{Q}(t)$ is conserved, therefore bounded for all time. Using these conservation laws, combining them with the global Sobolev inequalities, see Prop. 2.2.4 below, and taking advantage of the Bianchi equations, 2.2.13, we prove the following

Theorem 2.2.2 *Assume $\mathcal{Q}$ is finite at $t = 0$. Then,*

(i) In the exterior region[16] we have the following bounds for the various null components of the Weyl tensor

$$sup_{Ext}|r^{\frac{7}{2}}\alpha| \leq C_0 \ , \ sup_{Ext}|r\tau_-^{\frac{5}{2}}\underline{\alpha}| \leq C_0$$

[15] see Liouville Theorem 1.1.3

[16] The exterior region refers to the set of points of $\mathcal{M}$ such that $r \geq t$. Its complement will be called internal region or interior.

$$
\begin{aligned}
&sup_{Ext}|r^{\frac{7}{2}}\beta| \leq C_0\ ,\ sup_{Ext}|r^2\tau_-^{\frac{3}{2}}\underline{\beta}| \leq C_0 \qquad (2.2.15)\\
&sup_{Ext}|r^3\tau_-^{\frac{1}{2}}\sigma| \leq C_0\ ,\ sup_{Ext}|r^3\tau_-^{\frac{1}{2}}(\rho-\overline{\rho})| \leq C_0
\end{aligned}
$$

where $\overline{\rho}$ is the average of ρ on the spheres $S(t,u)$, $\tau_-^2 = 1+u^2$ and C_0 is a constant which depends on $\mathcal{Q}(t=0)$.

(ii) In the interior region

$$|W(t,x)| \leq c(1+t)^{-\frac{7}{2}}$$

(iii) The mass term $\overline{\rho}$ is in fact zero [17].

Since a result analogous to this for the full Einstein equations is at the heart of the proofs of the *C-K Theorem* and of the present *Main Theorem*, we give here the main ideas of the proof of Theorem 2.2.2. From the identities

$$
\begin{aligned}
Q(W)(e_3,e_3,e_3,e_3) &= 2|\underline{\alpha}|^2\\
Q(W)(e_4,e_4,e_4,e_4) &= 2|\alpha|^2\\
Q(W)(e_3,e_3,e_3,e_4) &= 4|\underline{\beta}|^2 \qquad (2.2.16)\\
Q(W)(e_3,e_4,e_4,e_4) &= 4|\beta|^2\\
Q(W)(e_3,e_3,e_4,e_4) &= 4(\rho^2+\sigma^2)
\end{aligned}
$$

we obtain, by a straightforward calculation,

$$
\begin{aligned}
Q(W)(K_0,K_0,T_0,T_0) &= \frac{1}{8}\underline{u}^4|\alpha|^2 + \frac{1}{8}u^4|\underline{\alpha}|^2 + \frac{1}{2}(\underline{u}^4 + \frac{1}{2}\underline{u}^2u^2)|\beta|^2\\
&+ \frac{1}{2}(\underline{u}^4 + u^4 + \underline{u}^2u^2)(\rho^2+\sigma^2) + \frac{1}{2}(u^4 + \frac{1}{2}\underline{u}^2u^2)|\underline{\beta}|^2\\
Q(W)(K_0,K_0,K_0,T_0) &= \frac{1}{8}\underline{u}^6|\alpha|^2 + \frac{1}{8}u^6|\underline{\alpha}|^2 + \frac{1}{4}(\underline{u}^6 + 3\underline{u}^4u^2)|\beta|^2 \qquad (2.2.17)\\
&+ \frac{3}{4}(\underline{u}^2+u^2)\underline{u}^2u^2(\rho^2+\sigma^2) + \frac{1}{4}(u^6 + 3\underline{u}^2u^4)|\underline{\beta}|^2
\end{aligned}
$$

We sketch two different methods of proving the estimates 2.2.15. The first based, on the maximal spacelike hypersurfaces $t = const$, corresponds to the method used in the proof of *C-K Theorem* while the second, based

[17] This is due to the fact that, relative to the "electro-magnetic" decomposition, $\text{div}E = 0$ and $\rho = E_{NN}$, with $N = \frac{x_i}{|x|}\partial_i$, see [Ch-Kl1]. In a general background spacetime, $\text{div}E$ has nontrivial source terms and consequently $\overline{\rho}$ fails to be zero. The asymptotic behavior of $\bar{\rho}$ is, in fact, tied to the nontriviality of the $\mathcal{ADM}$ mass.

on the null hyperurfaces $t-r=u$, $t+r=\underline{u}$, corresponds to the double null foliation approach of the *Main Theorem.*

In the first approach, based on the maximal foliation, the proof uses the conservation of the quantity $\mathcal{Q}(t)$, the null Bianchi equations 2.2.13 as well as the following form of the global Sobolev inequalities [18]

Proposition 2.2.4 *Let F be a smooth tensor field, tangent at each point to the corresponding $S=S(t,r)$. Denote by N the exterior unit normal to S, $\nabla\!\!\!\!/$ the induced covariant derivative and $\nabla\!\!\!\!/_N F$ the projection to S of the normal derivative $\nabla_N F$. We have*

Nondegenerate version:

$$\begin{aligned} sup_{S(t,r)}(r^{\frac{3}{2}}|F|) \;&\leq\; c\left(\int_{\Sigma_t}|F|^2+r^2|\nabla\!\!\!\!/ F|^2+r^2|\nabla\!\!\!\!/_N F|^2\right.\\ &\left.+\,r^4|\nabla\!\!\!\!/^2F|^2+r^4|\nabla\!\!\!\!/\nabla\!\!\!\!/_N F|^2\right)^{\frac{1}{2}} \qquad (2.2.18)\end{aligned}$$

Degenerate version:

$$\begin{aligned} sup_{S(t,r)}(r\tau_-^{\frac{1}{2}}|F|) \;&\leq\; c\left(\int_{\Sigma_t}|F|^2+r^2|\nabla\!\!\!\!/ F|^2+\tau_-^2|\nabla\!\!\!\!/_N F|^2\right.\\ &\left.+\,r^4|\nabla\!\!\!\!/^2F|^2+r^2\tau_-^2|\nabla\!\!\!\!/\nabla\!\!\!\!/_N F|^2\right)^{\frac{1}{2}} \qquad (2.2.19)\end{aligned}$$

In what follows we shall use all the results mentioned above in order to derive the aymptotic properties of α.

We start by applying Prop. 2.2.4 to $F=r^2\alpha$ and derive

$$\begin{aligned} sup_{\Sigma_t}(r^{\frac{7}{2}}|\alpha|) \;&\leq\; c\left(\int_{\Sigma_t}r^4|\alpha|^2+r^4|r\nabla\!\!\!\!/\alpha|^2+r^6|\nabla\!\!\!\!/_N\alpha|^2\right.\\ &\left.+\,r^4|r^2\nabla\!\!\!\!/^2\alpha|^2+r^6|r\nabla\!\!\!\!/\nabla\!\!\!\!/_N\alpha|^2\right)^{\frac{1}{2}}\end{aligned}$$

The integrals on the right hand side are controlled in terms of the quantity $\mathcal{Q}$ as follows:

i) The integrals $\int_{\Sigma_t}r^4|\alpha|^2$ and $\int_{\Sigma_t}r^4|r\nabla\!\!\!\!/\alpha|^2$ are both bounded by $\int_{\Sigma_t}Q(\hat{\mathcal{L}}_O W)(K_0,K_0,T_0,T_0)$. This follows from eq. 2.2.17 and the following simple identity, see [Ch-Kl1],

$$|\hat{\mathcal{L}}_O\alpha|^2\equiv\sum_i|\hat{\mathcal{L}}_{{}^{(i)}O}\alpha|^2=r^2|\nabla\!\!\!\!/\alpha|^2+4|\alpha|^2 \qquad (2.2.20)$$

[18]We do not give the proof of Proposition 2.2.4 because an analogous Proposition valid in a general spacetime is proved in any detail in Chapter 4, Proposition 4.1.4. See also [Ch-Kl], Proposition 3.2.3 .

ii) The integral $\int_{\Sigma_t} r^4|r^2\nabla\!\!\!/^2\alpha|^2$ is bounded by $\int_{\Sigma_t} Q(\hat{\mathcal{L}}_O^2 W)(K_0,K_0,T_0,T_0)$.

We are left with the integrals $\int_{\Sigma_t} r^4|r\nabla\!\!\!/_N\alpha|^2$ and $\int_{\Sigma_t} r^6|r\nabla\!\!\!/\nabla\!\!\!/_N\alpha|^2$. We indicate how to estimate the first, the second can, then, be dealt with in the same way. Observe that

$$\nabla_N\alpha = D_{T_0}\alpha - D_{e_3}\alpha$$

In view of this it suffices to estimate

$$\int_{\Sigma_t} r^6|D\!\!\!\!/_{T_0}\alpha|^2 \text{ and } \int_{\Sigma_t} r^6|D\!\!\!\!/_{e_3}\alpha|^2 \ .$$

iii) The integral $\int_{\Sigma_t} r^6|D\!\!\!\!/_{T_0}\alpha|^2$ is bounded by $\int_{\Sigma_t} Q(\hat{\mathcal{L}}_{T_0} W)(K_0,K_0,K_0,T_0)$. This can be checked again with the help of 2.2.17.

v) To bound the last integral, $\int_{\Sigma_t} r^6|D\!\!\!\!/_{e_3}\alpha|^2$, we have to use the null Bianchi equations, 2.2.13 to express $D\!\!\!\!/_{e_3}\alpha$ in terms of $\frac{1}{r}\alpha$ and $\nabla\!\!\!/\beta$. It follows immediately that these integrals are bounded by $\int_{\Sigma_t} Q(\hat{\mathcal{L}}_O W)(K_0,K_0,T_0,T_0)$.

The other components of the Weyl tensor can be treated in the same manner. The results in the interior region are much easier to derive, see [Ch-Kl1].

The proof of the asymptotic estimates of Theorem 2.2.2 described above is based on energy type estimates on the maximal spacelike hypersurfaces $t = const$. This is the main reason why a maximal spacelike foliation was used in [Ch-Kl].

In what follows we sketch a different approach to derive Theorem 2.2.2 using instead the double null foliation $t - r = u$, $t + r = \underline{u}$, see 2.2.9. The main idea of the new approach is to introduce some new quantities, analogous to $\mathcal{Q}(t)$, see 2.2.15, associated to both families of null hypersurfaces. We call these quantities "flux quantities" and we will use their boundedness in terms of the initial data.

More precisely, denoting λ, ν the values taken by the functions $u(p), \underline{u}(p)$ respectively, we define $V(\lambda,\nu)$ as the causal past of $S(\lambda,\nu) \equiv C(\lambda)\cap\underline{C}(\nu)$,

$$V(\lambda,\nu) = J^-(S(\lambda,\nu)) \ .$$

We call $\mathcal{K}$ the region of the Minkowski spacetime, $V(\lambda_0,\nu_*)$, for a fixed couple (λ_0,ν_*).
$\mathcal{K}$ lies in the future of the initial hypersurface $t=0$ and is foliated by the two families of null hypersurfaces $\{C(\lambda)$ and $\{\underline{C}(\nu)\}$ with λ and ν varying

in the finite intervals $[\lambda_1, \lambda_0]$ and $[\nu_0, \nu_*]$ respectively, where $\nu_0 = -\lambda_0$ and $\nu_* = -\lambda_1$. For simplicity we may assume $\lambda_0 = -\nu_0 = 0$. We shall also call [19] the null hypersurface $\underline{C}(\nu_*)$ "the last slice" of the spacetime region $\mathcal{K}$ under consideration and denote it by $\underline{C}_*$.
$\mathcal{K}$ lies outside the domain of influence of the origin [20] and in the causal past of the null hypersurface $\underline{C}_*$.

Remark: *In the proof of the "Main Theorem", ν_* is finite and the central part of the proof consists in showing that we can take the limit $\nu_* \to \infty$. Here we may, in fact, assume $\nu_* = \infty$. In this case $\mathcal{K}$ is the whole complement of the domain of depedence of the origin, $J^+(0)$.*

To define the quantity analogous to the conserved quantity $\mathcal{Q}(t)$ used in the previous proof, of Theorem 2.2.2, we go back to equation 2.2.8, see Proposition 2.2.3, which we integrate on $V(\lambda, \nu)$.
If X, Y, Z are conformal Killing vector fields, we derive the identity:

$$\begin{aligned}
&\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(W)(X,Y,Z,e_3) + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(W)(X,Y,Z,e_4) \\
&= \int_{\Sigma_0\cap V(\lambda,\nu)} Q(W)(X,Y,Z,T_0) \qquad (2.2.21)
\end{aligned}$$

Applying this identity to $\hat{\mathcal{L}}_{T_0}W$, $\hat{\mathcal{L}}_O W$, $\hat{\mathcal{L}}_O\hat{\mathcal{L}}_{T_0}W$, $\hat{\mathcal{L}}_S\hat{\mathcal{L}}_{T_0}W$, $\hat{\mathcal{L}}_O^2 W$, with X, Y, Z one of the conformal, timelike vectorfields T_0, K_0, we are led to the following quantities:

$$\begin{aligned}
\mathcal{Q}(\lambda,\nu) &= \mathcal{Q}_1(\lambda,\nu) + \mathcal{Q}_2(\lambda,\nu) \\
\underline{\mathcal{Q}}(\lambda,\nu) &= \underline{\mathcal{Q}}_1(\lambda,\nu) + \underline{\mathcal{Q}}_2(\lambda,\nu) \qquad (2.2.22)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{Q}_1(\lambda,\nu) &= \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_{T_0}W)(\bar{K},\bar{K},\bar{K},e_4) \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T_0,e_4) \\
\mathcal{Q}_2(\lambda,\nu) &= \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_{T_0}W)(\bar{K},\bar{K},\bar{K},e_4) \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T_0,e_4) \qquad (2.2.23)
\end{aligned}$$

[19] For reasons which become clear in the next subsections.
[20] The causal future of the origin $J^+(0)$.

and

$$\begin{aligned}\underline{\mathcal{Q}}_1(\lambda,\nu) &= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_{T_0}W)(\bar{K},\bar{K},\bar{K},e_3)\\ &\quad+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T_0,e_3)\\ \underline{\mathcal{Q}}_2(\lambda,\nu) &= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_{T_0}W)(\bar{K},\bar{K},\bar{K},e_3)\\ &\quad+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T_0,e_3) \qquad (2.2.24)\end{aligned}$$

In view of the identity 2.2.21 we infer that both flux quantities $\mathcal{Q}(\lambda,\nu)$ and $\underline{\mathcal{Q}}(\lambda,\nu)$ are bounded by $\mathcal{Q}(t=0)$. Assuming that the initial data are such that $\mathcal{Q}(t=0)$ is finite it follows that both quantities $\mathcal{Q}(\lambda,\nu)$ and $\underline{\mathcal{Q}}(\lambda,\nu)$ are finite and independent of the values of λ,ν. We have thus derived the following

Proposition 2.2.5 *Consider the spacetime region $\mathcal{K}$, as defined in the last remark, and assume that the data satisfy the condition $\mathcal{Q}_0 \equiv \mathcal{Q}(t=0) < \infty$. Then the following quantities are uniformly bounded for all $\lambda \leq 0$ and $\nu \geq 0$.*

$$\begin{aligned}&\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(K,K,T,e_4) \;,\; \int_{\underline{C}(\nu\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(K,K,T,e_3)\\ &\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(K,K,K,e_4) \;,\; \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(K,K,K,e_3)\\ &\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(K,K,T,e_4) \;,\; \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(K,K,T,e_3) \qquad (2.2.25)\\ &\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(K,K,K,e_4) \;,\; \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(K,K,K,e_3)\end{aligned}$$

In order to prove Theorem 2.2.2 we need, in addition to the above Proposition, the following analogue of Proposition 2.2.4

Proposition 2.2.6 *Let F be a smooth tensor field, tangent at each point to the corresponding $S(\lambda,\nu)$ passing through that point. The following estimates hold uniformly with regard to $\lambda \leq 0, \nu \geq 0$,*

$$\begin{aligned}sup_{S(\lambda,\nu)}(r^{\frac{3}{2}}|F|) &\leq c\Big[\left(\int_{S(\lambda,\nu_0)} r^4|F|^4\right)^{\frac{1}{4}} + \left(\int_{S(\lambda,\nu_0)} r^4|\nabla\!\!\!/ F|^4\right)^{\frac{1}{4}}\\ &\quad+\Big(\int_{C(\lambda)\cap V(\lambda,\nu)} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + r^2|D\!\!\!\!/_4 F|^2\\ &\quad+r^4|\nabla\!\!\!/^2 F|^2 + r^4|\nabla\!\!\!/ D\!\!\!\!/_4 F|^2\Big)^{\frac{1}{2}}\Big] \qquad (2.2.26)\end{aligned}$$

and

$$\begin{aligned} sup_{S(\lambda,\nu)}(r\tau_-^{\frac{1}{2}}|F|) &\leq c\Big[\left(\int_{S(\lambda,\nu_0)} r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} + \left(\int_{S(\lambda,\nu_0)} r^2\tau_-^2|r\nabla\!\!\!/ F|^4\right)^{\frac{1}{4}} \\ &+\left(\int_{C(\lambda)\cap V(\lambda,\nu)} (|F|^2 + r^2|\nabla\!\!\!/ F|^2 + \tau_-^2|\mathbf{D}\!\!\!\!/_4 F|^2 \right. \\ &\left. + r^4|\nabla\!\!\!/^2 F|^2 + r^2\tau_-^2|\nabla\!\!\!/ \mathbf{D}\!\!\!\!/_4 F|^2\right)^{\frac{1}{2}}\Big] \end{aligned} \tag{2.2.27}$$

The previous computation can also be done to express the sup norms in terms of integrals along the incoming null hypersurfaces $\underline{C}(\nu)$. The results are

$$\begin{aligned} sup_{S(\lambda,\nu)}(r^{\frac{3}{2}}|F|) &\leq c\Big[\left(\int_{S(\lambda_0,\nu)} r^4|F|^4\right)^{\frac{1}{4}} + \left(\int_{S(\lambda_0,\nu)} r^4|\nabla\!\!\!/ F|^4\right)^{\frac{1}{4}} \\ &+\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + r^2|\mathbf{D}\!\!\!\!/_3 F|^2 \right. \\ &\left. + r^4|\nabla\!\!\!/^2 F|^2 + r^4|\nabla\!\!\!/ \mathbf{D}\!\!\!\!/_3 F|^2\right)^{\frac{1}{2}}\Big] \end{aligned} \tag{2.2.28}$$

and

$$\begin{aligned} sup_{S(\lambda,\nu)}(r\tau_-^{\frac{1}{2}}|F|) &\leq c\Big[\left(\int_{S(\lambda_0,\nu)} r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} + \left(\int_{S(\lambda_0,\nu)} r^2\tau_-^2|r\nabla\!\!\!/ F|^4\right)^{\frac{1}{4}} \\ &+\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + \tau_-^2|\mathbf{D}\!\!\!\!/_3 F|^2 \right. \\ &\left. + r^4|\nabla\!\!\!/^2 F|^2 + r^2\tau_-^2|\nabla\!\!\!/ \mathbf{D}\!\!\!\!/_3 F|^2\right)^{\frac{1}{2}}\Big] \end{aligned} \tag{2.2.29}$$

We show how to use these new $\mathcal{Q}$ quantities, introduced above, and Proposition 2.2.6 to derive the asymptotic properties of $\underline{\alpha}$ in the flat case.

Asymptotic behaviour of $\underline{\alpha}$:

We observe that the quantities $Q(W)(K,K,T,e_4)$, $Q(W)(K,K,K,e_4)$, for an arbitrary Weyl field W, do not involve the null component $\underline{\alpha}$ of W. This follows easily from the expressions $T_0 = \frac{1}{2}(e_3 + e_4)$, $K_0 = \frac{1}{2}(u^2 e_3 + \underline{u}^2 e_4)$ as well as eqs. 2.2.17. We are therefore obliged to look at the integrals along $\underline{C}(\underline{u})$ in Prop. 2.2.5. On the other hand, according to Prop. 2.2.6, applied to $\tau_-^2\underline{\alpha}$, we have

$$
\begin{aligned}
sup_{S(\underline{u},u)}(r\tau_-^{\frac{5}{2}}|\underline{\alpha}|) &\leq c\Big[\left(\int_{S(u,\underline{u}_0)} r^2\tau_-^{10}|\underline{\alpha}|^4\right)^{\frac{1}{4}} + \left(\int_{S(u,\underline{u}_0)} r^2\tau_-^{10}|r\nabla\!\!\!/\underline{\alpha}|^4\right)^{\frac{1}{4}}\\
&+ \Big(\int_{\underline{C}(\underline{u}\cap V(u,\underline{u})} \tau_-^4|\underline{\alpha}|^2 + \tau_-^4|r\nabla\!\!\!/\underline{\alpha}|^2 + \tau_-^6|\mathbf{D}\!\!\!/_3\underline{\alpha}|^2\\
&+ \tau_-^4|r^2\nabla\!\!\!/^2\underline{\alpha}|^2 + \tau_-^6|r\nabla\!\!\!/\mathbf{D}\!\!\!/_3\underline{\alpha}|^2\Big)^{\frac{1}{2}}\Big] \qquad (2.2.30)
\end{aligned}
$$

i) In view of the identity 2.2.20 the integrals $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u}))}\tau_-^4|\underline{\alpha}|^2$ as well as $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u}))}\tau_-^4|r\nabla\!\!\!/\alpha|^2$ can be estimated by the bounded integral

$$\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(K_0,K_0,T_0,e_3)\ .$$

ii) Similarily the integral $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})}\tau_-^4|r^2\nabla\!\!\!/^2\alpha|^2$ can be estimated by the bounded integral $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u}))} Q(\hat{\mathcal{L}}_O^2 W)(K_0,K_0,T_0,e_3)$.

iii) We are left with the integral $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})}\tau_-^6|\mathbf{D}\!\!\!/_3\underline{\alpha}|^2$ as well as the integral $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})}\tau_-^6|r\nabla\!\!\!/\mathbf{D}\!\!\!/_3\underline{\alpha}|^2$. Observe that $\mathbf{D}\!\!\!/_3\underline{\alpha} = \mathbf{D}\!\!\!/_{T_0}\underline{\alpha} - \mathbf{D}\!\!\!/_4\underline{\alpha}$. In view of this it suffices to estimate $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})}\tau_-^6|\mathbf{D}\!\!\!/_{T_0}\underline{\alpha}|^2$ and $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})}\tau_-^6|\mathbf{D}\!\!\!/_4\underline{\alpha}|^2$. The first integral is bounded by $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_{T_0}W)(K_0,K_0,K_0,e_3)$. On the other side, to bound the last integral, $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})}\tau_-^6|\mathbf{D}\!\!\!/_4\underline{\alpha}|^2$, we have to use the null Bianchi equations,2.2.13, to express $\mathbf{D}\!\!\!/_4\underline{\alpha}$ in terms of $r^{-1}\underline{\alpha}$ and $\nabla\!\!\!/\underline{\beta}$. It follows that these integrals are bounded by $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(K_0,K_0,T_0,e_3)$. Finally, proceeding in the same way, one sees that $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})}\tau_-^6|r\nabla\!\!\!/\mathbf{D}\!\!\!/_3\underline{\alpha}|^2$ is bounded by the two integrals $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)(K_0,K_0,T_0,e_3)$ and $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(K_0,K_0,K_0,e_3)$.
Therefore we have obtained the asymptotic result for $\underline{\alpha}$ stated in Theorem 2.2.2. The other components of the Weyl tensor can be treated in the same manner.

2.3 Global non linear stability of the Minkowski spacetime

As we have mentioned earlier, the Bianchi equations provide the keystone in the overall strategy of the proofs of both *C-K Theorem* and the present

Main Theorem. They allow us to introduce the main energy type quantities similar to the $E_s[Du](t)$ energy type norms, see 2.1.28, introduced in the earlier discussion concerning global solutions of non linear wave equations.

More importantly they allow us to make an essential conceptual linearization of the Einstein equations. This consists in the following bootstrap scheme:

i) One can first assume given the spacetime with its well defined causal structure and study the Bianchi equations as a linear system on the given background spacetime. Unlike the case of the Minkowski spacetime we do not have any symmetry at our disposal and, therefore, no conserved quantities $\mathcal{Q}$. We can assume, however, and this will have to be justified as part of our overall bootstrap argument, that our background spacetime comes equipped with *approximate conformal Killing vector fields.* By this we mean in fact vector fields X whose traceless parts of their deformation tensors are small in a appropriate way. Using this we can construct quantities analogous to the $\mathcal{Q}$'s introduced in the Minkowski spacetime and discussed in the previous section. Instead of being conserved we need to prove that they remain bounded by an universal constant times their value on the initial hypersurface.

This leads, just as in the flat case, to precise asymptotic estimates for the various components of the Riemann tensor.

ii) To close the bootstrap we then proceed in the opposite way. We assume given a spacetime whose curvature tensor verifies the asymptotic properties obtained in step i) and deduce from them the assumptions concerning the causal structure made there.

The properties of the causal structure we construct have to be expressed relative to a foliation induced by two functions. In *C-K Theorem*, for example, one had to rely on a time function $t(p)$ whose levels are maximal spacelike hypersurfaces and an optical function $u(p)$, whose levels are the outgoing null hypersurfaces which play the role of the outgoing null cones in the Minkowski spacetime.

The optical function u is by far the more important one as all the radiation features of the Einstein equations depend heavily on it. The precise definition of u as a solution of the eikonal equation,

$$g^{\mu\nu}\partial_\mu u \partial_\nu u = 0 \ ,$$

allows us to treat the non trivial asymptotic properties of the causal structure of a spacetime with non vanishing $\mathcal{ADM}$ mass.

The maximal foliation seemed to be also important because of the traditional role played by the time t in deriving energy estimates as, for instance, in the case of the wave equations discussed in section 2.1. The non local features of the maximal foliation lead, however, to enormous technical complications which are not intrinsic to the real problem of evolution.

In the *Main Theorem* we rely instead on a double null foliation where u is defined as before and the second function $\underline{u}$ is defined symmetrically as an incoming solution of the eikonal equation, whose levels are incoming null hypersurfaces. This procedure is naturally adapted to the local hyperbolic features of the Einstein equations.

2.4 Structure of the work

In view of the previous discussion the plan of the remaining six chapters is as follows:

- Chapter 3 contains all the main geometric constructions, the definitions of the main quantities $\mathcal{Q}$, a precise formulation of the *Main Theorem* and a detailed description of the strategy of its proof.

  We start with a discussion of the double null foliation, the canonical null pairs and null frames and the associated Ricci coefficients. We then present the null decomposition of the Weyl tensor followed by the structure equations and the Bianchi equations expressed relative to our null frames. The structure equations relative to a double null foliation have been been previously derived by other authors, see for example **?? put it** and the references [21] therein.

  We introduce then the important notion of the canonical double null foliation defined in terms of initial data solutions of the *last slice* and of the *initial hypersurface* problems. We also review the main properties of the Bel-Robinson tensor. This, together with the definition of the vector fields $T, S, K_0, {}^{(i)}O$, related to the analogous vector fields introduced in Minkowski space, see subsection 2.2.1, allows us to define our main quantity $\mathcal{Q}$.

  In addition to the $\mathcal{Q}$ norm we introduce the other two fundamental family of norms, the $\mathcal{R}$ norms, which describe regularity and asymptotic properties of the null components of the Riemann tensor, and

[21] The first systematic use of null tetrads, not necessarily tied to foliations, goes back to E.T.Newman and R.Penrose, [Ne-Pe1].

the $\mathcal{O}$ norms which contain detailed regularity and asymptotic informations for the connection coefficients.

We introduce also a large family of norms describing the regularity and the asymptotic properties of the null components of the deformation tensors of $T, S, K_0, {}^{(i)}O$. These norms are controlled in terms of the $\mathcal{O}$ norms.

We state precise results concerning the relationship between $\mathcal{R}$, the $\mathcal{O}$ and the $\mathcal{Q}$ norms which form the heart of the proof of our main theorem.

These above mentioned results require precise assumptions on the initial data. We describe in which sense these data have to be small.

Finally we give the precise statement of the *Main Theorem* and give a detailed account of all the steps of the proof.

In the end of Chapter 3 we give, for comparison, a short review of the proof of *C-K Theorem*.

- Chapter 4 contains all the results concerning the $\mathcal{O}$ norms. They are obtained assuming that we control the $\mathcal{R}$ norms, as a *bootstrap assumption*, and expressed in terms of and initial conditions on Σ_0 and on the last slice. The crucial and delicate issue here is to control the regularity and asymptotic behavior of the null structure coefficients with respect to that of the null components of the curvature tensor. These require subtle estimates depending heavily on the geometric properties of the null structure equations introduced in Chapter 3. Though some of the main ideas we rely on are similar to those in [Ch-Kl] we encounter many additional difficulties as we have to estimate not only the null connection coefficients associated to the null hypersurfaces $C(u)$ but also those associated to the incoming null hypersurfaces[22] $\underline{C}(\underline{u})$; in fact these are heavily coupled in the null structure equations.

  In the last section of this Chapter we obtain the estimates of the rotation deformation tensors, based on the results of the previous sections.

- Chapter 5 is devoted to the control of the curvature tensor. Making appropriate smallness assumptions for the $\mathcal{O}$ norms of the connection

[22] In [Ch-Kl] these were replaced by the elliptic estimates of the geometric quantities associated to the maximal time foliation. The additional difficulties of treating the null structure equations are more than compensated by the avoidance of the very technical elliptic estimates and the gain in symmetry.

coefficients, we show how to control the $\mathcal{R}$ norms in terms of the $\mathcal{Q}$ norms.

- Chapter 6, in which we establish the boundedness of the $\mathcal{Q}$ norms, is central to the whole book. This requires the detailed analysis of the large number of *error terms* generated because of the nontriviality of the deformation tensors of the vector fields involved in the definition of the $\mathcal{Q}$ norms introduced[23] in Chapter 3, section 3.5.1.

- In Chapter 7 we discuss the solution of the so called *initial slice* and *final slice* problems. These are needed to define the *canonical double null foliation* of the spacetime region we construct. As in [Ch-Kl] the canonical null foliation plays a fundamental role in our approach; we explain this in more details in Chapter 3. The solution of the *initial slice problem* is a simplified version of the analogous result proved in [Ch-Kl], the *final slice problem* is however significantly different from the *last slice problem* discussed in [Ch-Kl] and we discuss it in detail.

- Chapter 8 is devoted to collect some conclusions on the asymptotic properties of these global solutions not discussed in the "Main Theorem". They do not differ significantly from those discussed in the last chapter of [Ch-Kl] except on the fact that, due to the kind of foliations we have used here, they are obtained in a much simpler way.

[23]Analogous to the definition 2.2.23 and 2.2.24 in the flat case.

Chapter 3

Definitions and results

3.1 Connection coefficients

3.1.1 Null second fundamental forms and torsion of a space-like 2-surface

Let S be a closed 2-dimensional surface embedded in a 3+1 dimensional spacetime $(\mathcal{M}, \mathbf{g})$. We assume that S has a compact filling by which we understand that there exists a Cauchy hypersurface Σ containing S and such that S is the boundary of a compact region of Σ.

Let γ be the induced metric on S,

$$\gamma(X, Y) = \mathbf{g}(X, Y) \tag{3.1.1}$$

for all $X, Y \in TS$, the tangent space to S. We denote by $d\mu_\gamma$ the area element and by ϵ_{ab} its components relative to an orthonormal frame $(e_a)_{a=1,2}$. We denote by $|S|$ the area and by $r(S)$ the radius of S,

$$r(S) = \sqrt{\frac{1}{4\pi}|S|} \tag{3.1.2}$$

Let $\nabla\!\!\!\!/$ be the induced connection on S and $\mathbf{K}$ its the Gauss curvature. We recall that, if $\mathbf{R}\!\!\!/$ is the intrinsic Riemann curvature tensor, and X, Y, Z three arbitrary vector fields tangent to S [1]:

$$\mathbf{R}\!\!\!/(X, Y)Z = (\gamma(Y, Z)X - \gamma(X, Z)Y)\,\mathbf{K}\ .$$

[1] Relative to an arbitrary orthonormal frame $(e_a)_{a=1,2}$ of S

$$\mathbf{R}\!\!\!/_{abcd} = (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\mathbf{K} \tag{3.1.3}$$

At every point p in S we consider the orthogonal complement $T_pS^\perp$ relative to $T_p\mathcal{M}$. This intersects the null cone through p along two null directions. Consider the future oriented half lines corresponding to these directions and their projections to the tangent space of Σ at p. The half line whose projection points towards the unbounded component of Σ will be called future outgoing, the other one future incoming, at p. Similarily we define the past incoming and past outgoing directions. The past incoming direction at p is complementary to the future outgoing while the past outgoing is complementary to the future incoming. Remark also that these definitions do not depend on the particular "fillings" of S, in other words they do not depend on the choice of the hypersurface Σ passing through S. At any point $p \in S$ we choose e_4, e_3 to be two future directed null vectors corresponding to the outgoing and incoming directions and subject to the normalization condition

$$g(e_4, e_3) = -2 \tag{3.1.4}$$

Here e_4 corresponds to the future outgoing direction while e_3 to the future incoming one.

Definition 3.1.1 *A smooth choice of such vectors will be called a null pair of S.*

According to our definitions a null pair is uniquely defined up to a scaling transformation:

$$e_4{}' = ae_4, \;\; e_3{}' = a^{-1}e_3 \tag{3.1.5}$$

for some smooth positive function a.
We should use systematically throughout the book the following notations

Definition 3.1.2 *Given a tensor U defined on $\mathcal{M}$ and tangent to S, at any point of S, we define $\not{\mathbf{D}}_4U$ and $\not{\mathbf{D}}_3U$ to be the projections to TS of $\mathbf{D}_4U$ and $\mathbf{D}_3U$.*

Definition 3.1.3 *Corresponding to any normalized null pair e_4, e_3 we define the null second fundamental forms of S to be the 2-covariant tensors on S:*

$$\chi(X,Y) = g(\mathbf{D}_Xe_4, Y), \;\; \underline{\chi}(X,Y) = g(\mathbf{D}_Xe_3, Y) \tag{3.1.6}$$

where X, Y are vector fields tangent to S and $\mathbf{D}$ denotes the connection on $(\mathcal{M}, \mathbf{g})$. Moreover we define the torsion of S to be the 1-form:

$$\zeta(X) = \frac{1}{2}g(\mathbf{D}_Xe_4, e_3) \tag{3.1.7}$$

Clearly, as $[X,Y] \in TS$, $\chi, \underline{\chi}$ are 2-covariant symmetric tensors [2] on S. Performing a scaling transformation of the form 3.1.5 we have

$$\chi' = a\chi \ , \ \underline{\chi}' = a^{-1}\underline{\chi} \tag{3.1.8}$$

Hence $\chi, \underline{\chi}$ are uniquely defined up to a transformation of the form 3.1.8. Under the same scaling transformation the torsion ζ transforms according to the formula

$$\zeta'(X) = \zeta(X) - a^{-1}X(a) \tag{3.1.9}$$

Remark:
The covariant derivatives, intrinsic to S, of χ and $\underline{\chi}$ are not invariant under the scaling transformation 3.1.5. Nevertheless the following tensors transform nicely under it:

$$\nabla\!\!\!/_X \chi + \zeta(X)\chi \quad , \quad \nabla\!\!\!/_X \underline{\chi} - \zeta(X)\underline{\chi}$$

where X is an arbitrary vectorfield on S. Indeed

$$\begin{aligned} \nabla\!\!\!/_X \chi' + \zeta'(X)\chi' &= a\left(\nabla\!\!\!/_X \chi + \zeta(X)\chi\right) \\ \nabla\!\!\!/_X \underline{\chi}' - \zeta'(X)\underline{\chi}' &= a^{-1}\left(\nabla\!\!\!/_X \chi - \zeta(X)\chi\right) \end{aligned}$$

We shall call the above quantities the conformal derivatives of χ, $\underline{\chi}$.

We denote by $\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$ the traces with respect to γ of χ, $\underline{\chi}$ and by $\hat{\chi}, \underline{\hat{\chi}}$ their traceless parts

$$\begin{aligned} \hat{\chi}(X,Y) &= \chi(X,Y) - \frac{1}{2}\mathrm{tr}\chi\gamma(X,Y) \\ \underline{\hat{\chi}}(X,Y) &= \underline{\chi}(X,Y) - \frac{1}{2}\mathrm{tr}\underline{\chi}\gamma(X,Y) \end{aligned} \tag{3.1.10}$$

Observe that the product $\mathrm{tr}\chi\mathrm{tr}\underline{\chi}$ is independent of the choice of the null pair.

---

[2] If S is the the standard sphere, $(x^1)^2+(x^2)^2+(x^3)^2 = r^2$, on the spacelike hypersurface $x^0 = const$ in Minkowski spacetime, the standard choice of the null pair is $e_4 = \partial_t + \partial_r$, $e_3 = \partial_t - \partial_r$. In this case $\mathrm{tr}\chi = -\mathrm{tr}\underline{\chi} = \frac{2}{r}$, $\hat{\chi} = \underline{\hat{\chi}} = 0$, $\zeta = 0$ and $K = \frac{1}{r^2}$.
In the Schwarzschild spacetime, where S is an orbit of the rotation group, a natural choice of a null pair is: $e_4 = \Phi^{-1}(\frac{\partial}{\partial t} + \frac{\partial}{\partial r_*}), e_3 = \Phi^{-1}(\frac{\partial}{\partial t} - \frac{\partial}{\partial r_*})$ with $r_* \equiv r + 2m\log(\frac{r}{2m} - 1)$, $\Phi^2 = \left(1 - \frac{2m}{r}\right)$. In this case, $\chi_{ab} = \delta_{ab}\frac{\Phi}{r}$, $\underline{\chi}_{ab} = -\delta_{ab}\frac{\Phi}{r}$.

Definition 3.1.4 *Given the one form ξ on S we define its Hodge dual* [3]*:*

$$ {}^\star\xi_a = \in_{ab} \xi^b $$

If ξ is a symmetric, traceless, 2-tensor we define the following left, ${}^\star\xi$, and right $\xi^\star$, Hodge duals:

$$ {}^\star\xi_{ab} = \in_{ac} \xi^c{}_b \ , \ \xi^\star_{ab} = \xi_a{}^c \in_{cb} $$

Remark: Clearly, if ξ is a one form, ${}^\star({}^\star\xi) = -\xi$. If ξ is a symmetric, traceless, 2-tensor observe that the tensors ${}^\star\xi$, $\xi^\star$ are also symmetric, traceless and satisfy

$$ {}^\star\xi = -\xi^\star \ , \ {}^\star({}^\star\xi) = -\xi. $$

Remark: Another simple but important property is the following one: Let ξ, η be 2-covariant symmetric traceless tensors, then:

$$ \xi_{ac}\eta_{cb} + \xi_{bc}\eta_{ca} = (\xi \cdot \eta)\delta_{ab} \tag{3.1.11} $$

We always decompose a symmetric 2-tensor ξ between its trace, $\text{tr}\xi = \delta^{ab}\xi_{ab}$, and traceless part. Thus if ξ_{ab} is such a tensor we write its traceless part:

$$ \hat{\xi}_{ab} = \xi_{ab} - \frac{1}{2}\text{tr}\xi\delta_{ab}. $$

Given $S \subset \mathcal{M}$ and the fixed null pair $\{e_4, e_3\}$ we can associate two triplets $\{\mathcal{N}, L, \phi_s\}$, $\{\underline{\mathcal{N}}, \underline{L}, \underline{\phi}_s\}$ as follows: starting with the vector field e_4 given on S, we introduce the one parameter flow $\phi_s(p) = l(s;p)$ where $l(s;p)$ denotes the null geodesic parametrized by the affine parameter [4] s with initial conditions:

$$ l(0) = p \ \ , \ (\frac{d}{ds}l)(0) = e_4|_p $$

We define L by

$$ \frac{d}{ds}l(s;p) = L(s;p) \ . $$

Clearly L satisfies:

$$ g(L, L) = 0 \quad \mathbf{D}_L L = 0 \ . $$

[3] here a, b are just coordinate indices

[4] This means that the the vector field L satisfies $Ls = 1$.

The flow $\{\phi_s\}$ generates, starting from S, a family of two dimensional surfaces $\{S_s\}$. The union of all future outgoing null geodesics initiating at points in S forms a three dimensional null hypersurface which we denote by $\mathcal{N}$. The diffeomorphism ϕ_t can be extended from points on S to any point q on $\mathcal{N}$: given $q = l(s,p) \in S_s$, ϕ_t moves q, along $l(s,p)$, as follows

$$\phi_t : \ \ \mathcal{N} \in q \longrightarrow \phi_t(q) = l(s+t;p) \in \mathcal{N} \ .$$

By replacing e_4 with e_3 we can repeat the same procedure and define the triplet $\{\underline{\mathcal{N}}, \underline{L}, \underline{\phi}_s\}$. Observe that the hypersurfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$ are independent on the particular choice of the null pair.

Definition 3.1.5 *Given S, we call $\mathcal{N}$ and $\underline{\mathcal{N}}$ the outgoing and incoming null hypersurfaces generated by S.*

Let X be a vector field defined on S tangent to it, $X \in TS$. We extend it to $\mathcal{N}$ (denoting it again by X) as follows

$$X_q \equiv d\phi_s \cdot X_p \ \ ,$$

where $p \in S$, $q = \phi_s(p) \in S_s$ and $d\phi_s$ is the differential of ϕ_s. The extension is such that $\phi_{s*}X = X$ holds for any s, where ϕ_{s*} is the standard push forward. According to the definiton of the Lie derivative, it is immediate to realize that on $\mathcal{N}$:

$$[L, X]_q = (\mathcal{L}_L X)_q = lim_{h\to 0}\frac{1}{h}[X_q - (\phi_{h*}X)_q] = 0 \ .$$

This implies that the flows ψ_t and ϕ_s commute, $\psi_t \circ \phi_s = \phi_s \circ \psi_t$, where ψ_t is the flow generated by the extended X.

Let $\mathcal{N}$ be the corresponding outgoing null hypersurface generated by S and U a tensor of type $\binom{0}{r}$ defined [5] on $\mathcal{N}$ and tangential to each S_s, we introduce the operation $\mathcal{D}U$ in the following way,

Definition 3.1.6 *At each point of S*

$$\mathcal{D}U = \frac{d}{ds}\phi_s^* U\Big|_{s=0} \tag{3.1.12}$$

where $\phi_s^* U$ is defined as usual

$$\phi_s^* U(X_p, .., Z_p) = U(d\phi_s \cdot X_p, ..., d\phi_s \cdot Z_p)$$

for $X_p, ..., Z_p$ in the tangent space to S.

[5] It is enough that U be defined in a neighborhood of S.

The same definition can be given for $\underline{\mathcal{D}}$ substituting L with $\underline{L}$ and $\mathcal{N}$ with $\underline{\mathcal{N}}$

$$\underline{\mathcal{D}}U = \frac{d}{ds}\underline{\phi}_s^* U\Big|_{s=0} \tag{3.1.13}$$

Remark that the operation $\mathcal{D}$ can be trivially extended to the whole of $\mathcal{N}$ and is intrinsic to $\mathcal{N}$. Observe also that $\mathcal{D}$ is essentially the Lie derivative $\mathcal{L}_L$. In fact if U is the restriction to $\mathcal{N}$ of a spacetime tensor field, then $\mathcal{D}U = \mathcal{L}_L U$. For example, denoting by γ the restriction of the spacetime metric $\mathbf{g}$ to the surfaces $S_s \subset \mathcal{N}$, we have

$$\begin{aligned}\mathcal{D}\gamma(X,Y) &= (\mathcal{L}_L\mathbf{g})(X,Y) = L(\mathbf{g}(X,Y)) - \mathbf{g}(\mathcal{L}_L X,Y) - \mathbf{g}(X,\mathcal{L}_L Y)\\ &= \mathbf{g}(\mathbf{D}_X L,Y) + \mathbf{g}(X,\mathbf{D}_Y L) = 2\chi(X,Y)\end{aligned}$$

and similarly for $\underline{\mathcal{D}}\gamma$. Therefore,

$$\mathcal{D}\gamma = 2\chi \quad , \quad \underline{\mathcal{D}}\gamma = 2\underline{\chi} \tag{3.1.14}$$

To stress the geometric and physical importance of χ and $\underline{\chi}$, it is appropriate to recall the following properties:

Let $|S|(s) = \int_S d\mu_{\gamma_s}$ be the area of S_s with γ_s the metric on S equal to the pullback by $(\phi_s)^*$ of g restricted to S_s. Then

$$\frac{d}{ds}|S|_{s=0} = \int_S \mathrm{tr}\chi \quad , \quad \frac{d}{d\underline{s}}|S|_{\underline{s}=0} = \int_S \mathrm{tr}\underline{\chi} \tag{3.1.15}$$

In other words $\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$ measure the change of area of S in the direction of e_4, and e_3 respectively. The null second fundamental forms χ and $\underline{\chi}$ measure also the change of the length of a curve Γ on S when mapped by ϕ_s on the surface S_s. In fact let $\Gamma : t \to \Gamma(t) \in S$ and let $\Gamma_s \equiv \phi_s(\Gamma)$. The length $|\Gamma|_s$ of Γ_s satisfies the following equations, where $V = \frac{d\Gamma}{dt}$,

$$\frac{d}{ds}|\Gamma|_{s=0} = \int \frac{\chi(V,V)}{|V|^2}dt \quad , \quad \frac{d}{d\underline{s}}|\Gamma|_{\underline{s}=0} = \int \frac{\underline{\chi}(V,V)}{|V|^2}dt \tag{3.1.16}$$

3.1.2 Null decomposition of the curvature tensor

Consider a surface S and a fixed null pair $\{e_4, e_3\}$. Associated with this we also consider a null frame $\{e_4, e_3, e_1, e_2\}$, where $\{e_1, e_2\}$ is an arbitrary orthonormal frame for TS. Remark that the quantities we define below depend only on the choice of the null pair. We express, at each point of

S, the various components of the Riemann curvature tensor of $(\mathcal{M}, \mathbf{g})$ with respect to it. We recall that the curvature tensor has the following symmetry properties:

$$\mathbf{R}_{\alpha\beta\gamma\delta} = -\mathbf{R}_{\beta\alpha\gamma\delta} = -\mathbf{R}_{\alpha\beta\delta\gamma} = \mathbf{R}_{\gamma\delta\alpha\beta}$$
$$\mathbf{R}_{\alpha\beta\gamma\delta} + \mathbf{R}_{\alpha\gamma\delta\beta} + \mathbf{R}_{\alpha\delta\beta\gamma} = 0$$

The curvature tensor has 20 independent components. Half of these components are taken into account by the Ricci curvature, the remaining ten components correspond to the conformal curvature tensor $\mathbf{C}$, see 1.1.14,

$$\begin{aligned}\mathbf{C}\ _{\alpha\beta\gamma\delta} &= \mathbf{R}_{\alpha\beta\gamma\delta} - \frac{1}{2}\left(g_{\alpha\gamma}\mathbf{R}_{\beta\delta} + \mathbf{g}_{\beta\delta}\mathbf{R}_{\alpha\gamma} - \mathbf{g}_{\beta\gamma}\mathbf{R}_{\alpha\delta} - \mathbf{g}_{\alpha\delta}\mathbf{R}_{\beta\gamma}\right)\\ &+ \frac{1}{6}(\mathbf{g}_{\alpha\gamma}\mathbf{g}_{\beta\delta} - \mathbf{g}_{\alpha\delta}\mathbf{g}_{\beta\gamma})\mathbf{R}\end{aligned}$$

The conformal curvature tensor $\mathbf{C}$ is the primary example of what we call a Weyl field namely a $\binom{0}{4}$ tensorfield W verifying all the symmetry properties of the Riemann curvature tensor:

$$\begin{aligned}&W_{\alpha\beta\gamma\delta} = -W_{\beta\alpha\gamma\delta} = -W_{\alpha\beta\delta\gamma} = W_{\gamma\delta\alpha\beta}\\ &W_{\alpha\beta\gamma\delta} + W_{\alpha\gamma\delta\beta} + W_{\alpha\delta\beta\gamma} = 0\end{aligned} \tag{3.1.17}$$

and, in addition,

$$g^{\alpha\gamma}W_{\alpha\beta\gamma\delta} = 0 \tag{3.1.18}$$

For a Weyl tensor field W the following definitions of left and right Hodge duals are equivalent:

$$^{\star}W_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}W^{\mu\nu}\ _{\gamma\delta}$$
$$W^{\star}_{\alpha\beta\gamma\delta} = W_{\alpha\beta}\ ^{\mu\nu}\frac{1}{2}\epsilon_{\mu\nu\gamma\delta}$$

where $\in^{\alpha\beta\gamma\delta}$ are the components of the volume element in $\mathcal{M}$. One can easily show that, $^{\star}W = W^{\star}$ is also a Weyl tensor field and that $^{\star}(^{\star}W) = -W$. Relative to the null frame we define the null components of the Weyl field:

Definition 3.1.7 *Let e_4, e_3 be the null pair of the null frame. Let W be a Weyl field and introduce the following tensor fields operating, at each $p \in S$, on the subspace TS_p of the tangent space $T\mathcal{M}_p$,*

$$
\begin{aligned}
&\alpha(W)(X,Y) = W(X,e_4,Y,e_4)\\
&\beta(W)(X) = \frac{1}{2}W(X,e_4,e_3,e_4)\\
&\rho(W) = \frac{1}{4}W(e_3,e_4,e_3,e_4)\\
&\sigma(W) = \frac{1}{4}\rho({}^\star W) = \frac{1}{4}{}^\star W(e_3,e_4,e_3,e_4)\\
&\underline{\beta}(W)(X) = \frac{1}{2}W(X,e_3,e_3,e_4)\\
&\underline{\alpha}(W)(X,Y) = W(X,e_3,Y,e_3)
\end{aligned}
\tag{3.1.19}
$$

where X, Y are arbitrary vectors tangent to S at p. We call the set

$$\alpha(W),\ \underline{\alpha}(W),\ \beta(W),\ \underline{\beta}(W),\ \rho(W),\ \sigma(W) \tag{3.1.20}$$

the null decomposition of W relative to e_4, e_3.

We easily check that, in view of 3.1.18, $\alpha(W), \underline{\alpha}(W)$ are symmetric traceless tensors, thus they have two independent components each. Together the total number of independent components of the set in 3.1.20 accounts for all the ten degrees of freedom of the Weyl tensorfield W.
The null components of W can be expressed in terms of the null decomposition, denoting $W_{\alpha\beta\gamma\delta} \equiv W(e_\alpha, e_\beta, e_\gamma, e_\delta)$, in the following way:

$$
\begin{aligned}
&W_{a33b} = -\underline{\alpha}_{ab}\ ,\ W_{a334} = 2\underline{\beta}_a\\
&W_{a44b} = -\alpha_{ab}\ ,\ W_{a443} = -2\beta_a\\
&W_{a3b4} = -\rho\delta_{ab} + \sigma\epsilon_{ab}\\
&W_{a3bc} = -{}^\star({}^\star W)_{a3bc} = \epsilon_{bc}{}^*\underline{\beta}_a\\
&W_{a4bc} = -{}^\star({}^\star W)_{a4bc} = -\epsilon_{bc}{}^*\beta_a\\
&\delta_{ab}W_{a3bc} = \underline{\beta}_c\ ,\ \delta_{ab}W_{a4bc} = -\beta_c\\
&W_{dcab}\delta_{da}\delta_{cb} = -2\rho\ ,\ W_{3434} = -4\rho\\
&W_{ab34} = 2\epsilon_{ab}\sigma
\end{aligned}
\tag{3.1.21}
$$

where ${}^*\alpha, {}^*\underline{\alpha}, {}^*\beta, {}^*\underline{\beta}$ are the Hodge duals of $\alpha, \underline{\alpha}, \beta, \underline{\beta}$ relative to TS_p. Thus, according to definition 3.1.4,

$$
\begin{aligned}
&\alpha({}^*W) = -{}^*\alpha(W)\ ,\ \beta({}^*W) = -{}^*\beta(W)\ ,\ \underline{\alpha}({}^*W) = {}^*\underline{\alpha}(W)\\
&\underline{\beta}({}^*W) = -{}^*\underline{\beta}(W)\ ,\ \rho({}^*W) = \sigma(W)\ ,\ \sigma({}^*W) = -\rho(W)\ .
\end{aligned}
$$

Relative to a rescaling of the null pair,

$$e_4 \to e_4' = ae_4 \ , \ e_3 \to e_3' = a^{-1}e_3 \ ,$$

the null components of W change according to

$$\begin{aligned} &\alpha' = a^2\alpha \quad \underline{\alpha}' = a^{-2}\underline{\alpha} \\ &\beta' = a\beta \quad \underline{\beta}' = a^{-1}\underline{\beta} \\ &\rho' = \rho, \quad \sigma' = \sigma \end{aligned} \tag{3.1.22}$$

In view of this we associate to the null components of W the following weights which we refer to as the "signature" of the corresponding component:

$$\begin{aligned} \text{sign}(\underline{\alpha}) &= -2 \quad \text{sign}(\alpha) &= 2 \\ \text{sign}(\underline{\beta}) &= -1 \quad \text{sign}(\beta) &= 1 \\ \text{sign}(\rho) &= 0 \quad \text{sign}(\sigma) &= 0 \end{aligned} \tag{3.1.23}$$

We also remark that, under the interchange of the components $3, 4$ in the null decomposition of W, we have:

$$\alpha \to \underline{\alpha} \ , \ \beta \to -\underline{\beta} \ , \ \rho \to \rho \ , \ \sigma \to -\sigma \tag{3.1.24}$$

.

Remark: Throughout the remaining of this Chapter $(\mathcal{M}, \mathbf{g})$ refers to an Einstein vacuum spacetime.

3.1.3 Null structure equations of a 2-surface S

The following equations associated to a fixed two surface S are a subset of the whole set of the null structure equations relative to the spacetime $(\mathcal{M}, \mathbf{g})$.

Proposition 3.1.1 *The Gauss curvature* $\mathbf{K}$ *of* S*, as well as the null second fundamental forms* $\chi, \underline{\chi}$ *and torsion* ζ *corresponding to a null pair* e_4, e_3 *verify the following null structure equations on* S*:*

1. *Gauss equation*

$$\mathbf{K} = -\frac{1}{4}tr\chi tr\underline{\chi} + \frac{1}{2}\hat{\chi} \cdot \hat{\underline{\chi}} - \rho$$

2. Null Codazzi equations

$$\not{d}iv\hat{\chi} + \hat{\chi}\cdot\zeta = \frac{1}{2}(\not{\nabla}tr\chi + \zeta\, tr\chi) - \beta$$
$$\not{d}iv\underline{\hat{\chi}} - \underline{\hat{\chi}}\cdot\zeta = \frac{1}{2}(\not{\nabla}tr\underline{\chi} - \zeta\, tr\underline{\chi}) + \underline{\beta}$$

3. Torsion equation

$$c\not{u}rl\zeta + \frac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}} = \sigma$$

The proof of this proposition is in the appendix to this chapter.

3.1.4 Integrable S-foliations of the spacetime

Assume that the spacetime $(\mathcal{M}, \mathbf{g})$ is foliated by a smooth, codimension two, foliation whose leaves are compact, spacelike, 2-surfaces diffeomorphic to S^2. We shall refer to it as an S-foliation of the spacetime.

Definition 3.1.8 *A tensor field on $(\mathcal{M}, \mathbf{g})$, which is tangent, at each point, to the leaf of the foliation passing through that point, is called S-tangent.*

At every point $p \in \mathcal{M}$ we consider the future incoming and outgoing null directions normal to the leaves of the foliation [6] and choose, correspondingly, a null pair e_4, e_3.
We introduce the following definition,

Definition 3.1.9 *An "adapted" null frame consists, in addition to the null pair e_3, e_4, of an orthonormal frame $\{e_a\}_{a=1,2}$ tangent to the two dimensional S-surfaces.*

Definition 3.1.10 *The S-foliation is said to be null outgoing, respectively null incoming, integrable if the distribution formed by the tangent spaces of S together with the null outgoing direction, respectively null incoming, is integrable. An S-foliation which is both null outgoing and incoming integrable is called double null integrable.*

Proposition 3.1.2 *An outgoing (incoming) null integrable foliation is locally given by the level hypersurfaces of a function u ($\underline{u}$) which verifies the eikonal equation*

$$g^{\mu\nu}\partial_\mu w \partial_\nu w = 0 \qquad (3.1.25)$$

[6] We will simply refer to them as the null directions of the foliation.

We shall refer to u and $\underline{u}$ as outgoing, respectively, incoming optical functions and denote their level surfaces by

$$C(\lambda) = \{p \in \mathcal{M} | u(p) = \lambda\}$$
$$\underline{C}(\nu) = \{p \in \mathcal{M} | \underline{u}(p) = \nu\} \tag{3.1.26}$$

Proof: If the S-foliation is null-outgoing integrable the distribution made by the linear span formed by TS and e_4,

$$p \in \mathcal{M} \longrightarrow \Delta_p \equiv \{TS \oplus e_4\}_p \ ,$$

is integrable which means that at each p there is a submanifold $\mathcal{N} \subset \mathcal{M}$ such that

$$T\mathcal{N}_p = \Delta_p \ .$$

Therefore the null hypersurface $\mathcal{N}$ can be expressed, locally, as the level hypersurface of a function u. It follows that the covariant vector n defined by $n_\mu = \partial_\mu u$ satisfies $n(e_a) = 0, \ n(e_4) = 0, \ a \in \{1,2\}$. Therefore $g^{\mu\nu} n_\nu = g^{\mu\nu}\partial_\nu u$ is a null vector field proportional to e_4; this implies

$$g^{\mu\nu}\partial_\mu u \partial_\nu u = 0 \ .$$

Everything goes in the same way for the null incoming integrable foliation, with the obvious substitutions, $p \in \mathcal{M} \longrightarrow \underline{\Delta}_p \equiv \{TS \oplus e_3\}_p$ and $\underline{\mathcal{N}}$ instead of $\mathcal{N}$.
The following corollary is an immediate consequence of Proposition 3.1.2

Corollary 3.1.1 *A double null integrable S-foliation, can be locally described by the level hypersurfaces $C(\lambda)$, $\underline{C}(\nu)$ associated to an outgoing optical function u and an incoming optical function $\underline{u}$. The leaves of the foliation take the form*

$$S(\lambda, \nu) = C(\lambda) \cap \underline{C}(\nu) \tag{3.1.27}$$

Definition 3.1.11 *The pair of foliations of the spacetime $(\mathcal{M}, \mathbf{g})$ defined by the null hypersurfaces $C(\lambda)$ and $\underline{C}(\nu)$ is called the "double null foliation" associated to u and $\underline{u}$.*

We associate to u, $\underline{u}$ the null geodesic vector fields,

$$L^\rho \equiv -g^{\rho\mu}\partial_\mu u \ \text{ and } \ \underline{L}^\rho \equiv -g^{\rho\mu}\partial_\mu \underline{u} \tag{3.1.28}$$

They satisfy

$$\mathbf{D}_L L = 0 \ , \ \mathbf{D}_{\underline{L}} \underline{L} = 0 \ .$$

We refer to $L, \underline{L}$ as a null geodesic pair of the double null foliation. At each point, L and $\underline{L}$ are proportional to e_4 and e_3, respectively.

Definition 3.1.12 *Given a double null foliation with associated null geodesic vector fields $L, \underline{L}$ we define its "spacetime lapse function" Ω by*

$$2\Omega^2 = -\mathbf{g}(L, \underline{L})^{-1} = -(g^{\rho\sigma}\partial_\rho u \partial_\sigma \underline{u})^{-1} \tag{3.1.29}$$

So far we have identified the null second fundamental forms and the torsion as natural geometric objects corresponding to a given two dimensional surface S embedded in the spacetime. We look now at the remaining connection coefficients, associated to an arbitrary S-foliation with a fixed null pair,

$$\begin{aligned} \xi_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_4} e_4, e_a) \quad , \quad \underline{\xi}_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_3} e_3, e_a) \\ \eta_a &= -\frac{1}{2}\mathbf{g}(\mathbf{D}_{e_3} e_a, e_4) \ , \ \underline{\eta}_a = -\frac{1}{2}\mathbf{g}(\mathbf{D}_{e_4} e_a, e_3) \\ \omega &= -\frac{1}{4}\mathbf{g}(\mathbf{D}_{e_4} e_3, e_4) \quad , \quad \underline{\omega} = -\frac{1}{4}\mathbf{g}(\mathbf{D}_{e_3} e_4, e_3) \end{aligned} \tag{3.1.30}$$

It is straightforward to check that, in the case of a double null integrable foliation, $\xi = \underline{\xi} = 0$.

The fact that a S-foliation is double null integrable does not depend on the choice of the null pair $\{e_4, e_3\}$. In fact $\{e_4, e_3\}$ can be subjected to a scaling transformation

$$e_4{}' = a e_4, \ \ e_3{}' = a^{-1} e_3 \ .$$

In the following we make a specific choice of a null pair.

Definition 3.1.13 *Given a double null foliation with its geodesic null pair $\{L, \underline{L}\}$ and lapse Ω we introduce,*

$$e_4 = \hat{N} = 2\Omega L \ , \ e_3 = \underline{\hat{N}} = 2\Omega \underline{L} \tag{3.1.31}$$

which we call the "normalized null pair of the foliation".

Indeed

$$\mathbf{g}(\hat{N}, \underline{\hat{N}}) = 4\Omega^2 \mathbf{g}(L, \underline{L}) = -2 \ .$$

Remark: Given a double null foliation, the scalar function Ω^2 and the normalized null pair $\{\hat{N}, \underline{\hat{N}}\}$ are uniquely determined. This definition implies that

$$\mathbf{D}_{\hat{N}} \hat{N} = (\mathbf{D}_{\hat{N}} \log \Omega) \hat{N} \ , \ \mathbf{D}_{\underline{\hat{N}}} \underline{\hat{N}} = (\mathbf{D}_{\underline{\hat{N}}} \log \Omega) \underline{\hat{N}} \ .$$

Recalling the definitions 3.1.30 of the connection coefficients we find

$$
\begin{aligned}
\underline{\xi}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{\hat{N}}}\underline{\hat{N}}, e_a) = \frac{1}{2}\underline{\hat{N}}(\log\Omega)\mathbf{g}(\underline{\hat{N}}, e_a) = 0 \\
\xi_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{\hat{N}}\hat{N}, e_a) = \frac{1}{2}\hat{N}(\log\Omega)\mathbf{g}(\hat{N}, e_a) = 0 \\
\underline{\omega} &= \frac{1}{4}\mathbf{g}(\mathbf{D}_{\underline{\hat{N}}}\underline{\hat{N}}, \hat{N}) = -\frac{1}{2}\mathbf{D}_3(\log\Omega) \\
\omega &= \frac{1}{4}\mathbf{g}(\mathbf{D}_{\hat{N}}\hat{N}, \underline{\hat{N}}) = -\frac{1}{2}\mathbf{D}_4(\log\Omega) \qquad (3.1.32) \\
\underline{\eta}_a &= -\zeta_a + \nabla\!\!\!\!/_a \log\Omega \ , \ \eta_a = \zeta_a + \nabla\!\!\!\!/_a \log\Omega \\
\zeta_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_a}\hat{N}, \underline{\hat{N}})
\end{aligned}
$$

Thus all the connection coefficients of a double null integrable foliation, can be expressed in terms of $\chi, \underline{\chi}, \zeta, \Omega$. We also remark that, under the interchange of the components 3, 4 in the connection coefficients, we have

$$(\chi, \underline{\chi}) \to (\underline{\chi}, \chi) \ , \ (\eta, \underline{\eta}) \to (\underline{\eta}, \eta) \ , \ (\omega, \underline{\omega}) \to (\underline{\omega}, \omega) \ , \ \zeta \to -\zeta \qquad (3.1.33)$$

The next definition introduces another important property of outgoing and incoming null integrable foliations.

Definition 3.1.14 *Consider an arbitrary S-foliation and a null outgoing vector field N normal to each S. N is said to be equivariant, relative to the foliation, if the leaves of the foliation are Lie transported by N. The same definition applies to a null incoming vector field $\underline{N}$.*

This means that, in the first case, the 1-parameter family of diffeomorphisms $\{\phi_t\}$ generated by N maps the leaves of the foliations into themselves and, in the second case, the same happens relatively to the 1-parameter family of diffeomorphisms $\{\underline{\phi}_t\}$ generated by $\underline{N}$.

Lemma 3.1.1 *Let ϕ_t be the 1-parameter family of diffeomorphisms generated by the equivariant vector field N mapping a given two surface S of the foliation onto another leaf S'. Let X be a S-tangent vector field, defined on $\mathcal{M}$, then $\phi_{t*}X$ is also S-tangent, at each point, to $S' = \phi_t(S)$ and so is $\mathcal{L}_N X = [N, X]$.*

Proof: Let $p \in S$ and $q = \phi_t(p)$ then:

$$
\begin{aligned}
(\phi_{t*}X)_q(f) &= d\phi_t \cdot X_p(f) = X_p(f \circ \phi_t) \qquad (3.1.34) \\
&= lim_{h\to 0}\frac{1}{h}[(f \circ \phi_t \circ \psi_h \circ \phi_t^{-1})(q) - f(q)]
\end{aligned}
$$

where ψ_h is the one parameter diffeomorphism generated by the vector field X and $(\phi_t \circ \psi_h \circ \phi_t^{-1})$ is a curve on S' whose tangent vector at q is $(\phi_{t*}X)_q$. Moreover, as

$$(\mathcal{L}_N X)_q = lim_{h\to 0}\frac{1}{h}[X_q - (\phi_{h*}X)_q] \ ,$$

it follows that $\mathcal{L}_N X = [N, X]$ is S tangent.

Lemma 3.1.2 *For a double null integral S-foliation the outgoing null vector field $N = 2\Omega^2 L$ and the incoming null vector field $\underline{N} = 2\Omega^2 \underline{L}$ are equivariant relative to it.*

Proof: Let $\{\phi_t\}$ be the one parameter family of diffeomorphisms, generated by a null vector field N_0 such that:

$$\underline{u}(\phi_t(p)) = \underline{u}(p) + t$$
$$u(\phi_t(p)) = u(p)$$

Hence

$$\frac{\partial}{\partial t}\underline{u}(\phi_t(p)) = 1 = \frac{\partial {\phi_t}^\mu}{\partial t}\frac{\partial}{\partial x^\mu}\underline{u}(p) = N(\underline{u})$$
$$\frac{\partial}{\partial t}u(\phi_t(p)) = 0 = \frac{\partial {\phi_t}^\mu}{\partial t}\frac{\partial}{\partial x^\mu}u(p) = N(u) \qquad (3.1.35)$$

Therefore there must exist a scalar function a such that $N_0 = aL$. Defining in the same way $\underline{\phi}_t$ as the diffeomorphism generated by the incoming null normal vector field $\underline{N}_0$, it follows also that $\underline{N}_0(u) = 1$, $\underline{N}_0(\underline{u}) = 0$ and, exactly as before, this implies the existence of a scalar function $\underline{a}$ such that $\underline{N}_0 = \underline{a}\ \underline{L}$. From 3.1.35 we have

$$1 = N_0(\underline{u}) = N_0^\mu \partial_\mu \underline{u} = -g_{\mu\nu}N_0^\mu(-g^{\nu\sigma}\partial_\sigma\underline{u}) = -g_{\mu\nu}N_0^\mu \underline{L}^\nu = -\underline{a}^{-1}\mathbf{g}(N_0, \underline{N}_0)$$

and from it

$$\mathbf{g}(N_0, \underline{N}_0) = -\underline{a} \ .$$

Repeating the same calculation, interchanging N_0 and $\underline{N}_0$, we have also

$$-a = \mathbf{g}(\underline{N}_0, N_0) = -\underline{a} \ .$$

From $N_0 = aL$, $\ \underline{N}_0 = a\underline{L}$ and 3.1.32 we obtain

$$a = 2\Omega^2 \qquad (3.1.36)$$

Finally, recalling eq. 3.1.31,

$$N_0 = \Omega\hat{N} = 2\Omega^2 L = N \quad , \quad \underline{N}_0 = \Omega\hat{\underline{N}} = 2\Omega^2\underline{L} = \underline{N} \tag{3.1.37}$$

Next lemma is a simple generalization of equation 3.1.15. It will be sistematically used in the next chapters.

Lemma 3.1.3 *Let S a two dimensional surface diffeomorphic to S^2 , for any scalar function f, the following equations hold:*

$$\begin{aligned}\frac{d}{d\underline{u}}\int_{S(u,\underline{u})} f d\mu_\gamma &= \int_{S(u,\underline{u})}\left(\frac{df}{d\underline{u}} + \Omega tr\chi f\right) d\mu_\gamma \\ \frac{d}{du}\int_{S(u,\underline{u})} f d\mu_\gamma &= \int_{S(u,\underline{u})}\left(\frac{df}{du} + \Omega tr\underline{\chi} f\right) d\mu_\gamma\end{aligned} \tag{3.1.38}$$

Proof: Explicit computation gives

$$\frac{d}{d\underline{u}}\int_{S(u,\underline{u})} f d\mu_\gamma = \int_{S(u,\underline{u})}\frac{df}{d\underline{u}} + lim_{\delta\to 0}\frac{1}{\delta}\left(\int_{S(u,\underline{u}+\delta)} f d\mu_\gamma - \int_{S(u,\underline{u})} f d\mu_\gamma\right) .$$

where $d\mu_\gamma$, the area form of $S(u,\underline{u})$, is the two form $d\mu_\gamma = \sqrt{g}dx^1 \wedge dx^2$. The null vector field $N = \Omega\hat{N}$ generates the diffeomorphism ϕ_δ sending $S(u,\underline{u})$ onto $S(u,\underline{u}+\delta)$. Let $q = \phi_\delta(p) \in S(u,\underline{u}+\delta)$ with $p \in S(u,\underline{u})$, then the following relation holds:

$$\int_{S_{(u,\underline{u}+\delta)}} f d\mu_\gamma = \int_{S_{(u,\underline{u})}} f d\mu_{\Phi^*_\delta\gamma}$$

It is easy to prove that

$$d\mu_{\Phi^*_\delta\gamma} = d\mu_\gamma + \delta\mathrm{tr}(\frac{1}{2}L_N\gamma)d\mu_\gamma + O(\delta^2) = (1+\delta\Omega\mathrm{tr}\chi)d\mu_\gamma + O(\delta^2)$$

and, from this relation, the first line of 3.1.38 follows. The second equation is proved exactly in the same way.
We recall the notion of "Fermi transported null frame"

Definition 3.1.15 *Given a null pair e_3, e_4, we say that a null frame $\{e_4, e_3, e_a\}$ is Fermi transported along $C(\lambda)$ if $\not{\mathbf{D}}_{e_4} e_a = 0$, we say that it is Fermi transported along $\underline{C}(\nu)$ if $\not{\mathbf{D}}_3 e_a = 0$.*

Remark: As $N, \underline{N}$ do not commute we cannot simultaneously have a null frame Fermi transported along both $C(\lambda)$ and $\underline{C}(\nu)$.

3.1.5 Null structure equations of a double null foliation

We assume that the spacetime $(\mathcal{M}, \mathbf{g})$ is foliated by a smooth S-foliation whose leaves are compact, spacelike, 2-surfaces diffeomorphic to S^2. We consider the null frame $\{e_4, e_3, e_1, e_2\}$, adapted to the S-foliation, and write the null structure equations satisfied by the connection coefficients with respect to it. Denoting the null frame and its dual basis

$$\{e_{(\alpha)}\} = \{e_\alpha\} = \{e_1, e_2, e_3, e_4\}$$
$$\{\theta^{(\alpha)}\} = \{\theta^\alpha\} = \{\theta^1, \theta^2, \theta^3, \theta^4\}$$

we define

$$\mathbf{D}_{e_\alpha} e_\beta \equiv \mathbf{\Gamma}^\gamma_{\alpha\beta} e_\gamma \ , \ \mathbf{R}(e_\alpha, e_\beta) e_\gamma \equiv \mathbf{R}^{\ \ \delta}_{\gamma\alpha\beta} \ e_\delta \tag{3.1.39}$$

The connection coefficients, introduced above, are the nonvanishing components of $\mathbf{\Gamma}^\gamma_{\alpha\beta}$. The connection 1-form and curvature 2-form are

$$\begin{aligned} \omega^\alpha_\beta &\equiv \mathbf{\Gamma}^\alpha_{\gamma\beta}\theta^\gamma \\ \mathbf{\Omega}^\alpha_\beta &\equiv \frac{1}{2}\mathbf{R}^\alpha_{\ \beta\gamma\delta}\theta^\gamma \wedge \theta^\delta \end{aligned} \tag{3.1.40}$$

They satisfy [7], the first and the second structure equations [8], see [Sp] ,

$$\begin{aligned} d\theta^\alpha &= -\omega^\alpha_\gamma \wedge \theta^\gamma \\ d\omega^\delta_\gamma &= -\omega^\delta_\sigma \wedge \omega^\sigma_\gamma + \mathbf{\Omega}^\delta_\gamma \end{aligned} \tag{3.1.41}$$

We shall now specialize to the case of a double null foliation, with the normalized null pair $\{\hat{N}, \underline{\hat{N}}\}$, see definition 3.1.13. The first structure equations, written explicitely in terms of the connection coefficients, take the form, see 3.1.30,

$$\begin{aligned} \mathbf{D}_a e_b &= \nabla\!\!\!\!/_a e_b + \frac{1}{2}\chi_{ab} e_3 + \frac{1}{2}\underline{\chi}_{ab} e_4 \\ \mathbf{D}_a e_3 &= \underline{\chi}_{ab} e_b + \zeta_a e_3 \quad , \quad \mathbf{D}_a e_4 = \chi_{ab} e_b - \zeta_a e_4 \\ \mathbf{D}_3 e_a &= \mathbf{D}\!\!\!\!/_3 e_a + \eta_a e_3 \quad , \quad \mathbf{D}_4 e_a = \mathbf{D}\!\!\!\!/_4 e_a + \underline{\eta}_a e_4 \\ \mathbf{D}_3 e_3 &= (\mathbf{D}_3 \log \Omega) e_3 \quad , \quad \mathbf{D}_3 e_4 = -(\mathbf{D}_3 \log \Omega) e_4 + 2\eta_b e_b \\ \mathbf{D}_4 e_4 &= (\mathbf{D}_4 \log \Omega) e_4 \quad , \quad \mathbf{D}_4 e_3 = -(\mathbf{D}_4 \log \Omega) e_3 + 2\underline{\eta}_b e_b \end{aligned} \tag{3.1.42}$$

[7] The structure equations can be stated in a more general framework for a general S-foliation as discussed in the appendix to this chapter or even in the absence of a foliation, see the general Newman-Penrose formalism, [Ne-Pe2].

[8] With the obvious modifications due to the Lorentzian metric.

We also state the following commutation relations which will be often used in the sequel

$$\begin{aligned}
&[\hat{N}, e_a] = \not{D}_4 e_a - \chi_{ab} e_b + (\not{\nabla}_a \log\Omega)\hat{N}\\
&[\hat{\underline{N}}, e_a] = \not{D}_3 e_a - \underline{\chi}_{ab} e_b + (\not{\nabla}_a \log\Omega)\hat{\underline{N}}\\
&[\hat{N}, \hat{\underline{N}}] = -(\mathbf{D}_4 \log\Omega)\hat{\underline{N}} + (\mathbf{D}_3 \log\Omega)\hat{N} - 4\zeta_b e_b
\end{aligned} \tag{3.1.43}$$

and from it

$$\begin{aligned}
&[N, e_a] = \Omega(\not{D}_4 e_a - \chi_{ab} e_b)\\
&[\underline{N}, e_a] = \Omega(\not{D}_3 e_a - \underline{\chi}_{ab} e_b)\\
&[N, \underline{N}] = -4\Omega^2 \zeta_b e_b
\end{aligned} \tag{3.1.44}$$

Recalling that all the connection coefficients of a double null foliation can be expressed relatively to $\chi, \underline{\chi}, \zeta, \Omega$, the second null structure equations take the following form:

Proposition 3.1.3 (Null structure equations) *The coefficients $\chi, \underline{\chi}, \zeta, \Omega$ associated to a double null foliation and normalized null pair $\{e_4 = \hat{N}, e_3 = \hat{\underline{N}}\}$, verify the following equations*

$$\begin{aligned}
&\not{D}_3\zeta + 2\underline{\chi}\cdot\zeta - \not{D}_3\not{\nabla}\log\Omega = -\underline{\beta}\\
&\not{D}_4\zeta + 2\chi\cdot\zeta + \not{D}_4\not{\nabla}\log\Omega = -\beta\\
&\not{D}_4\hat{\chi} + tr\chi\hat{\chi} - (\mathbf{D}_4\log\Omega)\hat{\chi} = -\alpha\\
&\mathbf{D}_4 tr\chi + \frac{1}{2}(tr\chi)^2 - (\mathbf{D}_4\log\Omega)tr\chi + |\hat{\chi}|^2 = 0\\
&\not{D}_3\hat{\underline{\chi}} + tr\underline{\chi}\hat{\underline{\chi}} - (\mathbf{D}_3\log\Omega)\hat{\underline{\chi}} = -\underline{\alpha}\\
&\mathbf{D}_3 tr\underline{\chi} + \frac{1}{2}(tr\underline{\chi})^2 - (\mathbf{D}_3\log\Omega)tr\underline{\chi} + |\hat{\underline{\chi}}|^2 = 0\\
&\not{D}_4\hat{\underline{\chi}} + \frac{1}{2}tr\chi\hat{\underline{\chi}} + \frac{1}{2}tr\underline{\chi}\hat{\chi} + (\mathbf{D}_4\log\Omega)\hat{\underline{\chi}} + \not{\nabla}\widehat{\otimes}\zeta - \zeta\widehat{\otimes}\zeta\\
&\qquad +2\zeta\widehat{\otimes}\not{\nabla}\log\Omega - (\not{\nabla}\widehat{\otimes}\not{\nabla})\log\Omega - \not{\nabla}\log\Omega\widehat{\otimes}\not{\nabla}\log\Omega = 0\\
&\mathbf{D}_4 tr\underline{\chi} + \frac{1}{2}tr\chi tr\underline{\chi} + (\mathbf{D}_4\log\Omega)tr\underline{\chi} + \hat{\chi}\cdot\hat{\underline{\chi}} + 2\not{d}iv\zeta - 2\not{\Delta}\log\Omega\\
&\qquad -2|\zeta|^2 - 4\zeta\cdot\not{\nabla}\log\Omega - 2|\not{\nabla}\log\Omega|^2 = 2\rho\\
&\not{D}_3\hat{\chi} + \frac{1}{2}tr\underline{\chi}\hat{\chi} + \frac{1}{2}tr\chi\hat{\underline{\chi}} + (\mathbf{D}_3\log\Omega)\hat{\chi} - \not{\nabla}\widehat{\otimes}\zeta - \zeta\widehat{\otimes}\zeta\\
&\qquad -2\zeta\widehat{\otimes}\not{\nabla}\log\Omega - (\not{\nabla}\widehat{\otimes}\not{\nabla})\log\Omega - \not{\nabla}\log\Omega\widehat{\otimes}\not{\nabla}\log\Omega = 0\\
&\mathbf{D}_3 tr\chi + \frac{1}{2}tr\underline{\chi} tr\chi + (\mathbf{D}_3\log\Omega)tr\chi + \hat{\underline{\chi}}\cdot\hat{\chi} - 2\not{d}iv\zeta - 2|\zeta|^2\\
&\qquad -2\not{\Delta}\log\Omega - 4\zeta\cdot\not{\nabla}\log\Omega - 2|\not{\nabla}\log\Omega|^2 = 2\rho
\end{aligned} \tag{3.1.45}$$

$$
\begin{aligned}
&\nabla\!\!\!/\, tr\underline{\chi} - d\!\!\!/iv\underline{\chi} + \zeta\cdot\underline{\chi} - \zeta tr\underline{\chi} = -\underline{\beta}\\
&\nabla\!\!\!/\, tr\chi - d\!\!\!/iv\chi - \zeta\cdot\chi + \zeta tr\chi = \beta\\
&cu\!\!\!/rl\zeta - \frac{1}{2}\hat{\underline{\chi}}\wedge\hat{\chi} = \sigma\\
&\mathbf{K} + \frac{1}{4}tr\chi tr\underline{\chi} - \frac{1}{2}\hat{\underline{\chi}}\cdot\hat{\chi} = -\rho
\end{aligned}
\tag{3.1.46}
$$

and, finally,

$$
\frac{1}{2}(\mathbf{D}_4\mathbf{D}_3\log\Omega + \mathbf{D}_3\mathbf{D}_4\log\Omega) + (\mathbf{D}_3\log\Omega)(\mathbf{D}_4\log\Omega) + 3|\zeta|^2 - |\nabla\!\!\!/\log\Omega|^2 = -\rho
\tag{3.1.47}
$$

Remark: Recall that $\mathbf{D}\!\!\!/_4, \mathbf{D}\!\!\!/_3$ are the projections of $\mathbf{D}_4, \mathbf{D}_3$ to TS. Moreover, given U, V two covariant S tangent vector fields, $U\widehat{\otimes}V$ is defined as twice the traceless part of their symmetric tensor product $U\otimes V$,

[Important modification of the, wrong, definition of $U\widehat{\otimes}V$, by a factor 2]

$$
(U\widehat{\otimes}V)_{ab} \equiv U_aV_b + U_bV_a - \delta_{ab}U\cdot V
\tag{3.1.48}
$$

Proof: See the appendix to this chapter, subsection 3.8.1.

3.1.6 The Einstein equations relative to a double null foliation

Among the complete set of structure equations, 3.1.45, 3.1.46, 3.1.47, we identify those which do not depend on the null components of the curvature tensor. They are the equations which correspond to $\mathbf{R}(e_\alpha, e_\beta) = 0$. In other words they can be interpreted as the "Einstein vacuum equations", expressed relatively to the double null foliation:

$$
\begin{cases}
\mathbf{D}_4\mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 + 2\omega\mathrm{tr}\chi + |\hat{\chi}|^2 = 0\\[2mm]
\mathbf{D}_4\mathrm{tr}\underline{\chi} + \mathrm{tr}\chi\mathrm{tr}\underline{\chi} - 2\omega\mathrm{tr}\underline{\chi} = -2\mathbf{K} + 2\mathrm{d}\!\!\!/\mathrm{iv}\,(-\zeta + \nabla\!\!\!/\log\Omega) + 2|-\zeta + \nabla\!\!\!/\log\Omega|^2\\[2mm]
\mathbf{D}\!\!\!/_4\hat{\underline{\chi}} - 2\omega\hat{\underline{\chi}} = \nabla\!\!\!/\widehat{\otimes}\underline{\eta} + \underline{\eta}\widehat{\otimes}\underline{\eta} - \frac{1}{2}(\mathrm{tr}\underline{\chi}\hat{\chi} + \mathrm{tr}\chi\hat{\underline{\chi}})\\[2mm]
\mathbf{D}\!\!\!/_4\zeta + \zeta\chi + \mathrm{tr}\chi\zeta = \mathrm{d}\!\!\!/\mathrm{iv}\,\chi - \nabla\!\!\!/\mathrm{tr}\chi - \mathbf{D}\!\!\!/_4\nabla\!\!\!/\log\Omega
\end{cases}
\tag{3.1.49}
$$

$$
\begin{cases}
\mathbf{D}_3\text{tr}\underline{\chi} + \frac{1}{2}\text{tr}\underline{\chi}^2 + 2\underline{\omega}\text{tr}\underline{\chi} + |\hat{\underline{\chi}}|^2 = 0 \\
\mathbf{D}_3\text{tr}\chi + \text{tr}\underline{\chi}\text{tr}\chi - 2\underline{\omega}\text{tr}\chi = -2\mathbf{K} + 2\,\text{d\!\!/iv}\,(\zeta + \nabla\!\!\!/ \log\Omega) + 2|\zeta + \nabla\!\!\!/ \log\Omega|^2 \\
\mathbf{D}\!\!\!\!/_3\hat{\chi} - 2\underline{\omega}\hat{\chi} = \nabla\!\!\!/ \widehat{\otimes}\eta + \eta\widehat{\otimes}\eta - \frac{1}{2}(\text{tr}\chi\hat{\underline{\chi}} + \text{tr}\underline{\chi}\hat{\chi}) \\
\mathbf{D}\!\!\!\!/_3\zeta + \zeta\underline{\chi} + \text{tr}\underline{\chi}\zeta = -\,\text{d\!\!/iv}\,\underline{\chi} + \nabla\!\!\!/ \text{tr}\underline{\chi} + \mathbf{D}\!\!\!\!/_3\nabla\!\!\!/ \log\Omega
\end{cases}
\tag{3.1.50}
$$

and

$$
\frac{1}{2}(\mathbf{D}_4\mathbf{D}_3\log\Omega + \mathbf{D}_3\mathbf{D}_4\log\Omega) + (\mathbf{D}_3\log\Omega)(\mathbf{D}_4\log\Omega) + 3|\zeta|^2 - |\nabla\!\!\!/ \log\Omega|^2
$$
$$
= \mathbf{K} + \frac{1}{4}\text{tr}\chi\text{tr}\underline{\chi} - \frac{1}{2}\hat{\underline{\chi}}\cdot\hat{\chi} \tag{3.1.51}
$$

Remark: the number of equations written in 3.1.49, 3.1.50, 3.1.51 is 13 instead of 10 as the independent components of the Ricci tensor. Therefore three of them are not independent. A careful look shows that the three equations $\mathbf{Ricci}(e_a, e_b) = 0$ can be written as the equations

$$
\mathbf{D}\!\!\!\!/_4\hat{\underline{\chi}} - 2\omega\hat{\underline{\chi}} = \nabla\!\!\!/ \widehat{\otimes}\underline{\eta} + \underline{\eta}\widehat{\otimes}\underline{\eta} - \frac{1}{2}(\text{tr}\underline{\chi}\hat{\chi} + \text{tr}\chi\hat{\underline{\chi}})
$$
$$
\mathbf{D}_4\text{tr}\underline{\chi} + \text{tr}\chi\text{tr}\underline{\chi} - 2\omega\text{tr}\underline{\chi} = -2\mathbf{K} + 2\,\text{d\!\!/iv}\,(-\zeta + \nabla\!\!\!/ \log\Omega) + 2|-\zeta + \nabla\!\!\!/ \log\Omega|^2
$$

or the equations

$$
\mathbf{D}\!\!\!\!/_3\hat{\chi} - 2\underline{\omega}\hat{\chi} = \nabla\!\!\!/ \widehat{\otimes}\eta + \eta\widehat{\otimes}\eta - \frac{1}{2}(\text{tr}\chi\hat{\underline{\chi}} + \text{tr}\underline{\chi}\hat{\chi})
$$
$$
\mathbf{D}_3\text{tr}\chi + \text{tr}\underline{\chi}\text{tr}\chi - 2\underline{\omega}\text{tr}\chi = -2\mathbf{K} + 2\,\text{d\!\!/iv}\,(\zeta + \nabla\!\!\!/ \log\Omega) + 2|\zeta + \nabla\!\!\!/ \log\Omega|^2
$$

restoring to 10 the total number of the Einstein equations.

To look at these equations as partial differential equations it is appropriate to rewrite them in terms of the $\mathcal{D}$, $\underline{\mathcal{D}}$ derivatives defined in subsection 3.1.1. Recalling definitions 3.1.12, 3.1.13, it follows immediately that

$$
\begin{aligned}
&\mathcal{D}\text{tr}\chi = \Omega\mathbf{D}_4\text{tr}\chi\ ,\ \mathcal{D}\text{tr}\underline{\chi} = \Omega\mathbf{D}_4\text{tr}\underline{\chi} \\
&\underline{\mathcal{D}}\text{tr}\chi = \Omega\mathbf{D}_3\text{tr}\chi\ ,\ \underline{\mathcal{D}}\text{tr}\underline{\chi} = \Omega\mathbf{D}_3\text{tr}\underline{\chi} \\
&\mathcal{D}\zeta = \Omega\,(\mathbf{D}\!\!\!\!/_4\zeta + \zeta\cdot\chi)\ ,\ \underline{\mathcal{D}}\zeta = \Omega\left(\mathbf{D}\!\!\!\!/_3\zeta + \zeta\cdot\underline{\chi}\right) \\
&\mathcal{D}\hat{\chi} = \Omega\,(\mathbf{D}\!\!\!\!/_4\hat{\chi} + 2\hat{\chi}\cdot\hat{\chi})\ ,\ \mathcal{D}\hat{\underline{\chi}} = \Omega\left(\mathbf{D}\!\!\!\!/_4\hat{\underline{\chi}} + 2\hat{\underline{\chi}}\cdot\hat{\chi}\right) \\
&\underline{\mathcal{D}}\hat{\chi} = \Omega\left(\mathbf{D}\!\!\!\!/_3\hat{\chi} + 2\hat{\chi}\cdot\hat{\underline{\chi}}\right)\ ,\ \underline{\mathcal{D}}\hat{\underline{\chi}} = \Omega\left(\mathbf{D}\!\!\!\!/_3\hat{\underline{\chi}} + 2\hat{\underline{\chi}}\cdot\hat{\underline{\chi}}\right)
\end{aligned}
\tag{3.1.52}
$$

Observe also that the equations for ζ along the $C(\lambda)$ and $\underline{C}(\nu)$ null hypersurfaces can be replaced by similar equations relative to η and $\underline{\eta}$ in view of the relations, see 3.1.32,

$$\eta = \zeta + \nabla\!\!\!/ \log\Omega \ , \ \underline{\eta} = -\zeta + \nabla\!\!\!/ \log\Omega \tag{3.1.53}$$

Thus the previous equations, 3.1.49, 3.1.50, take the following form

$$\begin{aligned}
&\mathcal{D}\mathrm{tr}\chi - (\mathcal{D}\log\Omega)tr\chi + \frac{1}{2}\Omega\mathrm{tr}\chi^2 + \Omega|\hat{\chi}|^2 = 0\\
&\mathcal{D}\mathrm{tr}\underline{\chi} + (\mathcal{D}\log\Omega)tr\underline{\chi} + \Omega\mathrm{tr}\chi\mathrm{tr}\underline{\chi} = 2\Omega\left(-\mathbf{K} + \mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\eta} + |\underline{\eta}|^2\right)\\
&\mathcal{D}\hat{\underline{\chi}} + (\mathcal{D}\log\Omega)\hat{\underline{\chi}} - 2\Omega\hat{\underline{\chi}}\cdot\hat{\chi} = \Omega\left(\nabla\!\!\!/\widehat{\otimes}\underline{\eta} + \underline{\eta}\widehat{\otimes}\underline{\eta} - \frac{1}{2}(tr\underline{\chi}\hat{\chi} + tr\chi\hat{\underline{\chi}})\right)\\
&\mathcal{D}\eta + \Omega\mathrm{tr}\chi\eta = \Omega(\mathrm{d}\!\!\!/\mathrm{iv}\,\chi - \nabla\!\!\!/\mathrm{tr}\chi) + \Omega\left(\hat{\chi}\cdot\nabla\!\!\!/\log\Omega + \frac{3}{2}\mathrm{tr}\chi\nabla\!\!\!/\log\Omega\right)
\end{aligned} \tag{3.1.54}$$

and

$$\begin{aligned}
&\underline{\mathcal{D}}\mathrm{tr}\underline{\chi} - (\underline{\mathcal{D}}\log\Omega)tr\underline{\chi} + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}^2 + \Omega|\hat{\underline{\chi}}|^2 = 0\\
&\underline{\mathcal{D}}\mathrm{tr}\chi + (\underline{\mathcal{D}}\log\Omega)tr\chi + \Omega\mathrm{tr}\underline{\chi}\mathrm{tr}\chi = 2\Omega\left(-\mathbf{K} + \mathrm{d}\!\!\!/\mathrm{iv}\,\eta + |\eta|^2\right)\\
&\underline{\mathcal{D}}\hat{\chi} + (\underline{\mathcal{D}}\log\Omega)\hat{\chi} - 2\Omega\hat{\chi}\cdot\hat{\underline{\chi}} = \Omega\left(\nabla\!\!\!/\widehat{\otimes}\eta + \eta\widehat{\otimes}\eta - \frac{1}{2}(tr\chi\hat{\underline{\chi}} + tr\underline{\chi}\hat{\chi})\right)\\
&\underline{\mathcal{D}}\underline{\eta} + \Omega\mathrm{tr}\underline{\chi}\underline{\eta} = \Omega(\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\chi} - \nabla\!\!\!/\mathrm{tr}\underline{\chi}) + \Omega\left(\hat{\underline{\chi}}\cdot\nabla\!\!\!/\log\Omega + \frac{3}{2}\mathrm{tr}\underline{\chi}\nabla\!\!\!/\log\Omega\right)
\end{aligned} \tag{3.1.55}$$

and

$$\mathcal{D}\underline{\mathcal{D}}\log\Omega + \underline{\mathcal{D}}\mathcal{D}\log\Omega = 2\Omega^2\left(\eta\cdot\underline{\eta} - 2|\eta - \nabla\!\!\!/\log\Omega|^2 + \mathbf{K} + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \frac{1}{2}\hat{\underline{\chi}}\cdot\hat{\chi}\right) \tag{3.1.56}$$

These equations form a closed system when supplemented by the equations 3.1.14,

$$\mathcal{D}\gamma = 2\chi \ , \ \underline{\mathcal{D}}\gamma = 2\underline{\chi}$$

Equations 3.1.54, 3.1.55, 3.1.56 and 3.1.14 can be expressed in terms of null coordinates by supplementing u and $\underline{u}$ with additional angular coordinates θ, ϕ.

We can define them as follows: we start with a fixed system of coordinates θ_0, ϕ_0 defined on $S(\lambda_0, \nu_*) = \underline{C}_* \cap C_0$, where $\underline{C}_* = \underline{C}(\underline{u} = \nu_*)$, $C_0 = C(u = \lambda_0)$ and define $\mathcal{M}$ as the causal past of $S(\lambda_0, \nu_*)$. Consider any other surface

$S(\lambda,\nu) \subset \mathcal{M}$. We can transport the coordinates from $S(\lambda_0,\nu_*)$ to $S(\lambda,\nu)$ in two different ways, using the flows ϕ_t and $\underline{\phi}_s$ associated to the equivariant null pair N and $\underline{N}$,

$$\begin{aligned} \theta|_{S(\lambda,\nu)} &= \theta_0(\underline{\phi}_s \circ \phi_t) \\ \phi|_{S(\lambda,\nu)} &= \phi_0(\underline{\phi}_s \circ \phi_t) \end{aligned} \tag{3.1.57}$$

or

$$\begin{aligned} \theta|_{S(\lambda,\nu)} &= \theta_0(\phi_t \circ \underline{\phi}_s) \\ \phi|_{S(\lambda,\nu)} &= \phi_0(\phi_t \circ \underline{\phi}_s) \end{aligned} \tag{3.1.58}$$

where $t = \nu_* - \nu, s = \lambda_0 - \lambda$. Recall that $\phi_{\nu_*-\nu}$ is the diffeomorphism from $S(\lambda,\nu)$ to $S(\lambda,\nu_*)$ and $\underline{\phi}_{\lambda_0-\lambda}$ is the diffeomorphism from $S(\lambda,\nu)$ to $S(\lambda_0,\nu)$. Observe that the two definitions differ; indeed since N and $\underline{N}$ do not commute

$$\phi_t \circ \underline{\phi}_s \neq \underline{\phi}_s \circ \phi_t \ .$$

Since N and $\underline{N}$ are equivariant [9] one of the two choice corresponds to write $N = \frac{\partial}{\partial \underline{u}}$ and the other one to $\underline{N} = \frac{\partial}{\partial u}$. Choosing $N = \frac{\partial}{\partial \underline{u}}$, we infer that $\underline{N}$ must have the form $\underline{N} = \frac{\partial}{\partial u} + X$. To determine the vector field X observe that [10], see 3.1.44,

$$[N,\underline{N}] = Z \equiv -4\Omega^2\zeta(e_b)e_b \tag{3.1.59}$$

On the other hand, from the previous expressions for N and $\underline{N}$,

$$[N,\underline{N}] = \left(\frac{\partial}{\partial \underline{u}}X^a\right)\frac{\partial}{\partial \omega^a} \tag{3.1.60}$$

where $\omega^1 = \theta, \omega^2 = \phi$. Therefore in view of equation 3.1.59 we can uniquely define the vector field X by [11]

$$\frac{\partial}{\partial \underline{u}}X^a = Z^a = -4\Omega^2\gamma^{ab}\zeta_b \ \ , \ \ X^a|_{\underline{C}_*} = 0 \tag{3.1.61}$$

With this choice of coordinates the metric $\mathbf{g}$ has the following expression

$$\mathbf{g}(\cdot,\cdot) = -2\Omega^2\left(dud\underline{u} + d\underline{u}du\right) + \gamma_{ab}(d\omega^a - X^a du)(d\omega^b - X^b du) \tag{3.1.62}$$

[9] Recall that $N(\underline{u}) = 1, N(u) = 0, \underline{N}(u) = 1, \underline{N}(\underline{u}) = 0$.
[10] We can also write $[\mathcal{D},\underline{\mathcal{D}}] = \mathcal{L}_{[N,\underline{N}]} = \mathcal{L}_Z$.
[11] Here a, b denote coordinates on S.

3.1.7 The characteristic initial value problem for the Einstein equations

In terms of the previous choices of coordinates we can now interpret equations 3.1.54 and 3.1.55 as equations for the six unknown γ_{ab}, X_a and Ω of the space time metric.

We consider given two null hypersurfaces we denote $C_*, \underline{C}_*$, both originated from a common two dimensional surface S, see definition 3.1.5. On $C_* \cup \underline{C}_*$ we assume prescribed γ_{ab}, X_a and Ω. How much freedom we have in assigning these six quantities will be discussed later on.
This means that on C_*, $\mathrm{tr}\chi$, $\hat{\chi}$ and ζ are automatically determined as, see 3.1.14 and 3.1.61, $2\chi = \mathcal{D}\gamma$ and $\zeta = \mathcal{D}X$. Of course, in view of the relations 3.1.53 and of the definitions 3.1.32, η, $\underline{\eta}$ and ω are also determined on C_*.

On $\underline{C}_*$ $\mathrm{tr}\underline{\chi}$, $\hat{\underline{\chi}}$, $\underline{\omega}$, η and $\underline{\eta}$ are automatically determined again as, see 3.1.14, $2\underline{\chi} = \underline{\mathcal{D}}\gamma$ and using definitions 3.1.32 for $\underline{\omega}, \eta, \underline{\eta}$. X is given equal zero, as allowed from our choice of coordinates, see 3.1.61, 3.1.62.
Moreover on $C_* \cup \underline{C}_*$ we can also determine the remaining connection coefficients, $\mathrm{tr}\chi$, $\hat{\chi}$ on $\underline{C}_*$ and $\mathrm{tr}\underline{\chi}$, $\hat{\underline{\chi}}$ on C_*. In fact, using equations 3.1.54 and 3.1.55,

$$\begin{aligned}
&\mathcal{D}\mathrm{tr}\underline{\chi} + (\mathcal{D}\log\Omega)tr\underline{\chi} + \Omega\mathrm{tr}\chi\mathrm{tr}\underline{\chi} = 2\Omega\left(-\mathbf{K} + \mathrm{d\!\!/iv}\,\underline{\eta} + |\underline{\eta}|^2\right)\\
&\mathcal{D}\hat{\underline{\chi}} + (\mathcal{D}\log\Omega)\hat{\underline{\chi}} - 2\Omega\hat{\underline{\chi}}\cdot\hat{\chi} = \Omega\left(\nabla\!\!\!/\widehat{\otimes}\underline{\eta} + \underline{\eta}\widehat{\otimes}\underline{\eta} - \frac{1}{2}(tr\underline{\chi}\hat{\chi} + tr\chi\hat{\underline{\chi}})\right)\\
&\underline{\mathcal{D}}\mathrm{tr}\chi + (\underline{\mathcal{D}}\log\Omega)tr\chi + \Omega\mathrm{tr}\underline{\chi}\mathrm{tr}\chi = 2\Omega\left(-\mathbf{K} + \mathrm{d\!\!/iv}\,\eta + |\eta|^2\right)\\
&\underline{\mathcal{D}}\hat{\chi} + (\underline{\mathcal{D}}\log\Omega)\hat{\chi} - 2\Omega\hat{\chi}\cdot\hat{\underline{\chi}} = \Omega\left(\nabla\!\!\!/\widehat{\otimes}\eta + \eta\widehat{\otimes}\eta - \frac{1}{2}(tr\chi\hat{\underline{\chi}} + tr\underline{\chi}\hat{\chi})\right)\ ,
\end{aligned}$$

these quantities are uniquely determined on C_* and $\underline{C}_*$, respectively, in terms of their values on S. Therefore, once the γ_{ab}, X_a and Ω metric commponents are assigned, all the null connection coefficients are specified, on $C_* \cup \underline{C}_*$.

To determine the spacetime $\mathcal{K}$, given the "initial data" on $C_* \cup \underline{C}_*$, we proceed in the following way. The incoming evolution equations 3.1.55 for χ, $\mathrm{tr}\underline{\chi}$, $\underline{\eta}$

$$\begin{aligned}
&\underline{\mathcal{D}}\mathrm{tr}\underline{\chi} - (\underline{\mathcal{D}}\log\Omega)tr\underline{\chi} + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}^2 + \Omega|\hat{\underline{\chi}}|^2 = 0\\
&\underline{\mathcal{D}}\mathrm{tr}\chi + (\underline{\mathcal{D}}\log\Omega)tr\chi + \Omega\mathrm{tr}\underline{\chi}\mathrm{tr}\chi = 2\Omega\left(-\mathbf{K} + \mathrm{d\!\!/iv}\,\eta + |\eta|^2\right)\\
&\underline{\mathcal{D}}\hat{\chi} + (\underline{\mathcal{D}}\log\Omega)\hat{\chi} - 2\Omega\hat{\chi}\cdot\hat{\underline{\chi}} = \Omega\left(\nabla\!\!\!/\widehat{\otimes}\eta + \eta\widehat{\otimes}\eta - \frac{1}{2}(tr\chi\hat{\underline{\chi}} + tr\underline{\chi}\hat{\chi})\right)
\end{aligned}$$

$$\underline{\mathcal{D}}\underline{\eta}+\Omega\mathrm{tr}\underline{\chi}\underline{\eta}=\Omega(\mathrm{d\!\!/iv}\,\underline{\chi}-\nabla\!\!\!/\mathrm{tr}\underline{\chi})+\Omega\left(\underline{\hat{\chi}}\cdot\nabla\!\!\!/\log\Omega+\frac{3}{2}\mathrm{tr}\underline{\chi}\nabla\!\!\!/\log\Omega\right)$$

and the equation 3.1.56, written as an evolution equation, along e_3, for ω,

$$\underline{\mathcal{D}}\omega-2\Omega\underline{\omega}\omega=\frac{3}{2}\Omega|\zeta|^2+\Omega\zeta\cdot\nabla\!\!\!/\log\Omega-\frac{1}{2}\Omega|\nabla\!\!\!/\log\Omega|^2-\frac{1}{2}\Omega\Big(\mathbf{K}+\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\frac{1}{2}\underline{\hat{\chi}}\cdot\hat{\chi}\Big) \quad (3.1.63)$$

allows us to determine, from the initial conditions on C_*, χ, $\mathrm{tr}\underline{\chi}$, $\underline{\eta}$ and ω.
Therefore it remains to determine $\underline{\hat{\chi}}$, $\underline{\omega}$ and η, then knowing η and $\underline{\eta}$, $\nabla\!\!\!/\log\Omega$ is also determined. This can be obtained from the outgoing evolution equations in 3.1.54 for $\underline{\hat{\chi}}$ and η,

$$\mathcal{D}\underline{\hat{\chi}}+(\mathcal{D}\log\Omega)\underline{\hat{\chi}}-2\Omega\underline{\hat{\chi}}\cdot\hat{\chi}=\Omega\Big(\nabla\!\!\!/\widehat{\otimes}\underline{\eta}+\underline{\eta}\widehat{\otimes}\underline{\eta}-\frac{1}{2}(tr\underline{\chi}\hat{\chi}+tr\chi\underline{\hat{\chi}})\Big)$$

$$\mathcal{D}\eta+\Omega\mathrm{tr}\chi\eta=\Omega(\mathrm{d\!\!/iv}\,\chi-\nabla\!\!\!/\mathrm{tr}\chi)+\Omega\left(\hat{\chi}\cdot\nabla\!\!\!/\log\Omega+\frac{3}{2}\mathrm{tr}\chi\nabla\!\!\!/\log\Omega\right)$$

and from equation 3.1.56, written as an evolution equation, along e_4, for $\underline{\omega}$,

$$\mathcal{D}\underline{\omega}-2\Omega\omega\underline{\omega}=\frac{3}{2}\Omega|\zeta|^2-\Omega\zeta\cdot\nabla\!\!\!/\log\Omega-\frac{1}{2}\Omega|\nabla\!\!\!/\log\Omega|^2-\frac{1}{2}\Omega\Big(\mathbf{K}+\frac{1}{4}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi-\frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}}\Big) \quad (3.1.64)$$

once their initial values are given on $\underline{C}_*$.
In fact all these equations have to be solved simultaneously since they are heavily coupled [12].

The initial data costraints
Observe that the initial metric components γ, X, Ω on $C_*\cup\underline{C}_*$ cannot be freely assigned. In fact let us recall the relations, see 3.1.14, 3.1.32 and 3.1.61,

$$\frac{\partial}{\partial\underline{u}}X^a=-4\Omega^2\zeta^a\ ,\ \frac{\partial}{\partial\underline{u}}\gamma_{ab}=2\chi_{ab}\ ,\ \frac{\partial}{\partial\underline{u}}\Omega=-2\Omega^2\omega \qquad (3.1.65)$$

and observe that, on C_*, $\mathrm{tr}\chi$ and ζ have to satisfy the equations [13]

$$\mathcal{D}\mathrm{tr}\chi-(\mathcal{D}\log\Omega)tr\chi+\frac{1}{2}\Omega\mathrm{tr}\chi^2+\Omega|\hat{\chi}|^2=0 \qquad (3.1.66)$$

$$\mathcal{D}\zeta+\Omega\mathrm{tr}\chi\zeta=\Omega(\mathrm{d\!\!/iv}\,\chi-\nabla\!\!\!/\mathrm{tr}\chi+\chi\cdot\nabla\!\!\!/\log\Omega)+(\nabla\!\!\!/\mathcal{D}\log\Omega+2(\mathcal{D}\log\Omega)\nabla\!\!\!/\log\Omega)$$

Therefore if we assigne on C_*, freely, Ω and the traceless part of $\gamma(\cdot,\cdot)$, $\hat{\gamma}_{ab}$ we know on C_* by simple derivation ω and $\hat{\chi}_{ab}$ and from the previous equations,

[12] See also on this subject [Ren].

[13] The outgoing equation for ζ can be derived immediately from the one for η.

3.1.66, $\mathrm{tr}\chi$ and ζ once they are given on S. Then by simple integration we also know on C_* the trace γ_{ab} and X_a provided they are specified on S. In conclusion the data we can assigne freely on C_* are Ω and the traceless part of the two dimensional metric, $\hat{\gamma}_{ab}$.
The situation is similar on $\underline{C}_*$, the analogous of the equations 3.1.66 are

$$\underline{\mathcal{D}}\mathrm{tr}\underline{\chi} - (\underline{\mathcal{D}}\log\Omega)tr\underline{\chi} + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}^2 + \Omega|\hat{\underline{\chi}}|^2 = 0 \tag{3.1.67}$$
$$\underline{\mathcal{D}}\zeta + \Omega\mathrm{tr}\underline{\chi}\zeta = \Omega(-\mathrm{d\!\!/iv}\,\underline{\chi} + \nabla\!\!\!\!/\,\mathrm{tr}\underline{\chi} - \underline{\chi}\cdot\nabla\!\!\!\!/\log\Omega) - (\nabla\!\!\!\!/\,\underline{\mathcal{D}}\log\Omega + 2(\underline{\mathcal{D}}\log\Omega)\nabla\!\!\!\!/\log\Omega)$$

Again we can assigne freely on $\underline{C}_*$ Ω and $\hat{\gamma}_{ab}$ and, knowing γ_{ab} and ζ_a on S, determine all the initial data on $\underline{C}_*$. The only relevant difference is that on $\underline{C}_*$ we can impose also $X_a = 0$.

Remark: The Einstein equations written in the form 3.1.54, 3.1.55 are highly non linear and manifestly non hyperbolic. The procedure to solve them, we have outlined above, is very formal. It can only be implemented locally, using a variant of the Cauchy Kowaleski method, in the class of analytic spacetimes. This procedure is, however, completely inadequate for studying global solutions. In fact in the proof of the *Main Theorem* we circumvent them completely by relying instead on the full set of the structure equations, 3.1.45, 3.1.46, 3.1.47, where the null components $\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}$ of the curvature tensor are treated as external sources. Indeed we will show that these curvature components can be estimated, separately, from the Bianchi equations as will be discussed in the remaining part of this chapter, see also the preliminary discussion in Chapter 2.

3.2 Bianchi equations in a Einstein vacuum spacetime

Definition 3.2.1 *We say that a Weyl tensor field is a solution of the Bianchi equation in* $(\mathcal{M}, \mathbf{g})$ *if, relative to the Levi-Civita connection* $\mathbf{D}$ *of* $\mathbf{g}$*, it verifies*

$$D_{[\sigma}W_{\gamma\delta]\alpha\beta} = 0 \tag{3.2.1}$$

Remarks:

a) If W verifies the equations 3.2.1 on a background spacetime $(\mathcal{M}, \mathbf{g})$, then it must also satisfy the compatibility condition, see [Ch7],

$$\mathbf{R}_{\mu}{}^{\alpha\beta\gamma\star}W_{\nu\alpha\beta\gamma} - \mathbf{R}_{\nu}{}^{\alpha\beta\gamma\star}W_{\mu\alpha\beta\gamma}\ .$$

b) The primary example of a solution of 3.2.1 is the Riemann curvature tensor of an Einstein vacuum spacetime $(\mathcal{M}, \mathbf{g})$.

We review the main properties of Weyl tensor fields and of the Bianchi equations 3.2.1, see also the extended discussion in [Ch-Kl].

Definition 3.2.2 *Given a Weyl field W and X an arbitrary vector field, we define the modified Lie derivative relative to X by*

$$\hat{\mathcal{L}}_X W = \mathcal{L}_X W - \frac{1}{2}{}^{(X)}[W] + \frac{3}{8} tr^{(X)}\pi W \tag{3.2.2}$$

where

$${}^{(X)}[W]_{\alpha\beta\gamma\delta} = {}^{(X)}\pi_\alpha^\mu W_{\mu\beta\gamma\delta} + {}^{(X)}\pi_\beta^\mu W_{\alpha\mu\gamma\delta} + {}^{(X)}\pi_\gamma^\mu W_{\alpha\beta\mu\delta} + {}^{(X)}\pi_\delta^\mu W_{\alpha\beta\gamma\mu} \tag{3.2.3}$$

and ${}^{(X)}\pi$ is the deformation tensor of X.

Proposition 3.2.1

I. The following four sets of equations are equivalent

$$D_{[\sigma} W_{\gamma\delta]\alpha\beta} = 0 \ , \ D^\mu W_{\mu\nu\alpha\beta} = 0$$
$$D^{\mu\star} W_{\mu\nu\alpha\beta} = 0 \ , \ D_{[\sigma}{}^\star W_{\gamma\delta]\alpha\beta} = 0 \ .$$

II. The Bianchi equations 3.2.1, are conformally covariant [14], see [Pe1], [Pe2] and also [Ch-Kl1], [Ch-Kl].

III. $\hat{\mathcal{L}}_X W$ is also a Weyl field and satisfies $\hat{\mathcal{L}}_X {}^*W = {}^*\hat{\mathcal{L}}_X W$.

Proof: See [Ch-Kl], Chapter 7.

The Bianchi equations 3.2.1 look complicated, nevertheless they are quite similar to the more familiar Maxwell equations. This is already obvious formally, but it becomes even more apparent if we decompose W into its "electric and magnetic" parts. Given vector fields X, Y we define $i_{(X,Y)}$ through the relation $(i_{(X,Y)}W)_{\mu\nu} = W_{\mu\alpha\nu\beta}X^\alpha Y^\beta$, then, with $X = Y = T$, define

$$E = i_{(T,T)} W \ , \ H = i_{(T,T)}{}^\star W. \tag{3.2.4}$$

These two covariant symmetric, traceless tensor fields E and H, tangent to the hyperplanes Σ_t, determine completely the Weyl tensor field. It is easy to

[14] This means that whenever we perform a conformal transformation ϕ of the spacetime $(\mathcal{M}, g)$ with $\tilde{g} = \phi_* g = \Lambda^2 g$, then $\tilde{W} = \Lambda^{-1}\phi_* W$ is a solution of the Bianchi equations for the spacetime $(\mathcal{M}, \tilde{g})$.

write the Bianchi equations for this decomposition and obtain the following "Maxwell-type" equations:

$$\begin{aligned}
&\Phi^{-1}\partial_t E + \mathrm{curl} H = \rho(E,H)\\
&\Phi^{-1}\partial_t H - \mathrm{curl} E = \sigma(E,H)\\
&\mathrm{div} E = k\wedge H\\
&\mathrm{div} H = -k\wedge E\ .
\end{aligned}$$

where ∇ is the covariant derivative with respect to Σ_t,

$$(\mathrm{div}E)_i = \nabla^j E_{ij}\ ,\ (\mathrm{curl}E)_{ij} = \epsilon_i^{\ lk}\nabla_l E_{kj}$$

and the corresponding expressions hold for H. The explicit expressions of $\rho(E,H)$ and $\sigma(E,H)$ can be found in [Ch-Kl], page 146.

The strong formal analogy with the Maxwell equations goes even further. In fact, just like the Maxwell equations, the Bianchi equations possess an analogous of the energy momentum tensor, the Bel-Robinson tensor, see [Bel].

Definition 3.2.3 *The Bel-Robinson tensor of the Weyl field W is the four covariant tensor field*

$$Q_{\alpha\beta\gamma\delta} = W_{\alpha\rho\gamma\sigma}W_{\beta\ \delta}^{\ \rho\ \sigma} + {}^\star W_{\alpha\rho\gamma\sigma}{}^\star W_{\beta\ \delta}^{\ \rho\ \sigma} \tag{3.2.5}$$

The Bel-Robinson tensor has the following important properties, which remind those of the energy momentum tensor of the Maxwell equations, see [Ch-Kl] Chapter 7,

Proposition 3.2.2
a) Q is symmetric and traceless relative to all pairs of indices.
b) Q satisfies the following positivity condition: $Q(X_1,X_2,X_3,X_4)$ is positive, unless $W=0$, for any timelike vector fields [15].
c) If W is a solution of the Bianchi equations then

$$D^\alpha Q_{\alpha\beta\gamma\delta} = 0 \tag{3.2.6}$$

Definition 3.2.4 *Given a vector field X we denote by ${}^{(X)}\pi \equiv \mathcal{L}_X g$ its the deformation tensor. We denote by ${}^{(X)}\hat{\pi} = g^{\mu\nu}\,{}^{(X)}\pi_{\mu\nu}$ its traceless part. They measure in a precise sense how much the diffeomorphism generated by X differs from an isometry or a conformal isometry.*

[15] We need this property when at most two of these vector fields are different, in which case the proof is straightforward, see [Ch-Kl].

Proposition 3.2.3 *Let $Q(W)$ be the Bel Robinson tensor of a Weyl field W and X, Y, Z a triplet of vector fields. We define the covariant vector field P associated to the triplet as*

$$P_\alpha = Q_{\alpha\beta\gamma\delta} X^\beta Y^\gamma Z^\delta. \tag{3.2.7}$$

Using all the symmetry properties of Q we have

$$\begin{aligned} DivP &= DivQ_{\beta\gamma\delta} X^\beta Y^\gamma Z^\delta \\ &+ \frac{1}{2} Q_{\alpha\beta\gamma\delta} \left({}^{(X)}\pi^{\alpha\beta} Y^\gamma Z^\delta + {}^{(Y)}\pi^{\alpha\gamma} X^\beta Z^\delta + {}^{(Z)}\pi^{\alpha\delta} X^\beta Y^\gamma \right) \end{aligned}$$

Recall, see Chapter 2, section 2.2, that this expression can be used, if X, Y, Z are Killing or conformal Killing vector fields, to construct conserved quantities.

We assume that the Einstein vacuum spacetime, or more precisely a region of it, is foliated by a smooth double null foliation. The Bianchi equations can be expressed in terms of the null components $\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}$ of the curvature tensor, according to the following proposition which extends equations 2.2.13 of Chapter 2.

Proposition 3.2.4 *Expressed relatively to an adapted null frame, the Bianchi equations take the form*

$$\begin{aligned}
\underline{\alpha}_4 &\equiv \not{\mathbf{D}}_4 \underline{\alpha} + \frac{1}{2} tr\chi \underline{\alpha} = -\not\nabla \widehat{\otimes} \underline{\beta} + \left[4\omega \underline{\alpha} - 3(\hat{\underline{\chi}}\rho - {}^\star\hat{\underline{\chi}}\sigma) + (\zeta - 4\underline{\eta}) \widehat{\otimes} \underline{\beta} \right] \\
\underline{\beta}_3 &\equiv \not{\mathbf{D}}_3 \underline{\beta} + 2tr\underline{\chi}\ \underline{\beta} = -\not{d}iv \underline{\alpha} - \left[2\underline{\omega}\underline{\beta} + (-2\zeta + \eta) \cdot \underline{\alpha} \right] \\
\underline{\beta}_4 &\equiv \not{\mathbf{D}}_4 \underline{\beta} + tr\chi \underline{\beta} = -\not\nabla \rho + \left[2\omega \underline{\beta} + 2\hat{\underline{\chi}} \cdot \beta + {}^\star\not\nabla \sigma - 3(\underline{\eta}\rho - {}^\star\underline{\eta}\sigma) \right] \\
\rho_3 &\equiv \mathbf{D}_3 \rho + \frac{3}{2} tr\underline{\chi}\rho = -\not{d}iv \underline{\beta} - \left[\frac{1}{2} \hat{\chi} \cdot \underline{\alpha} - \zeta \cdot \underline{\beta} + 2\eta \cdot \underline{\beta} \right] \\
\rho_4 &\equiv \mathbf{D}_4 \rho + \frac{3}{2} tr\chi\rho = \not{d}iv \beta - \left[\frac{1}{2} \hat{\underline{\chi}} \cdot \alpha - \zeta \cdot \beta - 2\underline{\eta} \cdot \beta \right] \qquad (3.2.8) \\
\sigma_3 &\equiv \mathbf{D}_3 \sigma + \frac{3}{2} tr\underline{\chi}\sigma = -\not{d}iv {}^\star\underline{\beta} + \left[\frac{1}{2} \hat{\chi} \cdot {}^\star\underline{\alpha} - \zeta \cdot {}^\star\underline{\beta} - 2\eta \cdot {}^\star\underline{\beta} \right] \\
\sigma_4 &\equiv \mathbf{D}_4 \sigma + \frac{3}{2} tr\chi\sigma = -\not{d}iv {}^\star\beta + \left[\frac{1}{2} \hat{\underline{\chi}} \cdot {}^\star\alpha - \zeta \cdot {}^\star\beta - 2\underline{\eta} \cdot {}^\star\beta \right] \\
\beta_3 &\equiv \not{\mathbf{D}}_3 \beta + tr\underline{\chi}\beta = \not\nabla \rho + \left[2\underline{\omega}\beta + {}^\star\not\nabla \sigma + 2\hat{\chi} \cdot \underline{\beta} + 3(\eta\rho + {}^\star\eta\sigma) \right] \\
\beta_4 &\equiv \not{\mathbf{D}}_4 \beta + 2tr\chi\beta = \not{d}iv \alpha - \left[2\omega\beta - (2\zeta + \underline{\eta})\alpha \right] \\
\alpha_3 &\equiv \not{\mathbf{D}}_3 \alpha + \frac{1}{2} tr\underline{\chi}\alpha = \not\nabla \widehat{\otimes} \beta + \left[4\underline{\omega}\alpha - 3(\hat{\chi}\rho + {}^\star\hat{\chi}\sigma) + (\zeta + 4\eta) \widehat{\otimes} \beta \right]
\end{aligned}$$

These equations are similar to the ones in the Minkowski spacetime. The terms in square brackets, absent in the flat case, are products between the Weyl null components and the connection coefficients [16].

3.3 Canonical double null foliation of the spacetime

In this section we introduce the concept of a canonical double foliation which plays an important role in the proof of the *Main Theorem.*
We start considering a bounded region of the spacetime, we denote by $\mathcal{K}$, whose boundary is identified by:

- A finite region of a spacelike hypersurface Σ_0; $\mathcal{K}$ is in the future of Σ_0.
- A portion of an incoming null hypersurface $\underline{C}_*$; $\underline{C}_* \cap \Sigma_0 \equiv S_*(\lambda_1)$ is diffeomorphic to S^2.
- A portion of an outgoing null hypersurface C_0; $C_0 \cap \Sigma_0 \equiv S_{(0)}(\nu_0)$ is diffeomorphic to S^2. Also $C_0 \cap \underline{C}_*$ is a two surface diffeomorphic to S^2.

A *double null foliation of* $\mathcal{K}$ is given by two optical functions u and $\underline{u}$ such that

$$\underline{C}_* = \{p \in \mathcal{K} | \underline{u}(p) = \nu_*\} \ , \ C_0 = \{p \in \mathcal{K} | u(p) = \lambda_0\}$$

with λ_0 and ν_* fixed constants [17].

A canonical double null foliation *of* $\mathcal{K}$ is a double null foliation such that the restriction of u on $\underline{C}_*$ and of $\underline{u}$ on Σ_0 are canonical in a sense which will be made precise in this section.

Remark: We shall refer in the sequel to $\Sigma_0 \cap \mathcal{K}$ as the "initial slice" and to $\underline{C}_* \cap \mathcal{K}$ as the "last slice".

[16] In the case of the Schwarzschild spacetime the only terms in parenthesis different from zero are those depending on $\omega, \underline{\omega}$.
[17] The reason of the notation $S_*(\lambda_1)$ and $S_{(0)}(\nu_0)$ will be clear after subsections 3.3.1 and 3.3.3.

3.3.1 Canonical foliation of the initial hypersurface

We consider foliations on regions of the initial hypersurface Σ_0 specified by a "radial" function $r(p) = w(p)$. By this we mean a differentiable real function defined on all points of this region, which takes values in an interval (σ, ∞) and verifies the following conditions:

a) w has no critical points.

b) The level surfaces $S_0(\nu) \equiv \{p \in \Sigma_0 | w(p) = \nu\}$ are diffeomorphic to the two dimensional spheres S^2.

Let $K \subset \Sigma_0$ be a compact set such that $\Sigma_0 \backslash K$ is diffeomorphic to the complement of the closed unit ball. Consider a radial foliation of $\Sigma_0 \backslash K$ given by the function $w(p)$. Its leaves are

$$S_0(\sigma) = \{p \in \Sigma_0 | w(p) = \sigma\}$$

We assume that ∂K is a leaf of the foliation, $\partial K = S_0(\sigma_0)$.
We choose on Σ_0 a moving frame, $\{\tilde{N}, e_a\}$, adapted to this foliation where $\tilde{N}^i = \frac{1}{|\partial w|} g^{ij} \partial_j w$ is the unit vector field defined on Σ_0 normal to each $S_0(\sigma)$. The metric on Σ_0 can be written

$$g(\cdot, \cdot) = a^2 dw^2 + \gamma_{ab} d\phi^a d\phi^b \tag{3.3.1}$$

and, with this choice of coordinates, $\tilde{N} = \frac{1}{a} \frac{\partial}{\partial w}$ and $a^{-2} = |\partial w|^2$.
Using Gauss and Codazzi-Mainardi equations relative to the surfaces $S_0(\nu)$ immersed in Σ_0 we obtain the following evolution equation [18] for $\mathrm{tr}\theta$,

$$\nabla_{\tilde{N}} \mathrm{tr}\theta + \frac{1}{2}(\mathrm{tr}\theta)^2 + a^{-1} \not{\Delta} a = -|\hat{\theta}|^2 - R_{\tilde{N}\tilde{N}} \tag{3.3.2}$$

which can be rewritten as

$$\nabla_{\tilde{N}} \mathrm{tr}\theta + \frac{1}{2}(\mathrm{tr}\theta)^2 = -(\not{\Delta} \log a + \rho) + \left[-|\not{\nabla} \log a|^2 - |\hat{\theta}|^2 + g(k)\right] \tag{3.3.3}$$

where ρ is the null component $-\frac{1}{4}\mathbf{R}_{3434}$ of the Riemann tensor [19], relative to the null pair $\{e'_4 = \tilde{N} + T_0, e'_3 = \tilde{N} - T_0\}$, and

$$g(k) \equiv k^2_{\tilde{N}\tilde{N}} + \sum_a |k_{e'_a \tilde{N}}|^2 \ .$$

[18] These are derived in Chapter 7, subsection 7.1.1, see also [Ch-Kl] Chapter 5.
[19] See also footnote 7 of subsection 7.1.3.

Definition 3.3.1 *A foliation on $\Sigma_0 \backslash K$, defined by a radial function $\underline{u}_{(0)}(p)$, is said to be canonical if $\underline{u}_{(0)}(p)$ is a solution to the "Initial slice problem" with initial condition on ∂K,*

$$|\nabla \underline{u}_{(0)}| = a^{-1} \ , \ \underline{u}_{(0)}|_{\partial K} = \nu_0$$
$$\not\Delta \log a = -(\rho - \overline{\rho}) \ , \ \overline{\log a} = 0 \tag{3.3.4}$$

The leaves of the canonical foliation are denoted

$$S_{(0)}(\nu) = \{p \in \Sigma_0 | \underline{u}_{(0)}(p) = \nu\} \tag{3.3.5}$$

The next theorem assures the local and global existence of a canonical foliation on $\Sigma_0 \backslash K$.

Theorem 3.3.1 *Under appropriate smallness assumptions on $\Sigma_0 \backslash K$ there exists a canonical foliation on $\Sigma_0 \backslash K$.*

The precise statement of Theorem 3.3.1 is given in section 3.7 and its proof is given in Chapter 7.

Remark: The basic reason which requires the introduction of the canonical foliation on $\Sigma_0 \backslash K$ is that we need to control θ up to third derivatives. Without the canonical foliation the control of the third derivatives of θ will require the control of g up to five derivatives and that of k up to four derivatives. This would lead to a stronger assumption on the initial data than necessary.

3.3.2 Foliations on the last slice

A foliation on the last slice $\underline{C}_*$ is specified giving a function u_* with the following properties:

a) u_* is a differentiable real function defined on all points of $\underline{C}_*$.

b) u_* has no critical points.

c) The level surfaces of $u_*(p)$, $S_*(\lambda) \equiv S(\lambda, \nu_*) = \{p \in \underline{C}_* | u_*(p) = \lambda\}$, are diffeomorphic to the two dimensional spheres S^2.

3.3.3 Canonical foliation of the last slice

The concept of the canonical foliation on the last slice is an important ingredient in the proof of *Main Theorem*, see also the discussion in [Kl-Ni].

We start defining the following functions we call mass aspect functions [20],

$$\mu = \mathbf{K} + \frac{1}{4}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi - \mathrm{d}\!\!/\mathrm{iv}\,\eta$$
$$\underline{\mu} = \mathbf{K} + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \mathrm{d}\!\!/\mathrm{iv}\,\underline{\eta} \tag{3.3.6}$$

We restrict our attention to $\underline{C}_*$ and its initial section $S_*(0)$. Let S be an arbitrary section of $\underline{C}_*$, there is a unique outgoing null normal L^* to S, conjugate to $\underline{L}$ such that $g(\underline{L}, L^*) = -2$. We recall, see 3.1.31, 3.1.32, that, in the normalized null frame $\{\hat{N} = \frac{1}{2\Omega}L^*\,,\ \hat{\underline{N}} = 2\Omega\underline{L}\}$,

$$\underline{\eta} = -\zeta + \nabla\!\!\!/ \log\Omega\ .$$

Therefore [21]

$$\underline{\eta}(X) = -\frac{1}{2}\mathbf{g}(\mathbf{D}_X L^*, \underline{L}) \tag{3.3.7}$$

follows easily, with $X \in TS$. Hence to obtain $\underline{\eta}$ the knowledge of Ω is not required. Once $\underline{C}_*$ and its null geodesic vector field $\underline{L}$ are given, $\underline{\eta}$ is uniquely defined by the section S. Clearly $\mathrm{tr}\chi\mathrm{tr}\underline{\chi}$ and the Gauss curvature $\mathbf{K}$ are also independent of Ω. Consequently the quantity, see 3.3.6,

$$\underline{\mu} = \mathbf{K} + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \mathrm{d}\!\!/\mathrm{iv}\,\underline{\eta}$$

is also independent of Ω. Consider a given scalar function u_* on $\underline{C}_*$ and let u be the outgoing solution of the eikonal equation such that $u|_{\underline{C}_*} = u_*$. Let L be the null geodesic vector field, $L^\mu = -g^{\mu\nu}\partial_\nu u$. The relation between the affine function v of $\underline{L}$ and the function $u_* = u|_{\underline{C}_*}$ is

$$\frac{du_*}{dv} = \underline{L}(u_*) = -g^{\mu\nu}\partial_\mu \underline{u}\partial_\nu u|_{\underline{C}_*} = -\mathbf{g}(\underline{L}, L)|_{\underline{C}_*} = (2\Omega^2)^{-1}\ .$$

We want to choose u_* on $\underline{C}_*$ such that the mass aspect function μ is constant on the surfaces

$$S_*(\lambda) = \{p \in \underline{C}_* | u_*(p) = \lambda\}\ , \tag{3.3.8}$$

the leaves of the foliation induced by u_* on $\underline{C}_*$. In other words μ satisfies the equation

$$\mu - \overline{\mu} = 0 \tag{3.3.9}$$

[20] These were first introduced in[Ch-Kl], Chapter 13.

[21] $\frac{1}{2}\mathbf{g}(\mathbf{D}_X L^*, \underline{L}) = \frac{1}{2}\mathbf{g}(\mathbf{D}_X \hat{N}, \hat{\underline{N}}) + \frac{1}{2}\Omega^{-1}(\mathbf{D}_X\Omega)\mathbf{g}(\hat{N}, \hat{\underline{N}}) = \zeta(X) - \nabla\!\!\!/_X \log\Omega = -\underline{\eta}(X)$.

with $\overline{\mu}$ the average of μ on $S_*(\lambda)$.
This can be viewed as an equation for Ω at each $S_*(\lambda)$. According to 3.3.6 and relation $\eta+\underline{\eta}=2\nabla\!\!\!/\log\Omega$, see 3.1.32, we have

$$\overline{\mu}+\underline{\mu}=\mu+\underline{\mu}=2\mathbf{K}+\frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-2\Delta\!\!\!/\log\Omega \tag{3.3.10}$$

Therefore

$$\begin{aligned}\Delta\!\!\!/\log\Omega &= \mathbf{K}+\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\frac{1}{2}(\underline{\mu}+\overline{\mu}) \\ &= \frac{1}{2}\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\eta}+\frac{1}{2}\left(\mathbf{K}-\overline{\mathbf{K}}+\frac{1}{4}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\chi\mathrm{tr}\underline{\chi}})\right)\end{aligned} \tag{3.3.11}$$

Observe that the right hand side of 3.3.11 does not depend on Ω [22].

Definition 3.3.2 *A foliation on the last slice given by the level sets of u_*, is said to be canonical if the functions u_* and Ω satisfy the following system of equations*

$$\begin{aligned}&\Delta\!\!\!/\log\Omega=\frac{1}{2}d\!\!\!/iv\underline{\eta}+\frac{1}{2}\left(\mathbf{K}-\overline{\mathbf{K}}+\frac{1}{4}(tr\chi tr\underline{\chi}-\overline{tr\chi tr\underline{\chi}})\right)\\ &\overline{\log 2\Omega}=0\\ &\frac{du_*}{dv}=(2\Omega^2)^{-1};\quad u_*|_{\underline{C}_*\cap\Sigma_0}=\lambda_1\end{aligned} \tag{3.3.12}$$

Remark that Ω is uniquely defined by the first two equations in 3.3.12.

The next theorem proves the existence of a canonical foliation on $\underline{C}_*$.

Theorem 3.3.2 *Assume given on $\underline{C}_*$ a background foliation whose connection coefficients and null curvature components satisfy "appropriate smallness assumptions". Then it is possible to construct, on the whole $\underline{C}_*$, a canonical foliation, close to the background one.*

---

[22]Observe that the mass aspect function $\underline{\mu}$ can be connected to the Hawking mass, defined by $2\frac{m_H}{r}=1+\frac{1}{16\pi}\int_S\mathrm{tr}\chi\mathrm{tr}\underline{\chi}d\mu_\gamma$, according to the following equation

$$\frac{8\pi m_H}{r}=\int_S\underline{\mu}d\mu_\gamma$$

Indeed integrating the first line of 3.3.10 we obtain $\overline{\mu}|S|+\int_S\underline{\mu}d\mu_\gamma=8\pi+\frac{1}{2}\int_S\mathrm{tr}\chi\mathrm{tr}\underline{\chi}d\mu_\gamma$, where $|S|=4\pi r^2$. On the other hand, from the second equation in 3.3.6, we have $\int_S\underline{\mu}d\mu_\gamma=4\pi+\frac{1}{4}\int_S\mathrm{tr}\chi\mathrm{tr}\underline{\chi}d\mu_\gamma$. Therefore the result follows.

Remark: As Theorem 3.3.2 plays an important role in the proof of the *Main Theorem*, we will state it again with all the details in section 3.7 after we have introduced the appropriate families of norms for the connection coefficients and the Riemann curvature tensor. The proof of Theorem 3.3.2 is given in Chapter 7.

We can now define the "*canonical double null foliation*" of the spacetime, a property which will be used in the Bootstrap assumption **B1** of the *Main Theorem*

Definition 3.3.3 *A "double null foliation of $\mathcal{K}$" is called "canonical" if:*
a) The $C(\lambda)$ null hypersurfaces are defined by $u(p)=\lambda$, where $\lambda\in[\lambda_1,\lambda_0]$; u is the incoming solution of the eikonal equation with "final data" given by the "canonical function" u_* on the last slice.
b) The $\underline{C}(\nu)$ null hypersurfaces are defined by $\underline{u}(p)=\nu$ where $\nu\in[\nu_0,\nu_*]$; $\underline{u}$ is the outgoing solution of the eikonal equation with the "initial data" given by the "canonical function" $\underline{u}_{(0)}$ on the initial hypersurface Σ_0.

Remark: *$\mathcal{K}$ is the causal past of $S(\lambda_0,\nu_*)$, in the future of Σ_0.*

The canonical double null foliation *of* $\mathcal{K}$ consists, therefore, of the $C(\lambda)$ null hypersurfaces, with $\lambda\in[\lambda_1,\lambda_0]$ and the $\underline{C}(\nu)$ null hypersurfaces with $\nu\in[\nu_0,\nu_*]$; each point $p\in\mathcal{K}$ belongs to one and only one pair of the hypersurfaces $C(\lambda)$ and $\underline{C}(\nu)$ [23].
Given this double null canonical foliation the two dimensional surfaces

$$S(\lambda,\nu)=C(\lambda)\cap\underline{C}(\nu)$$

define a codimension two double null integrable S-foliation.

Remark: The global spacetime of our Main Theorem will be constructed, by a continuity argument, with the help of a continous family of spacetime regions $\mathcal{K}$ each endowed with a canonical foliation. While the canonical foliation plays an essential part in our construction it has one undesirable feature; the foliations on Σ_0 induced by the two families of light cones $C(\lambda)$ and $\underline{C}(\nu)$ differ from each other. In particular the canonical surfaces on Σ_0 do not belong to the S-foliation associated to the double null foliation, $\{S(\lambda,\nu)=C(\lambda)\cap\underline{C}(\nu)\}$.
In order to correct for this we construct, in a small neighborhood of Σ_0, a different foliation which we denote the *initial layer foliation*. We shall discuss this in the next section.

[23] We use sometimes the more precise definition $C(\lambda,[\nu_a,\nu_b])$ and $\underline{C}(\nu,[\lambda_a,\lambda_b])$ where the interval where the functions $u(p)$ and $\underline{u}(p)$ vary is written expicitely.

3.3.4 Initial layer foliation

Starting with the canonical foliation on Σ_0, defined by the level hypersurfaces of $\underline{u}_{(0)}$, we consider the incoming null hypersurfaces $\underline{C}(\nu)$ and outgoing null hypersurfaces $C'(\lambda)$. More precisely:
a) The $C'(\lambda')$ null hypersurfaces are given by $u'(p)=\lambda'$, where $\lambda'\in[-\nu_0,-\nu_*]$; with u' the outgoing solution of the eikonal equation with initial condition $u'=-\underline{u}_{(0)}$ on Σ_0.
b) The $\underline{C}(\nu)$ null hypersurfaces are defined as before by $\underline{u}(p)=\nu$ where $\nu\in[\nu_0,\nu_*]$; with $\underline{u}$ the incoming solution of the eikonal equation with initial condition $\underline{u}=\underline{u}_{(0)}$ on the initial hypersurface Σ_0.
[$u'(p)+\underline{u}(p)\leq\delta_0$ has been substituted by $\frac{1}{2}(u'(p)+\underline{u}(p))\leq\delta_0$]
Consider the region of ${\mathcal K'}_{\delta_0}\subset\mathcal{K}$ specified by the condition

$$\frac{1}{2}(u'(p)+\underline{u}(p))\leq\delta_0 \tag{3.3.13}$$

Definition 3.3.4 *For a fixed δ_0, sufficiently small, we shall call ${\mathcal K'}_{\delta_0}$ the "initial layer region" of height δ_0. The double null foliation induced on ${\mathcal K'}_{\delta_0}$ by the optical functions u', $\underline{u}$ defined above is called the "initial layer foliation". Its two dimensional surfaces are denoted by*

$$S'(\lambda',\nu)=C'(\lambda')\cap\underline{C}(\nu) \tag{3.3.14}$$

Remarks:

i) The leaves of the canonical foliation of Σ_0, $S_{(0)}(\nu)$, belong to the *initial layer foliation.* More precisely

$$S_{(0)}(\nu)=S'(-\nu,\nu) \tag{3.3.15}$$

ii) Relative to the *initial layer foliation* we associate, as before, the normalized null pair $\hat{N}'=2\Omega'L'$, $\hat{\underline{N}}'=2\Omega'\underline{L}$ with

$$2{\Omega'}^2=-g(L',\underline{L})^{-1}=-(g^{\rho\sigma}\partial_\rho u'\partial_\sigma\underline{u})^{-1} \tag{3.3.16}$$

exactly as in definition 3.1.12 for Ω.
We shall also make use of the null equivariant pair $N'=2\Omega'^2L'$, $\underline{N}'=2\Omega'^2\underline{L}$.

Next Proposition shows that, given a double null foliation, it is possible to introduce a global time function and prove that the associated three dimensional spacelike hypersurfaces define a spacelike foliation.

Proposition 3.3.1 *Assume a double null foliation specified by the functions* $u(p), \underline{u}(p)$. *Let us define the "global time" function* $t(p) = \frac{1}{2}(u(p) + \underline{u}(p))$, *then the three dimensional spacelike hypersurfaces*

$$\tilde{\Sigma}_t \equiv \{p \in \mathcal{K} | t(p) = t\}$$

define a three dimensional spacelike foliation of $\mathcal{K}$. *Each two dimensional surface* $S(\lambda, \nu)$ *is immersed in the hypersurface* $\tilde{\Sigma}_t$ *with* $t = \frac{1}{2}(\lambda + \nu)$. *Moreover*

$$dt = -\frac{1}{4\Omega^2}(n + \underline{n}) \quad , \quad \frac{\partial}{\partial t} = (N + \underline{N}) \tag{3.3.17}$$

where $n, \underline{n}$ *are the one forms corresponding to* $N, \underline{N}$.
Finally, given the hypersurfaces $\tilde{\Sigma}_t$, *their second fundamental form* k *has the following expression in terms of the connection coefficients,*

$$k_{\tilde{N}\tilde{N}} = \omega + \underline{\omega} \ , \quad k_{e_a \tilde{N}} = \zeta_a \ , \ k_{e_a e_b} = -\frac{1}{2}(\chi_{ab} + \underline{\chi}_{ab}) \tag{3.3.18}$$

and, on each $\tilde{\Sigma}_t$, *the metric* $g_{ij}(t, x)$ *is given by the relation*

$$k_{ij} = -(2\Omega)^{-1} \partial_t g_{ij} \ .$$

It is easy to prove that the $\tilde{\Sigma}_t$ hypersurfaces are not maximal as, by direct computation, it follows that $\mathrm{tr}k \neq 0$ [24]. This implies that as, $\Sigma_0 \neq \tilde{\Sigma}_{t=0}$, the two dimensional surfaces $S_{(0)}(\underline{u}_{(0)} = \nu)$, $\nu \in [\nu_0, \nu_*]$, which foliate canonically Σ_0, see Definition 3.3.1, do not belong to the family $\{S(\lambda, \nu) = C(\lambda) \cap \underline{C}(\nu)\}$.

Using Proposition 3.3.1, it is possible to introduce a different spacelike foliation, adapted to the *initial layer foliation* whose spacelike hypersurfaces are defined by the "global time" function $t'(p) = \frac{1}{2}(u'(p) + \underline{u}(p))$. Its three dimensional spacelike hypersurfaces are

$$\Sigma'_{t'} \equiv \{p \in \mathcal{K} | t'(p) = t\}.$$

Obseve that Σ_0 is a leave of this foliation, $\Sigma_0 = \Sigma'_{t'=0}$.

Remark: As explained in a previous remark, we need the initial layer foliation to connect the initial hypersurface Σ_0 and its surfaces $\{S_{(0)}(\nu)\}$ with the canonical foliation of $\mathcal{K}$ and the surfaces $S(\lambda, \nu)$. This is discussed in detail in Chapter 4.

[24] Nevertheless it will follow from the results of the next chapters that $\mathrm{tr}k$ is small, see Theorem 3.7.3.

3.4 Deformation tensors

3.4.1 Approximate Killing and conformal Killing vector fields

The functions $u, \underline{u}$ of the double null foliation, together with the null pair $\{e_3 = N, e_4 = \underline{N}\}$, allow us to define the vector fields $T, S, K_0, \bar{K}$ analogous to the ones used earlier in the Minkowski spacetime, see [Ch-Kl1] and [Kl-Ni].

$$
\begin{aligned}
T &= \frac{1}{2}(e_3 + e_4) \ , \ S = \frac{1}{2}(ue_3 + \underline{u}e_4) \ , \\
K_0 &= \frac{1}{2}(u^2 e_3 + \underline{u}^2 e_4) \ , \ \bar{K} = \frac{1}{2}(\tau_+^2 e_4 + \tau_-^2 e_3)
\end{aligned} \tag{3.4.1}
$$

where [25]

$$
\tau_+ = (1 + \underline{u}^2)^{\frac{1}{2}} \ , \ \tau_- = (1 + u^2)^{\frac{1}{2}} \tag{3.4.2}
$$

Unlike the case of Minkowski spacetime, these vector fields are not conformal Killing. We show, however, that their deformation tensors, or rather their traceless parts, are asymptotically vanishing in a sufficient strong sense. We can also define approximate Killing, "angular momentum vector fields", ${}^{(i)}O$, $i \in \{1, 2, 3\}$, which play the same role as the angular momentum vector fields of Minkowski space. They are constructed, geometrically, as follows:

We start from the asymptotic region of the initial hypersurface Σ_0. There, in view of our strong asymptotic flatness assumptions, this manifold looks euclidean. We can thus define the canonical angular momentum vector fields at infinity and pull them back with the help of the diffeomorphism generated by the flow normal to the S surfaces [26] along Σ_0. The vector fields can the be "pushed forward", in the same way, along the last slice $\underline{C}_*$ using the diffeomorphism $\underline{\phi}_\tau$ generated by $\underline{N}$. Finally we pull them back, once more, along the hypersurfaces $C(\lambda)$, with the help of the diffeomorphism generated by null outgoing equivariant vector field N. These steps define the vector fields ${}^{(i)}O$ at any point [27] of our spacetime $\mathcal{K}$. By definition they are tangent to the S-foliation and commute with N. Moreover they satisfy

[25] Observe that in the sequel the vector fields K_0 and $\bar{K}$ can be both used as in the *Main Theorem* u is bounded from below.

[26] See [Kl-Ni] and Chapter 5 in[Ch-Kl].

[27] Let $q \in S(\lambda, \nu)$ be an arbitrary point of $\mathcal{K}$. As $S(\lambda, \nu)$ is diffeomorphic via $\phi_{\Delta=(\nu_* - \nu)}$ to $S(\lambda, \nu_*) \subset \underline{C}_*$, $\exists p \in S(\lambda, \nu_*)$ such that $q = \phi_\Delta^{-1}(p)$. We define the element O of the rotation group operating over q as $(O, q) \equiv \phi_\Delta^{-1}(O_*, p = \phi_\Delta(q))$ where (O, q) is a point of $S(\lambda, \nu)$, while $(O_*, p = \phi_\Delta(q))$ is the point of $S(\lambda, \nu_*)$ obtained applying O_* to the point p.

the canonical commutation relations. Thus, finally, the "extended" rotation generators, or angular vector fields ${}^{(i)}O$ satisfy [28]

$$
\begin{aligned}
&[{}^{(i)}O, {}^{(j)}O] = \epsilon_{ijk}{}^{(k)}O \\
&[N, {}^{(i)}O] = 0 \\
&\mathbf{g}({}^{(i)}O, e_4) = \mathbf{g}({}^{(i)}O, e_3) = 0.
\end{aligned}
\tag{3.4.3}
$$

All these steps are described in complete detail in Chapter 4, section 4.6 and in Chapter 7.

3.4.2 Deformation tensors of the vector fields T, S, K_0

We use the "adapted" null frame $\{\hat{N}, \underline{\hat{N}}, e_1, e_2\}$ associated to the canonical double null integral foliation introduced in the previous section.

Let X be a vector field on $\mathcal{K}$. If X were a Killing vector field then

$$
{}^{(X)}\pi \equiv L_X\mathbf{g} = 0
$$

would hold and the diffeomorphism generated by the integral curves of X would be an isometry of $(\mathcal{K}, g)$. If X is not a Killing vector field the previous relation does not hold, but, if the spacetime is not "too different" from the Minkowski spacetime, we expect to control the magnitude of some appropriate norms of the deformation tensor ${}^{(X)}\pi$. We recall that

$$
{}^{(X)}\pi_{\mu\nu} = D_\mu X_\nu + D_\nu X_\mu
$$

and its traceless part is

$$
{}^{(X)}\hat{\pi}_{\mu\nu} = {}^{(X)}\pi_{\mu\nu} - \frac{1}{4}g_{\mu\nu}tr\pi.
$$

In the null frame associated to the canonical foliation, therefore,

$$
\begin{aligned}
{}^{(X)}\pi_{ab} &= \mathbf{g}(\mathbf{D}_{e_a}X, e_b) + \mathbf{g}(\mathbf{D}_{e_b}X, e_a) \\
{}^{(X)}\pi_{4a} &= \mathbf{g}(\mathbf{D}_{\hat{N}}X, e_a) + \mathbf{g}(\mathbf{D}_{e_a}X, \hat{N}) \\
{}^{(X)}\pi_{3a} &= \mathbf{g}(\mathbf{D}_{\underline{\hat{N}}}X, e_a) + \mathbf{g}(\mathbf{D}_{e_a}X, \underline{\hat{N}}) \\
{}^{(X)}\pi_{34} &= \mathbf{g}(\mathbf{D}_{\underline{\hat{N}}}X, \hat{N}) + \mathbf{g}(\mathbf{D}_{\hat{N}}X, \underline{\hat{N}}) \\
{}^{(X)}\pi_{33} &= \mathbf{g}(\mathbf{D}_{\underline{\hat{N}}}X, \underline{\hat{N}}) + \mathbf{g}(\mathbf{D}_{\underline{\hat{N}}}X, \underline{\hat{N}}) \\
{}^{(X)}\pi_{44} &= \mathbf{g}(\mathbf{D}_{\hat{N}}X, \hat{N}) + \mathbf{g}(\mathbf{D}_{\hat{N}}X, \hat{N})
\end{aligned}
\tag{3.4.4}
$$

[28] The commutator $[{}^{(i)}O, \underline{N}] \in TS$ is different from zero, see subsection 4.6.1. This shows that one could have defined the rotation vector fields in a different way starting from the Σ_0 hypersurface and using the diffeomorphism $\underline{\phi}_t$ generated by $\underline{N}$.

and

$$\begin{aligned}
{}^{(X)}\hat{\pi}_{ab} &= {}^{(X)}\pi_{ab} - \tfrac{1}{4}\delta_{ab} tr\, {}^{(X)}\pi \\
{}^{(X)}\hat{\pi}_{4a} &= {}^{(X)}\pi_{4a} \\
{}^{(X)}\hat{\pi}_{3a} &= {}^{(X)}\pi_{3a} \\
{}^{(X)}\hat{\pi}_{34} &= {}^{(X)}\pi_{34} + \tfrac{1}{2} tr\, {}^{(X)}\pi \\
{}^{(X)}\hat{\pi}_{33} &= {}^{(X)}\pi_{33} \\
{}^{(X)}\hat{\pi}_{44} &= {}^{(X)}\pi_{44}
\end{aligned} \tag{3.4.5}$$

We denote the various components of the deformation tensors, with respect to a null frame, as

$$\begin{aligned}
{}^{(X)}\mathbf{i}_{ab} = {}^{(X)}\hat{\pi}_{ab} \;\; &; \;\; {}^{(X)}\mathbf{j} = {}^{(X)}\hat{\pi}_{34} \\
{}^{(X)}\mathbf{m}_a = {}^{(X)}\hat{\pi}_{4a} \;\; &; \;\; {}^{(X)}\underline{\mathbf{m}}_a = {}^{(X)}\hat{\pi}_{3a} \\
{}^{(X)}\mathbf{n} = {}^{(X)}\hat{\pi}_{44} \;\; &; \;\; {}^{(X)}\underline{\mathbf{n}} = {}^{(X)}\hat{\pi}_{33}
\end{aligned} \tag{3.4.6}$$

Their explicit expressions relative to the vectors defined in 3.4.1 are

$$\begin{aligned}
{}^{(T)}\mathbf{i}_{ab} &= \hat{\chi}_{ab} + \underline{\hat{\chi}}_{ab} + \frac{1}{2}\delta_{ab}\Big(\frac{1}{2}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) + (\omega + \underline{\omega})\Big) \\
{}^{(T)}\mathbf{j} &= \frac{1}{2}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) + (\omega + \underline{\omega}) \\
{}^{(T)}\mathbf{m}_a &= 2\underline{\eta}_a - \nabla\!\!\!/_a \log\Omega = \underline{\eta}_a - \zeta_a \\
{}^{(T)}\underline{\mathbf{m}}_a &= 2\eta_a - \nabla\!\!\!/_a \log\Omega = \eta_a + \zeta_a \\
{}^{(T)}\mathbf{n} &= -4\omega = 2\mathbf{D}_4 \log\Omega \\
{}^{(T)}\underline{\mathbf{n}} &= -4\underline{\omega} = 2\mathbf{D}_3 \log\Omega
\end{aligned} \tag{3.4.7}$$

$$\begin{aligned}
{}^{(S)}\mathbf{i}_{ab} &= \underline{u}\hat{\chi}_{ab} + u\underline{\hat{\chi}}_{ab} + \frac{1}{2}\delta_{ab}\Big(\frac{1}{2}(\underline{u}\mathrm{tr}\chi + u\mathrm{tr}\underline{\chi}) + (\underline{u}\omega + u\underline{\omega}) - \frac{1}{\Omega}\Big) \\
{}^{(S)}\mathbf{j} &= \frac{1}{2}(\underline{u}\mathrm{tr}\chi + u\mathrm{tr}\underline{\chi}) + (\underline{u}\omega + u\underline{\omega}) - \frac{1}{\Omega} \\
{}^{(S)}\mathbf{m}_a &= u(2\underline{\eta}_a - \nabla\!\!\!/_a \log\Omega) = u(\underline{\eta}_a - \zeta_a) = u\, {}^{(T)}\mathbf{m}_a \\
{}^{(S)}\underline{\mathbf{m}}_a &= \underline{u}(2\eta_a - \nabla\!\!\!/_a \log\Omega) = \underline{u}(\eta_a + \zeta_a) = \underline{u}\, {}^{(T)}\underline{\mathbf{m}}_a \\
{}^{(S)}\mathbf{n} &= u(-4\omega) = 2u\mathbf{D}_4 \log\Omega = u\, {}^{(T)}\mathbf{n} \\
{}^{(S)}\underline{\mathbf{n}} &= \underline{u}(-4\underline{\omega}) = 2\underline{u}\mathbf{D}_3 \log\Omega = \underline{u}\, {}^{(T)}\underline{\mathbf{n}}
\end{aligned} \tag{3.4.8}$$

$$ {}^{(K_0)}\mathbf{i}_{ab} = \underline{u}^2\hat{\chi}_{ab} + u^2\underline{\hat{\chi}}_{ab} + \frac{1}{2}\delta_{ab}\Big(\frac{1}{2}(\underline{u}^2\mathrm{tr}\chi + u^2\mathrm{tr}\underline{\chi}) + (\underline{u}^2\omega + u^2\underline{\omega}) - \frac{\underline{u}+u}{\Omega}\Big) $$

$$
\begin{aligned}
{}^{(K_0)}\mathbf{j} &= \frac{1}{2}(\underline{u}^2\mathrm{tr}\chi + u^2\mathrm{tr}\underline{\chi}) + (\underline{u}^2\omega + u^2\underline{\omega}) - \frac{\underline{u}+u}{\Omega}\\
{}^{(K_0)}\mathbf{m}_a &= u^2(2\underline{\eta}_a - \nabla\!\!\!/_a \log\Omega) = u^2(\underline{\eta}_a - \zeta_a) = u^2\,{}^{(T)}\mathbf{m}_a\\
{}^{(K_0)}\underline{\mathbf{m}}_a &= \underline{u}^2(2\eta_a - \nabla\!\!\!/_a \log\Omega) = \underline{u}^2(\eta_a + \zeta_a) = \underline{u}^2\,{}^{(T)}\underline{\mathbf{m}}_a\\
{}^{(K_0)}\mathbf{n} &= u^2(-4\omega) = 2u^2\mathbf{D}_4\log\Omega = u^2\,{}^{(T)}\mathbf{n}\\
{}^{(K_0)}\underline{\mathbf{n}} &= \underline{u}^2(-4\underline{\omega}) = 2\underline{u}^2\mathbf{D}_3\log\Omega = \underline{u}^2\,{}^{(T)}\underline{\mathbf{n}}
\end{aligned}
\tag{3.4.9}
$$

3.4.3 Rotation deformation tensors

In what follows we display the form of the various components of the deformation tensors associated to the rotation vector fields ${}^{(i)}O$. Recalling 3.4.4 and $N = \Omega\hat{N}$ we derive the following commutation relations

$$
[{}^{(i)}O, \hat{N}] = [{}^{(i)}O, \Omega^{-1}N] = -{}^{(i)}O(\log\Omega)\hat{N} = {}^{(i)}F\hat{N} \tag{3.4.10}
$$

where

$$
{}^{(i)}F \equiv -(\nabla\!\!\!/_c \log\Omega){}^{(i)}O_c\ .
$$

one can also easily check the following expressions

$$
\begin{aligned}
\mathbf{g}(\mathbf{D}_a{}^{(i)}O, e_4) &= -\chi_{ab}{}^{(i)}O_b\\
\mathbf{g}(\mathbf{D}_4{}^{(i)}O, e_a) &= \chi_{ab}{}^{(i)}O_b\\
\mathbf{g}(\mathbf{D}_4{}^{(i)}O, e_4) &= 0\\
\mathbf{g}(\mathbf{D}_a{}^{(i)}O, e_3) &= -\underline{\chi}_{ab}{}^{(i)}O_b\\
\mathbf{g}(\mathbf{D}_3{}^{(i)}O, e_3) &= 0\\
\mathbf{g}(\mathbf{D}_4{}^{(i)}O, e_3) &= -2\underline{\eta}_b{}^{(i)}O_b\\
\mathbf{g}(\mathbf{D}_3{}^{(i)}O, e_4) &= -2\eta_b{}^{(i)}O_b
\end{aligned}
\tag{3.4.11}
$$

Using these equations we compute explicitely the following components of the deformation tensor. Denoting ${}^{({}^{(i)}O)}\pi \equiv {}^{(i)}\pi$, we have

$$
\begin{aligned}
{}^{(i)}\pi_{44} &= 2\mathbf{g}(\mathbf{D}_4{}^{(i)}O, e_4) = 0\\
{}^{(i)}\pi_{4a} &= \mathbf{g}(\mathbf{D}_4{}^{(i)}O, e_a) + \mathbf{g}(\mathbf{D}_a{}^{(i)}O, e_4) = 0\\
{}^{(i)}\pi_{33} &= 2\mathbf{g}(\mathbf{D}_3{}^{(i)}O, e_3) = 0
\end{aligned}
\tag{3.4.12}
$$

The remaining components are denoted by

$$
\begin{aligned}
2{}^{(i)}H_{ab} &\equiv {}^{(i)}\pi_{ab} = \mathbf{g}(\mathbf{D}_a{}^{(i)}O, e_b) + \mathbf{g}(\mathbf{D}_b{}^{(i)}O, e_a)\\
4{}^{(i)}Z_a &\equiv {}^{(i)}\pi_{3a} = \mathbf{g}(\mathbf{D}_a{}^{(i)}O, e_3) + \mathbf{g}(\mathbf{D}_3{}^{(i)}O, e_a)\\
&= -{}^{(i)}O_b\underline{\chi}_{ab} + \underline{\hat{N}}({}^{(i)}O_a) + {}^{(i)}O_b\mathbf{g}(\mathbf{D}\!\!\!\!/_3 e_b, e_a)\\
4{}^{(i)}F &\equiv {}^{(i)}\pi_{34} = -2(\eta_b + \underline{\eta}_b){}^{(i)}O_b = -4(\nabla\!\!\!/_b \log\Omega){}^{(i)}O_b
\end{aligned}
\tag{3.4.13}
$$

In order to evaluate ${}^{(i)}F, {}^{(i)}Z_a, {}^{(i)}H_{ab}$ we use the evolution equations for these quantities. They are derived, togheter with those for their derivatives, in sections 4.6, 4.7 of Chapter 4.

3.5 The definitions of the fundamental norms

3.5.1 $\mathcal{Q}$ integral norms

Consider the spacetime $\mathcal{K}$ endowed with the *canonical double null foliation* $\{C(\lambda), \underline{C}(\nu)\}$ and the corresponding normalized null pair $\{e_4, e_3\}$.
Denoting by $Q(\mathbf{R})$ the Bell-Robinson tensor, see 3.2.5, associated to the curvature tensor $\mathbf{R}$ and saturating it with the vector fields K_0, T, see 3.4.1, e_4 and e_3 we obtain

$$\begin{aligned}
Q(\mathbf{R})(K_0,K_0,T,e_4) &= \frac{1}{4}\underline{u}^4|\alpha|^2 + \frac{1}{2}(\underline{u}^4 + 2\underline{u}^2u^2)|\beta|^2 + \frac{1}{2}(u^4 + 2\underline{u}^2u^2)(\rho^2+\sigma^2) \\
&\quad + \frac{1}{2}u^4|\underline{\beta}|^2 \\
Q(\mathbf{R})(K_0,K_0,T,e_3) &= \frac{1}{4}u^4|\underline{\alpha}|^2 + \frac{1}{2}(u^4 + 2\underline{u}^2u^2)|\underline{\beta}|^2 + \frac{1}{2}(\underline{u}^4 + 2\underline{u}^2u^2)(\rho^2+\sigma^2) \\
&\quad + \frac{1}{2}\underline{u}^4|\beta|^2 \qquad (3.5.1)
\end{aligned}$$

$$\begin{aligned}
Q(\mathbf{R})(K_0,K_0,K_0,e_4) &= \frac{1}{4}\underline{u}^6|\alpha|^2 + \frac{3}{2}\underline{u}^4u^2|\beta|^2 + \frac{3}{2}u^4\underline{u}^2(\rho^2+\sigma^2) + \frac{1}{2}u^6|\underline{\beta}|^2 \qquad (3.5.2) \\
Q(\mathbf{R})(K_0,K_0,K_0,e_3) &= \frac{1}{4}u^6|\underline{\alpha}|^2 + \frac{3}{2}u^4\underline{u}^2|\underline{\beta}|^2 + \frac{3}{2}\underline{u}^4u^2(\rho^2+\sigma^2) + \frac{1}{2}\underline{u}^6|\beta|^2
\end{aligned}$$

We also have

$$\begin{aligned}
Q(\mathbf{R})(K_0,K_0,T,T) &= \frac{1}{8}\underline{u}^4|\alpha|^2 + \frac{1}{8}u^4|\underline{\alpha}|^2 + \frac{1}{2}(\underline{u}^4 + \frac{1}{2}\underline{u}^2u^2)|\beta|^2 \\
&+ \frac{1}{2}(u^4 + 4\underline{u}^2u^2 + u^4)(\rho^2+\sigma^2) + \frac{1}{2}(u^4 + \frac{1}{2}u^2\underline{u}^2)|\underline{\beta}|^2 \\
Q(\mathbf{R})(K_0,K_0,K_0,T) &= \frac{1}{8}u^6|\underline{\alpha}|^2 + \frac{1}{8}u^6|\underline{\alpha}|^2 + \frac{1}{4}u^4(u^2 + 3\underline{u}^2)|\underline{\beta}|^2 \qquad (3.5.3) \\
&+ \frac{3}{4}(\underline{u}^2 + u^2)\underline{u}^2u^2(\rho^2+\sigma^2) + \frac{1}{4}\underline{u}^4(\underline{u}^2 + 3u^2)|\beta|^2
\end{aligned}$$

Using the vector fields $\bar{K}, S, T$, and ${}^{(i)}O$, see 3.4.1, and also we define the following "Energy-type" norms

$$\begin{aligned}
\mathcal{Q}(\lambda,\nu) &= \mathcal{Q}_1(\lambda,\nu) + \mathcal{Q}_2(\lambda,\nu) \\
\underline{\mathcal{Q}}(\lambda,\nu) &= \underline{\mathcal{Q}}_1(\lambda,\nu) + \underline{\mathcal{Q}}_2(\lambda,\nu) \qquad (3.5.4)
\end{aligned}$$

where, defining

$$V(\lambda,\nu) = J^-(S(\lambda,\nu)) \tag{3.5.5}$$

$$\begin{aligned}
\mathcal{Q}_1(\lambda,\nu) \;&\equiv\; \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_4)\\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\mathbf{R})(\bar{K},\bar{K},T,e_4)\\
\mathcal{Q}_2(\lambda,\nu) \;&\equiv\; \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_4)\\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2\mathbf{R})(\bar{K},\bar{K},T,e_4) \qquad (3.5.6)\\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_4)
\end{aligned}$$

$$\begin{aligned}
\underline{\mathcal{Q}}_1(\lambda,\nu) \;&\equiv\; \sup_{V(\lambda,\nu)\cap\Sigma_0}|r^3\overline{\rho}|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_3)\\
&\quad + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\mathbf{R})(\bar{K},\bar{K},T,e_3)\\
\underline{\mathcal{Q}}_2(\lambda,\nu) \;&\equiv\; \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_3)\\
&\quad + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2\mathbf{R})(\bar{K},\bar{K},T,e_3) \qquad (3.5.7)\\
&\quad + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_3)
\end{aligned}$$

We also introduce the quantity

$$\mathcal{Q}_{\mathcal{K}} \;\equiv\; \sup_{\{\lambda,\nu|S(\lambda,\nu)\subset\mathcal{K}\}}\{\mathcal{Q}(\lambda,\nu)+\underline{\mathcal{Q}}(\lambda,\nu)\}. \tag{3.5.8}$$

Similarily on the initial hypersurface Σ_0 we define

$$\mathcal{Q}_{\Sigma_0\cap\mathcal{K}} = \sup_{\{\lambda,\nu|S(\lambda,\nu)\subset\mathcal{K}\}}\{\mathcal{Q}_{1\,\Sigma_0\cap V(\lambda,\nu)}+\mathcal{Q}_{2\,\Sigma_0\cap V(\lambda,\nu)}\} \tag{3.5.9}$$

where

$$\begin{aligned}
\mathcal{Q}_{1\,\Sigma_0\cap V(\lambda,\nu)} \;&\equiv\; \int_{\Sigma_0\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},T)+\\
&\quad + \int_{\Sigma_0\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\mathbf{R})(\bar{K},\bar{K},T,T)\\
&\quad + \sup_{\Sigma_0\cap V(\lambda,\nu)}|r^3\overline{\rho}|^2 \qquad (3.5.10)
\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_{2\Sigma_0\cap V(\lambda,\nu)} \equiv & \int_{\Sigma_0\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},T) \\ & + \int_{\Sigma_0\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2\mathbf{R})(\bar{K},\bar{K},T,T) \\ & + \int_{\Sigma_0\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},T)\end{aligned} \tag{3.5.11}$$

Remark: By comparison to the quantities $\mathcal{Q}$ and $\underline{\mathcal{Q}}$ defined in Chapter 2, see 2.2.14, 2.2.23, 2.2.24, we have included the integrals containing $Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},T)$. These terms are needed for estimating α_{44} and $\underline{\alpha}_{33}$ which are generated in the error estimates of Chapter 6. One can view the quantities $\mathcal{Q}$, $\underline{\mathcal{Q}}$ defined in 3.5.7, 3.5.8, as containing the smallest number of terms needed for the proof of Theorem 3.7.10 (Theorem **M8**).

3.5.2 $\mathcal{R}$ norms for the Riemann null components

In the course of the proof of the *Main Theorem* we have to use a complicated quantity defined as a sum of weighted $L^2(C)$, $L^2(\underline{C})$ norms of the Riemann curvature tensor and their derivatives up to second order. These weighted $L^2(C)$ and $L^2(\underline{C})$ norms are imtimately connected with the "Energy-type $\mathcal{Q}$ integrals", defined in subsection 3.5.1.

The quantity $\mathcal{R}$ which enters in the statement of the *Main Theorem* is

$$\mathcal{R} \equiv \mathcal{R}_{[2]} + \underline{\mathcal{R}}_{[2]} \tag{3.5.12}$$

where

$$\begin{aligned}\mathcal{R}_{[2]} &= \mathcal{R}_{[1]} + \mathcal{R}_2\ ,\ \underline{\mathcal{R}}_{[2]} = \underline{\mathcal{R}}_{[1]} + \underline{\mathcal{R}}_2 \\ \mathcal{R}_{[1]} &= \mathcal{R}_{[0]} + \mathcal{R}_1\ ,\ \underline{\mathcal{R}}_{[1]} = \underline{\mathcal{R}}_{[0]} + \underline{\mathcal{R}}_1 \\ \mathcal{R}_{[0]} &= \mathcal{R}_0\ ,\ \underline{\mathcal{R}}_{[0]} = \underline{\mathcal{R}}_0 + \sup_{\mathcal{K}} r^3|\bar{\rho}|\end{aligned} \tag{3.5.13}$$

with

$$\begin{aligned}\mathcal{R}_0 &= \left(\mathcal{R}_0[\alpha]^2 + \mathcal{R}_0[\beta]^2 + \mathcal{R}_0[(\rho,\sigma)]^2 + \mathcal{R}_0[\underline{\beta}]^2\right)^{1/2} \\ \underline{\mathcal{R}}_0 &= \left(\underline{\mathcal{R}}_0[\beta]^2 + \underline{\mathcal{R}}_0[(\rho,\sigma)]^2 + \underline{\mathcal{R}}_0[\underline{\beta}]^2 + \underline{\mathcal{R}}_0[\underline{\alpha}]^2\right)^{1/2} \\ \mathcal{R}_1 &= \left(\mathcal{R}_1[\alpha]^2 + \mathcal{R}_1[\beta]^2 + \mathcal{R}_1[(\rho,\sigma)]^2 + \mathcal{R}_1[\underline{\beta}]^2\right)^{1/2} \\ \underline{\mathcal{R}}_1 &= \left(\underline{\mathcal{R}}_1[\beta]^2 + \underline{\mathcal{R}}_1[(\rho,\sigma)]^2 + \underline{\mathcal{R}}_1[\underline{\beta}]^2 + \underline{\mathcal{R}}_1[\underline{\alpha}]^2\right)^{1/2}\end{aligned} \tag{3.5.14}$$

$$\mathcal{R}_2 = \left(\mathcal{R}_2[\alpha]^2 + \mathcal{R}_2[\beta]^2 + \mathcal{R}_2[(\rho,\sigma)]^2 + \mathcal{R}_2[\underline{\beta}]^2\right)^{1/2}$$
$$\underline{\mathcal{R}}_2 = \left(\underline{\mathcal{R}}_2[\beta]^2 + \underline{\mathcal{R}}_2[(\rho,\sigma)]^2 + \mathcal{R}_2[\underline{\beta}]^2 + \underline{\mathcal{R}}_2[\underline{\alpha}]^2\right)^{1/2}$$

and

$$\mathcal{R}_{0,1,2}[w] \equiv \sup_{\mathcal{K}} \mathcal{R}_{0,1,2}[w](\lambda,\nu)$$
$$\underline{\mathcal{R}}_{0,1,2}[w] \equiv \sup_{\mathcal{K}} \underline{\mathcal{R}}_{0,1,2}[w](\lambda,\nu).$$

The terms $\mathcal{R}_{0,1,2}[w](u,\underline{u})$ denote the L^2 norms of the zero, first and second derivatives of the null component w, along the portion of null hypersurface $C(\lambda)\cap V(\lambda,\nu)$. Recall that $V(\lambda,\nu) = J^-(S(\lambda,\nu))$ is the causal past of $S(\lambda,\nu)$ relative to $\mathcal{K}$, whose boundary is formed by the union of the portions of the null hypersurfaces $C(\lambda)$ and $\underline{C}(\nu)$ lying in $V(\lambda,\nu)$ and by $J^-(S(\lambda,\nu))\cap \Sigma_0$.
An analogous definition holds for the terms $\underline{\mathcal{R}}_{0,1,2}[w](\lambda,\nu)$ relative to the null hypersurface $\underline{C}(\nu)\cap V(\lambda,\nu)$. We write all of them explicitely below,

a) L^2 norms for the zero derivatives of the Riemann components:

$$\begin{aligned}
\mathcal{R}_0[\alpha](\lambda,\nu) &= \|r^2\alpha\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
\mathcal{R}_0[\beta](\lambda,\nu) &= \|r^2\beta\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
\mathcal{R}_0[(\rho,\sigma)](\lambda,\nu) &= \|\tau_- r(\rho-\overline{\rho},\sigma-\overline{\sigma})\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
\mathcal{R}_0[\underline{\beta}](\lambda,\nu) &= \|\tau_-^2\underline{\beta}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
\underline{\mathcal{R}}_0[\beta](\lambda,\nu) &= \|r^2\beta\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
\underline{\mathcal{R}}_0[(\rho,\sigma)](\lambda,\nu) &= \|r^2(\rho-\overline{\rho},\sigma-\overline{\sigma})\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
\underline{\mathcal{R}}_0[\underline{\beta}](\lambda,\nu) &= \|\tau_- r\underline{\beta}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
\underline{\mathcal{R}}_0[\underline{\alpha}](\lambda,\nu) &= \|\tau_-^2\underline{\alpha}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}
\end{aligned} \tag{3.5.15}$$

b) L^2 norms for the first derivatives of the Riemann components:

$$\begin{aligned}
\mathcal{R}_1[\alpha](\lambda,\nu) &= \|r^3\nabla\!\!\!/\alpha\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|r^3\alpha_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
&\quad +\|r^3\alpha_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
\mathcal{R}_1[\beta](\lambda,\nu) &= \|r^3\nabla\!\!\!/\beta\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^2\beta_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
&\quad +\|r^3\beta_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
\mathcal{R}_1[(\rho,\sigma)](\lambda,\nu) &= \|\tau_- r^2\nabla\!\!\!/(\rho,\sigma)\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|r^3(\rho,\sigma)_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
&\quad +\|r\tau_-^2(\rho,\sigma)_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{R}_1[\underline{\beta}](\lambda,\nu) &= \|\tau_-^2 r \nabla\!\!\!/ \underline{\beta}\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^2 \underline{\beta}_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
\underline{\mathcal{R}}_1[\beta](\lambda,\nu) &= \|r^3 \nabla\!\!\!/ \beta\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|r^3 \beta_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
\underline{\mathcal{R}}_1[(\rho,\sigma)](\lambda,\nu) &= \|r^3 \nabla\!\!\!/ (\rho,\sigma)\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_- r^2 (\rho,\sigma)_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
\underline{\mathcal{R}}_1[\underline{\beta}](\lambda,\nu) &= \|\tau_- r^2 \nabla\!\!\!/ \underline{\beta}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_-^2 r \underline{\beta}_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
&\quad + \|r^3 \underline{\beta}_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
\underline{\mathcal{R}}_1[\underline{\alpha}](\lambda,\nu) &= \|\tau_-^2 r \nabla\!\!\!/ \underline{\alpha}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_-^3 \underline{\alpha}_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
&\quad + \|\tau_- r^2 \underline{\alpha}_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}
\end{aligned}
\tag{3.5.16}
$$

c) L^2 norms for the second derivatives of the Riemann components:

$$
\begin{aligned}
\mathcal{R}_2[\alpha](\lambda,\nu) &= \|r^4 \nabla\!\!\!/^2 \alpha\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|r^4 \nabla\!\!\!/ \alpha_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
&\quad + \|r^4 \nabla\!\!\!/ \alpha_4\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^3 \alpha_{33}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
&\quad + \|r^4 \alpha_{34}\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^3 \alpha_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
\mathcal{R}_2[\beta](\lambda,\nu) &= \|r^4 \nabla\!\!\!/^2 \beta\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^3 \nabla\!\!\!/ \beta_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
&\quad + \|r^4 \nabla\!\!\!/ \beta_4\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_-^2 r^2 \beta_{33}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
&\quad + \|r^4 \beta_{34}\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|r^4 \beta_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
\mathcal{R}_2[(\rho,\sigma)](u,\underline{u}) &= \|\tau_- r^3 \nabla\!\!\!/^2 (\rho,\sigma)\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_-^2 r^2 \nabla\!\!\!/ (\rho,\sigma)_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
&\quad + \|r^4 \nabla\!\!\!/ (\rho,\sigma)_4\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^3 (\rho,\sigma)_{34}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
&\quad + \|r^4 (\rho,\sigma)_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_-^3 r (\rho,\sigma)_{33}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
\mathcal{R}_2[\underline{\beta}](\lambda,\nu) &= \|\tau_-^2 r^2 \nabla\!\!\!/^2 \underline{\beta}\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^3 \nabla\!\!\!/ \underline{\beta}_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
&\quad + \|\tau_-^3 r \nabla\!\!\!/ \underline{\beta}_3\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_-^2 r^2 \underline{\beta}_{34}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\
&\quad + \|r^4 \underline{\beta}_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_-^3 r \underline{\beta}_{33}\|_{2,C(\lambda)\cap V(\lambda,\nu)}
\end{aligned}
\tag{3.5.17}
$$

$$
\begin{aligned}
\underline{\mathcal{R}}_2[\beta](\lambda,\nu) &= \|r^4 \nabla\!\!\!/^2 \beta\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|r^4 \nabla\!\!\!/ \beta_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
&\quad + \|r^4 \nabla\!\!\!/ \beta_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|r^4 \beta_{43}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
&\quad + \|\tau_- r^3 \beta_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_- r^3 \beta_{43}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
\underline{\mathcal{R}}_2[(\rho,\sigma)](\lambda,\nu) &= \|r^4 \nabla\!\!\!/^2 (\rho,\sigma)\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_- r^3 \nabla\!\!\!/ (\rho,\sigma)_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
&\quad + \|r^4 \nabla\!\!\!/ (\rho,\sigma)_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_- r^3 (\rho,\sigma)_{34}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
&\quad + \|\tau_-^2 r^2 (\rho,\sigma)_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|r^4 (\rho,\sigma)_{44}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
\underline{\mathcal{R}}_2[\underline{\beta}](\lambda,\nu) &= \|\tau_- r^3 \nabla\!\!\!/^2 \underline{\beta}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_-^2 r^2 \nabla\!\!\!/ \underline{\beta}_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
&\quad + \|r^4 \nabla\!\!\!/ \underline{\beta}_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_-^3 r \underline{\beta}_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\
&\quad + \|\tau_- r^3 \underline{\beta}_{34}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|r^4 \underline{\beta}_{44}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}
\end{aligned}
$$

$$\begin{aligned}\underline{\mathcal{R}}_2[\underline{\alpha}](\lambda,\nu) &= \|\tau_-^2 r^2 \nabla\!\!\!\!/^2 \underline{\alpha}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_-^3 r \nabla\!\!\!\!/ \underline{\alpha}_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\ &+\|\tau_- r^3 \nabla\!\!\!\!/ \underline{\alpha}_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_-^4 \underline{\alpha}_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\ &+\|\tau_-^2 r^2 \underline{\alpha}_{34}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_- r^3 \underline{\alpha}_{44}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\end{aligned}$$

Remark: The explicit definitions of $\alpha_3, \beta_4, \beta_3$.... were given in 3.2.8. The explicit definitions of $\alpha_{33}, \alpha_{34}, \beta_{34}$.... are given in Chapter 5.

In additions to the basic norms defined above we shall need some other curvature norms,

$$\begin{aligned}\mathcal{R}_0^\infty &\equiv \sup_{\mathcal{K}} \left[|r^{7/2}\alpha| + |r^{7/2}\beta| + |r^3 u^{\frac{1}{2}}(\rho-\overline{\rho},\sigma)|\right]\\ \underline{\mathcal{R}}_0^\infty &\equiv \sup_{\mathcal{K}} \left[|r^3\rho| + |r^2 u^{\frac{3}{2}}\underline{\beta}| + |r u^{\frac{5}{2}}\underline{\alpha}|\right] \qquad (3.5.18)\end{aligned}$$

$$\mathcal{R}_1^S = \sup_{p\in[2,4]} \mathcal{R}_1^{p,S}\ ,\ \underline{\mathcal{R}}_1^S = \sup_{p\in[2,4]} \underline{\mathcal{R}}_1^{p,S} \qquad (3.5.19)$$

where

$$\begin{aligned}\mathcal{R}_1^{p,S} &= \sup_{\mathcal{K}} \Big[|r^{\frac{9}{2}-\frac{2}{p}} \nabla\!\!\!\!/ \alpha|_{p,S(\lambda,\nu)} + |r^{\frac{9}{2}-\frac{2}{p}} \nabla\!\!\!\!/ \beta|_{p,S(u,\underline{u})} + |r^{4-\frac{2}{p}} u^{\frac{1}{2}} \nabla\!\!\!\!/ (\rho,\sigma)|_{p,S(\lambda,\nu)}\\ &\quad+|r^{\frac{9}{2}-\frac{2}{p}} \mathbf{D}\!\!\!\!/_4 \alpha|_{p,S(\lambda,\nu)}\Big] \qquad (3.5.20)\\ \underline{\mathcal{R}}_1^{p,S} &= \Big[|r^{4-\frac{2}{p}} u^{\frac{1}{2}} \nabla\!\!\!\!/ \underline{\beta}|_{p,S(\lambda,\nu)} + |r^{1-\frac{2}{p}} u^{\frac{5}{2}} \nabla\!\!\!\!/ \underline{\alpha}|_{p,S(\lambda,\nu)} + |r^{1-\frac{2}{p}} u^{\frac{5}{2}} \mathbf{D}\!\!\!\!/_3 \underline{\alpha}|_{p,S(\lambda,\nu)}\Big]\end{aligned}$$

The estimates for thes auxiliary norms can be obtained, using global Sobolev inequalities, in terms of the $\mathcal{R}$ norms introduced above [29].

3.5.3 $\mathcal{O}$ norms for the connection coefficients

In addition to the quantities $\mathcal{Q}$ and $\mathcal{R}$ the other basic quantity which enters in the statement of the *Main Theorem* is

$$\mathcal{O} \equiv \mathcal{O}_{[3]} + \underline{\mathcal{O}}_{[3]} \qquad (3.5.21)$$

where

$$\begin{aligned}\mathcal{O}_{[3]} &\equiv \left[\mathcal{O}_3 + \mathcal{O}_{[2]}\right]\\ \mathcal{O}_{[2]} &\equiv \left[\mathcal{O}_2 + \tilde{\mathcal{O}}_2(\underline{\omega})\right] + \mathcal{O}_{[1]} \qquad (3.5.22)\\ \mathcal{O}_{[1]} &\equiv \left[\mathcal{O}_1 + \sup_{p\in[2,4]} \tilde{\mathcal{O}}_1(\underline{\omega})\right] + \mathcal{O}_{[0]}^\infty\\ \mathcal{O}_{[0]}^\infty &\equiv \mathcal{O}_0^\infty + \sup_{\mathcal{K}} |r^2(\overline{\mathrm{tr}\chi} - \frac{2}{r})| + \sup_{\mathcal{K}} |r(\Omega - \frac{1}{2})|\end{aligned}$$

[29]With the exception of $\sup_{\mathcal{K}} |r^3\rho|$ in $\underline{\mathcal{R}}_0^\infty$.

and analogous definitions for the underlined quantities [30]

$$
\begin{aligned}
\underline{\mathcal{O}}_{[3]} &\equiv \left[\underline{\mathcal{O}}_3 + \underline{\mathcal{O}}_{[2]}\right] \\
\underline{\mathcal{O}}_{[2]} &\equiv \left[\underline{\mathcal{O}}_2 + \sup_{p\in[2,4]} \mathcal{O}_2^{p,S}(\omega)\right] + \underline{\mathcal{O}}_{[1]} \\
\underline{\mathcal{O}}_{[1]} &\equiv \left[\underline{\mathcal{O}}_1 + \sup_{p\in[2,4]} \mathcal{O}_0^{p,S}(\mathbf{D}_4\omega)\right] + \underline{\mathcal{O}}_{[0]}^{\infty} \\
\underline{\mathcal{O}}_{[0]}^{\infty} &\equiv \underline{\mathcal{O}}_0^{\infty} + \sup_{\mathcal{K}} |r\tau_-(\overline{\mathrm{tr}\underline{\chi}} + \frac{2}{r})|
\end{aligned} \tag{3.5.23}
$$

[Macros error (corrected) in 3.5.23]

The explicit forms of the various quantities in 3.5.22, 3.5.23 can be derived from the following definitions:

$$
\begin{aligned}
&\mathcal{O}_{0,1,2} \equiv \sup_{p\in[2,4]} \mathcal{O}_{0,1,2}^{p} \quad , \quad \underline{\mathcal{O}}_{0,1,2} \equiv \sup_{p\in[2,4]} \underline{\mathcal{O}}_{0,1,2}^{p} \\
&\mathcal{O}_3 \equiv \mathcal{O}_3^{p=2} \quad , \quad \underline{\mathcal{O}}_3 \equiv \underline{\mathcal{O}}_3^{p=2}
\end{aligned} \tag{3.5.24}
$$

The $\mathcal{O}_q^p$ norms have, for $q = 0, 1, 2$, the following expressions which depending on different connection coefficients:

$$
\begin{aligned}
\mathcal{O}_q^p &= \mathcal{O}_q^{p,S}(\mathrm{tr}\chi) + \mathcal{O}_q^{p,S}(\hat{\chi}) + \mathcal{O}_q^{p,S}(\eta) + \mathcal{O}_q^{p,S}(\underline{\omega}) \\
\underline{\mathcal{O}}_q^p &= \mathcal{O}_q^{p,S}(\mathrm{tr}\underline{\chi}) + \mathcal{O}_q^{p,S}(\underline{\hat{\chi}}) + \mathcal{O}_q^{p,S}(\underline{\eta}) + \mathcal{O}_q^{p,S}(\omega)
\end{aligned} \tag{3.5.25}
$$

In the $q = 3$ case we define

$$
\begin{aligned}
\mathcal{O}_3^{p=2} &= \mathcal{O}_3^{p=2}(\mathrm{tr}\chi) + \mathcal{O}_3^{p=2}(\hat{\chi}) + \mathcal{O}_3^{p=2}(\eta) + \mathcal{O}_3^{p=2}(\underline{\omega}) \\
\underline{\mathcal{O}}_3^{p=2} &= \mathcal{O}_3^{p=2}(\mathrm{tr}\underline{\chi}) + \mathcal{O}_3^{p=2}(\underline{\hat{\chi}}) + \mathcal{O}_3^{p=2}(\underline{\eta})
\end{aligned} \tag{3.5.26}
$$

Remark: We have sistematically assigned to the $\mathcal{O}$ norms those quantities which are estimated along the C null hypersurfaces and by $\underline{\mathcal{O}}$ norms those which are estimated along the $\underline{C}$ null hypresurfaces.

For an arbitrary connection coefficient X, we have,

$$
\mathcal{O}_q^{p,S}(X) \equiv \sup_{\mathcal{K}} \mathcal{O}_q^{p,S}(X)(\lambda,\nu) \ , \ \mathcal{O}_3^{p=2}(X) \equiv \sup_{\mathcal{K}} \mathcal{O}_3^{p=2}(X)(\lambda,\nu) \tag{3.5.27}
$$

The explicit expressions of all these norms are given in the following definitions:

[30] The reason why in the underlined norms the terms $\tilde{\mathcal{O}}_2(\omega)$ and $\tilde{\mathcal{O}}_1(\omega)$, analogous to $\tilde{\mathcal{O}}_2(\underline{\omega})$ and $\tilde{\mathcal{O}}_1(\underline{\omega})$, do not appear in the underlined norms 3.5.23, is discussed in Chapter 6, see subsections 6.3.5 and 6.3.7.

$q \leq 2:$

$$\begin{aligned}
\mathcal{O}_q^{p,S}(tr\chi)(\lambda,\nu) &= |r^{(2+q-\frac{2}{p})}\nabla\!\!\!\!/^q(\mathrm{tr}\chi-\overline{tr\chi})|_{p,S(\lambda,\nu)}\\
\mathcal{O}_q^{p,S}(tr\underline{\chi})(\lambda,\nu) &= |r^{(2+q-\frac{2}{p})}\nabla\!\!\!\!/^q(\mathrm{tr}\underline{\chi}-\overline{tr\underline{\chi}})|_{p,S(\lambda,\nu)}\\
\mathcal{O}_q^{p,S}(\hat{\chi})(\lambda,\nu) &= |r^{(2+q-\frac{2}{p})}\nabla\!\!\!\!/^q\hat{\chi}|_{p,S(\lambda,\nu)}\\
\mathcal{O}_q^{p,S}(\underline{\hat{\chi}})(\lambda,\nu) &= |r^{(1+q-\frac{2}{p})}\tau_-\nabla\!\!\!\!/^q\underline{\hat{\chi}}|_{p,S(\lambda,\nu)} \qquad (3.5.28)\\
\mathcal{O}_q^{p,S}(\eta)(\lambda,\nu) &= |r^{(2+q-\frac{2}{p})}\nabla\!\!\!\!/^q\eta|_{p,S(\lambda,\nu)}\\
\mathcal{O}_q^{p,S}(\underline{\eta})(\lambda,\nu) &= |r^{(2+q-\frac{2}{p})}\nabla\!\!\!\!/^q\underline{\eta}|_{p,S(\lambda,\nu)}\\
\mathcal{O}_q^{p,S}(\omega)(\lambda,\nu) &= |r^{(2+q-\frac{2}{p})}\nabla\!\!\!\!/^q\omega|_{p,S(\lambda,\nu)}\\
\mathcal{O}_q^{p,S}(\underline{\omega})(\lambda,\nu) &= |r^{(1+q-\frac{2}{p})}\tau_-\nabla\!\!\!\!/^q\underline{\omega}|_{p,S(\lambda,\nu)}\\
\mathcal{O}_q^{p,S}(\mathbf{D}_4\omega)(\lambda,\nu) &= |r^{(3+q-\frac{2}{p})}\nabla\!\!\!\!/^q\mathbf{D}_4\omega|_{p,S(\lambda,\nu)}\\
\mathcal{O}_q^{p,S}(\mathbf{D}_3\underline{\omega})(\lambda,\nu) &= |r^{(1+q-\frac{2}{p})}\tau_-^2\nabla\!\!\!\!/^q\mathbf{D}_3\underline{\omega}|_{p,S(\lambda,\nu)}
\end{aligned}$$

$q = 3:$

$$\begin{aligned}
\mathcal{O}_3^{p=2}(\hat{\chi})(\lambda,\nu) &= r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!\!/^3\hat{\chi}||_{L^2(\underline{C}(\nu;[\lambda_0,\lambda])}\\
\mathcal{O}_3^{p=2}(\mathrm{tr}\chi)(\lambda,\nu) &= r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!\!/^3\mathrm{tr}\chi||_{L^2(\underline{C}(\nu;[\lambda_0,\lambda])}\\
\mathcal{O}_3^{p=2}(\eta)(\lambda,\nu) &= r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!\!/^3\eta||_{L^2(\underline{C}(\nu;[\lambda_0,\lambda])}\\
\mathcal{O}_3^{p=2}(\underline{\omega})(\lambda,\nu) &= r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!\!/^3\underline{\omega}||_{L^2(\underline{C}(\nu;[\lambda_0,\lambda])} \qquad (3.5.29)\\
\mathcal{O}_3^{p=2}(\underline{\hat{\chi}})(\lambda,\nu) &= r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!\!/^3\underline{\hat{\chi}}||_{L^2(\underline{C}(\nu;[\lambda_0,\lambda])}\\
\mathcal{O}_3^{p=2}(\mathrm{tr}\underline{\chi})(\lambda,\nu) &= r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!\!/^3\mathrm{tr}\underline{\chi}||_{L^2(\underline{C}(\nu;[\lambda_0,\lambda])}\\
\mathcal{O}_3^{p=2}(\underline{\eta})(\lambda,\nu) &= r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!\!/^3\underline{\eta}||_{L^2(\underline{C}(\nu;[\lambda_0,\lambda])}
\end{aligned}$$

We also define the norms $\tilde{\mathcal{O}}_1(\underline{\omega})$, $\tilde{\mathcal{O}}_2(\underline{\omega})$. They involve the second null and mixed derivatives of $\underline{\omega}$ and will be needed in the proof of the *Main Theorem*,

$$\begin{aligned}
\tilde{\mathcal{O}}_1(\underline{\omega}) &\equiv ||\frac{1}{\sqrt{\tau_+}}\tau_-^2\mathbf{D}_3\underline{\omega}||_{L_2(C\cap\mathcal{K})} \qquad (3.5.30)\\
\tilde{\mathcal{O}}_2(\underline{\omega}) &\equiv ||\frac{1}{\sqrt{\tau_+}}r\tau_-^2\nabla\!\!\!\!/\mathbf{D}_3\underline{\omega}||_{L_2(C\cap\mathcal{K})}+||\frac{1}{\sqrt{\tau_+}}\tau_-^3\mathbf{D}_3^2\underline{\omega}||_{L_2(C\cap\mathcal{K})}
\end{aligned}$$

3.5.4 Norms on the initial layer region

In addition to the $\mathcal{O}$, $\mathcal{R}$ norms expressed relative to the canonical double null foliation [31] we shall also need similar norms defined in the initial layer region ${\mathcal{K}'}_{\delta_0}$, relative to the initial layer foliation. These norms have exactly the same expressions, to distinguish them from the main ones defined above we denote them by $\mathcal{O}'$, $\mathcal{R}'$.

3.5.5 $\mathcal{O}$ norms on the initial and final hypersurfaces

We consider now the previous $\mathcal{O}$ norms restricted [32] to the initial hypersurface, Σ_0, and to the last slice, $\underline{C}_*$. Observe that the $\mathcal{O}$ norms restricted to the initial hypersurface are tied to the *initial layer foliation* while those restricted to the last slice $\underline{C}_*$ are tied to the *canonical double null foliation.* The only difference with respect to the previous definitions is that some of the previous norms are absent as their restrictions on $\underline{C}_*$ or on Σ_0 are not needed.

$$
\begin{aligned}
&\mathcal{O}_{[1]}(\Sigma_0) \equiv \mathcal{O}_1(\Sigma_0) + \mathcal{O}^\infty_{[0]}(\Sigma_0)\\
&\underline{\mathcal{O}}_{[1]}(\Sigma_0) \equiv \left[\underline{\mathcal{O}}_1(\Sigma_0) + \sup_{p\in[2,4]} \mathcal{O}^{p,S}_0(\mathbf{D}_4\omega)(\Sigma_0)\right] + \underline{\mathcal{O}}^\infty_{[0]}(\Sigma_0)\\
&\mathcal{O}_{[2]}(\Sigma_0) = \mathcal{O}_2(\Sigma_0) + \mathcal{O}_{[1]}(\Sigma_0)\\
&\underline{\mathcal{O}}_{[2]}(\Sigma_0) = \left[\underline{\mathcal{O}}_2(\Sigma_0) + \sup_{p\in[2,4]} \left(\mathcal{O}^{p,S}_1(\mathbf{D}_4\omega) + \mathcal{O}^{p,S}_0(\mathbf{D}^2_4\omega)(\Sigma_0)\right)\right] + \underline{\mathcal{O}}_{[1]}(\Sigma_0)\\
&\mathcal{O}_{[1]}(\underline{C}_*) \equiv \left[\mathcal{O}_1(\underline{C}_*) + \sup_{p\in[2,4]} \mathcal{O}^{p,S}_0(\mathbf{D}_3\underline{\omega})(\underline{C}_*)\right] + \mathcal{O}^\infty_{[0]}(\underline{C}_*)\\
&\underline{\mathcal{O}}_{[1]}(\underline{C}_*) \equiv \underline{\mathcal{O}}_1(\underline{C}_*) + \underline{\mathcal{O}}^\infty_{[0]}(\underline{C}_*) \qquad (3.5.31)\\
&\mathcal{O}_{[2]}(\underline{C}_*) = \left[\mathcal{O}_2(\underline{C}_*) + \sup_{p\in[2,4]} \left(\mathcal{O}^{p,S}_1(\mathbf{D}_3\underline{\omega}) + \mathcal{O}^{p,S}_0(\mathbf{D}^2_3\underline{\omega})(\underline{C}_*)\right)\right] + \mathcal{O}_{[1]}(\underline{C}_*)\\
&\underline{\mathcal{O}}_{[2]}(\underline{C}_*) = \underline{\mathcal{O}}_2(\underline{C}_*) + \underline{\mathcal{O}}_{[1]}(\underline{C}_*)
\end{aligned}
$$

[Macroserror(corrected)in3.5.31]

where, for $q \leq 2$,

$$
\begin{aligned}
\mathcal{O}^{p,S}_q(\underline{C}_*)(X) &\equiv \sup_{\underline{C}_*} \mathcal{O}^{p,S}_q(X)(\lambda,\nu)\\
\mathcal{O}^{p,S}_q(\Sigma_0)(\underline{X}) &\equiv \sup_{\Sigma_0} \mathcal{O}^{p,S}_q(\underline{X})(\lambda,\nu) \qquad (3.5.32)
\end{aligned}
$$

[31] In Chapter 4, subsection 4.1.3, there is an accurate discussion about the region where these norms are defined.

[32] Although we refer here to norms on Σ_0 in reality we shall need these norms on $\Sigma_0 \backslash K$.

Finally we introduce the $\mathcal{O}_3(\underline{C}_*)$ and the $\underline{\mathcal{O}}_3(\Sigma_0)$ norms on the initial and final slice. They are defined in the following way:

$$\begin{aligned}\mathcal{O}_3(\underline{C}_*) &= \mathcal{O}_3(\underline{C}_*)(\mathrm{tr}\chi) + \mathcal{O}_3(\underline{C}_*)(\underline{\omega})\\ \underline{\mathcal{O}}_3(\Sigma_0) &= \mathcal{O}_3(\Sigma_0)(\mathrm{tr}\underline{\chi}) + \mathcal{O}_3(\Sigma_0)(\omega)\end{aligned} \tag{3.5.33}$$

where

$$\begin{aligned}\mathcal{O}_3(\underline{C}_*)(\mathrm{tr}\chi) &= r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!/^3\mathrm{tr}\chi||_{L^2(\underline{C}_*\cap V(\lambda,\nu))}\\ \mathcal{O}_3(\underline{C}_*)(\underline{\omega}) &= r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!/^3\underline{\omega}||_{L^2(\underline{C}_*\cap V(\lambda,\nu))}\\ \mathcal{O}_3(\Sigma_0)(\mathrm{tr}\underline{\chi}) &= r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!/^3\mathrm{tr}\underline{\chi}||_{L^2(\Sigma_0\cap V(\lambda,\nu))}\\ \mathcal{O}_3(\Sigma_0)(\omega) &= r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!/^3\omega||_{L^2(\Sigma_0\cap V(\lambda,\nu))}\end{aligned} \tag{3.5.34}$$

3.5.6 $\mathcal{D}$ norms for the rotation deformation tensors

We introduce the quantity

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 + \mathcal{D}_2 \tag{3.5.35}$$

where

$$\mathcal{D}_{0,1} \equiv \sup_{p\in[2,4]} \mathcal{D}^p_{0,1} \tag{3.5.36}$$

and

$$\mathcal{D}^p_0 = \mathcal{D}^{p,S}_0({}^{(i)}O) + \mathcal{D}^{p,S}_0({}^{(i)}F) + \mathcal{D}^{p,S}_0({}^{(i)}H) + \mathcal{D}^{p,S}_0({}^{(i)}Z) \tag{3.5.37}$$

$$\begin{aligned}\mathcal{D}^p_1 = \ & \mathcal{D}^{p,S}_1({}^{(i)}O) + \mathcal{D}^{p,S}_1({}^{(i)}F) + \mathcal{D}^{p,S}_1({}^{(i)}H) + \mathcal{D}^{p,S}_1({}^{(i)}Z)\\ &+\mathcal{D}^{p,S}_0(\mathbf{D}_3{}^{(i)}F) + \mathcal{D}^{p,S}_0(\mathbf{D}_3{}^{(i)}H) + \mathcal{D}^{p,S}_0(\mathbf{D}_3{}^{(i)}Z)\\ &+\mathcal{D}^{p,S}_0(\mathbf{D}_4{}^{(i)}F) + \mathcal{D}^{p,S}_0(\mathbf{D}_4{}^{(i)}H) + \mathcal{D}^{p,S}_0(\mathbf{D}_4{}^{(i)}Z)\end{aligned} \tag{3.5.38}$$

Finally, denotinjg by X any of the quantities ${}^{(i)}O$, ${}^{(i)}F$, ${}^{(i)}H$ and ${}^{(i)}Z$ and their derivatives,

$$\mathcal{D}^{p,S}_q(X) \equiv \sup_{\mathcal{K}} \mathcal{D}^{p,S}_q(X)(\lambda,\nu) \tag{3.5.39}$$

Explicitely we define:

$$\begin{aligned}\mathcal{D}^{p,S}_q({}^{(i)}O)(\lambda,\nu) &= |r^{(-1+q-\frac{2}{p})}\nabla\!\!\!/^{q\,(i)}O|_{p,S}(\lambda,\nu)\\ \mathcal{D}^{p,S}_q({}^{(i)}F)(\lambda,\nu) &= |r^{(1+q-\frac{2}{p})}\nabla\!\!\!/^{q\,(i)}F|_{p,S}(\lambda,\nu)\\ \mathcal{D}^{p,S}_q({}^{(i)}H)(\lambda,\nu) &= |r^{(1+q-\frac{2}{p})}\nabla\!\!\!/^{q\,(i)}H|_{p,S}(\lambda,\nu)\\ \mathcal{D}^{p,S}_q({}^{(i)}Z)(\lambda,\nu) &= |r^{(1+q-\frac{2}{p})}\nabla\!\!\!/^{q\,(i)}Z|_{p,S}(\lambda,\nu)\end{aligned} \tag{3.5.40}$$

$$\begin{aligned}
\mathcal{D}_q^{p,S}(\mathbf{D}_4{}^{(i)}F)(\lambda,\nu) &= |r^{(2+q-\frac{2}{p})}\nabla\!\!\!/^q\mathbf{D}_4{}^{(i)}F|_{p,S(\lambda,\nu)}\\
\mathcal{D}_q^{p,S}(\mathbf{D}_4{}^{(i)}H)(\lambda,\nu) &= |r^{(2+q-\frac{2}{p})}\nabla\!\!\!/^q\mathbf{D}_4{}^{(i)}H|_{p,S(\lambda,\nu)}\\
\mathcal{D}_q^{p,S}(\mathbf{D}_4{}^{(i)}Z)(\lambda,\nu) &= |r^{(2+q-\frac{2}{p})}\nabla\!\!\!/^q\mathbf{D}_4{}^{(i)}Z|_{p,S(\lambda,\nu)} \qquad (3.5.41)
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_q^{p,S}(\mathbf{D}_3{}^{(i)}F)(\lambda,\nu) &= |r^{(1+q-\frac{2}{p})}\tau_-\nabla\!\!\!/^q\mathbf{D}_3{}^{(i)}F|_{p,S(\lambda,\nu)}\\
\mathcal{D}_q^{p,S}(\mathbf{D}_3{}^{(i)}H)(\lambda,\nu) &= |r^{(1+q-\frac{2}{p})}\tau_-\nabla\!\!\!/^q\mathbf{D}_3{}^{(i)}H|_{p,S(\lambda,\nu)}\\
\mathcal{D}_q^{p,S}(\mathbf{D}_3{}^{(i)}Z)(\lambda,\nu) &= |r^{(1+q-\frac{2}{p})}\tau_-\nabla\!\!\!/^q\mathbf{D}_3{}^{(i)}Z|_{p,S(\lambda,\nu)} \qquad (3.5.42)
\end{aligned}$$

Their restrictions on the last slice $\underline{C}_*$[33] will be denoted by

$$\mathcal{D}_q^{p,S}(\underline{C}_*)(X) \equiv \sup_{\underline{C}_*}\mathcal{D}_q^{p,S}(X)(\lambda,\nu) \qquad (3.5.43)$$

It remains to define the norms appearing in $\mathcal{D}_2$. They are the more delicate ones as they depend on the third derivatives of the connection coefficients.

$$\mathcal{D}_2 = \sup_{\mathcal{K}}\mathcal{D}_2(\lambda,\nu) \qquad (3.5.44)$$

where, with $\epsilon > 0$,

$$\begin{aligned}
\mathcal{D}_2(\lambda,\nu) &= ||r\nabla\!\!\!/^2H||_{L^2(\underline{C}(\nu)\cap V(\lambda,\nu))} + ||r\nabla\!\!\!/^2Z||_{L^2(\underline{C}(\nu)\cap V(\lambda,\nu))}\\
&+ ||\frac{1}{\sqrt{r^{1-2\epsilon}}}r\nabla\!\!\!/\mathbf{D}\!\!\!/_3Z||_{L^2(\underline{C}(\nu)\cap V(\lambda,\nu))} \qquad (3.5.45)
\end{aligned}$$

Finally we denote $\mathcal{D}_2(\underline{C}_*) = \sup_{\underline{C}_*}\mathcal{D}_2(\lambda,\nu)$, its restriction on $\underline{C}_*$.

$\mathcal{D}$ norms relative to the *initial layer foliation*

In addition to the $\mathcal{D}$ norms expressed relative to the canonical double null foliation we shall aslo need similar norms defined, in the initial layer $\mathcal{K}'_{\delta_0}$, relative to the initial layer foliation. These norms have exactly the same expression, to distinguish them from the main ones defined above we denote them by $\mathcal{D}'$.

[33]For the deformation tensor norms we need only the restriction on the last slice, see for more details Chapter 4 and Chapter 6.

3.6 The initial data

3.6.1 Global initial data conditions

We restrict ourselves to initial data sets $\{\Sigma_0, g, k\}$ with Σ_0 diffeomorphic to R^3; moreover we assume they are strongly asymptotically flat in the following sense [34]

Definition 3.6.1 *An initial data set $\{\Sigma_0, g, k\}$ is "strongly asymptotically flat", see [Ch-Kl], eqs. (1.0.9a), (1.0.9b), if there exists a compact set B, such that its complement $\Sigma_0 \backslash B$ is diffeomorphic to the complement of the closed unit ball in R^3. Moreover there exists a coordinate system (x^1, x^2, x^3) defined in a neighborhood of infinity such that, as $r = \sqrt{\sum_{i=1}^3 (x^i)^2} \to \infty$, we have* [35]

$$\begin{aligned} g_{ij} &= (1 + 2M/r)\delta_{ij} + o_4(r^{-\frac{3}{2}}) \\ k_{ij} &= +o_3(r^{-\frac{5}{2}}) \end{aligned} \tag{3.6.1}$$

Remark: The *"Strong asymptotic flatness"*, guarantees that the ADM energy, linear momentum P_i and angular momentum J_i are well defined. Moreover from eq. 3.6.1 it follows that $P_i = 0;\ \ i \in \{1, 2, 3\}$, implying that we are placing ourselves in a center of mass frame. In this case $E = M$ and M is an invariant quantity greater than zero, due to the "positive energy theorem", [Sc-Yau1], [Sc-Yau2] and finite, see also [Ch-Kl] page 11.

In [Ch-Kl] the global smallness condition was defined with the help of the following quantity,

$$\begin{aligned} J_0(\{\Sigma_0, g, k\}; b) \;\;=\;\; & \sup_{\Sigma_0} \left\{ b^{-2}(d_0^2 + b^2)^3 |Ric|^2 \right\} + b^{-3} \left\{ \int_{\Sigma_0} \sum_{l=0}^{3} (d_0^2 + b^2)^{l+1} |\nabla^l k|^2 \right. \\ & \left. + \int_{\Sigma_0} \sum_{l=0}^{1} (d_0^2 + b^2)^{l+3} |\nabla^l B|^2 \right\} \end{aligned} \tag{3.6.2}$$

[34] Observe that this definition is stronger than the usual definition of an "asymptotically flat" initial data set : *An initial data set $\{\Sigma_0, g, k\}$ is "asymptotically flat" if there exists a compact set B, such that its complement $\Sigma_0 \backslash B$ is diffeomorphic to the complement of the closed unit ball in R^3. Moreover there exists a coordinate system (x^1, x^2, x^3) defined in a neighborhood of infinity such that as $r = \sqrt{\sum_{i=1}^3 (x^i)^2} \to \infty$ we have*

$$g_{ij} = (1 + 2M/r)\delta_{ij} + o(r^{-1}) \ .$$

[35] A function f is $o_m(r^{-k})$ if $\partial_l f = o(r^{-k-l})$ for $l = 0, 1, .., m$.

where d_0 is the Riemann geodesic distance from a fixed point O on Σ_0, Ric is the Ricci tensor and B is the tensor $B_{ij} = (curl\hat{R})_{ij}$, where $\hat{R}$ is the traceless part of the Ricci tensor. ∇ is the covariant derivative with respect to the metric on Σ_0 and b is a positive constant with the dimension of a length.

The g_{ij} components of the metric tensor have dimension zero in length unities, $[g_{ij}] = L^0$ and, from it, immediately, $[(Ricci)_{ij}] = L^{-2}$, $[B_{ij}] = L^{-3}$ and the extrinsic curvature k, $[k_{ij}] = L^{-1}$.
As b has the dimension of a length we can consider it as a "natural" length unit and define the new coordinates $\tilde{x}$, in these units, as $x = \tilde{x}b$. In these new coordinates the second and third terms of $J_0(\{\Sigma_0, g, k\}; b)$ do not change, that is they are invariant under the rescaling [36] $x \to \tilde{x} = \frac{x}{b}$. In fact we have

$$b^{-3}\left\{\int_{\Sigma_0}\sum_{l=0}^{3}(d_0^2+b^2)^{l+1}|\nabla^l k|^2 + \int_{\Sigma_0}\sum_{l=0}^{1}(d_0^2+b^2)^{l+3}|\nabla^l B|^2\right\}$$
$$= \int_{\Sigma_0}\sum_{l=0}^{3}(\tilde{d}_0^2+1)^{l+1}|\tilde{\nabla}^l\tilde{k}|^2 + \int_{\Sigma_0}\sum_{l=0}^{1}(\tilde{d}_0^2+1)^{l+3}|\tilde{\nabla}^l\tilde{B}|^2 \qquad (3.6.3)$$

The situation is different from the first term due to presence in Ric of a part depending on the mass M, see 3.6.1.
In fact the term $\{b^{-2}(d_0^2+b^2)^3|Ric(M)|^2\}$ is invariant under the rescaling [37]

$$x \to \tilde{x} = \frac{x}{b}\ ,\ M \to \tilde{M} = \frac{M}{b} \qquad (3.6.4)$$

This implies that, in view of the scale invariance property of the Einstein vacuum equations, the asymptotically flat initial data set specified by $\{\Sigma, g, k; M\}$ and the rescaled one specified by $\{\Sigma, \tilde{g}, \tilde{k}; \tilde{M}\}$, where $\tilde{g} = g$, $\tilde{k} = b^{-1}k$ and $\tilde{M} = b^{-1}M$, give rise to two equivalent developments [38] $\mathbf{g}$ and $\tilde{\mathbf{g}}$. Observe that by picking b large we can make $\tilde{M}$ arbitrarily small.

Remark: In the case of *"Strong asymptotic flatness"* the rescaling of M follows automatically from the rescaling of g and k and the explicit expressions of the ADM energy. In fact, as $\widetilde{\partial g} = b^{-1}\partial g$, $\tilde{E} = b^{-1}E$, where

$$E \equiv \frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r}\sum_{i,j}(\partial_i g_{ij} - \partial_j g_{ii})N^j dA \qquad (3.6.5)$$

[36] They are invariant with respect to a rescaling $x \to \tilde{x} = \frac{x}{\lambda}$ for an arbitrary λ. Of course $b \to \tilde{b} = \frac{b}{\lambda}$ and, if $\lambda = b$, $\tilde{b} = 1$.

[37] The need of rescaling also M is obvious in the Schwarzschild case where a direct computation gives $\{b^{-2}(d_0^2+b^2)^3|Ric(M)|^2\} = \{(\tilde{d}_0^2+1)^3|\tilde{Ric}(\tilde{M})|^2\}$.

[38] In other words if $(\tilde{g}_{ij}(\tilde{x}), \tilde{k}_{ij}(\tilde{x}))$ are a solution with initial data $\{\Sigma, \tilde{g}, \tilde{k}; \tilde{M}\}$, then $(g_{ij}(x), k_{ij}(x))$ are a solution with initial data $\{\Sigma, g, k; M\}$.

Therefore it is sufficient to define the global initial data smallness condition, see definition 3.6.3, with the help of the quantity [39]

$$\begin{aligned} J_0(\Sigma_0,g,k) &= \sup_{\Sigma_0}\left((d_0^2+1)^3|Ric|^2\right)+\int_{\Sigma_0}\sum_{l=0}^{3}(d_0^2+1)^{l+1}|\nabla^l k|^2\\ &+ \int_{\Sigma_0}\sum_{l=0}^{1}(d_0^2+1)^{l+3}|\nabla^l B|^2 \end{aligned} \tag{3.6.6}$$

Observe that by choosing b sufficiently large, the mass of $\{\Sigma_0,g,k\}$ can be chosen appropriately small.

Definition 3.6.2 *Given an initial data set $\{\Sigma_0,g,k\}$ and a compact set $K\subset\Sigma_0$ such that $\Sigma_0\backslash K$ is diffeomorphic to the complement of the closed unit ball in R^3, we define $J_K(\Sigma_0,g,k)$ as follows:*

- *We denote $\mathcal{G}$ the set of all the smooth extensions $(\tilde g,\tilde k)$ to the whole of Σ_0 of the data (g,k) restricted to $\Sigma_0\backslash K$, with $\tilde g$ Riemannian and $\tilde k$ a symmetric two tensor.*
- *We denote by $\tilde d_0$ the geodesic distance from a fixed point O in K relative to the metric $\tilde g$.*
- *We denote*

$$J_K(\Sigma_0,g,k)=\inf_{\mathcal{G}} J_0(\Sigma_0,\tilde g,\tilde k) \tag{3.6.7}$$

Definition 3.6.3 *We say that the initial data satisfy the "global smallness initial data condition" if for a sufficient small $\varepsilon>0$*

$$J_0(\Sigma_0,g,k)\le\varepsilon^2\ .$$

Definition 3.6.4 *Consider an initial data set $\{\Sigma_0,g,k\}$ where K is a compact set such that $\Sigma_0\backslash K$ is diffeomorphic to the complement of the closed unit ball in R^3. We say that the initial data set satisfy the "exterior global smallness condition" if, given $\varepsilon>0$ sufficiently small,*

$$J_K(\Sigma_0,g,k)\le\varepsilon^2\ .$$

[39] Where the ˜ have been suppressed.

Remarks:

a) An alternative definition of J_K could be given with the help of the geodesic distance function starting from the boundary of K.

b) Given an initial data set $\{\Sigma_0, g, k\}$ with $J_0(\Sigma_0, g, k) < \infty$ it should not be difficult to prove that for given $\varepsilon > 0$ sufficiently small we can find a sufficiently large compact set K such that $J_K(\Sigma_0, g, k) < \varepsilon^2$.

c) The same statement as in the previous remark should hold true for an arbitrary strongly asymptotically flat initial data set [40].

Osservazione 3.6.1 *Ad essere veramente precisi nell'enunciato del* Main Theorem *bisognerebbe dire che "The initial data set has a unique development $(\mathcal{M}, \mathbf{g})$, defined outside the domain of influence of K' where K' is acompact set diffeomorphic to B such that the distance between $\partial K'$ and ∂K is bounded by $c\varepsilon$ ". Questo deve essere verificato nella dimostrazione del "Main Theorem" e pertanto dovrebbe anche fare parte delle condizioni di Bootstrap. Per provare questo ci si deve rifare all'oscillation Lemma e mostrare che $S'(\lambda_0, \nu - 2\delta_0 - \lambda_0)$ and $S(\lambda_0, \nu - 2\delta_0 - \lambda_0)$ are very "near" , of order ϵ, and therefore going down from them with an "outgoing cone" to Σ_0 gives ∂K and $\partial K'$ respectively very near.*

3.7 The Main Theorem

Theorem 3.7.1 (Main Theorem) *Consider a strongly asymptotically flat, maximal, initial data set $\{\Sigma_0, g, k\}$ [41]. Assume that the initial data set satisfy the exterior global smallness condition, $J_K(\Sigma_0, g, k) < \varepsilon^2$, where K is a sufficiently large compact set $\subset \Sigma_0$ with $\Sigma_0 \backslash K$ diffeomorphic to $R^3 \backslash B$.*

The initial data set has a unique development $(\mathcal{M}, \mathbf{g})$, defined outside the domain of influence of K, with the following properties:

i) $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$ where $\mathcal{M}^+$ consists of the part of $\mathcal{M}$ which is in the future of $\Sigma_0 \backslash K$, $\mathcal{M}^-$ the one to the past.

ii) $(\mathcal{M}^+, g)$ can be foliated by a canonical double null foliation $\{C(\lambda), \underline{C}(\nu)\}$ *whose outgoing leaves $C(\lambda)$ are complete [42] for all $|\lambda| \leq |\lambda_0|$. The boundary*

[40] Observe that the finiteness of $J_0(\Sigma_0, g, k)$ or $J_K(\Sigma_0, g, k)$ is consistent with the stronger version of the asymptotic flatness assumption introduced in definition 3.6.1, called "strong asymptotic flatness".

[41] The requirement that Σ_0 be maximal is not essential. It can be avoided, as suggested in [Ch-Kl], starting with a local solution of the Einstein equations and using the result of [Ba] concerning the existence of a maximal hypersurface.

[42] By this we mean that the null geodesics generating $C(\lambda)$ can be indefinitely extended toward the future.

of K can be chosen to be the intersection of $C(\lambda_0)\cap\Sigma_0$.

iii) The norms $\mathcal{O}$, $\mathcal{D}$ and $\mathcal{R}$ are bounded by a constant $\leq c\varepsilon$.

iv) In particular the null Riemann components have the following asymptotic behaviour:

$$\begin{aligned}&\sup_{\mathcal{K}} r^{7/2}|\alpha| \leq C_0 \ , \ \sup_{\mathcal{K}} r|u|^{\frac{5}{2}}|\underline{\alpha}| \leq C_0\\&\sup_{\mathcal{K}} r^{7/2}|\beta| \leq C_0 \ , \ \sup_{\mathcal{K}} r^2|u|^{\frac{3}{2}}|\underline{\beta}| \leq C_0\\&\sup_{\mathcal{K}} r^3|\rho| \leq C_0 \ , \ \sup_{\mathcal{K}} r^3|u|^{\frac{1}{2}}|(\rho-\overline{\rho},\sigma)| \leq C_0\end{aligned} \tag{3.7.1}$$

with C_0 a constant depending on the initial data.

v) $(\mathcal{M}^-,g)$ satisfies the same properties as $(\mathcal{M}^+,g)$.

vi) If $J_0(\Sigma_0,g,k)$ is sufficiently small we should be able [43] to extend $(\mathcal{M},\mathbf{g})$ to a smooth, complete solution compatible with the global stability of the Minkowski space.

The proof of the *Main Theorem* which is given in section 3.7.9, hinges on a sequence of basic results which we state in subsections 3.7.2,...., 3.7.7, concerning estimates for the $\mathcal{O}$, $\mathcal{D}$, $\mathcal{R}$ and $\mathcal{Q}$- norms.
Their proofs are lengthy and form the content of the next four chapters. In the statements of the theorems given below c refers sistematically to a constant which is independent on all the main quantities appearing in the statement of the theorems.

3.7.1 Estimates for the initial layer foliation

Theorem 3.7.2 (Theorem M0) *Consider an initial data set which satisfies the exterior global smallness condition $J_K(\Sigma_0,g,k)\leq\varepsilon^2$, with ε sufficiently small. There exists an "initial layer foliation" on $\mathcal{K}'_{\delta_0}\subset\mathcal{K}$ of fixed height [44] $\delta_0<1$, such that the following estimates hold*

$$\begin{aligned}&{\mathcal{O}_{[3]}}' \leq c\varepsilon \ , \ {\underline{\mathcal{O}}_{[3]}}' \leq c\varepsilon\\&{\mathcal{R}_{[2]}}' \leq c\varepsilon \ , \ {\underline{\mathcal{R}}_{[2]}}' \leq c\varepsilon\end{aligned} \tag{3.7.2}$$

[43] We do not address this issue here, see discussion in Chapter 8.

[44] The initial layer region can in fact be extended to a height which is, at least, proportional to $1/\varepsilon_0$.

The proof of Theorem **M0** is discussed in Chapter 7.

Remark: Theorem **M0**, which describes the properties of the *initial layer foliation* is totally independent of the global results stated here and proved in the next chapters. Nevertheless the structure of its proof follows, in a far simpler local situation, all the main steps needed in the proof of Theorems **M1**,...,**M9**. We shall have a short discussion of the proof at the end of Chapter 7, after the proof of Theorems **M1**,...,**M9** has been completely addressed.

3.7.2 Estimates for the $\mathcal{O}$ norms in $\mathcal{K}$

Theorem 3.7.3 (Theorem M1) *Assume that, relative to the "double null canonical foliation" of $\mathcal{K}$,*

$$\mathcal{R} \leq \Delta \tag{3.7.3}$$

Moreover we assume that

$$\underline{\mathcal{O}}_{[3]}(\Sigma_0) \leq \mathcal{I}_0 \ , \ \mathcal{O}_{[3]}(\underline{C}_*) \leq \mathcal{I}_* \tag{3.7.4}$$

Then, if Δ, $\mathcal{I}_0$, $\mathcal{I}_*$ are sufficiently small, the following estimate holds

$$\mathcal{O} \leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta) \tag{3.7.5}$$

The proof of Theorem **M1** is in section 4.2

3.7.3 Estimates for the $\mathcal{D}$ norms in $\mathcal{K}$

Theorem 3.7.4 (Theorem M2) *Assume that, relative to a double null foliation of $\mathcal{K}$,*

$$\mathcal{R} \leq \Delta$$

Moreover we assume [45]

$$\mathcal{D}(\underline{C}_*) \leq \mathcal{I}_* \tag{3.7.6}$$

and the results of Theorem 3.7.3, then, if $\mathcal{I}_0$, $\mathcal{I}_*$, Δ are sufficiently small, the following estimate holds

$$\mathcal{D} \leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta) \tag{3.7.7}$$

The proof of Theorem **M2** is in section 4.7.

[45] The fact that we do not have to make assumptions for the $\mathcal{D}$ norms on Σ_0 follows from the construction of the rotation vector fields, see Chapter 4 section 4.6.

3.7.4 Estimates for the $\mathcal{O}$ norms on the initial hypersurface

Theorem 3.3.1 (Theorem M3) *Consider an initial data set which satisfies the exterior global smallness condition $J_K(\Sigma_0, g, k) \leq \varepsilon^2$, with ε sufficiently small. There exists a canonical foliation on $\Sigma_0 \backslash K$, such that the following estimates hold*

$$\mathcal{O}_{[3]}(\Sigma_0 \backslash K) \leq c\varepsilon \ , \ \underline{\mathcal{O}}_{[3]}(\Sigma_0 \backslash K) \leq c\varepsilon$$

The proof of Theorem **M3** is in subsection 7.1.3.

3.7.5 Estimates for the $\mathcal{O}$ norms and the $\mathcal{D}$ norms on the last slice

Theorem 3.7.5 (Theorem M4) *Consider a canonical foliation on $\underline{C}_*$, relative to which*

$$\mathcal{R} \leq \Delta \ .$$

Moreover we assume

$$\mathcal{O}_{[2]}(\underline{C}_* \cap \Sigma_0) + \mathcal{O}_3(\Sigma_0) + \underline{\mathcal{O}}_{[3]}(\Sigma_0) \leq \mathcal{I}_0.$$

If Δ, $\mathcal{I}_0$ are sufficiently small, then the following estimate holds

$$\underline{\mathcal{O}}_{[2]}(\underline{C}_*) + \mathcal{O}_{[3]}(\underline{C}_*) \leq c(\mathcal{I}_0 + \Delta)$$

Remark: A stronger version of Theorem 3.7.5 will be proved in Chapter 7, section 7.4.

The proof of Theorem **M4** is in subsection 3.5.5.

Theorem 3.7.6 (Theorem M5) *Consider a canonical foliation on $\underline{C}_*$, relative to which*

$$\mathcal{R} \leq \Delta \ .$$

Moreover we assume

$$\mathcal{O}_{[2]}(\underline{C}_* \cap \Sigma_0) + \mathcal{O}_3(\Sigma_0) + \underline{\mathcal{O}}_{[3]}(\Sigma_0) \leq \mathcal{I}_0.$$

If Δ, $\mathcal{I}_0$ are sufficiently small, then the following estimate holds

$$\mathcal{D}(\underline{C}_*) \leq c(\mathcal{I}_0 + \Delta)$$

The proof of Theorem **M5** is in section 7.5

Corollary 3.7.7 *If the double null foliation is canonical and Δ, $\mathcal{I}_0$ are sufficiently small, we have*

$$\mathcal{O} + \mathcal{D} \leq c\,(\mathcal{I}_0 + \Delta)$$

In addition we shall also need in the proof of the *Main Theorem* the following precise version of Theorem 3.3.2,

Theorem 3.3.2 (Theorem M6): *Assume given on $\underline{C}_*$ a radial foliation, not necessarily canonical, whose connection coefficients and null curvature components satisfy the inequalities*

$$\begin{aligned} \mathcal{R}'(\underline{C}_*) &\equiv {\mathcal{R}_{[2]}}'(\underline{C}_*) + {\underline{\mathcal{R}}_{[2]}}'(\underline{C}_*) \leq \epsilon_0' \\ \mathcal{O}'(\underline{C}_*) &\equiv {\underline{\mathcal{O}}_{[2]}}'(\underline{C}_*) + {\mathcal{O}_{[2]}}'(\underline{C}_*) \leq \epsilon_0' \end{aligned} \tag{3.7.8}$$

where ${\mathcal{R}_{[2]}}'(\underline{C}_*)$,${\underline{\mathcal{R}}_{[2]}}'(\underline{C}_*)$, ${\underline{\mathcal{O}}_{[2]}}'(\underline{C}_*)$, ${\mathcal{O}_{[2]}}'(\underline{C}_*)$ are the norms introduced in section 3.5, restricted to $\underline{C}_*$, relative to the radial foliation [46]. *Then there exists a canonical foliation, on $\underline{C}_*$ relative to which we have*

$$\begin{aligned} \mathcal{R}(\underline{C}_*) &\equiv \mathcal{R}_{[2]}(\underline{C}_*) + \underline{\mathcal{R}}_{[2]}(\underline{C}_*) \leq c\epsilon_0' \\ \mathcal{O}(\underline{C}_*) &\equiv \underline{\mathcal{O}}_{[2]}(\underline{C}_*) + \mathcal{O}_{[3]}(\underline{C}_*) \leq c\epsilon_0' \end{aligned} \tag{3.7.9}$$

In addition it can be shown that these two foliations remain close to each other in a sense which can be made precise.

The proof of Theorem **M6** is in section 7.3.

3.7.6 Estimates for the $\mathcal{R}$ norms

Theorem 3.7.8 (Theorem M7) *Assume that relative to a double null foliation on $\mathcal{K}$*

$$\mathcal{O}_{[2]} + \underline{\mathcal{O}}_{[2]} \leq \Gamma \tag{3.7.10}$$

Then, if Γ is sufficiently small, we have

$$\mathcal{R} \leq c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}} \tag{3.7.11}$$

The proof of Theorem **M7** is in Chapter 5.

Corollary 3.7.9 *Under the same assumptions of the previous theorem the following inequality holds:*

$$\mathcal{R}_0^{\infty} + \underline{\mathcal{R}}_0^{\infty} \leq c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}} \tag{3.7.12}$$

[46] These are the "appropriate smallness assumptions" of the first statement of the theorem.

3.7.7 Estimates for the $\mathcal{Q}$ integrals

Lemma 3.7.1 *The exterior global smallness assumptions $J_K(\Sigma_0, g, k) \leq \varepsilon^2$, with ε sufficiently small, imply*

$$\mathcal{Q}_{\Sigma_0 \cap \mathcal{K}} \leq c\varepsilon^2$$

Theorem 3.7.10 (Theorem M8) *Assume that relative to a double null foliation on $\mathcal{K}$,*

$$\mathcal{O} \leq \varepsilon_0 \ , \ \mathcal{R} \leq \varepsilon_0$$

with a constant ε_0 sufficiently small [47]*, then the following estimate holds:*

$$\mathcal{Q}_{\mathcal{K}} \leq c\mathcal{Q}_{\Sigma_0 \cap \mathcal{K}} \tag{3.7.13}$$

where c is a constant independent from ε_0.

The proof of Theorem **M8** is in Chapter 6.

3.7.8 Extension theorem

Theorem 3.7.11 (Theorem M9) *Consider the spacetime $\mathcal{K}(\lambda_0, \nu_*)$ together with its double null (canonical) foliation given by the functions u and $\underline{u}$ such that*

1) The norms $\mathcal{Q}$, $\mathcal{O}$, $\mathcal{R}$ are sufficiently small

$$\mathcal{Q} \leq \epsilon_0' \ , \ \mathcal{O} \leq \epsilon_0' \ , \ \mathcal{R} \leq \epsilon_0' \ .$$

2) The initial conditions on Σ_0 are such that

$$\mathcal{O}(\Sigma_0[\nu_*, \nu_* + \delta]) \leq \epsilon_0' \quad ,$$

where $\Sigma_0[\nu_*, \nu_* + \delta] \equiv \{p \in \Sigma_0 | u_{(0)}(p) \in [\nu_*, \nu_* + \delta]\}$.

Then we can extend the spacetime $\mathcal{K}(\lambda_0, \nu_*)$ and the double null foliation $\{u, \underline{u}\}$ to a larger spacetime $\mathcal{K}(\lambda_0, \nu_* + \delta)$, with δ sufficiently small, such that the extended norms, denoted $\mathcal{O}', \mathcal{R}'$ satisfy

$$\mathcal{O}' \leq c\epsilon_0' \ , \ \mathcal{R}' \leq c\epsilon_0' \ .$$

The proof of Theorem **M9** is in section 7.6.

[47] The assumption $\mathcal{R} \leq \varepsilon_0$ is needed to control the deformation tensors of the angular momentum vector fields. In fact the assumptions on $\mathcal{O}$ and on $\mathcal{R}$ imply, via Theorem 3.7.4, that $\mathcal{D} \leq c\varepsilon_0$.

3.7.9 Proof of the *Main Theorem*

Step 1: Using the result stated in Theorem **M3** we can construct a canonical foliation on $\Sigma_0\backslash K$ verifying

$$\mathcal{O}_{[3]}(\Sigma_0\backslash K) \le c\varepsilon\ ,\ \underline{\mathcal{O}}_{[3]}(\Sigma_0\backslash K) \le c\varepsilon\ .$$

We use this foliation to extend the rotation vector fields from the spacelike infinity to $\Sigma_0\backslash K$.

Step 2: We define $\mathcal{U}$ as the set of the values ν_1 such that there exists a spacetime $\mathcal{K} = \mathcal{K}(\lambda_0, \nu_1)$ with the following properties, we call "Bootstrap assumptions":

Bootstrap assumption B1:
The spacetime $\mathcal{K} = \mathcal{K}(\lambda_0, \nu_1)$ is foliated by a *canonical double null foliation*, as specified in Definition 3.3.2, made by the two families of null hypersurfaces $\{C(\lambda)\}$ and $\{\underline{C}(\nu)\}$, with λ and ν varying in the finite intervals $[\lambda_0, \lambda_1]$ and $[\nu_0, \nu_1]$ respectively.

Bootstrap assumption B2:

Relative to the *canonical double null foliation* of $\mathcal{K} = \mathcal{K}(\lambda_0, \nu_1)$ we have

$$\mathcal{O} \le \varepsilon_0 \quad , \quad \mathcal{R} \le \varepsilon_0$$

Step 3: We show that the set $\mathcal{U}$ is not empty. To do this we first construct a local solution, starting from the initial data on Σ_0, and prove the existence on it of a *canonical double null foliation*, satisfying **B1**. Then using the initial data assumptions, the properties of the local solution and the *canonical double null foliation* it is easy to check that also **B2** is satisfied. The two non trivial parts of this step are the actual local existence result which has been already discussed in detail in [Ch-Kl], Chapter 10, and the construction of the canonical foliation of the last slice [48] which has been discussed in section 3.3 and proved in Chapter 7. This completes the proof that $\mathcal{U}$ is not empty.

Step 4: This is the main step of the proof. Define ν_* to be the supremum of the set $\mathcal{U}$.
If $\nu_* = \infty$ the result is achieved. If ν_* is finite we may assume [49] $\nu_* \in \mathcal{U}$ and proceed in the following way:

[48] We remark that Theorem 3.3.2 concerning the existence of a canonical foliation on the last slice is used twice in the proof of the *Main Theorem*. The first time to prove that $\mathcal{U}$ is not empty, the second time in Step 7 to show that $\nu_* < \infty$ leads to a contradiction.

[49] In fact the argument below works for any fixed $\nu < \nu_*$ arbitrary close to ν_*.

i) We consider the region $\mathcal{K} = \mathcal{K}(\lambda_0, \nu_1)$. Making use of the properties **B1** and **B2**, for sufficiently small ε_0, we find, using Theorem **M8**, that the main quantity $\mathcal{Q}_{\mathcal{K}}$ is bounded by $c\mathcal{Q}_{\Sigma_0 \cap \mathcal{K}}$. As $\mathcal{Q}_{\Sigma_0 \cap \mathcal{K}}$ is expressed in terms of initial data it follows that, see Lemma 3.7.1,

$$\mathcal{Q}_{\mathcal{K}} \leq c\varepsilon^2. \tag{3.7.14}$$

ii) We use Theorems **M7**, **M8** to show that $\mathcal{R} \leq c\varepsilon$. Moreover recalling Theorem **M4** and Corollary 3.7.7 we find that

$$\mathcal{O} \leq c(\mathcal{I}_0 + \Delta) \ .$$

In view of the fact that $\mathcal{R} \leq c\varepsilon$ we can choose $\Delta \leq c\varepsilon$. Recall that $\mathcal{I}_0$ is an upper bound for $\mathcal{O}(\Sigma_0 \backslash K)$. In view of Step 1 we can choose $\mathcal{I}_0 \leq c\varepsilon$ and, therefore, we infer that $\mathcal{O} \leq c\varepsilon$.

iii) We summarize the previous steps: *Under the bootstrap assumptions*

$$\mathcal{O} \leq \varepsilon_0 \ , \mathcal{R} \leq \varepsilon_0 \ ,$$

with ε_0 sufficiently small, and assuming also the results of Step 1, in particular $\mathcal{I}_0 \leq c\varepsilon$, we have shown that

$$\mathcal{O} \leq c\varepsilon \ , \ \mathcal{R} \leq c\varepsilon \tag{3.7.15}$$

Therefore if ε is sufficiently small we obtain the improved estimate

$$\mathcal{O} \leq \frac{1}{2}\varepsilon_0 \ , \ \mathcal{R} \leq \frac{1}{2}\varepsilon_0 \tag{3.7.16}$$

Step 5: With the help of Theorem **M9**, for $\epsilon_0' = c\varepsilon$, the value on the right hand side of 3.7.15, we show that the spacetime $\mathcal{K}(\lambda_0, \nu_*)$ can be extended to a spacetime $\mathcal{K}(\lambda_0, \nu_* + \delta)$, for δ sufficiently small, foliated by a double null foliation which extends the *canonical double null foliation* of $\mathcal{K}(\lambda_0, \nu_*)$. Moreover, the norms $\mathcal{R}'$ and $\mathcal{O}'$, relative to the extended double null foliation, cannot become larger than $c^2\varepsilon$,

$$\mathcal{O}' \leq c^2\varepsilon \ , \ \mathcal{R}' \leq c^2\varepsilon \tag{3.7.17}$$

We remark that the extended double null foliation fails to be canonical on $\mathcal{K}(\lambda_0, \nu_* + \delta)$. In fact $\underline{u}$ is canonical on Σ_0 but the extended u fails to be canonical on the new last slice which we denote $\underline{C}_{**} \equiv \underline{C}(\nu_* + \delta)$.

Step 6: Finally we are able to show that the assumption $\nu_* < \infty$ leads to a contradiction. In fact the new spacetime $\mathcal{K}(\lambda_0, \nu_* + \delta)$ is a good candidate for

our family of spacetimes satisfying the bootstrap assumptions **B1** and **B2**. The only property still missing is that the extended function u be canonical on $\underline{C}_{**}$.
Using Theorem **M6** with $\epsilon_0' = c^2\varepsilon$ where $c^2\varepsilon$ is the constant on the right hand side of 3.7.17 we can construct a canonical foliation on $\underline{C}_{**}$ relative to which the new norm $\mathcal{R}(\underline{C}_{**})$ and $\mathcal{O}(\underline{C}_{**})$ verify

$$\mathcal{O}(\underline{C}_{**}) + \mathcal{R}(\underline{C}_{**}) \leq c^3\varepsilon \tag{3.7.18}$$

Starting with this new canonical foliation, on the new last slice, we extend it to the interior of the spacetime and thus obtain a new extended *canonical double null foliation* near the previous one. In view of the continuity properties of the propagation equations of the double null foliation we can check that, for small δ, the new norms $\mathcal{O}$ and $\mathcal{R}$ remain arbitrarily close to the old ones; in fact we show that these new norms satisfy

$$\mathcal{O} < 100c^3\varepsilon \ \ , \ \ \mathcal{R} < 100c^3\varepsilon \tag{3.7.19}$$

We shall prove this fact in the remark below. Therefore, for ε sufficiently small, we still have the inequalities

$$\mathcal{O} < \varepsilon_0 \ \ , \ \ \mathcal{R} < \varepsilon_0 \ .$$

We have, therefore, constructed the spacetime $\mathcal{K}(\lambda_0, \nu_* + \delta)$ verifying all the Bootstrap assumptions.
This proves that ν_* is not the supremum of the the set $\mathcal{U}$, contradicting our assumption. Therefore the only way to avoid a contradiction is that $\nu_* = \infty$.

Remark: In what follows we show in more detail how the new norms $\mathcal{O}$ and $\mathcal{R}$ defined in the extended spacetime $\mathcal{K}(\lambda_0, \nu_* + \delta)$ verify the inequality 3.7.19.
i) We start with the inequality 3.7.18 on $\underline{C}_{**}$

$$\mathcal{O}(\underline{C}_{**}) + \mathcal{R}(\underline{C}_{**}) \leq c^3\varepsilon \tag{3.7.20}$$

We also know that relative to the old foliation we have, on the whole spacetime $\mathcal{K}(\nu_* + \delta)$,

$$\mathcal{O}' \leq c^2\varepsilon \ , \ \mathcal{R}' \leq c^2\varepsilon \tag{3.7.21}$$

ii) To prove our result we first observe that we can pass from the norms $\mathcal{R}$ to the norms $\mathcal{R}'$ with the help of the following estimate, provided $\mathcal{O}, \mathcal{O}'$ are sufficiently small [50],

$$\mathcal{R} \leq \mathcal{R}' + c\mathcal{O}_{[2]} \cdot (1 + \mathcal{O}'_{[2]}) \cdot \mathcal{R}' + [\text{higher order terms}] \qquad (3.7.22)$$

This is proved in Chapter 7, Corollary 7.7.1. We then use the same bootstrap argument as in the proof of Theorem **M1**, see Chapter 4, Theorem 4.2.1. More precisely consider the region $\Delta(\lambda_2, \nu_2) \subset \mathcal{K}(\lambda_0, \nu_* + \delta)$ defined by

[A mistake here has been corrected; the right expression is "$\Delta(\lambda_2, \nu_2) \subset \mathcal{K}(\lambda_0, \nu_* + \delta)$" instead of "$\Delta(\lambda_2, \nu_2) \subset \mathcal{K}(\lambda_1, \nu_* + \delta)$" Moreover in 3.7.23 $(u(p), \underline{u}(p)) \in (\lambda_2, \overline{\lambda}_1] \times [\nu_*, \nu_2)$ instead of $(u(p), \underline{u}(p)) \in [\lambda_1, \lambda_2) \times (\nu_2, \nu_*]$.]

$$\Delta(\lambda_2, \nu_2) = \Big\{ p \in \mathcal{K}(\lambda_0, \nu_* + \delta) | \ (u(p), \underline{u}(p)) \in (\lambda_2, \overline{\lambda}_1] \times [\nu_*, \nu_2) \Big\} \qquad (3.7.23)$$

where $\overline{\lambda}_1 = u(p)|_{\underline{C}_{**} \cap \Sigma_0}$. Repeating the argument of Theorem 4.2.1 we obtain that, for any two dimensional surface S contained in Δ, we have the inequality

$$\mathcal{O}|_\Delta \leq c\Big(\mathcal{O}(\underline{C}_{**} \cap \Delta) + \mathcal{I}_0 + \mathcal{R}|_\Delta \Big) \qquad (3.7.24)$$

provided that $\mathcal{R}|_\Delta$ is sufficiently small. Therefore using 3.7.20, 3.7.22 and $\mathcal{I}_0 < c\varepsilon$, we conclude that

$$\mathcal{O}|_\Delta \leq c\Big(c^3\varepsilon + c\varepsilon + \mathcal{R}'|_\Delta + \mathcal{O}|_\Delta \cdot \mathcal{R}'|_\Delta + \mathcal{O}'|_\Delta \cdot \mathcal{R}'|_\Delta \Big) .$$

Now using the estimates 3.7.21, for $\mathcal{O}'$ and $\mathcal{R}'$ and taking ε sufficiently small,

$$\mathcal{O}|_\Delta \leq c\Big(c^3\varepsilon + c\varepsilon + \mathcal{R}'|_\Delta + \mathcal{O}'|_\Delta \cdot \mathcal{R}'|_\Delta \Big) \leq 4c^3\varepsilon \qquad (3.7.25)$$

Therefore in view of 3.7.22 we also have

$$\mathcal{R}|_\Delta \leq 4c^3\varepsilon \qquad (3.7.26)$$

By a standard continuity argument we can show that the region Δ can be chosen equal to the whole extended spacetime $\mathcal{K}(\lambda_0, \nu_* + \delta)$.

3.8 Appendix to Chapter 3

Proof of Proposition 3.1.1

We start considering the Gauss equation which expresses the Riemann tensor of the submanifold S, we denoted $\mathbf{R}\!\!\!/$, in terms of the Riemann tensor of the

[50] Since we proceed by a continuity argument, starting from the last slice, where $\mathcal{O}$ is small, this assumption is justified.

embedding manifold $(\mathcal{M}, \mathbf{g})$ and of the the null second fundamental forms $\chi, \underline{\chi}$ [51].

$$\not{R}^{\nu}_{\mu\rho\sigma} = \Pi^{\tau}_{\mu}\Pi^{\nu}_{\delta}\Pi^{\lambda}_{\rho}\Pi^{\zeta}_{\sigma}R^{\delta}_{\tau\lambda\zeta} - \frac{1}{2}(\chi^{\nu}_{\rho}\underline{\chi}_{\mu\sigma} - \chi^{\nu}{}_{\sigma}\underline{\chi}_{\mu\rho}) - \frac{1}{2}(\underline{\chi}^{\nu}{}_{\rho}\chi_{\mu\sigma} - \underline{\chi}^{\nu}{}_{\sigma}\chi_{\mu\rho}) \quad (3.8.1)$$

where $\Pi^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \frac{1}{2}({e_3}^{\mu}e_{4\nu} + {e_4}^{\mu}e_{3\nu})$ projects on the tangent space TS. Contracting the indices ν and ρ with respect to the metric g of $\mathcal{M}$, we obtain an expression for the Ricci tensor relative to S

$$\not{R}_{\mu\sigma} = R^{\delta}_{\tau\lambda\zeta}\Pi^{\tau}_{\mu}\Pi^{\lambda}_{\delta}\Pi^{\zeta}_{\sigma} - \frac{1}{2}(tr\chi\underline{\chi}_{\mu\sigma} + tr\underline{\chi}\chi^{\mu}{}_{\sigma}) + \frac{1}{2}\Big((\chi\cdot\underline{\chi})_{\mu\sigma} + (\underline{\chi}\cdot\chi)_{\mu\sigma}\Big)$$

and contracting again the indices μ, σ

$$\begin{aligned}
\not{R} &= R^{\delta}_{\tau\lambda\zeta}\Pi^{\lambda}{}_{\delta}\Pi^{\tau\zeta} - tr\chi \mathrm{tr}\underline{\chi} + \chi\cdot\underline{\chi} \\
&= \mathbf{R} + \mathbf{Ricci}(e_4, e_3) - \frac{1}{2}\mathbf{R}(e_4, e_3, e_4, e_3) - tr\chi\mathrm{tr}\underline{\chi} + \chi\cdot\underline{\chi}
\end{aligned}$$

In an Einstein vacuum manifold $\mathbf{R} = \mathbf{Ricci} = 0$ and the previous equation reduces to

$$\not{R} + tr\chi\mathrm{tr}\underline{\chi} - (\chi\cdot\underline{\chi}) = -\frac{1}{2}\mathbf{R}(e_4, e_3, e_4, e_3) = -2\rho \quad (3.8.2)$$

recalling the null decomposition of the Riemann tensor, which written, in terms of the scalar curvature of S, $K = \frac{1}{2}\not{R}$, becomes

$$\mathbf{K} = -\frac{1}{4}tr\chi\mathrm{tr}\underline{\chi} + \frac{1}{2}(\hat{\chi}\cdot\hat{\underline{\chi}}) - \rho \quad (3.8.3)$$

Proceeding in a similar way we compute the Codazzi equations which connect the S tangential derivatives of χ and $\underline{\chi}$ to the Riemann tensor of $\mathcal{M}$

[51] Let Y be a vector field $\in TS$, $\not\nabla Y$ its covariant derivative in S,

$$\not\nabla_{\mu}Y^{\rho} = \Pi^{\gamma}_{\mu}\Pi^{\rho}_{\beta}D_{\gamma}Y^{\beta}$$

$\not{R}$ is obtained computing the right hand side of the equation

$$\not{R}^{\nu}_{\mu\rho\sigma}Y^{\mu} = \not\nabla_{\rho}\not\nabla_{\sigma}Y^{\rho} - \not\nabla_{\sigma}\not\nabla_{\rho}Y^{\nu}\ .$$

As $\not\nabla_{\rho}\not\nabla_{\sigma}Y^{\rho} = \Pi^{\lambda}_{\rho}\Pi^{\zeta}_{\sigma}\Pi^{\nu}_{\delta}D_{\lambda}D_{\zeta}Y^{\delta} + \Pi^{\lambda}_{\rho}\Pi^{\nu}_{\delta}\Pi^{\tau}_{\sigma}(D_{\lambda}\Pi^{\tau}_{\sigma})D_{\zeta}Y^{\delta} + \Pi^{\lambda}_{\rho}\Pi^{\zeta}_{\sigma}\Pi^{\nu}_{\gamma}(D_{\lambda}\Pi^{\gamma}_{\delta})D_{\zeta}Y^{\delta}$

and $\quad \Pi^{\lambda}_{\rho}\Pi^{\zeta}_{\sigma}\Pi^{\nu}_{\gamma}(D_{\lambda}\Pi^{\gamma}_{\delta}) = \frac{1}{2}\Pi^{\zeta}_{\sigma}(\chi^{\nu}{}_{\rho}e_{3\delta} + \underline{\chi}^{\nu}{}_{\rho}e_{4\delta})$

the result follows.

and to $\chi, \underline{\chi}, \zeta$,

$$\not\nabla_\rho \chi_\mu^\rho - \not\nabla_\mu \chi_\rho^\rho = \frac{1}{2} R_{\tau\nu\delta\sigma} e_4^\tau e_3^\nu e_4^\delta \Pi_\mu^\sigma + tr\chi\zeta_a e_{a\mu} - \chi_{ab}\zeta_b e_{a\mu} \ .$$

As

$$\not\nabla_\rho \chi_\mu^\rho = (D_{e_b}\chi)_{ba} e_{a\mu} = (\not{d}\mathrm{iv}\,\chi)_a e_{a\mu}$$

we obtain

$$\not{d}\mathrm{iv}\,\chi_a + \chi_{ab}\zeta_b = \not\nabla_a \mathrm{tr}\chi + \zeta_a \mathrm{tr}\chi - \beta_a \tag{3.8.4}$$

and in the same way, with $\underline{\chi}$ instead of χ,

$$\not{d}\mathrm{iv}\,\chi_a - \underline{\chi}_{ab}\zeta_b = \not\nabla_a \mathrm{tr}\underline{\chi} - \zeta_a \mathrm{tr}\underline{\chi} + \underline{\beta}_a \tag{3.8.5}$$

3.8.1 Derivation of the structure equations

We give some example of the derivation of the explicit expressions of the structure equations, 3.1.49, 3.1.50.

We start from

$$\begin{aligned} \mathbf{R}^\delta_{\gamma\alpha\beta} &= \mathbf{\Omega}^\delta_\gamma(e_\alpha, e_\beta) = (d\omega^\delta_\gamma + \omega^\delta_\sigma \wedge \omega^\sigma_\gamma)(e_\alpha, e_\beta) \\ &= e_\alpha(\mathbf{\Gamma}^\delta_{\beta\gamma}) - e_\beta(\mathbf{\Gamma}^\delta_{\alpha\gamma}) + \mathbf{\Gamma}^\lambda_{\beta\gamma}\mathbf{\Gamma}^\delta_{\alpha\lambda} - \mathbf{\Gamma}^\lambda_{\alpha\gamma}\mathbf{\Gamma}^\delta_{\beta\lambda} - \omega^\delta_\gamma([e_\alpha, e_\beta]) \end{aligned} \tag{3.8.6}$$

and observe that [52]

$$\mathbf{R}_{\delta\gamma\alpha\beta} = <\mathbf{R}(e_\alpha, e_\beta)e_\gamma, \tilde{e_\delta}> = \#(\delta)\mathbf{R}^{\tilde\delta}_{\gamma\alpha\beta} \tag{3.8.7}$$

where

$$\#(\delta) = \begin{cases} -2 \ if \ \delta \in \{3,4\} \\ 1 \ if \ \delta \in \{1,2\} \end{cases} \tag{3.8.8}$$

choosing $\{(\delta,\gamma) = (a,3) \ , \ (\alpha,\beta) = (3,b)\}$ we obtain

$$\mathbf{R}_{a33b} = (\not{\mathbf{D}}_3\underline{\chi})_{ba} - 2(\not\nabla\underline{\xi})_{ba} + 2\underline{\omega}\underline{\chi}_{ba} + (\underline{\chi}_{\cdot c}\underline{\chi}_{\cdot c})_{ba} + 4\zeta_b\underline{\xi}_a - 2\eta_b\underline{\xi}_a - 2\underline{\eta}_a\underline{\xi}_b$$

Decomposing this equation in its traceless part and its trace part we obtain:

$$(\not{\mathbf{D}}_3\hat{\underline{\chi}})_{ba} + 2\underline{\omega}\hat{\underline{\chi}}_{ba} - (\not\nabla\widehat{\otimes}\underline{\xi})_{ba} + ((2\zeta - \underline{\eta} - \eta)\widehat{\otimes}\underline{\xi})_{ba} + \hat{\underline{\chi}}_{ba}\mathrm{tr}\underline{\chi} = \hat{\mathbf{R}}_{a33b}$$

$$\mathbf{D}_3\mathrm{tr}\underline{\chi} + 2\underline{\omega}\mathrm{tr}\underline{\chi} + |\hat{\underline{\chi}}|^2 + \frac{1}{2}(\mathrm{tr}\underline{\chi})^2 - 2\not{d}\mathrm{iv}\,\underline{\xi} - 2\underline{\xi}\dot{(}\eta + \underline{\eta} - 2\zeta) = \delta_{ba}\mathbf{R}_{a33b}$$

[52] In fact

$$\mathbf{R}_{\tilde\delta\gamma\alpha\beta} = R_{\mu\nu\rho\sigma}e^\nu_\gamma e^\rho_\alpha e^\sigma_\beta e^\mu_{\tilde\delta} = R^{\mu'}_{\nu\rho\sigma}e^\nu_\gamma e^\rho_\alpha e^\sigma_\beta(g_{\mu'\mu}e^\mu_{\tilde\delta}) = \#(\delta)\mathbf{R}^{\tilde\delta}_{\gamma\alpha\beta}$$

We consider now the indices $(\delta,\gamma)=(a,3)$, $(\alpha,\beta)=(4,b)$ and obtain

$$\mathbf{R}_{a34b}=(\not{D}_4\underline{\chi})_{ba}-2(\nabla\!\!\!/\underline{\eta})_{ba}+2\omega\underline{\chi}_{ba}+(\chi_{\cdot c}\underline{\chi}_{c\cdot})_{ba}-2\underline{\eta}_b\underline{\eta}_a-2\xi_b\underline{\xi}_a$$

Proceeding exactly as in the previous case, decomposing this equation in its trace part and its traceless part we obtain

$$(\not{D}_4\hat{\underline{\chi}})_{ba}-2\omega\hat{\underline{\chi}}_{ba}+(\widehat{\chi_{bc}\underline{\chi}_{ca}})-(\nabla\!\!\!/\widehat{\otimes}\underline{\eta})_{ba}-(\underline{\eta}\widehat{\otimes}\underline{\eta})_{ba}-(\xi\widehat{\otimes}\underline{\xi})_{ba}=\mathcal{S}(\widehat{\mathbf{R}}_{a34b})$$
$$\mathbf{D}_4\mathrm{tr}\underline{\chi}-2\omega\mathrm{tr}\underline{\chi}+\delta_{ba}(\chi_{bc}\underline{\chi}_{ca})-2\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\eta}-2|\underline{\eta}|^2-2\xi\cdot\underline{\xi}=\delta_{ba}\mathcal{S}(\mathbf{R}_{a34b})$$

Choosing $(\delta,\gamma)=(a,3)$, $(\alpha,\beta)=(b,c)$ it follows

$$(\nabla\!\!\!/_b\underline{\chi})_{ca}-(\nabla\!\!\!/_c\underline{\chi})_{ba}-(\zeta_b\underline{\chi}_{ca}-\zeta_c\underline{\chi}_{ba})=\mathbf{R}_{a3bc}$$

and again, considering the trace and the traceless part with respect to (a,c),

$$(\nabla\!\!\!/_b\hat{\underline{\chi}})_{ca}-(\widehat{\nabla\!\!\!/_c\underline{\chi}})_{ba}-(\zeta_b\hat{\underline{\chi}}_{ca}-\widehat{\zeta_c\underline{\chi}_{ba}})=\mathbf{R}_{a3bc}$$
$$\nabla\!\!\!/_b(\mathrm{tr}\underline{\chi})-(\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\chi})_b+(\zeta\cdot\underline{\chi})_b-\zeta_b\mathrm{tr}\underline{\chi}=\delta_{ac}\mathbf{R}_{a3bc}\ .$$

Choosing $(\delta,\gamma)=(a,3)$, $(\alpha,\beta)=(4,3)$ it follows

$$(\not{D}_4\underline{\xi})_a-(\not{D}_3\underline{\eta})_a+((\eta-\underline{\eta})\cdot\underline{\chi})_a-4\omega\underline{\xi}_a=\frac{1}{2}\mathbf{R}_{a343} \tag{3.8.9}$$

This result completes all the computations with $(\delta,\gamma)=(a,3)$.
Looking at the equations with $(\delta,\gamma)=(a,4)$ one realizes that most of the equations are not independent from the previous ones, the independent ones can be obtained just observing that substituting the index 3 with 4 amounts to change $\mathbf{D}_3$ and $\not{D}_3$ with $\mathbf{D}_4$ and $\not{D}_4$, the underlined Ricci coefficients with the non underlined ones, ζ with $-\zeta$ and viceversa. Therefore without extra computations choosing $(\delta,\gamma)=(a,4)$, $(\alpha,\beta)=(3,b)$ we obtain

$$\mathbf{R}_{a43b}=(\not{D}_3\chi)_{ba}-2(\nabla\!\!\!/\eta)_{ba}+2\omega\chi_{ba}+(\underline{\chi}_{\cdot c}\chi_{c\cdot})_{ba}-2\eta_b\eta_a-2\underline{\xi}_b\xi_a$$

Proceeding exactly as in the case $(\delta,\gamma)=(a,3)$, $(\alpha,\beta)=(4,b)$, decomposing this equation in its trace part and its traceless part we obtain

$$(\not{D}_3\hat{\chi})_{ba}-2\underline{\omega}\hat{\chi}_{ba}+(\widehat{\underline{\chi}_{bc}\chi_{ca}})-(\nabla\!\!\!/\widehat{\otimes}\eta)_{ba}-(\eta\widehat{\otimes}\eta)_{ba}-(\underline{\xi}\widehat{\otimes}\xi)_{ba}=\mathcal{S}(\hat{\mathbf{R}}_{a43b})$$
$$\mathbf{D}_3\mathrm{tr}\chi-2\underline{\omega}\mathrm{tr}\chi+\delta_{ba}(\underline{\chi}_{bc}\chi_{ca})-2\mathrm{d}\!\!\!/\mathrm{iv}\,\eta-2|\eta|^2-2\underline{\xi}\cdot\xi=\delta_{ba}\mathcal{S}(\mathbf{R}_{a43b})$$

We set $(\delta,\gamma)=(a,4)$, $(\alpha,\beta)=(4,b)$ and applying the substitutions described in (1.2.12) the following result holds

$$(\not{D}_4\hat{\chi})_{ba}+2\omega\hat{\chi}_{ba}-(\nabla\!\!\!/\widehat{\otimes}\xi)_{ba}-((2\zeta+\eta+\underline{\eta})\widehat{\otimes}\xi)_{ba}+\hat{\chi}_{ba}\mathrm{tr}\chi=\mathcal{S}(\hat{\mathbf{R}}_{a44b})$$
$$\mathbf{D}_4\mathrm{tr}\chi+2\omega\mathrm{tr}\chi+|\hat{\chi}|^2+\frac{1}{2}(\mathrm{tr}\chi)^2-2\mathrm{d}\!\!\!/\mathrm{iv}\,\xi-2\xi\cdot(\underline{\eta}+\eta+2\zeta)=\delta_{ba}\mathcal{S}(\mathbf{R}_{a44b})$$

As in the case of a double null integrable foliation the relations 3.1.32 hold, it is easy to obtain some of the structure equations 3.1.45, 3.1.46, 3.1.47.

Some remarks on the definition of the adapted null frame

It is possible to choose the null frame in such a way that it is transported along $C(\lambda)$, once defined on a generic $S(\lambda,\nu)$, remaining null orthonormal, that is such that $\mathbf{g}(e_a,e_b)=\delta_{ab}$ on the whole $C(\lambda)$. The analogous situation can be obtained, starting again from $S(\lambda,\nu)$ and extending the null orthonormal frame along $\underline{C}(\nu)$. In fact according to the equations:

$$\begin{aligned}\mathbf{D}_4 e_a &= \not{\mathbf{D}}_4 e_a + (-\zeta_a + \not\nabla_a \log\Omega)e_4 = \not{\mathbf{D}}_4 e_a + \eta_a e_4\\ \mathbf{D}_4 e_4 &= (\mathbf{D}_4 \log\Omega)e_4\\ \mathbf{D}_4 e_3 &= -(\mathbf{D}_4\log\Omega)e_3 + 2(-\zeta_b + \not\nabla_b \log\Omega)e_b\end{aligned} \tag{3.8.10}$$

if we impose that $\not{\mathbf{D}}_4 e_a = 0$ then we conclude that the null orthonormal frame $\{e_a, \hat{N}, \hat{\underline{N}}\}$ defined can be extended along $C(\lambda)$, once defined on a generic $S(\lambda,\nu)$, remaining null orthonormal, that is such that $g(e_a,e_b)=\delta_{ab}$ on the whole $C(\lambda)$. From equations 3.1.44 this implies:

$$\mathcal{L}_N e_a + \Omega\chi_{ac}e_c = 0 \tag{3.8.11}$$

Starting again from $S(\lambda,\nu)$ we can also extend the null orthonormal frame along $\underline{C}(\nu)$ using the equations [53]

$$\begin{aligned}\mathbf{D}_3 e_a &= \not{\mathbf{D}}_3 e_a + (\zeta_a + \not\nabla_a \log\Omega)e_3 = \not{\mathbf{D}}_3 e_a + \eta_a e_3\\ \mathbf{D}_3 e_3 &= (\mathbf{D}_3 \log\Omega)e_3\\ \mathbf{D}_3 e_4 &= (\mathbf{D}_3\log\Omega)e_4 + 2(\zeta_b + \not\nabla_b \log\Omega)e_b\end{aligned} \tag{3.8.12}$$

and, again, $g(e_a,e_b)=\delta_{ab}$ on the whole $\underline{C}(\nu)$ if we impose

$$\mathcal{L}_{\underline{N}} e_a + \Omega\underline{\chi}_{ac}e_c = 0. \tag{3.8.13}$$

3.8.2 Proof of Proposition 3.3.1

We repeat the statement of the Proposition, in a slightly more general way:

Proposition 3.3.1: *Let $\triangle_p$ be the three dimensional subspace of $T\mathcal{K}_p$ spanned by $TS_p \oplus \tilde{N}_p$, where $\tilde{N} \equiv \frac{1}{2}(N-\underline{N})$. Let us consider the three dimensional distribution on $\mathcal{K}$, $p \to \triangle_p$.*

[53] the null frames one builds extending the orthogonal vectors $\{e_a\}$ along the null hypersurfaces $\{C(\lambda)\}$ or $\{\underline{C}(\nu)\}$ are different and will be used in different situations.

This distribution is integrable. Moreover $\mathcal{K}$ is foliated by the three dimensional spacelike hypersurfaces

$$\tilde{\Sigma}_t \equiv \{p \in \mathcal{K} | t(p) = t\}$$

where $t(p) = \frac{1}{2}(u+\underline{u})$ and each two dimensional surface $S(\lambda,\nu)$ is immersed in the hypersurface $\tilde{\Sigma}_t$ with $t = \frac{1}{2}(\lambda+\nu)$. The "global time" function $t(p)$ satisfies

$$dt = -\frac{1}{4\Omega^2}(n+\underline{n}) \quad , \quad \frac{\partial}{\partial t} = (N+\underline{N})$$

where $n, \underline{n}$ are the one forms corresponding to $N, \underline{N}$. Finally, given the hypersurfaces $\tilde{\Sigma}_t$, their second fundamental form k has the following expression

$$k_{\tilde{N}\tilde{N}} = \omega + \underline{\omega} \ , \quad k_{e_a\tilde{N}} = \zeta_a \ , \ k_{e_a e_b} = -\frac{1}{2}(\chi_{ab} + \underline{\chi}_{ab})$$

and, on each $\tilde{\Sigma}_t$, the metric $g_{ij}(t,x)$ is given by the relation

$$k_{ij} = -(2\Omega)^{-1}\partial_t g_{ij} \ .$$

Proof: We observe that, denoting $\tilde{N} \equiv (N - \underline{N})$,

$$\begin{aligned} [\tilde{N}, e_a] &= ([N,e_a] - [\underline{N}, e_a]) = (\not{D}_4 e_a - \not{D}_3 e_a) - (\chi_{ab} - \underline{\chi}_{ab})e_b \\ [e_a, e_b] &= \not{\nabla}_a e_b - \not{\nabla}_b e_a \end{aligned} \tag{3.8.14}$$

Therefore, as a result of Frobenius theorem, see [Sp], vol.I, Chapter 6, the distribution $p \to \triangle_p$ is, locally, integrable. This implies that fixed a generic point $p \in \mathcal{K}$, there is a neighbourhood U of p such that, given $q \in U$ it is possible to define a submanifold $\tilde{\Sigma} \subset \mathcal{K}$, containing q, whose tangent space is, at each point $p' \in \tilde{\Sigma}$, $\triangle_{p'}$.
Let $t(p)$ be the function whose $\tilde{\Sigma}$ is a level surface, in the neighbourhood U we have

$$dt(\cdot) = (dt)_\mu dx^\mu(\cdot) = \alpha(N+\underline{N})_\mu dx^\mu(\cdot)$$

where α is a regular scalar function on U [54]. The result becomes valid in the whole $\mathcal{K}$ if we choose $\alpha = -\frac{1}{4\Omega^2}$. In fact, recalling 3.1.37,

$$-\frac{1}{4\Omega^2}(N+\underline{N})_\mu = -\frac{1}{2}(L+\underline{L})_\mu = \frac{1}{2}\partial_\mu(u+\underline{u}) = \partial_\mu t \tag{3.8.15}$$

[54] In principle one can define $\tilde{N} \equiv \frac{1}{2}(\hat{N} - \underline{\hat{N}})$ and given the vector field $T = \frac{1}{2}(\hat{N} + \underline{\hat{N}})$ build, locally, a time function t' just considering the flow of T. This is a local result and one does not know, apriori, if it holds globally.

so that

$$dt(\cdot)=\frac{\partial t}{\partial x^\mu}dx^\mu(\cdot)=\frac{1}{2}\frac{\partial(u+\underline{u})}{\partial x^\mu}dx^\mu(\cdot)\ .$$

Defining

$$\tilde{\Sigma}_t\equiv\{p\in\mathcal{K}|t(p)=t\} \tag{3.8.16}$$

we can choose t as a coordinate for $\mathcal{K}$ and define the vector field $\frac{\partial}{\partial t}$ as the vector field satisfying $dt(\frac{\partial}{\partial t})=1$. As

$$g^{\mu\nu}\frac{\partial t}{\partial x^\mu}\frac{\partial t}{\partial x^\nu}=-\frac{1}{4\Omega^2}$$

it follows immediately that

$$\left(\frac{\partial}{\partial t}\right)^\mu=(N+\underline{N})^\mu \tag{3.8.17}$$

This proves that a global time t and a global foliation using the spacelike hypersurfaces $\{\tilde{\Sigma}_t\}$ exists [55] while from the Frobenius theorem the result will be only local and not unique. The expression of the second fundamental form k in terms of the connection coefficients follows from a direct computation.

Remark: Observe that the spacetime foliation relative to the *canonical double null foliation* is not the one used in [Ch-Kl].as the $\tilde{\Sigma}_t$ hypersurfaces are not maximal. In fact in [Ch-Kl], page 268, the condition $\mathrm{tr}k=0$ is written as [56]

$$\delta=-\mathrm{tr}\eta_{(C.K.)}=-\sum_{a=1}^{2}k_{aa}\ .$$

Observing that in the present notations,

$$\sum_{a=1}^{2}k_{aa}=-(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})\quad\text{and}\quad\delta=\omega+\underline{\omega}\ ,$$

the maximality condition becomes

$$\omega+\underline{\omega}=\frac{1}{2}(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi}) \tag{3.8.18}$$

Equation 3.8.18 is not satisfied in the present approach. In fact it cannot be imposed in our foliation as Ω is already completely determined by the structure equations, the initial conditions on the Σ_0 hypersurface and those on the "last slice" $\underline{C}_*$

[55] Asymptotically, $\underline{u}$ is basically r_*+t, where r_* is the coordinate defined in the Schwarzschild spacetime.

[56] $\eta_{(C.K.)}$ is not the η connection coefficient we use here.

Chapter 4

Estimates for the connection coefficients

4.1 Preliminary results

4.1.1 Elliptic estimates for Hodge systems

We consider Hodge systems of equations defined on a compact two dimensional Riemann surface. We recall definition 3.1.4 of Chapter 3

Definition 3.1.4 *Given the one form* ξ *on* S *we define its Hodge dual* [1]*:*

$$^\star\xi_a = \in_{ab} \xi^b$$

Clearly $^\star(^\star\xi) = -\xi$. *If* ξ *is a symmetric, traceless, 2-tensor we define the following left,* $^\star\xi$, *and right* $\xi^\star$, *Hodge duals:*

$$^\star\xi_{ab} = \in_{ac} \xi^c{}_b \ , \ \xi^\star_{ab} = \xi_a{}^c \in_{cb}$$

Observe that the tensors $^\star\xi, \xi^\star$ *are also symmetric, traceless and satisfy*

$$^\star\xi = -\xi^\star \ , \ ^\star(^\star\xi) = -\xi.$$

We will need estimates for the following elliptic systems of equations:

$\mathbf{H}_0$: *Hodge system of type 0*

This refers to the scalar Poisson equation

$$\triangle\!\!\!/\,\phi = f$$

[1] here a, b are just coordinate indices

where f is an arbitrary scalar function on S and $\not\Delta$ is the Laplacian relative to the induced metric on S.

$\mathbf{H}_1$: *Hodge system of type 1*

This concerns one forms ξ verifying

$$\begin{aligned}\not{\mathrm{d}}\mathrm{iv}\,\xi &= f\\ \mathrm{c}\not{\mathrm{u}}\mathrm{rl}\,\xi &= f_*\end{aligned}$$

where f, f_* are given scalar functions on S and the operators $\not{\mathrm{d}}\mathrm{iv}$, $\mathrm{c}\not{\mathrm{u}}\mathrm{rl}$ are defined according to

$$\not{\mathrm{d}}\mathrm{iv}\,\xi = \not\nabla^a \xi_a \;\; ; \;\; \mathrm{c}\not{\mathrm{u}}\mathrm{rl}\,\xi = \in^{ab} \not\nabla_a \xi_b$$

$\mathbf{H}_2$: *Hodge system of type 2*

This concerns traceless symmetric two forms ξ satisfying

$$\not{\mathrm{d}}\mathrm{iv}\,\xi = f$$

where f is a given vector field and $\not{\mathrm{d}}\mathrm{iv}\,\xi$ is defined by

$$\not{\mathrm{d}}\mathrm{iv}\,\xi_a = \not\nabla^b \xi_{ab}$$

For these three systems of equations we have the following $L^2(S)$ estimates, see [Ch-Kl], Chapter 2.

Proposition 4.1.1 *On an arbitrary compact Riemannian manifold (S,γ), with K the Gauss curvature of S,*

i. *If ϕ is a solution of $\mathbf{H}_0$ the following estimate holds*

$$\int_S \{|\not\nabla^2\phi|^2 + K|\not\nabla\phi|^2\} = \int_S |f|^2$$

ii. *If the vector field ξ is a solution of $\mathbf{H}_1$ then*

$$\int_S \{|\not\nabla\xi|^2 + K|\xi|^2\} = \int_S \{|f|^2 + |f_*|^2\}$$

iii. *If the symmetric, traceless, 2-tensor ξ is a solution of $\mathbf{H}_2$ then*

$$\int_S \{|\not\nabla\xi|^2 + 2K|\xi|^2\} = 2\int_S |f|^2$$

Definition 4.1.1 *Assume (S,γ) an arbitrary compact Riemannian manifold with K its Gauss curvature, we introduce* [2] *the following quantities*

$$k_m = \min_S r^2 K \ , \ k_M = \max_S r^2 K \ , \ k_1 \equiv \left(\int_S |\nabla K|^2\right)^{\frac{1}{2}}$$

Proposition 4.1.2
Assume $k_m > 0$, then, if ξ is a solution of $\mathbf{H}_1$ or $\mathbf{H}_2$, the following inequalities hold,

$$\int_S \{|\nabla\xi|^2 + r^{-2}|\xi|^2\} = c_1 \int_S |f|^2$$

$$\int_S |\nabla^2\xi|^2 \quad \leq \quad c_2 \int_S \left(|\nabla f|^2 + r^{-2}|f|^2\right)$$

with c_1, c_2 two constants depending on k_m, k_M.
Assume, moreover, that k_1 is finite, then there exists a constant c_3, depending on k_m, k_M and k_1, such that

$$\int_S |\nabla^3\xi|^2 \quad \leq \quad c\int_S \left(|\nabla^2 f|^2 + r^{-2}|\nabla f|^2 + r^{-4}|f|^2\right)$$

where, in the $\mathbf{H}_1$ case, $f = (f, f_*)$ and $|f|^2 = |f|^2 + |f_*|^2$.

We will need also some L^p estimates for the above systems, which we recall from [Ch-Kl], Chapter 2.

Proposition 4.1.3 *Assume that S is an arbitrary compact 2-surface satisfying $k_m > 0$ and $k_M < \infty$, then the following statements hold*

i. Let ϕ be a solution to the Poisson equation $\mathbf{H_0}$ on S. There exists a constant c which depends only on ${k_m}^{-1}, k_M, p$ such that

$$|\nabla^2\phi|_{L^p} + r^{-1}|\nabla\phi|_{L^p} + r^{-2}|\phi - \bar{\phi}|_{L^p} \leq c|f|_{L^p}$$
$$|\nabla^3\phi|_{L^p} \leq c_p\left(|\nabla f|_{L^p} + r^{-1}|f|_{L^p}\right)$$

ii. Let ξ be a solution of either $\mathbf{H}_1$ or $\mathbf{H}_2$, then we have:

First derivatives estimates in L^p. There exists a constant c which depends only on ${k_m}^{-1}$, k_M and p such that, for all $2 \leq p < \infty$

$$\int_S (|\nabla\xi|^p + r^{-p}|\xi|^p) \leq c\int_S |f|^p$$

[2] r has been defined in 3.1.2.

Second derivatives estimates in L^p. *There exists a constant c which depends only on ${k_m}^{-1}$, k_M and p such that, for all $2 \leq p < \infty$*

$$\int_S |\nabla\!\!\!\!/^2 \xi|^p \leq c \int_S (|\nabla\!\!\!\!/ f|^p + r^{-p}|f|^p)$$

4.1.2 Global Sobolev inequalities

In this subsection we assume that the spacetime $\mathcal{K}$ has a double null foliation [3] and that the following assumptions hold

a) $\sup_{\mathcal{K}} |\text{tr}\chi - \frac{2}{r}| \leq \delta$, $\sup_{\mathcal{K}} |\text{tr}\underline{\chi} + \frac{2}{r}| \leq \delta$ with δ small.

b) $k_m > 0$ on any surface $S(u,\underline{u}) = C(u) \cap \underline{C}(\underline{u})$.

Moreover we use the notations

$$V(u,\underline{u}) = J^-(S(u,\underline{u})) \ , \ u_0 = u|_{\underline{C}(\underline{u})\cap\Sigma_0} \ , \ \underline{u}_0 = u|_{C(u)\cap\Sigma_0} \ .$$

Proposition 4.1.4 *Let F be a smooth, S-tangent tensor field [4]. The following nondegenerate version of the global Sobolev inequality along $C(u)$ holds true.*

$$\begin{aligned} sup_{S(u,\underline{u})}(r^{\frac{3}{2}}|F|) \ \leq \ & c\Big[(\int_{S(u,\underline{u}_0)} r^4|F|^4)^{\frac{1}{4}} + (\int_{S(u,\underline{u}_0)} r^4|r\nabla\!\!\!\!/ F|^4)^{\frac{1}{4}} \\ & +\Big(\int_{C(u)\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!\!/ F|^2 + r^2|\mathbf{D}\!\!\!\!/_4 F|^2 \\ & + r^4|\nabla\!\!\!\!/^2 F|^2 + r^4|\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_4 F|^2\Big)^{\frac{1}{2}} \Big] \end{aligned} \quad (4.1.1)$$

We also have the degenerate version,

$$\begin{aligned} sup_{S(u,\underline{u})}(r\tau_-^{\frac{1}{2}}|F|) \ \leq \ & c\Big[(\int_{S(u,\underline{u}_0)} r^2\tau_-^2|F|^4)^{\frac{1}{4}} + (\int_{S(u,\underline{u}_0)} r^2\tau_-^2|r\nabla\!\!\!\!/ F|^4)^{\frac{1}{4}} \\ & +\Big(\int_{C(u)\cap V(u,\underline{u})} (|F|^2 + r^2|\nabla\!\!\!\!/ F|^2 + \tau_-^2|\mathbf{D}\!\!\!\!/_4 F|^2 \\ & + r^4|\nabla\!\!\!\!/^2 F|^2 + r^2\tau_-^2|\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_4 F|^2\Big)^{\frac{1}{2}} \Big] \end{aligned} \quad (4.1.2)$$

[3] Observe that in the successive applications the spacetime $\mathcal{K}$ is foliated by two double null foliations, the *double null canonical foliation* and the *initial layer foliation* in the layer region near Σ_0. In that case u_0 and $\underline{u}_0$ have different expressions, see discussion in subsection 4.1.3.

[4] This means that at any point it is tangent to the 2-surface $S(u,\underline{u})$ passing through that point.

Analogous estimates are obtained along the incoming null hypersurfaces $\underline{C}(\underline{u})$,

$$
\begin{aligned}
sup_{S(u,\underline{u})}(r^{\frac{3}{2}}|F|) \quad \leq \quad & c\Big[\, (\int_{S(u_0,\underline{u})} r^4|F|^4)^{\frac{1}{4}} + (\int_{S(u_0,\underline{u})} r^4|r\nabla\!\!\!\!/ F|^4)^{\frac{1}{4}} \\
& +\Big(\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!\!/ F|^2 + r^2|\mathbf{D}\!\!\!\!/_3 F|^2 \\
& +r^4|\nabla\!\!\!\!/^2 F|^2 + r^4|\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3 F|^2\Big)^{\frac{1}{2}} \; \Big] \qquad (4.1.3)
\end{aligned}
$$

and

$$
\begin{aligned}
sup_{S(u,\underline{u})}(r\tau_-^{\frac{1}{2}}|F|) \quad \leq \quad & c\Big[(\int_{S(u_0,\underline{u})} r^2\tau_-^2|F|^4)^{\frac{1}{4}} + (\int_{S(u_0,\underline{u})} r^2\tau_-^2|r\nabla\!\!\!\!/ F|^4)^{\frac{1}{4}} \\
& +\Big(\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!\!/ F|^2 + \tau_-^2|\mathbf{D}\!\!\!\!/_3 F|^2 \\
& +r^4|\nabla\!\!\!\!/^2 F|^2 + r^2\tau_-^2|\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3 F|^2\Big)^{\frac{1}{2}} \; \Big] \qquad (4.1.4)
\end{aligned}
$$

Proof: The proof of inequalities 4.1.1, 4.1.2 is based on the following lemma.

Lemma 4.1.1 *Let* F *be a smooth tensorfield on* $\mathcal{K}$, *tangent to the two dimensional surfaces* $S(u,\underline{u})$ *at every point. Introduce the following quantities*

$$
\begin{aligned}
A(F) &\equiv \sup_{C(u)\cap V(u,\underline{u})} \left(\int_{S(u,\underline{u})} r^4|F|^4 \right)^{\frac{1}{4}} \\
B(F) &\equiv \left(\int_{C(u)\cap V(u,\underline{u})} r^6|F|^6 \right)^{\frac{1}{6}} \qquad (4.1.5) \\
E(F) &\equiv \left(\int_{C(u)\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!\!/ F|^2 + r^2|\mathbf{D}\!\!\!\!/_4 F|^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
A_*(F) &\equiv \sup_{C(u)\cap V(u,\underline{u})} \left(\int_{S(u,\underline{u})} r^2\tau_-^2|F|^4 \right)^{\frac{1}{4}} \\
B_*(F) &\equiv \left(\int_{C(u)\cap V(u,\underline{u})} r^4\tau_-^2|F|^6 \right)^{\frac{1}{6}} \qquad (4.1.6) \\
E_*(F) &\equiv \left(\int_{C(u)\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!\!/ F|^2 + \tau_-^2|\mathbf{D}\!\!\!\!/_4 F|^2 \right)^{\frac{1}{2}} .
\end{aligned}
$$

The following nondegenerate inequalities hold

$$B \leq c(I)A^{2/3}E^{1/3} \tag{4.1.7}$$

$$A \leq A_0 + c(I)B^{3/4}E^{1/4} \tag{4.1.8}$$

We also have the degenerate estimates,

$$B_* \leq c(I)A_*^{2/3}E_*^{1/3} \tag{4.1.9}$$

$$A_* \leq A_{*0} + c(I)B_*^{3/4}E_*^{1/4} \tag{4.1.10}$$

where

$$A_0 = \left(\int_{S(u,\underline{u}_0)} r^4|F|^4\right)^{\frac{1}{4}}\ ,\ A_{*0} = \left(\int_{S(u,\underline{u}_0)} r^2\tau_-^2|F|^4\right)^{\frac{1}{4}}\ ,$$

and $c(I)$ *is a constant depending on* $I = \sup_{C(u)} I(u,\underline{u})$, *where* $I(u,\underline{u})$ *is the isoperimetric constant of* $S(u,\underline{u})$.

The proof of inequalities 4.1.3, 4.1.4 is based on the analogous of Lemma 4.1.1

Lemma 4.1.2 *Let* G *be a smooth tensorfield on* $\mathcal{K}$ *tangent to the two dimensional surfaces* $S(u,\underline{u})$ *at every point. Introduce the following quantities*

$$\begin{aligned}
\underline{A}(F) &\equiv \sup_{\underline{C}(\underline{u})\cap V(u,\underline{u})}\left(\int_{S(u,\underline{u})} r^4|F|^4\right)^{\frac{1}{4}}\\
\underline{B}(F) &\equiv \left(\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} r^6|F|^6\right)^{\frac{1}{6}}\\
\underline{E}(F) &\equiv \left(\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + r^2|\mathbf{D}\!\!\!/_3F|^2\right)^{\frac{1}{2}}
\end{aligned} \tag{4.1.11}$$

and

$$\begin{aligned}
\underline{A}_{de.}(F) &\equiv \sup_{\underline{C}(\underline{u})\cap V(u,\underline{u})}\left(\int_{S(u,\underline{u})} r^2\tau_-^2|F|^4\right)^{\frac{1}{4}}\\
\underline{B}_{de.}(F) &\equiv \left(\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} r^4\tau_-^2|F|^6\right)^{\frac{1}{6}}\\
\underline{E}_{de.}(F) &\equiv \left(\int_{\underline{C}(\nu)\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + \tau_-^2|\mathbf{D}\!\!\!/_3F|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.1.12}$$

Then the following inequalities hold

$$\underline{B} \leq c(I)\underline{A}^{2/3}\underline{E}^{1/3} \tag{4.1.13}$$

$$\underline{A} \leq \underline{A}_0 + c(I)\underline{B}^{3/4}\underline{E}^{1/4} \tag{4.1.14}$$

and

$$\underline{B}_{de.} \leq c(I)\underline{A}_{de.}^{2/3}\underline{E}_{de.}^{1/3} \tag{4.1.15}$$

$$\underline{A}_{de.} \leq \underline{A}_{de.0} + c(I)\underline{B}_{de.}^{3/4}\underline{E}_{de.}^{1/4} \tag{4.1.16}$$

where

$$\underline{A}_0 = \left(\int_{S(u_0,\underline{u})} r^4|F|^4\right)^{\frac{1}{4}} \ , \ \underline{A}_{de.0} = \left(\int_{S(u_0,\underline{u})} r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} \tag{4.1.17}$$

Corollary 4.1.1 *Under the assumptions of Lemma 4.1.1 and Lemma 4.1.2 the following estimates hold*

$$\begin{aligned}
\left(\int_{S(u,\underline{u})} r^4|F|^4\right)^{\frac{1}{4}} &\leq \left(\int_{S(u,\underline{u}_0)} r^4|F|^4\right)^{\frac{1}{4}} \\
&\quad + c\left(\int_{C(u)\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + r^2|D\!\!\!\!/_4 F|^2\right)^{\frac{1}{2}} \\
\left(\int_{S(u,\underline{u})} r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} &\leq \left(\int_{S(u,\underline{u}_0)} r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} \\
&\quad + c\left(\int_{C(u)\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + \tau_-^2|D\!\!\!\!/_4 F|^2\right)^{\frac{1}{2}}
\end{aligned} \tag{4.1.18}$$

Also,

$$\begin{aligned}
\left(\int_{S(u,\underline{u})} r^4|F|^4\right)^{\frac{1}{4}} &\leq \left(\int_{S(u_0,\underline{u})} r^4|F|^4\right)^{\frac{1}{4}} \\
&\quad + c\left(\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + r^2|D\!\!\!\!/_3 F|^2\right)^{\frac{1}{2}} \\
\left(\int_{S(u,\underline{u})} r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} &\leq \left(\int_{S(u_0,\underline{u})} r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} \\
&\quad + c\left(\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + \tau_-^2|D\!\!\!\!/_3 F|^2\right)^{\frac{1}{2}}
\end{aligned} \tag{4.1.19}$$

The proof of Proposition 4.1.4 follows immediately from this corollary combined with the following form of the standard Sobolev inequalities for the sphere

Lemma 4.1.3 *Let G be a tensor field tangent to the spheres $S(u,\underline{u})$, then*

$$sup_{S(u,\underline{u})}|G| \leq cr^{-\frac{1}{2}}\left(\int_{S(u,\underline{u})}|G|^4 + r^4|\nabla\!\!\!/ G|^4\right)^{\frac{1}{4}} \tag{4.1.20}$$

Indeed, it suffices to apply this lemma to $G = rF$, or $G = r^{\frac{1}{2}}\tau_-^{\frac{1}{2}}F$ and then take Lemma 4.1.1 into account. We now present the main steps in the proof of the nondegenerate version of Lemma 4.1.1.

To prove 4.1.7 we recall the following version of the isoperimetric inequality, see [Cha], for a compact two dimensional surface S of strictly positive Gauss curvature:

$$\int_S(\Phi-\bar{\Phi})^2 \leq I(S)\Big(\int_S|\nabla\!\!\!/\Phi|\Big)^2 \tag{4.1.21}$$

where Φ is a scalar function on a sphere S in $\mathcal{K}$ and the isoperimetric constant $I(S)$ [5] can be bounded by a constant which depends only on k_M.

Applying 4.1.21 to the surfaces $S(u,\underline{u}) \subset C(u)\cap V(u,\underline{u})$ with $\Phi = |F|^3$ and using the Holder inequality we derive

$$\int_{S(u,\underline{u})}|F|^6 \leq c\Big(r^{-2}\int_{S(u,\underline{u})}|F|^4\Big)\Big(\int_{S(u,\underline{u})}|F|^2 + r^2|\nabla\!\!\!/ F|^2\Big). \tag{4.1.22}$$

Multiplying the equation 4.1.22 by r^6 and integrating with respect to $\underline{u}$ we easily derive 4.1.7. To obtain 4.1.8 we express, with the help of the divergence theorem, assuming that everywhere $\mathrm{tr}\chi$ is near to $\frac{2}{r}$, the integral $\int_{S(u,\underline{u})} r^4|F|^4$ in terms of an integral over $C(u)\cap V(u,\underline{u})$ and an integral over $S(u,\underline{u}_0)$. Applying also the Cauchy-Schwartz inequality this leads to

$$\begin{aligned}\int_{S(u,\underline{u})} r^4|F|^4 \quad &\leq \quad \int_{S(u,\underline{u}_0)} r^4|F|^4 \\ &+c\Big(\int_{C(u)\cap V(u,\underline{u})} r^6|F|^6\Big)^{\frac{1}{2}}\Big(\int_{C(u)\cap V(u,\underline{u})} r^2|D\!\!\!\!/_4 F|^2\Big)^{\frac{1}{2}}\end{aligned}$$

which proves 4.1.8.

[5] $I(S)^{-\frac{1}{2}} = \inf_\Gamma\left(L(\Gamma)/\min\{A(D_1),A(D_2)\}^{\frac{1}{2}}\right)$ where Γ is an arbitrary closed curve on $S(u,\underline{u})$, $L(\Gamma)$ its total length and $A(D_1)$, $A(D_2)$ the areas of the two components of S/Γ.

To prove the degenerate estimates 4.1.9, 4.1.10 of the Lemma 4.1.1 we proceed precisely in the same way with the quantities A_*, B_* and E_*. In this case the inequality 4.1.15 follows by multiplying 4.1.22 by $r^4\tau_-^2$ and integrating in $\underline{u}$. The corresponding inequality 4.1.10 follows, as in the nondegenerate case, by applying the divergence theorem to $\int_{S(u,\underline{u})} r^2\tau_-^2|F|^4$.

We conclude this subsection by recalling the Gronwall inequality, see [Ho], and the Evolution Lemma, which will be used, repeteadly in the following sections.

Lemma 4.1.4 (Gronwall inequality) *Let $f, g : [a,b) \to R$ be continuous and non negative. Assume*

$$f(t) \leq A + \int_a^t f(s)g(s)ds \ , \quad A \geq 0$$

then

$$f(t) \leq A \exp \int_a^t g(s)ds \ , \quad for \ \ t \in [a,b)$$

Lemma 4.1.5 (Evolution Lemma) *Consider the spacetime $\mathcal{K}$ foliated by a double null foliation.*

I) *Assume, with $\delta > 0$ sufficiently small,*

$$|\Omega tr\chi - \overline{\Omega tr\chi}| \leq \delta r^{-2} \tag{4.1.23}$$

Let U, F, be k-covariant S-tangent tensor fields verifying the outgoing evolution equation

$$\frac{dU_{a_1\ldots a_k}}{d\underline{u}} + \lambda_0 \Omega tr\chi U_{a_1\ldots a_k} = F_{a_1\ldots a_k} \tag{4.1.24}$$

with λ_0 is a non negative real number and

$$U_{a_1\ldots a_k} \equiv U(e_{a_1}, e_{a_2}, .., e_{a_k}) \ , \ F_{a_1\ldots a_k} \equiv F(e_{a_1}, e_{a_2}, .., e_{a_k})$$

the components relative to an arbitrary orthonormal frame on S. Denoting $\lambda_1 = 2(\lambda_0 - \frac{1}{p})$, we have, along $C(u)$,

$$|r^{\lambda_1} U|_{p,S}(u,\underline{u}) \leq c_0 \left(|r^{\lambda_1} U|_{p,S}(u,\underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*} |r^{\lambda_1} F|_{p,S}(u,\underline{u}')d\underline{u}' \right) \tag{4.1.25}$$

Here $\underline{u}_*$ is the value that the function $\underline{u}(p)$ assumes on $\underline{C}_*$.

II: *Assume, with $\delta > 0$ sufficiently small,*

$$|\Omega tr\underline{\chi} - \overline{\Omega tr\underline{\chi}}| \leq \delta r^{-1}\tau_-^{-1} \tag{4.1.26}$$

Let V, $\underline{F}$, be k-covariant S-tangent tensor fields verifying the incoming evolution equation

$$\frac{dV_{a_1\dots a_k}}{du} + \lambda_0 \Omega tr\underline{\chi} V_{a_1\dots a_k} = \underline{F}_{a_1\dots a_k} \tag{4.1.27}$$

Denoting $\lambda_1 = 2(\lambda_0 - \frac{1}{p})$, we have, along $\underline{C}(\underline{u})$,

$$|r^{\lambda_1}V|_{p,S}(u,\underline{u}) \leq c_0 \left(|r^{\lambda_1}V|_{p,S(u_0(\underline{u}),\underline{u})} + \int_{u_0(\underline{u})}^{u} |r^{\lambda_1}\underline{F}|_{p,S}(u',\underline{u})du' \right) \tag{4.1.28}$$

where $S(u_0(\underline{u}),\underline{u}) \equiv C(u_0(\underline{u})) \cap \underline{C}(\underline{u}) \subset \mathcal{K}$.

Remark: *Here $u_0(\underline{u}) \neq u|_{\underline{C}(\underline{u})\cap\Sigma_0}$. In the application of part II of the Lemma we will choose the two dimensional surface $S(u_0(\underline{u}),\underline{u})$ in a convenient way* [6]*.*

Proof: From Lemma 3.1.3 we have, for any scalar function f,

$$\frac{d}{d\underline{u}} \int_{S(u,\underline{u})} f d\mu_\gamma = \int_{S(u,\underline{u})} \left(\frac{df}{d\underline{u}} + \Omega \mathrm{tr}\chi f \right) d\mu_\gamma \tag{4.1.29}$$

In particular, setting $f = 1$, denoting $|S(u,\underline{u})|$ the area of $S(u,\underline{u})$ and with $\overline{h}$ the average of h over $S(u,\underline{u})$, we obtain:

$$\frac{d}{d\underline{u}}|S(u,\underline{u})| = |S(u,\underline{u})|\overline{\Omega\mathrm{tr}\chi}$$

and, from the definition 3.1.2, $r(u,\underline{u}) \equiv \sqrt{\frac{1}{4\pi}|S(u,\underline{u})|}$,

$$\frac{d}{d\underline{u}} r(u,\underline{u}) = \frac{r(u,\underline{u})}{2}\overline{\Omega\mathrm{tr}\chi} \tag{4.1.30}$$

Hence, for any function f and any real number λ,

$$\begin{aligned} \frac{d}{d\underline{u}} \left(\int_{S(u,\underline{u})} r^\lambda f d\mu_\gamma \right) &= \int_{S(u,\underline{u})} r^\lambda \left(\frac{df}{d\underline{u}} + (1 + \frac{\lambda}{2})\Omega\mathrm{tr}\chi f \right) \\ &- \frac{\lambda}{2} \int_{S(u,\underline{u})} r^\lambda f (\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}) \end{aligned} \tag{4.1.31}$$

[6]Such that it allows us to connect the norm on $S(u_0(\underline{u}),\underline{u})$ with the initial norms on $S_{(0)}(\underline{u}) \subset \Sigma_0$.

The equation satisfied by the tensor field U implies:

$$\frac{d}{d\underline{u}}|U|^p + \lambda_0 p\Omega\mathrm{tr}\chi|U|^p \le p|F||U|^{p-1}$$
$$\frac{d}{d\underline{u}}|U|^p + \lambda_0 p\Omega\mathrm{tr}\chi|U|^p \ge -p|F||U|^{p-1} \tag{4.1.32}$$

therefore setting $f = |U|^p$ in the equation 4.1.31 and

$$\lambda = \lambda_1 p \text{ where } \lambda_1 = \left(2\lambda_0 - \frac{2}{p}\right)$$

we obtain:

$$\begin{aligned}\frac{d}{d\underline{u}}\int_{S(u,\underline{u})} r^{\lambda_1 p}|U|^p d\mu_\gamma &= \int_{S(u,\underline{u})} r^{\lambda_1 p}\left(\frac{d}{d\underline{u}}|U|^p + \lambda_0 p\Omega\mathrm{tr}\chi|U|^p\right) d\mu_\gamma \\ &- \frac{\lambda_1 p}{2}\int_{S(u,\underline{u})} r^{\lambda_1 p}|U|^p(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi})d\mu_\gamma\end{aligned}$$

and, using the second inequality of 4.1.32:

$$\begin{aligned}\frac{d}{d\underline{u}}\int_{S(u,\underline{u})} r^{\lambda_1 p}|U|^p d\mu_\gamma &\ge -p\int_{S(u,\underline{u})} r^{\lambda_1 p}|F||U|^{p-1}d\mu_\gamma \\ &-\frac{\lambda_1 p}{2}\int_{S(u,\underline{u})} r^{\lambda_1 p}|U|^p(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi})d\mu_\gamma\end{aligned} \tag{4.1.33}$$

Applying the Holder inequality:

$$\int_{S(u,\underline{u})} r^{\lambda_1 p}|F||U|^{p-1} \le \left(\int_{S(u,\underline{u})} r^{\lambda_1 p}|F|^p\right)^{\frac{1}{p}}\left(\int_{S(u,\underline{u})} r^{\lambda_1 p}|U|^p\right)^{\frac{p-1}{p}} ,$$

we obtain

$$\begin{aligned}-\frac{d}{d\underline{u}}\int_{S(u,\underline{u})} r^{\lambda_1 p}|U|^p d\mu_\gamma &\le p\left(\int_{S(u,\underline{u})} r^{\lambda_1 p}|F|^p\right)^{\frac{1}{p}}\left(\int_{S(u,\underline{u})} r^{\lambda_1 p}|U|^p\right)^{\frac{p-1}{p}} \\ &+ \frac{|\lambda_1|p}{2}\int_{S(u,\underline{u})} r^{\lambda_1 p}|U|^p|\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}|d\mu_\gamma .\end{aligned}$$

We make now use of the first inequality 4.1.23 and derive the inequality

$$-\frac{d}{d\underline{u}}|r^{\lambda_1}U|_{p,S} \le \left(|r^{\lambda_1}F|_{p,S} + c\delta r^{-2}|r^{\lambda_1}U|_{p,S}\right) \tag{4.1.34}$$

which, upon integration in the interval $[\underline{u}, \underline{u}_*]$, yields

$$|r^{\lambda_1}U|_{p,S}(u,\underline{u}) \leq |r^{\lambda_1}U|_{p,S}(u,\underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*} |r^{\lambda_1}F|_{p,S}(u,\underline{u}')d\underline{u}' + (\sup_{[\underline{u},\underline{u}_*]} |r^{\lambda_1}U|_{p,S})\left(c\delta\int_{\underline{u}}^{\underline{u}_*} r^{-2}\right) \quad (4.1.35)$$

Choosing δ such that $\left(c\delta\int_{\underline{u}}^{\underline{u}_*} r^{-2}\right) \leq \delta' < 1$ we have, for fixed $\underline{u}$ and any $\underline{u}' \in [\underline{u}, \underline{u}_*]$,

$$|r^{\lambda_1}U|_{p,S}(u,\underline{u}') \leq |r^{\lambda_1}U|_{p,S}(u,\underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*} |r^{\lambda_1}F|_{p,S}(u,\underline{u}'')d\underline{u}'' + \delta' \sup_{[\underline{u},\underline{u}_*]} |r^{\lambda_1}U|_{p,S} \ .$$

Taking the sup with respect to $\underline{u}'$ in $[\underline{u}, \underline{u}_*]$, we obtain

$$(1-\delta')\sup_{[\underline{u},\underline{u}_*]} |r^{\lambda_1}U|_{p,S} \leq |r^{\lambda_1}U|_{p,S}(u,\underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*} |r^{\lambda_1}F|_{p,S}(u,\underline{u}')d\underline{u}' \quad (4.1.36)$$

from which (I) of the Evolution Lemma follows with $c_0 = \frac{1}{(1-\delta')}$.

To obtain (II) we proceed in the same way, using the differential inequalities

$$\begin{aligned} \frac{d}{du}|V|^p + \lambda_0 p\Omega\mathrm{tr}\underline{\chi}|V|^p &\leq p|\underline{F}||V|^{p-1} \\ \frac{d}{du}|V|^p + \lambda_0 p\Omega\mathrm{tr}\underline{\chi}|V|^p &\geq -p|\underline{F}||V|^{p-1} \end{aligned} \quad (4.1.37)$$

and assumption 4.1.26.

4.1.3 The initial layer foliation

In the previous section we have encountered a difficulty with part II) of the evolution Lemma 4.1.5 along the incoming null hypersurfaces $\underline{C}(\nu)$. The double null canonical foliation of $\mathcal{K}$ does not allow us to connect the surfaces $S(\lambda,\nu)$ with the surfaces $S_{(0)}(\nu)$ on Σ_0, see also Proposition 3.3.1.

We show in this subsection how to overcome this difficulty by using, in a small neighbourhood of Σ_0, the *initial layer foliation* introduced in Chapter 3, see definition 3.3.4 which we recall below.

The *initial layer foliation* is the double null foliation defined by the incoming null surfaces $\underline{C}(\underline{u})$, and the outgoing null hypersurfaces $C'(u')$ both intersecting Σ_0 along the canonical foliation $S_{(0)}(\nu)$. More precisely, see definition 3.3.4,
a) The $C'(\lambda')$ null hypersurfaces are given by $u'(p)=\lambda'$, where $\lambda'\in[-\nu_0,-\nu_*]$, with u' the outgoing solution of the eikonal equation with initial condition $u'=-\underline{u}_{(0)}$ on Σ_0.
b) The $\underline{C}(\nu)$ null hypersurfaces are defined as before by $\underline{u}(p)=\nu$ where $\nu\in[\nu_0,\nu_*]$, with $\underline{u}$ the incoming solution of the eikonal equation with initial condition $\underline{u}=\underline{u}_{(0)}$ on the initial hypersurface Σ_0.
Observe that $S_{(0)}(\nu)=C'(-\nu)\cap\underline{C}(\nu)$. The initial layer region $\mathcal{K}'_{\delta_0}\subset\mathcal{K}$ is specified by the condition

[Nella formula seguente, 4.1.38, $(u'(p)+\underline{u}(p))\leq\delta_0$ è stata sostituita da $\frac{1}{2}(u'(p)+\underline{u}(p))\leq\delta_0$.]

$$\frac{1}{2}(u'(p)+\underline{u}(p))\leq\delta_0 \tag{4.1.38}$$

As discussed in Chapter 3, see Proposition 3.3.1, the initial layer K'_{δ_0} comes also equipped with an adapted space like foliation, $\{\Sigma'_{t'}\}$ with $t'=\frac{1}{2}(u'+\underline{u})$. With this definition Σ_0, the initial hypersurface, satisfies $\Sigma_0=\Sigma'_{t'=0}$. Thus the height δ_0 of the layer K'_{δ_0} corresponds to the time interval $0\leq t'\leq\delta_0$.

We are now ready to state our main result concerning the compatibility between the canonical and initial layer foliations.

Lemma 4.1.6 (Oscillation Lemma) *Consider a space time region $\mathcal{K}$ with the "canonical double null foliation" generated by $u(p),\underline{u}(p)$. Consider also an initial layer region $\mathcal{K}'_{\delta_0}$, of height δ_0, with the "initial layer foliation" generated by $u'(p),\underline{u}(p)$. We make the following assumptions:*

- *On the surface $S'_*=\Sigma'_{\delta_0}\cap\underline{C}_*=S'(2\delta_0-\nu_*,\nu_*)$*

$$\left(\sup_{(p,p')\in S'_*}|u(p)-u(p')|\right)\leq\epsilon_0 \tag{4.1.39}$$

  Also,

[The assumptions on $g(L,L')$ have been changed. These are the right ones.]

$$|r^2\tau_-^{\frac{1}{2}}\eta|\leq\epsilon_0\ ,\ |r'^2\tau'_-\mathbf{g}(L',L)|\leq\epsilon_0\ ,\ |r'^3\tau'_-\nabla\!\!\!/\mathbf{g}(L',L)|\leq\epsilon_0 \tag{4.1.40}$$

- *On the initial hypersurface Σ_0,*

$$|r'^{\frac{5}{2}}\eta'|\leq\epsilon_0 \tag{4.1.41}$$

[In the following item "on $\mathcal{K}$" has been substituted with "on $\mathcal{K}/\mathcal{K}'_{\delta_0}$".]

- *On $\mathcal{K}/\mathcal{K}'_{\delta_0}$,*

$$\mathcal{O}^{\infty}_{[1]} + \underline{\mathcal{O}}^{\infty}_{[1]} \leq \epsilon_0 \tag{4.1.42}$$

[The conditions on $\nabla\!\!\!/ \mathbf{D}_{e_4} \log\Omega$ and $\nabla\!\!\!/ \mathbf{D}_{e_3} \log\Omega$ on $\mathcal{K}/\mathcal{K}'_{\delta_0}$ have been eliminated as we do not use anymore on $\mathcal{K}/\mathcal{K}'_{\delta_0}$ the stronger estimates for $\eta, \underline{\eta}, \eta', \underline{\eta}'$, although they are , of course, true. We use only the stronger estimates for $g(L, L')$.]

- *On the initial layer $\mathcal{K}'_{\delta_0}$,*

$$\mathcal{O}'^{\infty}_{[1]} + \underline{\mathcal{O}}'^{\infty}_{[1]} \leq \epsilon_0 \tag{4.1.43}$$

Then,

$$Osc(u)(\Sigma'_{\delta_0}) \equiv \sup_{\nu \in [\nu_0, \nu_*]} \left(\sup_{(p,p') \in S'(2\delta_0 - \nu, \nu)} |u(p) - u(p')| \right) \leq c\epsilon_0 \tag{4.1.44}$$

Remarks:

[The previous remark 2 has been eliminated. The third remark is now the second.]

1) The norms appearing in 4.1.40, 4.1.41, 4.1.42, and 4.1.43 are pointwise.
2) The assumptions 4.1.40 are verified in view of the canonicity of the foliation on the last slice $\underline{C}_*$, see Proposition 7.4.1 and Lemma 7.7.2.
3) The assumptions 4.1.41 are verified in view of the canonicity of the foliation on the initial slice Σ_0 and are used in Lemma 4.8.2.

Proof of the Oscillation Lemma: The detailed proof of the Oscillation Lemma is given in the appendix to this chapter.

Corollary 4.1.2 *Given an incoming null hypersurface $\underline{C}(\nu)$ there exists a two dimensional surface S relative to the "double null canonical foliation", belonging to $\underline{C}(\nu)$ and included in the initial layer region $\mathcal{K}'_{\delta_0}$, for any $\nu \in [\nu_0, \nu_*]$.*

Proof: We define

$$\tilde{\delta}_0 = \frac{1}{2} \inf_{\nu \in [\nu_0, \nu_*]} \left(\inf_{p \in S'(2\delta_0 - \nu, \nu)} (u(p) + \nu) \right) \tag{4.1.45}$$

From the Oscillation Lemma, it follows that

$$|\tilde{\delta}_0 - \delta_0| \leq c\epsilon_0 \tag{4.1.46}$$

$S(\lambda_0(\nu), \nu)$, with $\lambda_0(\nu) = 2\tilde{\delta}_0 - \nu$, is a two dimensional surface, relative to the "double null canonical foliation" included in the initial layer region $\mathcal{K}'_{\delta_0}$, for any $\nu \in [\nu_0, \nu_*]$. These are the surfaces which we refer to, in part II of the Evolution Lemma, see 4.1.28.

Remarks:
i) In Chapter 7, assuming tha initial and final slice endowed with a canonical foliation all the estimates in the assumptions of the Oscillation Lemma are proved with $\epsilon_0 \leq c\epsilon$.
ii) Recalling the definition of the $\tilde{\Sigma}_{\tilde{t}}$ spacelike hypersurfaces, associated to the *double null canonical foliation*, see Proposition 3.3.1, the previous lemma and its corollary implies that we can extend the double null canonical foliation to $\tilde{\Sigma}_{\tilde{\delta}_0}$ with $\tilde{\delta}_0 \geq \delta_0 - 2c\epsilon_0 > 0$, a little below Σ'_{δ_0}.
In other words we can find a spacelike hypersurface foliated by the $S(\lambda, \nu)$ two dimensional surfaces, relative to the *double null canonical foliation* contained in the initial layer region at a distance ϵ_0 from Σ'_{δ_0}.

We use Lemma 4.1.6 to express the estimates of part II of the Evolution Lemma, 4.1.28 in term of the initial data norms on $S_{(0)}$.
We rewrite, first, the estimates of Part II of the Evolution Lemma [7],

$$|r^{\lambda_1}V|_{p,S}(u,\underline{u}) \leq c_0 \left(|r^{\lambda_1}V|_{p,S(u_0(\underline{u}),\underline{u})} + \int_{u_0(\underline{u})}^{u} |r^{\lambda_1}\underline{F}|_{p,S}(u',\underline{u})du' \right) \quad (4.1.47)$$

In the next lemma we prove, using the Oscillation Lemma, how to control the difference between the norm $|r^{\lambda_1}V|_{p,S(u_0(\nu),\nu)}$ and the norm $|r'^{\lambda_1}V'|_{p,S_{(0)}(\nu)}$, where V' indicates the S-tangent tensor field analogous [8] to V, but relative to the *initial layer foliation*, see the Evolution Lemma.
Recall that the norm $|r^{\lambda_1}V|_{p,S(u_0(\nu),\nu)}$ refer to a surface $S(u_0(\nu),\nu)$ associated to the *double null canonical foliation* and $r = (\frac{1}{4\pi}|S(u_0(\nu),\nu)|)^{\frac{1}{2}}$, while $|r'^{\lambda_1}V'|_{p,S_{(0)}(\nu)}$ refers to a surface contained in Σ_0 associated to the *initial layer foliation* and, therefore, $r' = (\frac{1}{4\pi}|S_{(0)}(\nu)|)^{\frac{1}{2}}$.

Lemma 4.1.7 *Let V be a tensor field satisfying the evolution equation*

$$\mathbf{D}_N V + \lambda_0 \Omega tr\underline{\chi} V = \underline{F} \ .$$

Assume that on $\tilde{\Sigma}_{\tilde{\delta}_0}$ and on ${\Sigma'}_{\delta_0}$, with ϵ_0 sufficiently small, the following inequality holds, see **??**,

$$\begin{aligned} \mathcal{O}^{\infty}_{[1]} + \underline{\mathcal{O}}^{\infty}_{[1]} &\leq \epsilon_0 \\ \mathcal{O}'^{\infty}_{[1]} + \underline{\mathcal{O}}'^{\infty}_{[1]} &\leq \epsilon_0 \end{aligned} \quad (4.1.48)$$

[7] We use here the notation $u_0(\nu)$ instead of $\lambda_0(\nu)$ to avoid confusion with the exponents of the estimates like 4.1.47.

[8] V and V' are not the same tensor field as they are tangent to different two dimensional surfaces. Nevertheless V can be expressed in terms of S'-tangent tensor fields and viceversa. See also the proof of the next lemma.

and that, in the initial layer region,

$$\mathcal{O}_{[2]}{}' \leq \epsilon_0 \;,\; \underline{\mathcal{O}}_{[2]}{}' \leq \epsilon_0$$
$$\mathcal{R}_{[2]}{}' \leq \epsilon_0 \;,\; \underline{\mathcal{R}}_{[2]}{}' \leq \epsilon_0 \qquad (4.1.49)$$

then, with $\lambda_1 = 2\left(\lambda_0 - \frac{1}{p}\right)$,

$$|r^{\lambda_1}V|_{p,S(u_0(\underline{u}),\underline{u})} \leq c|r'^{\lambda_1}V'|_{p,S_{(0)}(\underline{u})} + c\epsilon_0 \qquad (4.1.50)$$

Remark: During the whole Chapter 4, where this result is used, the tensor field V describes the various underlined connection coefficients and their derivatives.

Proof: The proof of this lemma is in the appendix to this Chapter.

4.1.4 Comparison estimates for the function $r(u, \underline{u})$

In the proofs of this chapter and of Chapters 5 and 6 we often use some estimates which connect the function $r(u,\underline{u})$ with the functions $u(p)$ and $\underline{u}(p)$ and also with the functions $v(p)$ and $\underline{v}(p)$, the affine parameters of the null geodesics generating the null hypersurfaces $C(u)$ and $\underline{C}(\underline{u})$. We collect all these estimates here.

We recall that $u(p)$ and $\underline{u}(p)$ are solutions of the eikonal equation, the first one having, as "initial data" the function $u_*(p)$ defined on the "last slice" $\underline{C}_*$ solution of the "last slice problem", see Definition 3.3.2, and $\underline{u}(p)$ having as initial data the function $\underline{u}_{(0)}(p)$ defined on Σ_0, solution of the "initial slice problem", see Definition 3.3.1.

The first estimate for the function $r(u,\underline{u}) \equiv \sqrt{\frac{|S(u,\underline{u})|}{4\pi}}$ is provided from the following lemma[9]

Lemma 4.1.8 *Assume in the spacetime $\mathcal{K}$ the estimate, see 4.2.9,*

$$|r^2(\overline{\Omega tr\chi} - \frac{2}{r})| \leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0)$$
$$|r\tau_-(\overline{tr\underline{\chi}} + \frac{2}{r})| \leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0)$$

assume that on $\underline{C}_*$

$$|r(\lambda,\nu_*) - \frac{1}{2}(\nu_* - \lambda)| \leq c\mathcal{I}_* \log r(\lambda,\nu_*)$$

[9]We discuss here this lemma and the following one, although they require, to be proved, the results of Theorem 4.2.1. Of course the proof of these results do not depend on these Lemmas.

then if $(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)$ *is sufficently small, then* $r(u,\underline{u})$ *satisfies the following inequality*

$$|r(u,\underline{u})-\frac{1}{2}(\underline{u}-u)|\leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\log r(u,\underline{u}) \tag{4.1.51}$$

Moreover, there exists a constant c such that

$$c^{-1}\tau_+\leq r(u,\underline{u})\leq c\tau_+ \tag{4.1.52}$$

Proof: We integrate the equation, see 4.1.30,

$$\frac{d}{d\underline{u}}r(u,\underline{u})=\frac{1}{2}+\frac{r(u,\underline{u})}{2}\left(\overline{\Omega\mathrm{tr}\chi}-\frac{1}{r}\right)$$

along $C(u)$, from $\underline{u}=\nu_*$ to $\underline{u}$, obtaining

$$r(u,\nu_*)-r(u,\underline{u})=\frac{1}{2}(\nu_*-\underline{u})+\int_{\underline{u}}^{\nu_*}\frac{r}{2}\left(\overline{\Omega\mathrm{tr}\chi}-\frac{1}{r}\right)$$

which we rewrite as

$$r(u,\underline{u})-\frac{1}{2}(\underline{u}-u)=r(u,\nu_*)-\frac{1}{2}(\nu_*-u)-\int_{\underline{u}}^{\nu_*}\frac{r}{2}\left(\overline{\Omega\mathrm{tr}\chi}-\frac{1}{r}\right)$$

Using the assumptions we obtain

$$(r(u,\underline{u})-\frac{1}{2}(\underline{u}-u))+c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\log r(u,\underline{u})\leq c\mathcal{I}_*$$

which implies

$$|r(u,\underline{u})-\frac{1}{2}(\underline{u}-u)|\leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\log r(u,\underline{u}) \tag{4.1.53}$$

proving the first part [10] of the lemma. The second part follows immediately.

Remark: The result of Lemma 4.1.8

$$|r(u,\underline{u})-\frac{1}{2}(\underline{u}-u)|\leq c\epsilon_0\log r(u,\underline{u})$$

although sufficient for our purposes is not the optimal one. A better and more delicate result is stated in the following lemma, whose proof we do not report here [11],

[10] The assumption relative to the last slice is proved in Chapter 7.

[11] The proof is, anyway, an adapted version of Proposition 9.1.3 of [Ch-Kl].

Lemma 4.1.9 *Under appropriate assumptions, consistent with the bootstrap assumptions for $\mathcal{O}$ of Theorem* **M1**, *$r(u,\underline{u})$ satisfies the following inequalities*

$$|r(u,\underline{u})-(\underline{v}-\frac{u}{2})|\leq c(\mathcal{I}_*+\triangle_0)$$
$$|r(u,\underline{u})+(v-\frac{u}{2})|\leq c(\mathcal{I}_0+\triangle_0) \tag{4.1.54}$$

where $v,\underline{v}$ are the affine parameters of the null geodesics generating $C(u)$ and $\underline{C}(\underline{u})$ respectively.

4.2 Proof of Theorem M1

We are now ready to start proving Theorem **M1**. While we have structured the proof in a way which we believe it is optimal for the comprehension of the reader we omit giving detailed motivations for various important technical steps. For this we refer the reader to our review paper [Kl-Ni].

We divide Theorem **M1** in three theorems, the first referring to zero and first derivatives, the second one to second derivatives and the third one concerning third derivatives.

Remarks:

- It is important to observe that to prove Theorem **M1** relative to a double null foliation we need to introduce as an auxiliary assumption the result of the Oscillation Lemma. To remove this auxiliary assumption we need that $\mathcal{K}$ be endowed with a *double null canonical foliation.*

- It is important to realize that all the norm assumptions relative to the initial hypersurface Σ_0 are relative to the connection coefficients relative to the *initial layer foliation*, these norms are connected to those relative to the *double null canonical foliation* through the Oscillation Lemma and Lemma 4.1.7.

Theorem 4.2.1 *Assume that*

$$\mathcal{R}_0^\infty+\underline{\mathcal{R}}_0^\infty\leq\Delta_0$$
$$\mathcal{R}_1^S+\underline{\mathcal{R}}_1^S\leq\Delta_1 \tag{4.2.1}$$

and

$$\mathcal{O}_{[1]}(\underline{C}_*) \leq \mathcal{I}_* \ , \ \underline{\mathcal{O}}_{[1]}(\Sigma_0) \leq \mathcal{I}_0 \tag{4.2.2}$$

Assume further that in the initial layer region

$$\begin{aligned} {\mathcal{O}_{[1]}}' \leq \mathcal{I}_0 \ , \ {\underline{\mathcal{O}}_{[1]}}' \leq \mathcal{I}_0 \\ {\mathcal{R}_{[1]}}' \leq \Delta_0 \ , \ {\underline{\mathcal{R}}_{[1]}}' \leq \Delta_0 \end{aligned} \tag{4.2.3}$$

Assume finally that $\Delta_0, \ \Delta_1, \ \mathcal{I}_0, \ \mathcal{I}_*$ *are sufficiently small, then there exists a constant* [12] *such that the following estimates hold*

$$\mathcal{O}_{[1]} + \underline{\mathcal{O}}_{[1]} \leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1) \tag{4.2.4}$$

Proof: We present here the strategy of the proof. All the details are given in section 4.3.

We divide the proof of the theorem in the following steps:

i) We make the following additional bootstrap assumptions

$$\mathcal{O}^\infty_{[0]} + \underline{\mathcal{O}}^\infty_{[0]} \leq \Gamma_0 \tag{4.2.5}$$

$$Osc(u)(\Sigma'_{\delta_0}) \leq \Gamma_0 \tag{4.2.6}$$

with $\Gamma_0 > 0$ sufficiently small [13], then we prove that the following inequalities hold

$$\begin{aligned} \mathcal{O}_0 &\leq c\,(\mathcal{I}_* + \Delta_0) \\ \underline{\mathcal{O}}_0 &\leq c\,(\mathcal{I}_0 + \Delta_0) \\ \mathcal{O}_1 &\leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0) \\ \underline{\mathcal{O}}_1 &\leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0) \end{aligned} \tag{4.2.7}$$

$$\underline{\tilde{\mathcal{O}}}_1(\underline{\omega}) \leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1) \tag{4.2.8}$$

and, finally,

$$\begin{aligned} |r(\Omega - \frac{1}{2})| &\leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0) \\ |r^2(\overline{\mathrm{tr}\chi} - \frac{2}{r})| &\leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0) \\ |r\tau_-(\overline{\mathrm{tr}\underline{\chi}} + \frac{2}{r})| &\leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0) \end{aligned} \tag{4.2.9}$$

---

[12] We denote with c a constant which does not depend on the relevant parameters. It can be different in different estimates.

[13] Γ_0 must be such that $\Gamma_0^2 < (\mathcal{I}_0 + \mathcal{I}_* + \Delta_0) < \Gamma_0$.

This is the main part of the proof. The details will be given in the subsequent sections. Next steps ii) and iii) are standard.

Remark: the bootstrap assumptions 4.2.5 allow us to use all the preliminary results of section 4.1

ii) Using the estimates 4.2.7, 4.2.8 and the Sobolev Lemma 4.1.3, we infer that

$$\mathcal{O}^\infty_{[0]}+\underline{\mathcal{O}}^\infty_{[0]}\leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\leq\frac{\Gamma_0}{2} \tag{4.2.10}$$

provided that we choose $(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)$ sufficiently small.

iii) To remove the additional assumption 4.2.5 we consider the region, $\mathcal{S}(\Gamma_0)$ contained in $\mathcal{K}$ defined by the following properties [14]

a) $\mathcal{S}(\Gamma_0)=\{p\in\mathcal{K}|\ (u(p),\underline{u}(p))\in[\lambda_1,\lambda_2)\times(\nu_2,\nu_*]\}$.

b) In $\mathcal{S}(\Gamma_0)$, the following inequality holds

$$\mathcal{O}^\infty_{[0]}+\underline{\mathcal{O}}^\infty_{[0]}<\Gamma_0\ .$$

Using the result in ii) we infer that this region is, simultaneously, open and closed and, therefore, must coincide with the whole $\mathcal{K}$. From this result the estimates 4.2.7, 4.2.8 and 4.2.9 hold in $\mathcal{K}$ and the theorem follows.
The details of the implementation of step i) are given in section 4.3.

Remarks:

i) Instead of this bootstrap assumption, 4.2.5, we could have used a stronger bootstrap assumption involving the full norms $\mathcal{O}_{[3]}$ and $\underline{\mathcal{O}}_{[3]}$. Because of the importance of this result we prefer, however, this proof which emphasizes the fact that only the norms $\mathcal{O}_{[1]}$ and $\underline{\mathcal{O}}_{[1]}$ are needed to break the non linear structure of the null structure equations.

ii) It si easy to check that the bootstrap assumption 4.2.5 implies all the assumptions needed in the proofs of all the preliminary results of previous section.

Theorem 4.2.2 *Assume that*

$$\begin{aligned}&\mathcal{R}^\infty_0+\underline{\mathcal{R}}^\infty_0\leq\Delta_0\\&\mathcal{R}^S_1+\underline{\mathcal{R}}^S_1\leq\Delta_1\\&\mathcal{R}_2+\underline{\mathcal{R}}_2\leq\Delta_2\end{aligned} \tag{4.2.11}$$

[14]Recall that λ_1 is the value of $u(p)$ on $\underline{C}_*\cap\Sigma_0$, see Chapter 3.

and also

$$\mathcal{O}_{[2]}(\underline{C}_*) \leq \mathcal{I}_* \ , \ \underline{\mathcal{O}}_{[2]}(\Sigma_0) \leq \mathcal{I}_0 \tag{4.2.12}$$

Assume further that in the initial layer region

$$\begin{aligned} {\mathcal{O}_{[2]}}' \leq \mathcal{I}_0 \ , \ {\underline{\mathcal{O}}_{[2]}}' \leq \mathcal{I}_0 \\ {\mathcal{R}_{[2]}}' \leq \Delta_0 \ , \ {\underline{\mathcal{R}}_{[2]}}' \leq \Delta_0 \end{aligned} \tag{4.2.13}$$

Assuming finally that $\Delta_0, \ \Delta_1, \ \Delta_2, \ \mathcal{I}_0, \ \mathcal{I}_*$ *are sufficiently small, then there exists a generic constant* c *such that*

$$\mathcal{O}_{[2]} + \underline{\mathcal{O}}_{[2]} \leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1 + \Delta_2) \tag{4.2.14}$$

Proof: We present here the strategy of the proof. All the details are given in section 4.4. We divide the proof in four steps.

i) We assume 4.2.6 and the auxiliary bootstrap assumption

$$\mathcal{O}_1^\infty + \underline{\mathcal{O}}_1^\infty \leq \Gamma_1 \tag{4.2.15}$$

with Γ_1 sufficiently small. Then the following inequalities hold [15]

$$\begin{aligned} \mathcal{O}_2 \leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1) \\ \underline{\mathcal{O}}_2 \leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1) \end{aligned} \tag{4.2.16}$$

$$\tilde{\mathcal{O}}_2(\omega) \leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1 + \Delta_2) \tag{4.2.17}$$

ii) These inequalities, together with the estimates

$$\begin{aligned} \mathcal{O}_{[1]} \leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1) \\ \underline{\mathcal{O}}_{[1]} \leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1) \end{aligned}$$

proved in the previous theorem, allow us, applying Lemma 4.1.3, to estimate $\mathcal{O}_1^\infty + \underline{\mathcal{O}}_1^\infty$ in terms of $\mathcal{O}_{[1]} + \underline{\mathcal{O}}_{[1]}$, $\mathcal{O}_2 + \underline{\mathcal{O}}_2$. Therefore we obtain

$$\mathcal{O}_1^\infty + \underline{\mathcal{O}}_1^\infty \leq c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1) \tag{4.2.18}$$

so that choosing $\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1$ sufficiently small we infer that

$$\mathcal{O}_1^\infty + \underline{\mathcal{O}}_1^\infty \leq \frac{\Gamma_1}{2} \tag{4.2.19}$$

[15] To prove these estimates we also use the results of the previous theorem.

iii) Introducing again, as in Theorem 4.2.1, a region $\mathcal{S}(\Gamma_1)$ contained in $\mathcal{K}$ we repeat the previous argument and check that

$$\mathcal{O}_1^\infty + \underline{\mathcal{O}}_1^\infty \leq \Gamma_1 \tag{4.2.20}$$

holds in the whole spacetime $\mathcal{K}$. In view of this result the inequalities 4.2.16, 4.2.17, 4.2.18 hold in $\mathcal{K}$.
The details of the implementation of steps i), ii), iii) are given in section 4.4.

Theorem 4.2.3 *Assume that*

$$\begin{aligned} &\mathcal{R}_0^\infty + \underline{\mathcal{R}}_0^\infty \leq \Delta_0 \\ &\mathcal{R}_1^S + \underline{\mathcal{R}}_1^S \leq \Delta_1 \\ &\mathcal{R}_2 + \underline{\mathcal{R}}_2 \leq \Delta_2 \end{aligned} \tag{4.2.21}$$

and also

$$\mathcal{O}_{[3]}(\underline{C}_*) \leq \mathcal{I}_* \ , \ \underline{\mathcal{O}}_{[3]}(\Sigma_0) \leq \mathcal{I}_0 \tag{4.2.22}$$

Assuming further that $\Delta_0,\ \Delta_1,\ \Delta_2,\ \mathcal{I}_0,\ \mathcal{I}_*$ are sufficiently small, there exists a generic constant c such that

$$\mathcal{O}_3 + \underline{\mathcal{O}}_3 \leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1 + \Delta_2) \tag{4.2.23}$$

The proof of this theorem is discussed in section 4.5.

We remove now, to complete the proof of Theorem **M1**, the assumptions on the oscillation of u. In fact we have

Corollary 4.2.4 *Under the assumptions* [16] *of Theorem* **M1**, *relative to a "double null canonical foliation", the following inequality holds*

$$Osc(u)(\Sigma'_{\delta_0}) \leq (\mathcal{I}_0 + \mathcal{I}_* + \Delta_0) \leq \frac{\epsilon_0}{2} \tag{4.2.24}$$

provided we choose $\mathcal{I}_0, \mathcal{I}_*, \Delta_0$ sufficiently small. Therefore this allows us to conclude that $Osc(u)(\Sigma'_{\delta_0}) \leq \epsilon_0$ on the whole Σ'_{δ_0} and that we can reduce the height of the initial layer region to $\frac{\delta_0}{2}$.

[16] This implies stronger assumptions on the initial and last slice, see, in particular proposition 4.3.17 and corollary 4.4.1.

4.3 Proof of Theorem 4.2.1, estimates for the zero and first derivatives of the connection coefficents

We concentrate on the proof of part **(i** of the theorem and divide the proof in many steps.

4.3.1 Estimate for $\mathcal{O}^{p,S}_{0,1}(\mathrm{tr}\chi)$ and $\mathcal{O}^{p,S}_{0,1}(\hat\chi)$ with $p\in[2,4]$

[In Prop 4.3.1 the estimate for $\mathrm{tr}\chi$ requires an estimate of ζ done later on, therefore the estimate is postponed.]

[modification in 4.3.1]

Proposition 4.3.1 *Assuming 4.2.1, 4.2.2 and the bootstrap assumption 4.2.5 the following estimates hold,*

$$\begin{aligned}|r^{3-2/p}\nabla\!\!\!\!/\,\hat\chi|_{p,S}(u,\underline{u}) &\le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\ |r^{2-2/p}\hat\chi|_{p,S}(u,\underline{u}) &\le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\end{aligned}\tag{4.3.1}$$

Proof: We derive first an evolution equation for $\nabla\!\!\!\!/\,\mathrm{tr}\chi$ by differentiating the evolution equation of $\mathrm{tr}\chi$ along the outgoing direction, see 3.1.45,

$$\nabla\!\!\!\!/\,\mathbf{D}_4\mathrm{tr}\chi-\nabla\!\!\!\!/\,((\mathbf{D}_4\log\Omega)\mathrm{tr}\chi)=-\mathrm{tr}\chi\nabla\!\!\!\!/\,\mathrm{tr}\chi-\nabla\!\!\!\!/\,|\hat\chi|^2\tag{4.3.2}$$

Using the commutation relation, see 4.8.1 in the appendix to this chapter,

$$[\nabla\!\!\!\!/\,,\mathbf{D}\!\!\!\!/\,_4]\mathrm{tr}\chi=\chi\cdot\nabla\!\!\!\!/\,\mathrm{tr}\chi-(\zeta+\underline{\eta})\mathbf{D}_4\mathrm{tr}\chi$$

equation 4.3.2 becomes

$$\mathbf{D}\!\!\!\!/\,_4\nabla\!\!\!\!/\,\mathrm{tr}\chi+\chi\cdot\nabla\!\!\!\!/\,\mathrm{tr}\chi=(\nabla\!\!\!\!/\,\log\Omega)\mathbf{D}_4\mathrm{tr}\chi+\nabla\!\!\!\!/\,((\mathbf{D}_4\log\Omega)\mathrm{tr}\chi)-\mathrm{tr}\chi\nabla\!\!\!\!/\,\mathrm{tr}\chi-\nabla\!\!\!\!/\,|\hat\chi|^2$$

which we rewrite

$$\begin{aligned}&\mathbf{D}\!\!\!\!/\,_4\nabla\!\!\!\!/\,\mathrm{tr}\chi-(\mathbf{D}_4\log\Omega)\nabla\!\!\!\!/\,\mathrm{tr}\chi+\frac{1}{2}\mathrm{tr}\chi\nabla\!\!\!\!/\,\mathrm{tr}\chi=-\hat\chi\cdot\nabla\!\!\!\!/\,\mathrm{tr}\chi+(\nabla\!\!\!\!/\,\log\Omega)\mathbf{D}_4\mathrm{tr}\chi\\ &\quad+(\nabla\!\!\!\!/\,\mathbf{D}_4\log\Omega)\mathrm{tr}\chi-\mathrm{tr}\chi\nabla\!\!\!\!/\,\mathrm{tr}\chi-\nabla\!\!\!\!/\,|\hat\chi|^2\end{aligned}\tag{4.3.3}$$

Defining

$$U\equiv\Omega^{-1}\nabla\!\!\!\!/\,\mathrm{tr}\chi$$

and choosing a null frame such that the vector fields e_a satisfy $\mathbf{D}\!\!\!\!/\,_4e_a=0$, the previous equation becomes

$$\frac{d}{d\underline{u}}U_a+\frac{3}{2}\Omega\mathrm{tr}\chi U_a=F_a\tag{4.3.4}$$

where

$$F = -\Omega\hat{\chi}\cdot U - \nabla\!\!\!/|\hat{\chi}|^2 + (\nabla\!\!\!/\mathbf{D}_4\log\Omega)\mathrm{tr}\chi - (\nabla\!\!\!/\log\Omega)\left(\frac{1}{2}(\mathrm{tr}\chi)^2 - (\mathbf{D}_4\log\Omega)tr\chi + |\hat{\chi}|^2\right)$$

Equation 4.3.4 is not quite suited for our purposes as it leads to logarithmic divergences when we try to apply the Evolution Lemma to it[17], with $\lambda_1 = 3 - \frac{2}{p}$. To avoid this difficulty we introduce the tensor

$$\Psi\!\!\!/ = \Omega^{-1}\nabla\!\!\!/\mathrm{tr}\chi + \Omega^{-1}\mathrm{tr}\chi\zeta = U + \Omega^{-1}\mathrm{tr}\chi\zeta \tag{4.3.5}$$

Recalling, see 3.1.45, that ζ satisfies the equation

$$\mathbf{D}\!\!\!/_4\zeta = -\nabla\!\!\!/\mathbf{D}_4\log\Omega + \chi(\underline{\eta} - \zeta) - (\mathbf{D}_4\log\Omega)\nabla\!\!\!/\log\Omega - \beta$$

it is simple to see that $\Psi\!\!\!/$ verifies

$$\frac{d}{d\underline{u}}\Psi\!\!\!/_a + \frac{3}{2}\Omega\mathrm{tr}\chi\Psi\!\!\!/_a = F\!\!\!/_a \tag{4.3.6}$$

where

$$F\!\!\!/ \equiv -\Omega\hat{\chi}\cdot\Psi\!\!\!/ - \nabla\!\!\!/|\hat{\chi}|^2 - \eta|\hat{\chi}|^2 + \mathrm{tr}\chi\ \hat{\chi}\cdot\underline{\eta} - \mathrm{tr}\chi\beta \tag{4.3.7}$$

Applying the Evolution Lemma to the evolution equation 4.3.6 we obtain

$$|r^{3-\frac{2}{p}}\Psi\!\!\!/|_{p,S}(u,\underline{u}) \le c_0\left(|r^{3-\frac{2}{p}}\Psi\!\!\!/|_{p,S}(u,\underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*}|r^{3-\frac{2}{p}}F\!\!\!/|_{p,S}\right) \tag{4.3.8}$$

[17]Choosing $\lambda_1 = 3 - \frac{2}{p}$ we obtain, using the Evolution Lemma,

$$|r^{3-\frac{2}{p}}U|_{p,S}(u,\underline{u}) \le c_0(|r^{3-\frac{2}{p}}U|_{p,S}(u,\underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*}|r^{3-\frac{2}{p}}F|_{p,S})$$

where
$$\begin{aligned}|r^{3-\frac{2}{p}}F|_{p,S} \le{}& |r^{3-\frac{2}{p}}\Omega\hat{\chi}\cdot U|_{p,S} + |r^{3-\frac{2}{p}}\nabla\!\!\!/|\hat{\chi}|^2|_{p,S} + |r^{3-\frac{2}{p}}(\zeta+\underline{\eta})|\hat{\chi}|^2|_{p,S}\\ &+|r^{3-\frac{2}{p}}(\nabla\!\!\!/\mathbf{D}_4\log\Omega)\mathrm{tr}\chi|_{p,S} + \frac{1}{2}|r^{3-\frac{2}{p}}(\zeta+\underline{\eta})(\mathrm{tr}\chi)^2)|_{p,S}\end{aligned}$$

and, from the previous assumptions on $\mathrm{tr}\chi$, ζ, $\underline{\eta}$, it follows that the last term of $|r^{3-\frac{2}{p}}F|_{p,S}$ decays too slowly for the integral $\int_{\underline{u}}^{\underline{u}_*}|r^{3-\frac{2}{p}}F|_{p,S}$ to converge when $\underline{u}_* \to \infty$. This problem was already discussed and solved in [Ch-Kl]. Moreover if we assume for $\nabla\!\!\!/\mathbf{D}_4\log\Omega$ the expected asymptotic behaviour, which will be proved later on, one realizes that also the term $|r^{3-\frac{2}{p}}(\nabla\!\!\!/\mathbf{D}_4\log\Omega)\mathrm{tr}\chi|_{p,S}$ has a bad asymptotic behaviour.

It is easy to show that the integral in the right hand side is bounded for $\underline{u}_* \to \infty$. In fact,

$$\begin{aligned} |r^{3-\frac{2}{p}} \not{F}|_{p,S} &\leq |r^{3-\frac{2}{p}} \Omega\hat{\chi} \cdot \not{\Psi}|_{p,S} + |r^{3-\frac{2}{p}} \not\nabla |\hat{\chi}|^2|_{p,S} + \Big[|r^{3-\frac{2}{p}} \eta |\hat{\chi}|^2|_{p,S} \\ &+ |r^{3-\frac{2}{p}} \underline{\eta} \cdot \hat{\chi} \mathrm{tr}\chi|_{p,S} + |r^{3-\frac{2}{p}} \mathrm{tr}\chi\beta|_{p,S} \Big] \end{aligned} \tag{4.3.9}$$

Using the notation $\sup = \sup_{\mathcal{K}}$, the bootstrap assumption 4.2.5, and assumption 4.2.1 for β, we have

$$\begin{aligned} &|r^{3-\frac{2}{p}} \Omega\hat{\chi}\not{\Psi}|_{p,S} \leq \sup|\Omega| \sup|r^2\hat{\chi}||r^{3-\frac{2}{p}} \not{\Psi}|_{p,S} \frac{1}{r^2} \leq c(\Gamma_0)\Gamma_0 \frac{1}{r^2} |r^{3-\frac{2}{p}} \not{\Psi}|_{p,S} \\ &|r^{3-\frac{2}{p}} \not\nabla |\hat{\chi}|^2|_{p,S} \leq 2\sup|r^2\hat{\chi}||r^{3-\frac{2}{p}} \not\nabla\hat{\chi}|_{p,S} \frac{1}{r^2} \leq c(\Gamma_0)\Gamma_0 \frac{1}{r^2} |r^{3-\frac{2}{p}} \not\nabla\hat{\chi}|_{p,S} \\ &|r^{3-\frac{2}{p}} \eta|\hat{\chi}|^2|_{p,S} \leq \sup|r^2\hat{\chi}|^2 \sup|r^2\eta|_{p,S} \frac{1}{r^3} \leq \Gamma_0^3 \frac{1}{r^3} \\ &|r^{3-\frac{2}{p}} \underline{\eta} \cdot \hat{\chi}\mathrm{tr}\chi|_{p,S} \leq \sup|r^2\hat{\chi}| \sup|r\mathrm{tr}\chi| \sup|r^2\underline{\eta}| \frac{1}{r^2} \leq c(\Gamma_0)\Gamma_0^2 \frac{1}{r^2} \\ &|r^{3-\frac{2}{p}} \mathrm{tr}\chi\beta|_{p,S} \leq \sup|r\mathrm{tr}\chi| \sup|r^{\frac{7}{2}}\beta| \frac{1}{r^{\frac{3}{2}}} \leq c(\Gamma_0)\Delta_0 \frac{1}{r^{\frac{3}{2}}} \end{aligned} \tag{4.3.10}$$

where $c(\Gamma_0)$ is a constant depending on Γ_0 which can be bounded by $c(1+\Gamma_0)$. From these estimates the following inequality holds

$$\begin{aligned} \int_{\underline{u}}^{\underline{u}_*} |r^{3-\frac{2}{p}} \not{F}|_{p,S} &\leq c(\Gamma_0)\Gamma_0 \int_{\underline{u}}^{\underline{u}_*} \frac{1}{r^2} \left(|r^{3-\frac{2}{p}} \not{\Psi}|_{p,S} + |r^{3-\frac{2}{p}} \not\nabla\hat{\chi}|_{p,S} \right) \\ &+ c(\Gamma_0)(\Delta_0 + \Gamma_0^2 + \Gamma_0^3) \frac{1}{r^{\frac{1}{2}}} \end{aligned} \tag{4.3.11}$$

Using the final slice assumption $\mathcal{O}_{[1]}(\underline{C}_*) \leq \mathcal{I}_*$, to control $|r^{3-\frac{2}{p}} \not{\Psi}|_{p,S}(u, \underline{u}_*)$ we obtain, for $p \in [2, 4]$,

[The last term of 4.3.11 has been modified as $c(\Gamma_0)(\Delta_0 + \Gamma_0^2 + \Gamma_0^3)\frac{1}{r^{\frac{1}{2}}}$. Obvious subsequent modifications.]

$$\begin{aligned} |r^{3-\frac{2}{p}} \not{\Psi}|_{p,S}(u, \underline{u}) &\leq c(\Gamma_0) \left(\mathcal{I}_* + \Delta_0 + \Gamma_0^2 + \Gamma_0^3 \right) \\ &+ c(\Gamma_0)\Gamma_0 \int_{\underline{u}}^{\underline{u}_*} \frac{1}{r^2} \left(|r^{3-\frac{2}{p}} \not{\Psi}|_{p,S} + |r^{3-\frac{2}{p}} \not\nabla\hat{\chi}|_{p,S} \right) \end{aligned} \tag{4.3.12}$$

We then apply the elliptic L^p estimates of Proposition 4.1.3 to the the Codazzi equation, see eqs. 3.1.46, expressed relative to the tensor $\not{\Psi}$,

$$\not{\mathrm{d}}\mathrm{iv}\, \hat{\chi} + \zeta \cdot \hat{\chi} = \frac{1}{2} \Omega \not{\Psi} - \beta \tag{4.3.13}$$

and derive

$$\begin{aligned}|r^{3-2/p}\nabla\!\!\!/\hat{\chi}|_{p,S} &\leq c\left(|r^{3-\frac{2}{p}}\Omega\Psi\!\!\!/|_{p,S}+r^{-\frac{1}{2}}|r^{\frac{7}{2}-\frac{2}{p}}\beta|_{p,S}+r^{-1}|r^{4-\frac{2}{p}}\zeta\cdot\hat{\chi}|_{p,S}\right)\\ &\leq c\left(|r^{3-\frac{2}{p}}\Psi\!\!\!/|_{p,S}+r^{-\frac{1}{2}}\Delta_0+r^{-1}\Gamma_0^2\right) \qquad (4.3.14)\end{aligned}$$

[modification in 4.3.14] Substituting this estimate in the inequality 4.3.12 we obtain

$$|r^{3-\frac{2}{p}}\Psi\!\!\!/|_{p,S}(u,\underline{u})\leq c(\Gamma_0)\left(\mathcal{I}_*+\Delta_0+\Gamma_0^2+\Gamma_0^3\right)+c(\Gamma_0)\Gamma_0\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}|r^{3-\frac{2}{p}}\Psi\!\!\!/|_{p,S} \qquad (4.3.15)$$

[modification in 4.3.15] Finally we apply the Gronwall Lemma to 4.3.15 and assuming Γ_0 sufficiently small, we obtain

$$|r^{3-2/p}\Psi\!\!\!/|_{p,S}(u,\underline{u}) \leq c(\Gamma_0)\left(\mathcal{I}_*+\Delta_0+\Gamma_0^2+\Gamma_0^3\right) \qquad (4.3.16)$$

[modification in 4.3.16] This estimate together with the elliptic L^p estimates of Proposition 4.1.3 applied to 4.3.13 implies, for $p\in[2,4]$,

$$\begin{aligned}|r^{3-2/p}\nabla\!\!\!/\hat{\chi}|_{p,S}(u,\underline{u}) &\leq c(\Gamma_0)\left(\mathcal{I}_*+\Delta_0+\Gamma_0^2+\Gamma_0^3\right)\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\ |r^{2-2/p}\hat{\chi}|_{p,S}(u,\underline{u}) &\leq c(\Gamma_0)\left(\mathcal{I}_*+\Delta_0+\Gamma_0^2+\Gamma_0^3\right)\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\end{aligned}$$

completing the proof.

4.3.2 Estimate for $|r^{2-\frac{2}{p}}(\mathbf{tr}\chi-\overline{\mathbf{tr}\chi})|_{p,S}$ and $|r^{3-2/p}\nabla\!\!\!/\mathbf{tr}\chi|_{p,S}$, with $p\in[2,4]$

[La stima va scritta]

$$\begin{aligned}|r^{3-2/p}\nabla\!\!\!/\mathrm{tr}\chi|_{p,S} &\leq c(\Gamma_0)\left(\mathcal{I}_*+\Delta_0+\Gamma_0^2+\Gamma_0^3\right)\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\ |r^{2-\frac{2}{p}}(\mathrm{tr}\chi-\overline{\mathrm{tr}\chi})|_{p,S} &\leq c(\Gamma_0)\left(\mathcal{I}_*+\Delta_0+\Gamma_0^2+\Gamma_0^3\right)\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \qquad (4.3.17)\end{aligned}$$

Remark: To prove these estimates we need, in view of definition 4.3.5, an estimate of ζ which will be proved later on, see subsection 4.3.9. We do not report [18] the proof of 4.3.17 as in the estimates in subsection 4.3.3, we are going to prove a slightly stronger version which will be used in the *Main Theorem.*

[18] Given an estimate for $|r^{2-2/p}\zeta|_{p,S}$ we deduce an estimate for $|r^{3-2/p}\nabla\!\!\!/\mathrm{tr}\chi|_{p,S}$. Differentiating 4.3.5, we obtain $\Delta\!\!\!/\mathrm{tr}\chi=\mathrm{d}\!\!\!/\mathrm{iv}\,(\Omega\Psi\!\!\!/)-\mathrm{d}\!\!\!/\mathrm{iv}\,(\mathrm{tr}\chi\zeta)$, $\nabla\!\!\!/\Psi\!\!\!/$ can be easily estimated from its evolution equation, obtained differentiating 4.3.6. The result is

$$|r^{4-2/p}\nabla\!\!\!/\Psi\!\!\!/|_{p,S}(u,\underline{u})\leq c(\Gamma_0)\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Gamma_0^2+\Gamma_0^3\right)$$

Using the estimate already obtained for $\nabla\!\!\!/\mathrm{tr}\chi$ and the estimate for $\nabla\!\!\!/\zeta$ obtained from subsection 4.3.9 and applying Lemma 4.1.3 we obtain the result.

4.3.3 Estimate for $|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\mathbf{tr}\chi-\overline{\mathbf{tr}\chi})|_{p,S}$ with $p\in[2,4]$

Remarks: 1) Observe that this estimate we are going to prove is slightly stronger of the one suggested by the bootstrap assumption 4.2.5 and discussed in subsection 4.3.2.
2) The proof of this estimate, given in Proposition 4.3.14, implies a stronger assumption for $(\mathrm{tr}\chi-\overline{\mathrm{tr}\chi})$ on the last slice and an appropriate estimate for $(\Omega\mathbf{D}_4\log\Omega-\overline{\Omega\mathbf{D}_4\log\Omega})$. Therefore we delay its proof until this last estimate has been proved in Proposition 4.3.13.

4.3.4 Estimate for $|r^{2-\frac{2}{p}}\left(\overline{\Omega\mathbf{tr}\chi}-\frac{1}{r}\right)|_{p,S}$ with $p\in[2,4]$

Remark: As for the previous estimate, the estimate for $\left(\overline{\Omega\mathrm{tr}\chi}-\frac{1}{r}\right)$ requires, preliminary, an estimate for $\Omega\mathbf{D}_4\log\Omega$. Therefore we delay the proof of this result until this last estimate has been proved in Proposition 4.3.4.

4.3.5 Estimate for $\mathcal{O}^{p,S}_{0,1}(\mathbf{tr}\underline{\chi})$ and $\mathcal{O}^{p,S}_{0,1}(\hat{\underline{\chi}})$ with $p\in[2,4]$

Proposition 4.3.2 *Assuming 4.2.1, 4.2.2, 4.2.3 and the bootstrap assumptions 4.2.5, 4.2.6 the following estimates hold,*

$$
\begin{aligned}
&|r^{2-2/p}\tau_-\nabla\!\!\!/\hat{\underline{\chi}}|_{p,S}(u,\underline{u})\le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\
&|r^{1-2/p}\tau_-\hat{\underline{\chi}}|_{p,S}(u,\underline{u})\le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)
\end{aligned}
\tag{4.3.18}
$$

Proof: We proceed, basically, as in Proposition 4.3.1. We introduce the tensor field

$$
\underline{\psi\!\!\!/}=\Omega^{-1}\nabla\!\!\!/ tr\underline{\chi}-\Omega^{-1}tr\underline{\chi}\zeta \tag{4.3.19}
$$

and show that $\underline{\psi\!\!\!/}$ satisfies the evolution equation

$$
\frac{d}{du}\underline{\psi\!\!\!/}_a+\frac{3}{2}\Omega tr\underline{\chi}\underline{\psi\!\!\!/}_a=\underline{F\!\!\!/}_a \tag{4.3.20}
$$

where

$$
\underline{F\!\!\!/}\equiv-\Omega\hat{\underline{\chi}}\cdot\underline{\psi\!\!\!/}-\nabla\!\!\!/|\hat{\underline{\chi}}|^2-\underline{\eta}|\hat{\underline{\chi}}|^2+tr\underline{\chi}\ \hat{\underline{\chi}}\cdot\eta-tr\underline{\chi}\ \underline{\beta} \tag{4.3.21}
$$

satisifes

$$
\begin{aligned}
|r^{3-\frac{2}{p}}\underline{F\!\!\!/}|_{p,S}\ \ \le\ \ &|r^{3-\frac{2}{p}}\Omega\hat{\underline{\chi}}\cdot\underline{\psi\!\!\!/}|_{p,S}+|r^{3-\frac{2}{p}}\nabla\!\!\!/|\hat{\underline{\chi}}|^2|_{p,S}+\Big[|r^{3-\frac{2}{p}}\underline{\eta}|\hat{\underline{\chi}}|^2|_{p,S}\\
&+|r^{3-\frac{2}{p}}\eta\cdot\hat{\underline{\chi}}\mathrm{tr}\underline{\chi}|_{p,S}+|r^{3-\frac{2}{p}}\mathrm{tr}\underline{\chi}\ \underline{\beta}|_{p,S}\Big]\ .
\end{aligned}
$$

The various terms on the right hand side are estimated, with the help of the bootstrap assumption 4.2.5 and assumption 4.2.1 for $\underline{\beta}$,

$$
\begin{aligned}
&|r^{3-\frac{2}{p}}\Omega\hat{\underline{\chi}}\ \underline{\not{\Psi}}|_{p,S} \leq \sup|\Omega|\sup|r\tau_-^{\frac{3}{2}}\hat{\underline{\chi}}||r^{3-\frac{2}{p}}\underline{\not{\Psi}}|_{p,S}\frac{1}{r\tau_-^{\frac{3}{2}}} \leq c(\Gamma_0)\Gamma_0|r^{3-\frac{2}{p}}\underline{\not{\Psi}}|_{p,S}\frac{1}{r\tau_-^{\frac{3}{2}}}\\
&|r^{3-\frac{2}{p}}\not{\nabla}|\hat{\underline{\chi}}|^2|_{p,S} \leq 2\sup|r\tau_-^{\frac{3}{2}}\hat{\underline{\chi}}||r^{2-\frac{2}{p}}\tau_-\not{\nabla}\hat{\underline{\chi}}|_{p,S}\frac{1}{\tau_-^{\frac{5}{2}}} \leq c(\Gamma_0)\Gamma_0\frac{1}{\tau_-^{\frac{5}{2}}}|r^{2-\frac{2}{p}}\tau_-\not{\nabla}\hat{\underline{\chi}}|_{p,S}\\
&|r^{3-\frac{2}{p}}\underline{\eta}|\hat{\underline{\chi}}|^2|_{p,S} \leq \sup|r\tau_-^{\frac{3}{2}}\hat{\underline{\chi}}|^2\sup|r^2\underline{\eta}|_{p,S}\frac{1}{r\tau_-^3} \leq \Gamma_0^3\frac{1}{r\tau_-^3} \qquad (4.3.22)\\
&|r^{3-\frac{2}{p}}\eta\hat{\underline{\chi}}\mathrm{tr}\underline{\chi}|_{p,S} \leq \sup|r\tau_-^{\frac{3}{2}}\hat{\underline{\chi}}|\sup|r\mathrm{tr}\underline{\chi}|\sup|r^2\eta|\frac{1}{r\tau_-^{\frac{3}{2}}} \leq c(\Gamma_0)\Gamma_0^2\frac{1}{r\tau_-^{\frac{3}{2}}}\\
&|r^{3-\frac{2}{p}}\mathrm{tr}\underline{\chi}\ \underline{\beta}|_{p,S} \leq \sup|r\mathrm{tr}\underline{\chi}|\sup|r^2\tau_-^{\frac{3}{2}}\underline{\beta}|\frac{1}{\tau_-^{\frac{3}{2}}} \leq c(\Gamma_0)\Delta_0\frac{1}{\tau_-^{\frac{3}{2}}}
\end{aligned}
$$

[modification in 4.3.22] From these estimates we infer that

$$
\begin{aligned}
\int_{u_0(\underline{u})}^{u}|r^{3-\frac{2}{p}}\underline{\not{F}}|_{p,S} \quad &\leq \quad c(\Gamma_0)\Gamma_0\int_{u_0(\underline{u})}^{u}\left(\frac{1}{r\tau_-^{\frac{3}{2}}}|r^{3-\frac{2}{p}}\underline{\not{\Psi}}|_{p,S}+\frac{1}{\tau_-^{\frac{5}{2}}}|r^{2-\frac{2}{p}}\tau_-\not{\nabla}\hat{\underline{\chi}}|_{p,S}\right)\\
&+ \quad c(\Gamma_0)(\Delta_0+\Gamma_0^2+\Gamma_0^3)\left(\frac{1}{r\tau_-^2}+\frac{1}{r\tau_-^{\frac{1}{2}}}+\frac{1}{\tau_-^{\frac{1}{2}}}\right) \qquad (4.3.23)
\end{aligned}
$$

[modification in 4.3.23] where $u_0=u_0(\underline{u})=2\tilde{\delta}_0-\underline{u}$.

Using the initial assumption $\mathcal{O}_{[1]}(\Sigma_0)\leq\mathcal{I}_0$ to control $|r^{3-\frac{2}{p}}\underline{\not{\Psi}}|_{p,S_{(0)}}(\underline{u})$ and the results of Lemma 4.1.7 applied to $V=\underline{\not{\Psi}}$ we derive

$$
\begin{aligned}
|r^{3-\frac{2}{p}}\underline{\not{\Psi}}|_{p,S}(u,\underline{u}) \quad &\leq \quad c(\Gamma_0)\left(\Delta_0+\Gamma_0^2+\Gamma_0^3\right) \qquad (4.3.24)\\
&+ \quad c(\Gamma_0)\Gamma_0\int_{u_0}^{u}\left(\frac{1}{r\tau_-^{\frac{3}{2}}}|r^{3-\frac{2}{p}}\underline{\not{\Psi}}|_{p,S}+\frac{1}{\tau_-^{\frac{5}{2}}}|r^{2-\frac{2}{p}}\tau_-\not{\nabla}\hat{\underline{\chi}}|_{p,S}\right)
\end{aligned}
$$

[modification in 4.3.24] for $p\in[2,4]$. The main difference with the previous case is that the integration is made, here, along the incoming null hypersurfaces $\underline{C}$. We are now ready to apply the elliptic L^p estimates of Proposition 4.1.3 to the Hodge system, see 3.1.46,

$$\not{\mathrm{div}}\,\hat{\underline{\chi}}=\zeta\hat{\underline{\chi}}+\frac{1}{2}\Omega\underline{\not{\Psi}}-\underline{\beta} \qquad (4.3.25)$$

We thus obtain:

$$|r^{2-\frac{2}{p}}\tau_-\nabla\!\!\!/\hat{\underline{\chi}}|_{p,S} \leq c(\Gamma_0)\Big(|r^{3-\frac{2}{p}}\underline{\psi\!\!\!/}|_{p,S}\frac{\tau_-}{r}+\sup|r^{2-\frac{2}{p}}\tau_-^{3/2}\underline{\beta}|\frac{1}{\tau_-^{\frac{1}{2}}} + \sup|r^2\zeta|\sup|r\tau_-^{\frac{3}{2}}\hat{\underline{\chi}}|\frac{1}{r\tau_-^{\frac{1}{2}}}\Big)$$
$$\leq c(\Gamma_0)\Big(|r^{3-\frac{2}{p}}\underline{\psi\!\!\!/}|_{p,S}\frac{\tau_-}{r}+\frac{\Delta_0}{\tau_-^{\frac{1}{2}}}+\frac{\Gamma_0^2}{r\tau_-^{\frac{1}{2}}}\Big) \tag{4.3.26}$$

which substituted in 4.3.24 implies

$$|r^{3-\frac{2}{p}}\underline{\psi\!\!\!/}|_{p,S}(u,\underline{u}) \leq c(\Gamma_0)\Big(\Delta_0+\Gamma_0^2+\Gamma_0^3\Big)+c(\Gamma_0)\Gamma_0\left[\int_{u_0}^{u}\frac{1}{r\tau_-^{\frac{3}{2}}}|r^{3-\frac{2}{p}}\underline{\psi\!\!\!/}|_{p,S} + \int_{u_0}^{u}\frac{1}{\tau_-^{3}}\left(\frac{\Gamma_0^2}{r}+\Delta_0\right)\right] \tag{4.3.27}$$

and, by the Gronwall Lemma, [modification in 4.3.27]

$$|r^{3-2/p}\underline{\psi\!\!\!/}|_{p,S}(u,\underline{u}) \leq c(\Gamma_0)\Big(\Delta_0+\Gamma_0^2+\Gamma_0^3\Big) \tag{4.3.28}$$

From 4.3.28, going back to 4.3.26, we deduce [modification in 4.3.28 and in the next line elimination of "and to 4.3.19," and the first line of subsequent inequalities.]

$$|r^{1-2/p}\tau_-\hat{\underline{\chi}}|_{p,S}(u,\underline{u}) \leq c(\Gamma_0)\left(\mathcal{I}_0+\Delta_0+\Gamma_0^2+\Gamma_0^3\right) \leq c(\mathcal{I}_0+\Delta_0)$$
$$|r^{2-2/p}\tau_-\nabla\!\!\!/\hat{\underline{\chi}}|_{p,S}(u,\underline{u}) \leq c(\Gamma_0)\left(\mathcal{I}_0+\Delta_0+\Gamma_0^2+\Gamma_0^3\right) \leq c(\mathcal{I}_0+\Delta_0)$$

proving the proposition.

4.3.6 Estimate for $|r^{2-\frac{2}{p}}(\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\underline{\chi}})|_{p,S}$ and $|r^{3-2/p}\nabla\!\!\!/\mathrm{tr}\underline{\chi}|_{p,S}$, with $p\in[2,4]$

$$|r^{3-2/p}\nabla\!\!\!/\mathrm{tr}\underline{\chi}|_{p,S} \leq c(\Gamma_0)\left(\mathcal{I}_*+\Delta_0+\Gamma_0^2+\Gamma_0^3\right) \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)$$
$$|r^{2-\frac{2}{p}}(\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\underline{\chi}})|_{p,S} \leq c(\Gamma_0)\Big(\mathcal{I}_0+\Delta_0+\Gamma_0^2+\Gamma_0^3\Big) \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \tag{4.3.29}$$

Remark: To prove these estimates we need an estimate of ζ which will be proved later on, see subsection 4.3.9. We do not report [19] the proof of 4.3.29 as in the estimates in subsection 4.3.7 we are going to prove a slightly stronger version which will be used in the *Main Theorem.*

[19] Given an estimate for $|r^{2-2/p}\zeta|_{p,S}$ we deduce an estimate for $|r^{3-2/p}\nabla\!\!\!/\mathrm{tr}\underline{\chi}|_{p,S}$. Differ-

4.3.7 Estimate for $|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\mathbf{tr}\underline{\chi}-\overline{\mathbf{tr}\underline{\chi}})|_{p,S}$ with $p\in[2,4]$

Remarks:
1) Observe that, as in the case of $(\mathrm{tr}\chi-\overline{\mathrm{tr}\chi})$, the estimate we are going to prove is slightly stronger that the one suggested by the bootstrap assumption 4.2.5 [20] and implies that one.
2) The proof of this estimate, given in Proposition 4.3.14, implies a stronger assumption for $(\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\underline{\chi}})$ on the initial hypersurface and some estimate for $(\Omega\mathbf{D}_3\log\Omega-\overline{\Omega\mathbf{D}_3\log\Omega})$. Therefore we delay its proof until this last estimate has been proved in Proposition 4.3.13.

4.3.8 Estimate for $|r^{1-\frac{2}{p}}\tau_-\left(\overline{\Omega\mathbf{tr}\underline{\chi}}+\frac{1}{r}\right)|_{p,S}$ with $p\in[2,4]$

Remark: As for the previous estimate, the estimate for $\left(\overline{\Omega\mathrm{tr}\underline{\chi}}+\frac{1}{r}\right)$ requires preliminary an estimate for $\Omega\mathbf{D}_3\log\Omega$. Therefore we delay the proof of this result until this last estimate has been proved in Proposition 4.3.4.

4.3.9 Estimate for $\mathcal{O}^{p,S}_{0,1}(\eta)$, $\mathcal{O}^{p,S}_{0,1}(\underline{\eta})$, $p\in[2,4]$

Proposition 4.3.3 *Assuming 4.2.1, 4.2.2, 4.2.3 and 4.2.5, 4.2.6, then the following estimates hold,*

$$
\begin{aligned}
|r^{2-2/p}\eta|_{p,S}(u,\underline{u}) &\leq c\,(\mathcal{I}_*+\mathcal{I}_0+\Delta_0)\\
|r^{3-2/p}\nabla\!\!\!/\,\eta|_{p,S}(u,\underline{u}) &\leq c\,(\mathcal{I}_*+\mathcal{I}_0+\Delta_0)\\
|r^{2-2/p}\underline{\eta}|_{p,S}(u,\underline{u}) &\leq c\,(\mathcal{I}_*+\mathcal{I}_0+\Delta_0)\\
|r^{3-2/p}\nabla\!\!\!/\,\underline{\eta}|_{p,S}(u,\underline{u}) &\leq c\,(\mathcal{I}_*+\mathcal{I}_0+\Delta_0)
\end{aligned}
\tag{4.3.30}
$$

Proof: To obtain the norm estimates for η, $\underline{\eta}$ and their first tangential derivatives we recall the equations, see 3.1.45,

$$
\begin{aligned}
\mathbf{D}\!\!\!\!/\,_4\eta &= -\chi\cdot\eta+\chi\cdot\underline{\eta}-\beta\\
\mathbf{D}\!\!\!\!/\,_3\underline{\eta} &= -\underline{\chi}\cdot\underline{\eta}+\underline{\chi}\cdot\eta+\underline{\beta}\ .
\end{aligned}
$$

entiating 4.3.25, we obtain $\Delta\!\!\!\!/\,\mathrm{tr}\underline{\chi}=\mathrm{d}\!\!\!/\mathrm{iv}\,(\Omega\underline{\psi\!\!\!/})+\mathrm{d}\!\!\!/\mathrm{iv}\,(\mathrm{tr}\underline{\chi}\zeta)$, $\nabla\!\!\!\!/\,\underline{\psi\!\!\!/}$ can be easily estimated from its evolution equation, obtained differentiating 4.3.6. The result is

$$|r^{4-2/p}\nabla\!\!\!\!/\,\underline{\psi\!\!\!/}|_{p,S}(u,\underline{u})\leq c(\Gamma_0)\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Gamma_0^2+\Gamma_0^3\right)$$

Using the estimate already obtained for $\nabla\!\!\!\!/\,\mathrm{tr}\chi$ and the estimate for $\nabla\!\!\!\!/\,\zeta$ obtained from subsection 4.3.9 and applying Lemma 4.1.3 we obtain the result.

[20]Recall that $\mathcal{O}^{p,S}_{q=0}(tr\underline{\chi})(\lambda,\nu)=|r^2(\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\underline{\chi}})|_{p,S}$.

If we differentiate these equations to obtain estimates for $\nabla\!\!\!/ \eta$ or $\nabla\!\!\!/ \underline{\eta}$ it seems that their order of differentiability is the same as that of $\nabla\!\!\!/ \beta$, $\nabla\!\!\!/ \underline{\beta}$. To avoid this loss of derivatives we proceed as in [Ch-Kl], with the help of the "mass aspect functions" introduced in Chapter 2, eqs. 3.3.6,

$$\begin{aligned} \mu &= -\text{d}\!\!/\text{iv}\,\eta + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \rho \\ \underline{\mu} &= -\text{d}\!\!/\text{iv}\,\underline{\eta} + \frac{1}{2}\underline{\hat{\chi}}\cdot\hat{\chi} - \rho \end{aligned} \tag{4.3.31}$$

They satisfy the following lemma, proved by a direct computation:

Lemma 4.3.1 *The scalar functions $\mu, \underline{\mu}$ satisfy the following evolution equations*

$$\begin{aligned} \frac{d}{d\underline{u}}\mu + (\Omega tr\chi)\mu &= G + \frac{1}{2}(\Omega tr\chi)\underline{\mu} \\ \frac{d}{du}\underline{\mu} + (\Omega tr\underline{\chi})\underline{\mu} &= \underline{G} + \frac{1}{2}(\Omega tr\underline{\chi})\mu \end{aligned} \tag{4.3.32}$$

where

$$\begin{aligned} G &\equiv \Omega\underline{\hat{\chi}}\cdot(\nabla\!\!\!/\widehat{\otimes}\eta) + \Omega^2(\eta-\underline{\eta})\cdot\Psi\!\!\!/ - \frac{1}{4}\Omega tr\underline{\chi}|\hat{\chi}|^2 \\ &+ \frac{1}{2}\Omega tr\chi\left(-\hat{\chi}\cdot\underline{\hat{\chi}} + 2\rho - |\underline{\eta}|^2\right) + 2\Omega(\eta\cdot\hat{\chi}\cdot\underline{\eta} - \eta\cdot\beta) \\ \underline{G} &\equiv \Omega\underline{\hat{\chi}}\cdot(\nabla\!\!\!/\widehat{\otimes}\underline{\eta}) + \Omega^2(\underline{\eta}-\eta)\cdot\underline{\Psi\!\!\!/} - \frac{1}{4}\Omega tr\chi|\underline{\hat{\chi}}|^2 \\ &+ \frac{1}{2}\Omega tr\underline{\chi}\left(-\underline{\hat{\chi}}\cdot\hat{\chi} + 2\rho - |\eta|^2\right) + 2\Omega(\underline{\eta}\cdot\underline{\hat{\chi}}\cdot\eta + \underline{\eta}\cdot\underline{\beta}) \end{aligned} \tag{4.3.33}$$

Using the equations 4.3.32 it is possible to obtain estimates for $\mu, \underline{\mu}$ which together with the Hodge systems

$$\begin{aligned} \text{d}\!\!/\text{iv}\,\eta &= -\mu + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \rho \\ \text{c}\!\!/\text{url}\,\eta &= \sigma - \frac{1}{2}\underline{\hat{\chi}}\wedge\hat{\chi} \end{aligned} \tag{4.3.34}$$

$$\begin{aligned} \text{d}\!\!/\text{iv}\,\underline{\eta} &= -\underline{\mu} + \frac{1}{2}\underline{\hat{\chi}}\cdot\hat{\chi} - \rho \\ \text{c}\!\!/\text{url}\,\underline{\eta} &= -\sigma - \frac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}} \end{aligned} \tag{4.3.35}$$

allow to control the norms $\mathcal{O}^{p,S}_{0,1}(\eta)$, $\mathcal{O}^{p,S}_{0,1}(\underline{\eta})$. We choose, nevertheless, a slight different method which allows to obtain the result in a easier way

and, moreover, to obtain a somewhat improved estimate, if the assumptions on the initial and final slices are stronger.
We start introducing two different functions $\tilde{\mu}$ and $\underline{\tilde{\mu}}$,

$$\tilde{\mu}=(\mu-\overline{\mu})+\frac{1}{4}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\chi\mathrm{tr}\underline{\chi}})=-\mathrm{d}\!\!/\mathrm{iv}\,\eta+\frac{1}{2}(\chi\cdot\underline{\chi}-\overline{\chi\cdot\underline{\chi}})-(\rho-\overline{\rho})$$

$$\underline{\tilde{\mu}}=(\underline{\mu}-\overline{\underline{\mu}})+\frac{1}{4}(\mathrm{tr}\underline{\chi}\mathrm{tr}\chi-\overline{\mathrm{tr}\underline{\chi}\mathrm{tr}\chi})=-\mathrm{d}\!\!/\mathrm{iv}\,\underline{\eta}+\frac{1}{2}(\underline{\chi}\cdot\chi-\overline{\underline{\chi}\cdot\chi})-(\rho-\overline{\rho}) \quad (4.3.36)$$

relative to these functions η and $\underline{\eta}$ satisfy the following Hodge systems

$$\begin{aligned}\mathrm{d}\!\!/\mathrm{iv}\,\eta&=-\tilde{\mu}+\frac{1}{2}(\chi\cdot\underline{\chi}-\overline{\chi\cdot\underline{\chi}})-(\rho-\overline{\rho})\\ \mathrm{c}\!\!/\mathrm{url}\,\eta&=\sigma-\frac{1}{2}\underline{\hat{\chi}}\wedge\hat{\chi}\end{aligned} \quad (4.3.37)$$

$$\begin{aligned}\mathrm{d}\!\!/\mathrm{iv}\,\underline{\eta}&=-\underline{\tilde{\mu}}+\frac{1}{2}(\underline{\chi}\cdot\chi-\overline{\underline{\chi}\cdot\chi})-(\rho-\overline{\rho})\\ \mathrm{c}\!\!/\mathrm{url}\,\underline{\eta}&=-\sigma-\frac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}}\end{aligned} \quad (4.3.38)$$

The functions $\tilde{\mu}$ and $\underline{\tilde{\mu}}$ satisfy the following lemma,

Lemma 4.3.2 *$\tilde{\mu}$ and $\underline{\tilde{\mu}}$ verify the following evolution equations*

$$\begin{aligned}\frac{d}{d\underline{u}}\tilde{\mu}+(\Omega tr\chi)\tilde{\mu}&=(\Omega\tilde{F}-\overline{\Omega\tilde{F}})+(\Omega\tilde{H}-\overline{\Omega\tilde{H}}) \quad (4.3.39)\\ \frac{d}{du}\underline{\tilde{\mu}}+(\Omega tr\underline{\chi})\underline{\tilde{\mu}}&=(\Omega\underline{\tilde{F}}-\overline{\Omega\underline{\tilde{F}}})+(\Omega\underline{\tilde{H}}-\overline{\Omega\underline{\tilde{H}}})\end{aligned}$$

where

$$\begin{aligned}\tilde{F}&=\hat{\chi}\cdot(\nabla\!\!\!/\widehat{\otimes}\eta)+\Omega(\eta-\underline{\eta})\cdot\psi\!\!\!/+\frac{1}{2}tr\chi(|\eta|^2-|\underline{\eta}|^2)-\frac{1}{2}\left(tr\underline{\chi}|\hat{\chi}|^2+tr\chi(\hat{\chi}\cdot\underline{\hat{\chi}})\right)+2\eta\cdot\hat{\chi}\cdot\underline{\eta}\\ \tilde{H}&=\left(-tr\chi\frac{1}{2}\overline{\chi\cdot\underline{\chi}}-2\eta\cdot\beta+tr\chi(\rho+\overline{\rho})\right)\\ \underline{\tilde{F}}&=\underline{\hat{\chi}}\cdot(\nabla\!\!\!/\widehat{\otimes}\underline{\eta})+\Omega(\underline{\eta}-\eta)\cdot\underline{\psi\!\!\!/}+\frac{1}{2}\Omega tr\underline{\chi}(|\underline{\eta}|^2-|\eta|^2)-\frac{1}{2}\left(tr\chi|\underline{\hat{\chi}}|^2+tr\underline{\chi}(\underline{\hat{\chi}}\cdot\hat{\chi})\right)+2\underline{\eta}\cdot\underline{\hat{\chi}}\cdot\eta\\ \underline{\tilde{H}}&=\left(-tr\chi\frac{1}{2}\overline{\underline{\chi}\cdot\chi}+2\underline{\eta}\cdot\underline{\beta}+tr\underline{\chi}(\rho+\overline{\rho})\right)\end{aligned} \quad (4.3.40)$$

Proof: Recalling the definition

$$\tilde{\mu}=(\mu-\overline{\mu})+\frac{1}{4}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\chi\mathrm{tr}\underline{\chi}})\ ,$$

we observe that

$$\begin{aligned}
\mathbf{D}_4\left(\mu+\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\right) &= \mathbf{D}_4\mu+\frac{1}{4}\left(\mathrm{tr}\underline{\chi}\mathbf{D}_4\mathrm{tr}\chi+\mathrm{tr}\chi\mathbf{D}_4\mathrm{tr}\underline{\chi}\right)\\
&= \left[-\mathrm{tr}\chi\mu+\Omega^{-1}G+\frac{1}{2}\mathrm{tr}\chi\left(\frac{1}{2}\hat{\underline{\chi}}\cdot\hat{\chi}-\rho-\mathrm{d}\!\!/\mathrm{iv}\,\underline{\eta}\right)\right]\\
&+\frac{1}{4}\mathrm{tr}\underline{\chi}\left[(\mathbf{D}_4\log\Omega)\mathrm{tr}\chi-\frac{1}{2}(\mathrm{tr}\chi)^2-|\hat{\chi}|^2\right] \qquad (4.3.41)\\
&+\frac{1}{4}\mathrm{tr}\chi\left[-(\mathbf{D}_4\log\Omega)\mathrm{tr}\underline{\chi}-\frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\hat{\chi}\cdot\hat{\underline{\chi}}+2\mathrm{d}\!\!/\mathrm{iv}\,\underline{\eta}+2|\eta|^2+2\rho\right]
\end{aligned}$$

which can be rewritten as

$$\frac{d}{d\underline{u}}\left(\mu+\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\right)+(\Omega\mathrm{tr}\chi)\left(\mu+\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\right)=\Omega\tilde{F}+\Omega(\mathrm{tr}\chi\rho-2\eta\cdot\beta) \quad (4.3.42)$$

Posing $[\mu]\equiv\left(\mu+\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\right)$ we have

$$\begin{aligned}
\frac{d}{d\underline{u}}\overline{[\mu]} &= \frac{1}{|S|}\int_S\frac{d}{d\underline{u}}[\mu]-\frac{1}{|S|}\frac{\partial|S|}{\partial\underline{u}}\overline{[\mu]}+\frac{1}{|S|}\int_S\Omega\mathrm{tr}\chi[\mu]\\
&= \frac{1}{|S|}\int_S\Omega\left(-\mathrm{tr}\chi[\mu]+\tilde{F}+(\mathrm{tr}\chi\rho-2\eta\cdot\beta)\right)-(\overline{\Omega\mathrm{tr}\chi})\overline{[\mu]}+\overline{\Omega\mathrm{tr}\chi[\mu]}\\
&= \overline{\Omega\tilde{F}}+\overline{\Omega(\mathrm{tr}\chi\rho-2\eta\cdot\beta)}-(\overline{\Omega\mathrm{tr}\chi})\overline{[\mu]} \qquad (4.3.43)
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{d}{d\underline{u}}\overline{[\mu]}+(\Omega\mathrm{tr}\chi)\overline{[\mu]}=\overline{\Omega\tilde{F}}+\overline{\Omega(\mathrm{tr}\chi\rho-2\eta\cdot\beta)}+(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})\overline{[\mu]} \qquad (4.3.44)\\
&=\overline{\Omega\tilde{F}}-2\overline{\Omega\eta\cdot\beta}+(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})\frac{1}{2}\overline{\chi\cdot\underline{\chi}}+\overline{\Omega\mathrm{tr}\chi\rho}-(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})\overline{\rho}
\end{aligned}$$

Combining 4.3.41 and 4.3.44 we obtain

$$\begin{aligned}
\frac{d}{d\underline{u}}\tilde{\mu}+(\Omega\mathrm{tr}\chi)\tilde{\mu} &= (\Omega\tilde{F}-\overline{\Omega\tilde{F}})-(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})\frac{1}{2}\overline{\chi\cdot\underline{\chi}}-2(\Omega\eta\cdot\beta-\overline{\Omega\eta\cdot\beta})\\
&+(\Omega\mathrm{tr}\chi\rho-\overline{\Omega\mathrm{tr}\chi\rho})+(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})\overline{\rho} \qquad (4.3.45)
\end{aligned}$$

completing the proof of the first part of Lemma 4.3.2. The proof for $\underline{\tilde{\mu}}$ is exactly analogous to the one for $\tilde{\mu}$ and we do not report it here.
Using the evolution equations 4.3.39, the Evolution Lemma with $\lambda_1=2-\frac{2}{p}$, and the elliptic estimates of Proposition 4.1.3 we prove the following lemma:

Lemma 4.3.3 *Under the bootstrap assumption 4.2.5, if* $\Gamma_0 > 0$ *is sufficiently small, then the following inequality holds*

$$|r^{3-\frac{2}{p}}\tilde{\mu}|_{p,S}(u,\underline{u}) \leq c\left[|r^{3-\frac{2}{p}}\tilde{\mu}|_{p,S}(u,\underline{u}_*) + c(\Gamma_0)(\mathcal{I}_* + \Delta_0)\right] \quad (4.3.46)$$

and the corresponding one for $\underline{\mu}$*,*

$$|r^{3-\frac{2}{p}}\tilde{\underline{\mu}}|_{p,S}(u,\underline{u}) \leq c\left[|r^{3-\frac{2}{p}}\tilde{\underline{\mu}}|_{p,S}(u_0,\underline{u}) + c(\Gamma_0)(\mathcal{I}_0 + \Delta_0)\right] \quad (4.3.47)$$

Proof of Lemma 4.3.3: We apply the Evolution Lemma, with $\lambda_1 = 2 - \frac{2}{p}$, to equations 4.3.39 and derive

$$\begin{aligned}
|r^{2-\frac{2}{p}}\tilde{\mu}|_{p,S}(u,\underline{u}) \;&\leq\; c\Big(|r^{2-\frac{2}{p}}\tilde{\mu}|_{p,S}(u,\underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*}|r^{2-\frac{2}{p}}(\Omega\tilde{F} - \overline{\Omega\tilde{F}})|_{p,S} \\
&\quad + \int_{\underline{u}}^{\underline{u}_*}|r^{2-\frac{2}{p}}(\Omega\tilde{H} - \overline{\Omega\tilde{H}})|_{p,S}\Big) \quad (4.3.48)\\
|r^{2-\frac{2}{p}}\tilde{\underline{\mu}}|_{p,S}(u,\underline{u}) \;&\leq\; c(|r^{2-\frac{2}{p}}\tilde{\underline{\mu}}|_{p,S}(u_0,\underline{u}) + \int_{u_0}^{u}|r^{2-\frac{2}{p}}(\Omega\tilde{\underline{F}} - \overline{\Omega\tilde{\underline{F}}})|_{p,S}) \\
&\quad + \int_{u_0}^{u}|r^{2-\frac{2}{p}}(\Omega\tilde{\underline{H}} - \overline{\Omega\tilde{\underline{H}}})|_{p,S}\Big) \quad (4.3.49)
\end{aligned}$$

The various terms in $|r^{2-\frac{2}{p}}(\Omega\tilde{F} - \overline{\Omega\tilde{F}})|_{p,S}$, $|r^{2-\frac{2}{p}}(\Omega\tilde{\underline{F}} - \overline{\Omega\tilde{\underline{F}}})|_{p,S}$, are estimated using the bootstrap assumption 4.2.5, the estimates 4.3.16, 4.3.28 for $\not{\psi}$ and $\underline{\not{\psi}}$ and the assumption 4.2.1 for ρ, β and $\underline{\beta}$. We obtain

$$\begin{aligned}
&|r^{2-\frac{2}{p}}\Omega\hat{\chi}\cdot(\not{\nabla}\widehat{\otimes}\eta)|_{p,S} \leq c(\Gamma_0)\Gamma_0|r^{2-\frac{2}{p}}\not{\nabla}\eta|_{p,S}\frac{1}{r^2} \\
&|r^{2-\frac{2}{p}}\Omega^2(\eta - \underline{\eta})\cdot\not{\psi}|_{p,S} \leq c(\Gamma_0)\Gamma_0(\mathcal{I}_* + \Delta_0)\frac{1}{r^3} \\
&|r^{2-\frac{2}{p}}\Omega\mathrm{tr}\chi(|\underline{\eta}|^2 - |\eta|^2)|_{p,S} \leq c(\Gamma_0)\Gamma_0^2\frac{1}{r^3} \quad (4.3.50)\\
&|r^{2-\frac{2}{p}}\Omega\eta\cdot\hat{\chi}\cdot\underline{\eta}|_{p,S} \leq c(\Gamma_0)\Gamma_0^3\frac{1}{r^4} \\
&|r^{2-\frac{2}{p}}\left(\mathrm{tr}\underline{\chi}|\hat{\chi}|^2 + \mathrm{tr}\chi(\hat{\chi}\cdot\hat{\underline{\chi}})\right)|_{p,S} \leq c(\Gamma_0)\Gamma_0^2\frac{1}{r^2 u}
\end{aligned}$$

$$\begin{aligned}
&|r^{2-\frac{2}{p}}\Omega\hat{\underline{\chi}}\cdot(\not{\nabla}\widehat{\otimes}\underline{\eta})|_{p,S} \leq c(\Gamma_0)\Gamma_0|r^{2-\frac{2}{p}}\not{\nabla}\underline{\eta}|_{p,S}\frac{1}{ru} \\
&|r^{2-\frac{2}{p}}\Omega^2(\underline{\eta} - \eta)\cdot\underline{\not{\psi}}|_{p,S} \leq c(\Gamma_0)\Gamma_0(\mathcal{I}_0 + \Delta_0)\frac{1}{r^3}
\end{aligned}$$

$$|r^{2-\frac{2}{p}}\Omega\mathrm{tr}\underline{\chi}(|\underline{\eta}|^2-|\eta|^2)|_{p,S} \leq c(\Gamma_0)\Gamma_0^2\frac{1}{r^3} \tag{4.3.51}$$
$$|r^{2-\frac{2}{p}}\Omega\underline{\eta}\cdot\hat{\underline{\chi}}\cdot\eta|_{p,S} \leq c(\Gamma_0)\Gamma_0^3\frac{1}{r^3u}$$
$$|r^{2-\frac{2}{p}}\left(\mathrm{tr}\chi|\hat{\underline{\chi}}|^2+\mathrm{tr}\underline{\chi}(\hat{\underline{\chi}}\cdot\hat{\chi})\right)|_{p,S} \leq c(\Gamma_0)\Gamma_0^2\frac{1}{ru^2}$$

To estimate the various terms appearing in the explicit expressions of $|r^{2-\frac{2}{p}}(\Omega\tilde{H}-\overline{\Omega\tilde{H}})|_{p,S}$ and $|r^{2-\frac{2}{p}}(\Omega\tilde{\underline{H}}-\overline{\Omega\tilde{\underline{H}}})|_{p,S}$ we observe that

$$\begin{aligned}(\Omega\tilde{H}-\overline{\Omega\tilde{H}}) &= (\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})\frac{1}{2}\overline{\chi\cdot\underline{\chi}}-2(\Omega\eta\cdot\beta-\overline{\Omega\eta\cdot\beta})\\ &+ (\Omega\mathrm{tr}\chi\rho-\Omega\mathrm{tr}\chi\overline{\rho})+2(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})\overline{\rho}-\overline{(\Omega\mathrm{tr}\chi\rho-\Omega\mathrm{tr}\chi\overline{\rho})}\\ (\Omega\tilde{\underline{H}}-\overline{\Omega\tilde{\underline{H}}}) &= (\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})\frac{1}{2}\overline{\underline{\chi}\cdot\chi}+2(\Omega\underline{\eta}\cdot\underline{\beta}-\overline{\Omega\underline{\eta}\cdot\underline{\beta}})\\ &+ (\Omega\mathrm{tr}\underline{\chi}\rho-\Omega\mathrm{tr}\underline{\chi}\overline{\rho})+2(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})\overline{\rho}-\overline{(\Omega\mathrm{tr}\underline{\chi}\rho-\Omega\mathrm{tr}\underline{\chi}\overline{\rho})}\end{aligned}$$

Using the bootstrap assumption 4.2.5 and assumptions 4.2.1 for ρ, β and $\underline{\beta}$ we have

$$\begin{aligned}&|r^{2-\frac{2}{p}}(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})\overline{\chi\cdot\underline{\chi}}|_{p,S} \leq c(\Gamma_0)\Gamma_0^2\frac{1}{r^3u}\\ &|r^{2-\frac{2}{p}}(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})\overline{\underline{\chi}\cdot\chi}|_{p,S} \leq c(\Gamma_0)\Gamma_0^2\frac{1}{r^3u}\\ &|r^{2-\frac{2}{p}}(\Omega\eta\cdot\beta-\overline{\Omega\eta\cdot\beta})|_{p,S} \leq c(\Gamma_0)\Gamma_0\Delta_0\frac{1}{r^{\frac{7}{2}}}\\ &|r^{2-\frac{2}{p}}(\Omega\underline{\eta}\cdot\underline{\beta}-\overline{\Omega\underline{\eta}\cdot\underline{\beta}})|_{p,S} \leq c(\Gamma_0)\Gamma_0\Delta_0\frac{1}{r^2u^{\frac{3}{2}}} \qquad (4.3.52)\\ &|r^{2-\frac{2}{p}}(\Omega\mathrm{tr}\chi\rho-\Omega\mathrm{tr}\chi\overline{\rho})|_{p,S} \leq c(\Gamma_0)\Delta_0\frac{1}{r^2u^{\frac{1}{2}}}\\ &|r^{2-\frac{2}{p}}(\Omega\mathrm{tr}\underline{\chi}\rho-\Omega\mathrm{tr}\underline{\chi}\overline{\rho})|_{p,S} \leq c(\Gamma_0)\Delta_0\frac{1}{r^2u^{\frac{1}{2}}}\\ &|r^{2-\frac{2}{p}}(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})\overline{\rho}|_{p,S} \leq c(\Gamma_0)\Delta_0\frac{1}{r^3}\\ &|r^{2-\frac{2}{p}}(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})\overline{\rho}|_{p,S} \leq c(\Gamma_0)\Delta_0\frac{1}{r^3}\end{aligned}$$

Using the inequalities 4.3.50, 4.3.51 we derive the estimates,

$$|r^{2-\frac{2}{p}}(\Omega\tilde{F}-\overline{\Omega\tilde{F}})|_{p,S} \leq c(\Gamma_0)\left[\Gamma_0\frac{1}{r^2}|r^{2-\frac{2}{p}}\nabla\!\!\!/\,\eta|_{p,S}+\frac{1}{r^3}\left((\mathcal{I}_*+\Delta_0)+\Gamma_0^2\right)\right]$$
$$|r^{2-\frac{2}{p}}(\Omega\tilde{\underline{F}}-\overline{\Omega\tilde{\underline{F}}})|_{p,S} \leq c(\Gamma_0)\left[\Gamma_0\frac{1}{ru}|r^{2-\frac{2}{p}}\nabla\!\!\!/\,\underline{\eta}|_{p,S}+\frac{1}{r^2u}\left((\mathcal{I}_0+\Delta_0)+\Gamma_0^2\right)\right] \tag{4.3.53}$$

We estimate $|r^{2-\frac{2}{p}}\nabla\!\!\!/\eta|_{p,S}$, $|r^{2-\frac{2}{p}}\nabla\!\!\!/\underline{\eta}|_{p,S}$ with the help of the Hodge systems 4.3.37, 4.3.38. In view of Proposition 4.1.3 we derive

[The presence of $\mathcal{I}_* + \mathcal{I}_0$ in 4.3.54 arises from the term $\frac{1}{4}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \overline{\mathrm{tr}\chi\mathrm{tr}\underline{\chi}})$.]

$$\begin{aligned}
|r^{2-\frac{2}{p}}\nabla\!\!\!/\eta|_{p,S} &\leq c|r^{2-\frac{2}{p}}\tilde{\mu}|_{p,S} + c\frac{1}{r}|r^{3-\frac{2}{p}}(\rho-\overline{\rho},\sigma)|_{p,S} + \frac{1}{ru}\Gamma_0^2 + c\frac{1}{r}\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Gamma_0^2\right)\\
&\leq c|r^{2-\frac{2}{p}}\tilde{\mu}|_{p,S} + c\frac{1}{r}\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Gamma_0^2\right) + \frac{1}{ru}\Gamma_0^2\\
|r^{2-\frac{2}{p}}\nabla\!\!\!/\underline{\eta}|_{p,S} &\leq c|r^{2-\frac{2}{p}}\underline{\tilde{\mu}}|_{p,S} + c\frac{1}{r}|r^{3-\frac{2}{p}}(\rho,\sigma)|_{p,S} + \frac{1}{ru}\Gamma_0^2 + c\frac{1}{r}\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Gamma_0^2\right)\\
&\leq c|r^{2-\frac{2}{p}}\underline{\tilde{\mu}}|_{p,S} + c\frac{1}{r}\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Gamma_0^2\right) + \frac{1}{ru}\Gamma_0^2 \qquad (4.3.54)
\end{aligned}$$

Therefore inequalities 4.3.53 become

$$\begin{aligned}
|r^{2-\frac{2}{p}}(\Omega\tilde{F}-\overline{\Omega\tilde{F}})|_{p,S} &\leq c(\Gamma_0)\left[\Gamma_0\frac{1}{r^2}|r^{2-\frac{2}{p}}\tilde{\mu}|_{p,S} + \frac{1}{r^3}\left((\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0\Delta_0+\Gamma_0^2\right)\right]\\
|r^{2-\frac{2}{p}}(\Omega\underline{\tilde{F}}-\overline{\Omega\underline{\tilde{F}}})|_{p,S} &\leq c(\Gamma_0)\left[\Gamma_0\frac{1}{ru}|r^{2-\frac{2}{p}}\underline{\tilde{\mu}}|_{p,S} + \frac{1}{r^2u}\left((\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0\Delta_0+\Gamma_0^2\right)\right] \qquad (4.3.55)
\end{aligned}$$

Using the inequalities 4.3.52 we derive the estimates

$$\begin{aligned}
|r^{2-\frac{2}{p}}(\Omega\tilde{H}-\overline{\Omega\tilde{H}})|_{p,S} &\leq c(\Gamma_0)\left(\Gamma_0^2\frac{1}{r^2u}+\Gamma_0\Delta_0\frac{1}{r^{\frac{7}{2}}}+\Delta_0\frac{1}{r^2}\right)\\
|r^{2-\frac{2}{p}}(\Omega\underline{\tilde{H}}-\overline{\Omega\underline{\tilde{H}}})|_{p,S} &\leq c(\Gamma_0)\left(\Gamma_0^2\frac{1}{r^2u}+\Gamma_0\Delta_0\frac{1}{r^2u^{\frac{3}{2}}}+\Delta_0\frac{1}{r^2}\right) \qquad (4.3.56)
\end{aligned}$$

Using these estimates in 4.3.48, 4.3.49, Lemma 4.3.3 follows immediately from an application of the Gronwall Lemma.

Once the estimates for $\tilde{\mu}$ and $\underline{\tilde{\mu}}$, 4.3.46, 4.3.47, have been proved Proposition 4.3.3 follows applying again the estimates of Proposition 4.1.2 to the Hodge systems 4.3.37, 4.3.38.

The following subsections are dedicated to the estimates of $\log\Omega$ and its derivatives up to fourth order. Specifically we will control $\omega,\underline{\omega}$ and the following derivatives

$$\nabla\!\!\!/\omega, \nabla\!\!\!/\underline{\omega}, \mathbf{D}_3\underline{\omega}, \mathbf{D}_4\omega\ ,\ \mathbf{D}_3^2\underline{\omega}, \nabla\!\!\!/\mathbf{D}_3\underline{\omega}, \nabla\!\!\!/\mathbf{D}_4\omega\ ,\ \ \nabla\!\!\!/^2\omega, \nabla\!\!\!/^2\underline{\omega}, \nabla\!\!\!/^3\underline{\omega}\ .$$

[The estimate of $|r(\Omega-\frac{1}{2})|$ has been postponed as it requires the estimate of $\Omega\mathbf{D}_4\log\Omega$.]

4.3.10 Estimate for $\mathcal{O}_0^{p,S}(\omega)$ and $\mathcal{O}_0^{p,S}(\underline{\omega})$ with $p \in [2,4]$

Proposition 4.3.4 *Assuming 4.2.1, 4.2.2, 4.2.3, the bootstrap assumptions 4.2.5 and 4.2.6, then the following estimates hold,*

$$|r^{2-\frac{2}{p}}\Omega\mathbf{D}_4\log\Omega|_{p,S}(u,\underline{u}) \leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)$$
$$|r^{1-\frac{2}{p}}\tau_-\Omega\mathbf{D}_3\log\Omega|_{p,S}(u,\underline{u}) \leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \qquad (4.3.57)$$

Proof: To control ω and $\underline{\omega}$ we start from the equation 3.1.47 which we rewrite as

$$\mathbf{D}_3\mathbf{D}_4\log\Omega+\mathbf{D}_4\mathbf{D}_3\log\Omega = -2(\mathbf{D}_3\log\Omega)(\mathbf{D}_4\log\Omega)+2(\underline{\eta}\cdot\eta-2\zeta^2-\rho)$$

From, see Proposition 4.8.1 in the appendix to this chapter,

$$\mathbf{D}_4\mathbf{D}_3\log\Omega-\mathbf{D}_3\mathbf{D}_4\log\Omega = -4\zeta\cdot\nabla\!\!\!/\log\Omega\ ,$$

we derive

$$\mathbf{D}_3(\Omega\mathbf{D}_4\log\Omega) = \hat{F}-\Omega\rho$$
$$\mathbf{D}_4(\Omega\mathbf{D}_3\log\Omega) = \underline{\hat{F}}-\Omega\rho \qquad (4.3.58)$$

where $\hat{F},\underline{\hat{F}}$ are given by

$$\hat{F} \equiv 2\Omega\zeta\cdot\nabla\!\!\!/\log\Omega+\Omega(\underline{\eta}\cdot\eta-2\zeta^2)$$
$$\underline{\hat{F}} \equiv -2\Omega\zeta\cdot\nabla\!\!\!/\log\Omega+\Omega(\underline{\eta}\cdot\eta-2\zeta^2) \qquad (4.3.59)$$

Applying the Evolution Lemma to these equations we obtain

$$|r^{-\frac{2}{p}}\Omega\mathbf{D}_3\log\Omega|_{p,S}(u,\underline{u}) \leq c\left(|r^{-\frac{2}{p}}\Omega\mathbf{D}_3\log\Omega|_{p,S}(u,\underline{u}_*)+\int_{\underline{u}}^{\underline{u}_*}|r^{-\frac{2}{p}}\hat{F}|_{p,S}+\int_{\underline{u}}^{\underline{u}_*}|r^{-\frac{2}{p}}\Omega\rho|_{p,S}\right)$$
$$|r^{-\frac{2}{p}}\Omega\mathbf{D}_4\log\Omega|_{p,S}(u,\underline{u}) \leq c\left(|r^{-\frac{2}{p}}\Omega\mathbf{D}_4\log\Omega|_{p,S}(u_0,\underline{u})+\int_{u_0}^{u}|r^{-\frac{2}{p}}\underline{\hat{F}}|_{p,S}+\int_{u_0}^{u}|r^{-\frac{2}{p}}\Omega\rho|_{p,S}\right) \qquad (4.3.60)$$

Using assumptions 4.2.1 and the bootstrap assumption 4.2.5 we easily check that

$$|r^{-\frac{2}{p}}(\hat{F},\underline{\hat{F}})|_{p,S} \leq c(\Gamma_0)\Gamma_0^2\frac{1}{r^4}\ \ ,\ \ |r^{-\frac{2}{p}}\rho|_{p,S} \leq cr^{-3}|r^{3-\frac{2}{p}}\rho|_{p,S} \leq c\frac{\Delta_0}{r^3} \qquad (4.3.61)$$

Multiplying both sides of the first inequality in 4.3.60 by r^2 and those of the second one by $r\tau_-$, and using the results of Lemma 4.1.7 applied to $V = \mathbf{D}_4 \log \Omega$, we obtain

$$|r^{2-\frac{2}{p}}\Omega\mathbf{D}_4 \log \Omega|_{p,S}(u,\underline{u}) \le c\left(|r^{2-\frac{2}{p}}\Omega\mathbf{D}_4 \log \Omega|_{p,S_{(0)}}(\underline{u}) + c(\Gamma_0)(\Delta_0 + \Gamma_0^2)\right)$$
$$|r^{1-\frac{2}{p}}\tau_-\Omega\mathbf{D}_3 \log \Omega|_{p,S}(u,\underline{u}) \le c\left(|r^{1-\frac{2}{p}}\tau_-\Omega\mathbf{D}_3 \log \Omega|_{p,S}(u,\underline{u}_*) + c(\Gamma_0)(\Delta_0 + \Gamma_0^2)\right)$$

The exponents $(2 - \frac{2}{p})$ and $(1 - \frac{2}{p})$ are chosen in such a way that the norms

$$|r^{2-\frac{2}{p}}\Omega\mathbf{D}_4 \log \Omega|_{p,S_{(0)}}(\underline{u}) \ , \ |r^{1-\frac{2}{p}}\tau_-\Omega\mathbf{D}_3 \log \Omega|_{p,S}(u,\underline{u}_*)$$

[style modification] are controlled by the assumptions [21] 4.2.2. The final result are the following estimates:

$$|r^{2-\frac{2}{p}}\Omega\mathbf{D}_4 \log \Omega|_{p,S}(u,\underline{u}) \le c\left(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Gamma_0^2\right)$$
$$|r^{1-\frac{2}{p}}\tau_-\Omega\mathbf{D}_3 \log \Omega|_{p,S}(u,\underline{u}) \le c\left(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Gamma_0^2\right)$$

for any $p \in [2,4]$. Choosing $\Gamma_0^2 < \mathcal{I}_0 + \mathcal{I}_* + \Delta_0$ we infer

$$\sup_{p\in[2,4]} \left(\mathcal{O}_0^{p,S}(\omega) + \mathcal{O}_0^{p,S}(\underline{\omega})\right) \le c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0)$$

proving the proposition.

4.3.11 Estimate for $\sup|r(\Omega - \frac{1}{2})|$

Proposition 4.3.5 *Assuming 4.2.1, 4.2.2, 4.2.3 and 4.2.5, 4.2.6, then the following estimate holds,*

$$\sup|r(\Omega - \frac{1}{2})| \le c\,(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0) \tag{4.3.62}$$

Proof: We start from the inequalities 4.3.30. We recall that, as $\eta + \underline{\eta} = 2\nabla\!\!\!/ \log \Omega$,

$$\Delta\!\!\!/ \log \Omega = \frac{1}{2}\nabla\!\!\!/(\eta + \underline{\eta})$$

[21] Remark that, assuming $\mathcal{K}$ endowed with a *double null canonical foliation*, on $\underline{C}_*$ we control the stronger norm $|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}\Omega\mathbf{D}_3 \log \Omega|_{p,S_*}$, see definitions 7.4.3 in Chapter 7. Nevertheless this does not allow to obtain a stronger estimate for $\Omega\mathbf{D}_3 \log \Omega$ in the whole $\mathcal{K}$ due to the presence of the term $\Omega\rho$ in 4.3.58.

and use the elliptic estimates of Proposition 4.1.3 to obtain

$$|r^{1-2/p}(\log\Omega - \overline{\log\Omega})|_{p,S}(u,\underline{u}) \leq c|r^{3-2/p}\nabla\!\!\!/(\eta+\underline{\eta})|_{p,S} \leq c(\mathcal{I}_* + \mathcal{I}_0 + \Delta_0)$$
$$|r^{2-2/p}\nabla\!\!\!/\log\Omega|_{p,S}(u,\underline{u}) \leq c|r^{3-2/p}\nabla\!\!\!/(\eta+\underline{\eta})|_{p,S} \leq c(\mathcal{I}_* + \mathcal{I}_0 + \Delta_0) \quad (4.3.63)$$

From the first inequality, to control $\sup_{\mathcal{K}}|r(\Omega - \frac{1}{2})|$ we have to control $r\overline{\log\Omega}$. To do it we use the following lemma:

Lemma 4.3.4 *Assuming the last slice endowed with the canonical foliation, $\overline{\log\Omega}$ satisfies, along the null hypersurfaces $C(u)$, the following equation:*

$$\overline{\log 2\Omega}(u,\underline{u}) = -\int_{\underline{u}}^{\underline{u}_*}\frac{1}{|S|}\int_S \Omega tr\chi(\log\Omega - \overline{\log\Omega}) - \int_{\underline{u}}^{\underline{u}_*}\overline{\Omega\mathbf{D}_4\log\Omega} \quad (4.3.64)$$

Proof:

$$\begin{aligned}\frac{\partial}{\partial\underline{u}}\left(\frac{1}{|S|}\int_S\log\Omega\right) &= \frac{1}{|S|}\int_S\frac{\partial}{\partial\underline{u}}\log\Omega - \frac{1}{|S|}(\frac{\partial}{\partial\underline{u}}|S|)\overline{\log\Omega} + \frac{1}{|S|}\int_S\Omega tr\chi\log\Omega \\ &= \overline{\Omega\mathbf{D}_4\log\Omega} + \frac{1}{|S|}\int_S\Omega\mathrm{tr}\chi(\log\Omega - \overline{\log\Omega}) \qquad (4.3.65)\end{aligned}$$

where we used $\frac{\partial}{\partial\underline{u}}|S| = |S|\overline{\Omega\mathrm{tr}\chi}$.
Since the canonical foliation implies that, on the last slice, $\overline{\log 2\Omega} = 0$ the result is achieved.

Using Lemma 4.3.4 as well as eq. 4.3.63 we infer that

$$\begin{aligned}|\overline{\log 2\Omega}(u,\underline{u})| &\leq \int_{\underline{u}}^{\underline{u}_*}\frac{1}{|S|}\int_S\Omega|tr\chi||(\log\Omega - \overline{\log\Omega})| + \int_{\underline{u}}^{\underline{u}_*}|\overline{\Omega\mathbf{D}_4\log\Omega}| \\ &\leq \int_{\underline{u}}^{\underline{u}_*}\frac{1}{|S|}\left(\int_S|\Omega\mathrm{tr}\chi|^2\right)^{\frac{1}{2}}\left(\int_S|(\log\Omega - \overline{\log\Omega})|^2\right)^{\frac{1}{2}} + \int_{\underline{u}}^{\underline{u}_*}|\overline{\Omega\mathbf{D}_4\log\Omega}| \\ &\leq c(\Gamma_0)\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}|(\log\Omega - \overline{\log\Omega})|_{p=2,S} + \int_{\underline{u}}^{\underline{u}_*}|\overline{\Omega\mathbf{D}_4\log\Omega}| \\ &\leq c(\Gamma_0)(\mathcal{I}_* + \mathcal{I}_0 + \Delta_0)\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2} + c(\Gamma_0)\frac{1}{r}(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0) \\ &\leq c(\Gamma_0)\frac{1}{r}(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0) \qquad (4.3.66)\end{aligned}$$

Plugging this estimate back in eq. 4.3.63 we obtain

$$\begin{aligned}|r^{1-2/p}\log 2\Omega|_{p,S}(u,\underline{u}) &\leq |r^{1-2/p}\overline{\log 2\Omega}|_{p,S}(u,\underline{u}) + c(\Gamma_0)(\mathcal{I}_* + \mathcal{I}_0 + \Delta_0) \\ &\leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0)\end{aligned}$$

This result together with the second inequality of 4.3.63,

$$|r^{2-2/p}\nabla\!\!\!/ \log 2\Omega|_{p,S}(u,\underline{u}) \leq \frac{1}{2}|r^{2-2/p}(\eta+\underline{\eta})|_{p,S}(u,\underline{u}) \leq c\,(\mathcal{I}_*+\mathcal{I}_0+\Delta_0)$$

allows to conclude, using Lemma 4.1.3,

$$\sup|r\log 2\Omega| \leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \tag{4.3.67}$$

From

$$2(\Omega-\frac{1}{2}) = \exp(\log 2\Omega)-1 = \Sigma_{k=1}^{\infty}\frac{(\log 2\Omega)^k}{k!}$$

it follows that the inequality 4.3.67 implies the inequality

$$\sup|r(\Omega-\frac{1}{2})| \leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)$$

completing the proof.

4.3.12 Completion of the estimates for trχ and tr$\underline{\chi}$

Proposition 4.3.6 *Assume 4.2.1, 4.2.2 and the bootstrap assumption 4.2.5. With the help of the first result of Proposition 4.3.4,*

$$|r^{2-\frac{2}{p}}\Omega\mathbf{D}_4\log\Omega|_{p,S}(u,\underline{u}) \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \tag{4.3.68}$$

we prove

$$\sup|r^2(\overline{\Omega tr\chi}-\frac{1}{r})| \leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \tag{4.3.69}$$

Proof: The evolution equation for $\overline{\Omega\mathrm{tr}\chi}$ is the following one [22]:

$$\frac{d}{d\underline{u}}(\overline{\Omega\mathrm{tr}\chi})+\frac{1}{2}\Omega\mathrm{tr}\chi(\overline{\Omega\mathrm{tr}\chi}) = \frac{1}{2}(\overline{\Omega\mathrm{tr}\chi})V+\frac{1}{2}\overline{V^2}+\overline{E} \tag{4.3.70}$$

where

$$V \equiv \left(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi}\right)\ ,\ E \equiv \left[2\Omega\mathrm{tr}\chi(\Omega\mathbf{D}_4\log\Omega)-\Omega^2|\hat{\chi}|^2\right]\ .$$

[22]

$$\frac{d}{d\underline{u}}(\overline{\Omega\mathrm{tr}\chi})+\frac{1}{2}\Omega\mathrm{tr}\chi(\overline{\Omega\mathrm{tr}\chi}) = \frac{1}{2}\overline{(\Omega\mathrm{tr}\chi)^2}+\frac{1}{2}\Omega\mathrm{tr}\chi(\overline{\Omega\mathrm{tr}\chi})-(\overline{\Omega\mathrm{tr}\chi})^2+\overline{[2\Omega\mathrm{tr}\chi(\Omega\mathbf{D}_4\log\Omega)-\Omega^2|\hat{\chi}|^2]}$$
$$= \frac{1}{2}(\overline{\Omega\mathrm{tr}\chi})\left(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi}\right)+\frac{1}{2}\left(\overline{(\Omega\mathrm{tr}\chi)^2}-(\overline{\Omega\mathrm{tr}\chi})^2\right)+\overline{[2\Omega\mathrm{tr}\chi(\Omega\mathbf{D}_4\log\Omega)-\Omega^2|\hat{\chi}|^2]}$$

Moreover, see 4.1.30,

$$\frac{d}{d\underline{u}}\frac{1}{r}=-\frac{1}{r^2}\frac{\partial r}{\partial\underline{u}}=-\frac{1}{2r}\overline{\Omega\mathrm{tr}\chi}=-\frac{1}{2}\Omega\mathrm{tr}\chi\frac{1}{r}+\frac{1}{2r}V \tag{4.3.71}$$

and, denoting $W\equiv\left(\overline{\Omega\mathrm{tr}\chi}-\frac{1}{r}\right)$,

$$\frac{d}{d\underline{u}}W+\frac{1}{2}\Omega\mathrm{tr}\chi W=\frac{1}{2}WV+\frac{1}{2}\overline{V^2}+\overline{E} \tag{4.3.72}$$

We control $\overline{E}=[2\Omega\mathrm{tr}\chi(\Omega\mathbf{D}_4\log\Omega)-\Omega^2|\hat{\chi}|^2]$ using the bootstrap assumption 4.2.5 and assumption 4.3.68,

$$|r^{1-\frac{2}{p}}E|_{p,S}(u,\underline{u})\leq c(\Gamma_0)\left((\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0^2\right)\frac{1}{r^2}\ .$$

We now apply the Evolution Lemma to 4.3.72, with $\lambda_1=1-\frac{2}{p}$. Applying estimate 4.3.2 to V, Gronwall Lemma and multiplying both sides by r, we obtain, for $p\in[2,4]$, [corollary has been changed in estimate 4.3.2]

$$\left|r^{2-\frac{2}{p}}\left(\overline{\Omega\mathrm{tr}\chi}-\frac{1}{r}\right)\right|_{p,S}\leq c\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Gamma_0^2\right)\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\ .$$

Using Lemma 4.1.3, as $\nabla\!\!\!/\left(\overline{\Omega\mathrm{tr}\chi}-\frac{1}{r}\right)=0$, we obtain

$$\sup|r^2(\overline{\Omega\mathrm{tr}\chi}-\frac{1}{r})|\leq c\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0\right)$$

concluding the proof.

Proposition 4.3.7 *Assume 4.2.1, 4.2.2, 4.2.3, the bootstrap assumptions 4.2.5 and 4.2.6. With the help of the second result of Proposition 4.3.4,*

$$|r^{1-\frac{2}{p}}\tau_-\Omega\mathbf{D}_3\log\Omega|_{p,S}(u,\underline{u})\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \tag{4.3.73}$$

we prove,

$$\sup|r\tau_-(\overline{\Omega tr\underline{\chi}}+\frac{1}{r})|\leq c\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0\right) \tag{4.3.74}$$

Proof: We proceed as for $\left(\overline{\Omega\mathrm{tr}\chi}-\frac{1}{r}\right)$. In this case the evolution equation is, defining $\underline{W}\equiv\left(\overline{\Omega\mathrm{tr}\underline{\chi}}+\frac{1}{r}\right)$,

$$\frac{d}{du}\underline{W}+\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\underline{W}=\frac{1}{2}\underline{W}\ \underline{V}+\frac{1}{2}\overline{\underline{V}^2}+\overline{\underline{E}} \tag{4.3.75}$$

To control $\underline{E} \equiv \left[2\Omega\text{tr}\chi(\Omega\mathbf{D}_3\log\Omega) - \Omega^2|\hat{\underline{\chi}}|^2\right]$ we use the bootstrap assumption 4.2.5 and inequality 4.3.73

$$|r^{1-\frac{2}{p}}\underline{E}|_{p,S}(u,\underline{u}) \leq c(\Gamma_0)\left((\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0^2\right)\frac{1}{r\tau_-}\ .$$

[corollary has been changed in estimate 4.3.6 and the use of Lemma 4.1.7 has been recalled.]

We apply the Evolution Lemma to equation 4.3.75, with $\lambda_1 = 1-\frac{2}{p}$. Using estimate 4.3.6 to control $\underline{V}$, Gronwall Lemma and the results of Lemma 4.1.7 applied to $\underline{W}$, we obtain, for $p \in [2,4]$, multiplying both sides by τ_-,

$$\left|r^{1-\frac{2}{p}}\tau_-\left(\overline{\Omega\text{tr}\underline{\chi}}+\frac{1}{r}\right)\right|_{p,S} \leq c\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0\right) \tag{4.3.76}$$

In view of Lemma 4.1.3, we conclude that [23]

$$\sup|r\tau_-(\overline{\Omega\text{tr}\underline{\chi}}+\frac{1}{r})| \leq c\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0\right)$$

Next corollary, we state without proof, is an elementary consequence of previous results,

Corollary 4.3.1 *From Propositions 4.3.6, 4.3.7 and estimates 4.3.2, 4.3.6 the following inequality holds*

$$\sup|r\tau_-(tr\chi+tr\underline{\chi})| \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \tag{4.3.77}$$

4.3.13 Estimates for $\mathcal{O}_1^{p,S}(\omega)$ and $\mathcal{O}_1^{p,S}(\underline{\omega})$ with $p \in [2,4]$

[Assumptions are the extended ones as we refer to underlined and not underlined quantities]

Proposition 4.3.8 *Under the assumptions 4.2.1, 4.2.2 and the bootstrap assumptions 4.2.5, 4.2.6, the following estimates hold* [24]*, for any $p \in [2,4]$,*

$$|r^{3-\frac{2}{p}}\nabla\!\!\!/\Omega\mathbf{D}_4\log\Omega|_{p,S}(u,\underline{u}) \leq c\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1\right) \tag{4.3.78}$$
$$|r^{2-\frac{2}{p}}\tau_-\nabla\!\!\!/\Omega\mathbf{D}_3\log\Omega|_{p,S}(u,\underline{u}) \leq c\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1\right)$$

[23] Observe that the decay cannot be improved due to the term $2\Omega\text{tr}\chi(\Omega\mathbf{D}_3\log\Omega)$ in $\underline{E}$, see 4.3.75.

[24] The estimates of $\mathcal{O}_1^{p,S}(\omega)$ and $\mathcal{O}_1^{p,S}(\underline{\omega})$ discussed in this subsection can also be obtained in a different way together with the estimates of $\mathcal{O}_2^{p,S}(\omega)$ and $\mathcal{O}_2^{p,S}(\underline{\omega})$, see Proposition 4.4.1.

Proof: $\nabla\!\!\!/\,\omega=-\frac{1}{2}\nabla\!\!\!/\,\mathbf{D}_4\log\Omega$ and $\nabla\!\!\!/\,\underline{\omega}=-\frac{1}{2}\nabla\!\!\!/\,\mathbf{D}_3\log\Omega$ satisfy, along the $\underline{C}(\underline{u})$ and the $C(u)$ null hypersurfaces, the following evolution [25] equations, obtained deriving tangentially equations 4.3.58 and applying the commutation relations 4.8.2,

$$\begin{aligned}
\mathbf{D}\!\!\!\!/_3(\Omega\nabla\!\!\!/\,\mathbf{D}_4\log\Omega)+\frac{1}{2}tr\underline{\chi}(\Omega\nabla\!\!\!/\,\mathbf{D}_4\log\Omega) &= -\hat{\underline{\chi}}(\Omega\nabla\!\!\!/\,\mathbf{D}_4\log\Omega)+\nabla\!\!\!/\,\rho\\
&- (\mathbf{D}_4\log\Omega)(\Omega\nabla\!\!\!/\,\mathbf{D}_3\log\Omega)-\hat{H}\\
\mathbf{D}\!\!\!\!/_4(\Omega\nabla\!\!\!/\,\mathbf{D}_3\log\Omega)+\frac{1}{2}tr\chi(\Omega\nabla\!\!\!/\,\mathbf{D}_3\log\Omega) &= -\hat{\chi}(\Omega\nabla\!\!\!/\,\mathbf{D}_3\log\Omega)+\nabla\!\!\!/\,\rho \qquad (4.3.79)\\
&- (\mathbf{D}_3\log\Omega)(\Omega\nabla\!\!\!/\,\mathbf{D}_4\log\Omega)-\underline{\hat{H}}
\end{aligned}$$

where

$$\begin{aligned}
\hat{H} &= (\nabla\!\!\!/\log\Omega)\Big[-(\mathbf{D}_3\log\Omega)(\mathbf{D}_4\log\Omega)+2\zeta\cdot\nabla\!\!\!/\log\Omega+(\underline{\eta}\cdot\eta-2\zeta^2-\rho)\Big]\\
&\quad+\nabla\!\!\!/\Big[2\zeta\cdot\nabla\!\!\!/\log\Omega+(\underline{\eta}\cdot\eta-2\zeta^2)\Big]\\
\underline{\hat{H}} &= (\nabla\!\!\!/\log\Omega)\Big[-(\mathbf{D}_4\log\Omega)(\mathbf{D}_3\log\Omega)-2\zeta\cdot\nabla\!\!\!/\log\Omega+(\eta\cdot\underline{\eta}-2\zeta^2-\rho)\Big]\\
&\quad+\nabla\!\!\!/\Big[-2\zeta\cdot\nabla\!\!\!/\log\Omega+(\eta\cdot\underline{\eta}-2\zeta^2)\Big]
\end{aligned}$$

These evolution equations have to be estimated simultaneously. Using the bootstrap assumptions 4.2.5, 4.2.6 and assumptions 4.2.1, 4.2.2 we easily check [26] that, for $p\in[2,4]$,

$$\begin{aligned}
&|r^{-\frac{2}{p}}(\hat{H},\underline{\hat{H}})|_{p,S}\le cr^{-5}\Gamma_0\left(\Delta_0+(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0^2\right)\le cr^{-5}\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\
&|r^{4-\frac{2}{p}}\tau_-^{\frac{1}{2}}\nabla\!\!\!/\,\rho|_{p,S}\le\Delta_1
\end{aligned}$$

Also,

$$\begin{aligned}
|(\Omega\nabla\!\!\!/\,\mathbf{D}_3\log\Omega)(\mathbf{D}_4\log\Omega)|_{p,S}&\le r^{-4}\tau_-^{-1}\Gamma_0|r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\,\mathbf{D}_3\log\Omega)|_{p,S}\\
|(\Omega\nabla\!\!\!/\,\mathbf{D}_4\log\Omega)(\mathbf{D}_3\log\Omega)|_{p,S}&\le r^{-4}\tau_-^{-1}\Gamma_0|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\,\mathbf{D}_4\log\Omega)|_{p,S}\ .
\end{aligned}$$

Applying the Evolution Lemma to the first evolution equation 4.3.79, with $\lambda_1=1-\frac{2}{p}$, and then multiplying both sides by r^2 we obtain [27]

[In the first integral of the r.h.s. of 4.3.80, the factor $r^{-1}\tau_-^{-\frac{3}{2}}$ is substituted by $r^{-1}\tau_-^{-1}$]

[In the last term of the r.h.s. of 4.3.80 and of 4.3.81, the factor r^{-1} is wrong due to the presence of $\nabla\!\!\!/\,\rho$ and has to be substituted by $r^{-\frac{1}{2}}$]

[25] Choosing a null frame such that $\mathbf{D}\!\!\!\!/_3e_a=0$ or $\mathbf{D}\!\!\!\!/_4e_a=0$.

[26] Observe that the generic term of $\nabla\!\!\!/[2\zeta\cdot\nabla\!\!\!/\log\Omega+(\underline{\eta}\cdot\eta-2\zeta^2)]$ has to be estimated as a product $|A|_\infty|\nabla\!\!\!/ B|_{L^p(S)}$ which is bounded by $\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)$.

[27] Each time we use the evolution equations along the incoming null hypersurfaces we also use Lemma 4.1.7. As this is always done in the same way we do not repeat it anymore.

$$|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S}(u,\underline{u}) \le c\Big(|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S_{(0)}}(\underline{u})$$
$$+\Gamma_0\int_{u_0}^{u}\frac{1}{r\tau_-}|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S}+\Gamma_0\int_{u_0}^{u}\frac{1}{r\tau_-}|r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S}$$
$$+\frac{1}{r^{\frac{1}{2}}}\left(\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Delta_1\right)\Big) \quad (4.3.80)$$

Applying again the Evolution Lemma, with $\lambda_1 = 1-\frac{2}{p}$, to the second evolution equation 4.3.79 and then multiplying both sides by $r\tau_-$, we obtain

$$|r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S}(u,\underline{u}) \le c\Big(|r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S}(u,\underline{u}_*)$$
$$+\Gamma_0\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}|r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S}+\Gamma_0\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S}$$
$$+\frac{1}{r^{\frac{1}{2}}}\left(\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Delta_1\right)\Big) \quad (4.3.81)$$

Applying Gromwall Lemma to the inequalities 4.3.80 and 4.3.81 it follows

$$|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S}(u,\underline{u}) \le \Big(|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S_{(0)}}(\underline{u})$$
$$+\Gamma_0\int_{u_0}^{u}\frac{1}{r\tau_-}|r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S}+\frac{1}{r^{\frac{1}{2}}(u,\underline{u})}\left(\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Delta_1\right)\Big)$$
$$|r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S}(u,\underline{u}) \le \Big(|r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S}(u,\underline{u}_*)$$
$$+\Gamma_0\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S}+\frac{1}{r^{\frac{1}{2}}(u,\underline{u})}\left(\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Delta_1\right)\Big)$$

[In the last line of 4.3.82, the first integral does not appear and in the second integral the first factor r^{-2} has to be substituted by $r^{-1}\tau_-^{-1}$.

and, combining the two, we easily derive

$$|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S}(u,\underline{u}) \le c\left(\sup_{\Sigma_0\cap\mathcal{K}}|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S_{(0)}}(\underline{u})\right.$$
$$\left.+\Gamma_0\sup_{\underline{C}_*\cap\mathcal{K}}|r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S}(u,\underline{u}_*)+\frac{1}{r^{\frac{1}{2}}}(1+\Gamma_0)\left(\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Delta_1\right)\right)$$
$$+\Gamma_0^2\int_{u_0}^{u}\frac{1}{r\tau_-}\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S} \quad (4.3.82)$$

and the symmetric one. Choosing Γ_0 sufficiently small, and taking the sup of $|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S}$, we obtain

$$|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S}(u,\underline{u}) \le c\left(\sup_{\Sigma_0\cap\mathcal{K}}|r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|p,S_{(0)}(\underline{u})\right.$$

$$+\Gamma_0 \sup_{\underline{C}_*\cap\mathcal{K}} |r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S}(u,\underline{u}_*) + (1+\Gamma_0)\left(\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Delta_1\right)\Big) \tag{4.3.83}$$

and the symmetric one,

$$|r^{2-\frac{2}{p}}\tau_-\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega|_{p,S}(u,\underline{u}) \leq c\Big(\sup_{\underline{C}_*\cap\mathcal{K}} |r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S}(u,\underline{u}_*)$$
$$+\Gamma_0 \sup_{\Sigma_0\cap\mathcal{K}} |r^{3-\frac{2}{p}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|p,S_{(0)}(\underline{u}) + (1+\Gamma_0)\left(\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Delta_1\right)\Big) \tag{4.3.84}$$

Finally, making use of the initial and final slice assumptions 4.2.2

$$\mathcal{O}_{[1]}(\underline{C}_*) \leq \mathcal{I}_* \ , \ \underline{\mathcal{O}}_{[1]}(\Sigma_0) \leq \mathcal{I}_0 \ ,$$

and choosing Γ_0 sufficiently small, we derive, for any $p \in [2,4]$,

$$|r^{3-\frac{2}{p}}\nabla\!\!\!/\Omega\mathbf{D}_4\log\Omega|_{p,S}(u,\underline{u}) \leq c(\Gamma_0)\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1\right)$$
$$|r^{2-\frac{2}{p}}\tau_-\nabla\!\!\!/\Omega\mathbf{D}_3\log\Omega|_{p,S}(u,\underline{u}) \leq c(\Gamma_0)\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1\right)$$

proving the proposition.

The estimates 4.3.78 of Proposition 4.3.8 complete the control of the norms $\mathcal{O}_1^{p,S}(\omega)+\mathcal{O}_1^{p,S}(\underline{\omega})$, obtaining

$$\sup_{p\in[2,4]}\left(\mathcal{O}_1^{p,S}(\omega)+\mathcal{O}_1^{p,S}(\underline{\omega})\right) \leq c\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1\right) \ .$$

4.3.14 Estimate for $\mathcal{O}_0^{p,S}(\mathbf{D}_4\omega)$ and $\mathcal{O}_0^{p,S}(\mathbf{D}_3\underline{\omega})$ with $p \in [2,4]$

Osservazione 4.3.1 *It can be appropriate to observe that for $\mathbf{D}_4\omega$ and $\mathbf{D}_3\underline{\omega}$, as before for ω and $\underline{\omega}$, it is not possible to improve the decay by a factor $\tau_-^{\frac{1}{2}}$. The reason is clear looking at the evolution equations for ω and $\underline{\omega}$, 4.3.59, $\mathbf{D}_3(\Omega\mathbf{D}_4\log\Omega) = \hat{F} - \Omega\rho$, $\mathbf{D}_4(\Omega\mathbf{D}_3\log\Omega) = \hat{\underline{F}} - \Omega\rho$. It turns out that it is the term ρ which produces the decay $O(\frac{1}{r^3})$ and does not allow any stronger decay for ω and $\underline{\omega}$. In the case of $\mathbf{D}_4\omega$ and $\mathbf{D}_3\underline{\omega}$, in their evolution equations instead of ρ there is $\mathbf{D}_4\rho$ and $\mathbf{D}_3\rho$, which, looking at the Bianchi equations, give rise to $tr\chi\rho$ and $tr\underline{\chi}\rho$ which decay as $O(\frac{1}{r^4})$. The conditions of canonical foliation on $\underline{C}(\nu_*)$ and Σ_0 cannot, therefore, improve the asymptotic behaviour of the terms already present in the Schwarzschild spacetime. This remark has to be moved to Chapter 8.*

Proposition 4.3.9 *Under the assumptions of Theorem 4.2.1 and the bootstrap assumptions 4.2.5, 4.2.6, the following estimates hold, with* $p \in [2,4]$,

$$|r^{3-\frac{2}{p}}\mathbf{D}_4^2 \log\Omega|_{p,S} \leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)$$
$$|r^{1-\frac{2}{p}}\tau_-^2\mathbf{D}_3^2 \log\Omega|_{p,S} \leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1) \qquad (4.3.85)$$

Proof: To control the norms of $\mathbf{D}_3^2\log\Omega$ and $\mathbf{D}_4^2\log\Omega$ we derive, in the next lemma, their evolution equations along $C(u)$ and $\underline{C}(\underline{u})$,

Lemma 4.3.5 $(\Omega\mathbf{D}_4)^2\log\Omega$ *and* $(\Omega\mathbf{D}_3)^2\log\Omega$ *satisfy the following equations*

$$\mathbf{D}_3(\Omega\mathbf{D}_4)^2\log\Omega = M-\Omega^2\mathbf{D}_4\rho$$
$$\mathbf{D}_4(\Omega\mathbf{D}_3)^2\log\Omega = \underline{M}-\Omega^2\mathbf{D}_3\rho \qquad (4.3.86)$$

where

$$\begin{aligned} M &= 2\Omega\Big[2\zeta\cdot\nabla\!\!\!/\,\Omega\mathbf{D}_4\log\Omega+2(\Omega\mathbf{D}_4\log\Omega)\zeta\cdot\nabla\!\!\!/\log\Omega+(\Omega\mathbf{D}_4\log\Omega)(\underline{\eta}\cdot\eta-2\zeta^2)\\ &\quad+\Omega\eta\cdot\mathbf{D}_4\nabla\!\!\!/\log\Omega-(-3\zeta+\nabla\!\!\!/\log\Omega)\Big(\nabla\!\!\!/\,\Omega\mathbf{D}_4\log\Omega-\Omega\chi\cdot(\underline{\eta}-\zeta)\Big)\Big]\\ &+ 2\Omega\Big(-\Omega(-3\zeta+\nabla\!\!\!/\log\Omega)\beta-(\Omega\mathbf{D}_4\log\Omega)\rho\Big) \end{aligned} \qquad (4.3.87)$$

and $\underline{M}$ *is obtained by* M *with the obvious changes, see 3.1.33.*

The proof of Lemma 4.3.5 is in the appendix to this chapter, subsection 4.8.2.

In view of the assumptions of Theorem 4.2.1 and the bootstrap assumptions 4.2.5, it is easy to check that

$$|r^{5-\frac{2}{p}}M|_{p,S}(u,\underline{u}) \leq c(\Gamma_0)\Gamma_0\left((\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0^2\right) \qquad (4.3.88)$$

Moreover we control $|r^{4-\frac{2}{p}}\mathbf{D}_4\rho|_{p,S}$, for $p\in[2,4]$,

$$|r^{4-\frac{2}{p}}\mathbf{D}_4\rho|_{p,S} \leq c(\Gamma_0)(\Delta_0+\Delta_1) \qquad (4.3.89)$$

We apply the Evolution Lemma to the first equation of 4.3.86 and obtain, using initial and final slice assumptions 4.2.2, choosing Γ_0 and $(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)$ sufficiently small and multiplying both sides by r^3,

$$\begin{aligned} &|r^{3-\frac{2}{p}}\mathbf{D}_4^2\log\Omega|_{p,S}(u,\underline{u})\\ &\leq |r^{3-\frac{2}{p}}\mathbf{D}_4^2\log\Omega|_{p,S_{(0)}}(\underline{u})+c\left[(\Delta_0+\Delta_1)+\frac{1}{r}\Gamma_0\left((\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0^2\right)\right]\\ &\leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1) \end{aligned} \qquad (4.3.90)$$

The same calculation can be easily repeated for $\mathbf{D}_3^2 \log\Omega$, by interchanging the underlined quantities with those not underlined and viceversa, see 3.1.33. We obtain, for $p \in [2,4]$,

$$|r^{4-\frac{2}{p}}\tau_- \underline{M}|_{p,S}(u,\underline{u}) \leq c(\Gamma_0)\Gamma_0\left((\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0^2\right) \tag{4.3.91}$$

and observe that

$$|r^{3-\frac{2}{p}}\tau_- \mathbf{D}_3\rho|_{p,S} \leq c(\Gamma_0)(\Delta_0+\Delta_1)\ .$$

We apply, therefore, the Evolution Lemma to the second equation of 4.3.86 and obtain, using the initial and final slice assumptions 4.2.2 and multiplying both sides by $r\tau_-^2$,

$$\begin{aligned}|r^{1-\frac{2}{p}}\tau_-^2 \mathbf{D}_3^2 \log\Omega|_{p,S}(u,\underline{u}) &\leq |r^{1-\frac{2}{p}}\tau_-^2 \mathbf{D}_3^2 \log\Omega|_{p,S_{(0)}}(\underline{u})\\ &+ c\left[(\Delta_0+\Delta_1)+\frac{\tau_-}{r^2}\Gamma_0\left((\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0^2\right)\right]\\ &\leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\end{aligned} \tag{4.3.92}$$

for $p \in [2,4]$, completing Proposition 4.3.9.
The estimates 4.3.85 of Proposition 4.3.9 allow to control $\mathcal{O}_0^{p,S}(\mathbf{D}_4\omega)$ and $\underline{\mathcal{O}}_0^{p,S}(\mathbf{D}_3\underline{\omega})$, obtaining

$$\sup_{p\in[2,4]}\left(\mathcal{O}_0^{p,S}(\mathbf{D}_4\omega)+\underline{\mathcal{O}}_0^{p,S}(\mathbf{D}_3\underline{\omega})\right) \leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1) \tag{4.3.93}$$

Remark: *The control of the norm $\mathcal{O}_0^{p,S}(\mathbf{D}_3\underline{\omega})$ obtained in the previous estimate is not yet sufficient to control the "error terms", as discussed in the Chapter 6. In fact it would produce a logarithmic divergence. The control of the following norm, see 3.5.30,*

$$\underline{\tilde{\mathcal{O}}}_1(\underline{\omega}) \equiv ||\frac{1}{\sqrt{\tau_+}}\tau_-^2\mathbf{D}_3\underline{\omega}||_{L_2(C\cap\mathcal{K})}$$

avoids the problem.

4.3.15 Estimate for $\tilde{\mathcal{O}}_1(\underline{\omega})$ for $p \in [2,4]$

Proposition 4.3.10 *Under the assumptions of Theorem 4.2.1 and the bootstrap assumptions 4.2.5 the following estimate hold,*

$$||\frac{1}{\sqrt{\tau_+}}\tau_-^2\mathbf{D}_3\underline{\omega}||_{L_2(C\cap\mathcal{K})} \leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1) \tag{4.3.94}$$

Proof: It follows immediately from the following inequality

$$\begin{aligned}|r^{1-\frac{2}{p}}\tau_-^2\Omega^2\mathbf{D}_3^2\log\Omega|_{p,S}(u,\underline{u}) &\leq c\left[\frac{r(u,\underline{u})}{r(u,\underline{u}_*)}|r^{1-\frac{2}{p}}\tau_-^2\Omega^2\mathbf{D}_3^2\log\Omega|_{p,S}(u,\underline{u}_*)\right.\\ &+ \left. c\left((\Delta_0+\Delta_1)+\frac{\tau_-}{r^2}\Gamma_0\left((\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0^2\right)\right)\right]\end{aligned}$$

an elementary improvement of 4.3.92.

Proposition 4.3.10 completes the proof of part i) of Theorem 4.2.1.

Part ii) of Theorem 4.2.1 now follows immediately. In fact, once we have proved inequalities 4.2.7, 4.2.8, we prove inequality 4.2.10 using Lemma 4.1.3 and conclude immediately, recalling definitions 3.5.22 and 3.5.23, that

$$\begin{aligned}\mathcal{O}_0^\infty+\mathcal{O}_0^{\infty,S}(\omega)&\leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\ \underline{\mathcal{O}}_0^\infty+\underline{\mathcal{O}}_0^{\infty,S}(\underline{\omega})&\leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\end{aligned}\tag{4.3.95}$$

[The next estimates use the stronger assumptions on the initial and last slice satisfied by their canonical foliations, therefore "Remark:" is substituted with "Subsection:.]

This concludes the estimate 4.2.10 of Theorem 4.2.1 and its part ii).

4.3.16 Improved estimates under stronger assumptions on Σ_0 and $\underline{C}_*$

The estimates of the following propositions are proved making stronger assumptions for the various quantities on the initial hypersurface Σ_0 and on the last slice $\underline{C}_*$. These stronger assumptions are proved in Chapter 7 relative to a *double null canonical foliation.*

Proposition 4.3.11 *Assuming 4.2.1, the bootstrap assumption 4.2.5 and, on the last slice,*

$$|r^{3-2/p}\tau_-^{\frac{1}{2}}\not{\Psi}|_{p,S}(u,\underline{u}_*)\leq\mathcal{I}_*\tag{4.3.96}$$

then the following estimates hold,

$$\begin{aligned}|r^{3-2/p}\tau_-^{\frac{1}{2}}\not{\nabla}\hat{\chi}|_{p,S}(u,\underline{u})&\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\ |r^{2-2/p}\tau_-^{\frac{1}{2}}\hat{\chi}|_{p,S}(u,\underline{u})&\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\end{aligned}\tag{4.3.97}$$

Proof: The result is obtained as in Proposition 4.3.1 using the stronger assumption on the last slice.

Proposition 4.3.12 *Assuming 4.2.1, 4.2.3, the bootstrap assumptions 4.2.5, 4.2.6 and that, on the initial slice*

$$|r^{3-2/p}\tau_-^{\frac{1}{2}}\underline{\not{\psi}}|_{p,S_{(0)}}(\underline{u}) \leq \mathcal{I}_0 \tag{4.3.98}$$

then the following estimates hold,

$$|r^{2-2/p}\tau_-^{\frac{3}{2}}\nabla\!\!\!/\,\hat{\underline{\chi}}|_{p,S}(u,\underline{u}) \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)$$
$$|r^{1-2/p}\tau_-^{\frac{3}{2}}\hat{\underline{\chi}}|_{p,S}(u,\underline{u}) \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \tag{4.3.99}$$

Proof: The result is obtained as in Proposition 4.3.2, but using the stronger assumption, 4.3.98, on the initial slice.

Next result will be used in most of the following propositions,

Proposition 4.3.13 *Under the assumptions of Theorem 4.2.1 and the bootstrap assumptions 4.2.5, 4.2.6, assuming, moreover, on Σ_0 and on the last slice $\underline{C}_*$ the following inequalities,*

$$|r^{\frac{5}{2}-\frac{2}{p}}(\Omega\mathbf{D}_4\log\Omega-\overline{\Omega\mathbf{D}_4\log\Omega})|_{p,S_{(0)}}(\underline{u}) \leq \mathcal{I}_0 \tag{4.3.100}$$

$$|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}(\Omega\mathbf{D}_3\log\Omega-\overline{\Omega\mathbf{D}_3\log\Omega})|_{p,S}(u,\underline{u}_*) \leq \mathcal{I}_* \tag{4.3.101}$$

then, in $\mathcal{K}$,

$$|r^{\frac{5}{2}-\frac{2}{p}}\left(\Omega\mathbf{D}_4\log\Omega-\overline{\Omega\mathbf{D}_4\log\Omega}\right)|_{p,S}(u,\underline{u}) \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)$$
$$|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}(\Omega\mathbf{D}_3\log\Omega-\overline{\Omega\mathbf{D}_3\log\Omega})|_{p,S}(u,\underline{u}) \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \tag{4.3.102}$$

Proof: It is easy to prove from the first of eqs. 4.3.58 that

$$\mathbf{D}_3\overline{\Omega\mathbf{D}_4\log\Omega} = \overline{(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})\Omega\mathbf{D}_4\log\Omega}+\overline{\hat{F}}-\overline{\Omega\rho} \tag{4.3.103}$$

therefore, denoting $Y\equiv\Omega\mathbf{D}_4\log\Omega$, we derive the evolution equation

$$\frac{d}{du}(Y-\overline{Y}) = \Omega\left[(\hat{F}-\overline{\hat{F}})+(\Omega\rho-\overline{\Omega\rho})+\overline{(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})Y}\ \right] \tag{4.3.104}$$

As, due to assumptions 4.2.5 and 4.2.1,

$$|r^{4-\frac{2}{p}}\hat{F}|_{p,S} \leq c(\Gamma_0)\Gamma_0^2\ \ ,\ \ |r\tau_-(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})| \leq \Gamma_0\ \ ,\ \ |r^3\tau_-^{\frac{1}{2}}(\Omega\rho-\overline{\Omega\rho})| \leq c\Delta_0$$

we obtain

$$|r^{-\frac{2}{p}}(Y-\overline{Y})|_{p,S}(u,\underline{u}) \leq |r^{-\frac{2}{p}}(Y-\overline{Y})|_{p,S_{(0)}}(\underline{u}) + \frac{1}{r^{\frac{5}{2}}}c(\Gamma_0)\left(\Delta_0+\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0^2\right)$$

which implies, using assumption 4.3.100 and the results of Lemma 4.1.7 applied to $V=\mathbf{D}_4\log\Omega-\overline{\Omega\mathbf{D}_4\log\Omega}$,

$$|r^{\frac{5}{2}-\frac{2}{p}}\left(\Omega\mathbf{D}_4\log\Omega-\overline{\Omega\mathbf{D}_4\log\Omega}\right)|_{p,S}(u,\underline{u}) \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\ .$$

To prove the second part of the proposition we observe that, from the second equation in 4.3.58, it is easy to check that

$$\mathbf{D}_4\overline{\Omega\mathbf{D}_3\log\Omega} = \overline{(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})\Omega\mathbf{D}_3\log\Omega}+\overline{\hat{\underline{F}}}-\overline{\Omega\rho} \tag{4.3.105}$$

Denoting $\underline{Y}\equiv\Omega\mathbf{D}_3\log\Omega$, we obtain

$$\frac{d}{d\underline{u}}(\underline{Y}-\overline{\underline{Y}}) = \Omega\left[(\hat{\underline{F}}-\overline{\hat{\underline{F}}})+(\Omega\rho-\overline{\Omega\rho})+\overline{(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})\underline{Y}}\right] \tag{4.3.106}$$

As again,

$$|r^{4-\frac{2}{p}}\hat{\underline{F}}|_{p,S}\leq c(\Gamma_0)\Gamma_0^2\ ,\ \ |r^3\tau_-^{\frac{1}{2}}(\Omega\rho-\overline{\Omega\rho})|\leq c\Delta_0\ ,\text{and}\ \ |r^2(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})|\leq\Gamma_0\ ,$$

we conclude that

$$\begin{aligned}|r^{-\frac{2}{p}}(\underline{Y}-\overline{\underline{Y}})|_{p,S}(u,\underline{u}) \ \leq\ & c|r^{-\frac{2}{p}}(\underline{Y}-\overline{\underline{Y}})|_{p,S}(u,\underline{u}_*)\\ +\ & \frac{1}{r^2\tau_-^{\frac{1}{2}}}c(\Gamma_0)\left(\Delta_0+\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0^2\right)\end{aligned}$$

from which the second inequality of the proposition,

$$|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}(\Omega\mathbf{D}_3\log\Omega-\overline{\Omega\mathbf{D}_3\log\Omega})|_{p,S}(u,\underline{u})\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\ ,$$

follows, provided that, on the last slice, assumption 4.3.101 is satisfied.

Proposition 4.3.14 *Assume 4.2.1, the bootstrap assumption 4.2.5 and that, on the last slice* $\underline{C}_*$,

$$|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\left(\Omega tr\chi-\overline{\Omega tr\chi}\right)|_{p,S}(u,\underline{u}_*)\leq\mathcal{I}_* \tag{4.3.107}$$

then, with the help of the first inequality of 4.3.102 in Proposition 4.3.13,

$$|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\left(\Omega tr\chi-\overline{\Omega tr\chi}\right)|_{p,S}(u,\underline{u})\leq c\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0\right) \tag{4.3.108}$$

Proof: The evolution equation for $\text{tr}\chi$, see 3.1.45, can be rewritten as

$$\frac{d}{d\underline{u}}(\Omega\text{tr}\chi)+\frac{1}{2}\Omega\text{tr}\chi(\Omega\text{tr}\chi)=\left[2\Omega\text{tr}\chi(\Omega\mathbf{D}_4\log\Omega)-\Omega^2|\hat{\chi}|^2\right]$$

To control $(\text{tr}\chi-\overline{\text{tr}\chi})$ we derive the evolution equation for $\overline{\Omega\text{tr}\chi}$,

$$\frac{d}{d\underline{u}}\overline{\Omega\text{tr}\chi}=\frac{1}{2}\overline{(\Omega\text{tr}\chi)^2}-(\overline{\Omega\text{tr}\chi})^2+\overline{[2\Omega\text{tr}\chi(\Omega\mathbf{D}_4\log\Omega)-\Omega^2|\hat{\chi}|^2]}$$

and from it

$$\begin{aligned}\frac{d}{d\underline{u}}(\Omega\text{tr}\chi-\overline{\Omega\text{tr}\chi})&=-\left(\frac{1}{2}(\Omega\text{tr}\chi)^2+\frac{1}{2}\overline{(\Omega\text{tr}\chi)^2}-(\overline{\Omega\text{tr}\chi})^2\right)\\&+\left[2\Omega\text{tr}\chi(\Omega\mathbf{D}_4\log\Omega)-\Omega^2|\hat{\chi}|^2\right]-\overline{\left[2\Omega\text{tr}\chi(\Omega\mathbf{D}_4\log\Omega)-\Omega^2|\hat{\chi}|^2\right]}\end{aligned}$$

Denoting $V\equiv\left(\Omega\text{tr}\chi-\overline{\Omega\text{tr}\chi}\right)$, this equation can be written as

$$\begin{aligned}&\frac{d}{d\underline{u}}V+\Omega\text{tr}\chi V=\frac{1}{2}V^2-\overline{V^2}+2(\Omega\mathbf{D}_4\log\Omega)V-\left[\Omega^2|\hat{\chi}|^2-\overline{\Omega^2|\hat{\chi}|^2}\right]\quad(4.3.109)\\&+2(\overline{\Omega\text{tr}\chi})\left(\Omega\mathbf{D}_4\log\Omega-\overline{\Omega\mathbf{D}_4\log\Omega}\right)-\overline{2(\Omega\text{tr}\chi)\left(\Omega\mathbf{D}_4\log\Omega-\overline{\Omega\mathbf{D}_4\log\Omega}\right)}\end{aligned}$$

We use now the first inequality of 4.3.100 and the inequalities

$$|\Omega\text{tr}\chi|\leq c(\Gamma_0)\frac{1}{r}\ ,\ |r^{4-\frac{2}{p}}V^2|_{p,S}\leq\Gamma_0^2$$

which follow from the bootstrap assumptions 4.2.5 of Theorem 4.2.1. Applying the Evolution Lemma to the equation 4.3.109, with $\lambda_1=2-\frac{2}{p}$, we obtain, using assumption 4.3.107,

$$\begin{aligned}|r^{2-\frac{2}{p}}V|_{p,S}(u,\underline{u})&\leq c|r^{2-\frac{2}{p}}V|_{p,S}(u,\underline{u}_*)+c(\Gamma_0)\left(\Gamma_0^2\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}+(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^{\frac{3}{2}}}\right)\\&\leq c\left(|r^{2-\frac{2}{p}}V|_{p,S}(u,\underline{u}_*)+\frac{1}{r}\Gamma_0^2+\frac{1}{r^{\frac{1}{2}}}(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\right)\\&\leq c\tau_-^{-\frac{1}{2}}(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\qquad(4.3.110)\end{aligned}$$

choosing $(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)$ and Γ_0 sufficiently small.

Proposition 4.3.15 *Assume 4.2.1, 4.2.2, the bootstrap assumptions 4.2.5 and 4.2.6, and that, on the initial slice* Σ_0,

[Assumptions have been modified as we refer to underlined quantities]

$$|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\left(\Omega tr\underline{\chi}-\overline{\Omega tr\underline{\chi}}\right)|_{p,S_{(0)}}(\underline{u})\leq\mathcal{I}_0 \tag{4.3.111}$$

then, with the help of the second inequality of 4.3.102 in Proposition 4.3.13,

[Modification in the state

$$|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\left(\Omega tr\underline{\chi}-\overline{\Omega tr\underline{\chi}}\right)|_{p,S}(u,\underline{u})\leq c\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0\right) \tag{4.3.112}$$

Proof: The evolution equation for $\mathrm{tr}\underline{\chi}$, see 3.1.45, can be rewritten as

$$\frac{d}{du}(\Omega\mathrm{tr}\underline{\chi})+\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}(\Omega\mathrm{tr}\underline{\chi})=\left[2\Omega\mathrm{tr}\underline{\chi}(\Omega\mathbf{D}_3\log\Omega)-\Omega^2|\hat{\underline{\chi}}|^2\right]$$

To control $(\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\underline{\chi}})$ we derive the evolution equation for $\overline{\Omega\mathrm{tr}\underline{\chi}}$,

$$\frac{d}{du}(\overline{\Omega\mathrm{tr}\underline{\chi}})=\frac{1}{2}\overline{(\Omega\mathrm{tr}\underline{\chi})^2}-(\overline{\Omega\mathrm{tr}\underline{\chi}})^2+\overline{\left[2\Omega\mathrm{tr}\underline{\chi}(\Omega\mathbf{D}_3\log\Omega)-\Omega^2|\hat{\underline{\chi}}|^2\right]}$$

[Correction in 4.3.113, $\Omega\mathrm{tr}\underline{\chi}\underline{V}$ instead of $\Omega\mathrm{tr}\chi\underline{V}$]

so that, defining $\underline{V}=\left(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}}\right)$, we obtain

$$\frac{d}{du}\underline{V}+\Omega\mathrm{tr}\underline{\chi}\underline{V}=\frac{1}{2}\underline{V}^2-\overline{\underline{V}^2}+2(\Omega\mathbf{D}_3\log\Omega)\underline{V}-\left[\Omega^2|\hat{\underline{\chi}}|^2-\overline{\Omega^2|\hat{\underline{\chi}}|^2}\right] \tag{4.3.113}$$
$$+2(\overline{\Omega\mathrm{tr}\underline{\chi}})\left(\Omega\mathbf{D}_3\log\Omega-\overline{\Omega\mathbf{D}_3\log\Omega}\right)-\overline{2(\Omega\mathrm{tr}\underline{\chi})\left(\Omega\mathbf{D}_3\log\Omega-\overline{\Omega\mathbf{D}_3\log\Omega}\right)}$$

From the bootstrap assumptions 4.2.5 of Theorem 4.2.1, it follows

$$|\Omega\mathrm{tr}\underline{\chi}|\leq c(\Gamma_0)\frac{1}{r}\quad,\quad|\Omega(\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\underline{\chi}})|\leq c(\Gamma_0)\Gamma_0\frac{1}{r^2}\ .$$

We use the second inequality of 4.3.102 and, proceeding as in the case of $(\mathrm{tr}\chi-\overline{\mathrm{tr}\chi})$, applying the Evolution Lemma to the equation 4.3.113, with $\lambda_1=1-\frac{2}{p}$, using assumption 4.3.111 and the results of Lemma 4.1.7 applied to $V=\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\underline{\chi}}$ we conclude that, for $p\in[2,4]$,

[Lemma 4.1.7 recalled.]

$$|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\underline{\chi}})|_{p,S}\leq c\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0\right)$$

proving the proposition.

Proposition 4.3.16 *Assume 4.2.1, 4.2.2, the bootstrap assumptions 4.2.5 and 4.2.6, and that, on the initial and final slices,*

[Assumptions are the extended ones as we refer to underlined and not underlined quantities]

$$|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\left(\Omega tr\underline{\chi}-\overline{\Omega tr\underline{\chi}}\right)|_{p,S_{(0)}}(\underline{u})\leq\mathcal{I}_0$$
$$|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\left(\Omega tr\chi-\overline{\Omega tr\chi}\right)|_{p,S}(u,\underline{u}_*)\leq\mathcal{I}_*$$
$$|r^{3-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\underline{\mu}-\overline{\underline{\mu}})|_{p,S_{(0)}}(\underline{u})\leq c\mathcal{I}_0$$
$$|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\mu-\overline{\mu})|_{p,S}(u,\underline{u}_*)\leq c\mathcal{I}_*$$

then,

$$\begin{aligned}
|r^{2-2/p}\tau_-^{\frac{1}{2}}\eta|_{p,S}(u,\underline{u}) &\leq c\,(\mathcal{I}_*+\mathcal{I}_0+\Delta_0)\\
|r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!/\eta|_{p,S}(u,\underline{u}) &\leq c\,(\mathcal{I}_*+\mathcal{I}_0+\Delta_0)\\
|r^{2-2/p}\tau_-^{\frac{1}{2}}\underline{\eta}|_{p,S}(u,\underline{u}) &\leq c\,(\mathcal{I}_*+\mathcal{I}_0+\Delta_0) \qquad (4.3.114)\\
|r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!/\underline{\eta}|_{p,S}(u,\underline{u}) &\leq c\,(\mathcal{I}_*+\mathcal{I}_0+\Delta_0)
\end{aligned}$$

Proof: Recalling that

$$\begin{aligned}
\tilde{\mu} &= \left[(\mu-\overline{\mu})+\frac{1}{4}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\chi\mathrm{tr}\underline{\chi}})\right]\\
\underline{\tilde{\mu}} &= \left[(\underline{\mu}-\overline{\underline{\mu}})+\frac{1}{4}(\mathrm{tr}\underline{\chi}\mathrm{tr}\chi-\overline{\mathrm{tr}\underline{\chi}\mathrm{tr}\chi})\right] \qquad (4.3.115)
\end{aligned}$$

from the assumptions it follows that on the initial and final slice we have

$$\begin{aligned}
&|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\tilde{\mu}|_{p,S}(u,\underline{u}_*)\leq c\mathcal{I}_*\\
&|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\underline{\tilde{\mu}}|_{p,S_{(0)}}(\underline{u})\leq c\mathcal{I}_0 \qquad (4.3.116)
\end{aligned}$$

Using these estimates in Lemma 4.3.3, the results of Propositions 4.3.14, 4.3.15 we obtain the inequalities

$$\begin{aligned}
&|r^{3-\frac{2}{p}}\tau_-^{\frac{1}{2}}\tilde{\mu}|_{p,S}(u,\underline{u})\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\
&|r^{3-\frac{2}{p}}\tau_-^{\frac{1}{2}}\underline{\tilde{\mu}}|_{p,S}(u,\underline{u})\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \qquad (4.3.117)
\end{aligned}$$

The result follows using also propositions 4.3.11, 4.3.12 and applying Proposition 4.1.2 to the Hodge systems 4.3.37, 4.3.38. From the last proposition next corollary immediately follows,

Corollary 4.3.2 *Under the same assumptions as in Proposition 4.3.16,*

$$\begin{aligned}
&|r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!/ tr\chi|_{p,S}(u,\underline{u})\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\
&|r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!/ tr\underline{\chi}|_{p,S}(u,\underline{u})\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \qquad (4.3.118)
\end{aligned}$$

Proof: It follows immediatly from the relation $2\zeta=(\eta-\underline{\eta})$ and the definitions of $\psi\!\!\!/$ and $\underline{\psi\!\!\!/}$.

Proposition 4.3.17 *Assume 4.2.1, 4.2.2, 4.2.3, the bootstrap assumptions 4.2.5, 4.2.6. Assume, moreover, that on the last slice $\underline{C}_*$ and on the initial slice Σ_0 the following estimates hold*

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\nabla\!\!\!/\,\Omega\mathbf{D}_3\log\Omega|_{p,S}(u,\underline{u}_*)\leq\mathcal{I}_*\\
&|r^{3-\frac{2}{p}}\tau_-^{\frac{1}{2}}\nabla\!\!\!/\,\Omega\mathbf{D}_4\log\Omega|_{p,S_{(0)}}(\underline{u})\leq\mathcal{I}_0
\end{aligned}
\tag{4.3.119}
$$

then we derive, for any $p\in[2,4]$,

$$
\begin{aligned}
&|r^{3-\frac{2}{p}}\tau_-^{\frac{1}{2}}\nabla\!\!\!/\,\Omega\mathbf{D}_4\log\Omega|_{p,S}(u,\underline{u})\leq c(\Gamma_0)\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1\right)\\
&|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\nabla\!\!\!/\,\Omega\mathbf{D}_3\log\Omega|_{p,S}(u,\underline{u})\leq c(\Gamma_0)\left(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1\right)
\end{aligned}
\tag{4.3.120}
$$

Proof: The result is obtained proceeding as in Proposition 4.3.8, but using the stronger assumptions 4.3.119 on initial and last slices.

4.4 Proof of Theorem 4.2.2, estimates for the second derivatives of the connection coefficents

Let us examine the various second derivatives whose norm estimates are provided by Theorem 4.2.2.

1) Estimates for $\nabla\!\!\!/^2\mathrm{tr}\chi$, $\nabla\!\!\!/^2\hat{\chi}$

The estimates for these second derivatives are obtained exactly with the same procedure used for the corresponding first derivatives with the help of the equation for $\psi\!\!\!/$, 4.3.6, and the Codazzi equation, 4.3.13, which have to be differentiated once more in the angular direction.
As there are no new ideas required, just technical drudgery, we shall omit the proof.

2) Estimates for $\nabla\!\!\!/^2\mathrm{tr}\underline{\chi}$, $\nabla\!\!\!/^2\underline{\hat{\chi}}$

The same considerations as before apply. One uses the basic transport equation 4.3.19 and the Codazzi equation 4.3.25, which have to be differentiated to provide the appropriate equations.

3) Estimates for $\nabla\!\!\!/^2\eta$, $\nabla\!\!\!/^2\underline{\eta}$

The estimates are again obtained with the same procedure starting from the Hodge systems 4.3.37, 4.3.38 coupled with the transport equations 4.3.39 for $\tilde{\mu}$ and $\underline{\tilde{\mu}}$. We have again to take angular derivatives of these equations and proceed as done before.

Remark: Observe that also for the second derivatives $\nabla\!\!\!/^2\mathrm{tr}\chi$, $\nabla\!\!\!/^2\hat{\chi}$, $\nabla\!\!\!/^2\mathrm{tr}\underline{\chi}$, $\nabla\!\!\!/^2\hat{\underline{\chi}}$, $\nabla\!\!\!/^2\eta$, $\nabla\!\!\!/^2\underline{\eta}$, we do not discuss explicitely, better estimates can be obtained, provided stronger assumptions on the initial and final slices hold. In this case their $|\cdot|_{p,S}$ norms gain a factor $\tau_-^{\frac{1}{2}}$ exactly as proved for their zero and first derivatives in subsection 4.3.16.

4) The estimates for the angular derivatives and the null directions derivative of ω and $\underline{\omega}$ are, viceversa, more delicate. They require some new ideas and will be examined in full detail.

[The order of the two next subsections have been interchanged.]

4.4.1 Estimates for $\mathcal{O}_2^{p,S}(\omega)$ and $\mathcal{O}_2^{p,S}(\underline{\omega})$ for $p \in [2,4]$

Remark: *This subsection is devoted to the control of* $\nabla\!\!\!/^2\mathbf{D}_3\log\Omega$, $\nabla\!\!\!/^2\mathbf{D}_4\log\Omega$. *Nevertheless, proving these estimates it follows immediately that one obtains also an estimate for* $\nabla\!\!\!/\mathbf{D}_3\log\Omega$ *and* $\nabla\!\!\!/\mathbf{D}_4\log\Omega$ *which have been already estimated in a more direct way in subsection 4.3.13.*

Proposition 4.4.1 *Under the assumptions of Theorem 4.2.2 and the bootstrap assumptions 4.2.5 and 4.2.15, then, for any* $p \in [2,4]$,

[In view of what is said in the next remark, we erase from the statement of the proposition "using the results of Theorem 4.2.1, moreover "the bootstrap assumptions 4.2.21" is changed in "the bootstrap assumptions 4.2.5 and 4.2.15" .]

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\tau_-\nabla\!\!\!/\mathbf{D}_3\log\Omega|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\
&|r^{3-\frac{2}{p}}\nabla\!\!\!/\mathbf{D}_4\log\Omega|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\
&|r^{3-\frac{2}{p}}\tau_-\nabla\!\!\!/^2\mathbf{D}_3\log\Omega|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1) \qquad (4.4.1)\\
&|r^{4-\frac{2}{p}}\tau_-\nabla\!\!\!/^2\mathbf{D}_4\log\Omega|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)
\end{aligned}
$$

Remark: *In the proof of Theorem 4.2.2 we can, of course, rely on the results of Theorem 4.2.1. However since the terms to which the results of the previous theorem applies appear in non linear expressions, it suffices to use for them the weaker estimate* $\mathcal{O}^\infty_{[0]}+\underline{\mathcal{O}}^\infty_{[0]} \leq \Gamma_0$ *which is, in fact, the bootstrap assumptions of Theorem 4.2.1. As, moreover, we also make the new bootstrap assumption 4.2.15,* $\mathcal{O}^\infty_1+\underline{\mathcal{O}}^\infty_1 \leq \Gamma_1$, *this means that we can rely on the assumption* $\mathcal{O}^\infty_{[1]}+\underline{\mathcal{O}}^\infty_{[1]} \leq \Gamma_0+\Gamma_1$.

Proof: We discuss only the estimates for $\nabla\!\!\!/\mathbf{D}_3\log\Omega$ and $\nabla\!\!\!/^2\mathbf{D}_3\log\Omega$, as the estimates of $\nabla\!\!\!/\mathbf{D}_4\log\Omega$ and $\nabla\!\!\!/^2\mathbf{D}_4\log\Omega$ are obtained in the same way. The evolution equations for $\nabla\!\!\!/^2\mathbf{D}_3\log\Omega$ and $\nabla\!\!\!/^2\mathbf{D}_4\log\Omega$ are obtained in the following lemma, whose proof is obtained deriving tangentially the evolution equations for $\nabla\!\!\!/\mathbf{D}_3\log\Omega$ and $\nabla\!\!\!/\mathbf{D}_4\log\Omega$, see 4.3.79. The details are in the appendix to this chapter.

[formal modification, the evolution equation for $\nabla\!\!\!/^2\mathbf{D}_3\log\Omega$ and $\nabla\!\!\!/^2\mathbf{D}_4\log\Omega$ have been put in a lemma]

Lemma 4.4.1 $\nabla\!\!\!/^2\mathbf{D}_3\log\Omega$ *and* $\nabla\!\!\!/^2\mathbf{D}_4\log\Omega$ *satisfy the following evolution equations, denoting* $V=\nabla\!\!\!/\mathbf{D}_3\log\Omega$, $\underline{V}=\nabla\!\!\!/\mathbf{D}_4\log\Omega$,

$$\mathbf{D}\!\!\!\!/_4\nabla\!\!\!/_aV_b+tr\chi\nabla\!\!\!/_aV_b=-\left(\hat{\chi}_{ac}\nabla\!\!\!/_cV_b+\hat{\chi}_{bc}\nabla\!\!\!/_aV_c\right)+\Omega^{-1}\nabla\!\!\!/_a\nabla\!\!\!/_b(\Omega\mathbf{D}_4\mathbf{D}_3\log\Omega)+\underline{Q}_{ab}$$
$$\mathbf{D}\!\!\!\!/_3\nabla\!\!\!/_a\underline{V}_b+tr\underline{\chi}\nabla\!\!\!/_a\underline{V}_b=-\left(\hat{\underline{\chi}}_{ac}\nabla\!\!\!/_c\underline{V}_b+\hat{\underline{\chi}}_{bc}\nabla\!\!\!/_a\underline{V}_c\right)+\Omega^{-1}\nabla\!\!\!/_a\nabla\!\!\!/_b(\Omega\mathbf{D}_3\mathbf{D}_4\log\Omega)+Q_{ab} \quad (4.4.2)$$

where

$$\underline{Q}_{ab}=-\left[(\nabla\!\!\!/_a\log\Omega)\chi_{bc}V_c+(\nabla\!\!\!/_a\chi_{bc})V_c+\underline{\eta}_b\chi_{ac}V_c-\chi_{ab}\underline{\eta}_cV_c-e_4^\tau e_a^\rho\left([D_\tau,D_\rho]V_\sigma\right)e_b^\sigma\right]$$
$$Q_{ab}=-\left[(\nabla\!\!\!/_a\log\Omega)\underline{\chi}_{bc}\underline{V}_c+(\nabla\!\!\!/_a\underline{\chi}_{bc})\underline{V}_c+\eta_b\underline{\chi}_{ac}\underline{V}_c-\underline{\chi}_{ab}\eta_c\underline{V}_c-e_3^\tau e_a^\rho\left([D_\tau,D_\rho]\underline{V}_\sigma\right)e_b^\sigma\right]$$

Using directly this equation is not efficient. In fact from the explicit expression of $\Omega\mathbf{D}_4\mathbf{D}_3\log\Omega$, see 4.3.58,

$$\Omega\mathbf{D}_4\mathbf{D}_3\log\Omega=\Omega\left[-2\zeta\cdot\nabla\!\!\!/\log\Omega-(\mathbf{D}_3\log\Omega)(\mathbf{D}_4\log\Omega)+(\eta\cdot\underline{\eta}-2\zeta^2)\right]-\Omega\rho\ ,$$

we have

$$\Omega^{-1}\nabla\!\!\!/_a\nabla\!\!\!/_b(\Omega\mathbf{D}_4\mathbf{D}_3\log\Omega)=-\nabla\!\!\!/_a\nabla\!\!\!/_b\rho-(\mathbf{D}_4\log\Omega)(\nabla\!\!\!/_aV_b)+\underline{L}_{ab} \quad (4.4.3)$$

where

$$\begin{aligned}\underline{L}_{ab} \;=\;& \frac{1}{\Omega}\left[-(\nabla\!\!\!/_a\nabla\!\!\!/_b\Omega)\rho-2(\nabla\!\!\!/_a\Omega)\nabla\!\!\!/_b\rho\right]\\ &-\frac{1}{\Omega}\left[\nabla\!\!\!/_a\nabla\!\!\!/_b\left(\Omega(\mathbf{D}_3\log\Omega)(\mathbf{D}_4\log\Omega)\right)-\Omega(\mathbf{D}_4\log\Omega)\nabla\!\!\!/_aV_b\right]\\ &+\frac{1}{\Omega}\left[\nabla\!\!\!/_a\nabla\!\!\!/_b\left(\Omega(-2\zeta\cdot\nabla\!\!\!/\log\Omega+\eta\cdot\underline{\eta}-2\zeta^2)\right)\right]\end{aligned}$$

depends only on the first derivatives of the Riemann components [28] and second derivatives of the connection coefficients. Plugging this expression in the previous equation we obtain

$$\begin{aligned}\mathbf{D}_4(\Omega\nabla\!\!\!/_aV_b)+tr\chi(\Omega\nabla\!\!\!/_aV_b) \;=\;& -\left(\hat{\chi}_{ac}(\Omega\nabla\!\!\!/_cV_b)+\hat{\chi}_{bc}(\Omega\nabla\!\!\!/_aV_c)\right)\\ &+\;\Omega\left[\underline{Q}+\underline{L}\right]_{ab}-\Omega\nabla\!\!\!/_a\nabla\!\!\!/_b\rho \qquad (4.4.4)\end{aligned}$$

[28]This is true, notwithstanding the presence of $\nabla\!\!\!/_a\nabla\!\!\!/_b\mathbf{D}_4\log\Omega$ in $\underline{L}_{ab}$, as the same argument we are developing here holds for it.

The term $\Omega\nabla\!\!\!/_a\nabla\!\!\!/_b\rho$ makes impossible to obtain an estimate of $\Omega\nabla\!\!\!/_a V_b$ in the $|\cdot|_{p,S}$ norm [29]. We overcome this problem using the Bianchi equations to transform a tangential derivative in a null derivative, a procedure which has been repeatedly used in [Ch-Kl]. Using the equation[30], see 3.2.8,

$$\underline{\beta}_4 \equiv \mathbf{D}\!\!\!\!/_4\underline{\beta} + \mathrm{tr}\chi\underline{\beta} = -\nabla\!\!\!/\rho + {}^\star\nabla\!\!\!/\sigma + 2\hat{\underline{\chi}}\cdot\beta + 2\omega\underline{\beta} - 3(\underline{\eta}\rho - {}^\star\underline{\eta}\sigma)$$

we write $\nabla\!\!\!/_a\nabla\!\!\!/_b\rho$ in terms of $\mathbf{D}_4\nabla\!\!\!/_a\underline{\beta}_b$, plus "lower order" terms [31],

$$\mathbf{D}_4(\Omega\nabla\!\!\!/_a\underline{\beta}_b) + \frac{3}{2}\mathrm{tr}\chi\Omega\nabla\!\!\!/_a\underline{\beta}_b = -\Omega\nabla\!\!\!/_a\nabla\!\!\!/_b\rho + \Omega\nabla\!\!\!/_a{}^\star\nabla\!\!\!/_b\sigma - \hat{\underline{\chi}}_{ac}\Omega\nabla\!\!\!/_c\underline{\beta}_b + \underline{H}_{ab}\ ,$$

and the evolution equation for $\nabla\!\!\!/_a(V_b - \underline{\beta}_b)$ is

$$\begin{aligned}\mathbf{D}_4(\Omega\nabla\!\!\!/_a(V_b - \underline{\beta}_b)) + tr\chi(\Omega\nabla\!\!\!/_a(V_b - \underline{\beta}_b)) &= -\Big(\hat{\chi}_{ac}(\Omega\nabla\!\!\!/_c V_b) + \hat{\chi}_{bc}\Omega\nabla\!\!\!/_a V_c\Big)\\ &+\frac{1}{2}\mathrm{tr}\chi\Omega\nabla\!\!\!/_a\underline{\beta}_b + \Omega\left[\underline{Q} + \underline{L} + \underline{H}\right]_{ab} - \Omega\nabla\!\!\!/_a{}^\star\nabla\!\!\!/_b\sigma \quad (4.4.5)\end{aligned}$$

To achieve the result of avoiding, in the right hand side, the second derivatives of the Riemann tensor we have to get rid of the term $\nabla\!\!\!/{}^\star\nabla\!\!\!/\sigma$. This is obtained considering instead of $\Omega\nabla\!\!\!/_a(V_b - \underline{\beta}_b)$ the quantity

[formal modification, $\Omega\nabla\!\!\!/_a(V_b - \underline{\beta}_b)$ instead of $\Omega\nabla\!\!\!/_a V_b$.]

$$\underline{\psi\!\!\!/} \equiv \Omega(\mathrm{d}\!\!\!/\mathrm{iv}\, V - \mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\beta}) = -\Omega(2\triangle\!\!\!\!/\,\underline{\omega} + \mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\beta}) \quad (4.4.6)$$

which satisfies the evolution equation

$$\begin{aligned}\mathbf{D}_4\underline{\psi\!\!\!/} + \mathrm{tr}\chi\underline{\psi\!\!\!/} &= -2\hat{\chi}\cdot(\Omega\nabla\!\!\!/ V) + \frac{1}{2}\mathrm{tr}\chi\Omega\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\beta} + \Omega\mathrm{tr}(\underline{Q} + \underline{L} + \underline{H})\\ &= -2\hat{\chi}\cdot\Omega(\nabla\!\!\!/ V - \nabla\!\!\!/\underline{\beta}) - 2\hat{\chi}\cdot\Omega\nabla\!\!\!/\underline{\beta} + \frac{1}{2}\mathrm{tr}\chi\Omega\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\beta}\\ &\quad + \Omega\mathrm{tr}(\underline{Q} + \underline{L} + \underline{H}) \quad (4.4.7)\end{aligned}$$

The evolution equation for $\underline{\psi\!\!\!/}$ does not contain any second tangential derivative of ρ or σ and, therefore, can be estimated with the $|\cdot|_{p,S}$ norms, with $p\in[2,4]$. We obtain the following inequality

$$|r^{3-\frac{2}{p}}\tau_-\underline{\psi\!\!\!/}\,|_{p,S}(u,\underline{u}) \le c|r^{3-\frac{2}{p}}\tau_-\underline{\psi\!\!\!/}\,|_{p,S}(u,\underline{u}_*) + c\tau_-^{-\frac{1}{2}}(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0 + \Delta_1) \quad (4.4.8)$$

[29] We recall that, in this norm, we do not control the second derivatives of the Riemann components.

[30] Recall that ${}^\star\nabla\!\!\!/_a\sigma = \epsilon_{ac}\nabla\!\!\!/^c\sigma$.

[31] Here "lower order" is in the sense of order of derivatives.

We control $\Omega\nabla\!\!\!/_a V_b = \Omega\nabla\!\!\!/_a\nabla\!\!\!/_b\mathbf{D}_3\log\Omega$ using the estimate of $\underline{\psi\!\!\!/}$, 4.4.8, the elliptic estimates for the equation

$$\Delta\!\!\!/(\mathbf{D}_3\log\Omega) = \Omega^{-1}\underline{\psi\!\!\!/} \ + \mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\beta}\ ,$$

the estimate of $\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\beta}$ and the last slice estimate, derived from 4.2.12,

$$|r^{3-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/^2\mathbf{D}_3\log\Omega)|_{p,S}(u,\underline{u}_*) \leq \mathcal{I}_* \tag{4.4.9}$$

We obtain in this way the estimates which prove Proposition 4.4.1

$$\begin{aligned}&|r^{2-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\&|r^{3-\frac{2}{p}}\tau_-(\Omega\nabla\!\!\!/^2\mathbf{D}_3\log\Omega)|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\ .\end{aligned}$$

Corollary 4.4.1 *Assume inequalities 4.2.1, 4.2.5 and that on the last slice* $\underline{C}_*$ *and the initial slice* Σ_0 *the following estimate holds*

$$\begin{aligned}&|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}\nabla\!\!\!/^2\Omega\mathbf{D}_3\log\Omega|_{p,S}(u,\underline{u}_*) \leq \mathcal{I}_*\\&|r^{4-\frac{2}{p}}\tau_-^{\frac{1}{2}}\nabla\!\!\!/^2\Omega\mathbf{D}_4\log\Omega|_{p,S_{(0)}}(\underline{u}) \leq \mathcal{I}_0\end{aligned} \tag{4.4.10}$$

then we derive, for any $p\in[2,4]$,

$$\begin{aligned}&|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega)|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\&|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}(\Omega\nabla\!\!\!/^2\mathbf{D}_3\log\Omega)|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\&|r^{3-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega)|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\&|r^{4-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\Omega\nabla\!\!\!/^2\mathbf{D}_4\log\Omega)|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\end{aligned} \tag{4.4.11}$$

Proof: The result is obtained proceeding as in Proposition 4.4.1, but using the stronger assumption 4.4.10 on the last and initial slices.

4.4.2 Estimate for $\tilde{\mathcal{O}}_2(\underline{\omega})$ with $p\in[2,4]$

We recall the definition, see 3.5.30,

$$\tilde{\mathcal{O}}_2(\underline{\omega}) \equiv ||\frac{1}{\sqrt{\tau_+}}\tau_-^3\mathbf{D}_3^2\underline{\omega}||_{L_2(C\cap\mathcal{K})} + ||\frac{1}{\sqrt{\tau_+}}r\tau_-^2\nabla\!\!\!/\mathbf{D}_3\underline{\omega}||_{L_2(C\cap\mathcal{K})}$$

The terms in the right hand side are estimated in the following proposition.

Proposition 4.4.2 *Under the assumptions of Theorem 4.2.2 and the bootstrap assumptions 4.2.5 and 4.2.15, then, choosing* Γ_0 *and* Γ_1 *sufficiently small,*

ew of what is said in the k after proposition 4.4.1, ase from the statement of roposition "using the re- of Theorem 4.2.1, more- "the bootstrap assump- 4.2.21" is changed in "the rap assumptions 4.2.5 .2.15" .]

$$||\frac{1}{\sqrt{\tau_+}}r\tau_-^2\not\nabla\mathbf{D}_3\underline{\omega}||_{L_2(C\cap\mathcal{K})}\leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1+\Delta_2)\quad(4.4.12)$$

$$||\frac{1}{\sqrt{\tau_+}}\tau_-^3\mathbf{D}_3^2\underline{\omega}||_{L_2(C\cap\mathcal{K})}\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1+\Delta_2)\quad(4.4.13)$$

Proof: The result is a direct consequence of the following estimates

$$|r^{2-\frac{2}{p}}\tau_-^2\not\nabla\mathbf{D}_3^2\log\Omega|_{p=2,S}(u,\underline{u})\leq c\left[\frac{r(u,\underline{u})}{r(u,\underline{u}_*)}|r^{2-\frac{2}{p}}\tau_-^2\not\nabla\mathbf{D}_3^2\log\Omega|_{p=2,S}(u,\underline{u}_*)\right.$$
$$\left.+\frac{1}{r^{\frac{1}{2}}}(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1+\Delta_2)\right]\quad(4.4.14)$$

$$|r^{1-\frac{2}{p}}\tau_-^3(\Omega\mathbf{D}_3)^3\log\Omega|_{p=2,S}\leq c\left[\frac{r(u,\underline{u})}{r(u,\underline{u}_*)}|r^{1-\frac{2}{p}}\tau_-^3(\Omega\mathbf{D}_3)^3\log\Omega|_{p=2,S}(u,\underline{u}_*)\right.$$
$$\left.+\sqrt{\frac{\tau_-}{r}}\Delta_2+\frac{\tau_-}{r}(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1+\Delta_2)\right]\quad(4.4.15)$$

To prove inequality 4.4.14 we need the evolution equation for $\not\nabla\mathbf{D}_3^2\log\Omega$. This is obtained deriving tangentially the evolution equation for $(\Omega\mathbf{D}_3)^2\log\Omega$, see 4.3.86, and using the commutation relation

$$[\not\nabla_a,\not\mathbf{D}_4]f=-(\not\nabla\log\Omega)\mathbf{D}_4f+\chi_{ac}\not\nabla_cf\ ,$$

proved in the appendix to this chapter, Proposition 4.8.1. The result is

$$\mathbf{D}_4\Big(\not\nabla(\Omega\mathbf{D}_3)^2\log\Omega\Big)+\frac{1}{2}\mathrm{tr}\chi\Big(\not\nabla(\Omega\mathbf{D}_3)^2\log\Omega\Big)=\hat{\chi}\cdot\not\nabla(\Omega\mathbf{D}_3)^2\log\Omega-\Omega^2\not\nabla\mathbf{D}_3\rho+\underline{W}$$

where, with $\underline{M}$ defined in the previous subsection, see 4.3.87,

$$\underline{W}=(\not\nabla\log\Omega)\underline{M}-3\Omega^2(\not\nabla\log\Omega)\mathbf{D}_3\rho+\not\nabla\underline{M}\ .$$

Applying the Evolution Lemma, we derive

$$|r^{1-\frac{2}{p}}\tau_-^2(\not\nabla\Omega^2\mathbf{D}_3^2\log\Omega)|_{p,S}(u,\underline{u})\leq c\,\Big(|r^{1-\frac{2}{p}}\tau_-^2(\not\nabla\Omega^2\mathbf{D}_3^2\log\Omega)|_{p,S}(u,\underline{u}_*)$$
$$+\int_{\underline{u}}^{\underline{u}_*}|r^{1-\frac{2}{p}}\tau_-^2(\not\nabla\log\Omega)\Big(\underline{M}-3\Omega^2\mathbf{D}_3\rho\Big)|_{p,S}+\int_{\underline{u}}^{\underline{u}_*}|r^{1-\frac{2}{p}}\tau_-^2\not\nabla\underline{M}|_{p,S}$$
$$+\int_{\underline{u}}^{\underline{u}_*}|r^{1-\frac{2}{p}}\tau_-^2\Omega^2\not\nabla\mathbf{D}_3\rho|_{p,S}\Big)\quad(4.4.16)$$

The right hand side of 4.4.16 depends on $\nabla\!\!\!/ \mathbf{D}_3\rho$. As the second derivatives of the Riemann tensor are not bounded in the $|\cdot|_{p,S}$ norms, but only in the $L^2(C), L^2(\underline{C})$ norms, we consider inequality 4.4.16 with $p=2$, obtaining

$$\begin{aligned}
\int_{\underline{u}}^{\underline{u}_*} |r^{1-\frac{2}{p}}\tau_-^2(\nabla\!\!\!/ \log\Omega)\underline{M}|_{p=2,S} &\leq c\frac{\tau_-}{r^4}\Gamma_0^2\left((\mathcal{I}_0+\mathcal{I}_*+\Delta_0)+\Gamma_0+\Gamma_1\right)\\
\int_{\underline{u}}^{\underline{u}_*} |r^{1-\frac{2}{p}}\tau_-^2\nabla\!\!\!/ \underline{M}|_{p=2,S} &\leq c\frac{\tau_-}{r^3}\Big((\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)+\Gamma_0^2+\Gamma_0\Gamma_1\Big)\\
\int_{\underline{u}}^{\underline{u}_*} |r^{1-\frac{2}{p}}\tau_-^2(\nabla\!\!\!/ \log\Omega)\Omega^2\mathbf{D}_3\rho|_{p=2,S} &\leq c\frac{\tau_-}{r^3}\Gamma_0(\Delta_0+\Delta_1)\\
\int_{\underline{u}}^{\underline{u}_*} |r^{1-\frac{2}{p}}\tau_-^2\nabla\!\!\!/ \mathbf{D}_3\rho|_{p=2,S} &\leq c\frac{1}{r^{\frac{3}{2}}}(\Delta_1+\Delta_2) \qquad (4.4.17)
\end{aligned}$$

where the second inequality uses the estimate for $\nabla\!\!\!/ \underline{M}$ [32]

$$|r^{5-\frac{2}{p}}\tau_-\nabla\!\!\!/ \underline{M}|_{p,S} \leq c\Big((\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)+\Gamma_0^2+\Gamma_0\Gamma_1\Big) \qquad (4.4.18)$$

proved in Proposition 4.4.1. The last inequality in 4.4.17 is obtained in the following way:

$$\begin{aligned}
\int_{\underline{u}}^{\underline{u}_*} |r^{1-\frac{2}{p}}\tau_-^2\nabla\!\!\!/ \mathbf{D}_3\rho|_{p=2,S} &= \int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}\left(\int_{S(u,\underline{u}')}|r^2\tau_-^2\nabla\!\!\!/ \mathbf{D}_3\rho|^2\right)^{\frac{1}{2}}\\
&\leq \left(\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^4}\right)^{\frac{1}{2}}\left(\int_{\underline{u}}^{\underline{u}_*}\int_{S(u,\underline{u}')}|r^2\tau_-^2\nabla\!\!\!/ \mathbf{D}_3\rho|^2\right)^{\frac{1}{2}}\\
&\leq c\frac{1}{r^{\frac{3}{2}}}||r^2\tau_-^2\nabla\!\!\!/ \mathbf{D}_3\rho||_{L^2(C(u))} \leq c\frac{1}{r^{\frac{3}{2}}}\Delta_2
\end{aligned}$$

where the last line uses the estimate for $||r^2\tau_-^2\nabla\!\!\!/ \mathbf{D}_3\rho||_{L^2(C(u))}$ contained in the assumptions 4.2.11 of Theorem 4.2.2. From these estimates, observing that assumptions 4.2.22 imply, on the last slice, for $p\in[2,4]$,

$$|r^{2-2/p}\tau_-^2\nabla\!\!\!/ \mathbf{D}_3^2\log\Omega|_{p,S}(u,\underline{u}_*) \leq \mathcal{I}_* \ ,$$

the estimate 4.4.14 follows.

To prove inequality 4.4.15 we use the previous results and the commutation relation, see Proposition 4.8.1,

$$[\mathbf{D}_3,\mathbf{D}_4]f = (\mathbf{D}_4\log\Omega)\mathbf{D}_3 f - (\mathbf{D}_3\log\Omega)\mathbf{D}_4 f + 4\zeta\cdot\nabla\!\!\!/ f \ ,$$

[32] Apparently also $\nabla\!\!\!/ \underline{M}$ depends on the second derivatives of the Riemann tensor due to the term $\nabla\!\!\!/^2\mathbf{D}_3\log\Omega$ present in its expression. Nevertheless as discussed in Proposition 4.4.1, see also [Ch-Kl] page 373, we have the better estimate 4.4.18.

with f a scalar function. The evolution equation for $(\Omega\mathbf{D}_3)^3\log\Omega$ turns out to be

$$\mathbf{D}_4\left((\Omega\mathbf{D}_3)^3\log\Omega\right) = \underline{P} - \Omega^3\mathbf{D}_3^2\rho \tag{4.4.19}$$

where

$$\underline{P} = \left[\mathbf{D}_3(\Omega\underline{M}) - (\mathbf{D}_3\Omega^3)\mathbf{D}_3\rho - 4\Omega\zeta\cdot\nabla\!\!\!/(\Omega\mathbf{D}_3)^2\log\Omega\right] \tag{4.4.20}$$

As the right hand side of 4.4.19 depends on the second derivatives of the Riemann tensor, we can obtain only an estimate for $p=2$. Proceeding as in the previous proposition, see 4.4.16, we derive

$$\begin{aligned}|r^{-1}\tau_-^2(\Omega\mathbf{D}_3)^3\log\Omega|_{p=2,S}(u,\underline{u}) \quad &\leq \quad c\left(|r^{-1}\tau_-^2(\Omega\mathbf{D}_3)^3\log\Omega|_{p=2,S}(u,\underline{u}_*)\right.\\ &+ \quad \left.\int_{\underline{u}}^{\underline{u}_*}|r^{-1}\tau_-^2\underline{P}|_{p=2,S} + \int_{\underline{u}}^{\underline{u}_*}|r^{-1}\tau_-^2\mathbf{D}_3^2\rho|_{p=2,S}\right)\end{aligned} \tag{4.4.21}$$

The more delicate terms in $\underline{P}$ are $\zeta\cdot\nabla\!\!\!/(\Omega\mathbf{D}_3)^2\log\Omega$ and $(\mathbf{D}_3\log\Omega)\mathbf{D}_3\rho$. Considering the first contribution, we have

$$\int_{\underline{u}}^{\underline{u}_*}|r^{-1}\tau_-^2\zeta\cdot\nabla\!\!\!/(\Omega\mathbf{D}_3)^2\log\Omega|_{p=2,S} \leq c\Gamma_0\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}|r^{-1}\tau_-^2\nabla\!\!\!/(\Omega\mathbf{D}_3)^2\log\Omega|_{p=2,S} \ .$$

Recalling that, from inequality 4.4.16 and the subsequent estimates 4.4.17,

$$|r^{-1}\tau_-^2\nabla\!\!\!/(\Omega\mathbf{D}_3)^2\log\Omega|_{p=2,S} \leq c\frac{1}{r^2}(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1+\Delta_2)\ ,$$

we derive

$$\int_{\underline{u}}^{\underline{u}_*}|r^{-1}\tau_-^2\zeta\cdot\nabla\!\!\!/(\Omega\mathbf{D}_3)^2\log\Omega|_{p=2,S} \quad\leq\quad c\frac{1}{r^3}\Gamma_0(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1+\Delta_2)$$

To estimate the contribution due to the term $(\mathbf{D}_3\log\Omega)\mathbf{D}_3\rho$, we recall the estimate

$$\sup|\mathbf{D}_3\log\Omega| \leq c\frac{1}{r\tau_-}\Gamma_0$$

following from the bootstrap assumptions, obtaining

$$\int_{\underline{u}}^{\underline{u}_*}|r^{-1}\tau_-^2(\mathbf{D}_3\log\Omega)\mathbf{D}_3\rho|_{p=2,S} \quad\leq\quad c\Gamma_0\frac{1}{\tau_-}\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r}|r^{-1}\tau_-^2\mathbf{D}_3\rho|_{p=2,S}$$

The estimate

$$|r^{-\frac{2}{p}}\tau_-^2\mathbf{D}_3\rho|_{p=2,S} \leq c\frac{\tau_-}{r^3}(\Delta_0+\Delta_1)$$

follows from assumptions 4.2.11 [33], therefore

$$\int_{\underline{u}}^{\underline{u}_*}|r^{-1}\tau_-^2(\mathbf{D}_3\log\Omega)\mathbf{D}_3\rho|_{p=2,S} \leq c\frac{1}{r^3}\Gamma_0(\Delta_0+\Delta_1)\ .$$

Collecting these estimates we obtain

$$\int_{\underline{u}}^{\underline{u}_*}|r^{-1}\tau_-^2\underline{P}|_{p=2,S} \leq c\frac{1}{r^3}\Gamma_0\left(\mathcal{I}_*+\Delta_0+\Delta_1+\Delta_2\right) \tag{4.4.22}$$

The second integral in the right hand side of 4.4.21 is estimated as

$$\begin{aligned}\int_{\underline{u}}^{\underline{u}_*}|r^{-1}\tau_-^2\mathbf{D}_3^2\rho|_{p=2,S} &= \frac{1}{\sqrt{\tau_-}}\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}\left(\int_S|r\tau_-^{\frac{5}{2}}\mathbf{D}_3^2\rho|^2\right)^{\frac{1}{2}} \qquad (4.4.23)\\ &\leq c\frac{1}{\sqrt{\tau_-}}\left(\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^4}\right)^{\frac{1}{2}}\left(\int_{\underline{u}}^{\underline{u}_*}\int_S|r\tau_-^{\frac{5}{2}}\mathbf{D}_3^2\rho|^2\right)^{\frac{1}{2}}\\ &\leq c\frac{1}{\tau_-^{\frac{1}{2}}r^{\frac{3}{2}}}||r\tau_-^{\frac{5}{2}}\mathbf{D}_3^2\rho||_{L_2(C(u))} \leq c\frac{1}{\tau_-^{\frac{1}{2}}r^{\frac{3}{2}}}\Delta_2\end{aligned}$$

where the inequality in the last line follows from the assumptions 4.2.11.

This completes the proof of inequality 4.4.15. Inequality 4.4.13, also follows immediately recalling that, from assumptions 4.2.22, we have, on the last slice,

$$|r^{1-2/p}\tau_-^3(\Omega\mathbf{D}_3)^3\log\Omega)|_{p=2,S}(u,\underline{u}_*) \leq \mathcal{I}_*\ .$$

Remark: The estimates of Propositions 4.4.1, 4.4.2 complete the more delicate part of the control of the second derivatives of the connection coefficients and, therefore, of Theorem 4.2.2. In the next section we provide the estimates concerning the third derivatives of the connection coefficients. These estimates are needed to control the $\mathcal{D}$ norms introduced in subsection 3.5.6 and estimated in section 4.7.

[33] In fact from the Bianchi equations we have, for $p\in[2,4]$, $|r^{-\frac{2}{p}}\tau_-^2\mathbf{D}_3\rho|_{p=2,S} \leq \frac{\tau_-^2}{r^4}|r^{4-\frac{2}{p}}tr\underline{\chi}\rho|_{p=2,S}+\frac{\sqrt{\tau_-}}{r^3}|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta}|_{p=2,S} \leq c\frac{\tau_-}{r^3}(\Delta_0+\Delta_1)$.

4.5 Proof of Theorem 4.2.3, control of third derivatives of the connection coefficents

The proof of Theorem 4.2.3 is achieved once we prove the following proposition,

Proposition 4.5.1 *Under the assumptions of Theorem 4.2.3 and using the results of Theorems 4.2.1, 4.2.2, denoting* $\Delta \equiv \Delta_0 + \Delta_1 + \Delta_2$, *it follows*

$$
\begin{aligned}
r^{\frac{1}{2}}(u,\underline{u})||r^3 \nabla\!\!\!\!/^3 \hat{\chi}||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} &\leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta) \\
r^{\frac{1}{2}}(u,\underline{u})||r^3 \nabla\!\!\!\!/^3 tr\chi||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} &\leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta) \\
r^{\frac{1}{2}}(u,\underline{u})||r^3 \nabla\!\!\!\!/^3 \eta||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} &\leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta) \\
r^{\frac{1}{2}}(u,\underline{u})||r^3 \nabla\!\!\!\!/^3 \underline{\omega}||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} &\leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta) \\
r^{\frac{1}{2}}(u,\underline{u})||r^3 \nabla\!\!\!\!/^3 \underline{\hat{\chi}}||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} &\leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta) \\
r^{\frac{1}{2}}(u,\underline{u})||r^3 \nabla\!\!\!\!/^3 tr\underline{\chi}||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} &\leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta)
\end{aligned}
$$

Remark: Proposition 4.5.1 does not require the the foliations on the initial and final slices be canonical. In that case, as discussed in Chapter 7, one can prove the boundedness of slightly stronger norms.

Proof: All these norms are estimated essentially in the same way. We give only a detailed account of the first estimate.

Estimate of $r^{\frac{1}{2}}(u,\underline{u})||r^3 \nabla\!\!\!\!/^3 \hat{\chi}||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))}$

From the definition, $\psi\!\!\!/ = \Omega^{-1}(\nabla\!\!\!\!/ \mathrm{tr}\chi + \mathrm{tr}\chi\zeta)$, see 4.3.5, and the Codazzi equation, see 4.3.13,

$$\mathrm{d}\!\!\!/\mathrm{iv}\,\hat{\chi} + \zeta\cdot\hat{\chi} = \frac{1}{2}\Omega\psi\!\!\!/ - \beta$$

it follows that $\nabla\!\!\!\!/^3\hat{\chi}$ and $\nabla\!\!\!\!/^3\mathrm{tr}\chi$ are controlled in terms of $\nabla\!\!\!\!/^2\psi\!\!\!/$.

Let us consider first $\nabla\!\!\!\!/^3\hat{\chi}$. Applying Proposition 4.1.2 to the Codazzi equation, we derive

$$|r^{3-\frac{2}{p}}\nabla\!\!\!\!/^3\hat{\chi}|_{p=2,S} \leq c\left(|r^{3-\frac{2}{p}}\nabla\!\!\!\!/^2\psi\!\!\!/|_{p=2,S} + |r^{3-\frac{2}{p}}\nabla\!\!\!\!/^2\beta|_{p=2,S}\right) + [\cdots] \quad (4.5.1)$$

where $[\cdots]$ does not contain second derivatives of the Riemann tensor and behaves as $O(r^{-\frac{5}{2}})$. To estimate [34] $\nabla\!\!\!\!/^2\psi\!\!\!/$ we introduce the scalar quantity

[The factor $O(r^{-2}\tau_-^{-\frac{1}{2}})$ was incorrect and has been substituted by $O(r^{-\frac{5}{2}})$.]

$$\not{\Psi} \equiv \mathrm{d}\!\!\!/\mathrm{iv}\,\Psi + \Omega^{-1}\mathrm{tr}\chi\rho \tag{4.5.2}$$

and write its evolution equation [35] along the outgoing null hypersurfaces

$$\frac{d}{d\underline{u}}\not{\Psi} + 2\Omega\mathrm{tr}\chi\not{\Psi} = -2\Omega\hat{\chi}\not{\Psi} - 4\hat{\chi}\cdot\nabla\!\!\!\!/\,\beta + \mathcal{U}_1 \tag{4.5.3}$$

where $\mathcal{U}_1$ denotes all the terms which do not depend on the first derivatives of the Riemann tensor and have an appropriate asymptotic behaviour[36]. The one form Ψ satisfies the following Hodge system,

$$\begin{aligned}\mathrm{d}\!\!\!/\mathrm{iv}\,\Psi &= \not{\Psi} - \Omega^{-1}\mathrm{tr}\chi\rho\\ \mathrm{c}\!\!\!/\mathrm{url}\,\Psi &= \Omega^{-1}\mathrm{tr}\chi\,\mathrm{c}\!\!\!/\mathrm{url}\,\zeta + \zeta\cdot{}^*\Psi\end{aligned} \tag{4.5.4}$$

Applying Proposition 4.1.2 to the Hodge system 4.5.4 we obtain

$$|r^{3-\frac{2}{p}}\nabla\!\!\!\!/^2\Psi|_{p=2,S} \le c\left(|r^{3-\frac{2}{p}}\nabla\!\!\!\!/\not{\Psi}|_{p=2,S} + |\mathrm{tr}\chi|_{\infty,S}|r^{3-\frac{2}{p}}\nabla\!\!\!\!/\rho|_{p=2,S}\right) + [\cdots] \tag{4.5.5}$$

where, again, $[\cdots]$ indicates terms with a better asymptotic behaviour and which can be estimated in the $L^p(S)$ norms. Putting together 4.5.5 and 4.5.1 we derive

$$\begin{aligned}|r^{4-\frac{2}{p}}\nabla\!\!\!\!/^3\hat{\chi}|_{p=2,S} &\le c\left(|r^{4-\frac{2}{p}}\nabla\!\!\!\!/^2\Psi|_{p=2,S} + |r^{4-\frac{2}{p}}\nabla\!\!\!\!/^2\beta|_{p=2,S}\right) + [\cdots]\\ &\le c\left(|r^{4-\frac{2}{p}}\nabla\!\!\!\!/\not{\Psi}|_{p=2,S} + |r^{4-\frac{2}{p}}\nabla\!\!\!\!/^2\beta|_{p=2,S}\right) + [\cdots]\end{aligned} \tag{4.5.6}$$

[The factor $O(r^{-1}\tau_-^{-\frac{1}{2}})$ was incorrect and has been substituted by $O(r^{-\frac{3}{2}})$.]

where $[\cdots] = O(r^{-\frac{3}{2}})$ and does not depend on second Riemann derivatives. These terms will be, hereafter, neglected. Using this estimate we write,

$$\int_{\underline{C}(\underline{u},[u_0,u])}|r^3\nabla\!\!\!\!/^3\hat{\chi}|^2 \le c\int_{u_0}^{u}\frac{1}{r^2}|r^{5-\frac{2}{p}}\nabla\!\!\!\!/\not{\Psi}|^2_{p=2,S} + c\int_{u_0}^{u}|r^{4-\frac{2}{p}}\nabla\!\!\!\!/^2\beta|^2_{p=2,S} \tag{4.5.7}$$

[34] We follow here the discussion in Chapter 13 of [Ch-Kl].

[35] The evolution equation is obtained deriving tangentially the evolution equation of Ψ and using the commutation relations in Proposition 4.8.1. The term $\Omega^{-1}\mathrm{tr}\chi\rho$, added in the definition of $\not{\Psi}$, is necessary to cancel the term $-\mathrm{tr}\chi\,\mathrm{d}\!\!\!/\mathrm{iv}\,\beta$ which appears in the right hand side of the evolution equation for $\mathrm{d}\!\!\!/\mathrm{iv}\,\Psi$ and which prevents the integrability of the evolution equation for $\mathrm{d}\!\!\!/\mathrm{iv}\,\Psi$, $\frac{d}{d\underline{u}}(\mathrm{d}\!\!\!/\mathrm{iv}\,\Psi) + 2\Omega\mathrm{tr}\chi(\mathrm{d}\!\!\!/\mathrm{iv}\,\Psi) = -\mathrm{tr}\chi\,\mathrm{d}\!\!\!/\mathrm{iv}\,\beta - 2\Omega\hat{\chi}\cdot(\nabla\!\!\!\!/\Psi) - 2\chi_{cd}\Delta\!\!\!\!/\chi_{cd} + \mathcal{U}_2$, where $\mathcal{U}_2$ depends on $\nabla\!\!\!\!/\chi, \Psi, \underline{\eta}, \beta$ and decays as $O(r^{-6})$.

[36] The term $-4\Omega\hat{\chi}\cdot\nabla\!\!\!\!/\beta$, although decaying properly, as $O(r^{-\frac{13}{2}})$, has been written explicitely as it is the only term depending on the first derivative of the curvature tensor. It arises from the term $-2\chi_{cd}\Delta\!\!\!\!/\chi_{cd}$ appearing in the evolution equation of $\mathrm{d}\!\!\!/\mathrm{iv}\,\Psi$, deriving the Codazzi equation, see 3.1.46.

The second integral of 4.5.7 satisfies, using assumptions 4.2.11 of Theorem 4.2.2, see also 3.5.17,

$$\int_{u_0}^{u} |r^{4-\frac{2}{p}} \nabla\!\!\!/^2 \beta|^2_{p=2,S} \leq \int_{\underline{C}(\underline{u},[u_0,u])} r^2 |r^2 \nabla\!\!\!/^2 \beta|^2 \leq c\frac{1}{r^2}\underline{\mathcal{R}}_2[\beta]^2 \leq c\frac{1}{r^2}\Delta_2^2 \quad (4.5.8)$$

To estimate the first integral in the right hand side of 4.5.7 we write the evolution of $\nabla\!\!\!/ \Psi\!\!\!/$ along the outgoing null direction. A long, but straightforward computation gives, deriving tangentially the evolution equation for $\Psi\!\!\!/$,

$$\frac{d}{d\underline{u}}\nabla\!\!\!/ \Psi\!\!\!/ + \frac{5}{2}\Omega \mathrm{tr}\chi \nabla\!\!\!/ \Psi\!\!\!/ = -4\hat{\chi}\nabla\!\!\!/ \Psi\!\!\!/ - 4\hat{\chi}\nabla\!\!\!/^2\beta + \mathcal{U}_3 \quad (4.5.9)$$

where $\mathcal{U}_3$ denotes all the terms which do not depend on the second derivatives of the Riemann tensor and, therefore, can be estimated in the $L^p(S)$ norms. These terms are easier to treat, have the appropriate asymptotic behaviour, and, hereafter, will be omitted. Applying the Evolution Lemma and Gronwall Lemma to 4.5.9 we obtain, using the previous results on $\mathcal{O}_{[2]}$,

$$\begin{aligned}
|r^{5-\frac{2}{p}}\nabla\!\!\!/ \Psi\!\!\!/|_{p=2,S}(u,\underline{u}') &\leq c|r^{5-\frac{2}{p}}\nabla\!\!\!/ \Psi\!\!\!/|_{p=2,S}(u,\underline{u}_*) + c\int_{\underline{u}'}^{\underline{u}_*} |r^{5-\frac{2}{p}}\hat{\chi}\nabla\!\!\!/^2\beta|_{p=2,S} \\
&\leq c|r^{5-\frac{2}{p}}\nabla\!\!\!/ \Psi\!\!\!/|_{p=2,S}(u,\underline{u}_*) + c\Gamma_0\int_{\underline{u}'}^{\underline{u}_*} |r^{3-\frac{2}{p}}\nabla\!\!\!/^2\beta|_{p=2,S} \\
&\leq c|r^{5-\frac{2}{p}}\nabla\!\!\!/ \Psi\!\!\!/|_{p=2,S}(u,\underline{u}_*) + c\Gamma_0\Delta_2\frac{1}{r^{\frac{3}{2}}} \quad (4.5.10)
\end{aligned}$$

where the last integral of 4.5.10 has been estimated, recalling 3.5.17, as

[The factor $r^{-\frac{3}{2}}$ was missing in the next line and has been put in.]

$$\int_{\underline{u}'}^{\underline{u}_*} |r^{3-\frac{2}{p}}\nabla\!\!\!/^2\beta|_{p=2,S} \leq \left(\int_{\underline{u}'}^{\underline{u}_*}\frac{1}{r^2}\right)^{\frac{1}{2}}\left(\int_{C(u)\cap V(u,\underline{u}')} r^2|r^2\nabla\!\!\!/^2\beta|^2\right)^{\frac{1}{2}} \leq c\Delta_2\frac{1}{r^{\frac{3}{2}}}$$

In conclusion we have

[The final factor $c\Delta_2 r(u,\underline{u})^{-\frac{1}{2}}$ of 4.5.11 is wrong, the correct one is $c\Delta_2\frac{1}{r(u,\underline{u})}(1+\Gamma_0\frac{1}{r})$.]

$$\begin{aligned}
||r^3\nabla\!\!\!/^3\hat{\chi}||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))} &\leq \left(\int_{u_0}^{u} du'\frac{1}{r^2}|r^{5-\frac{2}{p}}\nabla\!\!\!/ \Psi\!\!\!/|^2_{p=2,S}(u',\underline{u}_*)\right)^{\frac{1}{2}} \\
&+ c\Delta_2\frac{1}{r(u,\underline{u})}(1+\Gamma_0\frac{1}{r(u,\underline{u})}) \quad (4.5.11) \\
&\leq ||r^3\nabla\!\!\!/ \Psi\!\!\!/||_{L^2(\underline{C}_*\cap V(u,\underline{u}))} + c\Delta_2\frac{1}{r(u,\underline{u})}
\end{aligned}$$

The estimate is achieved, once we observe that the final slice assumption of Theorem **M1**, $\mathcal{O}_3(\underline{C}_*) \leq \mathcal{I}_*$, implies the inequality

$$r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla\!\!\!/ \Psi\!\!\!/||_{L^2(\underline{C}_*\cap V(u,\underline{u}))} \leq \mathcal{I}_* \ .$$

The estimate of the norm $||r^3\nabla\!\!\!\!/^3\mathrm{tr}\chi||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))}$ is immediately reduced to the previous one using the identity

$$\Delta\!\!\!\!/\mathrm{tr}\chi = \Omega \mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\psi} - \mathrm{tr}\chi \mathrm{d}\!\!\!/\mathrm{iv}\,\zeta - \zeta\nabla\!\!\!\!/\mathrm{tr}\chi + (\nabla\!\!\!\!/\log\Omega)\mathrm{tr}\chi\zeta$$

and Proposition 4.1.3.

[Next estimate is modified as it is now adapted to the new estimates for $\tilde{\mu}$ and $\underline{\tilde{\mu}}$.]

Estimate of $r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla\!\!\!\!/^3\eta||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))}$

We apply Proposition 4.1.2 to the Hodge system, see 4.3.37,

$$\begin{aligned}\mathrm{d}\!\!\!/\mathrm{iv}\,\eta &= -\tilde{\mu} + \frac{1}{2}(\chi\cdot\underline{\chi} - \overline{\chi\cdot\underline{\chi}}) - (\rho - \overline{\rho})\\ \mathrm{c}\!\!\!/\mathrm{url}\,\eta &= \sigma - \frac{1}{2}\underline{\hat{\chi}}\wedge\hat{\chi}\end{aligned}$$

obtaining

$$|r^{4-\frac{2}{p}}\nabla\!\!\!\!/^3\eta|_{p=2,S} \le c\Big(|r^{4-\frac{2}{p}}\nabla\!\!\!\!/^2\tilde{\mu}|_{p=2,S} + |r^{4-\frac{2}{p}}\nabla\!\!\!\!/^2(\rho,\sigma)|_{p=2,S}\Big) + c\frac{1}{r}(\mathcal{I}_0+\mathcal{I}_*+\Delta)(1+\Gamma_0\frac{1}{\tau_-})\ .$$

Using this inequality it follows

$$\begin{aligned}\int_{u_0}^{u}|r^{4-\frac{2}{p}}\nabla\!\!\!\!/^3\eta|^2_{p=2,S}(u',\underline{u}) \ &\le\ c\int_{u_0}^{u}\frac{1}{r^2}|r^{5-\frac{2}{p}}\nabla\!\!\!\!/^2\tilde{\mu}|^2_{p=2,S} + c\int_{u_0}^{u}|r^{4-\frac{2}{p}}\nabla\!\!\!\!/^2(\rho,\sigma)|^2_{p=2,S}\\ &+\ c\frac{1}{r}(\mathcal{I}_0+\mathcal{I}_*+\Delta)^2 \qquad (4.5.12)\end{aligned}$$

The last integral is estimated using assumptions 4.2.11 of Theorem 4.2.2, see also 3.5.17,

$$\begin{aligned}&\int_{u_0}^{u}|r^{4-\frac{2}{p}}\nabla\!\!\!\!/^2(\rho,\sigma)|^2_{p=2,S} = \frac{1}{r^2(u,\underline{u})}\int_{\underline{C}(\underline{u};[u_0,u])}r^4|r^2\nabla\!\!\!\!/^2(\rho,\sigma)|^2\\ &\le c\frac{1}{r^2(u,\underline{u})}\underline{\mathcal{R}}_2[(\rho,\sigma)](u,\underline{u}) \le c\frac{1}{r^2}\Delta_2^2 \qquad (4.5.13)\end{aligned}$$

To estimate the first integral in the right hand side of 4.5.12 we have to control $|r^{5-\frac{2}{p}}\nabla\!\!\!\!/^2\tilde{\mu}|_{p=2,S}$. The estimates for $|r^{5-\frac{2}{p}}\nabla\!\!\!\!/^2\tilde{\mu}|_{p=2,S}$ are obtained in a similar way as in Lemma 4.3.3 for $\tilde{\mu}$ and $\underline{\tilde{\mu}}$. The evolution equation for $\nabla\!\!\!\!/^2\tilde{\mu}$ can be written as

$$\frac{d}{d\underline{u}}(\nabla\!\!\!\!/^2\tilde{\mu}) + 2(\Omega\mathrm{tr}\chi)(\nabla\!\!\!\!/^2\tilde{\mu}) = \tilde{F}_2 - 2\Omega\eta\cdot\nabla\!\!\!\!/^2\beta \qquad (4.5.14)$$

where $\tilde{F}_2$ does not depend on the second derivatives of the Riemann tensor and, therefore, can be estimated with the $|\cdot|_{p,S}$ norms. Applying to

this evolution equation the Evolution Lemma and Gronwall's inequality we
plete estimate 4.5.15.] obtain,

$$
\begin{aligned}
|r^{4-\frac{2}{p}}\nabla\!\!\!/^2\tilde{\mu}|_{p=2,S}(u,\underline{u}) \quad &\leq \quad c\left(|r^{4-\frac{2}{p}}\nabla\!\!\!/^2\tilde{\mu}|_{p=2,S}(u,\underline{u}_*)+\int_{\underline{u}}^{\underline{u}_*}|r^3\tilde{F}_2|_{p=2,S}\right.\\
&+\quad \left.\sup|2\Omega\eta|\int_{\underline{u}}^{\underline{u}_*}|r^3\nabla\!\!\!/^2\beta|_{p=2,S}\right) \qquad (4.5.15)
\end{aligned}
$$

Moreover on the last slice the canonical foliation of $\underline{C}_*$ implies, see 3.3.9,

$$
|r^{4-\frac{2}{p}}\nabla\!\!\!/^2\tilde{\mu}|_{p=2,S}(u,\underline{u}_*)\leq|r^{4-\frac{2}{p}}\nabla\!\!\!/^2(\mathrm{tr}\chi\mathrm{tr}\underline{\chi})|_{p=2,S}(u,\underline{u}_*)\leq c\frac{1}{r}\mathcal{I}_* \quad (4.5.16)
$$

The estimates 4.5.15 and 4.5.16 allow to control the integral of $|r^{4-\frac{2}{p}}\nabla\!\!\!/^2\mu|^2_{p=2,S}$ and, therefore, $||r^3\nabla\!\!\!/^2\mu||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))}$ completing the estimate of $\nabla\!\!\!/^3\eta$. The estimates for the remaining $\mathcal{O}_3$ norms proceed in the same way and we do not discuss them here.

4.6 Rotation tensors estimates

The rotation vector fields ${}^{(i)}O$ form the Lie algebra of the rotation group $SO(3)$. They satisfy the commutation relations

$$
[{}^{(i)}O,{}^{(j)}O]=\epsilon_{ijk}{}^{(k)}O\ ,\ i,j,k\in\{1,2,3\}
$$

They are defined on the tangent space TS_p, for any $p\in\mathcal{M}$.

As the spacetime $\mathcal{K}$ is not flat but, in a appropriate sense, "nearly" flat we expect that the rotation vector fields will produce a set of diffeomorphisms which are "nearly" isometries and, therefore, that the deformation tensors associated to these fields have small norms. This will be crucial in the error estimates of Chapter 6 where we need to control the norms of the various components of the Riemann curvature tensor in terms of the initial data

To define these vector fields we start by transporting the "canonical" generators of the rotation group defined at spacelike infinity of Σ_0, backward along Σ_0 up to the surface

$$
S_{(0)}(\underline{u}_*)\equiv\underline{C}_*\cap\Sigma_0
$$

using the diffeomorphism induced by the flow normal to our canonical foliation. We then continue to transport them along the null hypersurface $\underline{C}_*$

using the diffeomorphism $\underline{\phi}_t$, restricted to $\underline{C}_*$, generated by the equivariant null vector field $\underline{N}=\frac{d}{d\underline{u}_*}$. On $\underline{C}_*$ we denote them ${}^{(i)}O_*$, $i=1,2,3$.
Finally, starting from any surface $S(u,\underline{u}_*)$ foliating $\underline{C}_*$, we use the diffeomorphism ϕ_t generated by N to transport the rotation vector fields to any surface $S(u,\underline{u})$ of $\mathcal{K}$.

The discussion about the costruction of the ${}^{(i)}O$ vector fields on the initial hypersurface, Σ_0, and on the last slice, $\underline{C}_*$, is done in Chapter 7. Here we show how to extend the vector fields ${}^{(i)}O$ from the last slice to the whole $\mathcal{K}$.

4.6.1 Technical construction

In what follows we start [37] with the rotation fields ${}^{(i)}O_*$ defined on $\underline{C}_*$. They satisfy, on any $S(u,\underline{u}_*)$ surface,

$$[{}^{(i)}O_*,{}^{(j)}O_*]=\epsilon_{ijk}{}^{(k)}O_*\ ,\ ijk\in\{1,2,3\}\ .$$

Let $q\in S(u,\underline{u})$ be a generic point of $\mathcal{K}$. As $S(u,\underline{u})$ is diffeomorphic via ϕ_Δ, $\Delta=\underline{u}_*-\underline{u}$, to $S(u,\underline{u}_*)$, there exists a $p\in S(u,\underline{u}_*)$ such that $q=\phi_\Delta^{-1}(p)$. We define the element O of the rotation group operating over q in the following way [38]:

$$(O;q)\equiv\phi_\Delta^{-1}(O_*;p)$$

where $(O;q)$ is a point of $S(u,\underline{u})$ and $(O_*;p)$ is the point of $S(u,\underline{u}_*)$ obtained applying O_* to the point p. This extension of the action of the rotation group to the whole of $\mathcal{K}$ satisfies

$$O=\phi_t^{-1}O\phi_t$$

This implies that the generators ${}^{(i)}O$ satisfy

$$[N,{}^{(i)}O]=0$$

where ${}^{(i)}O$ is the extension of ${}^{(i)}O_*$ to $\mathcal{K}$. From the previous definitions we easily check that

$$[{}^{(i)}O,{}^{(j)}O]=\epsilon_{ijk}{}^{(k)}O\ ,\ ijk\in\{1,2,3\}\ .$$

[37] The construction on the initial hypersurface and final slice is discussed in Chapter 7

[38] At the differential level we can define the extension in the following way:

$${}^{(i)}O\equiv\phi_{*-\Delta}{}^{(i)}O_*\ .$$

As ${}^{(i)}O \in TS_q$ we have also $g({}^{(i)}O, e_4) = g({}^{(i)}O, e_3) = 0$. In conclusion the generators ${}^{(i)}O$, defined on the whole $\mathcal{K}$, satisfy

$$\begin{aligned} &[{}^{(i)}O, {}^{(j)}O] = \epsilon_{ijk}{}^{(k)}O \\ &[N, {}^{(i)}O] = 0 \\ &g({}^{(i)}O, e_4) = g({}^{(i)}O, e_3) = 0 \end{aligned} \tag{4.6.1}$$

Moreover, as $N = \Omega\hat{N} = \Omega e_4$, it follows

$$[{}^{(i)}O, e_4] = {}^{(i)}F e_4 \tag{4.6.2}$$

where

$${}^{(i)}F \equiv -{}^{(i)}O_c(\nabla\!\!\!\!/_c \log\Omega)\ ,\ {}^{(i)}O_c \equiv g({}^{(i)}O, e_c) \tag{4.6.3}$$

Proposition 4.6.1 *The quantities* ${}^{(i)}O_a$ *and* $\nabla\!\!\!\!/_b{}^{(i)}O_a \equiv (\nabla\!\!\!\!/{}^{(i)}O)_{ab}$ *satisfy the following evolution equations*

$$\frac{d}{d\underline{u}}{}^{(i)}O_b = \Omega\chi_{bc}{}^{(i)}O_c \tag{4.6.4}$$

$$\frac{d}{d\underline{u}}(\nabla\!\!\!\!/{}^{(i)}O)_{ab} = \Omega\left(\hat{\chi}_{bc}(\nabla\!\!\!\!/_a{}^{(i)}O)_c - \hat{\chi}_{ac}(\nabla\!\!\!\!/_c{}^{(i)}O)_b\right) + (\mathcal{F}_1)_{ab} \tag{4.6.5}$$

where

$$\begin{aligned} (\mathcal{F}_1)_{ab} \equiv\ & \Omega\Big[{}^{(i)}O_c(\chi_{cb}\underline{\eta}_a - \chi_{ca}\underline{\eta}_b) + {}^{(i)}O_c R_{4acb} + {}^{(i)}O_c\chi_{cb}\zeta_a \\ & +\chi_{ab}(\underline{\eta}_c{}^{(i)}O_c) + {}^{(i)}O_c(\nabla\!\!\!\!/_a\chi)_{cb}\Big] \end{aligned} \tag{4.6.6}$$

Proof: From $[N, {}^{(i)}O] = 0$ we infer that

$$\Omega\mathbf{D}_4{}^{(i)}O = {}^{(i)}O_c(\nabla\!\!\!\!/_c\log\Omega)N + \Omega\mathbf{D}_{{}^{(i)}O}e_4 \tag{4.6.7}$$

and, choosing a moving frame satisfying $\mathbf{D}\!\!\!\!/_4 e_b = 0$,

$$\frac{d}{d\underline{u}}{}^{(i)}O_b = \Omega\chi_{bc}{}^{(i)}O_c$$

To obtain an evolution equation for $\nabla\!\!\!\!/_b{}^{(i)}O_a \equiv (\nabla\!\!\!\!/{}^{(i)}O)_{ab}$. we start from equation 4.6.7, which we rewrite as

$$\mathbf{D}_4{}^{(i)}O = {}^{(i)}O_c\chi_{cb}e_b + {}^{(i)}O_c\underline{\eta}_c e_4 \tag{4.6.8}$$

Using the commutation relations in the appendix to this chapter, see Proposition 4.8.1, we derive

$$\begin{aligned}\frac{d}{d\underline{u}}(\nabla\!\!\!/ \,{}^{(i)}O)_{ab} &= \Omega\Big[\hat{\chi}_{bc}(\nabla\!\!\!/_a{}^{(i)}O)_c - \hat{\chi}_{ac}(\nabla\!\!\!/_c{}^{(i)}O)_b + {}^{(i)}O_c(\chi_{cb}\underline{\eta}_a - \chi_{ca}\underline{\eta}_b)\\ &\quad + {}^{(i)}O_c R_{4acb} + {}^{(i)}O_c\chi_{cb}\zeta_a + \chi_{ab}(\underline{\eta}_c{}^{(i)}O_c) + {}^{(i)}O_c(\nabla\!\!\!/_a\chi)_{cb}\Big]\end{aligned}$$

In view of 4.6.1 and 4.6.8 we have

$$\begin{aligned}&g(\mathbf{D}_a{}^{(i)}O, e_4) = -\chi_{ab}{}^{(i)}O_b\\ &g(\mathbf{D}_4{}^{(i)}O, e_a) = \chi_{ab}{}^{(i)}O_b\\ &g(\mathbf{D}_4{}^{(i)}O, e_4) = 0\\ &g(\mathbf{D}_a{}^{(i)}O, e_3) = -\underline{\chi}_{ab}{}^{(i)}O_b\\ &g(\mathbf{D}_3{}^{(i)}O, e_3) = 0\\ &g(\mathbf{D}_4{}^{(i)}O, e_3) = -2\underline{\eta}_b{}^{(i)}O_b\\ &g(\mathbf{D}_3{}^{(i)}O, e_4) = -2\eta_b{}^{(i)}O_b\end{aligned} \tag{4.6.9}$$

Using this we compute some components of the deformation tensor relative to the rotation vector fields. Denoting ${}^{({}^{(i)}O)}\pi \equiv {}^{(i)}\pi$, we obtain

$$\begin{aligned}&{}^{(i)}\pi_{44} = 2g(\mathbf{D}_4{}^{(i)}O, e_4) = 0\\ &{}^{(i)}\pi_{33} = 2g(\mathbf{D}_3{}^{(i)}O, e_3) = 0\\ &{}^{(i)}\pi_{4a} = g(\mathbf{D}_4{}^{(i)}O, e_a) + g(\mathbf{D}_a{}^{(i)}O, e_4) = 0\\ &{}^{(i)}\pi_{34} = -2(\eta_b + \underline{\eta}_b){}^{(i)}O_b = -4(\nabla\!\!\!/_b \log\Omega){}^{(i)}O_b = 4{}^{(i)}F\end{aligned} \tag{4.6.10}$$

the remaining components are

$$\begin{aligned}&{}^{(i)}\pi_{ab} = g(\mathbf{D}_a{}^{(i)}O, e_b) + g(\mathbf{D}_b{}^{(i)}O, e_a) \equiv 2{}^{(i)}H_{ab}\\ &{}^{(i)}\pi_{3a} = g(\mathbf{D}_a{}^{(i)}O, e_3) + g(\mathbf{D}_3{}^{(i)}O, e_a) \equiv 4{}^{(i)}Z_a\end{aligned} \tag{4.6.11}$$

Observe that ${}^{(i)}Z_a$ can be written

$${}^{(i)}Z_a = \frac{1}{4}\left(-{}^{(i)}O_b\underline{\chi}_{ab} + e_3({}^{(i)}O_a) + {}^{(i)}O_b g(\mathbf{D}\!\!\!\!/_3 e_b, e_a)\right)$$

and, in view of $[{}^{(i)}O, e_3] = -4{}^{(i)}Z_b e_b + {}^{(i)}F e_3$, we have

$$[{}^{(i)}O, \underline{N}] = -4\Omega{}^{(i)}Z_b e_b \tag{4.6.12}$$

To control the quantities ${}^{(i)}H_{ab}, {}^{(i)}Z_b$ we derive their evolution equations along the $C(u)$ hypersurfaces.

Proposition 4.6.2 *The quantities ${}^{(i)}H_{ab}$ and ${}^{(i)}Z_b$ satisfy the following evolution equations*

$$
\begin{aligned}
\frac{d}{d\underline{u}}{}^{(i)}H_{ab} &= -\Omega\left(\hat{\chi}_{ac}{}^{(i)}H_{cb}+\hat{\chi}_{bc}{}^{(i)}H_{ca}\right)+\Omega\chi_{ab}(\nabla\!\!\!/_c\log\Omega){}^{(i)}O_c\\
&\quad+\Omega\left(\hat{\chi}_{bc}\nabla\!\!\!/_a{}^{(i)}O_c+\hat{\chi}_{ac}\nabla\!\!\!/_b{}^{(i)}O_c\right)+\Omega{}^{(i)}O_c(\nabla\!\!\!/_c\chi)_{ab}\\
\frac{d}{d\underline{u}}{}^{(i)}Z_a &= \frac{1}{2}\Omega tr\chi{}^{(i)}Z_a+\Omega\hat{\chi}_{ab}{}^{(i)}Z_b+2\Omega\omega{}^{(i)}Z_a+\Omega\Big[\frac{1}{2}(L_{{}^{(i)}O}-{}^{(i)}F)(\zeta+\eta))_a\\
&\quad+\frac{1}{2}\nabla\!\!\!/_a{}^{(i)}F-2\zeta_b{}^{(i)}H_{ab}-\frac{1}{2}(\eta-\underline{\eta})_a{}^{(i)}F)\Big]
\end{aligned}
\tag{4.6.13}
$$

Proof: The proof is a long but direct computation and is reported in the appendix to this chapter.

4.6.2 Derivatives of the rotation deformation tensors

The following relations hold

$$
\begin{aligned}
&(\mathbf{D}_4{}^{(i)}\pi)_{44}=0\ ,\ (\mathbf{D}_a{}^{(i)}\pi)_{44}=0\\
&(\mathbf{D}_3{}^{(i)}\pi)_{44}=0\ ,\ (\mathbf{D}_4{}^{(i)}\pi)_{a4}=0\\
&(\mathbf{D}_b{}^{(i)}\pi)_{a4}=-2(\chi_{ba}{}^{(i)}F+\chi_{bc}{}^{(i)}H_{ca})\\
&(\mathbf{D}_3{}^{(i)}\pi)_{a4}=-4(\eta_a{}^{(i)}F+\eta_b{}^{(i)}H_{ab})\\
&(\mathbf{D}_4{}^{(i)}\pi)_{34}=4\mathbf{D}_4{}^{(i)}F\\
&(\mathbf{D}_a{}^{(i)}\pi)_{34}=4(\nabla\!\!\!/_a{}^{(i)}F-\chi_{ab}{}^{(i)}Z_b)\\
&(\mathbf{D}_3{}^{(i)}\pi)_{34}=4(\mathbf{D}_3{}^{(i)}F-2\eta_b{}^{(i)}Z_b)\\
&(\mathbf{D}_4{}^{(i)}\pi)_{33}=-16\underline{\eta}_b{}^{(i)}Z_b\\
&(\mathbf{D}_a{}^{(i)}\pi)_{33}=-8\underline{\chi}_{ab}{}^{(i)}Z_b\\
&(\mathbf{D}_3{}^{(i)}\pi)_{33}=0\\
&(\mathbf{D}_4{}^{(i)}\pi)_{ab}=2\mathbf{D}_4{}^{(i)}H_{ab}=2\Omega^{-1}\frac{d}{d\underline{u}}{}^{(i)}H_{ab}\\
&(\mathbf{D}_c{}^{(i)}\pi)_{ab}=2(\nabla\!\!\!/_c{}^{(i)}H)_{ab}-2\chi_{ca}{}^{(i)}Z_b-2\chi_{cb}{}^{(i)}Z_a\\
&(\mathbf{D}_3{}^{(i)}\pi)_{ab}=2(\mathbf{D}_3{}^{(i)}H)_{ab}-4(\eta_a{}^{(i)}Z_b+\eta_b{}^{(i)}Z_a)\\
&(\mathbf{D}_4{}^{(i)}\pi)_{a3}=4\Omega^{-2}\frac{d}{d\underline{u}}(\Omega{}^{(i)}Z_a)-4(\underline{\eta}_a{}^{(i)}F+\underline{\eta}_b{}^{(i)}H_{ba})\\
&(\mathbf{D}_b{}^{(i)}\pi)_{a3}=4(\nabla\!\!\!/_b{}^{(i)}Z)_a-2\underline{\chi}_{ba}{}^{(i)}F-2\underline{\chi}_{bc}{}^{(i)}H_{ca}-4\zeta_b{}^{(i)}Z_a\\
&(\mathbf{D}_3{}^{(i)}\pi)_{a3}=4(\mathbf{D}_3{}^{(i)}Z)_a-4(\mathbf{D}_3\log\Omega){}^{(i)}Z_a
\end{aligned}
\tag{4.6.14}
$$

Most of the terms in the right hand side have already been estimated or their estimates follow immediately from the previous results. The derivative terms we still need to control are:

$$\mathbf{D}_3{}^{(i)}H\ ,\ \nabla\!\!\!/_c{}^{(i)}H\ ,\ \mathbf{D}_3{}^{(i)}Z\ ,\ \nabla\!\!\!/_b{}^{(i)}Z\ .$$

Not all of them are independent, in fact a long, but straightforward computation gives

$$\begin{aligned}(\mathbf{D}_3{}^{(i)}\pi)_{ab} &= (\mathbf{D}_a{}^{(i)}\pi)_{3b} + (\mathbf{D}_b{}^{(i)}\pi)_{3a} + {}^{(i)}O_c(R_{3abc} + R_{3bac})\\ &+ 2((\nabla\!\!\!/_a{}^{(i)}O)_c\hat{\chi}_{cb} + (\nabla\!\!\!/_b{}^{(i)}O)_c\underline{\hat{\chi}}_{ca}) + tr\underline{\chi}((\nabla\!\!\!/_a{}^{(i)}O)_b + (\nabla\!\!\!/_b{}^{(i)}O)_a)\\ &+ {}^{(i)}O_c((\nabla\!\!\!/_a\underline{\chi})_{bc} + (\nabla\!\!\!/_b\underline{\chi})_{ac}) - 2\underline{\chi}_{ab}\underline{\eta}_c{}^{(i)}O_c - {}^{(i)}O_c(\zeta_a\underline{\chi}_{bc} + \zeta_b\underline{\chi}_{ac})\end{aligned}$$

This equation allows to write $\mathbf{D}_3{}^{(i)}H_{ab}$ in terms of $\nabla\!\!\!/_a{}^{(i)}Z_b$ and of $\nabla\!\!\!/_b{}^{(i)}Z_a$ so that, finally, we have only to control $\nabla\!\!\!/_c{}^{(i)}H$, $\mathbf{D}_3{}^{(i)}Z$, $\nabla\!\!\!/_b{}^{(i)}Z$.

Proposition 4.6.3 *The quantities $\nabla\!\!\!/_c{}^{(i)}H_{ab}$, $\nabla\!\!\!/_c{}^{(i)}Z_b$ and $\mathbf{D}_3{}^{(i)}Z_b$ satisfy the following evolution equations*

$$\begin{aligned}\frac{d}{d\underline{u}}(\nabla\!\!\!/_c{}^{(i)}H_{ab}) + \frac{1}{2}\Omega tr\chi(\nabla\!\!\!/_c{}^{(i)}H_{ab}) &= -\Omega\hat{\chi}_{ad}(\nabla\!\!\!/^{(i)}H)_{cdb} + \hat{\chi}_{ad}\left[(\nabla\!\!\!/^{(i)}H)_{bdc} - (\nabla\!\!\!/^{(i)}H)_{dbc}\right]\\ &\quad + \hat{\chi}_{bd}\left[(\nabla\!\!\!/^{(i)}H)_{adc} - (\nabla\!\!\!/^{(i)}H)_{dac}\right] + \mathcal{H}_1\\ \frac{d}{d\underline{u}}(\nabla\!\!\!/_a{}^{(i)}Z_b) &= -\Omega\hat{\chi}_{ac}\nabla\!\!\!/_c{}^{(i)}Z_b + \Omega\hat{\chi}_{bc}\nabla\!\!\!/_a{}^{(i)}Z_c + \mathcal{Z}_1 \qquad (4.6.15)\\ \frac{d}{d\underline{u}}(\mathbf{D}_3{}^{(i)}Z_b) &= 4\Omega\omega(\mathbf{D}_3{}^{(i)}Z_b) - 4\Omega(\zeta\cdot\nabla\!\!\!/)^{(i)}Z_b + \Omega\mathbf{R}_{bc43}{}^{(i)}Z_c\\ &\quad + \frac{1}{2}\Omega tr\chi(\mathbf{D}_3{}^{(i)}Z_b) + \Omega\hat{\chi}_{bc}(\mathbf{D}_3{}^{(i)}Z_c) + \mathcal{Z}_{11} \qquad (4.6.16)\end{aligned}$$

where

$$\begin{aligned}\mathcal{H}_1 &= \Omega\Big\{(\nabla\!\!\!/\log\Omega)_c\mathbf{D}_{e_4}({}^{(i)}H_{ab}) + \left[(\mathbf{R}_{ad4c} - \underline{\eta}_a\chi_{dc})^{(i)}H_{db} + (\mathbf{R}_{bd4c} - \underline{\eta}_b\chi_{cd})^{(i)}H_{da}\right]\\ &- \left[(\nabla\!\!\!/\hat{\chi})_{cad}{}^{(i)}H_{db} + (\nabla\!\!\!/\hat{\chi})_{cbd}{}^{(i)}H_{da}\right] + (L_O\nabla\!\!\!/\hat{\chi})_{cab} + \frac{1}{2}\delta_{ab}\nabla\!\!\!/_c(L_O tr\chi) - (\nabla\!\!\!/^{(i)}F\chi)_{cab}\Big\}\end{aligned} \qquad (4.6.17)$$

$$\begin{aligned}\mathcal{Z}_1 &= \Omega\Big\{(\nabla\!\!\!/_a\log\Omega)\mathbf{D}\!\!\!\!/_N{}^{(i)}Z_b + \Big[(\nabla\!\!\!/_a\chi_{bc})^{(i)}Z_c + \frac{1}{2}L_O\nabla\!\!\!/_a(\zeta+\eta)_b\\ &+ \frac{1}{2}\nabla\!\!\!/_a\nabla\!\!\!/_b{}^{(i)}F - \frac{1}{2}\left(\nabla\!\!\!/_a{}^{(i)}F(\zeta+\eta)\right) - 2(\nabla\!\!\!/_a\zeta_c)^{(i)}H_{cb} - 2\zeta_c(\nabla\!\!\!/_a{}^{(i)}H_{cb})\end{aligned}$$

$$-\frac{1}{2}\nabla\!\!\!/_a({}^{(i)}F(\zeta-\underline{\eta})_b)\Big]+\Big[R_{bca4}{}^{(i)}Z_c-\Omega\underline{\eta}_b\chi_{ac}{}^{(i)}Z_c+\Omega\chi_{ab}\underline{\eta}_c{}^{(i)}Z_c\Big]$$
$$+\frac{1}{2}\left(\nabla\!\!\!/_a{}^{(i)}H_{cb}+\nabla\!\!\!/_b{}^{(i)}H_{ca}-\nabla\!\!\!/_c{}^{(i)}H_{ab}\right)(\zeta+\eta)_c\Big\} \tag{4.6.18}$$

$$\mathcal{Z}_{11}=\Big\{2(\mathbf{D}_3\Omega\omega)^{(i)}Z_b+\frac{1}{2}(\mathbf{D}_3\Omega tr\chi)^{(i)}Z_b+(\mathbf{D}_3\Omega\hat{\chi}_{bc})^{(i)}Z_c \tag{4.6.19}$$
$$+\mathbf{D}_3\left(\Omega\left[\frac{1}{2}(L_{{}^{(i)}O}-{}^{(i)}F)(\zeta+\eta))_a+\frac{1}{2}\nabla\!\!\!/_a{}^{(i)}F-2\zeta_b{}^{(i)}H_{ab}-\frac{1}{2}(\eta-\underline{\eta})_a{}^{(i)}F)\right]\right)\Big\}$$

Proof: The proof is a long but direct computation and is reported in the appendix to this chapter.

4.7 Proof of Theorem 3.7.4, estimates for the $\mathcal{D}$ norms of the rotation deformation tensors

We discuss how to control the $\mathcal{D}_0$ and $\mathcal{D}_1$ norms relative to the zero and first order derivatives and the $\mathcal{D}_2$ norms for the second order derivatives in two different propositions, as their estimates are somewhat different.

Proposition 4.7.1 *Assume, on $\underline{C}_*$, the following estimates, for $p\in[2,4]$,*

$$|r^{-1}{}^{(i)}O|_{p,S}(u,\underline{u}_*)\le c\mathcal{I}_* \quad,\quad |r^{-\frac{2}{p}}\nabla\!\!\!/{}^{(i)}O|_{p,S}(u,\underline{u}_*)\le c\mathcal{I}_*$$
$$|r{}^{(i)}H_{ab}|_{p,S}(u,\underline{u}_*)\le c\mathcal{I}_* \quad,\quad |r^{2-\frac{2}{p}}\nabla\!\!\!/{}^{(i)}H_{ab}|_{p,S}(u,\underline{u}_*)\le c\mathcal{I}_*$$
$$|r{}^{(i)}Z_a|_{p,S}(u,\underline{u}_*)\le c\mathcal{I}_* \quad,\quad |r^{2-\frac{2}{p}}\nabla\!\!\!/{}^{(i)}Z_{ab}|_{p,S}(u,\underline{u}_*)\le c\mathcal{I}_*$$
$$|r^{1-\frac{2}{p}}\tau_-\mathbf{D}_3{}^{(i)}H_{ab}|_{p,S}\le c\mathcal{I}_*\ ,\ |r^{1-\frac{2}{p}}\tau_-\mathbf{D}_3{}^{(i)}Z_a|_{p,S}\le c\mathcal{I}_*$$
$$|r^{1-\frac{2}{p}}\tau_-\mathbf{D}_3{}^{(i)}F|_{p,S}\le c\mathcal{I}_*$$

then, in view of the results of Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.2.3, we prove the following inequalities, for $p\in[2,4]$,

$$\begin{aligned}
&|r^{(-1-\frac{2}{p})}{}^{(i)}O|_{p,S}\le c(\mathcal{I}_*+\Delta_0)\\
&|r^{(1-\frac{2}{p})}{}^{(i)}F|_{p,S}\le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\
&|r^{(1-\frac{2}{p})}{}^{(i)}H|_{p,S}\le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\
&|r^{(1-\frac{2}{p})}{}^{(i)}Z|_{p,S}\le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)
\end{aligned} \tag{4.7.1}$$

$$
\begin{aligned}
&|r^{-\frac{2}{p}}\nabla\!\!\!\!/^{(i)}O|_{p,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\
&|r^{(2-\frac{2}{p})}\nabla\!\!\!\!/^{(i)}F|_{p,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\
&|r^{(2-\frac{2}{p})}\nabla\!\!\!\!/^{(i)}H|_{p,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\
&|r^{(2-\frac{2}{p})}\nabla\!\!\!\!/^{(i)}Z|_{p,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)
\end{aligned}
\tag{4.7.2}
$$

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\mathbf{D}_4{}^{(i)}H_{ab}|_{p,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\
&|r^{2-\frac{2}{p}}\mathbf{D}_4{}^{(i)}Z_a|_{p,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\
&|r^{2-\frac{2}{p}}\mathbf{D}_4{}^{(i)}F|_{p,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)
\end{aligned}
\tag{4.7.3}
$$

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\tau_-\mathbf{D}_3{}^{(i)}H_{ab}|_{p,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\
&|r^{1-\frac{2}{p}}\tau_-\mathbf{D}_3{}^{(i)}Z_a|_{p,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\
&|r^{1-\frac{2}{p}}\tau_-\mathbf{D}_3{}^{(i)}F|_{p,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)
\end{aligned}
\tag{4.7.4}
$$

Moreover from the inequalities 4.7.1, 4.7.2 the following estimates hold

$$
\begin{aligned}
&|r^{-1(i)}O|_{\infty,S} \le c(\mathcal{I}_*+\Delta_0)\\
&|r^{(i)}F|_{\infty,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\
&|r^{(i)}H_{ab}|_{\infty,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\
&|r^{(i)}Z_a|_{\infty,S} \le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)
\end{aligned}
\tag{4.7.5}
$$

Proof: Applying the Evolution Lemma to the equations 4.6.4 and 4.6.5, we obtain the following inequalities

$$
\begin{aligned}
|r^{(-1-\frac{2}{p})(i)}O|_{p,S}(u,\underline{u}) &\le c\left(|r^{(-1-\frac{2}{p})(i)}O|_{p,S}(u,\underline{u}_*)+\int_{\underline{u}}^{\underline{u}_*}\Omega|\hat{\chi}|_{\infty,S}|r^{(-1-\frac{2}{p})(i)}O|_{p,S}\right)\\
|r^{-\frac{2}{p}}\nabla\!\!\!\!/^{(i)}O|_{p,S}(u,\underline{u}) &\le c\left(|r^{-\frac{2}{p}}\nabla\!\!\!\!/^{(i)}O|_{p,S}(u,\underline{u}_*)+\int_{\underline{u}}^{\underline{u}_*}\Omega|\hat{\chi}|_{\infty,S}|r^{-\frac{2}{p}}\nabla\!\!\!\!/^{(i)}O|_{p,S}\right.\\
&\qquad\left.+\int_{\underline{u}}^{\underline{u}_*}|r^{-\frac{2}{p}}\mathcal{F}_1|_{p,S}\right)
\end{aligned}
\tag{4.7.6}
$$

where, see 4.6.9,

$$
|r^{-\frac{2}{p}}\mathcal{F}_1|_{p,S} \le c\frac{(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)}{r^3}|^{(i)}O|_{p,S} \le c\frac{(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)}{r^2}\mathcal{I}_*\ .
$$

Using the Gronwall Lemma, we obtain the following inequalities

$$|r^{(-1-\frac{2}{p})}{}^{(i)}O|_{p,S}(u,\underline{u}) \leq c|r^{(-1-\frac{2}{p})}{}^{(i)}O|_{p,S}(u,\underline{u}_*) \leq c\mathcal{I}_* \tag{4.7.7}$$
$$|r^{-\frac{2}{p}}\nabla\!\!\!\!/ {}^{(i)}O|_{p,S}(u,\underline{u}) \leq |r^{-\frac{2}{p}}\nabla\!\!\!\!/ {}^{(i)}O|_{p,S}(u,\underline{u}_*) + c\frac{(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)}{r}\mathcal{I}_*$$

and, applying Lemma 4.1.3,

$$|r^{-1}{}^{(i)}O|_{\infty,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\ .$$

Using the explicit expression of the Lie derivatives with respect to ${}^{(i)}O$ and the previous results for the connection coefficients we can prove the following inequalities [39]

$$|r^2 L_{{}^{(i)}O} tr\chi|_{\infty,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)$$
$$|r^2 L_{{}^{(i)}O}\hat{\chi}|_{\infty,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)$$

and, in view of 4.6.10 , we also obtain

$$|r\ {}^{(i)}F|_{\infty,S} = |r\ {}^{(i)}O\nabla\!\!\!\!/ \log\Omega|_{\infty,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\ .$$

To derive the estimates 4.7.1, 4.7.2 for ${}^{(i)}H$ and ${}^{(i)}Z$ we use their evolution equations 4.6.13. Thus, in the case of ${}^{(i)}H$, we derive, using the Evolution Lemma,

$$\begin{aligned}\frac{d}{d\underline{u}}|r^{-\frac{2}{p}}{}^{(i)}H|_{p,S} &\leq c\frac{(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)}{r^2}\Big(|r^{-\frac{2}{p}}{}^{(i)}H|_{p,S} + (|r^3\nabla\!\!\!\!/\hat{\chi}|_{\infty,S} + |r^2\hat{\chi}|_{\infty,S} \\ &\qquad + |r^3\nabla\!\!\!\!/ \mathrm{tr}\chi|_{\infty,S})\Big) \\ &\leq \frac{(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)}{r^2}\Big(|r^{-\frac{2}{p}}{}^{(i)}H|_{p,S} + c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\Big)\ .\end{aligned}$$

Integrating from $\underline{u}$ to $\underline{u}_*$ and applying the Gronwall Lemma, gives the inequality [40]

$$|r^{1-\frac{2}{p}}{}^{(i)}H|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0) \tag{4.7.8}$$

[39] In fact this requires also the control of the norms $|r^{-\frac{2}{p}}\nabla\!\!\!\!/ {}^{(i)}O|_{\infty,S}$ and, at its turn, this requires also the control of $|r^{1-\frac{2}{p}}\nabla\!\!\!\!/^{2}{}^{(i)}O|_{p,S}$ which can be proven in tha same way as for $|r^{-\frac{2}{p}}\nabla\!\!\!\!/ {}^{(i)}O|_{p,S}$, simply deriving once more eq. 4.6.5.

[40] Observe that the different asymptotic behaviour of ${}^{(i)}H$ and $\nabla\!\!\!\!/ {}^{(i)}O$ is due to the different estimates they satisfy on $\underline{C}_*$.

Proceeding in the same way, starting from eq. 4.6.15, we obtain

$$\frac{d}{d\underline{u}}|r^{1-\frac{2}{p}}\nabla\!\!\!/^{(i)}H|_{p,S} \leq c\frac{(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)}{r^2}\left(|r^{1-\frac{2}{p}}\nabla\!\!\!/^{(i)}H|_{p,S}+\frac{1}{r}(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\right)$$

which, again, integrating from $\underline{u}$ to $\underline{u}_*$ and applying the Gronwall Lemma, gives the inequality

$$|r^{2-\frac{2}{p}}\nabla\!\!\!/^{(i)}H|_{p,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\ .$$

Collecting these results together and using Lemma 4.1.3 we conclude

$$|r^{(i)}H|_{\infty,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1) \tag{4.7.9}$$

Proceeding exactly in the same way for ${}^{(i)}Z$ we obtain

$$\begin{aligned}|r^{1-\frac{2}{p}}{}^{(i)}Z|_{p,S} &\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\ |r^{2-\frac{2}{p}}\nabla\!\!\!/^{(i)}Z|_{p,S} &\leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\end{aligned}$$

and, therefore,

$$|r^{(i)}Z_a|_{\infty,S} \leq c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\ .$$

The estimates 4.7.3 are obtained writing their explicit expressions and estimating them using the previous results. The estimates 4.7.4 require to use the evolution equation for $\mathbf{D}_3{}^{(i)}Z$, see 4.6.15, and then proceed as before.

Proposition 4.7.1 allows to estimate the components of the traceless part of the rotation deformation tensor [41],

$$\begin{aligned}{}^{(O)}\mathbf{i}_{ab} &\equiv {}^{(i)}\hat{\pi}_{ab} = 2^{(i)}H_{ab}-2^{-1}\delta_{ab}({}^{(i)}H_{aa}-2^{(i)}F)\\ {}^{(O)}\mathbf{j} &\equiv {}^{(i)}\hat{\pi}_{34} = (2^{(i)}F+{}^{(i)}H_{aa})\\ {}^{(O)}\underline{\mathbf{m}}_a &\equiv {}^{(i)}\hat{\pi}_{3a} = 4^{(i)}Z_a\\ {}^{(O)}\mathbf{m}_a &\equiv {}^{(i)}\hat{\pi}_{4a} = 0\\ {}^{(O)}\underline{\mathbf{n}} &\equiv {}^{(i)}\hat{\pi}_{33} = 2g(\mathbf{D}_3{}^{(i)}O,e_3) = 0\\ {}^{(O)}\mathbf{n} &\equiv {}^{(i)}\hat{\pi}_{44} = 2g(\mathbf{D}_4{}^{(i)}O,e_4) = 0\end{aligned} \tag{4.7.10}$$

proving the following result,

[41]Recall that ${}^{(i)}\hat{\pi}_{ab} = {}^{(i)}\pi_{ab}-\frac{1}{4}\delta_{ab}tr\pi$ and $tr^{(i)}\pi = {}^{(i)}\pi_{aa}-{}^{(i)}\pi_{34}$.

Corollary 4.7.1 *In $\mathcal{K}$, the following inequalities hold:*

$$\begin{aligned}
&|r^{1-\frac{2}{p}}({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|_{p,S}\le c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\
&|r^{2-\frac{2}{p}}\nabla\!\!\!/({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|_{p,S}\le c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\
&|r^{2-\frac{2}{p}}\mathbf{D}\!\!\!\!/_4({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|_{p,S}\le c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\
&|r^{1-\frac{2}{p}}\tau_-\mathbf{D}\!\!\!\!/_3({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|_{p,S}\le c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)
\end{aligned}\tag{4.7.11}$$

The first line for $p\in[2,\infty)$ [42], the other ones for $p\in[2,4]$.

The next proposition provides us with the estimates for the $\mathcal{D}_2$ norm, see definition 3.5.45. The $\mathcal{D}_2$ norm collects the norms of the second derivatives of the rotation deformation tensors which will be used in chapter 6 for the estimates of the "Error terms".

Proposition 4.7.2 *Assume that, on Σ_0 the following inequality holds*

$$||r\nabla\!\!\!/^2H||_{L^2(\underline{C}_*\cap V(u,\underline{u}_*))}=\left(\int_{u_{(0)}(\underline{u}_*)}^{u}du'|r^{2-\frac{2}{p}}\nabla\!\!\!/^2H|^2_{p=2,S}(u',\underline{u}_*)\right)^{\frac{1}{2}}\le c\mathcal{I}_*$$

then in view of the results of Theorems 4.2.1, 4.2.2, 4.2.3 and of Proposition 4.7.1 the following estimates hold, the last one for $\delta>\epsilon>0$

$$||r\nabla\!\!\!/^2H||_{L^2(\underline{C}(\underline{u}')\cap V(u,\underline{u}'))}\le c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta)\tag{4.7.12}$$

$$||r\nabla\!\!\!/^2Z||_{L^2(\underline{C}(\underline{u}')\cap V(u,\underline{u}'))}\le c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta)\tag{4.7.13}$$

$$||\frac{r}{\sqrt{r^{1-2\epsilon}}}\nabla\!\!\!/\mathbf{D}\!\!\!\!/_3Z||_{L^2(\underline{C}(\underline{u}')\cap V(u,\underline{u}'))}\le c\frac{1}{u\sqrt{r^{1-2\delta}}}\,(\mathcal{I}_0+\mathcal{I}_*+\Delta)\tag{4.7.14}$$

where we denoted $\Delta\equiv\Delta_0+\Delta_1+\Delta_2$.

Proof of inequality 4.7.12:

To estimate $\nabla\!\!\!/^{2\,(i)}H$ we use its evolution equation, obtained deriving tangentially the one for $\nabla\!\!\!/^{(i)}H$, see 4.6.15. It can be written as

$$\frac{d}{d\underline{u}}(\nabla\!\!\!/^2H)+\Omega\mathrm{tr}\chi(\nabla\!\!\!/^2H)=\hat{\chi}(\nabla\!\!\!/^2H)+(L_O\nabla\!\!\!/^2\chi)+\mathcal{H}_2\tag{4.7.15}$$

[42] The estimates hold also for $p=\infty$, but, in this case, their bound depends also on $\triangle_1$, see 4.7.8, 4.7.9.

where $\mathcal{H}_2$ collects all the terms which do not depend on third order derivatives of the connection coefficients and, therefore, can be estimated in the $L^p(S)$ norms, with $p \in [2,4]$ [43]. Applying to 4.7.15 the Evolution Lemma and Gronwall Lemma we obtain

$$|r^{2-\frac{2}{p}}\nabla^2 H|_{p=2,S}(u,\underline{u}) \leq c\Big(|r^{2-\frac{2}{p}}\nabla^2 H|_{p=2,S}(u,\underline{u}_*) \qquad (4.7.16)$$
$$+\int_{\underline{u}}^{\underline{u}_*} d\underline{u}' \left[|r^{2-\frac{2}{p}}L_O\nabla^2\chi|_{p=2,S} + |r^{2-\frac{2}{p}}\mathcal{H}_2|_{p=2,S}\right](u,\underline{u}')\Big)$$

where the integral with $\mathcal{H}_2$, depending on lower derivatives, does not give problems and, hereafter, will be neglected. Substituting 4.7.16 in the left hand side of 4.7.12 we have [44]

$$||r\nabla^2 H||^2_{L^2(\underline{C}(\underline{u}')\cap V(u,\underline{u}'))} = \int_{u_0(\underline{u}')}^{u} du'|r^{2-\frac{2}{p}}\nabla^2 H|^2_{p=2,S}(u',\underline{u})$$
$$\leq \int_{u_0(\underline{u}')}^{u} du'\left[|r^{2-\frac{2}{p}}\nabla^2 H|^2_{p=2,S}(u',\underline{u}_*) + \left(\int_{\underline{u}'}^{\underline{u}_*} d\underline{u}''|r^{3-\frac{2}{p}}\nabla^3\chi|_{p=2,S}(u',\underline{u}'')\right)^2\right]$$
$$\leq c\mathcal{I}_*^2 + c\int_{u_0(\underline{u}')}^{u} du'\left(\int_{\underline{u}'}^{\underline{u}_*}\frac{1}{r(u',\underline{u}')}\right)\left(\int_{\underline{u}'}^{\underline{u}_*} d\underline{u}''|r^{4-\frac{2}{p}}\nabla^3\chi|^2_{p=2,S}(u',\underline{u}'')\right)$$
$$\leq c\mathcal{I}_*^2 + c\int_{u_0(\underline{u}')}^{u} du'\frac{1}{r(u',\underline{u}')}||r^3\nabla^3\chi||^2_{L^2(C(u',[\underline{u}',\underline{u}_*]))}$$
$$\leq c\mathcal{I}_*^2 + c\left(\sup_u r^{\frac{1}{2}}(u,\underline{u}')||r^3\nabla^3\chi||_{L^2(C(u,[\underline{u}',\underline{u}_*]))}\right)^2\left(\int_{u_0(\underline{u}')}^{u} du'\frac{1}{r^2(u',\underline{u}')}\right) \qquad (4.7.17)$$
$$\leq c\mathcal{I}_*^2 + c\left(\sup_u r^{\frac{1}{2}}(u,\underline{u}')||r^3\nabla^3\chi||_{L^2(C(u,[\underline{u}',\underline{u}_*]))}\right)^2 \leq c\mathcal{I}_*^2 + (\mathcal{I}_0+\mathcal{I}_*+\Delta)^2$$

using the result of Proposition 4.5.1.

Proof of inequality 4.7.13:

To estimate $\nabla^{2\,(i)}Z$ we look at its evolution equation, obtained deriving tangentially the evolution equation for $\nabla^{(i)}Z$, see 4.6.15,

$$\frac{d}{d\underline{u}}(\nabla^2 Z) + \frac{1}{2}\Omega\mathrm{tr}\chi(\nabla^2 Z) = \Omega\hat{\chi}(\nabla^2 Z) + L_O\nabla^2(\zeta+\eta) + \nabla^3 F + (\zeta+\eta)\nabla^2 H + \mathcal{Z}_2$$

[43]Moreover they also have the appropriate asymptotic behaviour to control their integration.

[44]Here ϵ_0 is a small constant which bounds $c\mathcal{I}_*$. In general with ϵ_0 we denote a small constant satisfying $c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta) \leq \epsilon_0$.

where $\mathcal{Z}_2$ collects all the terms which do not depend on third order derivatives of the connection coefficients and, therefore, can be estimated in the $L^p(S)$ norms, with $p \in [2,4]$. Therefore we will neglect it in the sequel. Applying the Evolution Lemma and Gronwall Lemma to this evolution equation we obtain,

$$
\begin{aligned}
& |r^{2-\frac{2}{p}}\nabla\!\!\!\!/^2 Z|_{p=2,S}(u,\underline{u}) \leq c\Big(|r^{2-\frac{2}{p}}\nabla\!\!\!\!/^2 Z|_{p=2,S}(u,\underline{u}_*) \\
& + \int_{\underline{u}}^{\underline{u}_*}\Big[|r^{2-\frac{2}{p}}L_O\nabla\!\!\!\!/^2\eta|_{p=2,S} + |r^{2-\frac{2}{p}}L_O\nabla\!\!\!\!/^2\underline{\eta}|_{p=2,S} + |r^{3-\frac{2}{p}}\nabla\!\!\!\!/^3(\eta+\underline{\eta})|_{p=2,S}\Big] \\
& + \int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}|r^{2-\frac{2}{p}}\nabla\!\!\!\!/^2 H|_{p=2,S}\Big) \\
\leq\ & c\Big(|r^{2-\frac{2}{p}}\nabla\!\!\!\!/^2 H|_{p=2,S}(u,\underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*}|r^{3-\frac{2}{p}}\nabla\!\!\!\!/^3(\eta+\underline{\eta})|_{p=2,S} + \int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}|r^{3-\frac{2}{p}}\nabla\!\!\!\!/^3\chi|_{p=2,S}\Big)
\end{aligned}
$$

The last integral in the last line can be treated as in the previous estimate and gives a better contribution due to the factor r^{-2}. Therefore we are left with the inequality

$$
\begin{aligned}
& ||r\nabla\!\!\!\!/^2 Z||^2_{L^2(\underline{C}(\underline{u}')\cap V(u,\underline{u}'))} \leq \int_{u_0(\underline{u}')}^{u} du'|r^{2-\frac{2}{p}}\nabla\!\!\!\!/^2 Z|^2_{p=2,S}(u',\underline{u}_*) \\
& + \int_{u_0(\underline{u}')}^{u} du'\Big(\int_{\underline{u}'}^{\underline{u}_*} d\underline{u}''|r^{3-\frac{2}{p}}\nabla\!\!\!\!/^3(\eta+\underline{\eta})|_{p=2,S}(u',\underline{u}'')\Big)^2 + c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta)^2
\end{aligned}
\tag{4.7.18}
$$

The estimate of the last integral is obtained by a straightforward application of Schwartz inequality and of the result of Proposition 4.5.1.

Proof of inequality 4.7.14:

To estimate $\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3{}^{(i)}Z$ we have to look at its evolution equation, obtained applying $\mathbf{D}\!\!\!\!/_3$ to the evolution equation for $\nabla\!\!\!\!/^{(i)}Z$, see 4.6.15,

$$
\frac{d}{d\underline{u}}(\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3 Z) = \Omega\Big[\hat{\chi}(\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3 Z) + \frac{1}{2}\mathcal{L}_O\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3(\zeta+\eta) + \frac{1}{2}\nabla\!\!\!\!/^2\mathbf{D}\!\!\!\!/_3 F - 2(\zeta+\eta)\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3 H\Big] + \mathcal{S}_2
$$

where $\mathcal{S}_2$ collects all the terms which do not depend on third order derivatives of the connection coefficients and, therefore, can be estimated in the $L^p(S)$ norms, with $p \in [2,4]$. Between the terms in parenthesis the more delicate ones are: $r\nabla\!\!\!\!/^3\underline{\omega}$ and $r\nabla\!\!\!\!/^2\underline{\beta}$ present in the explicit expressions of $\mathcal{L}_O\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3(\zeta+\eta)$ and $\nabla\!\!\!\!/^2\mathbf{D}\!\!\!\!/_3 F$ [45]. Considering only these terms and applying

[45] All the terms in $[\cdots]$ of 4.7.19 have to be estimated in the L^2-norms, but we focus our attention on these ones as they have the slowest decay.

the Evolution Lemma and Gronwall Lemma to the evolution equation we obtain [46]

$$|r^{(\frac{3}{2}+\epsilon)-\frac{2}{p}}\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3 Z|_{p=2,S}(u',\underline{u}') \leq c\int_{\underline{u}'}^{\underline{u}_*}\left[|r^{(\frac{5}{2}+\epsilon)-\frac{2}{p}}\nabla\!\!\!\!/^3\underline{\omega}|_{p=2,S}+\frac{1}{r^2}|r^{(\frac{5}{2}+\epsilon)-\frac{2}{p}}\nabla\!\!\!\!/^2\underline{\beta}|_{p=2,S}\right]$$

Therefore we have

$$\begin{aligned}|r^{(\frac{3}{2}+\epsilon)-\frac{2}{p}}\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3 Z|^2_{p=2,S}(u',\underline{u}') &\leq 2\left(\int_{\underline{u}'}^{\underline{u}_*}|r^{(\frac{5}{2}+\epsilon)-\frac{2}{p}}\nabla\!\!\!\!/^3\underline{\omega}|_{p=2,S}(u'\underline{u}'')\right)^2\\ &+ 2\left(\int_{\underline{u}'}^{\underline{u}_*}|r^{(\frac{5}{2}+\epsilon)-\frac{2}{p}}\nabla\!\!\!\!/^2\underline{\beta}|_{p=2,S}(u'\underline{u}'')\right)^2 \quad (4.7.19)\end{aligned}$$

and, from it,

$$\begin{aligned}&||\frac{r}{\sqrt{r^{1-2\epsilon}}}\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3 Z||^2_{L^2(\underline{C}(\underline{u}')\cap V(u,\underline{u}'))} \leq c\int_{u_0(\underline{u}')}^{u}du'\left(\int_{\underline{u}'}^{\underline{u}_*}d\underline{u}''|r^{(\frac{5}{2}+\epsilon)-\frac{2}{p}}\nabla\!\!\!\!/^3\underline{\omega}|_{p=2,S}(u'\underline{u}'')\right)^2\\ &\quad +c\int_{u_0(\underline{u}')}^{u}du'\left(\int_{\underline{u}'}^{\underline{u}_*}d\underline{u}''|r^{(\frac{5}{2}+\epsilon)-\frac{2}{p}}\nabla\!\!\!\!/^2\underline{\beta}|_{p=2,S}(u'\underline{u}'')\right)^2 \equiv c\,(\mathcal{I}_3+\mathcal{I}_4) \qquad (4.7.20)\end{aligned}$$

To estimate the $\mathcal{I}_4$ integral we apply Schwartz inequality and obtain, choosing $\eta > 0$,

$$\begin{aligned}\mathcal{I}_4 &\leq c\int_{\underline{u}'}^{\underline{u}_*}d\underline{u}''\left(\int_{\underline{u}'}^{\underline{u}_*}d\underline{u}''\frac{1}{r^{1+2\eta}}\right)\left(\int_{u_0(\underline{u}')}^{u}du'|r^{(3+(\epsilon+\eta))-\frac{2}{p}}\nabla\!\!\!\!/^2\underline{\beta}|^2_{p=2,S}(u'\underline{u}'')\right)\\ &\leq c\frac{1}{u^2}\int_{\underline{u}'}^{\underline{u}_*}d\underline{u}''\frac{1}{r^{2-2\epsilon}(u'\underline{u}'')}\int_{\underline{C}(\underline{u}'';[u_0,u])}r^2u'^2|\underline{\beta}(\hat{\mathcal{L}}_O^2W)|^2 \qquad (4.7.21)\\ &\leq c\frac{1}{u^2r^{(1-2\epsilon)}(u,\underline{u}')}||\tau_-r^3\nabla\!\!\!\!/^2\underline{\beta}||^2_{L^2} \leq c\frac{\Delta_2^2}{u^2r^{(1-2\epsilon)}} \leq c\,(\mathcal{I}_0+\mathcal{I}_*+\Delta)^2\,\frac{1}{u^2r^{(1-2\epsilon)}}\end{aligned}$$

where in the last line we used Proposition 5.1.5 of Chapter 5, see **??**. Proceeding in the same way for $\mathcal{I}_3$, we obtain

$$\begin{aligned}\mathcal{I}_3 &\leq \int_{u_0(\underline{u}')}^{u}du'\left(\int_{\underline{u}'}^{\underline{u}_*}d\underline{u}''\frac{1}{r^{1+2\eta}}\right)\left(\int_{\underline{u}'}^{\underline{u}_*}d\underline{u}''|r^{(3+(\epsilon+\eta))-\frac{2}{p}}\nabla\!\!\!\!/^3\underline{\omega}|^2_{p=2,S}(u'\underline{u}'')\right)\\ &\leq c\int_{u_0(\underline{u}')}^{u}du'\int_{\underline{u}'}^{\underline{u}_*}d\underline{u}''|r^{(3+(\epsilon+\eta))-\frac{2}{p}}\nabla\!\!\!\!/^3\underline{\omega}|^2_{p=2,S}(u'\underline{u}'')\end{aligned}$$

[46]the term $|r^{(\frac{3}{2}+\epsilon)-\frac{2}{p}}\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3 Z|_{p=2,S}(u,\underline{u}_*)$ is missing as in Chapter 7 it is proved, see Proposition 7.5.1, that $Z=0$ on $\underline{C}_*$.

$$\leq c\int_{\underline{u}'}^{\underline{u}_*} d\underline{u}'' \int_{u_0(\underline{u}')}^{u} du' |r^{(3+(\epsilon+\eta))-\frac{2}{p}} \nabla\!\!\!\!/^3 \underline{\omega}|^2_{p=2,S}(u'\underline{u}'')$$
$$\leq c\int_{\underline{u}'}^{\underline{u}_*} d\underline{u}'' \frac{1}{r^{2(1-(\epsilon+\eta))}} \int_{u_0(\underline{u}')}^{u} du' |r^{4-\frac{2}{p}} \nabla\!\!\!\!/^3 \underline{\omega}|^2_{p=2,S}(u'\underline{u}'')$$
$$\leq c\left(\sup_{\underline{u}} r^{\frac{1}{2}}(u,\underline{u})||r^3\nabla\!\!\!\!/^3\underline{\omega}||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))}\right)^2 \leq (\mathcal{I}_0+\mathcal{I}_*+\Delta)^2 \quad (4.7.22)$$

using the results of Proposition 4.5.1.

4.8 Appendix to Chapter 4

4.8.1 Some commutation relations

We collect in the following proposition the proof of some commutation relations which will be repeatedly used in various estimates.

Proposition 4.8.1

a) *Let f be a scalar function, then* [47]

$$[\mathbf{D}_3,\mathbf{D}_4]f = (\mathbf{D}_4\log\Omega)\mathbf{D}_3 f - (\mathbf{D}_3\log\Omega)\mathbf{D}_4 f + 4\zeta\cdot\nabla\!\!\!\!/ f$$
$$[\mathbf{D}\!\!\!\!/_4,\nabla\!\!\!\!/]f = (\zeta+\underline{\eta})\mathbf{D}_4 f - \chi\cdot\nabla\!\!\!\!/ f \quad (4.8.1)$$

b) *Let V be a vector field tangent to S, then* [48]

$$\begin{aligned}[\mathbf{D}\!\!\!\!/_4,\nabla\!\!\!\!/_a]V_b &= -\chi_{ac}\nabla\!\!\!\!/_c V_b - \underline{\eta}_b\chi_{ac}V_c + \chi_{ab}(\underline{\eta}\cdot V)\\ &+ (\nabla\!\!\!\!/_a\log\Omega)\mathbf{D}\!\!\!\!/_4 V_b + ([D_\tau,D_\rho]V_\sigma)\, e_4^\tau e_a^\rho e_b^\sigma\\ [\mathbf{D}\!\!\!\!/_3,\nabla\!\!\!\!/_a]V_b &= -\underline{\chi}_{ac}\nabla\!\!\!\!/_c V_b - \eta_b\underline{\chi}_{ac}V_c + \underline{\chi}_{ab}(\eta\cdot V) \quad (4.8.2)\\ &+ (\nabla\!\!\!\!/_a\log\Omega)\mathbf{D}\!\!\!\!/_3 V_b + ([D_\tau,D_\rho]V_\sigma)\, e_3^\tau e_a^\rho e_b^\sigma\end{aligned}$$

[Manca l'espressione analoga per $[\mathbf{D}\!\!\!\!/_3,\nabla\!\!\!\!/_a]V_b$. Aggiungere.]

c) *Let f be a scalar function, then* [49]

$$\begin{aligned}[\Delta\!\!\!\!/,\mathbf{D}_3]f &= -\Big[-\eta_a\underline{\chi}_{ab}\nabla\!\!\!\!/_b f + tr\underline{\chi}\eta_b\nabla\!\!\!\!/_b f - 2\underline{\chi}_{ab}(\nabla\!\!\!\!/\nabla\!\!\!\!/ f)_{ab}\\ &\quad - \zeta_a\underline{\chi}_{ab}\nabla\!\!\!\!/_b f - (d\!\!\!/iv\underline{\chi})_b\nabla\!\!\!\!/_b f - \underline{\beta}_b\nabla\!\!\!\!/_b f\Big]\\ &\quad -\Big[(\nabla\!\!\!\!/_a\log\Omega)(\mathbf{D}_3\nabla\!\!\!\!/ f)_a - \zeta_a(\mathbf{D}_3\nabla\!\!\!\!/ f)_a + \eta_a\nabla\!\!\!\!/_a\mathbf{D}_3 f\\ &\quad +(\Delta\!\!\!\!/\log\Omega)\mathbf{D}_3 f + \zeta_a(\nabla\!\!\!\!/_a\log\Omega)\mathbf{D}_3 f\Big] \quad (4.8.3)\end{aligned}$$

[47] Of course $[\mathbf{D}\!\!\!\!/_4,\nabla\!\!\!\!/]f = \mathbf{D}\!\!\!\!/_4\nabla\!\!\!\!/ f - \nabla\!\!\!\!/\mathbf{D}_4 f$.

[48] Where $[\mathbf{D}\!\!\!\!/_4,\nabla\!\!\!\!/_a]V_b \equiv (\mathbf{D}\!\!\!\!/_4\nabla\!\!\!\!/ V - \nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_4 V)_{ab}$.

[49] $\Delta\!\!\!\!/ f \equiv e_a^\mu e_a^\nu\nabla\!\!\!\!/_\mu\nabla\!\!\!\!/_\nu f$. On a scalar function f, $\nabla\!\!\!\!/_\mu f = \Pi_\mu^\sigma D_\sigma f$ and, on a vector field X tangent to S, $\nabla\!\!\!\!/_\mu X^\nu = \Pi_\mu^\sigma\Pi_\rho^\nu D_\sigma X^\rho$ where Π is the tensor projecting on TS. Finally, assuming that the vector fields $\{e_a\}$ satisfy $\mathbf{D}\!\!\!\!/_3 e_a = 0$ we have: $\mathbf{D}_3\Delta\!\!\!\!/ f = e_a^\mu e_a^\nu\mathbf{D}_3\nabla\!\!\!\!/_\mu\nabla\!\!\!\!/_\nu f$.

d) *Let X be a vector field tangent to S, then*

$$
\begin{aligned}
[\not{D}_3, \not{div}]X &= -\underline{\chi}_{ac}\not\nabla_c X_a + \not\nabla_a \log\Omega \not{D}_3 X_a - \eta_a \underline{\chi}_{ac} X_c + tr\underline{\chi}(\eta\cdot X)\\
&\quad + e_a^\rho e_a^\sigma e_3^\tau [D_\tau, D_\rho]X_\sigma\\
[\not{D}_4, \not{div}]X &= -\chi_{ac}\not\nabla_c X_a + \not\nabla_a \log\Omega \not{D}_4 X_a - \underline{\eta}_a \chi_{ac} X_c + tr\chi(\underline{\eta}\cdot X)\\
&\quad + e_a^\rho e_a^\sigma e_4^\tau [D_\tau, D_\rho]X_\sigma \qquad (4.8.4)
\end{aligned}
$$

[correzione nella 4.8.4, tr$\chi(\underline{\eta}\cdot X)$ al posto di tr$\chi(\underline{\eta}\cdot\chi)$ e tr$\underline{\chi}(\eta\cdot X)$ al posto di tr$\underline{\chi}(\eta\cdot\underline{\chi})$]

Proof: To prove the second line of 4.8.1 we recall the definitions [50]

$$
\begin{aligned}
\not\nabla_a \mathbf{D}_4 f &= e_a^\mu D_\mu \mathbf{D}_4 f = (\mathbf{D}_{e_a} e_4)^\sigma D_\sigma f + e_a^\mu e_4^\sigma D_\mu D_\sigma f\\
\not{D}_4 \not\nabla_a f &\equiv (\not{D}_4 \not\nabla_\mu f) e_a^\mu = \mathbf{D}_4(\Pi_\mu^\rho D_\rho f) e_a^\mu\\
&= e_a^\mu (\mathbf{D}_4 \Pi)_\mu^\rho D_\rho f + e_a^\mu e_4^\sigma D_\sigma D_\mu f \qquad (4.8.5)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(\not\nabla_a \mathbf{D}_4 - \not{D}_4 \not\nabla_a) f = (\mathbf{D}_{e_a} e_4)^\sigma D_\sigma f - e_a^\mu (\mathbf{D}_4\Pi)_\mu^\rho D_\rho f + e_a^\mu e_4^\sigma [D_\sigma, D_\mu] f\\
&= -\frac{1}{2}(\mathbf{D}_{e_a} e_4)^\nu e_{3\nu} e_4^\sigma D_\sigma f + (\mathbf{D}_{e_a} e_4)^\nu e_{c\nu} e_c^\sigma D_\sigma f - e_a^\mu (\mathbf{D}_4\Pi)_\mu^\rho D_\rho f\\
&= -\frac{1}{2} g(\mathbf{D}_{e_a} e_4, e_3)\mathbf{D}_4 f + g(\mathbf{D}_{e_a} e_4, e_c)\mathbf{D}_c f - e_a^\mu (\mathbf{D}_4\Pi)_\mu^\rho D_\rho f\\
&= -\zeta_a \mathbf{D}_4 f + \chi_{ac}\not\nabla_c f - e_a^\mu (\mathbf{D}_4\Pi)_\mu^\rho D_\rho f \qquad (4.8.6)
\end{aligned}
$$

the proof is completed observing that $e_a^\mu \mathbf{D}_4 \Pi_\mu^\rho = \underline{\eta}_a e_4^\rho$. To prove 4.8.2 observe that

$$
\begin{aligned}
\not{D}_4 \not\nabla_a V_b &\equiv e_a^\mu e_b^\nu \mathbf{D}_4 \Pi_\mu^\rho \Pi_\nu^\sigma D_\rho V_\sigma = e_a^\mu (\mathbf{D}_4 \Pi_\mu^\rho) e_b^\sigma D_\rho V_\sigma\\
&+ e_a^\rho e_b^\nu (\mathbf{D}_4 \Pi_\nu^\sigma) D_\rho V_\sigma + e_a^\rho e_b^\sigma \mathbf{D}_4 D_\rho V_\sigma \qquad (4.8.7)
\end{aligned}
$$

which using the relations

$$
e_a^\mu (\mathbf{D}_4 \Pi_\mu^\rho) = \underline{\eta}_a e_4^\rho \ , \ e_b^\nu (\mathbf{D}_4 \Pi_\nu^\sigma) = \underline{\eta}_b e_4^\sigma \qquad (4.8.8)
$$

can be written as

$$
\not{D}_4 \not\nabla_a V_b = \underline{\eta}_a \not{D}_4 V_b - \underline{\eta}_b \chi_{ac} V_c + e_a^\rho e_b^\sigma e_4^\tau D_\rho D_\tau V_\sigma + e_a^\rho e_b^\sigma e_4^\tau [D_\tau, D_\rho] V_\sigma \qquad (4.8.9)
$$

[50] We always use the notations $\mathbf{D}W_a \equiv (\mathbf{D}W)_a$, $\not\nabla W_a \equiv (\not\nabla W)_a$, unless it can generate confusion.

On the other side

$$\begin{aligned}\nabla\!\!\!/_a \mathbf{D}\!\!\!/_4 V_b &\equiv e_a^\mu e_b^\nu \left(\Pi_\mu^\rho \Pi_\nu^\sigma D_\rho \Pi_\sigma^\tau \mathbf{D}_4 V_\tau\right) = e_a^\rho e_b^\sigma \left(D_\rho \Pi_\sigma^\tau \mathbf{D}_4 V_\tau\right) \\ &= e_a^\rho (D_\rho \Pi_\sigma^\tau) e_b^\sigma \mathbf{D}_4 V_\tau + e_a^\rho e_b^\sigma D_\rho e_4^\tau D_\tau V_\sigma \qquad (4.8.10)\end{aligned}$$

and, as

$$e_a^\rho (D_\rho \Pi_\sigma^\tau) e_b^\sigma = \frac{1}{2}\underline{\chi}_{ab} e_4^\tau + \frac{1}{2}\chi_{ab} e_3^\tau \qquad (4.8.11)$$

we obtain

$$\nabla\!\!\!/_a \mathbf{D}\!\!\!/_4 V_b = -\chi_{ab}(\underline{\eta}\cdot V) + e_a^\rho e_b^\sigma (D_\rho e_4^\tau) D_\tau V_\sigma + e_a^\rho e_b^\sigma e_4^\tau D_\rho D_\tau V_\sigma \qquad (4.8.12)$$

Moreover, as

$$e_a^\rho e_b^\sigma (D_\rho e_4^\tau) D_\tau V_\sigma = \chi_{ac} \nabla\!\!\!/_c V_b - \zeta_a \mathbf{D}\!\!\!/_4 V_b \ , \qquad (4.8.13)$$

4.8.12 can be rewritten as

$$\nabla\!\!\!/_a \mathbf{D}\!\!\!/_4 V_b = -\chi_{ab}(\underline{\eta}\cdot V) + \chi_{ac}\nabla\!\!\!/_c V_b - \zeta_a \mathbf{D}\!\!\!/_4 V_b + e_a^\rho e_b^\sigma e_4^\tau D_\rho D_\tau V_\sigma \qquad (4.8.14)$$

which togheter with 4.8.9 proves relation 4.8.2.

To prove 4.8.3 observe that

$$\begin{aligned}[\mathbf{D}_3, \triangle\!\!\!/]f &= \Big\{ e_a^\mu (\mathbf{D}_3\Pi)_\mu^\sigma e_a^\tau D_\sigma \Pi_\tau^\lambda D_\lambda f + e_a^\sigma e_a^\nu (\mathbf{D}_3\Pi)_\nu^\tau D_\sigma \Pi_\tau^\lambda D_\lambda f \\ &+ e_a^\sigma e_a^\tau D_\sigma (\mathbf{D}_3\Pi)_\tau^\lambda D_\lambda f - e_a^\sigma e_a^\tau (D_\sigma e_3)^\delta D_\delta \Pi_\tau^\lambda D_\lambda f \\ &- e_a^\sigma e_a^\tau D_\sigma \Pi_\tau^\lambda (D_\lambda e_3)^\delta D_\delta f - e_a^\sigma e_a^\tau e_3^\delta R^\gamma{}_{\tau\delta\sigma} \nabla\!\!\!/_\gamma f \Big\} \qquad (4.8.15)\end{aligned}$$

where the six terms in brackets have the following expressions

$$\begin{aligned} e_a^\mu (\mathbf{D}_3\Pi)_\mu^\sigma e_a^\tau D_\sigma \Pi_\tau^\lambda D_\lambda f &= \eta_a \mathbf{D}_3 (\nabla\!\!\!/ f)_a \\ e_a^\sigma e_a^\nu (\mathbf{D}_3\Pi)_\nu^\tau D_\sigma \Pi_\tau^\lambda D_\lambda f &= -\eta_a \underline{\chi}_{ab} \nabla\!\!\!/_b f \\ e_a^\sigma e_a^\tau D_\sigma (\mathbf{D}_3\Pi)_\tau^\lambda D_\lambda f &= tr\underline{\chi}\eta_b \nabla\!\!\!/_b f + \eta_a \nabla\!\!\!/_a \mathbf{D}_3 f + (\mathrm{d}\!\!\!/\mathrm{iv}\,\eta)\mathbf{D}_3 f \\ -e_a^\sigma e_a^\tau (D_\sigma e_3)^\delta D_\delta \Pi_\tau^\lambda D_\lambda f &= -\underline{\chi}_{ab} (\nabla\!\!\!/\nabla\!\!\!/ f)_{ab} - \zeta_a (\mathbf{D}_3 \nabla\!\!\!/ f)_a \\ -e_a^\sigma e_a^\tau D_\sigma \Pi_\tau^\lambda (D_\lambda e_3)^\delta D_\delta f &= -\underline{\chi}_{ab} (\nabla\!\!\!/\nabla\!\!\!/ f)_{ab} - (\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\chi})_b \nabla\!\!\!/_b f \\ &\quad -(\mathrm{d}\!\!\!/\mathrm{iv}\,\zeta)\mathbf{D}_3 f - \zeta_a (\mathbf{D}_3 \nabla\!\!\!/ f)_a \\ &\quad -\zeta_a \underline{\chi}_{ab} \nabla\!\!\!/_b f + \zeta_a (\nabla\!\!\!/_a \log\Omega)\mathbf{D}_3 f \\ -e_a^\sigma e_a^\tau e_3^\delta R^\gamma{}_{\tau\delta\sigma} \nabla\!\!\!/_\gamma f &= -\underline{\beta}_a \nabla\!\!\!/_a f \qquad (4.8.16)\end{aligned}$$

Collecting all these results together we obtain the result. To prove the first line of 4.8.4, observe that

$$\not{D}_3 \not{d}\mathrm{iv}\, X = e_a^\mu(\mathbf{D}_3\Pi_\mu^\rho)e_a^\sigma D_\rho X_\sigma + e_a^\rho e_a^\nu(\mathbf{D}_3\Pi_\nu^\sigma)D_\rho X_\sigma + e_a^\rho e_b^\sigma \mathbf{D}_3 D_\rho X_\sigma$$

and, as

$$e_a^\mu(\mathbf{D}_3\Pi_\mu^\rho) = \eta_a e_3^\rho \ , \ e_a^\nu(\mathbf{D}_3\Pi_\nu^\sigma) = \eta_a e_3^\sigma \tag{4.8.17}$$

the result follows.

4.8.2 Proof of Lemma 4.3.5

[Title of subsection modified.]

We start from

$$\mathbf{D}_3(\Omega\mathbf{D}_4)^2 \log\Omega = \Omega\mathbf{D}_4\mathbf{D}_3(\Omega\mathbf{D}_4\log\Omega) + [\mathbf{D}_3, \Omega\mathbf{D}_4](\Omega\mathbf{D}_4\log\Omega) \tag{4.8.18}$$

As $[\mathbf{D}_3, \Omega\mathbf{D}_4]f = (\mathbf{D}_3\Omega)\mathbf{D}_4 f + \Omega[\mathbf{D}_3, \mathbf{D}_4]f$, using 4.8.1,

$$\begin{aligned}
[\mathbf{D}_3, \Omega\mathbf{D}_4]f &= (\mathbf{D}_3\Omega)\mathbf{D}_4 f + \Omega\Big((\mathbf{D}_4\log\Omega)\mathbf{D}_3 f - (\mathbf{D}_3\log\Omega)\mathbf{D}_4 f + 4\zeta\cdot\not\nabla f\Big)\\
&= (\mathbf{D}_4\log\Omega)\Omega\mathbf{D}_3 f + 4\Omega\zeta\cdot\not\nabla f
\end{aligned}$$

and, posing $f = \Omega\mathbf{D}_4\log\Omega$,

$$\begin{aligned}
[\mathbf{D}_3, \Omega\mathbf{D}_4](\Omega\mathbf{D}_4\log\Omega) &= 4\Omega\zeta\cdot\not\nabla(\Omega\mathbf{D}_4\log\Omega) - (\Omega\mathbf{D}_3\log\Omega)\mathbf{D}_4(\Omega\mathbf{D}_4\log\Omega)\\
&\quad +(\Omega\mathbf{D}_4\log\Omega)\mathbf{D}_3(\Omega\mathbf{D}_4\log\Omega) + (\mathbf{D}_3\Omega)\mathbf{D}_4(\Omega\mathbf{D}_4\log\Omega)\\
&= (\Omega\mathbf{D}_4\log\Omega)\mathbf{D}_3(\Omega\mathbf{D}_4\log\Omega) + 4\Omega\zeta\cdot\not\nabla(\Omega\mathbf{D}_4\log\Omega)\ .
\end{aligned}$$

Substituting in 4.8.18,

$$\begin{aligned}
\mathbf{D}_3(\Omega\mathbf{D}_4)^2\log\Omega &= \Omega\mathbf{D}_4\mathbf{D}_3(\Omega\mathbf{D}_4\log\Omega) + (\Omega\mathbf{D}_4\log\Omega)\mathbf{D}_3(\Omega\mathbf{D}_4\log\Omega)\\
&\quad +4\Omega\zeta\cdot\not\nabla(\Omega\mathbf{D}_4\log\Omega)
\end{aligned} \tag{4.8.19}$$

As, see 4.3.58,

$$\mathbf{D}_3(\Omega\mathbf{D}_4\log\Omega) = 2\Omega\zeta\cdot\not\nabla\log\Omega + \Omega(\underline{\eta}\cdot\eta - 2\zeta^2 - \rho)$$

eq. 4.8.19 becomes

$$\begin{aligned}
&\mathbf{D}_3\left((\Omega\mathbf{D}_4)^2\log\Omega\right) = \Omega\mathbf{D}_4\left[2\Omega\zeta\cdot\not\nabla\log\Omega + \Omega(\underline{\eta}\cdot\eta - 2\zeta^2 - \rho)\right]\\
&+(\Omega\mathbf{D}_4\log\Omega)\left[2\Omega\zeta\cdot\not\nabla\log\Omega + \Omega(\underline{\eta}\cdot\eta - 2\zeta^2 - \rho)\right] + 4\Omega\zeta\cdot\not\nabla(\Omega\mathbf{D}_4\log\Omega)
\end{aligned}$$

$$
\begin{aligned}
&= \Omega^2\Big[2\mathbf{D}_4(\zeta\cdot\nabla\!\!\!/\log\Omega)+\mathbf{D}_4(\underline{\eta}\cdot\eta-2\zeta^2-\rho)\Big]\\
&+ 2(\Omega\mathbf{D}_4\log\Omega)\Big[2\Omega\zeta\cdot\nabla\!\!\!/\log\Omega+\Omega(\underline{\eta}\cdot\eta-2\zeta^2-\rho)\Big]+4\Omega\zeta\cdot\nabla\!\!\!/(\Omega\mathbf{D}_4\log\Omega)\\
&= \Big[4\Omega\zeta\cdot\nabla\!\!\!/(\Omega\mathbf{D}_4\log\Omega)+2(\Omega\mathbf{D}_4\log\Omega)\Big(2\Omega\zeta\cdot\nabla\!\!\!/\log\Omega+\Omega(\underline{\eta}\cdot\eta-2\zeta^2-\rho)\Big)\Big]\\
&+ \Big[\Omega^2\Big(2\mathbf{D}_4\zeta\cdot\nabla\!\!\!/\log\Omega+\mathbf{D}_4(\underline{\eta}\cdot\eta-2\zeta^2)\Big)\Big]-\Omega^2\mathbf{D}_4\rho\\
&= [1]+[2]-\Omega^2\mathbf{D}_4\rho
\end{aligned}
\tag{4.8.20}
$$

Using the evolution equation for ζ, see 3.1.45,

$$
\mathbf{D}\!\!\!/_4\zeta=-2\chi\cdot\zeta-\mathbf{D}\!\!\!/_4\nabla\!\!\!/\log\Omega-\beta
$$

and the commutation relation, see 4.8.1,

$$
[\mathbf{D}\!\!\!/_4,\nabla\!\!\!/]\log\Omega=(\zeta+\underline{\eta})\mathbf{D}_4\log\Omega-\chi\cdot\nabla\!\!\!/\log\Omega
$$

it is easy to show that [2] can be rewritten as

$$
[2]=\Omega^2\Big[(6\zeta-2\nabla\!\!\!/\log\Omega)\Omega^{-1}\Big(\nabla\!\!\!/\Omega\mathbf{D}_4\log\Omega+\Omega\chi\cdot(\zeta-\underline{\eta})+\Omega\beta\Big)+2\eta\mathbf{D}_4\nabla\!\!\!/\log\Omega\Big]
$$

so that finally

$$
\mathbf{D}_3\Big((\Omega\mathbf{D}_4)^2\log\Omega\Big)=M-\Omega^2\mathbf{D}_4\rho
$$

and

$$
\begin{aligned}
M \;=\; 2\Omega\Big\{&\Big[2\zeta\cdot\nabla\!\!\!/\Omega\mathbf{D}_4\log\Omega+2(\Omega\mathbf{D}_4\log\Omega)\zeta\cdot\nabla\!\!\!/\log\Omega\\
&+(\Omega\mathbf{D}_4\log\Omega)(\underline{\eta}\cdot\eta-2\zeta^2)+\Omega\eta\cdot\mathbf{D}_4\nabla\!\!\!/\log\Omega\\
&-(-3\zeta+\nabla\!\!\!/\log\Omega)\Big(\nabla\!\!\!/\Omega\mathbf{D}_4\log\Omega-\Omega\chi\cdot(\underline{\eta}-\zeta)\Big)\Big]\\
&+\Big[-\Omega(-3\zeta+\nabla\!\!\!/\log\Omega)\beta-(\Omega\mathbf{D}_4\log\Omega)\rho\Big]\Big\}
\end{aligned}
\tag{4.8.21}
$$

4.8.3 Proof of Lemma 4.4.1

Starting from

$$
\mathbf{D}\!\!\!/_4(\nabla\!\!\!/V)=\nabla\!\!\!/\mathbf{D}\!\!\!/_4V+[\mathbf{D}\!\!\!/_4,\nabla\!\!\!/]V
\tag{4.8.22}
$$

with $V = \nabla\!\!\!/ \mathbf{D}_3 \log\Omega$ and using 4.8.2 to compute $[D\!\!\!\!/_4, \nabla\!\!\!/]V$, we obtain

$$\begin{aligned}
D\!\!\!\!/_4 \nabla\!\!\!/_a V_b &= -\chi_{ac}\nabla\!\!\!/_c V_b + \nabla\!\!\!/_a D\!\!\!\!/_4 V_b + (\nabla\!\!\!/_a \log\Omega) D\!\!\!\!/_4 V_b \\
&\quad + \left\{ -\underline{\eta}_b (\chi\cdot V)_a + \chi_{ab}(\underline{\eta}\cdot V) + e_4^\tau e_a^\rho \left([D_\tau, D_\rho]V_\sigma\right) e_b^\sigma \right\} \\
&= -\chi_{ac}\nabla\!\!\!/_c V_b + \frac{1}{\Omega}\nabla\!\!\!/_a \Omega D\!\!\!\!/_4 V_b \\
&\quad + \left\{ -\underline{\eta}_b (\chi\cdot V)_a + \chi_{ab}(\underline{\eta}\cdot V) + e_4^\tau e_a^\rho \left([D_\tau, D_\rho]V_\sigma\right) e_b^\sigma \right\}
\end{aligned} \tag{4.8.23}$$

Recalling that

$$\begin{aligned}
D\!\!\!\!/_4 V_b &= \nabla\!\!\!/_b \mathbf{D}_4\mathbf{D}_3 \log\Omega + (\nabla\!\!\!/_b \log\Omega)\mathbf{D}_4\mathbf{D}_3\log\Omega - \chi_{bc}V_c \\
&= \frac{1}{\Omega}\nabla\!\!\!/_b \Omega \mathbf{D}_4\mathbf{D}_3\log\Omega - \chi_{bc}V_c
\end{aligned} \tag{4.8.24}$$

and plugging it in the previous equation we obtain

$$\begin{aligned}
D\!\!\!\!/_4 \nabla\!\!\!/_a V_b &= -\chi_{ac}\nabla\!\!\!/_c V_b + \frac{1}{\Omega}\nabla\!\!\!/_a (\nabla\!\!\!/_b \Omega \mathbf{D}_4\mathbf{D}_3\log\Omega - \Omega\chi_{bc}V_c) \\
&\quad + \left\{ -\underline{\eta}_b (\chi\cdot V)_a + \chi_{ab}(\underline{\eta}\cdot V) + e_4^\tau e_a^\rho \left([D_\tau, D_\rho]V_\sigma\right) e_b^\sigma \right\} \\
&= -\chi_{ac}\nabla\!\!\!/_c V_b - \chi_{bc}\nabla\!\!\!/_a V_c + \frac{1}{\Omega}\nabla\!\!\!/_a\nabla\!\!\!/_b \Omega \mathbf{D}_4\mathbf{D}_3\log\Omega \\
&\quad - [(\nabla\!\!\!/_a \log\Omega)\chi_{bc}V_c + (\nabla\!\!\!/_a \chi_{bc})V_c] \\
&\quad + \left\{ -\underline{\eta}_b (\chi\cdot V)_a + \chi_{ab}(\underline{\eta}\cdot V) + e_4^\tau e_a^\rho \left([D_\tau, D_\rho]V_\sigma\right) e_b^\sigma \right\}
\end{aligned} \tag{4.8.25}$$

and finally

$$\begin{aligned}
D\!\!\!\!/_4 \nabla\!\!\!/_a V_b + \mathrm{tr}\chi \nabla\!\!\!/_a V_b &= \frac{1}{\Omega}\nabla\!\!\!/_a\nabla\!\!\!/_b \Omega \mathbf{D}_4\mathbf{D}_3\log\Omega - (\hat{\chi}_{ac}\nabla\!\!\!/_c V_b + \hat{\chi}_{bc}\nabla\!\!\!/_a V_c) \\
&\quad - \Big[(\nabla\!\!\!/_a \log\Omega)\chi_{bc}V_c + (\nabla\!\!\!/_a \chi_{bc})V_c + \underline{\eta}_b \chi_{ac}V_c \\
&\quad - \chi_{ab}(\underline{\eta}_c V_c) - e_4^\tau e_a^\rho \left([D_\tau, D_\rho]V_\sigma\right) e_b^\sigma \Big]
\end{aligned} \tag{4.8.26}$$

The commutation relation $[\mathbf{D}_3, \mathbf{d\!\!\!/iv}]\underline{\beta}$

From eq. 4.8.7, with the obvious substitutions, $a = b$, $V = \underline{\beta}$, $4 \to 3$, we obtain

$$D\!\!\!\!/_3 \mathrm{d\!\!\!/iv}\,\underline{\beta} = e_a^\mu(\mathbf{D}_3\Pi_\mu^\rho)e_a^\sigma D_\rho \underline{\beta}_\sigma + e_a^\rho e_a^\nu(\mathbf{D}_3\Pi_\nu^\sigma)D_\rho\underline{\beta}_\sigma + e_a^\rho e_b^\sigma \mathbf{D}_3 D_\rho \underline{\beta}_\sigma$$

It is easy to show that

$$e^\mu_a(\mathbf{D}_3\Pi^\rho_\mu) = \eta_a e^\rho_3\ ,\ \ e^\nu_a(\mathbf{D}_3\Pi^\sigma_\nu) = \eta_a e^\sigma_3 \tag{4.8.27}$$

therefore

$$\not{D}_3 \not{d}\text{iv}\,\underline{\beta} = \eta_a \not{D}_3 \underline{\beta}_a - \eta_a \underline{\chi}_{ac}\underline{\beta}_c + e^\rho_a e^\sigma_a e^\tau_3 [D_\tau, D_\rho]\underline{\beta}_\sigma + \not{d}\text{iv}\,\mathbf{D}_3\underline{\beta} \tag{4.8.28}$$

4.8.4 Proof of Proposition 4.6.2

To prove the first line of Proposition 4.6.2 we observe, see 4.6.11, that

$$^{(i)}H_{ab} = \frac{1}{2}\left((\not{\nabla}_a {}^{(i)}O)_b + (\not{\nabla}_b {}^{(i)}O)_a\right)\ .$$

therefore, using the evolution equation for ${}^{(i)}O$, we obtain immediately,

$$\begin{aligned}
\frac{d}{d\underline{u}}{}^{(i)}H_{ab} &= \frac{\Omega}{2}\Big[\Big(\hat{\chi}_{bc}(\not{\nabla}_a {}^{(i)}O)_c + \hat{\chi}_{ac}(\not{\nabla}_b {}^{(i)}O)_c\Big) - \Big(\hat{\chi}_{ac}(\not{\nabla}_c {}^{(i)}O)_b + \hat{\chi}_{bc}(\not{\nabla}_c {}^{(i)}O)_a\Big) \\
&\quad + {}^{(i)}O_c\left((\not{\nabla}_a\chi)_{cb} + (\not{\nabla}_b\chi)_{ca}\right) + {}^{(i)}O_c\left(\chi_{cb}\zeta_a + \chi_{ca}\zeta_b\right) + 2\chi_{ab}(\underline{\eta}_c {}^{(i)}O_c) \\
&\quad + {}^{(i)}O_c\left(R_{4acb} + R_{4bca}\right)\Big]
\end{aligned} \tag{4.8.29}$$

A simple computation shows the explicit expression of $(R_{4acb} + R_{4bca})$,

$$R_{4acb} + R_{4bca} = -(\not{\nabla}_a\chi)_{cb} - (\not{\nabla}_b\chi)_{ca} + 2(\not{\nabla}_c\chi)_{ab} + 2\zeta_c\chi_{ba} - (\chi_{cb}\zeta_a + \chi_{ca}\zeta_b)$$

which, substituted in 4.8.29, gives immediately

$$\begin{aligned}
\frac{d}{d\underline{u}}{}^{(i)}H_{ab} &= -\Omega\left(\hat{\chi}_{ac}{}^{(i)}H_{cb} + \hat{\chi}_{bc}{}^{(i)}H_{ca}\right) + \Omega\chi_{ab}(\not{\nabla}_c \log\Omega){}^{(i)}O_c \\
&\quad + \Omega\left(\hat{\chi}_{bc}\not{\nabla}_a {}^{(i)}O_c + \hat{\chi}_{ac}\not{\nabla}_b {}^{(i)}O_c\right) + \Omega{}^{(i)}O_c(\not{\nabla}_c\chi)_{ab}
\end{aligned}$$

To prove the second line of Proposition 4.6.2 we denote $W = \mathcal{L}_O g$ and write, omitting the indices ${}^{(i)}$,

$$\begin{aligned}
\frac{d}{d\underline{u}}W(e_a,e_3) &= \Omega\mathcal{L}_{e_4}W(e_a,e_3) = \Omega\left[(\mathcal{L}_{e_4}W)(e_a,e_3) + W([e_4,e_a],e_3) + W(e_a,[e_4,e_3])\right] \\
&= \Omega\left[(\mathcal{L}_{e_4}W)(e_a,e_3) - \chi_{ab}W(e_b,e_3) + (\not{\nabla}_a\log\Omega)W(e_4,e_3)\right. \\
&\quad \left. -(\mathbf{D}_4\log\Omega)W(e_a,e_3) - 4\zeta_b W(e_a,e_b)\right]
\end{aligned} \tag{4.8.30}$$

which can be rewritten as

$$\frac{d}{d\underline{u}}Z_a = \Omega\left[\frac{1}{4}(\mathcal{L}_{e_4}W)(e_a,e_3) - \chi_{ab}Z_b + (\not{\nabla}_a\log\Omega)F - (\mathbf{D}_4\log\Omega)Z_a - 2\zeta_b H_{ab}\right] \tag{4.8.31}$$

The first term in the right hand side of 4.8.31 can be rewritten as [51]

$$
\begin{aligned}
\mathcal{L}_{e_4} W &= \mathcal{L}_{e_4}\mathcal{L}_O g = \mathcal{L}_O\mathcal{L}_{e_4} g + \mathcal{L}_{[e_4,O]} g \\
&= \mathcal{L}_O\mathcal{L}_{e_4} g - \mathcal{L}_{[O,e_4]} g = \mathcal{L}_O\mathcal{L}_{e_4} g - \mathcal{L}_{Fe_4} g \\
&= \mathcal{L}_O\mathcal{L}_{e_4} g - F(\mathcal{L}_{e_4} g)(e_a,e_3) - 2\nabla\!\!\!/_a F \\
&= \mathcal{L}_O\mathcal{L}_{e_4} g - 2(\zeta+\eta)_a F - 2\nabla\!\!\!/_a F \qquad (4.8.32)
\end{aligned}
$$

Plugging 4.8.32 in 4.8.31 we obtain

$$
\begin{aligned}
\frac{d}{d\underline{u}} Z_a + \Omega(\mathbf{D}_4 \log\Omega) Z_a + \Omega\chi_{ab} Z_b &= \frac{\Omega}{4}\left[(\mathcal{L}_O\mathcal{L}_{e_4} g)(e_a,e_3) - 2(\zeta+\eta)_a F + 2\nabla\!\!\!/_a F\right] \\
&+ \Omega\left[(\nabla\!\!\!/_a \log\Omega) F - 2\zeta_b H_{ab}\right] \qquad (4.8.33)
\end{aligned}
$$

The first term in the right hand side of 4.8.33 can be written as [52]

$$
(\mathcal{L}_O\mathcal{L}_{e_4} g)(e_a,e_3) = 2\left(\mathcal{L}_O(\zeta+\eta)\right)_a - (\mathcal{L}_{e_4} g)_{a\nu}[O,e_3]^\nu \qquad (4.8.34)
$$

and as, see 4.6.12,

$$
[O,e_3] = -4Z_b e_b - O_c(\nabla\!\!\!/_c \log\Omega) e_3
$$

we obtain

$$
(\mathcal{L}_O\mathcal{L}_{e_4} g)(e_a,e_3) = 2\left((\mathcal{L}_O - F)(\zeta+\eta)\right)_a + 8\chi_{ab} Z_b \qquad (4.8.35)
$$

Inserting this last expression in 4.8.33 we obtain the expected result

$$
\begin{aligned}
\frac{d}{d\underline{u}}(\Omega^{(i)} Z_a) &= \Omega\chi_{ab}(\Omega^{(i)} Z_b) + \Omega^2\left[\frac{1}{2}(\mathcal{L}_{{}^{(i)}O} - {}^{(i)}F)(\zeta+\eta))_a\right. \\
&+ \left.\frac{1}{2}\nabla\!\!\!/_a {}^{(i)}F - 2\zeta_b {}^{(i)}H_{ab} - \frac{1}{2}(\eta-\underline{\eta})_a {}^{(i)}F)\right] \qquad (4.8.36)
\end{aligned}
$$

---

[51]Using the equation

$$
([\mathcal{L}_{e_4}, \mathcal{L}_{{}^{(i)}O}] g)_{\mu\nu} = (\mathcal{L}_{-({}^{(i)}Fe_4)} g)_{\mu\nu} = -{}^{(i)}F(\mathcal{L}_{e_4} g)_{\mu\nu} - (D_\mu {}^{(i)}F) g_{4\nu} - (D_\nu {}^{(i)}F) g_{\mu 4}
$$

which follows from the relation $\mathcal{L}_X\mathcal{L}_Y U - \mathcal{L}_Y\mathcal{L}_X U = \mathcal{L}_{[X,Y]} U$ and the commutation relation $[{}^{(i)}O, e_4] = {}^{(i)}F e_4$.

[52]Writing

$$
\begin{aligned}
(\mathcal{L}_O\mathcal{L}_{e_4} g)(e_a,e_3) &= ((\mathcal{L}_O\mathcal{L}_{e_4} g)_{\mu\nu} e_3^\nu \Pi^\mu_\sigma)\, e_a^\sigma = [\mathcal{L}_O((\mathcal{L}_{e_4} g)_{\mu\nu} e_3^\nu \Pi^\mu_\sigma)]\, e_a^\sigma \\
&- [(\mathcal{L}_{e_4} g)_{\mu\nu}(\mathcal{L}_O\Pi)^\mu_\sigma e_3^\nu + (\mathcal{L}_{e_4} g)_{\mu\nu}\Pi^\mu_\sigma(\mathcal{L}_O e_3)^\nu]\, e_a^\sigma\ .
\end{aligned}
$$

4.8.5 Proof of Proposition 4.6.3

We start from the equality

$$\frac{d}{d\underline{u}}(\nabla\!\!\!/_c H_{ab}) = \Omega\left[(\nabla\!\!\!/ D\!\!\!\!/_4 H)_{cab} + ([D\!\!\!\!/_4, \nabla\!\!\!/]H)_{cab}\right] \tag{4.8.37}$$

choosing the null frame in such a way that $D\!\!\!\!/_4 e_a = 0$. A standard computation gives

$$\begin{aligned}([D\!\!\!\!/_4, \nabla\!\!\!/]H)_{cab} &= [\cdots]^{\mu\nu\rho}_{cab} D_\mu H_{\nu\rho} - (\mathbf{D}_c e_4)^\sigma (D_\sigma H_{\nu\rho}) e^\nu_a e^\rho_b \\ &\quad + e^\mu_c e^\nu_a e^\rho_b {e_4}^\sigma (D_\sigma D_\mu - D_\mu D_\sigma) H_{\nu\rho}\end{aligned} \tag{4.8.38}$$

where

$$[\cdots]^{\mu\nu\rho}_{cab} = (\mathbf{D}_4\Pi)^\mu_c e^\nu_a e^\rho_b + e^\mu_c (\mathbf{D}_4\Pi)^\nu_a e^\rho_b + e^\mu_c e^\nu_a (\mathbf{D}_4\Pi)^\rho_b$$

As

$$(\mathbf{D}_4\Pi)^\mu_c = \underline{\eta}_c e^\mu_4$$

it follows immediately that

$$[\cdots]^{\mu\nu\rho}_{cab} D_\mu H_{\nu\rho} = \underline{\eta}_c \mathbf{D}_4(H_{ab}) - \underline{\eta}_a \chi_{cd} H_{db} - \underline{\eta}_b \chi_{cd} H_{ad}\ .$$

Moreover

$$e^\mu_c e^\nu_a e^\rho_b {e_4}^\sigma (D_\sigma D_\mu - D_\mu D_\sigma) H_{\nu\rho} = (R_{ad4c} H_{db} + R_{bd4c} H_{da})$$

and

$$-(\mathbf{D}_c e_4)^\sigma (D_\sigma H)_{ab} = -\chi_{cd} \nabla\!\!\!/_d H_{ab} + \zeta_c \mathbf{D}_4(H_{ab})\ .$$

Therefore the left hand side of 4.8.38 becomes

$$\begin{aligned}([D\!\!\!\!/_4, \nabla\!\!\!/]H)_{cab} &= -\chi_{cd}(\nabla\!\!\!/_d H_{ab}) + (\nabla\!\!\!/ \log\Omega)\mathbf{D}_4(H_{ab}) \\ &+ \left[(R_{ad4c} - \underline{\eta}_a \chi_{dc}) H_{db} + (R_{bd4c} - \underline{\eta}_b \chi_{cd}) H_{da}\right]\end{aligned} \tag{4.8.39}$$

which substituted in eq. 4.8.37 gives

$$\begin{aligned}\frac{d}{d\underline{u}}(\nabla\!\!\!/_c H_{ab}) + \Omega\chi_{cd}(\nabla\!\!\!/_d H_{ab}) &= \Omega(\nabla\!\!\!/ D\!\!\!\!/_4 H)_{cab} + \Omega(\nabla\!\!\!/_c \log\Omega)(\mathbf{D}_4 H)_{ab} \\ &+ \Omega\left[(R_{ad4c} - \underline{\eta}_a \chi_{dc}) H_{db} + (R_{bd4c} - \underline{\eta}_b \chi_{cd}) H_{da}\right]\end{aligned} \tag{4.8.40}$$

Starting from [53]

$$\mathbf{D}_4 H_{\mu\nu} = -(\hat{\chi}_{\mu d} H_{d\nu} + \hat{\chi}_{\nu d} H_{d\mu}) + (\mathcal{L}_O - F)\hat{\chi}_{\mu\nu} + \frac{1}{2}\Pi_{\mu\nu}(\mathcal{L}_O - F)tr\chi \ ,$$

we compute explicitely

$$\begin{aligned}(\nabla\!\!\!\!/_c \mathbf{D}\!\!\!\!/_4 H)_{ab} &= -\left((\nabla\!\!\!\!/_c \hat{\chi}_{ad})H_{db} + (\nabla\!\!\!\!/_c \hat{\chi}_{bd})H_{da}\right) - (\hat{\chi}_{ad}\nabla\!\!\!\!/_c H_{db} + \hat{\chi}_{bd}\nabla\!\!\!\!/_c H_{da}) \\ &\quad +(\nabla\!\!\!\!/_c(\mathcal{L}_O - F)\hat{\chi})_{ab} + \frac{1}{2}\Pi_{\mu\nu}(\nabla\!\!\!\!/_c(\mathcal{L}_O - F)tr\chi) \qquad (4.8.41)\end{aligned}$$

inserting this last expression in eq. 4.8.40 we obtain

$$\begin{aligned}\frac{d}{d\underline{u}}(\nabla\!\!\!\!/_c H_{ab}) + \frac{1}{2}\Omega\mathrm{tr}\chi(\nabla\!\!\!\!/_c H_{ab}) &= -\Omega\left(\hat{\chi}_{ad}\nabla\!\!\!\!/_c H_{db} + \hat{\chi}_{bd}\nabla\!\!\!\!/_c H_{da} + \hat{\chi}_{cd}\nabla\!\!\!\!/_d H_{ab}\right) \\ &\quad +\Omega\Big\{(\nabla\!\!\!\!/ \log\Omega)_c \mathbf{D}_4(H_{ab}) \\ &\quad + \Big[(R_{ad4c} - \underline{\eta}_a\chi_{dc})H_{db} + (R_{bd4c} - \underline{\eta}_b\chi_{cd})H_{da}\Big] \\ &\quad - \Big((\nabla\!\!\!\!/_c \hat{\chi}_{ad})H_{db} + (\nabla\!\!\!\!/_c \hat{\chi}_{bd})H_{da}\Big) \\ &\quad +(\nabla\!\!\!\!/(\mathcal{L}_O - F)\hat{\chi})_{ab} + \frac{1}{2}\delta_{ab}(\nabla\!\!\!\!/(\mathcal{L}_O - F)tr\chi)\Big\} \qquad (4.8.42)\end{aligned}$$

Which we rewrite in a more formal way as

$$\begin{aligned}\frac{d}{d\underline{u}}(\nabla\!\!\!\!/_c H_{ab}) + \frac{1}{2}\Omega\mathrm{tr}\chi(\nabla\!\!\!\!/_c H_{ab}) &= -\Omega\hat{\chi}_{ad}(\nabla\!\!\!\!/ H)_{cdb} + \Omega\Big\{(\nabla\!\!\!\!/ \log\Omega)_c \mathbf{D}_4(H_{ab}) \\ &\quad + \Big[(R_{ad4c} - \underline{\eta}_a\chi_{dc})H_{db} + (R_{bd4c} - \underline{\eta}_b\chi_{cd})H_{da}\Big] \\ &\quad - \Big((\nabla\!\!\!\!/\hat{\chi})_{cad}H_{db} + (\nabla\!\!\!\!/\hat{\chi})_{cbd}H_{da}\Big) \\ &\quad - \Big(\hat{\chi}_{bd}(\nabla\!\!\!\!/ H)_{cda} + \hat{\chi}_{cd}(\nabla\!\!\!\!/ H)_{dab}\Big) \\ &\quad +(\mathcal{L}_O\nabla\!\!\!\!/\hat{\chi})_{cab} + \frac{1}{2}\delta_{ab}\nabla\!\!\!\!/_c(\mathcal{L}_O tr\chi) - (\nabla\!\!\!\!/ F\chi)_{cab}\Big\} \\ &\quad +\Omega([\nabla\!\!\!\!/, \mathcal{L}_O]\hat{\chi})_{cab} \qquad (4.8.43)\end{aligned}$$

[53]Starting from eq. 4.6.13,

$$(\mathbf{D}_4 H)_{ab} = -(\hat{\chi}_{ad}H_{db} + \hat{\chi}_{bd}H_{da}) + \left((\mathcal{L}_O - F)\hat{\chi}\right)_{ab} + \frac{1}{2}\delta_{ab}(\mathcal{L}_O - F)tr\chi \ .$$

As the following commutation relation holds

$$([\nabla\!\!\!/,\mathcal{L}_O]\hat{\chi})_{cab} = \hat{\chi}_{bd}\left(\nabla\!\!\!/_a H_{dc}+\nabla\!\!\!/_c H_{da}-\nabla\!\!\!/_d H_{ac}\right) + \hat{\chi}_{ad}\left(\nabla\!\!\!/_b H_{dc}+\nabla\!\!\!/_c H_{db}-\nabla\!\!\!/_d H_{bc}\right) \quad (4.8.44)$$

substituting in the previous expression we obtain the result of Proposition 4.6.3

$$\begin{aligned}\frac{d}{d\underline{u}}(\nabla\!\!\!/_c H_{ab})+\frac{1}{2}\Omega\mathrm{tr}\chi(\nabla\!\!\!/_c H_{ab}) &= -\Omega\hat{\chi}_{ad}(\nabla\!\!\!/ H)_{cdb}\\ &+\hat{\chi}_{ad}\left[(\nabla\!\!\!/ H)_{bdc}-(\nabla\!\!\!/ H)_{dbc}\right]+\hat{\chi}_{bd}\left[(\nabla\!\!\!/ H)_{adc}-(\nabla\!\!\!/ H)_{dac}\right]\\ &+\Omega\Big\{(\nabla\!\!\!/\log\Omega)_c\mathbf{D}_4(H_{ab})\\ &+\Big[(R_{ad4c}-\underline{\eta}_a\chi_{dc})H_{db}+(R_{bd4c}-\underline{\eta}_b\chi_{cd})H_{da}\Big]\\ &-\Big[(\nabla\!\!\!/\hat{\chi})_{cad}H_{db}+(\nabla\!\!\!/\hat{\chi})_{cbd}H_{da}\Big]\\ &+(\mathcal{L}_O\nabla\!\!\!/\hat{\chi})_{cab}+\frac{1}{2}\delta_{ab}\nabla\!\!\!/_c(\mathcal{L}_O tr\chi)-(\nabla\!\!\!/ F\chi)_{cab}\Big\} \quad (4.8.45)\end{aligned}$$

The proofs of the second and third line equations of Proposition 4.6.3 are similar and we do not report them here.

4.8.6 Proof of the Oscillation Lemma

We repeat here the statement of the lemma.

Lemma 4.1.6 (Oscillation Lemma) *Consider a space time region $\mathcal{K}$ with the "canonical double null foliation" generated by $u(p),\underline{u}(p)$. Consider also an initial layer region ${\mathcal{K}'}_{\delta_0}$, of height δ_0, with the "initial layer foliation" generated by $u'(p),\underline{u}(p)$. We make the following assumptions:*

- *On the surface $S'_*=\Sigma'_{\delta_0}\cap\underline{C}_*=S'(2\delta_0-\nu_*,\nu_*)$*

$$\left(\sup_{(p,p')\in S'_*}|u(p)-u(p')|\right)\leq\epsilon_0 \quad (4.8.46)$$

Also,

[The assumptions on $g(l,L')$ have been changed. These ones are the right ones.]

$$|r^2\tau_-^{\frac{1}{2}}\eta|\leq\epsilon_0\ ,\ |r'^2\tau'_-\mathbf{g}(L',L)|\leq\epsilon_0\ ,\ |r'^3\tau'_-\nabla\!\!\!/\mathbf{g}(L',L)|\leq\epsilon_0 \quad (4.8.47)$$

- *On the initial hypersurface Σ_0,*

$$|r'^{\frac{5}{2}}\eta'|\leq\epsilon_0 \quad (4.8.48)$$

[In the next item "on $\mathcal{K}$" has been substituted with "on $\mathcal{K}/\mathcal{K}'_{\delta_0}$".]

- *On* $\mathcal{K}/\mathcal{K}'_{\delta_0}$,

$$\mathcal{O}^\infty_{[1]} + \underline{\mathcal{O}}^\infty_{[1]} \leq \epsilon_0 \tag{4.8.49}$$

[The conditions on $\nabla\!\!\!/\mathbf{D}_{e_4}\log\Omega$ and $\nabla\!\!\!/\mathbf{D}_{e_3}\log\Omega$ on $\mathcal{K}/\mathcal{K}'_{\delta_0}$ have been eliminated as we do not use anymore on $\mathcal{K}/\mathcal{K}'_{\delta_0}$ the stronger estimates for $\eta, \underline{\eta}, \eta', \underline{\eta}'$, although they are , of course, true. We use only the stronger estimates for $g(L, L')$.]

- *On the initial layer* $\mathcal{K}'_{\delta_0}$,

$$\mathcal{O}'^\infty_{[1]} + \underline{\mathcal{O}}'^\infty_{[1]} \leq \epsilon_0 \tag{4.8.50}$$

Then,

$$Osc(u)(\Sigma'_{\delta_0}) \equiv \sup_{\nu\in[\nu_0,\nu_*]}\left(\sup_{(p,p')\in S'(2\delta_0-\nu,\nu)}|u(p)-u(p')|\right) \leq c\epsilon_0 \tag{4.8.51}$$

Remarks:

[The previous remark 2 has been eliminated. The third remark is now the second.]

1) The norms appearing in 4.8.47, 4.8.48, 4.8.49, and 4.8.50 are pointwise.
2) The assumptions 4.8.47 are verified in view of the canonicity of the foliation on the last slice $\underline{C}_*$, see Proposition 7.4.1 and Lemma 7.7.2.
3) The assumptions 4.8.48 are verified in view of the canonicity of the foliation on the initial slice Σ_0 and are used in Lemma 4.8.2.

Proof: The proof requires a bootstrap mechanism which has been used many times in this chapter . We assume that the oscillation of u is bounded by a small quantity, Γ'_0,

$$Osc(u)(\Sigma'_{\delta_0}) \leq \Gamma'_0 \tag{4.8.52}$$

and prove the better inequality

$$Osc(u)(\Sigma'_{\delta_0}) \leq c\epsilon_0 \leq \frac{\Gamma'_0}{2} \tag{4.8.53}$$

Denoting $\Sigma'_{\delta_0}[r'_*, r'_1]$ the portion of Σ'_{δ_0} where 4.8.52 is satisfied, the inequality 4.8.53 allows us to conclude that $\Sigma'_{\delta_0}[r'_*, r'_1]$ coincides with the whole Σ'_{δ_0} and therefore 4.8.52 holds everywhere.

Let $\{e'_4, e'_3, e'_a\}$ be the normalized null frame adapted to the *initial layer foliation* with

$$e'_4 = 2\Omega' L' \ ,\ e'_3 = 2\Omega' \underline{L} \tag{4.8.54}$$

where Ω' is given by

$$(2\Omega'^2)^{-1} = -\mathbf{g}(L', \underline{L}) = -(g^{\rho\sigma}\partial_\rho u' \partial_\sigma \underline{u})$$

as in definition 3.1.12, see also 3.3.16. Let us define

$$\tilde{N}' \equiv \frac{1}{2}(e'_4 - e'_3)\ ,\ T' \equiv \frac{1}{2}(e'_4 + e'_3) \tag{4.8.55}$$

$\tilde{N}'$ is a unit vector field tangent to Σ'_{δ_0}, orthogonal to the leaves $S'(2\delta_0-\nu,\nu)$, contained in Σ'_{δ_0}, and T' is the unit vector field normal to Σ'_{δ_0}. Introducing the functions

$$t' = \frac{1}{2}(\underline{u} + u')\ ,\ r' = \frac{1}{2}(\underline{u} - u') \tag{4.8.56}$$

we observe that, see Proposition 3.3.1,

$$T' = \frac{1}{2\Omega'}\frac{\partial}{\partial t'}\ ,\ \tilde{N}' = \frac{1}{2\Omega'}\frac{\partial}{\partial r'} \tag{4.8.57}$$

Therefore, denoting $r'_* = r'|_{\underline{C}_*\cap\Sigma'_{\delta_0}}$ and introducing [54] the angular coordinates $\phi = \{\phi^1, \phi^2\}$ on Σ'_{δ_0},

[In 4.8.58 the term $\frac{\partial u}{\partial r'}(r'',\phi)$ has been substituted by the term $\frac{\partial u}{\partial r'}(\gamma(r''))$.]

$$\begin{aligned} u(r',\phi) &= u(r'_*,\phi) - \int_{r'}^{r'_*}\frac{\partial u}{\partial r'}(\gamma(r'',\phi)) = u(r'_*,\phi) - \int_{r'}^{r'_*} 2\Omega'\tilde{N}'(u)(r'',\phi) \\ &= u(r'_*,\phi) - \int_{r'}^{r'_*}\Omega'\left(e'_4(u) - e'_3(u)\right) \end{aligned} \tag{4.8.58}$$

where $\gamma(r'',\phi)$ is the integral curve starting from $p=(r'_*,\phi) \in S'(2\delta_0-\nu_*,\nu_*)$, with tangent vector $\tilde{N}'$.
To estimate the right hand side we express the null frame $\{e'_4, e'_3, e'_a\}$ adapted to the *initial layer foliation* in term of the null frame relative to the *double null canonical foliation*, $\{e_4, e_3, e_a\}$ where

$$e_4 = \hat{N} = 2\Omega L\ ,\ e_3 = \underline{\hat{N}} = 2\Omega\underline{L} \tag{4.8.59}$$

It is easy to show, imposing that both frames are null frames, that

$$\begin{aligned} e'_4 &= \frac{\Omega}{\Omega'}e_4 + \Omega'\Omega(-2\mathbf{g}(L',L))e_3 - 2\Omega(-2\mathbf{g}(L',L))^{\frac{1}{2}}\hat{\sigma}_a e_a \\ e'_3 &= \frac{\Omega'}{\Omega}e_3 \\ e'_a &= e_a - \Omega'(-2\mathbf{g}(L',L))^{\frac{1}{2}}\hat{\sigma}_a e_3 \end{aligned} \tag{4.8.60}$$

[54] The angular coordinates ϕ are introduced in the standard way starting from the surface $S'_* = \underline{C}_* \cap \Sigma'_{\delta_0}$ and moving along the radial curves with tangent vector field $\tilde{N}'$.

and

$$\begin{aligned} e_4 &= \frac{\Omega'}{\Omega}e'_4 + \Omega\Omega'(-2\mathbf{g}(L',L))e'_3 + 2\Omega'(-2\mathbf{g}(L',L))^{\frac{1}{2}}\hat{\sigma}_a e'_a \\ e_3 &= \frac{\Omega}{\Omega'}e'_3 \\ e_a &= e'_a + \Omega(-2\mathbf{g}(L',L))^{\frac{1}{2}}\hat{\sigma}_a e'_3 \end{aligned} \tag{4.8.61}$$

where the vectors $\hat{\sigma}_a$ satisfy $|\hat{\sigma}|^2 = \sum_a \hat{\sigma}_a^2 = 1$. Recall also that

$$e_4(u) = \Omega^{-1}N(u) = 0\ ,\ e_a(u) = 0\ ,\ e_3(u) = \Omega^{-1}\underline{N}(u) = \Omega^{-1} \tag{4.8.62}$$

where N is the outgoing equivariant vector field relative to the *double null canonical foliation.* Therefore the last line of 4.8.58 can be rewritten as

$$u(r',\phi) = u(r'_*,\phi) - \int_{r'}^{r'_*}\Omega'\left(e'_4(u) - \frac{\Omega'}{\Omega^2}\right)(r'',\phi) \tag{4.8.63}$$

[In the second line of 4.8.64 $\int_\phi^{\phi'}\frac{\partial}{\partial\phi''}F(\phi'')$ is modified in $\int_{\gamma(\phi,\phi')}\frac{d}{ds}F(\sigma(s))$ where $\sigma(s)$ is a curve on S^2 from ϕ to ϕ'.]

The oscillation of $u(r',\phi)$ on the surface $S'(2\delta_0-\nu,\nu)$ satisfies

$$\begin{aligned} &\sup_{p,p'\in S'(2\delta_0-\nu,\nu)}|u(p)-u(p')| = \sup_{\phi,\phi'\in S^2}|u(r'(\nu),\phi)-u(r'(\nu),\phi')| \\ &\le \sup_{\phi,\phi'\in S^2}\left[|u(r'_*,\phi)-u(r'_*,\phi')| + \int_{s(\phi)}^{s(\phi')}\frac{d}{ds}\left(\int_{r'}^{r'_*}\Omega'\left(e'_4(u)-\frac{\Omega'}{\Omega^2}\right)(r'',\sigma(s))\right)\right] \\ &\le \sup_{\phi,\phi'\in S^2}|u(r'_*,\phi)-u(r'_*,\phi')| + c\left(\sup_{(a,\phi\in S^2)}\left|\int_{r'}^{r'_*}\frac{\partial}{\partial\phi^a}\Omega'\left(e'_4(u)-\frac{\Omega'}{\Omega^2}\right)(r'',\phi)\right|\right) \\ &\le \epsilon_0 + c\left(\sup_{(a,\phi\in S^2)}\left|\int_{r'}^{r'_*}r''\nabla\!\!\!\!/'_a\left[\Omega'\left(e'_4(u)-\frac{\Omega'}{\Omega^2}\right)(r'',\phi)\right]dr''\right|\right) \end{aligned} \tag{4.8.64}$$

where in the first inequality $\sigma(s)$ is a regular curve on S^2 from ϕ to ϕ'. We are left with estimating the integrals on the right hand side of 4.8.64,

$$\int_{r'}^{r'_*}r''\nabla\!\!\!\!/'_a\left(\Omega' e'_4(u)\right) - \int_{r'}^{r'_*}r''\nabla\!\!\!\!/'_a\left(\frac{\Omega'^2}{\Omega^2}\right) \equiv \textbf{(I)}+\textbf{(II)} \tag{4.8.65}$$

which in the next lemma are estimated in terms of the connection coefficients relative to to the canonical and the initial layer foliations and of the quantities $g(L,L')$ and $\nabla\!\!\!\!/' g(L,L')$.

Lemma 4.8.1 *The integrals in 4.8.65 satisfy the following inequalities*

$$\textbf{(I)} = \int_{r'}^{r'_*}r''\nabla\!\!\!\!/'_a(\Omega' e'_4(u)) \le 2\int_{r'}^{r'_*}r''|\Omega'^2(\eta'+\underline{\eta}')||\mathbf{g}(L',L)| + 2\int_{r'}^{r'_*}r''\Omega'^2|\nabla\!\!\!\!/'_a\mathbf{g}(L',L)|$$

$$\textbf{(II)} = \int_{r'}^{r'_*}r''\nabla\!\!\!\!/'_a\frac{\Omega'^2}{\Omega^2} \le \int_{r'}^{r'_*}\frac{\Omega'^2}{\Omega^2}r''(-2\mathbf{g}(L',L))^{\frac{1}{2}}\left(\Omega|\underline{\chi}'| + |\eta'-\eta| + 4|\Omega'\underline{\omega} + \Omega\underline{\omega}'|\right) \tag{4.8.66}$$

Proof: To estimate **(I)** observe that

$$\mathbf{(I)} = \int_{r'}^{r'_*} r'' \nabla\!\!\!/'_a (\Omega' e'_4(u)) = \int_{r'}^{r'_*} r'' \nabla\!\!\!/'_a \Big[\Omega' \Big(\frac{\Omega}{\Omega'} e_4(u) + \Omega'\Omega(-2\mathbf{g}(L',L)) e_3(u) - 2\Omega(-2\mathbf{g}(L',L))^{\frac12} \hat{\sigma}_a e_a(u)\Big)\Big] = \int_{r'}^{r'_*} r'' \nabla\!\!\!/'_a \Big(\Omega'^2(-2\mathbf{g}(L',L))\Big) \tag{4.8.67}$$

where the last equality follows from 4.8.62. Finally

$$\begin{aligned}\int_{r'}^{r'_*} r'' \nabla\!\!\!/'_a \Big(\Omega'^2(-2\mathbf{g}(L',L))\Big) &= 2\int_{r'}^{r'_*} r''\Omega'^2 \nabla\!\!\!/'_a \log\Omega'(-2\mathbf{g}(L',L)) - 2\int_{r'}^{r'_*} r''\Omega'^2 (\nabla\!\!\!/'_a \mathbf{g}(L',L))\\ &= \int_{r'}^{r'_*} r''\Omega'^2(\eta' + \underline{\eta}')(-2\mathbf{g}(L',L)) - 2\int_{r'}^{r'_*} r''\Omega'^2(\nabla\!\!\!/'_a \mathbf{g}(L',L))\\ &\le 2\int_{r'}^{r'_*} r''|\Omega'^2(\eta' + \underline{\eta}')||\mathbf{g}(L',L)| + 2\int_{r'}^{r'_*} r''\Omega'^2|\nabla\!\!\!/'_a \mathbf{g}(L',L)| \end{aligned} \tag{4.8.68}$$

To estimate **(II)** we write

$$\begin{aligned}&\int_{r'}^{r'_*} r'' \nabla\!\!\!/'_a \frac{\Omega'^2}{\Omega^2} = 2\int_{r'}^{r'_*} r'' \left(\frac{\Omega'^2}{\Omega^2}\nabla\!\!\!/'_a \log\Omega' - \frac{\Omega'^2}{\Omega^2}\nabla\!\!\!/'_a \log\Omega\right)\\ &= 2\int_{r'}^{r'_*} r'' \left(\frac{\Omega'^2}{\Omega^2}\frac12(\eta' + \underline{\eta}')_a - \frac{\Omega'^2}{\Omega^2}\Big(\nabla\!\!\!/_a \log\Omega - \Omega'(-2\mathbf{g}(L',L))^{\frac12}\hat{\sigma}_a \mathbf{D}_{e_3}\log\Omega\Big)\right)\\ &= \int_{r'}^{r'_*} \frac{\Omega'^2}{\Omega^2} r'' \Big(\Big[(\eta' + \underline{\eta}')_a - (\eta + \underline{\eta})_a\Big] - 4\Omega'(-2\mathbf{g}(L',L))^{\frac12}\hat{\sigma}_a \underline{\omega}\Big) \end{aligned} \tag{4.8.69}$$

We write the term $\Big[(\eta' + \underline{\eta}')_a - (\eta + \underline{\eta})_a\Big]$, expressing $(\eta + \underline{\eta})$ in terms of the primed quantities. A long, but easy computation gives

$$\begin{aligned}\underline{\eta}_a &= \underline{\eta}'_a + \Omega(-2\mathbf{g}(L,L'))^{\frac12}\hat{\sigma}_b \underline{\chi}'_{ba}\\ \eta_a &= \eta'_a + (-2\mathbf{g}(L,L'))^{\frac12}\hat{\sigma}_a(\Omega'\underline{\omega} + \Omega\underline{\omega}') - \Omega\hat{\sigma}_a \mathbf{D}_{e'_3}(-2\mathbf{g}(L,L'))^{\frac12}\end{aligned} \tag{4.8.70}$$

Finally the explicit computation of $\mathbf{D}_{e'_3}(-2\mathbf{g}(L,L'))^{\frac12}$ gives, as discussed in Chapter 7, see Lemma 7.7.2,

$$\mathbf{D}_{e'_3}(-2\mathbf{g}(L,L'))^{\frac12} = \Omega^{-1}(-2\mathbf{g}(L,L'))^{\frac12}\left[2(\Omega'\underline{\omega} + \Omega\underline{\omega}') - \hat{\sigma}\cdot(\eta' - \eta)\right] \tag{4.8.71}$$

Using 4.8.70 and 4.8.71 in the right hand side of 4.8.69 we estimate **(II)** as

$$\begin{aligned}&\int_{r'}^{r'_*} r'' \nabla\!\!\!/'_a \frac{\Omega'^2}{\Omega^2} = \int_{r'}^{r'_*} \frac{\Omega'^2}{\Omega^2} r'' \Big(\Big[(\eta' + \underline{\eta}')_a - (\eta + \underline{\eta})_a\Big] - 4\Omega'(-2\mathbf{g}(L',L))^{\frac12}\hat{\sigma}_a \underline{\omega}\Big)\\ &= \int_{r'}^{r'_*} \frac{\Omega'^2}{\Omega^2} r''(-2\mathbf{g}(L',L))^{\frac12}\hat{\sigma}_a\Big(-\Omega\underline{\chi}'_{ba}\hat{\sigma}_b + \hat{\sigma}\cdot(\eta' - \eta) - 4(\Omega'\underline{\omega} + \Omega\underline{\omega}')\Big)\\ &\le \int_{r'}^{r'_*} \frac{\Omega'^2}{\Omega^2} r''(-2\mathbf{g}(L',L))^{\frac12}\Big(\Omega|\underline{\chi}'| + |\eta' - \eta| + 4|\Omega'\underline{\omega} + \Omega\underline{\omega}'|\Big)\end{aligned} \tag{4.8.72}$$

Therefore the estimates for the Oscillation Lemma require the estimates for $\mathbf{g}(L', L)$ and $\nabla\!\!\!/'\mathbf{g}(L', L)$ along Σ'_{δ_0}. These estimates are obtained writing the evolution equations for these quantities along Σ'_{δ_0}. the result is in the following lemma.

[The assumptions of the lemma are modified and are relative only to $\mathbf{g}(L, L')$ and $\nabla\!\!\!/'\mathbf{g}(L, L')$, see also the new statement of the Oscillation Lemma.]

Lemma 4.8.2 *Under the assumptions of the Oscillation Lemma, the following estimates hold on* Σ'_{δ_0}

$$|r'^3\mathbf{g}(L', L)| \leq c\epsilon_0 \ , \ |r'^4\nabla\!\!\!/'\mathbf{g}(L', L)| \leq c\epsilon_0 \tag{4.8.73}$$

Proof: Using equations 4.8.60 and 4.8.61 a long computation gives

$$\begin{aligned}\frac{d}{dr'}\mathbf{g}(L', L) &= \Omega(-2\mathbf{g}(L, L'))\frac{\mathrm{tr}\chi}{2} + \Big\{\Omega(-2\mathbf{g}(L, L'))\hat{\sigma}_a\hat{\sigma}_b\hat{\chi}_{ab}\\ &+2\frac{\Omega}{\Omega'}(-2\mathbf{g}(L, L'))(\Omega'\underline{\omega} + \Omega\underline{\omega}') - 2\Omega\Omega'(-2\mathbf{g}(L, L'))^{\frac{3}{2}}\hat{\sigma}\cdot\eta\\ &+2\Omega'^2\Omega(-2\mathbf{g}(L, L'))^2\underline{\omega} + \frac{\Omega'}{\Omega}(-2\mathbf{g}(L, L'))^{\frac{1}{2}}\hat{\sigma}\cdot(\eta - \eta')\Big\}\end{aligned} \tag{4.8.74}$$

The first term in the right hand side can be written as

$$\begin{aligned}\Omega(-2\mathbf{g}(L, L'))\frac{\mathrm{tr}\chi}{2} &= -\Omega\mathbf{g}(L, L')\left(\frac{1}{2}(\mathrm{tr}\chi - \mathrm{tr}\underline{\chi}) + \frac{1}{2}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi})\right)\\ &= -\frac{1}{2}\mathrm{tr}\theta\mathbf{g}(L, L') + \left(\Omega - \frac{1}{2}\right)\frac{\mathrm{tr}\theta}{2}(-2\mathbf{g}(L, L')) + \frac{\Omega}{4}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi})(-2\mathbf{g}(L, L'))\end{aligned} \tag{4.8.75}$$

where

$$\mathrm{tr}\theta = \frac{1}{2}(\mathrm{tr}\chi - \mathrm{tr}\underline{\chi}) = \gamma^{ab}\mathbf{g}(\mathbf{D}_{e_a}\tilde{N}, e_b) \tag{4.8.76}$$

and the evolution equation 4.8.74 becomes

$$\begin{aligned}&\frac{d}{dr'}\mathbf{g}(L', L) + \frac{1}{2}\mathrm{tr}\theta\mathbf{g}(L, L') = \left[\left(\Omega - \frac{1}{2}\right)\frac{\mathrm{tr}\theta}{2}(-2\mathbf{g}(L, L')) + \frac{\Omega}{4}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi})(-2\mathbf{g}(L, L'))\right]\\ &+ \Big\{\Omega(-2\mathbf{g}(L, L'))\hat{\sigma}_a\hat{\sigma}_b\hat{\chi}_{ab} + 2\frac{\Omega}{\Omega'}(-2\mathbf{g}(L, L'))(\Omega'\underline{\omega} + \Omega\underline{\omega}') - 2\Omega\Omega'(-2\mathbf{g}(L, L'))^{\frac{3}{2}}\hat{\sigma}\cdot\eta\\ &\quad +2\Omega'^2\Omega(-2\mathbf{g}(L, L'))^2\underline{\omega} + \frac{\Omega'}{\Omega}(-2\mathbf{g}(L, L'))^{\frac{1}{2}}\hat{\sigma}\cdot(\eta - \eta')\Big\}\end{aligned} \tag{4.8.77}$$

To apply the analogous of the Evolution Lemma to the equation 4.8.77 we have still to replace $\mathrm{tr}\theta$, defined in 4.8.78, with

$$\mathrm{tr}\theta' = \frac{1}{2}(\mathrm{tr}\chi' - \mathrm{tr}\underline{\chi}') = \gamma^{ab}\mathbf{g}(\mathbf{D}_{e'_a}\tilde{N}', e'_b) \tag{4.8.78}$$

as $\mathrm{tr}\theta'$ is the second fundamental form associated to the $S'(2\delta_0-\nu,\nu)$ surfaces foliating Σ'_{δ_0}, while $\mathrm{tr}\theta$ refers to the $S(\lambda,\nu)$ two dimensional surfaces of the double null canonical foliation. Making this substitution, equation 4.8.77 becomes

$$\begin{aligned}
&\frac{d}{dr'}\mathbf{g}(L',L)+\frac{\mathrm{tr}\theta'}{2}\mathbf{g}(L,L')\\
&=\left[\frac{1}{4}(\mathrm{tr}\theta-\mathrm{tr}\theta')(-2\mathbf{g}(L,L'))+\left(\Omega-\frac{1}{2}\right)\frac{\mathrm{tr}\theta}{2}(-2\mathbf{g}(L,L'))+\frac{\Omega}{4}(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})(-2\mathbf{g}(L,L'))\right]\\
&+\left\{\Omega(-2\mathbf{g}(L,L'))\left(\hat{\sigma}_a\hat{\sigma}_b\hat{\chi}_{ab}+2\frac{\Omega}{\Omega'}(\Omega'\underline{\omega}+\Omega\underline{\omega}')\right)-2\Omega\Omega'(-2\mathbf{g}(L,L'))^{\frac{3}{2}}\hat{\sigma}\cdot\eta\right.\\
&\quad\left.+2\Omega'^2\Omega(-2\mathbf{g}(L,L'))^2\underline{\omega}\right\}+\frac{\Omega'}{\Omega}(-2\mathbf{g}(L,L'))^{\frac{1}{2}}\hat{\sigma}\cdot(\eta-\eta')
\end{aligned}\tag{4.8.79}$$

which we rewrite as

$$\begin{aligned}
\frac{d}{dr'}\mathbf{g}(L',L)+\frac{\mathrm{tr}\theta'}{2}\mathbf{g}(L',L) &= (-2\mathbf{g}(L',L))^{\frac{1}{2}}F_{\frac{1}{2}}+(-2\mathbf{g}(L',L))F_1\\
&+ (-2\mathbf{g}(L',L))^{\frac{3}{2}}F_{\frac{3}{2}}+(-2\mathbf{g}(L',L))^2F_2
\end{aligned}$$

where

$$\begin{aligned}
F_{\frac{1}{2}} &= \frac{\Omega'}{\Omega}\hat{\sigma}\cdot(\eta-\eta')\\
F_1 &= \left[\frac{1}{4}\left(\mathrm{tr}\theta-\frac{2}{r}\right)+\frac{1}{2}\left(\frac{1}{r}-\frac{1}{r'}\right)+\frac{1}{4}\left(\frac{2}{r'}-\mathrm{tr}\theta'\right)+\left(\Omega-\frac{1}{2}\right)\frac{\mathrm{tr}\theta}{2}\right.\\
&\quad\left.+\frac{\Omega}{4}(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})+\left(\Omega\hat{\sigma}_a\hat{\sigma}_b\hat{\chi}_{ab}+2\frac{\Omega}{\Omega'}(\Omega'\underline{\omega}+\Omega\underline{\omega}')\right)\right]\\
F_{\frac{3}{2}} &= -2\Omega\Omega'\hat{\sigma}\cdot\eta\\
F_2 &= +2\Omega'^2\Omega\underline{\omega}
\end{aligned}\tag{4.8.80}$$

Using assumptions 4.1.42, 4.1.43 for the Oscillation Lemma and the auxiliary assumption 4.8.52 it follows [55]

$$\begin{aligned}
|F_1|(r',\phi) &\le c\frac{(\Gamma'_0+c\epsilon_0)}{r'^2}\log r'\\
|F_{\frac{3}{2}}|(r',\phi) &\le c\frac{\epsilon_0}{r'^{\frac{5}{2}}}\\
|F_2|(r',\phi) &\le c\frac{\epsilon_0}{r'^2}\\
|F_{\frac{1}{2}}|(r',\phi) &\le c\frac{\epsilon_0}{r'^{\frac{5}{2}}}
\end{aligned}\tag{4.8.81}$$

[55] While r' is defined in 4.8.56, the variable r introduced in 4.8.80 is defined analogously by $r=\frac{1}{2}(\underline{u}-u)$. They differ logarithmically from the standard definition of r given in 3.1.2.

Observe that to estimate $|F_{\frac{1}{2}}|(r',\phi)$ and $|F_{\frac{3}{2}}|(r',\phi)$ we used the following estimates for η and η' on Σ'_{δ_0},

$$|r^{\frac{5}{2}}\eta| \leq c\epsilon_0 \ , \ |r'^{\frac{5}{2}}\eta'| \leq c\epsilon_0 \tag{4.8.82}$$

These estimates follow from the assumptions 4.1.40, 4.1.41 and [56] Proposition 4.3.16. Using assumptions 4.8.81 the following inequality holds

$$\begin{aligned}\frac{d}{dr'}\mathbf{g}(L',L)+\frac{\mathrm{tr}\theta'}{2}\mathbf{g}(L',L) &\leq \frac{(\Gamma'_0+c\epsilon_0)}{r'^2}(-\mathbf{g}(L',L))+\frac{c\epsilon_0}{r'^{\frac{5}{2}}}(-\mathbf{g}(L',L))^{\frac{3}{2}}\\ &+\frac{c\epsilon_0}{r'^2}(-\mathbf{g}(L',L))^2+\frac{c\epsilon_0}{r'^{\frac{5}{2}}}(-\mathbf{g}(L',L))^{\frac{1}{2}}\end{aligned} \tag{4.8.83}$$

Applying the Evolution Lemma to the evolution inequality on Σ'_{δ_0} we obtain

$$\begin{aligned}&|r'^{1-\frac{2}{p}}\mathbf{g}(L,L')|_{p,S'}(\overline{r}') \leq |r'^{1-\frac{2}{p}}\mathbf{g}(L,L')|_{p,S'}(r'_*)+\int_{\overline{r}'}^{r'_*}\frac{(\Gamma'_0+c\epsilon_0)}{r'^2}|r'^{1-\frac{2}{p}}\mathbf{g}(L,L')|_{p,S'}\\ &+\int_{\overline{r}'}^{r'_*}\frac{c\epsilon_0}{r'^{\frac{5}{2}}}|r'^{1-\frac{2}{p}}\mathbf{g}(L,L')|_{p,S'}|\mathbf{g}^{\frac{1}{2}}(L,L')|_{\infty,S'}+\int_{\overline{r}'}^{r'_*}\frac{c\epsilon_0}{r'^2}|r'^{1-\frac{2}{p}}\mathbf{g}(L,L')|_{p,S'}|\mathbf{g}(L,L')|_{\infty,S'}\\ &+\int_{\overline{r}'}^{r'_*}\frac{c\epsilon_0}{r'^{\frac{5}{2}}}r'|\mathbf{g}^{\frac{1}{2}}(L,L')|_{\infty,S'}\end{aligned} \tag{4.8.84}$$

We use again a bootstrap mechanism. We assume that, on Σ'_{δ_0}, $|\mathbf{g}(L,L')|_{\infty,S'}$ satisfies

$$|\mathbf{g}(L,L')|_{\infty,S'} \leq \frac{\tilde{\Gamma}}{r'^3} \tag{4.8.85}$$

with $\epsilon_0 < \tilde{\Gamma} < c\epsilon_0$, and we prove that

$$|\mathbf{g}(L,L')|_{\infty,S'} \leq \frac{1}{2}\frac{\tilde{\Gamma}}{r'^3} \tag{4.8.86}$$

. This allows to conclude that

$$|\mathbf{g}(L,L')|_{\infty,S'} \leq \frac{c\epsilon_0}{r'^3} \tag{4.8.87}$$

on the whole Σ'_{δ_0}. To prove inequality 4.8.86 we start applying Gronwall Lemma to 4.8.84 obtaining

$$|r'^{1-\frac{2}{p}}\mathbf{g}(L,L')|_{p,S'}(\overline{r}') \ \leq \ \left(|r'^{1-\frac{2}{p}}\mathbf{g}(L,L')|_{p,S'}(r'_*)+\int_{\overline{r}'}^{r'_*}\frac{c\epsilon_0}{r'^4}r'\tilde{\Gamma}\right)$$

[56]To estimate η' we use a slightly modified version of Proposition 4.3.16.

$$\leq \left(|r'^{1-\frac{2}{p}}\mathbf{g}(L,L')|_{p,S'}(r'_*) + \frac{c\epsilon_0\tilde{\Gamma}^{\frac{1}{2}}}{r'^2}\right)$$
$$\leq \left(\frac{\epsilon_0}{r'^2} + \frac{c\epsilon_0\tilde{\Gamma}^{\frac{1}{2}}}{r'^2}\right) \leq \frac{c\epsilon_0}{r'^2} \tag{4.8.88}$$

To complete the proof we have only to obtain the analogous estimate for $|r'^{2-\frac{2}{p}}\nabla\!\!\!\!/\,'\mathbf{g}(L,L')|_{p,S'}(\overline{r}')$, deriving tangentially the previous evolution equation, and, finally, apply Lemma 4.1.3.

Once Lemma 4.8.2 is proved we complete the proof of the Oscillation Lemma using the result of this lemma to estimate the right hand sides of 4.8.66.

4.8.7 Proof of Lemma 4.1.7

Lemma 4.1.7 controls the difference between the norm $|r^{\lambda_1}V|_{p,S(u_0(\nu),\nu)}$ and the norm $|r'^{\lambda_1}V'|_{p,S_{(0)}(\nu)}$. Observe that the norm $|r^{\lambda_1}V|_{p,S(u_0(\nu),\nu)}$ refers to a surface $S(u_0(\nu),\nu)$ associated to the *double null canonical foliation* with $r = (\frac{1}{4\pi}|S(u_0(\nu),\nu)|)^{\frac{1}{2}}$, while $|r'^{\lambda_1}V'|_{p,S_{(0)}(\nu)}$ refers to a two dimensional surface contained in Σ_0 associated to the *initial layer foliation* and, therefore, $r' = (\frac{1}{4\pi}|S_{(0)}(\nu)|)^{\frac{1}{2}}$.
We recall also that this result, completing part II of the Evolution Lemma, is applied, in this chapter, to the estimates of the underlined connection coefficients.
Let, therefore, V be a connection coefficient which satisfies the evolution equation

$$\mathbf{D}_N V + \lambda_0 \Omega \mathrm{tr}\underline{\chi} V = \underline{F}$$

Using the relations 4.8.60, 4.8.61, between the normalized null frame adapted to the *double null canonical foliation* and the one relative to the *initial layer foliation* it is possible to express the connection coefficient V in terms of a linear combination of the connection coefficients relative to the *initial layer foliation*. The explicit expressions for $\underline{\chi}, \underline{\eta}$ and ω are:

[The expression for $\underline{\eta}$ and ω have been corrected.]

$$\begin{aligned}
\underline{\chi}_{ab} &= \frac{\Omega}{\Omega'}\underline{\chi}'_{ab} \\
\underline{\eta}_a &= \underline{\eta}'_a + \Omega(-2\mathbf{g}(L',L))^{\frac{1}{2}}\hat{\sigma}_b\underline{\chi}'_{ab} \\
\omega &= \frac{\Omega'}{\Omega}\omega' - \Omega'(-2\mathbf{g}(L',L))^{\frac{1}{2}}\hat{\sigma}_a\underline{\eta}' - \frac{\Omega\Omega'}{2}(-2\mathbf{g}(L',L))\hat{\sigma}_a\hat{\sigma}_b\underline{\chi}'_{ab}
\end{aligned} \tag{4.8.89}$$

Let us prove the lemma in the case $V = \omega$, the proof in the other cases proceeds exactly along the same lines. We observe that, using the results of the Oscillation Lemma and of Lemma 4.8.2,

$$\begin{aligned}|r^{\lambda_1}\omega|^p_{p,S(u_0(\nu),\nu)} &= \int_{S(u_0(\nu),\nu)}|r^{\lambda_1}\omega|^p \leq c\int_{S(u_0(\nu),\nu)}|r'^{\lambda_1}\omega'|^p \\ +c\Bigg[\epsilon_0^{\frac{1}{2}}\int_{S(u_0(\nu),\nu)}|r'^{(\lambda_1-\frac{3}{2})}\underline{\eta}'|^p &+ \epsilon_0\int_{S(u_0(\nu),\nu)}|r'^{(\lambda_1-3)}\underline{\chi}'|^p\Bigg] \quad (4.8.90)\end{aligned}$$

It is easy to realize that the terms in the $[\cdot\cdot]$ brackets can be treated as corrections. Therefore we fix the attention on the first term, $\int_{S(u_0(\nu),\nu)}|r'^{\lambda_1}\omega'|^p$, which we rewrite in the following way

$$\int_{S(u_0(\nu),\nu)}|r'^{\lambda_1}\omega'|^p \leq c\int_{S^2} d\phi r'^2|r'^{\lambda_1}\omega'|^p(u'(\phi),\phi) \quad (4.8.91)$$

choosing $u'(\phi)$ in such a way that

$$S(u_0(\nu),\nu) = \{p \in \underline{C}(\nu)|u'(p) = u'(\phi)\}\ .$$

The connection coefficient $\omega' = -\frac{1}{2}\mathbf{D}_{e'_4}\log\Omega'$ satisfies in the initial layer region the following evolution equation, see subsection 4.3.10,

$$\mathbf{D}'_3(\Omega'\mathbf{D}'_4\log\Omega') = \hat{F}' - \Omega'\rho' = F' \quad (4.8.92)$$

where $\hat{F}'$ is given by

$$\hat{F}' \equiv 2\Omega'\zeta'\cdot\nabla\!\!\!/'\log\Omega' + \Omega'(\underline{\eta}'\cdot\eta' - 2\zeta'^2) \quad (4.8.93)$$

The scalar function $|\omega'|^p(u',\phi)$ satisfies the evolution equation

$$\frac{d|\omega'|^p}{du'} = \frac{p}{2}|\omega'|^{p-1}\frac{\omega'}{|\omega'|}F'$$

and, therefore,

$$\frac{d|\omega'|}{du'}(u',\phi) \leq \frac{1}{2}|F'|(u',\phi) \quad (4.8.94)$$

Integrating with respect to u', at fixed ϕ, starting from

$$S_{(0)}(\underline{u}) = \{p \in \underline{C}(\underline{u})|u'(p) = u'_0\}\ ,$$

where $u'_0 = -\underline{u}_{(0)}$, we obtain, multiplying both sides with r'^{λ_1},

$$|r'^{\lambda_1}\omega'|(u'(\phi),\phi) \leq c\left(|r'^{\lambda_1}\omega'|(u'_0,\phi) + \int_{u'_0}^{u'(\phi)}du''|r'^{\lambda_1}F'|(u'',\phi)\right)\ .$$

Substituting this last expression in 4.8.91 we obtain

$$\int_{S(u_0(\nu),\nu)} |r'^{\lambda_1}\omega'|^p = \int_{S^2} d\phi r'^2 |r'^{\lambda_1}\omega'|^p(u'(\phi),\phi) \tag{4.8.95}$$
$$\leq c\int_{S^2} d\phi r'^2 |r'^{\lambda_1}\omega'|^p(u'_0,\phi) + c\int_{S^2} d\phi r'^2 \left(\int_{u'_0}^{u'(\phi)} du''|r'^{\lambda_1}F'|(u'',\phi)\right)^p$$
$$\leq c|r'^{\lambda_1}\omega'|^p_{p,S_{(0)}}(\nu) + c\left|\left|\left(\int_{u'_0}^{u'_S} du'' r'^2|r'^{\lambda_1}F'|(u'',\phi)\right)\right|\right|^p_{L^p(S^2)}$$

where, in the last inequality we have chosen

[The definition of u'_S has been modified.]

$$u'_S(\nu) = 2\delta_0 - \nu \tag{4.8.96}$$

and used that δ_0 is small to bring r' inside the integral. Applying Minkowski inequality to the norm in the last line of 4.8.95, for $p \in [2,4]$, we obtain

$$|r'^{\lambda_1}\omega'|_{p,S(u_0(\underline{u}),\underline{u})} \leq c|r'^{\lambda_1}\omega'|_{p,S_{(0)}(\underline{u})} + c\int_{u'_0}^{u'_S}\left|r'^{\lambda_1}F'\right|_{p,S'}(u'',\underline{u})du'' \tag{4.8.97}$$

The choice of λ_1 is dictated by the explicit expression of F' and by the estimate for ω' on the initial slice Σ_0, it is $\lambda_1 = 2 - \frac{2}{p}$, as expected. The estimate for integral $\int_{u'_0}^{u'_S}\left|r^{\lambda_1}F'\right|_{p,S}(u'',\underline{u})du''$ is exactly of the same type as all the estimates done in the rest of this chapter, with the only difference that all the quantities are here relative to the *initial layer foliation*. Its estimate is, therefore, a repetition of the previous ones.

The estimates for the remaining terms of 4.8.90 proceeds in the same way, the final result is, therefore,

[Some simplification here due to the fact that the term $\underline{\omega}$ is, in fact, absent.]

$$|r^{\lambda_1}\omega|_{p,S(u_0(\underline{u}),\underline{u})} \leq c|r'^{\lambda_1}\omega'|_{p,S_{(0)}(\underline{u})} + c\epsilon_0 \tag{4.8.98}$$

proving the lemma.

Chapter 5

Estimates for the Riemann curvature tensor

This chapter is devoted to the proof of Theorem **M7** concerning the estimate of the $\mathcal{R}$ norms in terms of the fundamental quantities $\mathcal{Q}$, $\underline{\mathcal{Q}}$. These quantities can be expressed, according to 3.5.1, as weighted integrals of the null components of $\hat{\mathcal{L}}_O\mathbf{R}$ $\hat{\mathcal{L}}_T\mathbf{R}$, $\hat{\mathcal{L}}_O^2\mathbf{R}$, $\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\mathbf{R}$ and $\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\mathbf{R}$, along the null hypersurfaces $C(\lambda)$ and $\underline{C}(\nu)$. We recall their explicit expressions:

$$\mathcal{Q}(\lambda,\nu) = \mathcal{Q}_1(\lambda,\nu) + \mathcal{Q}_2(\lambda,\nu)$$
$$\underline{\mathcal{Q}}(\lambda,\nu) = \underline{\mathcal{Q}}_1(\lambda,\nu) + \underline{\mathcal{Q}}_2(\lambda,\nu)$$

$$\begin{aligned}
\mathcal{Q}_1(\lambda,\nu) &\equiv \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_4) \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\mathbf{R})(\bar{K},\bar{K},T,e_4) \\
\mathcal{Q}_2(\lambda,\nu) &\equiv \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_4) \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2\mathbf{R})(\bar{K},\bar{K},T,e_4) \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_4)
\end{aligned}$$

$$\begin{aligned}
\underline{\mathcal{Q}}_1(\lambda,\nu) &\equiv \sup_{V(\lambda,\nu)\cap\Sigma_0} |r^3\overline{\rho}|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_3) \\
&\quad + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\mathbf{R})(\bar{K},\bar{K},T,e_3)
\end{aligned}$$

$$\begin{aligned}\underline{\mathcal{Q}}_2(\lambda,\nu) \;\equiv\; & \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_3)\\ & + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2\mathbf{R})(\bar{K},\bar{K},T,e_3)\\ & + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},e_3)\end{aligned}$$

with $T, S, O, \bar{K}$ the vector fields defined in Chapter 3, see 3.4.1.

Osservazione 5.0.1 *Si osservi che il teorema* **M7** *è dimostrato relativamente ad una "double null foliation" generica di* $\mathcal{K}$*. Nel caso che ci interessa lo applicheremo alla porzione di* $\mathcal{K}$ *sopra la superficie* $\tilde{\Sigma}_{\tilde{\delta}_0}$*, vedi anche la discussione all'inizio della sezione 6.2. Pertanto, come discusso là, con* Σ_0 *si intende* $\tilde{\Sigma}_{\tilde{\delta}_0}$*. Forse ricordare questo con una footnote sarebbe opportuno.*

Theorem M7 *Assume that relative to a double null foliation on* $\mathcal{K}$

$$\mathcal{O}_{[2]} + \underline{\mathcal{O}}_{[2]} \le \epsilon_0$$

Then, if ϵ_0 *is sufficiently small, we have*

$$\mathcal{R} \le c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}} \tag{5.0.1}$$

where c is a positive constant.

Proof: According to the statement of Theorem **M7**, we make use of the assumption $\mathcal{O}_{[2]} + \underline{\mathcal{O}}_{[2]} \le \epsilon_0$ and prove the following result,

$$\begin{aligned}&\mathcal{R}_{[1]} \le [\mathcal{Q}]_1 + \epsilon_0(\mathcal{R}_{[0]} + \underline{\mathcal{R}}_{[0]})\\ &\underline{\mathcal{R}}_{[1]} \le [\mathcal{Q}]_1 + \epsilon_0(\mathcal{R}_{[0]} + \underline{\mathcal{R}}_{[0]})\\ &\mathcal{R}_2 \le [\mathcal{Q}]_1 + [\mathcal{Q}]_2 + \epsilon_0(\mathcal{R}_{[1]} + \underline{\mathcal{R}}_{[1]})\\ &\underline{\mathcal{R}}_2 \le [\mathcal{Q}]_1 + [\mathcal{Q}]_2 + \epsilon_0(\mathcal{R}_{[1]} + \underline{\mathcal{R}}_{[1]})\end{aligned} \tag{5.0.2}$$

where $[\mathcal{Q}]_{1,2} \equiv \mathcal{Q}_{1,2} + \underline{\mathcal{Q}}_{1,2}$.
From these estimates we conclude that, for sufficiently small ϵ_0, $\mathcal{R}_{[2]} + \underline{\mathcal{R}}_{[2]}$ is bounded, by $c\mathcal{Q}_{\mathcal{K}}$. We then use these results, together with the global Sobolev inequalities, see subsection 4.1.2, to derive the estimates for $\mathcal{R}_0^\infty$, $\underline{\mathcal{R}}_0^\infty$, $\mathcal{R}_1^S$ and $\underline{\mathcal{R}}_1^{p,S}$.

Discussion: *All the main ideas in the proof of Theorem* **M7** *are already present in the flat case* [1]*, see Theorem 2.2.2. We view the proof of the flat*

[1] The only exception are the estimates for $\overline{\rho}$ and $\overline{\sigma}$ which are trivial in the flat case.

case as a prerequisite to understand Theorem **M7**. *The non trivial character of the background spacetime, controlled by the smallness assumption $\mathcal{O} \leq \epsilon_0$, introduces quadratic or higher order terms which have, roughly, the following structure:*

1. *Linear relative to the null components of the curvature tensor.*
2. *Linear relative to the connection coefficients $tr\chi - \frac{2}{r}$, $tr\underline{\chi} + \frac{2}{r}$, $\hat{\chi}$, $\hat{\underline{\chi}}$, η, $\underline{\eta}$, ω, $\underline{\omega}$.*
3. *Linear relative to the deformation tensors and the Lie coefficients of the vector fields ${}^{(i)}O, S, T$.*

These correction terms have two different sources:

- *The corrections to the null Bianchi equations, 3.2.8, due to the non flat character of the spacetime; this is reflected in the presence of the terms $tr\chi - \frac{2}{r}$, $tr\underline{\chi} + \frac{2}{r}$, $\hat{\chi}$, $\hat{\underline{\chi}}$, η, $\underline{\eta}$, ω, $\underline{\omega}$.* [Slight modification of the first item, ω, $\underline{\omega}$ added.
- *The terms generated by commuting $\hat{\mathcal{L}}_{{}^{(i)}O}, \hat{\mathcal{L}}_S, \hat{\mathcal{L}}_T$ with the null decomposition of the Riemann curvature tensor.*

It is because of this structure that these corrections contribute to the terms of the form $\epsilon_0\mathcal{R}$ in the right hand side of inequalities 5.0.2. These correction terms, unlikeky the error terms discussed in the next chapter, are very easy to treat. We shall show in detail how to handle them in some examples and later simply ignore them.

5.1 Preliminary tools

We collect in this section a large number of definitions and a family of propositions and lemmas, without proofs, which will be used more and more in the rest of the chapter.

Definition 5.1.1 *Let X be a vector field in the family $\{T, S, {}^{(i)}O\}$, the Lie coefficients of X: ${}^{(X)}P, {}^{(X)}\underline{P}$, ${}^{(X)}Q$, ${}^{(X)}\underline{Q}$, ${}^{(X)}M$, ${}^{(X)}\underline{M}$, ${}^{(X)}N$, ${}^{(X)}\underline{N}$ are defined through the following commutation relations, see [Ch-Kl], Proposition 7.3.1,*

$$
\begin{aligned}
[X, e_3] &= {}^{(X)}\underline{P}_b e_b + {}^{(X)}\underline{M} e_3 + {}^{(X)}\underline{N} e_4 \\
[X, e_4] &= {}^{(X)}P_b e_b + {}^{(X)}N e_3 + {}^{(X)}M e_4 \qquad (5.1.1) \\
[X, e_a] &= \Pi[X, e_a] + \frac{1}{2}{}^{(X)}Q_a e_3 + {}^{(X)}\underline{Q}_a e_4
\end{aligned}
$$

their explicit expressions are, see also [Ch-Kl], equation (7.3.6b),

$$\begin{aligned}
\underline{P}_a &= \mathbf{g}(\mathbf{D}_X e_3, e_a) - \mathbf{D}_3 X_a \ , \ P_a = \mathbf{g}(\mathbf{D}_X e_4, e_a) - \mathbf{D}_4 X_a \\
\underline{Q}_a &= \mathbf{g}(\mathbf{D}_X e_3, e_a) + \mathbf{D}_a X_3 \ , \ Q_a = \mathbf{g}(\mathbf{D}_X e_4, e_a) + \mathbf{D}_a X_4 \qquad (5.1.2) \\
\underline{M}_a &= -\frac{1}{2}\mathbf{g}(\mathbf{D}_X e_3, e_4) + \frac{1}{2}\mathbf{D}_3 X_4 \ , \ M_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_X e_4, e_3) + \frac{1}{2}\mathbf{D}_4 X_3 \\
\underline{N}_a &= \frac{1}{2}\mathbf{D}_3 X_3 \ , \ N_a = \frac{1}{2}\mathbf{D}_4 X_4
\end{aligned}$$

The Lie coefficients of the vector field X originate when we commute $\hat{\mathcal{L}}_X$ with the null decomposition of the Riemann curvature tensor. The result of this commutation is expressed in the following proposition, see also [Ch-Kl], Proposition 7.3.1,

Proposition 5.1.1 *Let W be an arbitrary Weyl tensor. Consider the null components $\alpha(W), ..., \underline{\alpha}(W)$ as well as the null components $\alpha(\hat{\mathcal{L}}_X W), ..., \underline{\alpha}(\hat{\mathcal{L}}_X W)$. Let $\not{\hat{\mathcal{L}}}_X$ be the projection on $S(\lambda, \nu)$ of the Lie derivative $\mathcal{L}_X$, $\not{\hat{\mathcal{L}}}_X \alpha$, $\not{\hat{\mathcal{L}}}_X \underline{\alpha}$ the traceless parts of tensors $\hat{\mathcal{L}}\alpha$, $\hat{\mathcal{L}}\underline{\alpha}$, then the following relations hold*

[Corrections of the expression for α and $\underline{\alpha}$, $(-2^{(X)}M - \frac{1}{8}\text{tr}^{(X)}\pi)$ goes into $(-(^{(X)}M + {}^{(X)}\underline{M}) + \frac{1}{8}\text{tr}^{(X)}\pi)$]

$$\begin{aligned}
\alpha(\hat{\mathcal{L}}_X W)_{ab} &= \not{\hat{\mathcal{L}}}_X \alpha(W)_{ab} + \left(-(^{(X)}M + {}^{(X)}\underline{M}) + \frac{1}{8}tr^{(X)}\pi\right)\alpha(W)_{ab} \\
&- \left({}^{(X)}P_a + {}^{(X)}Q_a\right)\beta(W)_b - \left({}^{(X)}P_b + {}^{(X)}Q_b\right)\beta(W)_a \\
&+ \delta_{ab}(^{(X)}P + {}^{(X)}Q)\cdot\beta(W) \\
\beta(\hat{\mathcal{L}}_X W)_a &= \not{\mathcal{L}}_X \beta(W)_a - \frac{1}{2}{}^{(X)}\hat{\pi}_{ab}\beta(W)_b + \left(-{}^{(X)}M - \frac{1}{8}tr^{(X)}\pi\right)\beta(W)_a \\
&- \frac{3}{4}\left({}^{(X)}P_a + {}^{(X)}Q_a\right)\rho(W) - \frac{3}{4}\in_{ab}\left({}^{(X)}P_b + {}^{(X)}Q_b\right)\sigma(W) \\
&- \frac{1}{4}\left({}^{(X)}\underline{P}_b + {}^{(X)}\underline{Q}_b\right)\alpha(W)_{ab} \\
\rho(\hat{\mathcal{L}}_X W) &= \mathcal{L}_X \rho(W) - \frac{1}{8}tr^{(X)}\pi\rho(W) \\
&- \frac{1}{2}\left({}^{(X)}\underline{P}_a + {}^{(X)}\underline{Q}_a\right)\beta(W)_a + \frac{1}{2}\left({}^{(X)}P_a + {}^{(X)}Q_a\right)\underline{\beta}(W)_a \\
\sigma(\hat{\mathcal{L}}_X W) &= \mathcal{L}_X \sigma(W) - \frac{1}{8}tr^{(X)}\pi\sigma(W) \\
&+ \frac{1}{2}\left({}^{(X)}\underline{P}_a + {}^{(X)}\underline{Q}_a\right){}^*\beta(W)_a + \frac{1}{2}\left({}^{(X)}P_a + {}^{(X)}Q_a\right){}^*\underline{\beta}(W)_a \\
\underline{\beta}(\hat{\mathcal{L}}_X W)_a &= \not{\mathcal{L}}_X \underline{\beta}(W)_a - \frac{1}{2}{}^{(X)}\hat{\pi}_{ab}\underline{\beta}(W)_b + \left(-{}^{(X)}\underline{M} - \frac{1}{8}tr^{(X)}\pi\right)\underline{\beta}(W)_a \\
&+ \frac{3}{4}\left({}^{(X)}\underline{P}_a + {}^{(X)}\underline{Q}_a\right)\rho(W) - \frac{3}{4}\in_{ab}\left({}^{(X)}\underline{P}_b + {}^{(X)}\underline{Q}_b\right)\sigma(W)
\end{aligned}$$

$$
\begin{aligned}
& + \quad \frac{1}{4}\left({}^{(X)}P_b + {}^{(X)}Q_b\right)\underline{\alpha}(W)_{ab} \\
\underline{\alpha}(\hat{\mathcal{L}}_X W)_{ab} &= \hat{\not{\mathcal{L}}}_X \underline{\alpha}(W)_{ab} + \left(-({}^{(X)}\underline{M} + {}^{(X)}M) + \frac{1}{8}tr{}^{(X)}\pi\right)\underline{\alpha}(W)_{ab} \\
&+ \left({}^{(X)}\underline{P}_a + {}^{(X)}\underline{Q}_a\right)\underline{\beta}(W)_b + \left({}^{(X)}\underline{P}_b + {}^{(X)}\underline{Q}_b\right)\underline{\beta}(W)_a \\
&- \delta_{ab}({}^{(X)}\underline{P} + {}^{(X)}\underline{Q})\cdot\underline{\beta}(W) \qquad (5.1.3)
\end{aligned}
$$

Proof: We sketch the proof for the first equation of 5.1.3, a detailed discussion is in [Ch-Kl], Proposition 7.3.1. We compute $\alpha(\mathcal{L}_X W)_{ab} = (\mathcal{L}_X W)_{4a4b}$ obtaining

$$
\begin{aligned}
\mathcal{L}_X\alpha(W)_{ab} &= \alpha(\mathcal{L}_X W)_{ab} + 2{}^{(X)}M\alpha(W)_{ab} + \left({}^{(X)}P_a + {}^{(X)}Q_a\right)\beta(W)_b \\
&+\left({}^{(X)}P_b + {}^{(X)}Q_b\right)\beta(W)_a - 2\delta_{ab}P\cdot\beta(W) - 2{}^{(X)}N\rho(W)\delta_{ab} \qquad (5.1.4)
\end{aligned}
$$

Recalling the definition of $\hat{\mathcal{L}}_X$, 3.2.2, it follows that

$$
\alpha(\mathcal{L}_X W)_{ab} = \alpha(\hat{\mathcal{L}}_X W)_{ab} - \frac{3}{8}\mathrm{tr}{}^{(X)}\pi\alpha(W)_{ab} + \frac{1}{2}{}^{(X)}[W]_{a4b4} \qquad (5.1.5)
$$

where, see 3.2.3,

$$
\begin{aligned}
{}^{(X)}[W]_{a4b4} &= \left({}^{(X)}\hat{\pi}_{ac}\alpha(W)_{cb} + {}^{(X)}\hat{\pi}_{bc}\alpha(W)_{ca}\right) + \mathrm{tr}{}^{(X)}\pi\alpha(W)_{ab} \qquad (5.1.6) \\
&- {}^{(X)}\hat{\pi}_{34}\alpha(W)_{ab} + {}^{(X)}\hat{\pi}_{44}\rho(W)\delta_{ab} - 2\delta_{ab}{}^{(X)}\hat{\pi}_{4c}\beta(W)_c
\end{aligned}
$$

Substituting 5.1.6 in 5.1.5 and using the explicit expressions of the Lie coeeficients, 5.1.2, we obtain the result.

In the course of the various estimates of this chapter we will use systematically the fact that the rotation vector fields, defined in Chapter 3, section 3.4, satisfy the following lemma, see also [Ch-Kl], Proposition 7.5.3 page 178,

Lemma 5.1.1 *The rotation vector fields ${}^{(i)}O$ defined in Chapter 3, section 3.4 satisfy the following properties,*

Property 1: *Given an S-tangent tensor field f on $\mathcal{M}$ there exists a constant c_0 such that*

$$
c_0^{-1}\int_{S(\lambda,\nu)} r^2|\nabla\!\!\!/ f|^2 \leq \int_{S(\lambda,\nu)}|\mathcal{L}_O f|^2 \leq c_0\int_{S(\lambda,\nu)}(|f|^2 + r^2|\nabla\!\!\!/ f|^2) \qquad (5.1.7)
$$

where $|\mathcal{L}_O f|^2 \equiv \sum_i |\mathcal{L}_{{}^{(i)}O} f|^2$.

Property 2: *The Lie coefficients* ${}^{({}^{(i)}O)}P, {}^{({}^{(i)}O)}Q, {}^{({}^{(i)}O)}\underline{Q}, {}^{({}^{(i)}O)}N$ *of the rotation vector fields* ${}^{(i)}O$ *are identically zero in* $\mathcal{K}$. *If* f *is a one form or a traceless symmetric two covariant tensor tangent to the surfaces* $S(\lambda,\nu)$ *the following inequality holds*

$$c_0^{-1}\int_{S(\lambda,\nu)}|f|^2 \leq \int_{S(\lambda,\nu)}|\mathcal{L}_O f|^2 \tag{5.1.8}$$

In the L^2 estimates for the first and second derivatives of the Riemann null components we will often use the following relationships,

$$\begin{aligned}
\not{\mathcal{L}}_T\alpha_{ab} &= \not{\mathbf{D}}_T\alpha_{ab} + \Big(\alpha_{ac}\cdot(\chi+\underline{\chi})_{cb} + \alpha_{bc}\cdot(\chi+\underline{\chi})_{ca}\Big) + (\omega+\underline{\omega})\alpha_{ab} \\
&= \not{\mathbf{D}}_T\alpha_{ab} + \delta_{ab}\alpha\cdot(\hat{\chi}+\underline{\hat{\chi}}) + \Big((\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) + (\omega+\underline{\omega})\Big)\alpha_{ab}
\end{aligned} \tag{5.1.9}$$

During the chapter we give only the main ideas of the various estimates and we do not discuss all the technical details[2]. They can be easily recovered using Chapter 7 of [Ch-Kl]. The estimates of the various Lie coefficients of the X vector fields and their derivatives can be obtained from their explict expressions in terms of the null connection coefficients and of the estimates proved in Chapter 4.

5.1.1 L^2 estimates for the zero derivatives

We recall the definitions of the $\mathcal{R}$ norms, see 3.5.13,

$$\mathcal{R}_{[0]} = \mathcal{R}_0\ ,\ \underline{\mathcal{R}}_{[0]} = \underline{\mathcal{R}}_0 + \sup_{\mathcal{K}} r^3|\bar{\rho}|$$

with

$$\begin{aligned}
\mathcal{R}_0 &= \Big(\mathcal{R}_0[\alpha]^2 + \mathcal{R}_0[\beta]^2 + \mathcal{R}_0[(\rho,\sigma)]^2 + \mathcal{R}_0[\underline{\beta}]^2\Big)^{1/2} \\
\underline{\mathcal{R}}_0 &= \Big(\underline{\mathcal{R}}_0[\beta]^2 + \underline{\mathcal{R}}_0[(\rho,\sigma)]^2 + \underline{\mathcal{R}}_0[\underline{\beta}]^2 + \underline{\mathcal{R}}_0[\underline{\alpha}]^2\Big)^{1/2}
\end{aligned}$$

and

$$\mathcal{R}_{0,1,2}[w] \equiv \sup_{\mathcal{K}}\mathcal{R}_{0,1,2}[w](\lambda,\nu)\ ,\ \underline{\mathcal{R}}_{0,1,2}[w] \equiv \sup_{\mathcal{K}}\underline{\mathcal{R}}_{0,1,2}[w](\lambda,\nu).$$

[2]Some of them are examinated in the appendix to this chapter.

Proposition 5.1.2 *Under the assumptions of Theorem* **M7** *the following inequalities hold*

$$\mathcal{R}_0[\alpha]^2(\lambda,\nu) \le c\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2{\mathcal{R}_{[0]}}^2$$
$$\mathcal{R}_0[\beta]^2(\lambda,\nu) \le c\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2{\mathcal{R}_{[0]}}^2$$
$$\mathcal{R}_0[(\rho,\sigma)]^2(\lambda,\nu) \le c\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2{\mathcal{R}_{[0]}}^2$$
$$\mathcal{R}_0[\underline{\beta}]^2(\lambda,\nu) \le c\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2{\mathcal{R}_{[0]}}^2 \quad (5.1.10)$$

$$\underline{\mathcal{R}}_0[\beta]^2(\lambda,\nu) \le c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)$$
$$+c\epsilon_0\left[\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\right.$$
$$\left.+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2{\mathcal{R}_{[0]}}^2\right] + c\epsilon_0^2{\underline{\mathcal{R}}_{[0]}}^2$$
$$\underline{\mathcal{R}}_0[(\rho,\sigma)]^2(\lambda,\nu) \le c\int_{\underline{C}(\nu)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3) + c\epsilon_0^2{\underline{\mathcal{R}}_{[0]}}^2$$
$$\underline{\mathcal{R}}_0[\underline{\beta}]^2(u,\underline{u}) \le c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3) + c\epsilon_0^2{\underline{\mathcal{R}}_{[0]}}^2$$
$$\underline{\mathcal{R}}_0[\underline{\alpha}]^2(u,\underline{u}) \le c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3) + c\epsilon_0^2{\underline{\mathcal{R}}_{[0]}}^2 \quad (5.1.11)$$

Proof:

1. $\mathcal{R}_0[\alpha](\lambda,\nu) = \|r^2\alpha\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

To control $\|r^2\alpha\|_{2,C(\lambda)\cap V(\lambda,\nu)}$ we first use the inequality 5.1.8, Property 2 of Lemma 5.1.1, to infer

$$\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(W)|^2 \le c\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\mathcal{L}_O\alpha(W)|^2 \quad (5.1.12)$$

then we express $\mathcal{L}_O\alpha(W)$ in terms of $\alpha(\mathcal{L}_O W)$ plus correction terms, see equation 5.1.4, and finally we write $\alpha(\mathcal{L}_O W)$ in terms of $\alpha(\hat{\mathcal{L}}_O W)$ using the relationship 5.1.5, with $X = O$. From these equations we obtain

$$(\mathcal{L}_X\alpha(W))_{ab} \;=\; \alpha(\hat{\mathcal{L}}_X W)_{ab} + \left(\frac{1}{8}\mathrm{tr}^{(X)}\pi + 2^{(X)}M\right)\alpha(W)_{ab} \quad (5.1.13)$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\delta_{ab}{}^{(X)}\hat{\pi}\cdot\alpha(W) - \frac{1}{2}{}^{(X)}\hat{\pi}_{34}\alpha(W)_{ab} - \delta_{ab}(P+Q)\cdot\beta(W) \\
&+ \quad \Big(({}^{(X)}P_a + {}^{(X)}Q_a)\beta(W)_b + ({}^{(X)}P_b + {}^{(X)}Q_b)\beta(W)_a\Big)
\end{aligned}
$$

In the case $X = O$, we have, recalling[3] equations 5.1.2 and Property 2 of Lemma 5.1.1,

$$
\begin{aligned}
&{}^{({}^iO)}M = {}^{({}^iO)}\underline{M} = -{}^{(i)}O_b \nabla\!\!\!/_b \log\Omega \\
&{}^{({}^iO)}P_a = {}^{({}^iO)}\underline{Q}_a = {}^{({}^iO)}Q_a = {}^{({}^iO)}\underline{N} = {}^{({}^iO)}N = 0
\end{aligned}
\tag{5.1.14}
$$

and the previous expression 5.1.13 can be written as

$$
\begin{aligned}
(\mathcal{L}_O\alpha(W))_{ab} &= \alpha(\hat{\mathcal{L}}_O W)_{ab} + \left(\frac{1}{8}\mathrm{tr}^{(O)}\pi + 2^{(O)}M\right)\alpha(W)_{ab} \\
&+ \quad \frac{1}{2}\delta_{ab}{}^{(O)}\hat{\pi}\cdot\alpha(W) - \frac{1}{2}{}^{(O)}\hat{\pi}_{34}\alpha(W)_{ab} \\
&= \alpha(\hat{\mathcal{L}}_O W)_{ab} - \frac{1}{8}\mathrm{tr}^{(O)}\pi\alpha(W)_{ab} + \frac{1}{2}\delta_{ab}{}^{(O)}\hat{\pi}\cdot\alpha(W)
\end{aligned}
\tag{5.1.15}
$$

Using 5.1.15, inequality 5.1.12 can be rewritten as

$$
\begin{aligned}
\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(W)|^2 &\le c\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(\hat{\mathcal{L}}_O W)|^2 \\
&+ \int_{C(\lambda)\cap V(\lambda,\nu)} r^4 Qr[({}^{(O)}M, \mathrm{tr}^{(O)}\pi, {}^{(O)}\hat{\pi}); \alpha(W)]^2
\end{aligned}
\tag{5.1.16}
$$

where $Qr[A,B]$ denotes a quadratic term linear in A and B. From the estimates for the sup norms of ${}^{(O)}M$, $\mathrm{tr}^{(O)}\pi$ and ${}^{(O)}\hat{\pi}$, discussed in Chapter 4, see subsection 4.7 [4],

$$
\sup_{\mathcal{K}}|r\mathrm{tr}^{(O)}\pi| \ , \ \sup_{\mathcal{K}}|r^{(O)}\hat{\pi}| \ , \ \sup_{\mathcal{K}}|r^{(O)}M| \le c\epsilon_0
$$

we obtain

$$
\int_{C(\lambda)\cap V(\lambda,\nu)} r^4 Qr[({}^{(O)}M, \mathrm{tr}^{(O)}\pi, {}^{(O)}\hat{\pi}); \alpha(W)]^2 \le c\frac{\epsilon_0^2}{r(\lambda,\nu)^2}\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(W)|^2
\tag{5.1.17}
$$

which substituted in 5.1.16 gives

$$
\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(W)|^2 \le c\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(\hat{\mathcal{L}}_O W)|^2 + [Correction]^2
$$

[3]See also the definitions of the deformation tensors in Chapter 3, see subsection 3.4.3.
[4]The estimate for ${}^{(O)}M$ can be easily obtained from the results of subsection 4.7.

where the $[Correction]$ term satisfies the inequality

$$[Correction] \leq c\epsilon_0 \left(\int_{C(\lambda)\cap V(\lambda,\nu)} r^4 |\alpha(W)|^2 \right)^{\frac{1}{2}} \leq c\epsilon_0 \mathcal{R}_{[0]} \qquad (5.1.18)$$

Collecting these results we obtain

$$\int_{C(\lambda)\cap V(\lambda,\nu)} r^4 |\hat{\mathcal{L}}_O \alpha(W)|^2 \leq c \int_{C(\lambda)\cap V(\lambda,\nu)} r^4 |\alpha(\hat{\mathcal{L}}_O W)|^2 + c\epsilon_0^2 {\mathcal{R}_{[0]}}^2$$

and the first term in the right hand side is controlled by the $\mathcal{Q}$-integral $\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K}, \bar{K}, T, e_4)$.

2. $\mathcal{R}_0[\beta](\lambda,\nu) = \|r^2 \beta\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

We use first Lemma 5.1.1 to infer

$$\int_{C(\lambda)\cap V(\lambda,\nu)} r^4 |\beta(W)|^2 \leq c \int_{C(\lambda)\cap V(\lambda,\nu)} r^4 |\mathcal{L}_O \beta(W)|^2$$

then, using Proposition 5.1.1, we express $\mathcal{L}_O \beta(W)$ in terms of $\beta(\mathcal{L}_O W)$ plus correction terms and $\beta(\mathcal{L}_O W)$ in terms of $\beta(\hat{\mathcal{L}}_O W)$ and correction terms, obtaining

$$\mathcal{L}_O \beta(W) = \beta(\hat{\mathcal{L}}_O W) + \frac{1}{2} {}^{(O)}\hat{\pi} \cdot \beta(W) + ({}^{(O)}M + \frac{1}{8} \mathrm{tr}\, {}^{(O)}\pi) \beta(W) + \frac{1}{4} {}^{(O)}\underline{P} \cdot \alpha(W) \ .$$

Proceeding as in the previous case we prove that the correction terms satisfy the inequality

$$[Correction] \leq c\epsilon_0 \left[\left(\int_{C(\lambda)\cap V(\lambda,\nu)} r^4 |\beta(W)|^2 \right)^{\frac{1}{2}} + \left(\int_{C(\lambda)\cap V(\lambda,\nu)} r^4 |\alpha(W)|^2 \right)^{\frac{1}{2}} \right] \leq c\epsilon_0 \mathcal{R}_{[0]}$$

so that, finally,

$$\int_{C(\lambda)\cap V(\lambda,\nu)} r^4 |\beta(W)|^2 \leq c \int_{C(\lambda)\cap V(\lambda,\nu)} r^4 |\beta(\hat{\mathcal{L}}_O W)|^2 + c\epsilon_0^2 {\mathcal{R}_{[0]}}^2$$

and the integral in the right hand side is controlled by the $\mathcal{Q}$-integral $\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K}, \bar{K}, T, e_4)$.

3. $\mathcal{R}_0[(\rho,\sigma)](\lambda,\nu)=\|\tau_- r(\rho-\overline{\rho},\sigma-\overline{\sigma})\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

The control of $\rho-\overline{\rho}$ and of $\sigma-\overline{\sigma}$ is obtained in a similar way. From the Poincaré inequality,

$$\int_S(\Phi-\bar{\Phi})^2\leq c\int_S|r\nabla\!\!\!\!/\,\Phi|^2\ ,n$$

the following inequality holds

$$\int_{C(\lambda)\cap V(\lambda,\nu)}\tau_-^2r^2|\rho-\overline{\rho}|^2\leq c\int_{C(\lambda)\cap V(\lambda,\nu)}\tau_-^2r^2|r\nabla\!\!\!\!/\,\rho|^2 \tag{5.1.19}$$

We estimate $|r\nabla\!\!\!\!/\,\rho(W)|$ in terms of $|\mathcal{L}_O\rho(W)|$, using Lemma 5.1.1, and, repeating the same steps done for $\beta(W)$, we obtain

$$\mathcal{L}_O\rho(W)=\rho(\hat{\mathcal{L}}_OW)+\frac{1}{8}\mathrm{tr}^{(O)}\pi\rho(W)+\frac{1}{2}{}^{(O)}\underline{P}\cdot\beta(W)\ .$$

Therefore

$$\int_{C(\lambda)\cap V(\lambda,\nu)}\tau_-^2r^2|(\rho-\overline{\rho})(W)|^2\leq c\int_{C(\lambda)\cap V(\lambda,\nu)}\tau_-^2r^2|\rho(\hat{\mathcal{L}}_OW)|^2+[Correction]^2$$

and

$$\begin{aligned}[Correction]\ &\leq\ c\left[\left(\int_{C(\lambda)\cap V(\lambda,\nu)}\tau_-^2r^2|\mathrm{tr}^{(O)}\pi|^2|\rho(W)|^2\right)^{\frac{1}{2}}\right. \\ &+\ \left.\left(\int_{C(\lambda)\cap V(\lambda,\nu)}\tau_-^2r^2|{}^{(O)}\underline{P}|^2|\beta(W)|^2\right)^{\frac{1}{2}}\right]\leq c\epsilon_0\mathcal{R}_{[0]}\end{aligned} \tag{5.1.20}$$

so that, in conclusion,

$$\int_{C(\lambda)\cap V(\lambda,\nu)}\tau_-^2r^2|(\rho-\overline{\rho})(W)|^2\leq c\int_{C(\lambda)\cap V(\lambda,\nu)}\tau_-^2r^2|\rho(\hat{\mathcal{L}}_OW)|^2+c\epsilon_0^2{\mathcal{R}_{[0]}}^2$$

and the integral in the right hand side is controlled by the $\mathcal{Q}$-integral $\int_{C(\lambda)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_4)$.

4. $\mathcal{R}_0[\underline{\beta}](\lambda,\nu)=\|\tau_-^2\underline{\beta}\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

Using again Lemma 5.1.1 and controlling the correction terms exactly as before, it follows that

$$\int_{C(\lambda)\cap V(\lambda,\nu)}\tau_-^4|\underline{\beta}(W)|^2\leq c\int_{C(\lambda)\cap V(\lambda,\nu)}\tau_-^4|\underline{\beta}(\hat{\mathcal{L}}_OW)|^2+c\epsilon_0^2{\mathcal{R}_{[0]}}^2$$

and the integral in the right hand side is controlled by the $\mathcal{Q}$-integral $\int_{C(\lambda)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_4)$.

5. $\underline{\mathcal{R}}_0[\beta](\lambda,\nu)=\|r^2\beta\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

Using again Property 2 of Lemma 5.1.1, it follows that,

$$\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^4|\beta(W)|^2 \le c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^4|\beta(\mathcal{L}_O W)|^2$$

Applying Proposition 5.1.1, we then express $\mathcal{L}_O\beta(W)$ in terms of $\beta(\mathcal{L}_O W)$ plus correction terms and $\beta(\mathcal{L}_O W)$ in terms of $\beta(\hat{\mathcal{L}}_O W)$ and correction terms,

$$\mathcal{L}_O\beta(W)=\beta(\hat{\mathcal{L}}_O W)+\frac{1}{2}{}^{(O)}\hat{\pi}\cdot\beta(W)+({}^{(O)}M+\frac{1}{8}\mathrm{tr}\,{}^{(O)}\pi)\beta(W)+\frac{1}{4}{}^{(O)}\underline{P}\cdot\alpha(W)$$

so that finally

$$\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^4|\beta(W)|^2 \le c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^4|\beta(\mathcal{L}_O W)|^2+[Correction]^2$$

As in the previous case the correction terms satisfy the inequality

$$[Correction]\le c\epsilon_0\left[\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^2|\beta(W)|^2\right)^{\frac{1}{2}}+\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^2|\alpha(W)|^2\right)^{\frac{1}{2}}\right]$$

Observe that the estimate of the second integral of the previous inequality has to be performed differently. Its bound is provided in Proposition 5.1.4 and, from this result we obtain, with $\epsilon<1$,

$$\begin{aligned}\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^2|\alpha(W)|^2 \;&\le\; \frac{1}{r^{1+\epsilon}}\left[\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\right.\\ &\left.+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},T,e_4)+c\epsilon_0^2{\mathcal{R}_{[0]}}^2\right]\end{aligned}$$

Finally

$$\begin{aligned}\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^4|\beta(W)|^2 \;&\le\; c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)+c\epsilon_0^2{\underline{\mathcal{R}}_{[0]}}^2\\ &+\; c\epsilon_0\left[\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\right.\\ &+\; \left.\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},T,e_4)+c\epsilon_0^2{\mathcal{R}_{[0]}}^2\right]\end{aligned}$$

6. $\underline{\mathcal{R}}_0[(\rho,\sigma)](\lambda,\nu)=\|r^2(\rho-\overline{\rho},\sigma-\overline{\sigma})\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

The estimate proceeds exactly as that for $\mathcal{R}_0[(\rho,\sigma)](\lambda,\nu)$, but with a better weight, due to the expression of $Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_3)$, see 3.5.1. The $[Correction]$ term is controlled as in the previous cases, the final result is

$$\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^4|(\rho-\overline{\rho})(W)|^2\leq c\int_{\underline{C}(\nu)\cap V(u,\underline{u})} r^4|\rho(\hat{\mathcal{L}}_OW)|^2+c\epsilon_0^2\underline{\mathcal{R}}_{[0]}{}^2$$

and the integral in the right hand side is controlled by the $\mathcal{Q}$-integral $\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_3)$.

7. $\underline{\mathcal{R}}_0[\underline{\beta}](\lambda,\nu)=\|\tau_-r\underline{\beta}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

Proceeding as in the estimate of $\mathcal{R}_0[\underline{\beta}](\lambda,\nu)$ we obtain

$$\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^2r^2|\underline{\beta}(W)|^2\leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^2r^2|\underline{\beta}(\hat{\mathcal{L}}_OW)|^2+c\epsilon_0^2\underline{\mathcal{R}}_{[0]}{}^2 \tag{5.1.21}$$

and the the integral in the right hand side is controlled by the $\mathcal{Q}$-integral $\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_3)$.

8. $\underline{\mathcal{R}}_0[\underline{\alpha}](\lambda,\nu)=\|\tau_-^2\underline{\alpha}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

The estimate goes as the one for $\mathcal{R}_0[\alpha](\lambda,\nu)$, interchanging the underlined and not underlined quantities and substituting e_4 with e_3; also the estimate of the correction terms goes in the same way. The final result is

$$\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^4|\underline{\alpha}(W)|^2\leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^4|\underline{\alpha}(\hat{\mathcal{L}}_OW)|^2+c\epsilon_0^2\underline{\mathcal{R}}_{[0]}{}^2$$

and the integral in the right hand side is controlled by the $\mathcal{Q}$-integral $\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_3)$.

5.1.2 L^2 estimates for the first derivatives

Proposition 5.1.3 *Under the assumptions of Theorem* **M7** *the following inequalities hold*

$$\mathcal{R}_1[\alpha]^2(\lambda,\nu)\ \ \leq\ \ c\Bigg(\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_4)$$

$$
\begin{aligned}
&\qquad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\Bigg) + c\epsilon_0^2 {\mathcal{R}_{[0]}}^2 \\
\mathcal{R}_1[\beta]^2(\lambda,\nu) &\leq c\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2 {\mathcal{R}_{[0]}}^2 \\
\mathcal{R}_1[(\rho,\sigma)]^2(\lambda,\nu) &\leq c\Bigg(\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) \\
&\qquad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\Bigg) + c\epsilon_0^2 {\mathcal{R}_{[0]}}^2 \\
\mathcal{R}_1[\underline{\beta}]^2(\lambda,\nu) &\leq c\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2 {\mathcal{R}_{[0]}}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\underline{\mathcal{R}}_1[\beta]^2(\lambda,\nu) &\leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3) \\
&\quad + c\epsilon_0\Bigg[\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2 {\mathcal{R}_{[0]}}^2\Bigg] + c\epsilon_0^2 {\underline{\mathcal{R}}_{[0]}}^2 \\
\underline{\mathcal{R}}_1[(\rho,\sigma)]^2(\lambda,\nu) &\leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3) \\
\underline{\mathcal{R}}_1[\underline{\beta}]^2(\lambda,\nu) &\leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3) \qquad (5.1.22) \\
\underline{\mathcal{R}}_1[\underline{\alpha}]^2(\lambda,\nu) &\leq c\Bigg(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3) + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)\Bigg)
\end{aligned}
$$

Proof:

1. $\mathcal{R}_1[\alpha](\lambda,\nu) = \|r^3\nabla\!\!\!/\alpha\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|r^3\alpha_3\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|r^3\alpha_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

Proceeding as in subsection 5.1.1, the various terms of $\mathcal{R}_1[\alpha](\lambda,\nu)$ are controlled in the following way:

a) Using Lemma 5.1.1 $\|r^3\nabla\!\!\!/\alpha\|^2_{2,C(\lambda)\cap V(\lambda,\nu)}$ is, first, bounded in the following way:

$$
\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\nabla\!\!\!/\alpha(W)|^2 \leq c\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\mathcal{L}_O\alpha(W)|^2 \ .
$$

Then we proceed as in the previous estimate of $\|r^2\alpha\|^2_{2,C(\lambda)\cap V(\lambda,\nu)}$ obtaining

$$
\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\mathcal{L}_O\alpha(W)|^2 \quad\leq\quad \int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(\hat{\mathcal{L}}_O W)|^2 + [Correction]
$$

$$\leq \int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(\hat{\mathcal{L}}_O W)|^2 + c\epsilon_0^2 {\mathcal{R}_{[0]}}^2$$

and the right hand side integral is bounded by $\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)$.

b) $\|r^3\alpha_3\|^2_{2,C(\lambda)\cap V(\lambda,\nu)}$ is controlled using the Bianchi equation, see 3.2.8,

$$\alpha_3 \equiv \mathbf{D\!\!\!/}_3\alpha + \frac{1}{2}\mathrm{tr}\underline{\chi}\alpha = \nabla\!\!\!\!/\,\widehat{\otimes}\beta + \left[4\underline{\omega}\alpha - 3(\hat{\chi}\rho + {}^{\star}\hat{\chi}\sigma) + (\zeta + 4\eta)\widehat{\otimes}\beta\right] \quad (5.1.23)$$

Neglecting the terms in square brackets we are left to estimate

$$\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\nabla\!\!\!\!/\,\beta(W)|^2 \ .$$

This term, using Lemma 5.1.1, is bounded in the following way

$$\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\nabla\!\!\!\!/\,\beta(W)|^2 \leq c\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\mathcal{L}_O\beta(W)|^2$$
$$\leq c\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\beta(\hat{\mathcal{L}}_O W)|^2 + c\epsilon_0^2 {\mathcal{R}_{[0]}}^2$$

where the second inequality has been already obtained in the estimate of $\mathcal{R}_0[\beta](\lambda,\nu)$. To control the terms in the square brackets of the Bianchi null equation 5.1.23, we observe that, recalling the assumption $\mathcal{O} \leq \epsilon_0$ on the Ricci coefficients, see Theorem **M7**, it is easy to recognize that these terms are small and being quadratic, linear in the Ricci coefficients and in the null Riemann components, have the same structure as the correction terms [5] discussed in subsection 5.1.1. Therefore we write

$$\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\alpha_3|^2 \leq c\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\nabla\!\!\!\!/\,\beta(W)|^2 + [Correction]^2$$
$$\leq c\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\nabla\!\!\!\!/\,\beta(W)|^2 + c\epsilon_0^2 {\mathcal{R}_{[0]}}^2$$

and the integral appearing in the right hand side is controlled by $\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)$.

c) $\|r^3\alpha_4\|^2_{2,C(\lambda)\cap V(\lambda,\nu)}$ is estimated in a different way as the Bianchi equations do not provide an evolution equation for α, along the forward null hypersurfaces. We write, therefore, $\mathbf{D\!\!\!/}_4\alpha = -\mathbf{D\!\!\!/}_3\alpha + 2\mathbf{D\!\!\!/}_T\alpha$ and use the Bianchi equation along the $C(\lambda)$ null hypersurfaces, as we did before, to control the term $\|r^3\alpha_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}$.

[5] Moreover they have a better asymptotic behaviour.

To control the term $\|r^3 \not{D}_T \alpha(W)\|_{2,C(\lambda)\cap V(\lambda,\nu)}$, we express first $\not{D}_T \alpha(W)$ in terms of $\mathcal{L}_T \alpha(W)$ plus corrections, using the relation 5.1.9,

$$\not{D}_T \alpha(W) = \mathcal{L}_T \alpha(W) + {}^{(T)}H \cdot \alpha(W) \tag{5.1.24}$$

where

$$\left({}^{(T)}H \cdot \alpha(W)\right)_{ab} = \delta_{ab}\alpha \cdot (\hat{\chi} + \underline{\hat{\chi}}) + \left((\text{tr}\chi + \text{tr}\underline{\chi}) + (\omega + \underline{\omega})\right)\alpha_{ab}$$

Moreover using the estimates of subsection 6.1.1, it follows that ${}^{(T)}H$ satisfies

$$\sup |r^{(T)}H| \leq c\epsilon_0 \ .$$

We apply Proposition 5.1.1, see 5.1.13, with $X = T$, and observe that, using again the estimates of subsection 6.1.1 we have

$$\sup\left\{|r^{(T)}P|, |r^{(T)}Q|, |r^{(T)}M|, |r^{(T)}\pi|\right\} \leq c\epsilon_0 \tag{5.1.25}$$

We end, therefore, with the following inequality

$$\int_{C(\lambda)\cap V(\lambda,\nu)} r^6 |\not{D}_T \alpha(W)|^2 \leq \int_{C(\lambda)\cap V(\lambda,\nu)} r^6 |\alpha(\hat{\mathcal{L}}_T W)|^2 + [Correction]^2$$

where the first integral in the right hand side is bounded by the $\mathcal{Q}$-integral $\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K}, \bar{K}, \bar{K}, e_4)$ and the $[Correction]$ term has the same structure as discussed before and, therefore, satisfies

$$[Correction] \leq c\epsilon_0^2 \mathcal{R}_{[0]}{}^2 \ .$$

All these estimates imply

$$\begin{aligned} \mathcal{R}_1[\alpha]^2(\lambda,\nu) \ &\leq \ c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K}, \bar{K}, T, e_4)\right. \\ &+ \left.\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K}, \bar{K}, \bar{K}, e_4)\right) + c\epsilon_0^2 \mathcal{R}_{[0]}{}^2 \end{aligned} \tag{5.1.26}$$

2. $\mathcal{R}_1[\beta](\lambda,\nu) = \|r^3 \not{\nabla}\beta\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^2 \beta_3\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|r^3 \beta_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

The control of these norms proceeds as in the previous case. The main difference is that for β we have the evolution equations both along the forward and backward null directions, see 3.2.8,

$$\begin{aligned} \beta_3 &\equiv \not{D}_3\beta + \text{tr}\underline{\chi}\beta = \not{\nabla}\rho + \left[2\underline{\omega}\beta + {}^\star\not{\nabla}\sigma + 2\hat{\chi} \cdot \underline{\beta} + 3(\eta\rho + {}^\star\eta\sigma)\right] \\ \beta_4 &\equiv \not{D}_4\beta + 2\text{tr}\chi\beta = \not{d}\text{iv}\,\alpha - \left[2\omega\beta - (2\zeta + \underline{\eta})\alpha\right] \ . \end{aligned}$$

Using these equations, taking into account the main terms and controlling the corrections in the square brackets as previously discussed, we obtain for $\mathcal{R}_1[\beta](\lambda,\nu)$ the bound [6]

$$\mathcal{R}_1[\beta]^2(\lambda,\nu) \leq c\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2{\mathcal{R}_{[0]}}^2 \quad (5.1.27)$$

3. $\mathcal{R}_1[(\rho,\sigma)](\lambda,\nu) = \|\tau_- r^2\nabla\!\!\!\!/(\rho,\sigma)\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|r^3(\rho,\sigma)_4\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|r\tau_-^2(\rho,\sigma)_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

The norm $\|\tau_- r^2\nabla\!\!\!\!/(\rho,\sigma)\|_{2,C(\lambda)\cap V(\lambda,\nu)}$ has been estimated before in the estimate of $\mathcal{R}_0[(\rho,\sigma)]$ and it is controlled, apart from corrections, by the $\mathcal{Q}$-integral $\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)$.

The norm $\|r^3(\rho,\sigma)_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}$ is estimated using the Bianchi equations

$$\rho_4 \equiv \mathbf{D}_4\rho + \frac{3}{2}\mathrm{tr}\chi\rho = \mathrm{d}\!\!\!/\mathrm{iv}\,\beta - \left[\frac{1}{2}\hat{\underline{\chi}}\cdot\alpha - \zeta\cdot\beta - 2\underline{\eta}\cdot\beta\right]$$
$$\sigma_4 \equiv \mathbf{D}_4\sigma + \frac{3}{2}\mathrm{tr}\chi\sigma = -\mathrm{c}\!\!\!/\mathrm{url}\,{}^\star\beta + \left[\frac{1}{2}\hat{\underline{\chi}}\cdot{}^\star\alpha - \zeta\cdot{}^\star\beta - 2\underline{\eta}\cdot{}^\star\beta\right]$$

and, apart from the estimate of the terms in the square bracket which produce standard correction terms controlled by $c\epsilon_0^2{\mathcal{R}_{[0]}}^2$, we are reduced to control the integral $\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\nabla\!\!\!\!/\beta|^2$ which, as already discussed in the previous estimate, is bounded by $\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)$.

The norm $\|r\tau_-^2(\rho,\sigma)_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}$ has to be estimated in a slightly different way [7]. Using the decomposition

$$\rho_3 = 2\mathbf{D}_T\rho - \rho_4 + \frac{3}{2}\left(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}\right)\rho$$

we are reduced to the estimate of the norms

$$||r\tau_-^2\mathbf{D}_T\rho||_{2,C(\lambda)\cap V(\lambda,\nu)},\ ||r\tau_-^2\rho_4||_{2,C(\lambda)\cap V(\lambda,\nu)},\ ||r\tau_-^2(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})\rho||_{2,C(\lambda)\cap V(\lambda,\nu)}\ .$$

[6] The weight $\tau_- r^2$ for the $L^2(C)$ norm of β_3 is due to the presence of $\nabla\!\!\!\!/\rho$ in the Bianchi equation for β_3.

[7] In fact if we use the Bianchi equations

$$\rho_3 \equiv \mathbf{D}_3\rho + \frac{3}{2}\mathrm{tr}\underline{\chi}\rho = -\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\beta} - \left[\frac{1}{2}\hat{\chi}\cdot\underline{\alpha} - \zeta\cdot\underline{\beta} + 2\eta\cdot\underline{\beta}\right]$$
$$\sigma_3 \equiv \mathbf{D}_3\sigma + \frac{3}{2}\mathrm{tr}\underline{\chi}\sigma = -\mathrm{c}\!\!\!/\mathrm{url}\,{}^\star\underline{\beta} + \left[\frac{1}{2}\hat{\chi}\cdot{}^\star\underline{\alpha} - \zeta\cdot{}^\star\underline{\beta} - 2\eta\cdot{}^\star\underline{\beta}\right]$$

the component $\underline{\alpha}$ appears in the bracket terms. As $\underline{\alpha}$ never appears in the $\mathcal{Q}$-integrals along $C(\lambda)$, this will require to estimate differently the integral $\int_{C(\lambda)\cap V(\lambda,\nu)} r^{-2}\tau_-^4|\underline{\alpha}|^2$. This is, nevertheless, possible and will be discussed in Proposition 5.1.4.

The second and the third ones are immediately bounded by

$$\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2 {\mathcal{R}_{[0]}}^2 ,$$

using the previous estimate for $\|r^3(\rho,\sigma)_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}$, the estimates[8] of $\mathcal{R}_0[(\rho,\sigma)](\lambda,\nu)$ and observing that $(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) = O(r^{-1}\tau_-^{-1})$, see Corollary 4.3.1.

The norm $||r\tau_-^2 \mathbf{D}_T\rho||_{2,C(\lambda)\cap V(\lambda,\nu)}$ is estimated in the same way as the norm $\|r^3 \not{\mathbf{D}}_T \alpha(W)\|_{2,C(\lambda)\cap V(\lambda,\nu)}$. First we observe that

$$||r\tau_-^2 \mathbf{D}_T\rho||_{2,C(\lambda)\cap V(\lambda,\nu)} = ||r\tau_-^2 \mathcal{L}_T\rho||_{2,C(\lambda)\cap V(\lambda,\nu)} ,$$

then we apply Proposition 5.1.1, for $X = T$, and observe that, using again the estimates of subsection 6.1.1, see also 5.1.25,

$$\sup\left\{|r{}^{(T)}\!P|, |r{}^{(T)}\!\underline{P}|, |r{}^{(T)}\!Q|, |r{}^{(T)}\!\underline{Q}|, |r{}^{(T)}\!M|, |r{}^{(T)}\!\pi|\right\} \leq c\epsilon_0 .$$

In conclusion we obtain the inequality

$$\int_{C(\lambda)\cap V(\lambda,\nu)} r^2\tau_-^4|\not{\mathbf{D}}_T\rho(W)|^2 \leq c\int_{C(\lambda)\cap V(\lambda,\nu)} r^2\tau_-^4|\rho(\hat{\mathcal{L}}_T(W))|^2 + [Correction]^2 ,$$

where the first integral in the right hand side is bounded by the $\mathcal{Q}$-integral $\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)$ and the correction terms has the same structure as discussed before and satisfies $[Correction] \leq c\epsilon_0\mathcal{R}_{[0]}$. Finally

$$\begin{aligned} \mathcal{R}_1[(\rho,\sigma)]^2(\lambda,\nu) \quad &\leq \quad c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\right. \\ &+ \quad \left.\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\right) + c\epsilon_0^2{\mathcal{R}_{[0]}}^2 \end{aligned} \tag{5.1.28}$$

4. $\mathcal{R}_1[\underline{\beta}](\lambda,\nu) = \|\tau_-^2 r \not{\nabla}\underline{\beta}\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^2 \underline{\beta}_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

To control the norm $\|\tau_-^2 r\not{\nabla}\underline{\beta}\|_{2,C(\lambda)\cap V(\lambda,\nu)}$ we use first Lemma 5.1.1 to obtain

$$\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^4 r^2|\not{\nabla}\beta(W)|^2 \leq c\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^4|\mathcal{L}_O\beta(W)|^2$$

[8] more precisely we need also the norm estimate for $\overline{\rho}$ which is discussed later on in this chapter.

The integral on the right hand side has been already estimated when we controlled $\mathcal{R}_0[\underline{\beta}](\lambda,\nu)$ so that, finally,

$$\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^4 r^2 |\nabla\!\!\!/ \beta(W)|^2 \leq c\int_{C(\lambda)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2 \mathcal{R}_{[0]}{}^2 \ .$$

The norm $\|\tau_- r^2 \underline{\beta}_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}$ is controlled using the Bianchi equation

$$\underline{\beta}_4 \equiv \mathbf{D}\!\!\!\!/_4\underline{\beta} + \mathrm{tr}\chi\underline{\beta} = -\nabla\!\!\!/\rho + {}^\star\nabla\!\!\!/\sigma + \left[2\omega\underline{\beta} + 2\hat{\underline{\chi}}\cdot\beta + -3(\underline{\eta}\rho - {}^\star\underline{\eta}\sigma)\right] \ .$$

Apart from the terms in square brackets, this amounts to control the integrals $\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\nabla\!\!\!/(\rho,\sigma)|^2$. These integrals are bounded, as in the estimate of $\mathcal{R}_0[(\rho,\sigma)](\lambda,\underline{\nu})$, by $\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)+[Correction]^2$. The terms in square bracket appearing in the Bianchi equation produce, as already discussed in the case of $\|r^3\alpha_3\|^2_{2,C(\lambda)\cap V(\lambda,\nu)}$, some terms with the same structure as the $[Correction]$ term [9] and therefore can be inglobed in it. In conclusion,

$$\mathcal{R}_1[\underline{\beta}]^2(\lambda,\nu) \leq c\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2 \mathcal{R}_{[0]}{}^2 \quad (5.1.29)$$

5. $\underline{\mathcal{R}}_1[\beta](\lambda,\nu) = \|r^3\nabla\!\!\!/\beta\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|r^3\beta_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

The norm $\|r^3\nabla\!\!\!/\beta\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$ has been already bounded during to estimate of $\underline{\mathcal{R}}_0[\beta](u,\underline{u})$. The result is

$$\begin{aligned}\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^6|\nabla\!\!\!/\beta|^2 \ &\leq\ c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3) + c\epsilon_0^2 \underline{\mathcal{R}}_{[0]}{}^2 \\ &+\ c\epsilon_0\left[\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\right. \\ &+\ \left.\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2\mathcal{R}_{[0]}{}^2\right]\end{aligned}$$

The norm $\|r^3\beta_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$ is controlled using the Bianchi equation

$$\beta_3 \equiv \mathbf{D}\!\!\!\!/_3\beta + \mathrm{tr}\underline{\chi}\beta = \nabla\!\!\!/\rho + \left[2\underline{\omega}\beta + {}^\star\nabla\!\!\!/\sigma + 2\hat{\chi}\cdot\underline{\beta} + 3(\eta\rho + {}^\star\eta\sigma)\ .\right]$$

Repeating the previous arguments, apart from corrections, we have to control the integral $\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^6|\nabla\!\!\!/\rho|^2$ which is bounded, see the estimate of

[9] Moreover they have a better asymptotic behaviour.

$\underline{\mathcal{R}}_0[\rho]$, by $\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)$. Therefore, in conclusion,

$$\begin{aligned}\underline{\mathcal{R}}_1[\beta]^2(\lambda,\nu) &\leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3) + c\epsilon_0^2\underline{\mathcal{R}}_{[0]}{}^2\\ &+ c\epsilon_0\left[\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\right.\\ &+ \left.\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},T,e_4) + c\epsilon_0^2\mathcal{R}_{[0]}{}^2\right]\end{aligned} \tag{5.1.30}$$

6. $\underline{\mathcal{R}}_1[(\rho,\sigma)](\lambda,\nu) = \|r^3\nabla\!\!\!/(\rho,\sigma)\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_- r^2(\rho,\sigma)_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

The estimate of $\underline{\mathcal{R}}_1[(\rho,\sigma)](\lambda,\nu)$ proceeds exactly as the one of $\mathcal{R}_1[(\rho,\sigma)](\lambda,\nu)$ with the obvious substitutions.
The norm $\|r^3\nabla\!\!\!/(\rho,\sigma)\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$ has been already bounded in the estimate of $\underline{\mathcal{R}}_0[(\rho,\sigma)]$ and it is controlled, apart from correction terms, by $\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)$.
The norm $\|\tau_- r^2(\rho,\sigma)_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$ is estimated using the Bianchi equations

$$\begin{aligned}\rho_3 \equiv \mathbf{D}_3\rho + \frac{3}{2}\mathrm{tr}\underline{\chi}\rho &= -\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\beta} - \left[\frac{1}{2}\hat{\chi}\cdot\underline{\alpha} - \zeta\cdot\underline{\beta} + 2\eta\cdot\underline{\beta}\right]\\ \sigma_3 \equiv \mathbf{D}_3\sigma + \frac{3}{2}\mathrm{tr}\underline{\chi}\sigma &= -\mathrm{d}\!\!\!/\mathrm{iv}\,{}^\star\underline{\beta} + \left[\frac{1}{2}\hat{\chi}\cdot{}^\star\underline{\alpha} - \zeta\cdot{}^\star\underline{\beta} - 2\eta\cdot{}^\star\underline{\beta}\right]\end{aligned}$$

and, again, this implies that, apart from the corrections arising from the terms in the square brackets, which we have already discussed, we have to control $\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^2 r^4|\nabla\!\!\!/\underline{\beta}|^2$, bounded by $\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)$ plus corrections. Therefore, finally,

$$\underline{\mathcal{R}}_1[(\rho,\sigma)]^2(\lambda,\nu) \leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3) + c\epsilon_0^2\underline{\mathcal{R}}_{[0]}{}^2 \tag{5.1.31}$$

7. $\underline{\mathcal{R}}_1[\underline{\beta}](\lambda,\nu) = \|\tau_- r^2\nabla\!\!\!/\underline{\beta}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_-^2 r\underline{\beta}_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|r^3\underline{\beta}_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

The estimate of the norm $\underline{\mathcal{R}}_1[\underline{\beta}](\lambda,\nu)$ proceeds exactly as the one of $\mathcal{R}_1[\beta](\lambda,\nu)$ with the obvious changes. the final result is

$$\underline{\mathcal{R}}_1[\underline{\beta}]^2(\lambda,\nu) \leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3) + c\epsilon_0^2\underline{\mathcal{R}}_{[0]}{}^2 \tag{5.1.32}$$

8. $\underline{\mathcal{R}}_1[\underline{\alpha}](\lambda,\nu)=\|\tau_-^2 r\nabla\!\!\!/\,\underline{\alpha}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}+\|\tau_-^3\underline{\alpha}_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}+\|\tau_- r^2\underline{\alpha}_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$.

The estimate of the norm $\underline{\mathcal{R}}_1[\underline{\alpha}](\lambda,\nu)$ proceeds exactly as the one of $\mathcal{R}_1[\alpha](\lambda,\nu)$ with the obvious changes. the final result is

$$\begin{aligned}\underline{\mathcal{R}}_1[\underline{\alpha}]^2(\lambda,\nu) \quad &\leq \quad c\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)\right. \qquad (5.1.33)\\ &\left.+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)\right)+c\epsilon_0^2\underline{\mathcal{R}}_{[0]}{}^2\end{aligned}$$

Remarks: Some observations relative to the previous proposition, are now appropriate.

1) The quantity $\mathcal{R}_1[\underline{\beta}]$ does not contain an L^2 norm for $\underline{\beta}_3$. Nevertheless proceeding as in the case of $(\rho,\sigma)_3$, one can easily [10] bound $\|\tau_-^3\underline{\beta}_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}$. A better result can be obtained using the Bianchi equation

$$\underline{\beta}_3\equiv \mathbf{D}\!\!\!\!/_3\underline{\beta}+2\text{tr}\underline{\chi}\;\underline{\beta}=-\text{d}\!\!\!/\text{iv}\,\underline{\alpha}-\left[2\underline{\omega}\underline{\beta}+(-2\zeta+\eta)\cdot\underline{\alpha}\right]\;.$$

The problem here is the presence of $\text{d}\!\!\!/\text{iv}\,\underline{\alpha}$ in the right hand side. As $\underline{\alpha}$ never appears in the $\mathcal{Q}$-integrals along $C(\lambda)$, we have to estimate differently an integral like $\int_{C(\lambda)\cap V(\lambda,\nu)} r^\sigma\tau_-^\delta|\nabla\!\!\!/\,\underline{\alpha}|^2$. The final result, stated in Proposition 5.1.4, is the control of $\|r^\lambda\tau_-^{\frac{5}{2}}\underline{\beta}_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}$, with $\lambda<\frac{1}{2}$.

2) In $\underline{\mathcal{R}}_1[\beta]$ we would like to control also the norm $\|r^3\beta_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$. The estimate we obtain is slightly weaker, in fact we control $\|r^\lambda\beta_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$ with $\lambda<3$. This result can be obtained either writing

$$\beta_4=2\mathbf{D}\!\!\!\!/_T\beta-\beta_3+(2\text{tr}\chi+\text{tr}\underline{\chi})\beta$$

and then estimating the corresponding norms[11], either using the Bianchi equation [12]

$$\beta_4\equiv\mathbf{D}\!\!\!\!/_4\beta+2\text{tr}\chi\beta=\text{d}\!\!\!/\text{iv}\,\alpha+\left[-2\omega\beta+(2\zeta+\underline{\eta})\cdot\alpha\right]\;.$$

This result is discussed in Proposition 5.1.4.

[10] A weaker estimate than the one in [Ch-Kl]

[11] In this case we cannot obtain $\lambda=3$ due to the corrections which arises when we express the norm of $\mathbf{D}\!\!\!\!/_T\beta(W)$ in terms of the norm of $\beta(\hat{\mathcal{L}}_T W)$.

[12] In this case we cannot reach $\lambda=3$ due to the presence of the component $\text{d}\!\!\!/\text{iv}\,\alpha$.

3) Under the assumptions of Theorem **M7** it is possible to control also the norm $\|r^\lambda(\rho,\sigma)_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$ with $\lambda<3$, see Proposition 5.1.4. To obtain it we have to use [13] the Bianchi equations

$$\begin{aligned}\rho_4 &\equiv \mathbf{D}_4\rho+\frac{3}{2}\mathrm{tr}\chi\rho = \mathrm{d}\!\!/\mathrm{iv}\,\beta-\left[\frac{1}{2}\underline{\hat{\chi}}\cdot\alpha-\zeta\cdot\beta-2\underline{\eta}\cdot\beta\right]\\ \sigma_4 &\equiv \mathbf{D}_4\sigma+\frac{3}{2}\mathrm{tr}\chi\sigma = -\mathrm{d}\!\!/\mathrm{iv}\,{}^\star\beta+\left[\frac{1}{2}\underline{\hat{\chi}}\cdot{}^\star\alpha-\zeta\cdot{}^\star\beta-2\underline{\eta}\cdot{}^\star\beta\right]\ .\end{aligned}$$

We cannot reach the value $\lambda=3$ due to the presence of α in the right hand side.

5.1.3 Some extra L^2 norms for the zero and first derivatives of the Riemann components

As observed in the last remarks of the previous section, we can control in terms of the $\mathcal{Q}$-integrals introduced in subsection 2.6.3, some other L^2 norms of zero and first derivatives of Riemann components. We collect these estimates in the next proposition and discuss its proof in the appendix. The main strategy is to use the estimates for the $|\cdot|_{p,S}$ norms in terms of the appropriate $L^2(C,\underline{C})$ norms, see Corollary 4.1.1.

Proposition 5.1.4 *Under the assumptions of Theorem* **M7**, *we have the following L^2 norm estimates,*

$$\begin{aligned}&||r^\delta\tau_-^{\frac{5}{2}}\underline{\alpha}||_{2,C(\lambda)\cap V(\lambda,\nu)}\leq c\left(\mathcal{Q}_{\mathcal{K}}+\epsilon_0(\mathcal{R}_{[0]}+\underline{\mathcal{R}}_{[0]})\right) &&,\ \textit{for}\ \delta<-\tfrac{1}{2}\\ &||r^\delta\tau_-^{\frac{5}{2}}\nabla\!\!\!\!/\,\underline{\alpha}||_{2,C(\lambda)\cap V(\lambda,\nu)}\leq c\left(\mathcal{Q}_{\mathcal{K}}+\epsilon_0(\mathcal{R}_{[0]}+\underline{\mathcal{R}}_{[0]})\right) &&,\ \textit{for}\ \delta<\tfrac{1}{2}\\ &\|r^\delta\tau_-^2\underline{\alpha}_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}\leq c\left(\mathcal{Q}_{\mathcal{K}}+\epsilon_0(\mathcal{R}_{[0]}+\underline{\mathcal{R}}_{[0]})\right) &&,\ \textit{for}\ \delta\leq 1\\ &||r^\delta\tau_-^{\frac{7}{2}}\underline{\alpha}_3||_{2,C(\lambda)\cap V(\lambda,\nu)}\leq c\left(\mathcal{Q}_{\mathcal{K}}+\epsilon_0(\mathcal{R}_{[0]}+\underline{\mathcal{R}}_{[0]})\right) &&,\ \textit{for}\ \delta<-\tfrac{1}{2}\\ &\|r^\delta\tau_-^{\frac{5}{2}}\underline{\beta}_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}\leq c\left(\mathcal{Q}_{\mathcal{K}}+\epsilon_0(\mathcal{R}_{[0]}+\underline{\mathcal{R}}_{[0]})\right) &&,\ \textit{for}\ \delta<\tfrac{1}{2}\end{aligned}\tag{5.1.34}$$

$$\begin{aligned}&||r^\delta\alpha||_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\leq c\left(\mathcal{Q}_{\mathcal{K}}+\epsilon_0(\mathcal{R}_{[0]}+\underline{\mathcal{R}}_{[0]})\right) &&,\ \textit{for}\ \delta<2\\ &\|r^\delta\nabla\!\!\!\!/\,\alpha\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\leq c\left(\mathcal{Q}_{\mathcal{K}}+\epsilon_0(\mathcal{R}_{[0]}+\underline{\mathcal{R}}_{[0]})\right) &&,\ \textit{for}\ \delta<3\\ &\|r^\delta\alpha_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\leq c\left(\mathcal{Q}_{\mathcal{K}}+\epsilon_0(\mathcal{R}_{[0]}+\underline{\mathcal{R}}_{[0]})\right) &&,\ \textit{for}\ \delta<3\end{aligned}$$

[13] In this case we cannot proceed as we have done for the norm $\|r^3(\rho,\sigma)_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}$.

$$
\begin{aligned}
&||r^\delta \alpha_4||_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} \leq c\left(\mathcal{Q}_{\mathcal{K}} + \epsilon_0(\mathcal{R}_{[0]} + \underline{\mathcal{R}}_{[0]})\right) \quad , \textit{for } \delta < 3\\
&\|r^\delta \beta_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} \leq c\left(\mathcal{Q}_{\mathcal{K}} + \epsilon_0(\mathcal{R}_{[0]} + \underline{\mathcal{R}}_{[0]})\right) \quad , \textit{for } \delta < 3\\
&\|r^\delta (\rho,\sigma)_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} \leq c\left(\mathcal{Q}_{\mathcal{K}} + \epsilon_0(\mathcal{R}_{[0]} + \underline{\mathcal{R}}_{[0]})\right) \quad , \textit{for } \delta < 3
\end{aligned}
\tag{5.1.35}
$$

Proof: See the appendix to this chapter.

5.1.4 Control of the L^2 norms of the second derivatives of the Riemann components

Proposition 5.1.5 *Under the assumptions of Theorem* **M7** *the following inequalities hold*

$$
\begin{aligned}
\mathcal{R}_2[\alpha]^2(\lambda,\nu) \;\leq\; & c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4)\right.\\
&+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\\
&\left.+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\right) + c(1+\epsilon_0^2){\mathcal{R}_{[1]}}^2\\
\mathcal{R}_2[\beta]^2(\lambda,\nu) \;\leq\; & c\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4) + c(1+\epsilon_0^2){\mathcal{R}_{[1]}}^2\\
\mathcal{R}_2[(\rho,\sigma)]^2(\lambda,\nu) \;\leq\; & c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4)\right.\\
&\left.+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\right) + c(1+\epsilon_0^2){\mathcal{R}_{[1]}}^2\\
\mathcal{R}_2[\underline{\beta}]^2(\lambda,\nu) \;\leq\; & \left(\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4)\right.\\
&+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\\
&\left.+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\right) + c(1+\epsilon_0^2){\mathcal{R}_{[1]}}^2
\end{aligned}
$$

and

$$
\underline{\mathcal{R}}_2[\beta]^2(\lambda,\nu) \;\leq\; c\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3)\right.
$$

$$
\begin{aligned}
&\quad + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)\Bigg) + c(1+\epsilon_0^2)\underline{\mathcal{R}}_{[1]}{}^2 \\
\underline{\mathcal{R}}_2[(\rho,\sigma)]^2(\lambda,\nu) &\leq c\Bigg(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3) \\
&\quad + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)\Bigg) + c(1+\epsilon_0^2)\underline{\mathcal{R}}_{[1]}{}^2 \\
\underline{\mathcal{R}}_2[\underline{\beta}]^2(\lambda,\nu) &\leq c\Bigg(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3) \\
&\quad + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)\Bigg) + c(1+\epsilon_0^2)\underline{\mathcal{R}}_{[1]}{}^2 \\
\underline{\mathcal{R}}_2[\underline{\alpha}]^2(\lambda,\nu) &\leq c\Bigg(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3) \qquad (5.1.36)\\
&\quad + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3) \\
&\quad + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)\Bigg) + c(1+\epsilon_0^2)\underline{\mathcal{R}}_{[1]}{}^2
\end{aligned}
$$

Proof: The proof is in the appendix to this chapter.

Remarks: To understand in detail where the various integrals composing $\mathcal{Q}_2$ and $\underline{\mathcal{Q}}_2$ are needed, we make the following observations:

i) $\int_{C(u)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4)$ and $\int_{\underline{C}(\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3)$ are used to control the integrals

$$
\begin{aligned}
&\int_{C(u)} r^4\tau_+^{\gamma}\tau_-^{\delta}|\nabla\!\!\!/^2(\alpha,\beta,\rho,\sigma,\underline{\beta})|^2 \\
&\int_{\underline{C}(\underline{u})} r^4\tau_+^{\gamma}\tau_-^{\delta}|\nabla\!\!\!/^2(\beta,\rho,\sigma,\underline{\beta},\underline{\alpha})|^2 \qquad (5.1.37)
\end{aligned}
$$

with $\gamma+\delta=4$.

ii) The presence of the integral norms

$$
\int_{C(u)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\ ,\ \int_{\underline{C}(\underline{u})} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)
$$

and of the integral norms

$$
\int_{C(u)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\ ,\ \int_{\underline{C}(\underline{u})} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)
$$

is due to two different reasons. The first one is that the Bianchi evolution equations for the Riemann components α and $\underline{\alpha}$ exist only relative to the e_3 and e_4 directions, respectively. This implies that the terms $\mathbf{D}_4\alpha$ and $\mathbf{D}_3\underline{\alpha}$, in the integrand, have to be trasformed into $\mathbf{D}_3\alpha + 2\mathbf{D}_T\alpha$ and $\mathbf{D}_4\underline{\alpha} - 2\mathbf{D}_T\alpha$ and that the Bianchi equations can be applied only to the first part of them.

The second reason is that the norm integrals along $C(u)$ do not contain terms with $\underline{\alpha}(W)$ and the norm integrals along $\underline{C}(\underline{u})$ do not contain terms with $\alpha(W)$.

Recalling the definition of signature given in Chapter 3, see 3.1.23, it follows that, when we are considering te norms along $C(u)$, each time the signature of the integrand is $+3$, giving signature $+1$ to $\mathbf{D}_4$ and -1 to $\mathbf{D}_3$, then the integrand is of type $\nabla\!\!\!/ \mathbf{D}_4\alpha$ or, using the Bianchi equations, contains a term of this type. This implies that, to estimate it, we have to estimate an integral norm of $\nabla\!\!\!/ \mathbf{D}_T\alpha$ which requires $\int_{C(u)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)$ to be bounded.

If, viceversa, the signature is $+4$, the integrand must contain $\mathbf{D}_4^2\alpha$ and, to estimate this term, we have to express again $\mathbf{D}_4$ as $\mathbf{D}_3 + 2\mathbf{D}_T$. This implies that, finally, we have to consider integrals of $\mathbf{D}_T^2\alpha$ which require [14], to be controlled, the norms $\int_{C(u)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)$.

The second reason shows up in the following way: when the integrand of a $C(u)$ integral norm has signature -2, it contains a $\mathbf{D}_3\underline{\beta}$ or a $\mathbf{D}_3^2(\rho,\sigma)$ term which produces, using the Bianchi equations, a $\nabla\!\!\!/ \underline{\alpha}(W)$ term. This term cannot be estimated in a straightforward way using the $\mathcal{Q}_2$ norms[15]. Therefore, in this case, the strategy is to substitute $\mathbf{D}_3$ with $-\mathbf{D}_4 + 2\mathbf{D}_T$ which, again, implies that we have to use the $\int_{C(u)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)$ norm to control the terms containing $\mathbf{D}_T\underline{\beta}$ or $\nabla\!\!\!/ \mathbf{D}_T\underline{\beta}$. If the integrand signature is -3, repeating the previous argument, it follows that we have to control also terms derived twice with respect to T like, for instance, $\mathbf{D}_T^2\underline{\beta}$, which again require, to be controlled, the norms $\int_{C(u)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)$.

Exactly the same discussion holds for the integrals along $\underline{C}(\underline{u})$ with all the signatures interchanged. These integrals norms are needed, in this case, for the integrands of signature $+2$, $+3$ and -3, -4.

[14]The reason why the integral norm $\int_{C(u)} Q(\hat{\mathcal{L}}_T^2 W)(\bar{K},\bar{K},\bar{K},e_4)$ is not sufficient is connected to the weight of the integrand.

[15]Nevertheless see Proposition 5.1.4.

5.1.5 Asymptotic behaviour of the null Riemann components

It is an immediate consequence of Corollary 4.1.1 and of Propositions 5.1.2, 5.1.3, 5.1.5 that the sup norms in the inequalities 3.7.1 are bounded by $\mathcal{Q}_{\mathcal{K}}$,

$$\begin{aligned}
&\sup_{\mathcal{K}} r^{7/2}|\alpha| \leq c\mathcal{Q}_{\mathcal{K}}\ ,\ \sup_{\mathcal{K}} r|u|^{\frac{5}{2}}|\underline{\alpha}| \leq c\mathcal{Q}_{\mathcal{K}}\\
&\sup_{\mathcal{K}} r^{7/2}|\beta| \leq c\mathcal{Q}_{\mathcal{K}}\ ,\ \sup_{\mathcal{K}} r^2|u|^{\frac{3}{2}}|\underline{\beta}| \leq c\mathcal{Q}_{\mathcal{K}}\\
&\sup_{\mathcal{K}} r^3|\rho| \leq c\mathcal{Q}_{\mathcal{K}}\ ,\ \sup_{\mathcal{K}} r^3|u|^{\frac{1}{2}}|(\rho-\overline{\rho},\sigma)| \leq c\mathcal{Q}_{\mathcal{K}}
\end{aligned}$$

Theorem **M8**, proved in the next chapter, completes the proof of 3.7.1.

5.1.6 The asymptotic behaviour of $\overline{\rho}$

We prove the following lemma,

Lemma 5.1.2 *Assuming that the connection coefficients satisfy the inequality*

$$\mathcal{O} \leq \epsilon_0\ ,$$

the average of ρ on the two dimensional surfaces $S(\lambda,\nu)$,

$$\overline{\rho}(\lambda,\nu) = \frac{1}{|S(\lambda,\nu)|}\int_{S(\lambda,\nu)} \rho\ \ ,$$

satisfies the following estimates:

$$\sup_{\mathcal{K}} |r^3\overline{\rho}| \leq \sup_{\mathcal{K}\cap\Sigma_0} |r^3\overline{\rho}| + c\left(\sup_{\mathcal{K}} \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)\right)^{\frac{1}{2}} + c\epsilon_0\underline{\mathcal{R}}_{[0]}$$

Proof: proceeding as in Lemma 4.3.4 we obtain that $\overline{\rho}$ satisfies the following evolution equation

$$\frac{d}{d\lambda}\overline{\rho} = -\overline{\Omega tr\underline{\chi}}\,\overline{\rho} + \frac{1}{|S(\lambda,\nu)|}\int_{S(\lambda,\nu)}\left(\frac{d\rho}{d\lambda} + \Omega tr\underline{\chi}\rho\right)$$

Using the Bianchi equation 3.2.4 for ρ along the $\underline{C}(\nu)$ null hypersurfaces

$$\frac{d}{d\lambda}\rho + \frac{3}{2}\Omega tr\underline{\chi}\rho = \Omega(-\mathrm{d\!/\!iv}\,\underline{\beta} - 2\eta\cdot\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\underline{\alpha} + \zeta\cdot\underline{\beta})$$

it follows denoting $f \equiv r^3\overline{\rho}$,

$$\begin{aligned}\frac{d}{d\lambda}f &= \frac{d}{d\lambda}r^3\overline{\rho} = r^3\left(\frac{3}{2}\overline{\Omega\mathrm{tr}\underline{\chi}}\,\overline{\rho} + \frac{d}{d\lambda}\overline{\rho}\right)\\ &= \frac{1}{2}\frac{r^3}{|S(\lambda,\nu)|}\int_{S(\lambda,\nu)}(\overline{\Omega tr\underline{\chi}} - \Omega\mathrm{tr}\underline{\chi})(\rho-\overline{\rho}) \qquad (5.1.38)\\ &+ \frac{r^3}{|S(\lambda,\nu)|}\int_{S(\lambda,\nu)}\Omega\left(-\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta} - 2\eta\cdot\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\underline{\alpha} + \zeta\cdot\underline{\beta}\right)\end{aligned}$$

From the assumptions on the Ricci coefficients

$$\left|\frac{d}{d\lambda}f\right| \leq c\left(\int_{S(\lambda,\nu)}\frac{1}{r}|\rho-\overline{\rho}| + \int_{S(\lambda,\nu)}r|\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta}| + \int_{S(\lambda,\nu)}\frac{1}{r}|\underline{\beta}| + \int_{S(\lambda,\nu)}\frac{1}{r}|\underline{\alpha}|\right)$$

so that, integrating along $\underline{C}(\nu)\cap V(\lambda,\nu)$,

$$\begin{aligned}\int_{\lambda_0}^{\lambda}d\lambda'|\frac{d}{d\lambda}f|(\lambda',\nu) &\leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}O(\frac{1}{r^3})|r^2(\rho-\overline{\rho})| + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)}O(\frac{1}{r\lambda'})|r^2\lambda'\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta}|\\ &+ \int_{\underline{C}(\nu)\cap V(\lambda,\nu)}O(\frac{1}{r\lambda'^2})|\lambda'^2\underline{\alpha}| + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)}O(\frac{1}{r^2\lambda'})|r\lambda'\underline{\beta}|\ .\end{aligned}$$

Applying the Schwartz inequality, we obtain

$$\begin{aligned}\int_{\lambda_0}^{\lambda}d\lambda'|\frac{d}{d\lambda}f|(\lambda',\nu) &\leq c\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^4|(\rho-\overline{\rho})|^2\right)^{\frac{1}{2}} + \left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^4\lambda'^2|\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta}|^2\right)^{\frac{1}{2}}\\ &+ \left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}\lambda'^4|\underline{\alpha}|^2\right)^{\frac{1}{2}} + \left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^2\lambda'^2|\underline{\beta}|^2\right)^{\frac{1}{2}}\end{aligned}$$

Using the estimates for the L^2 weighted norms of the Riemann null components, proved in Propositions 5.1.2 and 5.1.3,

$$\begin{aligned}&\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^4|(\rho-\overline{\rho})|^2 \leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)\\ &\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^4\lambda'^2|\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta}|^2 \leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)\\ &\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^2\lambda'^2|\underline{\beta}|^2 \leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)\\ &\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}\lambda'^4|\underline{\alpha}|^2 \leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)\end{aligned}$$

and taking the sup over $\mathcal{K}$ the proof is completed. Using the previous result, see Proposition 5.1.5,

$$\begin{aligned}\sup_{\mathcal{K}}|r^3\tau_-^{\frac{1}{2}}(\rho-\overline{\rho})| &\leq c\sup_{\mathcal{K}}\left[\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)\right)^{\frac{1}{2}}\right.\\ &+ \left.\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3)\right)^{\frac{1}{2}}\right] \qquad (5.1.39)\end{aligned}$$

we conclude that

$$\begin{aligned}\sup_{\mathcal{K}}|r^3\rho| &\leq \sup_{\mathcal{K}}|r^3(\rho-\overline{\rho})|+\sup_{\mathcal{K}}|r^3\overline{\rho}|\\ &\leq \sup_{r\in\mathcal{K}\cap\Sigma_0}|r^3\overline{\rho}|+c\sup_{\mathcal{K}}\left[\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)\right)^{\frac{1}{2}}\right.\\ &\left.+\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3)\right)^{\frac{1}{2}}\right] \qquad (5.1.40)\end{aligned}$$

5.2 Appendix to Chapter 5

5.2.1 Proof of Proposition 5.1.4

We recall

Corollary 4.1.1: *Under the assumptions of Lemma 4.1.1 and Lemma 4.1.2 the following estimates hold*

$$\begin{aligned}\left(\int_{S(\lambda,\nu)}r^4|F|^4\right)^{\frac{1}{4}} &\leq \left(\int_{C(\lambda)\cap\Sigma_0}r^4|F|^4\right)^{\frac{1}{4}}\\ &+c\left(\int_{C(\lambda)\cap V(\lambda,\nu)}|F|^2+r^2|\nabla\!\!\!/ F|^2+r^2|\mathbf{D}\!\!\!\!/_4F|^2\right)^{\frac{1}{2}}\\ \left(\int_{S(\lambda,\nu)}r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} &\leq \left(\int_{C(\lambda)\cap\Sigma_0}r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} \qquad (5.2.1)\\ &+c\left(\int_{C(\lambda)\cap V(\lambda,\nu)}|F|^2+r^2|\nabla\!\!\!/ F|^2+\tau_-^2|\mathbf{D}\!\!\!\!/_4F|^2\right)^{\frac{1}{2}}\end{aligned}$$

and

$$\left(\int_{S(\lambda,\nu)}r^4|F|^4\right)^{\frac{1}{4}} \leq \left(\int_{\underline{C}(\nu)\cap\Sigma_0}r^4|F|^4\right)^{\frac{1}{4}}$$

$$+c\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}|F|^2+r^2|\nabla\!\!\!/ F|^2+r^2|\mathbf{D}\!\!\!\!/_3F|^2\right)^{\frac{1}{2}}$$

$$\left(\int_{S(\lambda,\nu)}r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} \leq \left(\int_{\underline{C}(\nu)\cap\Sigma_0}r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} \tag{5.2.2}$$

$$+c\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}|F|^2+r^2|\nabla\!\!\!/ F|^2+\tau_-^2|\mathbf{D}\!\!\!\!/_3F|^2\right)^{\frac{1}{2}}$$

Proof of inequalities 5.1.34: We discuss the proof of the more relevant inequalities in 5.1.34. The other ones can be easily deduced with the same technique.

i) $\|r^\delta\tau_-^{\frac{5}{2}}\underline{\alpha}\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

From the second equation of 5.2.1 we have

$$|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}\underline{\alpha}|_{p=4,S}\equiv\left(\int_{S(\lambda,\nu)}r^2\tau_-^{10}|\underline{\alpha}|^4\right)^{\frac{1}{4}}\leq\left(\int_{\underline{C}(\nu)\cap\Sigma_0}r^2\tau_-^{10}|\underline{\alpha}|^4\right)^{\frac{1}{4}}$$

$$+c\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}\tau_-^4|\underline{\alpha}|^2+r^2\tau_-^4|\nabla\!\!\!/\underline{\alpha}|^2+\tau_-^6|\mathbf{D}\!\!\!\!/_3\underline{\alpha}|^2\right)^{\frac{1}{2}}$$

Moreover

$$\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}(\tau_-^4|\underline{\alpha}|^2+r^2\tau_-^4|\nabla\!\!\!/\underline{\alpha}|^2)\leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_3)\ .$$

To estimate $\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}\tau_-^6|\mathbf{D}\!\!\!\!/_3\underline{\alpha}|^2$ we express $\underline{\alpha}_3$ in terms of $\underline{\alpha}_4$ and $\mathbf{D}\!\!\!\!/_T$:

$$\underline{\alpha}_3=2\mathbf{D}\!\!\!\!/_T\underline{\alpha}-\underline{\alpha}_4+(\frac{5}{2}\mathrm{tr}\underline{\chi}+\frac{1}{2}\mathrm{tr}\chi)\underline{\alpha}\ .$$

As $\mathrm{tr}\underline{\chi}$, $\mathrm{tr}\chi=O(\frac{1}{r})$ we are left with estimating

$$||\tau_-^3\mathbf{D}\!\!\!\!/_T\underline{\alpha}||_{2,\underline{C}(\nu)\cap V(\lambda,\nu)},\ ||\tau_-^3\underline{\alpha}_4||_{2,\underline{C}(\nu)\cap V(\lambda,\nu)},\ ||\tau_-^2\underline{\alpha}||_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\ .$$

The only term we have to estimate is the first one. In fact the second one can be bounded by $||\tau_-r^2\underline{\alpha}_4||_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$ which has been bounded in Proposition 5.1.3 and the third one is controlled in Proposition 5.1.2. Apart from correction terms, this amounts to estimate $\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}\tau_-^6|\underline{\alpha}(\hat{\mathcal{L}}_TW)|^2$ which is controlled by $\int_{\underline{C}\cap V}Q(\hat{\mathcal{L}}_TW)(\bar{K},\bar{K},\bar{K},e_3)$.

In conclusion

$$\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}\left(\tau_-^4|\underline{\alpha}|^2+r^2\tau_-^4|\nabla\!\!\!\!/\,\underline{\alpha}|^2+\tau_-^6|\mathbf{D}\!\!\!\!/_3\underline{\alpha}|^2\right)$$
$$\leq c\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_3)+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_TW)(\bar{K},\bar{K},\bar{K},e_3)$$

Therefore

$$\int_{C(\lambda)\cap V(\lambda,\nu)}r^{2\delta}\tau_-^5|\underline{\alpha}|^2=\int_{\nu_0}^{\nu}\int_{S(\lambda,\nu')}r^{2\delta}\tau_-^5|\underline{\alpha}|^2$$
$$\leq c\int_{\nu_0}^{\nu}\left(\int_{S(\lambda,\nu')}\right)^{\frac{1}{2}}\left[\left(\int_{S(\lambda,\nu')}r^{4\delta}\tau_-^{10}|\underline{\alpha}|^4\right)^{\frac{1}{4}}\right]^2$$
$$\leq c\int_{\nu_0}^{\nu}r(\lambda,\nu')^{2\delta}\left[\left(\int_{S(\lambda,\nu')}r^2\tau_-^{10}|\underline{\alpha}|^4\right)^{\frac{1}{4}}\right]^2$$
$$\leq c\int_{\nu_0}^{\nu}r(\lambda,\nu')^{2\delta}\left(\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_3)\right.$$
$$\left.+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_TW)(\bar{K},\bar{K},\bar{K},e_3)\right)$$

In conclusion, if $\delta<-\frac{1}{2}$, we have

$$||r^\delta\tau_-^{\frac{5}{2}}\underline{\alpha}||_{2,C(\lambda)\cap V(\lambda,\nu)} \leq c\sup_{\nu'\in V(\lambda,\nu)\cap\Sigma_0}\left(\int_{\underline{C}(\nu')\cap V(\lambda,\nu')}Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_3)\right.$$
$$\left.+\int_{\underline{C}(\nu')\cap V(\lambda,\nu')}Q(\hat{\mathcal{L}}_TW)(\bar{K},\bar{K},\bar{K},e_3)\right)^{\frac{1}{2}}$$

ii. $\|r^\delta\tau_-^\sigma\underline{\beta}_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

To estimate this term we use the Bianchi equation

$$\underline{\beta}_3\equiv\mathbf{D}\!\!\!\!/_3\underline{\beta}+2\mathrm{tr}\underline{\chi}\ \underline{\beta}=-\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\alpha}-\left[2\underline{\omega}\underline{\beta}+(-2\zeta+\eta)\cdot\underline{\alpha}\right]\ .$$

it comes out that the main term to estimate is $\|r^\delta\tau_-^\sigma\nabla\!\!\!\!/\,\underline{\alpha}\|_{2,C(\lambda)\cap V(\lambda,\nu)}$. From its estimate, which can be done as for the estimate of $||r^\delta\tau_-^{\frac{5}{2}}\underline{\alpha}||_{2,C(\lambda)\cap V(\lambda,\nu)}$, we conclude that this term is bounded for $\delta<\frac{1}{2},\sigma\leq\frac{5}{2}$. Therefore

$$\|r^\delta\tau_-^\sigma\underline{\beta}_3\|_{2,C(\lambda)\cap V(\lambda,\nu)} \leq c\sup_{\nu'\in V(\lambda,\nu)\cap\Sigma_0}\left(\int_{\underline{C}(\nu')\cap V(\lambda,\nu')}Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_3)\right.$$

$$
+\int_{\underline{C}(\nu')\cap V(\lambda,\nu')} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3)
+\int_{\underline{C}(\nu')\cap V(\lambda,\nu')} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)\Bigg)^{\frac{1}{2}}
$$

for $\delta<\frac{1}{2}$, $\sigma\le\frac{5}{2}$.

iii. $\|r^\delta\tau_-^\sigma\alpha\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

This term is estimated in the same way as the term $\|r^\delta\tau_-^\sigma\underline{\alpha}\|_{2,C(\lambda)\cap V(\lambda,\nu)}$ with the appropriate weights. Therefore we have, for $\delta<2$,

$$
\begin{aligned}
||r^\delta\alpha||_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} \le\ & c\sup_{\lambda'\in V(\lambda,\nu)\cap\Sigma_0}\left(\int_{C(\lambda')\cap V(\lambda',\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\right.\\
&\left.+\int_{C(\lambda')\cap V(\lambda',\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\right)^{\frac{1}{2}}
\end{aligned}
$$

iv. $\|r^\delta\tau_-^\sigma\beta_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

The estimate of this term proceeds as that of $\|r^\delta\tau_-^\sigma\underline{\beta}_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}$. We conclude that

$$
\begin{aligned}
\|r^\lambda\beta_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} \le\ & c\sup_{\lambda'\in V(\lambda,\nu)\cap\Sigma_0}\left(\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\right.\\
&+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4)\\
&\left.+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\right)^{\frac{1}{2}}
\end{aligned}
$$

with $\lambda<3$.

v. $\|r^\delta\tau_-^\sigma(\rho,\sigma)_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

This term is estimated differently [16] from $\|r^\delta\tau_-^\sigma(\rho,\sigma)_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}$. From the Bianchi equation

$$
\rho_4\equiv\mathbf{D}_4\rho+\frac{3}{2}\mathrm{tr}\chi\rho=\mathrm{d}\!\!/\mathrm{iv}\,\beta-\left[\frac{1}{2}\underline{\hat{\chi}}\cdot\alpha-\zeta\cdot\beta-2\underline{\eta}\cdot\beta\right]
$$

[16] If one tries to proceed as done for $\|r^\delta\tau_-^\sigma(\rho,\sigma)_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}$ one gets a weaker result.

we observe that the main term to control is $\|r^\delta\tau_-^\sigma\nabla\!\!\!/\beta\|_{2,C(\lambda)\cap V(\lambda,\nu)}$ which has been bounded in the estimate of the norm $\underline{\mathcal{R}}_1[\beta](u,\underline{u})$, by the integral $\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)$ for $\lambda\leq 3$, $\sigma\leq 0$. For the main correction term we have to estimate $\|r^{(\delta-1)}\tau_-^{(\sigma-\frac{3}{2})}\alpha\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$. From the previous estimate of $\|r^\delta\tau_-^\sigma\alpha\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$ it follows that

$$\begin{aligned}||r^{\delta-1}\tau_-^{\sigma-\frac{3}{2}}\alpha||_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} \quad\leq\quad & c\sup_{\lambda'\in V(\lambda,\nu)\cap\Sigma_0}\left(\int_{C(\lambda')\cap V(\lambda',\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\right.\\ &\left.+\int_{C(\lambda')\cap V(\lambda',\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\right)^{\frac{1}{2}}\end{aligned}$$

with $\delta<3$ and $\sigma\leq\frac{3}{2}$, so that finally we have

$$\begin{aligned}\|r^\delta\tau_-^\sigma(\rho,\sigma)_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} \quad\leq\quad & c\sup_{\lambda'\in V(\lambda,\nu)\cap\Sigma_0}\left(\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\right.\\ &+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\\ &\left.+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4)\right)^{\frac{1}{2}}\end{aligned}$$

for $\delta<3$ and $\sigma\leq 0$.

5.2.2 Proof of Proposition 5.1.5

We give the main ideas of the proof, neglecting all the correction terms coming from the commutation relations, the relation between the covariant derivatives and the $\mathcal{L}_X$ derivatives and those between the $\mathcal{L}_X$ and the modified $\hat{\mathcal{L}}_X$ derivatives. At the end we discuss, in some specific case, the structure of the correction terms for the second derivatives. Therefore the equality sign appearing in the proof has always to be interpreted as "equal apart from correction terms"[17].

1. $\mathcal{R}_2[\alpha]$**:**

$$\begin{aligned}\mathcal{R}_2[\alpha](\lambda,\nu) &= \|r^4\nabla\!\!\!/^2\alpha\|_{2,C(\lambda)\cap V(\lambda,\nu)}+\|r^4\nabla\!\!\!/\alpha_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\ &+\|r^4\nabla\!\!\!/\alpha_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}+\|\tau_-r^3\alpha_{33}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\ &+\|r^4\alpha_{34}\|_{2,C(\lambda)\cap V(\lambda,\nu)}+\|\tau_-r^3\alpha_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\end{aligned}$$

[17] The results here are exact if we interpret the Riemann components as the components of a Weyl tensor in the Minkowski space as the background space. We also omit the numerical coefficients in front of the various integrals.

Estimate for $\|r^4 \nabla\!\!\!/ \alpha_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

$$\begin{aligned}
\|r^4 \nabla\!\!\!/ \alpha_4\|^2_{2,C(\lambda)\cap V(\lambda,\nu)} &= \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \alpha_4(W)|^2 \\
&= \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \alpha_3(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \mathbf{D}\!\!\!\!/_T \alpha(W)|^2 \\
&= \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \alpha_3(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^6 |\alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)|^2 \\
&\leq \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4) \\
&+ \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4) \qquad (5.2.3)
\end{aligned}$$

Estimate for $\|\tau_- r^3 \alpha_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

$$\begin{aligned}
&\|\tau_- r^3 \alpha_{44}\|^2_{2,C(\lambda)\cap V(\lambda,\nu)} = \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_{44}(W)|^2 \\
&= \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_{43}(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |(\mathbf{D}\!\!\!\!/_T \alpha)_4(W)|^2 \\
&= \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_{43}(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_4(\hat{\mathcal{L}}_T W)|^2 \\
&= \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_{43}(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_3(\hat{\mathcal{L}}_T W)|^2 \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\mathbf{D}\!\!\!\!/_T \alpha(\hat{\mathcal{L}}_T W)|^2 \\
&= \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_{43}(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\nabla\!\!\!/ \beta(\hat{\mathcal{L}}_T W)|^2 \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 \frac{1}{\tau_+^2} |\mathbf{D}\!\!\!\!/_S \alpha(\hat{\mathcal{L}}_T W)|^2 \\
&= \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_{43}(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\beta(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)|^2 \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\alpha(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)|^2 \\
&\leq \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4) + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4) \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4) \qquad (5.2.4)
\end{aligned}$$

Remark: *To estimate* $\|\tau_- r^3\alpha_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)}$ *we are obliged to use the* $\mathcal{Q}$ *norm* $\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)$.

2. $\mathcal{R}_2[\beta]$**:**

$$\begin{aligned}\mathcal{R}_2[\beta](\lambda,\nu) = &\|r^4\nabla\!\!\!/^2\beta\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^3\nabla\!\!\!/\beta_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\ &+\|r^4\nabla\!\!\!/\beta_4\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_-^2 r^2\beta_{33}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\ &+\|r^4\beta_{34}\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|r^4\beta_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\end{aligned}$$

Estimate for $\|r^4\beta_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

$$\begin{aligned}\|r^4\beta_{44}\|^2_{2,C(\lambda)\cap V(\lambda,\nu)} &= \int_{C(\lambda)\cap V(\lambda,\nu)} r^8|\beta_{44}(W)|^2 = \int_{C(\lambda)\cap V(\lambda,\nu)} r^8|\nabla\!\!\!/\alpha_4(W)|^2\\ &= \int_{C(\lambda)\cap V(\lambda,\nu)} r^8|\nabla\!\!\!/\alpha_3(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8|\nabla\!\!\!/ \mathbf{D}\!\!\!/_T\alpha(W)|^2\\ &= \int_{C(\lambda)\cap V(\lambda,\nu)} r^8|\nabla\!\!\!/\alpha_3(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\alpha(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)|^2\\ &\le \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4)\\ &+ \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4) \qquad (5.2.5)\end{aligned}$$

3. $\mathcal{R}_2[(\rho,\sigma)]$**:**

$$\begin{aligned}\mathcal{R}_2[(\rho,\sigma)](u,\underline{u}) = &\|\tau_- r^3\nabla\!\!\!/^2(\rho,\sigma)\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_-^2 r^2\nabla\!\!\!/(\rho,\sigma)_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\ &+\|r^4\nabla\!\!\!/(\rho,\sigma)_4\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^3(\rho,\sigma)_{34}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\\ &+\|r^4(\rho,\sigma)_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_-^3 r(\rho,\sigma)_{33}\|_{2,C(\lambda)\cap V(\lambda,\nu)}\end{aligned}$$

Estimate for $\|\tau_-^3 r(\rho,\sigma)_{33}\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

$$\begin{aligned}&\|\tau_-^3 r(\rho,\sigma)_{33}\|^2_{2,C(\lambda)\cap V(\lambda,\nu)}\\ &= \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|(\rho,\sigma)_{33}(W)|^2 = \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/\underline{\beta}_3(W)|^2\\ &= \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/\underline{\beta}_4(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/ \mathbf{D}\!\!\!/_T\underline{\beta}(W)|^2\\ &= \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/^2(\rho,\sigma)(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/\underline{\beta}(\hat{\mathcal{L}}_T W)|^2\end{aligned}$$

$$
\begin{aligned}
&= \int_{C(\lambda)\cap V(\lambda,\nu)} \frac{\tau_-^6}{r^2}|(\rho,\sigma)(\hat{\mathcal{L}}_O^2 W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6|\underline{\beta}(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)|^2 \\
&\leq \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4) + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)
\end{aligned}
\tag{5.2.6}
$$

4. $\mathcal{R}_2[\underline{\beta}]$**:**

$$
\begin{aligned}
\mathcal{R}_2[\underline{\beta}](\lambda,\nu) &= \|\tau_-^2 r^2 \nabla\!\!\!/^2\underline{\beta}\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_- r^3\nabla\!\!\!/\underline{\beta}_4\|_{2,C(\lambda)\cap V(\lambda,\nu)} \\
&+ \|\tau_-^3 r\nabla\!\!\!/\underline{\beta}_3\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_-^2 r^2\underline{\beta}_{34}\|_{2,C(\lambda)\cap V(\lambda,\nu)} \\
&+ \|r^4\underline{\beta}_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)} + \|\tau_-^3 r\underline{\beta}_{33}\|_{2,C(\lambda)\cap V(\lambda,\nu)}
\end{aligned}
$$

Estimate for $\|\tau_- r^3\nabla\!\!\!/\underline{\beta}_3\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

$$
\begin{aligned}
&\|\tau_-^3 r\nabla\!\!\!/\underline{\beta}_3\|^2_{2,C(\lambda)\cap V(\lambda,\nu)} = \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/\underline{\beta}_3|^2 \\
&= \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/\underline{\beta}_4|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/\mathbf{D}\!\!\!\!/_T\underline{\beta}|^2 \\
&= \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/^2\rho(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/\underline{\beta}(\hat{\mathcal{L}}_T W)|^2 \\
&= \int_{C(\lambda)\cap V(\lambda,\nu)} \frac{\tau_-^6}{r^2}|\rho(\hat{\mathcal{L}}_O^2 W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6|\underline{\beta}(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)|^2 \qquad (5.2.7) \\
&\leq \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4) + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)
\end{aligned}
$$

Estimate for $\|\tau_-^3 r\underline{\beta}_{33}\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

$$
\begin{aligned}
&\|\tau_-^3 r\underline{\beta}_{33}\|^2_{2,C(\lambda)\cap V(\lambda,\nu)} = \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\underline{\beta}_{33}|^2 = \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\underline{\beta}_{44}|^2 \\
&+ \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\mathbf{D}\!\!\!\!/_T\underline{\beta}_4|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\mathbf{D}\!\!\!\!/_T^2\underline{\beta}|^2 \\
&= \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/^2\beta|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\mathbf{D}\!\!\!\!/_T\nabla\!\!\!/\rho(W)|^2 \\
&+ \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6 r^2|\underline{\beta}(\hat{\mathcal{L}}_T^2 W)|^2 = \int_{C(\lambda)\cap V(\lambda,\nu)} \frac{\tau_-^6}{r^2}|\beta(\hat{\mathcal{L}}_O^2 W)|^2 \\
&+ \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6|\rho(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^6|\underline{\beta}(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)|^2
\end{aligned}
$$

$$\leq \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4) + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)$$
$$+\int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4) \tag{5.2.8}$$

5. $\underline{\mathcal{R}}_2[\beta]$**:**

$$\underline{\mathcal{R}}_2[\beta](\lambda,\nu) = \|r^4 \nabla\!\!\!/^2 \beta\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|r^4 \nabla\!\!\!/ \beta_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|r^4 \nabla\!\!\!/ \beta_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$$
$$+\|r^4 \beta_{43}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_- r^3 \beta_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_- r^3 \beta_{43}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$$

Estimate for $\|r^4 \nabla\!\!\!/ \beta_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

$$\|r^4 \nabla\!\!\!/ \beta_4\|^2_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} = \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \beta_4(W)|^2$$
$$= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \beta_3(W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \mathbf{D}\!\!\!\!/_T \beta(W)|^2$$
$$= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \beta_3(W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \beta(\hat{\mathcal{L}}_T W)|^2$$
$$= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \beta_3(W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} r^6 |\beta(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)|^2 \tag{5.2.9}$$
$$\leq \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3) + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)$$

Estimate for $\|\tau_- r^3 \beta_{44}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

$$\|\tau_- r^3 \beta_{44}\|^2_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} = \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\beta_{44}(W)|^2$$
$$= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\beta_{33}(W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\mathbf{D}\!\!\!\!/_T \beta_3(W)|^2$$
$$+ \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\mathbf{D}\!\!\!\!/_T^2 \beta(W)|^2$$
$$= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\beta_{33}(W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\rho(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)|^2$$
$$+ \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\beta(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)|^2$$
$$\leq \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3) + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)$$
$$\int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3) \tag{5.2.10}$$

6. $\underline{\mathcal{R}}_2[(\rho,\sigma)]$**:**

$$\begin{aligned}\underline{\mathcal{R}}_2[(\rho,\sigma)](\lambda,\nu) &= \|r^4\nabla\!\!\!/^2(\rho,\sigma)\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}+\|\tau_-r^3\nabla\!\!\!/(\rho,\sigma)_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\ &+ \|r^4\nabla\!\!\!/(\rho,\sigma)_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}+\|\tau_-r^3(\rho,\sigma)_{34}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\ &+ \|\tau_-^2r^2(\rho,\sigma)_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}+\|r^4(\rho,\sigma)_{44}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\end{aligned}$$

Estimate for $\|r^4(\rho,\sigma)_{44}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

$$\begin{aligned}&\|r^4(\rho,\sigma)_{44}\|^2_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}=\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^8|(\rho,\sigma)_{44}(W)|^2\\ &=\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^8|(\rho,\sigma)_{34}(W)|^2+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^8|\left(\mathbf{D}\!\!\!\!/_T(\rho,\sigma)\right)_4(W)|^2\\ &=\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^8|(\rho,\sigma)_{34}(W)|^2+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^8|(\rho,\sigma)_4(\hat{\mathcal{L}}_TW)|^2\\ &=\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^8|(\rho,\sigma)_{34}(W)|^2+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^8|\nabla\!\!\!/\beta(\hat{\mathcal{L}}_TW)|^2\\ &=\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^8|(\rho,\sigma)_{34}(W)|^2+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}r^6|\beta(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_TW)|^2 \qquad (5.2.11)\\ &\le\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_O^2W)(\bar{K},\bar{K},T,e_3)+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_TW)(\bar{K},\bar{K},\bar{K},e_3)\end{aligned}$$

7. $\underline{\mathcal{R}}_2[\underline{\beta}]$**:**

$$\begin{aligned}\underline{\mathcal{R}}_2[\underline{\beta}](\lambda,\nu) &= \|\tau_-r^3\nabla\!\!\!/^2\underline{\beta}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}+\|\tau_-^2r^2\nabla\!\!\!/\underline{\beta}_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\ &+ \|r^4\nabla\!\!\!/\underline{\beta}_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}+\|\tau_-^3r\underline{\beta}_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\ &+ \|\tau_-r^3\underline{\beta}_{34}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}+\|r^4\underline{\beta}_{44}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\end{aligned}$$

Estimate for $\|\tau_-^3r\underline{\beta}_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

$$\begin{aligned}&\|\tau_-^3r\underline{\beta}_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}=\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}\tau_-^6r^2|\underline{\beta}_{33}(W)|^2\\ &=\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}\tau_-^6r^2|\nabla\!\!\!/\underline{\alpha}_4(W)|^2+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}\tau_-^6r^2|\nabla\!\!\!/\mathbf{D}\!\!\!\!/_T\underline{\alpha}(W)|^2\\ &=\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}\tau_-^6r^2|\nabla\!\!\!/^2\underline{\beta}(W)|^2+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}\tau_-^6r^2|\nabla\!\!\!/\underline{\alpha}(\hat{\mathcal{L}}_TW)|^2\end{aligned}$$

$$= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \frac{\tau_-^6}{r^2}|\underline{\beta}(\hat{\mathcal{L}}_O^2 W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^6|\underline{\beta}(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)|^2 \tag{5.2.12}$$
$$\leq \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3) + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},T,e_3)$$

8. $\underline{\mathcal{R}}_2[\underline{\alpha}]$**:**

$$\begin{aligned}\underline{\mathcal{R}}_2[\underline{\alpha}](\lambda,\nu) &= \|\tau_-^2 r^2 \nabla\!\!\!/^2\underline{\alpha}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_-^3 r \nabla\!\!\!/\underline{\alpha}_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\ &+ \|\tau_- r^3 \nabla\!\!\!/\underline{\alpha}_4\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_-^4 \underline{\alpha}_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\\ &+ \|\tau_-^2 r^2 \underline{\alpha}_{34}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} + \|\tau_- r^3 \underline{\alpha}_{44}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}\end{aligned}$$

Estimate for $\|\tau_-^3 r\nabla\!\!\!/\underline{\alpha}_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

$$\|\tau_-^3 r\nabla\!\!\!/\underline{\alpha}_3\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} = \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/\underline{\alpha}_3(W)|^2$$
$$= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/ \mathbf{D}\!\!\!\!/_T\underline{\alpha}(W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/\underline{\alpha}_4(W)|^2$$
$$= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/\underline{\alpha}(\hat{\mathcal{L}}_T W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^6 r^2|\nabla\!\!\!/^2\underline{\beta}(W)|^2$$
$$= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^6|\underline{\alpha}(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \frac{\tau_-^6}{r^2}|\underline{\beta}(\hat{\mathcal{L}}_O^2 W)|^2 \tag{5.2.13}$$
$$\leq \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3) + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)$$

Estimate for $\|\tau_-^2 r^2\underline{\alpha}_{34}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

$$\|\tau_-^2 r^2\underline{\alpha}_{34}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} = \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^4 r^4|\underline{\alpha}_{34}(W)|^2$$
$$= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^4 r^4|\nabla\!\!\!/\underline{\beta}_3(W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^4 r^4|\nabla\!\!\!/^2\underline{\alpha}(W)|^2 \tag{5.2.14}$$
$$= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^4|\underline{\alpha}(\hat{\mathcal{L}}_O^2 W)|^2 \leq \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3)$$

Estimate for $\|\tau_-^4\underline{\alpha}_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)}$

$$\|\tau_-^4\underline{\alpha}_{33}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} = \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^8|\underline{\alpha}_{33}(W)|^2 = \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^8|\underline{\alpha}_{44}(W)|^2$$

$$
\begin{aligned}
&+ \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^8|\not{D}_T\not{D}_T\underline{\alpha}(W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^8|(\not{D}_T\underline{\alpha})_4(W)|^2 \\
&= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^8|\underline{\alpha}_{44}(W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^8|\underline{\alpha}(\hat{\mathcal{L}}_T^2W)|^2 \\
&+ \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^8|\not{\nabla}\underline{\beta}(\hat{\mathcal{L}}_TW)|^2 = \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^8|\underline{\alpha}_{44}(W)|^2 \\
&+ \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^8|\underline{\alpha}(\hat{\mathcal{L}}_T^2W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \frac{\tau_-^8}{r^2}|\underline{\beta}(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_TW)|^2 \\
&= \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \tau_-^8|\underline{\alpha}_{44}(W)|^2 + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \frac{\tau_-^8}{r^2}|\underline{\alpha}(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_TW)|^2 \\
&+ \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} \frac{\tau_-^8}{r^2}|\underline{\beta}(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_TW)|^2 \le \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2W)(\bar{K},\bar{K},T,e_3) \qquad (5.2.15) \\
&+ \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_TW)(\bar{K},\bar{K},\bar{K},e_3) + \int_{\underline{C}(\nu)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_TW)(\bar{K},\bar{K},\bar{K},e_3)
\end{aligned}
$$

We end the appendix discussing at length some of the more delicate estimates of Proposition 5.1.5. The detailed estimates for the other norms are easier and proceed along the same lines.

1.1 $\|r^4\not{\nabla}^2\alpha\|_{2,C(\lambda)\cap V(\lambda,\nu)}$

Using Proposition 7.5.3 of [Ch-Kl] it follows

$$
\begin{aligned}
&\int_{C(\lambda)\cap V(\lambda,\nu)} r^8|\not{\nabla}^2\alpha(W)|^2 \le c\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\mathcal{L}_O\not{\nabla}\alpha(W)|^2 \qquad (5.2.16) \\
&= c\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\not{\nabla}\mathcal{L}_O\alpha(W)|^2 + c\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|[\mathcal{L}_O,\not{\nabla}]\alpha(W)|^2
\end{aligned}
$$

We use the relation, see 5.1.15,

$$
\mathcal{L}_O\alpha(W) \;=\; \alpha(\hat{\mathcal{L}}_OW) + S({}^{(O)}\pi,{}^{(O)}M)\cdot\alpha(W)
$$

where

$$
S({}^{(O)}\pi,{}^{(O)}M)_{abcd} = -\frac{1}{8}\mathrm{tr}^{(O)}\pi\delta_{ac}\delta_{bd} + \frac{1}{2}\delta_{ab}{}^{(O)}\mathbf{i}_{cd}
$$

to rewrite the first integral and estimate it as

$$
\begin{aligned}
&\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\not{\nabla}\mathcal{L}_O\alpha(W)|^2 = \int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\not{\nabla}\alpha(\hat{\mathcal{L}}_OW)|^2 \\
&+\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\not{\nabla}[S({}^{(O)}\pi,{}^{(O)}M)\cdot\alpha(W)]|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\mathcal{L}_O\alpha(\hat{\mathcal{L}}_OW)|^2+\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\nabla\!\!\!/ S({}^{(O)}\pi,{}^{(O)}M)|^2|\alpha(W)|^2\right.\\
&\left.+\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|S({}^{(O)}\pi,{}^{(O)}M)|^2|\nabla\!\!\!/\alpha(W)|^2\right)\\
&\leq c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(\hat{\mathcal{L}}_O^2W)|^2+\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|S({}^{(O)}\pi,{}^{(O)}M)|^2|\alpha(\hat{\mathcal{L}}_OW)|^2\right.\\
&\left.+\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|\nabla\!\!\!/ S({}^{(O)}\pi,{}^{(O)}M)|^2|\alpha(W)|^2+\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|S({}^{(O)}\pi,{}^{(O)}M)|^2|\nabla\!\!\!/\alpha(W)|^2\right)\\
&\leq c\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(\hat{\mathcal{L}}_O^2W)|^2+c\frac{\epsilon_0^2}{r^2}\left(\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(\hat{\mathcal{L}}_OW)|^2\right.\\
&\left.+\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(W)|^2\right)\leq c\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(\hat{\mathcal{L}}_O^2W)|^2+c\epsilon_0^2{\mathcal{R}_{[0]}}^2 \qquad (5.2.17)
\end{aligned}
$$

where we used the following estimates for $S({}^{(O)}\pi,{}^{(O)}M)$ which can be easily deduced from Chapter 3, section 3.7, for $p\in[2,4]$,

$$
\begin{aligned}
&\sup_{\mathcal{K}}|rS({}^{(O)}\pi,{}^{(O)}M)|\leq c\epsilon_0\\
&\sup_{\mathcal{K}}|r^{2-\frac{2}{p}}\nabla\!\!\!/ S({}^{(O)}\pi,{}^{(O)}M)|_{p,S}\leq c\epsilon_0 \qquad (5.2.18)
\end{aligned}
$$

To estimate the second integral of 5.2.17, $\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|[\mathcal{L}_O,\nabla\!\!\!/]\alpha(W)|^2$, we observe that, see 4.8.44, and Corollary 16.1.31 of [Ch-Kl]

$$
\begin{aligned}
([\nabla\!\!\!/,L_O]\alpha)_{abc} &= (\nabla\!\!\!/_bH_{da}+\nabla\!\!\!/_aH_{db}-\nabla\!\!\!/_dH_{ba})\,\alpha_{dc}\\
&+ (\nabla\!\!\!/_bH_{da}+\nabla\!\!\!/_aH_{dc}-\nabla\!\!\!/_dH_{ca})\,\alpha_{db}
\end{aligned}
$$

As, from the estimates of Chapter 3, section 3.7, the following inequalities hold, $p\in[2,4]$,

$$
|[\mathcal{L}_O,\nabla\!\!\!/]\alpha(W)|^2_{p,S}\leq c\frac{\epsilon_0^2}{r^2}|\alpha(W)|^2_{p,S}
$$

we obtain

$$
\int_{C(\lambda)\cap V(\lambda,\nu)} r^6|[\mathcal{L}_O,\nabla\!\!\!/]\alpha(W)|^2\leq c\epsilon_0^2\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(W)|^2\leq c\epsilon_0^2{\mathcal{R}_{[0]}}^2
$$

so that, finally,

$$
\|r^4\nabla\!\!\!/^2\alpha\|^2_{2,C(\lambda)\cap V(\lambda,\nu)}\leq c\int_{C(\lambda)\cap V(\lambda,\nu)} r^4|\alpha(\hat{\mathcal{L}}_O^2W)|^2+c\epsilon_0^2{\mathcal{R}_{[0]}}^2
$$

1.2 $\|r^4 \nabla\!\!\!/ \alpha_4\|_{2,C(\lambda)\cap V(\lambda,\nu)}$.

$$
\begin{aligned}
&\|r^4 \nabla\!\!\!/ \alpha_4\|^2_{2,C(\lambda)\cap V(\lambda,\nu)} = \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \alpha_4(W)|^2 \\
&\leq c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \alpha_3(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \mathbf{D}\!\!\!\!/\,_T \alpha(W)|^2 \right.\\
&\left. + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ (\frac{5}{2}\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\underline{\chi})\alpha(W)|^2 \right)\\
&\leq c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \alpha_3(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \mathbf{D}\!\!\!\!/\,_T \alpha(W)|^2 \right.\\
&\left. + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ (\mathrm{tr}\chi, \mathrm{tr}\underline{\chi})|^2 |\alpha(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |(\mathrm{tr}\chi, \mathrm{tr}\underline{\chi})|^2 |\nabla\!\!\!/ \alpha(W)|^2 \right)\\
&\leq c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \alpha_3(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \mathbf{D}\!\!\!\!/\,_T \alpha(W)|^2 \right.\\
&\left. + \int_{C(\lambda)\cap V(\lambda,\nu)} r^6 |\nabla\!\!\!/ \alpha(W)|^2 + \epsilon_0^2 \int_{C(\lambda)\cap V(\lambda,\nu)} r^2 |\alpha(W)|^2 \right)\\
&\leq c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \alpha_3(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \mathbf{D}\!\!\!\!/\,_T \alpha(W)|^2 + {\mathcal{R}_1}^2 + \epsilon_0^2 {\mathcal{R}_{[0]}}^2 \right)
\end{aligned}
\tag{5.2.19}
$$

where we used the relation $\alpha_4 = -\alpha_3 + 2\mathbf{D}\!\!\!\!/\,_T\alpha + (\frac{5}{2}\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\underline{\chi})\alpha$ and the estimates of Chapter 3 for $\mathrm{tr}\chi, \mathrm{tr}\underline{\chi}, \nabla\!\!\!/ \mathrm{tr}\chi, \nabla\!\!\!/ \mathrm{tr}\underline{\chi}$. Using the Bianchi equation

$$\alpha_3 = \nabla\!\!\!/ \widehat{\otimes}\beta + [4\underline{\omega}\alpha - 3(\hat{\chi}\rho + {}^\star\hat{\chi}\sigma) + (\zeta + 4\eta)\widehat{\otimes}\beta]$$

and the estimates on the Ricci coefficients proved in Chapter 3 we obtain

$$
\begin{aligned}
\int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \alpha_3(W)|^2 &\leq c\int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/^2 \beta(W)|^2 + c\epsilon_0^2 {\mathcal{R}_{[1]}}^2 \\
&\leq c\mathcal{R}_2[\beta]^2 + c\epsilon_0^2 {\mathcal{R}_{[1]}}^2
\end{aligned}
$$

to estimate the last integral of 5.2.19 we recall that, see 5.1.24 and Proposition 5.1.1,

$$\mathbf{D}\!\!\!\!/\,_T \alpha(W) = \mathcal{L}_T \alpha(W) + {}^{(T)}H \cdot \alpha(W) \ .$$

Moreover, using Lemma 5.1.1, we write

$$\nabla\!\!\!/ \mathcal{L}_T \alpha(W) \ = \ \nabla\!\!\!/ \left(\alpha(\hat{\mathcal{L}}_T W) + G_1({}^{(T)}M, {}^{(T)}\pi)\alpha(W) + G_2({}^{(T)}P, {}^{(T)}Q)\beta(W)\right)$$

$$
\begin{aligned}
&= \nabla\!\!\!/ \alpha(\hat{\mathcal{L}}_T W + (\nabla\!\!\!/ G_1({}^{(T)}M, {}^{(T)}\pi))\alpha(W) + G_1({}^{(T)}M, {}^{(T)}\pi)(\nabla\!\!\!/ \alpha(W)) \\
&+ \left(\nabla\!\!\!/ G_2({}^{(T)}P, {}^{(T)}Q)\right)\beta(W) + G_2({}^{(T)}P, {}^{(T)}Q)\nabla\!\!\!/ \beta(W)
\end{aligned}
$$

and, from it,

$$
\begin{aligned}
&\int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \mathbf{D}\!\!\!/_T \alpha(W)|^2 \leq \\
&\leq \int_{C(\lambda)\cap V(\lambda,\nu)} r^6 |\alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |{}^{(O)}G_1|^2 |\alpha(\hat{\mathcal{L}}_T W)|^2 \\
&+ \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ {}^{(T)}H|^2 |\alpha(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |{}^{(T)}H|^2 |\nabla\!\!\!/ \alpha(W)|^2 \\
&+ \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ {}^{(T)}G_1|^2 |\alpha(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |{}^{(T)}G_1|^2 |\nabla\!\!\!/ \alpha(W)|^2 \\
&+ \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ {}^{(T)}G_2|^2 |\beta(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |{}^{(T)}G_2|^2 |\nabla\!\!\!/ \beta(W)|^2
\end{aligned}
$$

Using the estimates, previously proved,

$$
\begin{aligned}
&\sup\left(|r^{(T)}H| + \sup|r^{(O)}G_1|\right) \leq c\epsilon_0 \\
&\sup\left(|r^{(T)}P| + |r^{(T)}Q| + |r^{(T)}M| + |r^{(T)}\pi|\right) \leq c\epsilon_0 \\
&|r^{2-\frac{2}{p}} \nabla\!\!\!/ {}^{(T)}H|_{p,S} \leq c\epsilon_0 \\
&|r^{2-\frac{2}{p}} \nabla\!\!\!/ {}^{(T)}P|_{p,S} + |r^{2-\frac{2}{p}} \nabla\!\!\!/ {}^{(T)}Q|_{p,S} + |r^{2-\frac{2}{p}} \nabla\!\!\!/ {}^{(T)}M|_{p,S} + |r^{2-\frac{2}{p}} \nabla\!\!\!/ {}^{(T)}\pi|_{p,S} \leq c\epsilon_0
\end{aligned}
$$

for $p \in [2,4]$, we obtain

$$
\int_{C(\lambda)\cap V(\lambda,\nu)} r^8 |\nabla\!\!\!/ \mathbf{D}\!\!\!/_T \alpha(W)|^2 \leq \int_{C(\lambda)\cap V(\lambda,\nu)} r^6 |\alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)|^2 + c\epsilon_0^2 {\mathcal{R}_{[1]}}^2
$$

and, finally,

$$
\begin{aligned}
&\|r^4 \nabla\!\!\!/ \alpha_4\|^2_{2,C(\lambda)\cap V(\lambda,\nu)} \leq \int_{C(\lambda)\cap V(\lambda,\nu)} r^6 |\alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)|^2 + c\mathcal{R}_2[\beta]^2 + c(1+\epsilon_0^2){\mathcal{R}_{[1]}}^2 \\
&\leq \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\bar{K}, \bar{K}, \bar{K}, e_4) + c\mathcal{R}_2[\beta]^2 + c(1+\epsilon_0^2){\mathcal{R}_{[1]}}^2 \\
&\leq \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K}, \bar{K}, T, e_4) + \int_{C(\lambda)\cap V(\lambda,\nu)} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\bar{K}, \bar{K}, \bar{K}, e_4) \\
&\quad + c(1+\epsilon_0^2){\mathcal{R}_{[1]}}^2
\end{aligned}
$$

where we used the estimate of Proposition 5.1.5 for the $\mathcal{R}_2[\beta]^2$ term.

1.3 $\|\tau_- r^3 \alpha_{44}\|_{2,C(\lambda)\cap V(\lambda,\nu)}$ **.** Starting from the relationship, easy to derive[18],

$$\alpha_{44} = -\alpha_{34} - 2\not{D}_3\not{D}_T\alpha + 4\not{D}_T^2\alpha + \frac{5}{2}\mathrm{tr}\chi\alpha_3 + \frac{1}{2}\left[5\not{D}_4(\mathrm{tr}\chi\alpha) + \not{D}_4(\mathrm{tr}\underline{\chi}\alpha)\right]$$

it follows

$$\begin{aligned}
&\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_{44}(W)|^2 \le c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_{34}(W)|^2\right.\\
&+\left.\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\not{D}_3\not{D}_T\alpha(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\not{D}_T^2\alpha(W)|^2\right)\\
&+\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\alpha_3(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\alpha_4(W)|^2 + c\epsilon_0^2 \mathcal{R}_{[0]}{}^2\\
&\le c\left(\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_{34}(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\not{D}_3\not{D}_T\alpha(W)|^2\right.\\
&\quad+\left.\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\not{D}_T^2\alpha(W)|^2\right) + c(1+\epsilon_0^2)\mathcal{R}_{[1]}{}^2 \qquad (5.2.20)
\end{aligned}$$

Let us examine the second integral of 5.2.20, $\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\not{D}_3\not{D}_T\alpha(W)|^2$. From 5.1.24 and Proposition 5.1.1,

$$\not{D}_T\alpha(W) = \mathcal{L}_T\alpha(W) + {}^{(T)}H\cdot\alpha(W)\ .$$

Moreover, using Lemma 5.1.1, we can write

$$\not{D}_3\mathcal{L}_T\alpha(W) = \not{D}_3\alpha(\hat{\mathcal{L}}_T W) + \not{D}_3\left(G_1({}^{(T)}M, {}^{(T)}\pi)\alpha(W) + G_2({}^{(T)}P, {}^{(T)}Q)\beta(W)\right)$$

and derive the inequality

$$\begin{aligned}
&\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\not{D}_3\not{D}_T\alpha(W)|^2 \le \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\not{D}_3\alpha(\hat{\mathcal{L}}_T W)|^2\\
&+c\epsilon_0^2\left(\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\not{D}_3\alpha(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} r^4 |\alpha(W)|^2\right)
\end{aligned}$$

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$$\begin{aligned}
\alpha_{44} &= \not{D}_4\alpha_4 + \frac{7}{2}\mathrm{tr}\chi\alpha_4 = \not{D}_4[-\alpha_3 + 2\not{D}_T\alpha + \frac{1}{2}(5\mathrm{tr}\chi + \mathrm{tr}\underline{\chi})\alpha]\\
&= -\not{D}_4\alpha_3 - 2\not{D}_3\not{D}_T\alpha + 4\not{D}_T^2\alpha + \frac{1}{2}\left[5\not{D}_4(\mathrm{tr}\chi\alpha) + \not{D}_4(\mathrm{tr}\underline{\chi}\alpha)\right]\\
&= -\alpha_{34} - 2\not{D}_3\not{D}_T\alpha + 4\not{D}_T^2\alpha + \frac{5}{2}\mathrm{tr}\chi\alpha_3 + \frac{1}{2}\left[5\not{D}_4(\mathrm{tr}\chi\alpha) + \not{D}_4(\mathrm{tr}\underline{\chi}\alpha)\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\mathbf{D}\!\!\!/_3 \alpha(\hat{\mathcal{L}}_T W)|^2 + c\epsilon_0^2 {\mathcal{R}_{[1]}}^2 \\
&\leq c \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\nabla\!\!\!\!/\, \beta(\hat{\mathcal{L}}_T W)|^2 + c \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\alpha(\hat{\mathcal{L}}_T W)|^2 \\
&\quad + c\epsilon_0^2 \left({\mathcal{R}_0}^2(\hat{\mathcal{L}}_T W) + {\mathcal{R}_{[1]}}^2 \right) \\
&\leq c \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\beta(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)|^2 + c \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\alpha(\hat{\mathcal{L}}_T W)|^2 \\
&\quad + c\epsilon_0^2 \left({\mathcal{R}_0}^2(\hat{\mathcal{L}}_T W) + {\mathcal{R}_{[1]}}^2 \right) \ .
\end{aligned}
$$

The third integral is estimated in a similar way, the main difference being that its main term, $\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\mathbf{D}\!\!\!/_4 \alpha(\hat{\mathcal{L}}_T W)|^2$, has to be estimated using the expression $\mathbf{D}\!\!\!/_4 \alpha = \frac{1}{\tau_+} \mathbf{D}\!\!\!/_S \alpha - 2\frac{u}{\tau_+} \mathbf{D}\!\!\!/_3 \alpha$, in the following way

$$
\begin{aligned}
\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\mathbf{D}\!\!\!/_4 \alpha(\hat{\mathcal{L}}_T W)|^2 \ &\leq \ \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\mathbf{D}\!\!\!/_S \alpha(\hat{\mathcal{L}}_T W)|^2 \\
&+ \ \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\mathbf{D}\!\!\!/_3 \alpha(\hat{\mathcal{L}}_T W)|^2
\end{aligned}
$$

The only term left to estimate is

$$
\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\mathbf{D}\!\!\!/_S \alpha(\hat{\mathcal{L}}_T W)|^2 \leq \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\alpha(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)|^2 + \epsilon_0^2 {\mathcal{R}_0}^2(\hat{\mathcal{L}}_T W) \ .
$$

Collecting all these estimates togheter we infer that

$$
\begin{aligned}
&\int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_{44}(W)|^2 \\
&\leq c \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^6 |\alpha_{34}(W)|^2 + \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\beta(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)|^2 \\
&+ \int_{C(\lambda)\cap V(\lambda,\nu)} \tau_-^2 r^4 |\alpha(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)|^2 + c(1+\epsilon_0^2) \left[{\mathcal{R}_0}^2(\hat{\mathcal{L}}_T W) + {\mathcal{R}_{[1]}}^2 \right]
\end{aligned}
$$

concluding the estimate.

Chapter 6

The error estimates

In this chapter we assume the spacetime $\mathcal{K}$ foliated by a double null canonical foliation, verifying the assumptions

$$\mathcal{O} \leq \epsilon_0 \ , \ \mathcal{D} \leq \epsilon_0 \tag{6.0.1}$$

and we are going to make use of the inequality, proven in Theorem 3.7.8,

$$\mathcal{R} \leq c\mathcal{Q}_{\mathcal{K}} \tag{6.0.2}$$

The main result of the chapter is the proof of Theorem 3.7.10, which we restate below,

Theorem 3.7.10 *Under the assumptions 6.0.1 and 6.0.2 with ϵ_0 sufficiently small, then the following estimate holds:*

$$\mathcal{Q}_{\mathcal{K}} \leq c_1 \mathcal{Q}_{\Sigma_0 \cap \mathcal{K}} \tag{6.0.3}$$

with c_1 a constant independent from ϵ_0.

Remark: Observe that the true assumptions of Theorem 3.7.10 stated [1] in Chapter 3 imply the assumptions stated here.

To prove this result we need to control the quantity

$$\mathcal{E}(u,\underline{u}) \equiv (\mathcal{Q}+\underline{\mathcal{Q}})(u,\underline{u}) - \mathcal{Q}_{\Sigma_0 \cap V(u,\underline{u})}$$

which we call *Error term*, for all values of u and $\underline{u}$ on $\mathcal{K}$. Using the expression, see Proposition 3.2.3,

$$\begin{aligned} DivP &= DivQ_{\beta\gamma\delta}X^\beta Y^\gamma Z^\delta \\ &+ \frac{1}{2}Q^{\alpha\beta\gamma\delta}\left({}^{(X)}\pi_{\alpha\beta}Y_\gamma Z_\delta + {}^{(Y)}\pi_{\alpha\beta}Z_\gamma X_\delta + {}^{(Z)}\pi_{\alpha\beta}X_\gamma Y_\delta\right) \end{aligned} \tag{6.0.4}$$

[1] The assumption $\mathcal{R} \leq \epsilon_0$ stated in the Chapter 3 version of this theorem is needed to control the deformation tensors of the angular momentum vector fields, see Theorem 3.7.4.

and Stokes theorem, it follows that

$$\left\{\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(W)(X,Y,Z,e_3) + \int_{C(u)\cap V(u,\underline{u})} Q(W)(X,Y,Z,e_4) - \int_{\Sigma_0\cap V(u,\underline{u})} Q(W)(X,Y,Z,T)\right\}$$
$$= \int_{V_{(u,\underline{u})}} \Big[DivQ(W)_{\beta\gamma\delta}X^\beta Y^\gamma Z^\delta + \frac{1}{2}Q^{\alpha\beta\gamma\delta}(W)\left({}^{(X)}\pi_{\alpha\beta}Y_\gamma Z_\delta + {}^{(Y)}\pi_{\alpha\beta}Z_\gamma X_\delta + {}^{(Z)}\pi_{\alpha\beta}X_\gamma Y_\delta\right)\Big] \tag{6.0.5}$$

Therefore

$$\mathcal{E}(u,\underline{u}) \equiv \mathcal{E}_1(u,\underline{u}) + \mathcal{E}_2(u,\underline{u})$$

is a sum of terms like the right hand side of 6.0.5 where W is replaced by $\hat{\mathcal{L}}_T W$, $\hat{\mathcal{L}}_O W$, $\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W$, $\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W$, $\hat{\mathcal{L}}_O^2 W$ and X,Y,Z take values in $\{T,\bar{K}\}$. $\mathcal{E}_1$ and $\mathcal{E}_2$ have the explicit expressions [2]:

$$\begin{aligned}\mathcal{E}_1(u,\underline{u}) &= \int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta)\\ &+\int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma T^\delta)\\ &+\frac{3}{2}\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)\\ &+\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)\\ &+\frac{1}{2}\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)\end{aligned} \tag{6.0.6}$$

$$\begin{aligned}\mathcal{E}_2(u,\underline{u}) &= \int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_O^2 W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma T^\delta)\\ &+\int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta)\\ &+\int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta)\end{aligned}$$

[2] Differently from Chapter 3 we do not distinguish here between the functions $u,\underline{u}$ and the values λ,ν they can assume, as in this chapter no ambiguity can arise.

$$
\begin{aligned}
&+\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta) \\
&+\frac{1}{2}\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta) \\
&+\frac{3}{2}\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta) \qquad (6.0.7)\\
&+\frac{3}{2}\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta)
\end{aligned}
$$

The estimates of these terms are algebraically quite involved, the final result, however, is very simple. We shall show that

$$\mathcal{E}(u,\underline{u}) \leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}} \qquad (6.0.8)$$

with c a generic constant. This implies

$$\mathcal{Q}_{\mathcal{K}} \leq \frac{1}{1-c\epsilon_0}\mathcal{Q}_{\Sigma_0\cap\mathcal{K}}$$

which, choosing ϵ_0 sufficiently small, concludes the proof of the theorem. The next sections are devoted to the detailed estimates of the error terms required to prove eq. 6.0.8.

Discussion 6.0.1 *The estimates of the spacetime integrals appearing in 6.0.6 and 6.0.7 are the most sensitive part of the proof of "Main Theorem". To understand how this is done it is useful to remember the discussion, concerning global existence for non linear wave equations in Chapter 2. To estimate the error terms, appearing in the derivation of energy estimates for the model problem 2.1.23, we had to introduce the commuting vector fields 2.1.27, define the generalized energy norms 2.1.29, use the global Sobolev inequalities 2.1.30 to derive decay estimates. These allowed us to prove 2.1.32, which implies the desired global existence result for $n > 3$. In dimension $n = 3$ we had, in addition, to rely on the special structure of nonlinear terms, called the "null condition". All these elements, except the last, were already incorporated in our discussion of the proof of "Main Theorem". To estimate the error terms 6.0.6, 6.0.7 we need also to use the special structure of these terms. Just as in the simple case of the null condition for nonlinear wave equation, we have to make sure, by carefully decomposing all the terms appearing in the above integrals in terms of their null components, that the slowest decaying components are counterbalanced by terms which decay fast.*

For this reason we need to know the precise asymptotic behavior of all components of $\mathbf{R}$*, and its derivatives as well as those of the various deformation tensors. The behavior of the null components of the deformation tensors depends crucially on that of the null Ricci coefficients.*

6.1 Definitions and prerequisites

To estimate the first two integrals of $\mathcal{E}_1(u,\underline{u})$ [3]

$$\int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta)\ ,\ \int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma T^\delta)$$

we have to compute explicitely $DivQ(\hat{\mathcal{L}}_X W)$ with $X=T,O$. Denoting

$$D(X,W)\equiv DivQ(\hat{\mathcal{L}}_X W)$$

it follows, by a straightforward calculation, see also [Ch-Kl] equation (8.1.3.c), that

$$\begin{aligned} D(X,W)(\bar{K},\bar{K},T) &= \frac{1}{8}\tau_+^4(D(X,W)_{444}+D(X,W)_{344}) \\ &\quad +\frac{1}{4}\tau_+^2\tau_-^2(D(X,W)_{344}+D(X,W)_{334}) \\ &\quad +\frac{1}{8}\tau_-^4(D(X,W)_{334}+D(X,W)_{333}) \end{aligned} \tag{6.1.1}$$

$$\begin{aligned} D(X,W)(\bar{K},\bar{K},\bar{K}) &= \frac{1}{8}\tau_+^6 D(X,W)_{444}+\frac{3}{8}\tau_+^4\tau_-^2 D(X,W)_{344} \\ &\quad +\frac{3}{8}\tau_+^2\tau_-^4 D(X,W)_{334}+\frac{1}{8}\tau_-^6 D(X,W)_{333} \end{aligned} \tag{6.1.2}$$

where

$$\begin{aligned} D(X,W)_{444} &= 4\alpha(\hat{\mathcal{L}}_X W)\cdot\Theta(X,W)-8\beta(\hat{\mathcal{L}}_X W)\cdot\Xi(X,W) \\ D(X,W)_{443} &= 8\rho(\hat{\mathcal{L}}_X W)\Lambda(X,W)+8\sigma(\hat{\mathcal{L}}_X W)K(X,W) \\ &\quad +8\beta(\hat{\mathcal{L}}_X W)\cdot I(X,W) \\ D(X,W)_{334} &= 8\rho(\hat{\mathcal{L}}_X W)\underline{\Lambda}(X,W)-8\sigma(\hat{\mathcal{L}}_X W)\underline{K}(X,W) \\ &\quad -8\underline{\beta}(\hat{\mathcal{L}}_X W)\cdot\underline{I}(X,W) \\ D(X,W)_{333} &= 4\underline{\alpha}(\hat{\mathcal{L}}_X W)\cdot\underline{\Theta}(X,W)+8\underline{\beta}(\hat{\mathcal{L}}_X W)\cdot\underline{\Xi}(X,W) \end{aligned} \tag{6.1.3}$$

[3] The following expressions are used also, with slight modifications, to estimate the first three integrals of $\mathcal{E}_2(u,\underline{u})$.

where

$$\Lambda(X,W)\ ,\ K(X,W)\ ,\ I(X,W)\ ,\ \Theta(X,W)\ ,\ \Xi(X,W)$$
$$\underline{\Lambda}(X,W)\ ,\ \underline{K}(X,W)\ ,\ \underline{I}(X,W)\ ,\ \underline{\Theta}(X,W)\ ,\ \underline{\Xi}(X,W)$$

are the null components [4] of

$$J(X,W)_{\beta\gamma\delta} \equiv D^{\alpha}(\hat{\mathcal{L}}_X W)_{\alpha\beta\gamma\delta}\ ,$$

and [5]

$$\begin{aligned}
&\Lambda(J)=\frac{1}{4}J_{434}\ ,\ \underline{\Lambda}(J)=\frac{1}{4}J_{343}\ ,\ \Xi(J)_a=\frac{1}{2}J_{44a}\ ,\ \underline{\Xi}(J)_a=\frac{1}{2}J_{33a} \qquad (6.1.4)\\
&I(J)_a=\frac{1}{2}J_{34a}\ ,\ \underline{I}(J)_a=\frac{1}{2}J_{43a}\ ,\ K(J)=\frac{1}{4}\in^{ab}J_{4ab}\ ,\ \underline{K}(J)=\frac{1}{4}\in^{ab}J_{3ab}\\
&\Theta(J)_{ab}=J_{a4b}+J_{b4a}-(\delta^{cd}J_{c4d})\delta_{ab}\ ,\ \underline{\Theta}(J)_{ab}=J_{a3b}+J_{b3a}-(\delta^{cd}J_{c3d})\delta_{ab}
\end{aligned}$$

where we used the relations

$$\begin{array}{llcll}
\Lambda(J^\star) & = K(J) & ; & \underline{\Lambda}(J^\star) & = -\underline{K}(J)\\
K(J^\star) & = -\Lambda(J) & ; & \underline{K}(J^\star) & = \underline{\Lambda}(J)\\
\Xi(J^\star) & = -{}^\star\Xi(J) & ; & \underline{\Xi}(J^\star) & = {}^\star\underline{\Xi}(J)\\
I(J^\star) & = -{}^\star I(J) & ; & \underline{I}(J^\star) & = {}^\star\underline{I}(J)\\
\Theta(J^\star) & = -{}^\star\Theta(J) & ; & \underline{\Theta}(J^\star) & = {}^\star\underline{\Theta}(J)
\end{array} \qquad (6.1.5)$$

Finally [6], $J(X,W)$ can be decomposed in three different parts [7]:

$$J(X;W)=J^1(X;W)+J^2(X;W)+J^3(X;W)\ ,$$

where
$$\begin{aligned}
J^1(X;W)_{\beta\gamma\delta} &= \frac{1}{2}{}^{(X)}\hat{\pi}^{\mu\nu}\mathbf{D}_\nu W_{\mu\beta\gamma\delta}\\
J^2(X;W)_{\beta\gamma\delta} &= \frac{1}{2}{}^{(X)}p_\lambda W^{\lambda}{}_{\beta\gamma\delta} \qquad (6.1.6)\\
J^3(X;W)_{\beta\gamma\delta} &= \frac{1}{2}\left({}^{(X)}q_{\alpha\beta\lambda}W^{\alpha\lambda}{}_{\gamma\delta}+{}^{(X)}q_{\alpha\gamma\lambda}W^{\alpha}{}_{\beta}{}^{\lambda}{}_{\delta}+{}^{(X)}q_{\alpha\delta\lambda}W^{\alpha}{}_{\beta\gamma}{}^{\lambda}\right)
\end{aligned}$$

---

[4] If X is not a Killing or a conformal Killing vector field, $J(X,W)$ is different from zero even if W satisfies the homogeneous Bianchi equations.

[5] We remark also that $J_{a4b}=\Theta(J)_{ab}-\Lambda\delta_{ab}+K\in_{ab}$, $J_{a3b}=\underline{\Theta}(J)_{ab}-\underline{\Lambda}\delta_{ab}+\underline{K}\in_{ab}$, $J_{abc}=\in_{bc}({}^\star I(J)_a+{}^\star\underline{I}(J)_a)$

[6] See also Proposition 7.1.2 and equation 8.1.2b of [Ch-Kl].

[7] To estimate $\mathcal{E}_2(u,\underline{u})$ it is necessary to consider also the divergence of the second Lie derivatives of the Weyl field, $J(X,Y,W)_{\beta\gamma\delta}=D^\alpha(\hat{\mathcal{L}}_Y\hat{\mathcal{L}}_X W)_{\alpha\beta\gamma\delta}$. We will give their explicit expressions later on.

and

$$
\begin{aligned}
{}^{(X)}p_\lambda &= \mathbf{D}^{\alpha\,(X)}\hat{\pi}_{\alpha\gamma} \\
{}^{(X)}q_{\alpha\beta\gamma} &= \mathbf{D}^{\beta\,(X)}\hat{\pi}_{\gamma\alpha} - \mathbf{D}^{\gamma\,(X)}\hat{\pi}_{\beta\alpha} - \frac{1}{3}\left({}^{(X)}p_\gamma g_{\alpha\beta} - {}^{(X)}p_\beta g_{\alpha\gamma}\right)
\end{aligned}
\tag{6.1.7}
$$

It follows that the various factors $\Theta(X,W),\ \Xi(X,W),\ \Lambda(X,W),....$ of 6.1.3 can also be decomposed in three parts, depending which part of $J(X,W)$ they are connected to.
All these null components of the Weyl current can be explicitely written in terms of the null components of the Riemann tensor and of its first derivatives, the null components of the traceless part of the deformation tensors and their derivatives ${}^{(X)}p$, ${}^{(X)}q$ which appear in the expressions of $J^1(X,W), J^2(X,W), J^3(X,W)$. Recalling the null decomposition of the deformation tensors, see 3.4.6,

$$
\begin{aligned}
{}^{(X)}\mathbf{i}_{ab} &= {}^{(X)}\hat{\pi}_{ab} \ ; \quad {}^{(X)}\mathbf{j} = {}^{(X)}\hat{\pi}_{34} \\
{}^{(X)}\mathbf{m}_a &= {}^{(X)}\hat{\pi}_{4a} \ ; \quad {}^{(X)}\underline{\mathbf{m}}_a = {}^{(X)}\hat{\pi}_{3a} \\
{}^{(X)}\mathbf{n} &= {}^{(X)}\hat{\pi}_{44} \ ; \quad {}^{(X)}\underline{\mathbf{n}} = {}^{(X)}\hat{\pi}_{33}
\end{aligned}
$$

the explicit expressions of the components of $J(X,W)$ are [8],

$$
\begin{aligned}
\underline{\Xi}(J^1) &= \mathrm{Qr}\left[{}^{(X)}\mathbf{i}\,;\ \nabla\!\!\!/\underline{\alpha}\right] + \mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\ \underline{\alpha}_3\right] + \mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\ \underline{\alpha}_4\right] \\
&+ \mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\ \nabla\!\!\!/\underline{\beta}\right] + \mathrm{Qr}\left[{}^{(X)}\mathbf{j}\,;\ \underline{\beta}_3\right] + \mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{n}}\,;\ \underline{\beta}_4\right] \\
&+ \mathrm{tr}\underline{\chi}\left(\mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\ \underline{\alpha}\right] + \mathrm{Qr}\left[({}^{(X)}\mathbf{i}, {}^{(X)}\mathbf{j})\,;\ \underline{\beta}\right] + \mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\ (\rho,\sigma)\right]\right) \\
&+ \mathrm{tr}\chi\left(\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\ \underline{\alpha}\right] + \mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{n}}\,;\ \underline{\beta}\right]\right) + \mathrm{l.o.t.}
\end{aligned}
\tag{6.1.8}
$$

$$
\begin{aligned}
\underline{\Theta}(J^1) &= \mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\ \nabla\!\!\!/\underline{\alpha}\right] + \mathrm{Qr}\left[{}^{(X)}\mathbf{n}\,;\ \underline{\alpha}_3\right] + \mathrm{Qr}\left[{}^{(X)}\mathbf{j}\,;\ \underline{\alpha}_4\right] \\
&+ \mathrm{Qr}\left[{}^{(X)}\mathbf{i}\,;\ \nabla\!\!\!/\underline{\beta}\right] + \mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\ \underline{\beta}_3\right] + \mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\ \underline{\beta}_4\right] \\
&+ \mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\ \nabla\!\!\!/(\rho,\sigma)\right] + \mathrm{Qr}\left[{}^{(X)}\mathbf{j}\,;\ (\rho_3,\sigma_3)\right] + \mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{n}}\,;\ (\rho_4,\sigma_4)\right] \\
&+ \mathrm{tr}\underline{\chi}\left(\mathrm{Qr}\left[{}^{(X)}\mathbf{n}\,;\ \underline{\alpha}\right] + \mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\ \underline{\beta}\right] + \mathrm{Qr}\left[({}^{(X)}\mathbf{i}, {}^{(X)}\mathbf{j})\,;\ (\rho,\sigma)\right]\right. \\
&+ \left.\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\ \beta\right]\right)
\end{aligned}
$$

[8]See Proposition 8.1.4 of [Ch-Kl]. Qr [;] is a generic notation for any quadratic form with coefficients which depend only on the induced metric and area form of $S(u,\underline{u})$. We note also that the terms which are boxed below are in fact vanishing; we include them to emphasize the importance of the corresponding cancellations.

$$
\begin{aligned}
&+ \quad \mathrm{tr}\chi\Big(\boxed{\mathrm{Qr}\big[{}^{(X)}\mathbf{i}\,;\,\underline{\alpha}\big]} + \mathrm{Qr}\big[{}^{(X)}\mathbf{j}\,;\,\underline{\alpha}\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\underline{\beta}\big] \\
&+ \quad \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{n}}\,;\,(\rho,\sigma)\big]\Big) + \mathrm{l.o.t.} \qquad (6.1.9)
\end{aligned}
$$

$$
\begin{aligned}
\underline{\Lambda}(J^1) &= \mathrm{Qr}\big[{}^{(X)}\mathbf{i}\,;\,\nabla\!\!\!/\underline{\beta}\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,\underline{\beta}_3\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\underline{\beta}_4\big] \\
&+ \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\nabla\!\!\!/(\rho,\sigma)\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{j}\,;\,(\rho_3,\sigma_3)\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{n}}\,;\,(\rho_4,\sigma_4)\big] \\
&+ \mathrm{tr}\underline{\chi}\Big(\mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,\underline{\beta}\big] + \mathrm{Qr}\big[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,(\rho,\sigma)\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\beta\big]\Big) \\
&+ \mathrm{tr}\chi\Big(\mathrm{Qr}\big[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,\underline{\alpha}\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\underline{\beta}\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{n}}\,;\,(\rho,\sigma)\big]\Big) \\
&+ \mathrm{l.o.t.} \qquad (6.1.10)
\end{aligned}
$$

$$
\begin{aligned}
\underline{K}(J^1) &= \mathrm{Qr}\big[{}^{(X)}\mathbf{i}\,;\,\nabla\!\!\!/\underline{\beta}\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,\underline{\beta}_3\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\underline{\beta}_4\big] \\
&+ \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\nabla\!\!\!/(\rho,\sigma)\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{j}\,;\,(\rho_3,\sigma_3)\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{n}}\,;\,(\rho_4,\sigma_4)\big] \\
&+ \mathrm{tr}\underline{\chi}\Big(\mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,\underline{\beta}\big] + \mathrm{Qr}\big[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,(\rho,\sigma)\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\beta\big]\Big) \\
&+ \mathrm{tr}\chi\Big(\mathrm{Qr}\big[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,\underline{\alpha}\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\underline{\beta}\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{n}}\,;\,(\rho,\sigma)\big]\Big) \\
&+ \mathrm{l.o.t.} \qquad (6.1.11)
\end{aligned}
$$

$$
\begin{aligned}
\underline{I}(J^1) &= \mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,\nabla\!\!\!/\underline{\beta}\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{n}\,;\,\underline{\beta}_3\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{j}\,;\,\underline{\beta}_4\big] \\
&+ \mathrm{Qr}\big[{}^{(X)}\mathbf{i}\,;\,\nabla\!\!\!/(\rho,\sigma)\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,(\rho_3,\sigma_3)\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,(\rho_4,\sigma_4)\big] \\
&+ \mathrm{tr}\underline{\chi}\Big(\mathrm{Qr}\big[{}^{(X)}\mathbf{n}\,;\,\underline{\beta}\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,(\rho,\sigma)\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{i}\,;\,\beta\big]\Big) \\
&+ \mathrm{tr}\chi\Big(\mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,\underline{\alpha}\big] + \mathrm{Qr}\big[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,\underline{\beta}\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,(\rho,\sigma)\big]\Big) \\
&+ \mathrm{l.o.t.} \qquad (6.1.12)
\end{aligned}
$$

$$
\begin{aligned}
\Xi(J^1) &= \mathrm{Qr}\big[{}^{(X)}\mathbf{i}\,;\,\nabla\!\!\!/\alpha\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\alpha_4\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,\alpha_3\big] \\
&+ \mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,\nabla\!\!\!/\beta\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{j}\,;\,\beta_4\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{n}\,;\,\beta_3\big] \\
&+ \mathrm{tr}\chi\Big(\mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\alpha\big] + \mathrm{Qr}\big[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,\beta\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,(\rho,\sigma)\big]\Big) \\
&+ \mathrm{tr}\underline{\chi}\Big(\mathrm{Qr}\big[{}^{(X)}\mathbf{m}\,;\,\alpha\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{n}\,;\,\beta\big]\Big) + \mathrm{l.o.t.} \qquad (6.1.13)
\end{aligned}
$$

$$
\Theta(J^1) = \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{m}}\,;\,\nabla\!\!\!/\alpha\big] + \mathrm{Qr}\big[{}^{(X)}\underline{\mathbf{n}}\,;\,\alpha_4\big] + \mathrm{Qr}\big[{}^{(X)}\mathbf{j}\,;\,\alpha_3\big]
$$

$$
\begin{aligned}
&+\quad \mathrm{Qr}\left[{}^{(X)}\mathbf{i}\,;\,\not\nabla\beta\right]+\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\,\beta_4\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\beta_3\right]\\
&+\quad \mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\not\nabla(\rho,\sigma)\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{j}\,;\,(\rho_4,\sigma_4)\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{n}\,;\,(\rho_3,\sigma_3)\right]\\
&+\quad \mathrm{tr}\chi\Big(\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{n}}\,;\,\alpha\right]+\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\,\beta\right]+\mathrm{Qr}\left[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,(\rho,\sigma)\right]\\
&+\quad \mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\underline{\beta}\right]\Big)+\mathrm{tr}\underline{\chi}\Big(\boxed{\quad\mathrm{Qr}\left[{}^{(X)}\mathbf{i}\,;\,\alpha\right]\quad}+\mathrm{Qr}\left[{}^{(X)}\mathbf{j}\,;\,\alpha\right]\\
&+\quad \mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\beta\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{n}\,;\,(\rho,\sigma)\right]\Big)+\mathrm{l.o.t.}
\end{aligned}
\tag{6.1.14}
$$

$$
\begin{aligned}
\Lambda(J^1)\;=&\quad \mathrm{Qr}\left[{}^{(X)}\mathbf{i}\,;\,\not\nabla\beta\right]+\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\,\beta_4\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\beta_3\right]\\
&+\quad \mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\not\nabla(\rho,\sigma)\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{j}\,;\,(\rho_4,\sigma_4)\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{n}\,;\,(\rho_3,\sigma_3)\right]\\
&+\quad \mathrm{tr}\chi\Big(\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\,\beta\right]+\mathrm{Qr}\left[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,(\rho,\sigma)\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\underline{\beta}\right]\Big)\\
&+\quad \mathrm{tr}\underline{\chi}\Big(\mathrm{Qr}\left[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,\alpha\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\beta\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{n}\,;\,(\rho,\sigma)\right]\Big)\\
&+\quad \mathrm{l.o.t.}
\end{aligned}
\tag{6.1.15}
$$

$$
\begin{aligned}
K(J^1)\;=&\quad \mathrm{Qr}\left[{}^{(X)}\mathbf{i}\,;\,\not\nabla\beta\right]+\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\,\beta_4\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\beta_3\right]\\
&+\quad \mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\not\nabla(\rho,\sigma)\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{j}\,;\,(\rho_4,\sigma_4)\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{n}\,;\,(\rho_3,\sigma_3)\right]\\
&+\quad \mathrm{tr}\chi\Big(\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\,\beta\right]+\mathrm{Qr}\left[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,(\rho,\sigma)\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\underline{\beta}\right]\Big)\\
&+\quad \mathrm{tr}\underline{\chi}\Big(\mathrm{Qr}\left[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,\alpha\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,\beta\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{n}\,;\,(\rho,\sigma)\right]\Big)\\
&+\quad \mathrm{l.o.t.}
\end{aligned}
\tag{6.1.16}
$$

$$
\begin{aligned}
I(J^1)\;=&\quad \mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\,\not\nabla\beta\right]+\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{n}}\,;\,\beta_4\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{j}\,;\,\beta_3\right]\\
&+\quad \mathrm{Qr}\left[{}^{(X)}\mathbf{i}\,;\,\not\nabla(\rho,\sigma)\right]+\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\,(\rho_4,\sigma_4)\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,(\rho_3,\sigma_3)\right]\\
&+\quad \mathrm{tr}\chi\Big(\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{n}}\,;\,\beta\right]+\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\,(\rho,\sigma)\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{i}\,;\,\underline{\beta}\right]\Big)\\
&+\quad \mathrm{tr}\underline{\chi}\Big(\mathrm{Qr}\left[{}^{(X)}\underline{\mathbf{m}}\,;\,\alpha\right]+\mathrm{Qr}\left[({}^{(X)}\mathbf{i},{}^{(X)}\mathbf{j})\,;\,\beta\right]+\mathrm{Qr}\left[{}^{(X)}\mathbf{m}\,;\,(\rho,\sigma)\right]\Big)\\
&+\quad \mathrm{l.o.t.}
\end{aligned}
\tag{6.1.17}
$$

Remark: *The terms which we denote by l.o.t. are cubic with respect to* ${}^{(X)}\hat{\pi}$, W *and the connection coefficients* $\eta,\underline{\eta},\omega,\underline{\omega},\chi,\hat{\underline{\chi}}$ *and linear with regard to each of them separetely. They are manifestly of lower order by comparison to all other terms both in regard to their asymptotic behavior along the outgoing null hypersurfaces and to the order of differentiabilty relative to* W.

The null decomposition of J^2 is given by:

$$
\begin{aligned}
\underline{\Xi}(J^2) &= \mathrm{Qr}\left[{}^{(X)}\not p\,;\,\underline{\alpha}\right] + \mathrm{Qr}\left[{}^{(X)}p_3\,;\,\underline{\beta}\right]\\
\underline{\Theta}(J^2) &= \mathrm{Qr}\left[{}^{(X)}p_4\,;\,\underline{\alpha}\right] + \mathrm{Qr}\left[{}^{(X)}\not p\,;\,\underline{\beta}\right] + \mathrm{Qr}\left[{}^{(X)}p_3\,;\,(\rho,\sigma)\right]\\
\underline{\Lambda}(J^2) &= \mathrm{Qr}\left[{}^{(X)}\not p\,;\,\underline{\beta}\right] + \mathrm{Qr}\left[{}^{(X)}p_3\,;\,(\rho,\sigma)\right]\\
\underline{K}(J^2) &= \mathrm{Qr}\left[{}^{(X)}\not p\,;\,\underline{\beta}\right] + \mathrm{Qr}\left[{}^{(X)}p_3\,;\,(\rho,\sigma)\right]\\
\underline{I}(J^2) &= \mathrm{Qr}\left[{}^{(X)}p_4\,;\,\underline{\beta}\right] + \mathrm{Qr}\left[{}^{(X)}\not p\,;\,(\rho,\sigma)\right] \qquad (6.1.18)\\
I(J^2) &= \mathrm{Qr}\left[{}^{(X)}p_3\,;\,\beta\right] + \mathrm{Qr}\left[{}^{(X)}\not p\,;\,(\rho,\sigma)\right]\\
K(J^2) &= \mathrm{Qr}\left[{}^{(X)}\not p\,;\,\beta\right] + \mathrm{Qr}\left[{}^{(X)}p_4\,;\,(\rho,\sigma)\right]\\
\Lambda(J^2) &= \mathrm{Qr}\left[{}^{(X)}\not p\,;\,\beta\right] + \mathrm{Qr}\left[{}^{(X)}p_4\,;\,(\rho,\sigma)\right]\\
\Theta(J^2) &= \mathrm{Qr}\left[{}^{(X)}p_3\,;\,\alpha\right] + \mathrm{Qr}\left[{}^{(X)}\not p\,;\,\beta\right] + \mathrm{Qr}\left[{}^{(X)}p_4\,;\,(\rho,\sigma)\right]\\
\Xi(J^2) &= \mathrm{Qr}\left[{}^{(X)}\not p\,;\,\alpha\right] + \mathrm{Qr}\left[{}^{(X)}p_4\,;\,\beta\right]
\end{aligned}
$$

and the null decomposition of J^3 by:

$$
\begin{aligned}
\underline{\Xi}(J^3) &= \mathrm{Qr}\left[\underline{\alpha}\,;\,(I,\underline{I})({}^{(X)}q)\right] + \mathrm{Qr}\left[\underline{\beta}\,;\,(\underline{K},\underline{\Lambda},\underline{\Theta})({}^{(X)}q)\right] + \mathrm{Qr}\left[(\rho,\sigma)\,;\,\underline{\Xi}({}^{(X)}q)\right]\\
\underline{\Theta}(J^3) &= \mathrm{Qr}\left[\underline{\alpha}\,;\,K({}^{(X)}q)\right] + \mathrm{Qr}\left[\underline{\alpha}\,;\,\Lambda({}^{(X)}q)\right] + \boxed{\mathrm{Qr}\left[\underline{\alpha}\,;\,\Theta({}^{(X)}q)\right]} \qquad (6.1.19)\\
&\quad +\mathrm{Qr}\left[\underline{\beta}\,;\,(I,\underline{I})({}^{(X)}q)\right] + \mathrm{Qr}\left[(\rho,\sigma)\,;\,\underline{\Theta}({}^{(X)}q)\right] + \boxed{\mathrm{Qr}\left[\beta\,;\,\underline{\Xi}({}^{(X)}q)\right]}\\
\underline{\Lambda}(J^3) &= \mathrm{Qr}\left[\underline{\alpha}\,;\,\Theta({}^{(X)}q)\right] + \boxed{\mathrm{Qr}\left[\underline{\beta}\,;\,(I,\underline{I})({}^{(X)}q)\right]}\\
&\quad +\mathrm{Qr}\left[(\rho,\sigma)\,;\,(\underline{K},\underline{\Lambda})({}^{(X)}q)\right] + \mathrm{Qr}\left[\beta\,;\,\underline{\Xi}({}^{(X)}q)\right]\\
\underline{K}(J^3) &= \mathrm{Qr}\left[\underline{\alpha}\,;\,\Theta({}^{(X)}q)\right] + \boxed{\mathrm{Qr}\left[\underline{\beta}\,;\,(I,\underline{I})({}^{(X)}q)\right]}\\
&\quad +\mathrm{Qr}\left[(\rho,\sigma)\,;\,(\underline{K},\underline{\Lambda})({}^{(X)}q)\right] + \mathrm{Qr}\left[\beta\,;\,\underline{\Xi}({}^{(X)}q)\right] \qquad (6.1.20)\\
\underline{I}(J^3) &= \mathrm{Qr}\left[\underline{\alpha}\,;\,\Xi({}^{(X)}q)\right] + \mathrm{Qr}\left[\underline{\beta}\,;\,(K,\Lambda,\Theta)({}^{(X)}q)\right]\\
&\quad +\mathrm{Qr}\left[(\rho,{}^{(X)}\sigma)\,;\,(I,\underline{I})({}^{(X)}q)\right] + \mathrm{Qr}\left[\beta\,;\,(\underline{K},\underline{\Lambda},\underline{\Theta})({}^{(X)}q)\right]\\
&\quad + \boxed{\mathrm{Qr}\left[\alpha\,;\,\underline{\Xi}({}^{(X)}q)\right]}\\
I(J^3) &= \boxed{\mathrm{Qr}\left[\underline{\alpha}\,;\,\Xi({}^{(X)}q)\right]} + \mathrm{Qr}\left[\underline{\beta}\,;\,(K,\Lambda,\Theta)({}^{(X)}q)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\mathrm{Qr}\left[(\rho,{}^{(X)}\sigma)\,;\,(I,\underline{I})({}^{(X)}q)\right]+\mathrm{Qr}\left[\beta\,;\,(\underline{K},\underline{\Lambda},\underline{\Theta})({}^{(X)}q)\right]\\
&+\mathrm{Qr}\left[\alpha\,;\,\underline{\Xi}({}^{(X)}q)\right] \qquad (6.1.21)
\end{aligned}
$$

$$
\begin{aligned}
K(J^3) &= \mathrm{Qr}\left[\alpha\,;\,\underline{\Theta}({}^{(X)}q)\right]+\boxed{\mathrm{Qr}\left[\beta\,;\,(I,\underline{I})({}^{(X)}q)\right]}\\
&+\mathrm{Qr}\left[(\rho,\sigma)\,;\,(K,\Lambda)({}^{(X)}q)\right]+\mathrm{Qr}\left[\underline{\beta}\,;\,\Xi({}^{(X)}q)\right]\\
\Lambda(J^3) &= \mathrm{Qr}\left[\alpha\,;\,\underline{\Theta}({}^{(X)}q)\right]+\boxed{\mathrm{Qr}\left[\beta\,;\,(I,\underline{I})({}^{(X)}q)\right]}\\
&+\mathrm{Qr}\left[(\rho,\sigma)\,;\,(K,\Lambda)({}^{(X)}q)\right]+\mathrm{Qr}\left[\underline{\beta}\,;\,\Xi({}^{(X)}q)\right] \qquad (6.1.22)\\
\Theta(J^3) &= \mathrm{Qr}\left[\alpha\,;\,\underline{K}({}^{(X)}q)\right]+\mathrm{Qr}\left[\alpha\,;\,\underline{\Lambda}({}^{(X)}q)\right]+\boxed{\mathrm{Qr}\left[\alpha\,;\,\underline{\Theta}({}^{(X)}q)\right]}\\
&+\mathrm{Qr}\left[\beta\,;\,(I,\underline{I})({}^{(X)}q)\right]+\mathrm{Qr}\left[(\rho,\sigma)\,;\,\Theta({}^{(X)}q)\right]+\boxed{\mathrm{Qr}\left[\underline{\beta}\,;\,\Xi({}^{(X)}q)\right]}\\
\Xi(J^3) &= \mathrm{Qr}\left[\alpha\,;\,(I,\underline{I})({}^{(X)}q)\right]+\mathrm{Qr}\left[\beta\,;\,(K,\Lambda,\Theta)({}^{(X)}q)\right]+\mathrm{Qr}\left[(\rho,\sigma)\,;\,\Xi({}^{(X)}q)\right]
\end{aligned}
$$
(6.1.23)

The above expressions for the currents $J^2(X,W)$ and $J^3(X,W)$ depend on the null components of ${}^{(X)}p$ and ${}^{(X)}q$. They are:

$$
\begin{aligned}
{}^{(X)}p_3 &= \mathrm{d}\!\!/\mathrm{iv}\,{}^{(X)}\underline{\mathbf{m}}-\frac{1}{2}(\mathbf{D}\!\!\!/_4\,{}^{(X)}\underline{\mathbf{n}}+\mathbf{D}\!\!\!/_3\,{}^{(X)}\mathbf{j})+(2\underline{\eta}+\eta-\zeta)\cdot{}^{(X)}\underline{\mathbf{m}} \qquad (6.1.24)\\
&-\hat{\chi}\cdot{}^{(X)}\mathbf{i}-\frac{1}{2}tr\chi(tr{}^{(X)}\mathbf{i}+{}^{(X)}\mathbf{j})-\frac{1}{2}tr\underline{\chi}\,{}^{(X)}\mathbf{n}-(\mathbf{D}_3\log\Omega)\,{}^{(X)}\mathbf{n}
\end{aligned}
$$

$$
\begin{aligned}
{}^{(X)}p_4 &= \mathrm{d}\!\!/\mathrm{iv}\,{}^{(X)}\mathbf{m}-\frac{1}{2}(\mathbf{D}\!\!\!/_3\,{}^{(X)}\mathbf{n}+\mathbf{D}\!\!\!/_4\,{}^{(X)}\mathbf{j})+(2\eta+\underline{\eta}+\zeta)\cdot{}^{(X)}\mathbf{m} \qquad (6.1.25)\\
&-\underline{\hat{\chi}}\cdot{}^{(X)}\mathbf{i}-\frac{1}{2}tr\underline{\chi}(tr{}^{(X)}\mathbf{i}+{}^{(X)}\mathbf{j})-\frac{1}{2}tr\chi\,{}^{(X)}\underline{\mathbf{n}}-(\mathbf{D}_4\log\Omega)\,{}^{(X)}\underline{\mathbf{n}}
\end{aligned}
$$

$$
\begin{aligned}
{}^{(X)}p\!\!\!/ &= \nabla\!\!\!\!/_c\,{}^{(X)}\mathbf{i}-\frac{1}{2}(\mathbf{D}\!\!\!/_4\,{}^{(X)}\underline{\mathbf{m}}+\mathbf{D}\!\!\!/_3\,{}^{(X)}\mathbf{m})-\frac{1}{2}(\mathbf{D}_4\log\Omega)\,{}^{(X)}\underline{\mathbf{m}}-\frac{1}{2}(\mathbf{D}_3\log\Omega)\,{}^{(X)}\mathbf{m}\\
&+\frac{1}{2}{}^{(X)}\mathbf{j}(\eta+\underline{\eta})+{}^{(X)}\mathbf{i}\cdot(\eta+\underline{\eta})-\frac{3}{4}tr\chi\,{}^{(X)}\underline{\mathbf{m}}-\frac{3}{4}tr\underline{\chi}\,{}^{(X)}\mathbf{m}-\frac{1}{2}\underline{\hat{\chi}}\cdot{}^{(X)}\mathbf{m}\\
&-\frac{1}{2}\hat{\chi}\cdot{}^{(X)}\underline{\mathbf{m}} \qquad (6.1.26)
\end{aligned}
$$

The null components of ${}^{(X)}q$ are, introducing the same notation used in 6.1.4, to denote the various null components,

$$
\begin{aligned}
\Lambda({}^{(X)}q) &= \frac{1}{4}\left(\mathbf{D}_3{}^{(X)}\mathbf{n} - 2(\mathbf{D}_3\log\Omega)^{(X)}\mathbf{n} - 4\eta\cdot{}^{(X)}\mathbf{m}\right)\\
&- \frac{1}{4}\left(\mathbf{D}_4{}^{(X)}\mathbf{j} - 2\underline{\eta}\cdot{}^{(X)}\mathbf{m}\right) + \frac{2}{3}{}^{(X)}p_4\\
K({}^{(X)}q)_{ab} &= \frac{1}{2}\left(\nabla\!\!\!/_a{}^{(X)}\mathbf{m}_b - \nabla\!\!\!/_b{}^{(X)}\mathbf{m}_a\right) + \frac{1}{2}\left(\zeta_a{}^{(X)}\mathbf{m}_b - \zeta_b{}^{(X)}\mathbf{m}_a\right)\\
&- \frac{1}{2}\left(\hat{\chi}_{ac}{}^{(X)}\mathbf{i}_{cb} - \hat{\chi}_{bc}{}^{(X)}\mathbf{i}_{ca}\right)\\
\Xi({}^{(X)}q)_a &= \frac{1}{2}\mathbf{D}\!\!\!\!/_4{}^{(X)}\mathbf{m}_a - \frac{1}{2}\nabla\!\!\!/_a{}^{(X)}\mathbf{n} - \frac{1}{2}\underline{\eta}_a{}^{(X)}\mathbf{n} - \frac{1}{2}(\mathbf{D}_4\log\Omega)^{(X)}\mathbf{m}_a\\
&+ \frac{1}{2}\mathrm{tr}\chi^{(X)}\mathbf{m}_a + \hat{\chi}_{ac}{}^{(X)}\mathbf{m}_c \qquad (6.1.27)\\
I({}^{(X)}q)_a &= \frac{1}{2}\mathbf{D}\!\!\!\!/_4{}^{(X)}\underline{\mathbf{m}}_a - \frac{1}{2}\nabla\!\!\!/_a{}^{(X)}\mathbf{j} - \frac{1}{2}(\mathbf{D}_4\log\Omega)^{(X)}\underline{\mathbf{m}}_a + \frac{1}{4}\mathrm{tr}\chi^{(X)}\underline{\mathbf{m}}_a\\
&+ \frac{1}{2}\hat{\chi}_{ac}{}^{(X)}\underline{\mathbf{m}}_c + \frac{1}{4}\mathrm{tr}\underline{\chi}^{(X)}\mathbf{m}_a + \frac{1}{2}\underline{\hat{\chi}}_{ac}{}^{(X)}\mathbf{m}_c - \frac{1}{2}\underline{\eta}_c{}^{(X)}\mathbf{i}_{ca} + \frac{3}{2}{}^{(X)}p\!\!\!/_a\\
\Theta({}^{(X)}q)_{ab} &= 2\left(\mathbf{D}\!\!\!\!/_4{}^{(X)}\mathbf{i}_{ab} - \frac{1}{2}\delta_{ab}\mathrm{tr}(\mathbf{D}\!\!\!\!/_4{}^{(X)}\mathbf{i})\right) - \left(\nabla\!\!\!/_a{}^{(X)}\mathbf{m}_b + \nabla\!\!\!/_b{}^{(X)}\mathbf{m}_a - \delta_{ab}\nabla\!\!\!/_c{}^{(X)}\mathbf{m}_c\right)\\
&- 2\left(\underline{\eta}_a{}^{(X)}\mathbf{m}_b + \underline{\eta}_b{}^{(X)}\mathbf{m}_a - \delta_{ab}\underline{\eta}_c{}^{(X)}\mathbf{m}_c\right) - \left(\zeta_a{}^{(X)}\mathbf{m}_b + \zeta_b{}^{(X)}\mathbf{m}_a - \delta_{ab}\zeta_c{}^{(X)}\mathbf{m}_c\right)\\
&+ \mathrm{tr}\chi^{(\hat{X})}\mathbf{i}_{ab} + \hat{\chi}_{ab}\mathrm{tr}^{(X)}\mathbf{i} + \underline{\hat{\chi}}_{ab}{}^{(X)}\mathbf{n} + \hat{\chi}_{ab}{}^{(X)}\mathbf{j}
\end{aligned}
$$

and the underlined quantities are obtained with the standard substitutions,

$$
\begin{aligned}
\underline{\Lambda}({}^{(X)}q) &= \frac{1}{4}\left(\mathbf{D}_4{}^{(X)}\underline{\mathbf{n}} - 2(\mathbf{D}_4\log\Omega)^{(X)}\underline{\mathbf{n}} - 4\underline{\eta}\cdot{}^{(X)}\underline{\mathbf{m}}\right) - \frac{1}{4}\left(\mathbf{D}_3{}^{(X)}\mathbf{j} - 2\eta\cdot{}^{(X)}\underline{\mathbf{m}}\right) + \frac{2}{3}{}^{(X)}p_3\\
\underline{K}({}^{(X)}q)_{ab} &= \frac{1}{2}\left(\nabla\!\!\!/_a{}^{(X)}\underline{\mathbf{m}}_b - \nabla\!\!\!/_b{}^{(X)}\underline{\mathbf{m}}_a\right) - \frac{1}{2}\left(\zeta_a{}^{(X)}\underline{\mathbf{m}}_b - \zeta_b{}^{(X)}\underline{\mathbf{m}}_a\right) - \frac{1}{2}\left(\underline{\hat{\chi}}_{ac}{}^{(X)}\mathbf{i}_{cb} - \underline{\hat{\chi}}_{bc}{}^{(X)}\mathbf{i}_{ca}\right)\\
\underline{\Xi}({}^{(X)}q)_a &= \frac{1}{2}\mathbf{D}\!\!\!\!/_3{}^{(X)}\underline{\mathbf{m}}_a - \frac{1}{2}\nabla\!\!\!/_a{}^{(X)}\underline{\mathbf{n}} - \frac{1}{2}\eta_a{}^{(X)}\underline{\mathbf{n}} - \frac{1}{2}(\mathbf{D}_3\log\Omega)^{(X)}\underline{\mathbf{m}}_a + \frac{1}{2}\mathrm{tr}\underline{\chi}^{(X)}\underline{\mathbf{m}}_a + \underline{\hat{\chi}}_{ac}{}^{(X)}\underline{\mathbf{m}}_c\\
\underline{I}({}^{(X)}q)_a &= \frac{1}{2}\mathbf{D}\!\!\!\!/_3{}^{(X)}\mathbf{m}_a - \frac{1}{2}\nabla\!\!\!/_a{}^{(X)}\mathbf{j} - \frac{1}{2}(\mathbf{D}_3\log\Omega)^{(X)}\mathbf{m}_a + \frac{1}{4}\mathrm{tr}\underline{\chi}^{(X)}\mathbf{m}_a\\
&+ \frac{1}{2}\underline{\hat{\chi}}_{ac}{}^{(X)}\mathbf{m}_c + \frac{1}{4}\mathrm{tr}\chi^{(X)}\underline{\mathbf{m}}_a + \frac{1}{2}\hat{\chi}_{ac}{}^{(X)}\underline{\mathbf{m}}_c - \frac{1}{2}\eta_c{}^{(X)}\mathbf{i}_{ca} + \frac{3}{2}{}^{(X)}p\!\!\!/_a \qquad (6.1.28)\\
\underline{\Theta}({}^{(X)}q)_{ab} &= 2\left(\mathbf{D}\!\!\!\!/_3{}^{(X)}\mathbf{i}_{ab} - \frac{1}{2}\delta_{ab}\mathrm{tr}(\mathbf{D}\!\!\!\!/_3{}^{(X)}\mathbf{i})\right) - \left(\nabla\!\!\!/_a{}^{(X)}\underline{\mathbf{m}}_b + \nabla\!\!\!/_b{}^{(X)}\underline{\mathbf{m}}_a - \delta_{ab}\nabla\!\!\!/_c{}^{(X)}\underline{\mathbf{m}}_c\right)\\
&- 2\left(\eta_a{}^{(X)}\underline{\mathbf{m}}_b + \eta_b{}^{(X)}\underline{\mathbf{m}}_a - \delta_{ab}\eta_c{}^{(X)}\underline{\mathbf{m}}_c\right) + \left(\zeta_a{}^{(X)}\underline{\mathbf{m}}_b + \zeta_b{}^{(X)}\underline{\mathbf{m}}_a - \delta_{ab}\zeta_c{}^{(X)}\underline{\mathbf{m}}_c\right)\\
&+ \mathrm{tr}\underline{\chi}^{(X)}\mathbf{i}_{ab} + \underline{\hat{\chi}}_{ab}\mathrm{tr}^{(X)}\mathbf{i} + \hat{\chi}_{ab}{}^{(X)}\underline{\mathbf{n}} + \underline{\hat{\chi}}_{ab}{}^{(X)}\mathbf{j}
\end{aligned}
$$

6.1.1 Estimates for the T, S, $\bar{K}$ deformation tensors

In Chapter 4 we have proved the following results concerning the $\mathcal{O}$ norms, see Theorems 4.2.1, 4.2.2, 4.2.3,

$$\begin{aligned}
&\mathcal{O}_0+\underline{\mathcal{O}}_0\le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0)\\
&\mathcal{O}_{[1]}+\underline{\mathcal{O}}_{[1]}\le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1)\\
&\mathcal{O}_{[2]}+\underline{\mathcal{O}}_{[2]}\le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1+\Delta_2)\\
&\mathcal{O}_3+\underline{\mathcal{O}}_3\le c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1+\Delta_2)
\end{aligned}$$

They justify, together with Theorem 3.7.4, assumptions 6.0.1, provided we choose $c(\mathcal{I}_0+\mathcal{I}_*+\Delta_0+\Delta_1+\Delta_2)\le\epsilon_0$.

Based on these assumptions we state now a sequence of propositions concerning the components of deformation tensors associated to T, S, $\bar{K}$.

Proposition 6.1.1 *Under the assumptions 6.0.1, the following estimates hold, for any $S\subset\mathcal{K}$ with $p\in[2,4]$,*

$$\begin{aligned}
&|r^{1-\frac{2}{p}}\tau_-{}^{(T)}\mathbf{i}|_{p,S}\le c\epsilon_0\\
&|r^{1-\frac{2}{p}}\tau_-{}^{(T)}\mathbf{j}|_{p,S}\le c\epsilon_0\\
&|r^{2-\frac{2}{p}}({}^{(T)}\mathbf{m},{}^{(T)}\underline{\mathbf{m}})|_{p,S}\le c\epsilon_0\\
&|r^{2-\frac{2}{p}}{}^{(T)}\mathbf{n}|_{p,S}\le c\epsilon_0\\
&|r^{1-\frac{2}{p}}\tau_-{}^{(T)}\underline{\mathbf{n}}|_{p,S}\le c\epsilon_0
\end{aligned}\tag{6.1.29}$$

$$\begin{aligned}
&|r^{2-\frac{2}{p}}\tau_-\nabla\!\!\!/\,{}^{(T)}\mathbf{i}|_{p,S}\le c\epsilon_0\\
&|r^{2-\frac{2}{p}}\nabla\!\!\!/\,{}^{(T)}\mathbf{j}|_{p,S}\le c\epsilon_0\\
&|r^{3-\frac{2}{p}}\nabla\!\!\!/\,({}^{(T)}\mathbf{m},{}^{(T)}\underline{\mathbf{m}})|_{p,S}\le c\epsilon_0\\
&|r^{3-\frac{2}{p}}\nabla\!\!\!/\,{}^{(T)}\mathbf{n}|_{p,S}\le c\epsilon_0\\
&|r^{2-\frac{2}{p}}\tau_-\nabla\!\!\!/\,{}^{(T)}\underline{\mathbf{n}}|_{p,S}\le c\epsilon_0
\end{aligned}\tag{6.1.30}$$

$$\begin{aligned}
&|r^{1-\frac{2}{p}}\tau_-^2\mathcal{D}\!\!\!/_3{}^{(T)}\mathbf{i}|_{p,S}\le c\epsilon_0\\
&|r^{1-\frac{2}{p}}\tau_-^2\mathcal{D}\!\!\!/_3{}^{(T)}\mathbf{j}|_{p,S}\le c\epsilon_0\\
&|r^{2-\frac{2}{p}}\tau_-\mathcal{D}\!\!\!/_3({}^{(T)}\mathbf{m},{}^{(T)}\underline{\mathbf{m}})|_{p,S}\le c\epsilon_0\\
&|r^{3-\frac{2}{p}}\mathcal{D}\!\!\!/_3{}^{(T)}\mathbf{n}|_{p,S}\le c\epsilon_0\\
&|r^{1-\frac{2}{p}}\tau_-^2\mathcal{D}\!\!\!/_3{}^{(T)}\underline{\mathbf{n}}|_{p,S}\le c\epsilon_0
\end{aligned}\tag{6.1.31}$$

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\tau_-\not{\mathbf{D}}_4{}^{(T)}\mathbf{i}|_{p,S} \leq c\epsilon_0 \\
&|r^{2-\frac{2}{p}}\tau_-\not{\mathbf{D}}_4{}^{(T)}\mathbf{j}|_{p,S} \leq c\epsilon_0 \\
&|r^{3-\frac{2}{p}}\not{\mathbf{D}}_4({}^{(T)}\mathbf{m}, {}^{(T)}\underline{\mathbf{m}})|_{p,S} \leq c\epsilon_0 \\
&|r^{3-\frac{2}{p}}\not{\mathbf{D}}_4{}^{(T)}\mathbf{n}|_{p,S} \leq c\epsilon_0 \\
&\||r^{3-\frac{2}{p}}\not{\mathbf{D}}_4{}^{(T)}\underline{\mathbf{n}}|_{p,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.32}
$$

Corollary 6.1.1 *Under the previous assumptions the following inequalities hold*

$$
\begin{aligned}
&|r\tau_-{}^{(T)}\mathbf{i}|_{\infty,S} \leq c\epsilon_0 \\
&|r\tau_-{}^{(T)}\mathbf{j}|_{\infty,S} \leq c\epsilon_0 \\
&|r^2({}^{(T)}\mathbf{m}, {}^{(T)}\underline{\mathbf{m}})|_{\infty,S} \leq c\epsilon_0 \\
&|r^{2(T)}\mathbf{n}|_{\infty,S} \leq c\epsilon_0 \\
&|r\tau_-{}^{(T)}\underline{\mathbf{n}}|_{\infty,S} \leq c\epsilon_0 \\
&|r^{3-\frac{2}{p}}\not{\mathbf{D}}_3{}^{(T)}\mathbf{n}|_{\infty,S} \leq c\epsilon_0 \\
&|r^{3-\frac{2}{p}}\not{\mathbf{D}}_4{}^{(T)}\underline{\mathbf{n}}|_{\infty,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.33}
$$

Proposition 6.1.2 *Assuming the results of Theorems 4.2.1, 4.2.2 the following estimates hold, for any $S \subset \mathcal{K}$, for $p \in [2,4]$,*

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\frac{1}{\log r}{}^{(S)}\mathbf{i}|_{p,S} \leq c\epsilon_0 \\
&|r^{1-\frac{2}{p}}\frac{1}{\log r}{}^{(S)}\mathbf{j}|_{p,S} \leq c\epsilon_0 \\
&|r^{2-\frac{2}{p}}{}^{(S)}\mathbf{m}|_{p,S} \leq c\tau_-\epsilon_0 \\
&|r^{1-\frac{2}{p}}{}^{(S)}\underline{\mathbf{m}}|_{p,S} \leq c\epsilon_0 \\
&|r^{2-\frac{2}{p}}{}^{(S)}\mathbf{n}|_{p,S} \leq c\tau_-\epsilon_0 \\
&|r^{-\frac{2}{p}}\tau_-{}^{(S)}\underline{\mathbf{n}}|_{p,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.34}
$$

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\not{\nabla}{}^{(S)}\mathbf{i}|_{p,S} \leq c\epsilon_0 \\
&|r^{2-\frac{2}{p}}\not{\nabla}{}^{(S)}\mathbf{j}|_{p,S} \leq c\epsilon_0 \\
&|r^{3-\frac{2}{p}}\not{\nabla}{}^{(S)}\mathbf{m}|_{p,S} \leq c\tau_-\epsilon_0 \\
&|r^{2-\frac{2}{p}}\not{\nabla}{}^{(S)}\underline{\mathbf{m}}|_{p,S} \leq c\epsilon_0 \\
&|r^{3-\frac{2}{p}}\not{\nabla}{}^{(S)}\mathbf{n}|_{p,S} \leq c\tau_-\epsilon_0 \\
&|r^{1-\frac{2}{p}}\tau_-\not{\nabla}{}^{(S)}\underline{\mathbf{n}}|_{p,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.35}
$$

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\frac{\tau_-}{\log r}\not{D}_3{}^{(S)}\mathbf{i}|_{p,S} \leq c\epsilon_0\\
&|r^{1-\frac{2}{p}}\frac{\tau_-}{\log r}\not{D}_3{}^{(S)}\mathbf{j}|_{p,S} \leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\not{D}_3{}^{(S)}\mathbf{m}|_{p,S} \leq c\epsilon_0\\
&|r^{1-\frac{2}{p}}\tau_-\not{D}_3{}^{(S)}\underline{\mathbf{m}}|_{p,S} \leq c\epsilon_0\\
&|r^{-\frac{2}{p}}\tau_-^2\not{D}_3{}^{(S)}\underline{\mathbf{n}}|_{p,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.36}
$$

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\frac{\tau_-}{\log r}\not{D}_4{}^{(S)}\mathbf{i}|_{p,S} \leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\frac{\tau_-}{\log r}\not{D}_4{}^{(S)}\mathbf{j}|_{p,S} \leq c\epsilon_0\\
&|r^{3-\frac{2}{p}}\not{D}_4{}^{(S)}\mathbf{m}|_{p,S} \leq c\tau_-\epsilon_0\\
&|r^{2-\frac{2}{p}}\not{D}_4{}^{(S)}\underline{\mathbf{m}}|_{p,S} \leq c\epsilon_0\\
&|r^{3-\frac{2}{p}}\not{D}_4{}^{(S)}\mathbf{n}|_{p,S} \leq c\tau_-\epsilon_0
\end{aligned}
\tag{6.1.37}
$$

and, for $p \in [2,\infty]$,

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\not{D}_3{}^{(S)}\mathbf{n}|_{p,S} \leq c\epsilon_0\\
&|r^{1-\frac{2}{p}}\tau_-\not{D}_4{}^{(S)}\underline{\mathbf{n}}|_{p,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.38}
$$

Proof: We examine only the estimate of $|r^{1-\frac{2}{p}}{}^{(S)}\mathbf{i}|_{p,S}$ to explain the logarithmic factor present in its estimate, the other estimates follow immediately from those relative to the connection coefficients. From the explicit expression of ${}^{(S)}\mathbf{i}$, see 3.4.8,

$$
{}^{(S)}\mathbf{i}_{ab} = \underline{u}\hat{\chi}_{ab} + u\hat{\underline{\chi}}_{ab} + \frac{1}{2}\delta_{ab}(\frac{1}{2}(\underline{u}\mathrm{tr}\chi + u\mathrm{tr}\underline{\chi}) + (\underline{u}\omega + u\underline{\omega}) - \frac{1}{\Omega})
$$

one realizes that the part more delicate to control is $\frac{1}{2}\left[(\underline{u}tr\chi + utr\underline{\chi}) - \frac{1}{\Omega}\right]$. Using inequality 4.3.77,

$$
\begin{aligned}
\frac{1}{2}\left[\underline{u}(tr\chi + utr\underline{\chi}) - \Omega^{-1}\right] &= (2\Omega)^{-1}\left[\left(\Omega\frac{(\underline{u}-u)}{4}(tr\chi - tr\underline{\chi}) - 1\right)\right.\\
&\qquad\left. + \left(\Omega\frac{(\underline{u}+u)}{4}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi})\right)\right]\\
&\leq (2\Omega)^{-1}\left[\Omega\frac{(\underline{u}-u)}{4}(tr\chi - tr\underline{\chi}) - 1\right] + c\epsilon_0\frac{1}{r}
\end{aligned}
$$

The first term in the right hand side can be rewritten as

$$
\begin{aligned}
\frac{1}{2\Omega}\left[\Omega\frac{(\underline{u}-u)}{4}(tr\chi - tr\underline{\chi}) - 1\right] &= \frac{1}{2\Omega}\left[\frac{\Omega(\underline{u}-u)}{4}\left(\mathrm{tr}\chi - \frac{2}{\Omega(\underline{u}-u)}\right)\right]\\
&- \frac{1}{2\Omega}\left[\frac{\Omega(\underline{u}-u)}{4}\left(\mathrm{tr}\underline{\chi} + \frac{2}{\Omega(\underline{u}-u)}\right)\right]
\end{aligned}
$$

and the two terms in the right hand side are estimated in the same way. Let us consider the first one,

$$\begin{aligned}\left(\mathrm{tr}\chi-\frac{2}{\Omega(\underline{u}-u)}\right) &= \left(\mathrm{tr}\chi-\frac{2}{r}\right)+\left(\frac{2}{r}-\frac{2}{\Omega(\underline{u}-u)}\right)\\ &= \epsilon_0 O\left(\frac{1}{r^2}\right)+2\left(\frac{1}{r}-\frac{1}{\Omega(\underline{u}-u)}\right)\\ &= \epsilon_0 O\left(\frac{1}{r^2}\right)+\frac{2}{\Omega r}\left[\left(\Omega-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{r}{\underline{u}-u}\right)\right]\\ &= \epsilon_0 O\left(\frac{1}{r^2}\right)+\frac{1}{r(\underline{u}-u)}\left[\frac{\underline{u}-u}{2}-r\right]\end{aligned}$$

and we are left to control $\left|\frac{\underline{u}-u}{2}-r\right|$. From the results of Chapter 4, see Lemma 4.1.8, we obtain $\left|\frac{\underline{u}-u}{2}-r\right| \le c\epsilon_0 \log r$ so that, finally,

$$\frac{1}{2}\left[(\underline{u}tr\chi+utr\underline{\chi})-\frac{1}{\Omega}\right] \le c\epsilon_0\frac{\log r}{r} \tag{6.1.39}$$

and, from it, $|r^{1-\frac{2}{p}}\frac{1}{\log r}{}^{(S)}\mathbf{i}|_{p,S} \le c\epsilon_0$.

Corollary 6.1.2 *Under the previous assumptions the following inequalities hold, for any $S\subset\mathcal{K}$,*

$$\begin{aligned}&|\tfrac{r}{\log r}{}^{(S)}\mathbf{i}|_{\infty,S} \le c\epsilon_0\\ &|\tfrac{r}{\log r}{}^{(S)}\mathbf{j}|_{\infty,S} \le c\epsilon_0\\ &|r^{2}{}^{(S)}\mathbf{m}|_{\infty,S} \le c\tau_-\epsilon_0\\ &|r{}^{(S)}\underline{\mathbf{m}}|_{\infty,S} \le c\epsilon_0\\ &|r^{2}{}^{(S)}\mathbf{n}|_{\infty,S} \le c\tau_-\epsilon_0\\ &|\tau_-{}^{(S)}\underline{\mathbf{n}}|_{\infty,S} \le c\epsilon_0\end{aligned} \tag{6.1.40}$$

Proposition 6.1.3 *Assuming the results of Theorems 4.2.1, 4.2.2 the following estimates hold, for any $S\subset\mathcal{K}$ and $p\in[2,4]$,*

$$\begin{aligned}&|r^{-\frac{2}{p}}\tfrac{1}{\log r}{}^{(\bar{K})}\mathbf{i}|_{p,S} \le c\epsilon_0\\ &|r^{-\frac{2}{p}}\tfrac{1}{\log r}{}^{(\bar{K})}\mathbf{j}|_{p,S} \le c\epsilon_0\\ &|r^{2-\frac{2}{p}}{}^{(\bar{K})}\mathbf{m}|_{p,S} \le c\tau_-^2\epsilon_0\\ &|r^{-\frac{2}{p}}{}^{(\bar{K})}\underline{\mathbf{m}}|_{p,S} \le c\epsilon_0\\ &|r^{2-\frac{2}{p}}{}^{(\bar{K})}\mathbf{n}|_{p,S} \le c\tau_-^2\epsilon_0\\ &|r^{-\frac{2}{p}}\tau_-{}^{(\bar{K})}\underline{\mathbf{n}}|_{p,S} \le cr\epsilon_0\end{aligned} \tag{6.1.41}$$

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\nabla\!\!\!/\,{}^{(\bar{K})}\mathbf{i}|_{p,S} \leq c\epsilon_0\\
&|r^{1-\frac{2}{p}}\nabla\!\!\!/\,{}^{(\bar{K})}\mathbf{j}|_{p,S} \leq c\epsilon_0\\
&|r^{3-\frac{2}{p}}\nabla\!\!\!/\,{}^{(\bar{K})}\mathbf{m}|_{p,S} \leq c\tau_-^2\epsilon_0\\
&|r^{1-\frac{2}{p}}\nabla\!\!\!/\,{}^{(\bar{K})}\underline{\mathbf{m}}|_{p,S} \leq c\epsilon_0\\
&|r^{3-\frac{2}{p}}\nabla\!\!\!/\,{}^{(\bar{K})}\mathbf{n}|_{p,S} \leq c\tau_-^2\epsilon_0\\
&|r^{-\frac{2}{p}}\tau_-\nabla\!\!\!/\,{}^{(\bar{K})}\underline{\mathbf{n}}|_{p,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.42}
$$

$$
\begin{aligned}
&|r^{-\frac{2}{p}}\frac{\tau_-}{\log r}\mathbf{D}\!\!\!\!/_3{}^{(\bar{K})}\mathbf{i}|_{p,S} \leq c\epsilon_0\\
&|r^{-\frac{2}{p}}\frac{\tau_-}{\log r}\mathbf{D}_3{}^{(\bar{K})}\mathbf{j}|_{p,S} \leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\mathbf{D}\!\!\!\!/_3{}^{(\bar{K})}\mathbf{m}|_{p,S} \leq c\tau_-\epsilon_0\\
&|r^{-\frac{2}{p}}\tau_-\mathbf{D}\!\!\!\!/_3{}^{(\bar{K})}\underline{\mathbf{m}}|_{p,S} \leq c\epsilon_0\\
&|r^{-\frac{2}{p}}\tau_-^2\mathbf{D}\!\!\!\!/_3{}^{(\bar{K})}\underline{\mathbf{n}}|_{p,S} \leq cr\epsilon_0
\end{aligned}
\tag{6.1.43}
$$

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\frac{\tau_-}{\log r}\mathbf{D}\!\!\!\!/_4{}^{(\bar{K})}\mathbf{i}|_{p,S} \leq c\epsilon_0\\
&|r^{1-\frac{2}{p}}\frac{\tau_-}{\log r}\mathbf{D}\!\!\!\!/_4{}^{(\bar{K})}\mathbf{j}|_{p,S} \leq c\epsilon_0\\
&|r^{3-\frac{2}{p}}\mathbf{D}\!\!\!\!/_4{}^{(\bar{K})}\mathbf{m}|_{p,S} \leq c\tau_-\epsilon_0\\
&|r^{1-\frac{2}{p}}\mathbf{D}\!\!\!\!/_4{}^{(\bar{K})}\underline{\mathbf{m}}|_{p,S} \leq c\epsilon_0\\
&|r^{3-\frac{2}{p}}\mathbf{D}\!\!\!\!/_4{}^{(\bar{K})}\mathbf{n}|_{p,S} \leq c\tau_-^2\epsilon_0
\end{aligned}
\tag{6.1.44}
$$

and, for $p \in [2,\infty]$,

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\mathbf{D}\!\!\!\!/_3{}^{(\bar{K})}\mathbf{n}|_{p,S} \leq c\tau_-\epsilon_0\\
&|r^{-\frac{2}{p}}\mathbf{D}\!\!\!\!/_4{}^{(\bar{K})}\underline{\mathbf{n}}|_{p,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.45}
$$

Proof: Again, the more delicate K_0-deformation tensor[9] components to estimate are, see 3.4.9,

$$
\begin{aligned}
{}^{(K_0)}\mathbf{i}_{ab} &= \underline{u}^2\hat{\chi}_{ab} + u^2\hat{\underline{\chi}}_{ab} + \frac{1}{2}\delta_{ab}\left(\frac{1}{2}(\underline{u}^2\text{tr}\chi + u^2\text{tr}\underline{\chi}) + (\underline{u}^2\omega + u^2\underline{\omega}) - \frac{\underline{u}+u}{\Omega}\right)\\
{}^{(K_0)}\mathbf{j} &= \frac{1}{2}(\underline{u}^2\text{tr}\chi + u^2\text{tr}\underline{\chi}) + (\underline{u}^2\omega + u^2\underline{\omega}) - \frac{\underline{u}+u}{\Omega}
\end{aligned}
$$

Looking at the expression of ${}^{(K_0)}\mathbf{i}$, the part more delicate to control is $\frac{1}{2}\left[(\underline{u}^2\text{tr}\chi + u^2\text{tr}\underline{\chi}) - \frac{1}{\Omega}(u+\underline{u})\right]$. Proceeding as in the case of ${}^{(S)}\mathbf{i}$, we estimate it, using inequality 4.3.77, and Lemma 4.1.8

$$
\frac{1}{2}(\underline{u}^2\text{tr}\chi + u^2\text{tr}\underline{\chi}) - \frac{1}{\Omega}(u+\underline{u}) = \frac{1}{2}(\underline{u}^2 - u^2)\text{tr}\chi + \frac{1}{2}u^2(\text{tr}\chi + \text{tr}\underline{\chi}) - \frac{1}{\Omega}(u+\underline{u})
$$

[9] Recall that we can use indifferently, here, K_0 or $\bar{K}$.

$$
\begin{aligned}
&= \frac{1}{2}u^2(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) + (\underline{u} + u)\left[\frac{1}{2}(\underline{u} - u)\mathrm{tr}\chi - 2\right] + O(\epsilon_0) \\
&= O(r)\left[\frac{(\underline{u} - u)}{r} - 2\right] + \epsilon_0\left[1 + O(\tau_-^2 r^{-2})\right] = \epsilon_0\left[O(1) + O(\log r)\right]
\end{aligned}
$$

All the remaining estimates follow easily from the corresponding estimates of the connection coefficients.

Corollary 6.1.3 *Under the previous assumptions the following inequalities hold, for any $S \subset \mathcal{K}$,*

$$
\begin{aligned}
&|\tfrac{1}{\log r}{}^{(\bar{K})}\mathbf{i}|_{\infty,S} \leq c\epsilon_0 \\
&|\tfrac{1}{\log r}{}^{(\bar{K})}\mathbf{j}|_{\infty,S} \leq c\epsilon_0 \\
&|r^{2\,(\bar{K})}\mathbf{m}|_{\infty,S} \leq c\tau_-^2\epsilon_0 \\
&|{}^{(\bar{K})}\underline{\mathbf{m}}|_{\infty,S} \leq c\epsilon_0 \\
&|r^{2\,(\bar{K})}\mathbf{n}|_{\infty,S} \leq c\tau_-^2\epsilon_0 \\
&|\tau_-{}^{(\bar{K})}\underline{\mathbf{n}}|_{\infty,S} \leq cr\epsilon_0
\end{aligned}
\tag{6.1.46}
$$

Proposition 6.1.4 *From the results of Proposition 6.1.1 and from the explicit expressions of ${}^{(T)}p_3$, ${}^{(T)}p_4$ and ${}^{(T)}\not{p}$, we obtain, for $p \in [2,4]$, the following estimates, for any $S \subset \mathcal{K}$,*

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\tau_-^{2\,(T)}p_3|_{p,S} \leq c\epsilon_0 \\
&|r^{2-\frac{2}{p}}\tau_-{}^{(T)}p_4|_{p,S} \leq c\epsilon_0 \\
&|r^{2-\frac{2}{p}}\tau_-{}^{(T)}\not{p}_a|_{p,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.47}
$$

Proposition 6.1.5 *From the results of Proposition 6.1.1 and from the explicit expressions of $\not{\nabla}{}^{(T)}p_3$, $\not{\nabla}{}^{(T)}p_4$ and $\not{\nabla}{}^{(T)}\not{p}$, we obtain, for $p \in [2,4]$, the following estimates, for any $S \subset \mathcal{K}$,*

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\tau_-^2\not{\nabla}{}^{(T)}p_3|_{p,S} \leq c\epsilon_0 \\
&|r^{3-\frac{2}{p}}\tau_-\not{\nabla}{}^{(T)}p_4|_{p,S} \leq c\epsilon_0 \\
&|r^{3-\frac{2}{p}}\tau_-\not{\nabla}{}^{(T)}\not{p}_a|_{p,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.48}
$$

In addition to the estimates of Propositions 6.1.4, 6.1.5 we shall also need, for the T deformation tensors, the following proposition:

Proposition 6.1.6 *From the results of Proposition 6.1.1 and of Propositions 4.3.10 and 4.4.2 the following estimates hold*

$$||\frac{1}{\sqrt{\tau_+}}\tau_-^2\,{}^{(T)}p_3||_{L_2(C\cap\mathcal{K})} \le c\epsilon_0 \tag{6.1.49}$$
$$||\frac{1}{\sqrt{\tau_+}}\tau_-^2 r\nabla\!\!\!\!/\,{}^{(T)}p_3||_{L_2(C\cap\mathcal{K})} \le c\epsilon_0$$
$$||\frac{1}{\sqrt{\tau_+}}\tau_-^2\hat{\mathcal{L}}_S{}^{(T)}p_3||_{L_2(C\cap\mathcal{K})} \le c\epsilon_0$$

Proof: From the explicit expression of ${}^{(T)}p_3$, see 6.1.24, it follows immediately that its more delicate term is $\mathbf{D}_3^2\log\Omega$, then one refers to Propositions 4.3.10 and 4.4.2.

Proposition 6.1.7 *From the results of Proposition 6.1.2 and from the explicit expressions of ${}^{(S)}p_3$, ${}^{(S)}p_4$ and ${}^{(S)}p\!\!\!/$, we obtain, for $p\in[2,4]$, the following estimates, for any $S\subset\mathcal{K}$,*

$$\begin{aligned}&|r^{1-\frac{2}{p}}\frac{\tau_-}{\log r}{}^{(S)}p_3|_{p,S}\le c\epsilon_0\\ &|r^{1-\frac{2}{p}}\frac{\tau_-}{\log r}{}^{(S)}p_4|_{p,S}\le c\epsilon_0\\ &|r^{2-\frac{2}{p}}{}^{(S)}p\!\!\!/_a|_{p,S}\le c\epsilon_0\end{aligned} \tag{6.1.50}$$

The various components of ${}^{(T)}q$ and of ${}^{(S)}q$ satisfy the following estimates:

Proposition 6.1.8 *From the results of Proposition 6.1.1 and from the explicit expressions of ${}^{(T)}q$ we obtain, for $p\in[2,4]$, the following estimates, for any $S\subset\mathcal{K}$,*

$$\begin{aligned}&|r^{2-\frac{2}{p}}\tau_-\Lambda({}^{(T)}q)|_{p,S}\le c\epsilon_0\\ &|r^{3-\frac{2}{p}}K({}^{(T)}q)|_{p,S}\le c\epsilon_0\\ &|r^{3-\frac{2}{p}}\Xi({}^{(T)}q)|_{p,S}\le c\epsilon_0\\ &|r^{2-\frac{2}{p}}\tau_-I({}^{(T)}q)|_{p,S}\le c\epsilon_0\\ &|r^{2-\frac{2}{p}}\tau_-\Theta({}^{(T)}q)|_{p,S}\le c\epsilon_0\\ &|r^{4-\frac{2}{p}}\nabla\!\!\!\!/ K({}^{(T)}q)|_{p,S}\le c\epsilon_0\end{aligned} \tag{6.1.51}$$

and

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\tau_-^2\underline{\Lambda}({}^{(T)}q)|_{p,S} \leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\tau_-\underline{K}({}^{(T)}q)|_{p,S} \leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\tau_-\underline{\Xi}({}^{(T)}q)|_{p,S} \leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\tau_-\underline{I}({}^{(T)}q)|_{p,S} \leq c\epsilon_0\\
&|r^{1-\frac{2}{p}}\tau_-^2\underline{\Theta}({}^{(T)}q)|_{p,S} \leq c\epsilon_0\\
&|r^{3-\frac{2}{p}}\tau_-\nabla\!\!\!/\underline{K}({}^{(T)}q)|_{p,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.52}
$$

Corollary 6.1.4 *Using the Sobolev Lemma 4.1.3, the sup norms for $K({}^{(T)}q)$ and $\underline{K}({}^{(T)}q)$ are bounded*

$$
\begin{aligned}
&|r^3K({}^{(T)}q)|_{\infty,S} \leq c\epsilon_0\\
&|r^2\tau_-\underline{K}({}^{(T)}q)|_{\infty,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.53}
$$

Remark: Recall that the norms associated to the T vector field with the slowest asymptotic behaviour in r are

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\tau_-{}^{(T)}\mathbf{i}|_{p,S}\ ,\ |r^{1-\frac{2}{p}}\tau_-{}^{(T)}\mathbf{j}|_{p,S}\ ,\ |r^{1-\frac{2}{p}}\tau_-{}^{(T)}\mathbf{n}|_{p,S}\\
&|r^{1-\frac{2}{p}}\tau_-^2\mathbf{D}\!\!\!\!/_3{}^{(T)}\mathbf{i}|_{p,S}\ ,\ |r^{1-\frac{2}{p}}\tau_-^2\mathbf{D}\!\!\!\!/_3{}^{(T)}\mathbf{j}|_{p,S}\ ,\ |r^{1-\frac{2}{p}}\tau_-^2\mathbf{D}\!\!\!\!/_3{}^{(T)}\underline{\mathbf{n}}|_{p,S}\\
&|r^{1-\frac{2}{p}}\tau_-^2{}^{(T)}p_3|_{p,S}\ ,\ |r^{1-\frac{2}{p}}\tau_-^2\underline{\Lambda}({}^{(T)}q)|_{p,S}\ ,\ |r^{1-\frac{2}{p}}\tau_-^2\underline{\Theta}({}^{(T)}q)|_{p,S}
\end{aligned}
$$

Examining all these terms we observe that the slow decay of these quantities originates from the behaviour of $\mathbf{D}_3\log\Omega$ or $\mathbf{D}_3^2\log\Omega$ on the last slice, see subsection 3.5.5.

Proposition 6.1.9 *From the results of Proposition 6.1.2 and from the explicit expressions of ${}^{(S)}q$ we obtain, for $p \in [2,4]$, the following estimates, for any $S \subset \mathcal{K}$,*

$$
\begin{aligned}
&|r^{2-\frac{2}{p}}\frac{\tau_-}{\log r}\Lambda({}^{(S)}q)|_{p,S} \leq c\epsilon_0\\
&|r^{3-\frac{2}{p}}\frac{1}{\log r}K({}^{(S)}q)|_{p,S} \leq c\epsilon_0\\
&|r^{3-\frac{2}{p}}\Xi({}^{(S)}q)|_{p,S} \leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\frac{\tau_-}{\log r}I({}^{(S)}q)|_{p,S} \leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\frac{\tau_-}{\log r}\Theta({}^{(S)}q)|_{p,S} \leq c\epsilon_0\\
&|r^{4-\frac{2}{p}}\frac{1}{\log r}\nabla\!\!\!/ K({}^{(S)}q)|_{p,S} \leq c\epsilon_0
\end{aligned}
\tag{6.1.54}
$$

and

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\frac{\tau_-^2}{\log r}\underline{\Lambda}({}^{(S)}q)|_{p,S}\leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\frac{\tau_-}{\log r}\underline{K}({}^{(S)}q)|_{p,S}\leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\tau_-\underline{\Xi}({}^{(S)}q)|_{p,S}\leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\frac{\tau_-}{\log r}\underline{I}({}^{(S)}q)|_{p,S}\leq c\epsilon_0\\
&|r^{1-\frac{2}{p}}\frac{\tau_-^2}{\log r}\underline{\Theta}({}^{(S)}q)|_{p,S}\leq c\epsilon_0\\
&|r^{3-\frac{2}{p}}\frac{\tau_-}{\log r}\nabla\!\!\!/\underline{K}({}^{(S)}q)|_{p,S}\leq c\epsilon_0
\end{aligned}
\tag{6.1.55}
$$

6.1.2 Estimates for the rotation deformation tensors

We recall the result proven in Chapter 4, section 4.6:

Corollary 4.7.1 *In $\mathcal{K}$, the following inequalities hold:*

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|_{p,S}\leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\nabla\!\!\!/({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|_{p,S}\leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}\mathbf{D}\!\!\!\!/_4({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|_{p,S}\leq c\epsilon_0\\
&|r^{1-\frac{2}{p}}\tau_-\mathbf{D}\!\!\!\!/_3({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|_{p,S}\leq c\epsilon_0
\end{aligned}
$$

The first line for any $p\in[2,\infty]$ and the other ones for $p\in[2,4]$.
Moreover

$$
{}^{(O)}\mathbf{n},{}^{(O)}\underline{\mathbf{n}},{}^{(O)}\mathbf{m}=0\ .
$$

The following propositions provide the estimates for the various components of ${}^{(O)}p$ and ${}^{(O)}q$. They are a consequence of the estimates for the connection coefficients proven in Theorems 4.2.1, 4.2.2 and of Corollary 4.7.1.

Proposition 6.1.10 *Assuming the results of Theorems 4.2.1, 4.2.2 and of Corollary 4.7.1, the following estimates hold, for any $p\in[2,4]$, for any $S\subset\mathcal{K}$,*

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\tau_-{}^{(O)}p_3|_{p,S}\leq c\epsilon_0\\
&|r^{2-\frac{2}{p}}({}^{(O)}p_4,{}^{(O)}p\!\!\!/)|_{p,S}\leq c\epsilon_0
\end{aligned}
\tag{6.1.56}
$$

Proposition 6.1.11 *Assuming the results of Theorems 4.2.1, 4.2.2 and of Corollary 4.7.1, the following estimates hold*

$$
\begin{aligned}
&||\hat{\mathcal{L}}_O{}^{(O)}p_3||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))}\leq c\epsilon_0\\
&||\hat{\mathcal{L}}_O{}^{(O)}p\!\!\!/||_{L^2(\underline{C}(\underline{u})\cap V(u,\underline{u}))}\leq c\epsilon_0\\
&||\frac{1}{\sqrt{\tau_+}}r\hat{\mathcal{L}}_O{}^{(O)}p_4||_{L^2(C(u)\cap V(u,\underline{u}))}\leq c\epsilon_0
\end{aligned}
$$

Proof: It is enough to observe that the more delicate term for $\hat{\mathcal{L}}_O{}^{(O)}p_3$ is $\nabla\!\!\!\!/^{2(i)}Z_a$ and for $\hat{\mathcal{L}}_O{}^{(O)}p\!\!\!/$ is $\nabla\!\!\!\!/^{2(i)}H_{ab}$ and then use Proposition 4.7.2. The last inequality is easier to obtain and we do not report it here.

Proposition 6.1.12 *Under the same assumptions as in Proposition 6.1.11 the following estimates hold*

$$\begin{aligned}
&\sup_{\mathcal{K}} |r^{2-\frac{2}{p}}\Lambda(^{(O)}q)|_{p,S} \leq c\epsilon_0\\
&\sup_{\mathcal{K}} |r^{3-\frac{2}{p}}K(^{(O)}q)|_{p,S} \leq c\epsilon_0\\
&\sup_{\mathcal{K}} |r^{2-\frac{2}{p}}I(^{(O)}q)|_{p,S} \leq c\epsilon_0\\
&\sup_{\mathcal{K}} |r^{2-\frac{2}{p}}\Theta(^{(O)}q)|_{p,S} \leq c\epsilon_0\\
&\qquad \Xi(^{(O)}q) = 0
\end{aligned} \tag{6.1.57}$$

and

$$\begin{aligned}
&\sup_{\mathcal{K}} |r^{1-\frac{2}{p}}\tau_-\underline{\Lambda}(^{(O)}q)|_{p,S} \leq c\epsilon_0\\
&\sup_{\mathcal{K}} |r^{2-\frac{2}{p}}\underline{K}(^{(O)}q)|_{p,S} \leq c\epsilon_0\\
&\sup_{\mathcal{K}} |r^{1-\frac{2}{p}}\underline{\Xi}(^{(O)}q)|_{p,S} \leq c\epsilon_0\\
&\sup_{\mathcal{K}} |r^{2-\frac{2}{p}}\underline{I}(^{(O)}q)|_{p,S} \leq c\epsilon_0\\
&\sup_{\mathcal{K}} |r^{1-\frac{2}{p}}\tau_-\underline{\Theta}(^{(O)}q)|_{p,S} \leq c\epsilon_0
\end{aligned} \tag{6.1.58}$$

for $p \in [2,4]$.

Proposition 6.1.13 *Under the same assumptions as in Proposition 6.1.11 the following estimates hold, the first with* $\delta > \epsilon > 0$,

$$\begin{aligned}
&||\frac{1}{\sqrt{r^{1-2\epsilon}}}\hat{\mathcal{L}}_O\underline{\Xi}(^{(O)}q)||_{L^2(\underline{C}(\underline{u}')\cap V(u,\underline{u}'))} \leq c\frac{1}{u\sqrt{r^{1-2\delta}}}\epsilon_0\\
&||\hat{\mathcal{L}}_O(I(^{(O)}q),\underline{I}(^{(O)}q))||_{L^2(\underline{C}(\underline{u}')\cap V(u,\underline{u}'))} \leq c\epsilon_0\\
&||\hat{\mathcal{L}}_O\underline{\Lambda}(^{(O)}q)||_{L^2(\underline{C}(\underline{u}')\cap V(u,\underline{u}'))} \leq c\epsilon_0\\
&||\hat{\mathcal{L}}_O\underline{K}(^{(O)}q)||_{L^2(\underline{C}(\underline{u}')\cap V(u,\underline{u}'))} \leq c\epsilon_0\\
&||\hat{\mathcal{L}}_O\underline{\Theta}(^{(O)}q)||_{L^2(\underline{C}(\underline{u}')\cap V(u,\underline{u}'))} \leq c\epsilon_0
\end{aligned} \tag{6.1.59}$$

Proof: For the first inequality it is enough to observe that the more delicate term for $\hat{\mathcal{L}}_O\underline{\Xi}(^{(O)}q)$ is $r\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3 Z$ and then use Proposition 4.7.2. An analogous argument holds for the remaining inequalities.

Remark: Some of the most delicate error terms appear in connection to the highest derivatives of the rotation deformation tensors ${}^{(O)}\hat{\pi}$. Indeed, as discussed in detail in section 4.6, unlike all other deformation tensors, the second derivatives of ${}^{(O)}\hat{\pi}_{ab}$ and ${}^{(O)}\hat{\pi}_{a3}$ involve the third derivatives of the connection coefficients. This is the reason why it is crucial to show that the $\mathcal{D}_2$ norms depend only on the second derivatives [10] of the curvature tensor and not on the third derivatives as it may first have appeared from the structure equations. In the appendix to this chapter we recall precisely where the third derivatives of the connection coefficients appear.

6.2 The error terms $\mathcal{E}_1$

In Chapter 3, subsection 3.3.4, we have introduced above Σ_0 a narrow region $\mathcal{K}'_{\delta_0}$ called "initial layer region" endowed with a different foliation, the *initial layer foliation* which fits appropriately with the initial hypersurface Σ_0. The hypersurface Σ'_{δ_0} is the upper boundary of $\mathcal{K}'_{\delta_0}$. Moreover the Oscillation Lemma shows that we can define an hypersurface $\tilde{\Sigma}_{\tilde{\delta}_0}$, see Corollary **??**, associated to the *double null canonical foliation* at a distance $c\epsilon_0$ from Σ'_{δ_0}. All the estimates done in this Chapter are relative to the *double null canonical foliation* and the "initial hypersurface" is, in this case, $\tilde{\Sigma}_{\tilde{\delta}_0}$, we call simply Σ_0 in the sequel.

To complete the proof of Theorem **M8** we have then to estimate $\mathcal{Q}_{\tilde{\Sigma}_{\tilde{\delta}_0}\cap\mathcal{K}}$ in terms of $\mathcal{Q}_{\Sigma_0\cap\mathcal{K}}$. This is an immediate consequence of Theorem **M0** and the Oscillation Lemma.

Remark: It is easy to realize that the $\frac{1}{\log r}$ factors appearing in the various estimates of the deformation tensors relative to the S and K_0 vector fields appearing in Proposition 6.1.2, Corollary 6.1.2, Proposition 6.1.3, Corollary 6.1.3, Proposition 6.1.7 and Proposition 6.1.9 will not play any role in the subsequent estimates and, therefore, will be hereafter disregarded.

6.2.1 Estimate of $\int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta,\bar{K}^\gamma,\bar{K}^\delta)$

This requires to estimate the following four integrals:

$$B_1 \equiv \int_{V_{(u,\underline{u})}} \tau_+^6 D(T,W)_{444} \quad , \; B_2 \equiv \int_{V_{(u,\underline{u})}} \tau_+^4\tau_-^2 D(T,W)_{344}$$
$$B_3 \equiv \int_{V_{(u,\underline{u})}} \tau_+^2\tau_-^4 D(T,W)_{334} \; , \; B_4 \equiv \int_{V_{(u,\underline{u})}} \tau_-^6 D(T,W)_{333}$$

[10] See Proposition 4.7.2

From equations 6.1.3, to estimate B_1 we have to control the integrals:

$$\begin{aligned}
&\textstyle\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_T W) \cdot \Theta(T,W) \\
&\textstyle\int_{V_{(u,\underline{u})}} \tau_+^6 \beta(\hat{\mathcal{L}}_T W) \cdot \Xi(T,W)
\end{aligned} \tag{6.2.1}$$

to estimate B_2 we have to control the integrals:

$$\begin{aligned}
&\textstyle\int_{V_{(u,\underline{u})}} \tau_+^4 \tau_-^2 \rho(\hat{\mathcal{L}}_T W) \Lambda(T,W) \\
&\textstyle\int_{V_{(u,\underline{u})}} \tau_+^4 \tau_-^2 \sigma(\hat{\mathcal{L}}_T W) K(T,W) \\
&\textstyle\int_{V_{(u,\underline{u})}} \tau_+^4 \tau_-^2 \beta(\hat{\mathcal{L}}_T W) \cdot I(T,W)
\end{aligned} \tag{6.2.2}$$

to estimate B_3 we have to control the integrals:

$$\begin{aligned}
&\textstyle\int_{V_{(u,\underline{u})}} \tau_+^2 \tau_-^4 \rho(\hat{\mathcal{L}}_T W) \underline{\Lambda}(T,W) \\
&\textstyle\int_{V_{(u,\underline{u})}} \tau_+^2 \tau_-^4 \sigma(\hat{\mathcal{L}}_T W) \underline{K}(T,W) \\
&\textstyle\int_{V_{(u,\underline{u})}} \tau_+^2 \tau_-^4 \underline{\beta}(\hat{\mathcal{L}}_T W) \cdot \underline{I}(T,W)
\end{aligned} \tag{6.2.3}$$

to estimate B_4 we have to control the integrals:

$$\begin{aligned}
&\textstyle\int_{V_{(u,\underline{u})}} \tau_-^6 \underline{\alpha}(\hat{\mathcal{L}}_T W) \cdot \underline{\Theta}(T,W) \\
&\textstyle\int_{V_{(u,\underline{u})}} \tau_-^6 \underline{\beta}(\hat{\mathcal{L}}_T W) \cdot \underline{\Xi}(T,W)
\end{aligned} \tag{6.2.4}$$

We estimate in details the integrals appearing in B_1, those relative to the other groups B_2, B_3, B_4 have lower weights in τ_+ and, therefore, are easier to treat.

Estimate of the B_1 integrals

From the decomposition $J(X;W) = J^1(X;W) + J^2(X;W) + J^3(X;W)$, see eq. 6.1.6, it follows

$$\begin{aligned}
\Theta(T,W) &= \Theta^{(1)}(T,W) + \Theta^{(2)}(T,W) + \Theta^{(3)}(T,W) \\
\Xi(T,W) &= \Xi^{(1)}(T,W) + \Xi^{(2)}(T,W) + \Xi^{(3)}(T,W)
\end{aligned}$$

and we write the two B_1 integrals as sums of three terms:

$$\begin{aligned}
\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_T W) \cdot \Theta(T,W) &= \sum_{i=1}^{3} \int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_T W) \cdot \Theta^{(i)}(T,W) \\
\int_{V_{(u,\underline{u})}} \tau_+^6 \beta(\hat{\mathcal{L}}_T W) \cdot \Xi(T,W) &= \sum_{i=1}^{3} \int_{V_{(u,\underline{u})}} \tau_+^6 \beta(\hat{\mathcal{L}}_T W) \cdot \Xi^{(i)}(T,W)
\end{aligned}$$

Proposition 6.2.1 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\left|\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_T W)\cdot\Theta(T,W)\right| \leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}$$
$$\left|\int_{V_{(u,\underline{u})}} \tau_+^6 \beta(\hat{\mathcal{L}}_T W)\cdot\Xi(T,W)\right| \leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}$$

Proof: We discuss in detail the first integral, the estimate of the second one is similar. Using the coarea formulas

$$\int_{V(u,\underline{u})} F = \int_{u_0}^{u} du' \int_{C(u')\cap V(u,\underline{u})} F$$
$$\int_{V(u,\underline{u})} F = \int_{\underline{u}_0}^{\underline{u}} d\underline{u}' \int_{\underline{C}(\underline{u}')\cap V(u,\underline{u})} F \tag{6.2.5}$$

and Schwartz inequality we write

$$\left|\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_T W)\cdot\Theta(T,W)\right|$$
$$\leq c\int_{u_0}^{u} du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6|\alpha(\hat{\mathcal{L}}_T W)|^2\right)^{\frac{1}{2}} \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6|\Theta(T,W)|^2\right)^{\frac{1}{2}}$$
$$\leq c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\left[\sum_{i=1}^{3}\int_{u_0}^{u} du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6|\Theta^{(i)}(T,W)|^2\right)^{\frac{1}{2}}\right] \tag{6.2.6}$$

where we used the definition of $\mathcal{R}$ given in Chapter 3 and the inequality 6.0.2. The various terms in the right hand side associated to the currents J^1, J^2, J^3 are estimated separately [11]. The result is obtained proving the following

[11] The integrals depending on J^1 are estimated differently than those depending on J^2, J^3. The reason is that $\Theta^{(1)}(T,W)$ and $\Xi^{(1)}(T,W)$, see 6.1.8,..,6.1.17, are quadratic expressions depending linearly on the various components of the deformation tensor ${}^{(T)}\hat{\pi}$ and on the zero and first derivatives of the null Riemann components. Therefore, in this case, the components of the deformation tensor are estimated with their sup norms. On the other side the terms $\Theta^{(2,3)}(T,W), \Xi^{(2,3)}(T,W)$ associated to the J^2, J^3 currents, see 6.1.18,..,6.1.23, are quadratic expressions depending linearly on the deformation tensor ${}^{(T)}\hat{\pi}$, on its first derivatives and on the non derived null Riemann components. Therefore, in this case, the first derivative of the deformation tensor are estimated in the $|\cdot|_{p,S}$ norms, with $p\in[2,4]$, and the Riemann components with their sup norms.

Lemma 6.2.1 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^6|\Theta^{(1)}(T,W)|^2\right)^{\frac{1}{2}}\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^{\frac{3}{2}}}$$
$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^6|\Theta^{(2)}(T,W)|^2\right)^{\frac{1}{2}}\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^2}\qquad(6.2.7)$$
$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^6|\Theta^{(3)}(T,W)|^2\right)^{\frac{1}{2}}\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^2}$$

Proof: All the various terms composing the first integral of 6.2.7 are estimated in the same way [12]. We discuss the first term

$$\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^6|{}^{(T)}\underline{\mathbf{m}}|^2|\nabla\!\!\!/\alpha(W)|^2$$

Using Corollary 6.1.1 for ${}^{(T)}\underline{\mathbf{m}}$,

$$\begin{aligned}\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^6|{}^{(T)}\underline{\mathbf{m}}|^2|\nabla\!\!\!/\alpha(W)|^2 &\leq c\frac{1}{u'^4}\left(\sup_{V(u,\underline{u})}|r^{2(T)}\underline{\mathbf{m}}|\right)^2\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|\nabla\!\!\!/\alpha(W)|^2\\ &\leq c\frac{1}{u'^4}\left(\sup_{V(u,\underline{u})}|r^{2(T)}\underline{\mathbf{m}}|\right)^2\mathcal{Q}_{\mathcal{K}}\leq c\frac{1}{u'^4}\epsilon_0^2\mathcal{Q}_{\mathcal{K}}\ .\end{aligned}$$

To control the second integral of 6.2.7, recalling

$$\Theta^{(2)}(T,W)=\mathrm{Qr}\left[{}^{(T)}p_3\,;\,\alpha\right]+\mathrm{Qr}\left[{}^{(T)}p\!\!\!/;\,\beta\right]+\mathrm{Qr}\left[{}^{(T)}p_4\,;\,(\rho,\sigma)\right]\ ,$$

we have to estimate the integrals

$$\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|{}^{(T)}p_3|^2|\alpha(W)|^2$$
$$\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|{}^{(T)}p\!\!\!/|^2|\beta(W)|^2\qquad(6.2.8)$$
$$\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|{}^{(T)}p_4|^2|(\rho,\sigma)(W)|^2$$

[12] in fact only the term depending on ρ produces the factor $u'^{-\frac{3}{2}}$, all the other ones behave better, as $O(u'^{-2})$.

The first one is the more delicate, as ${}^{(T)}p_3$ has the slowest asymptotic decay, see 6.1.47,

$$
\begin{aligned}
&\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|{}^{(T)}p_3|^2|\alpha(W)|^2\right)^{\frac{1}{2}} \leq \frac{1}{u'^2}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}|\tau_-^2\,{}^{(T)}p_3|^2\tau_+^6|\alpha(W)|^2\right)^{\frac{1}{2}}\\
&\leq \frac{1}{u'^2}\left(\sup_{\mathcal{K}}\tau_+^{\frac{7}{2}}|\alpha(W)|\right)\left(\int_{\underline{u}_0}^{\underline{u}}\frac{1}{\tau_+}|\tau_-^2\,{}^{(T)}p_3|^2_{p=2,S}\right)^{\frac{1}{2}}\\
&\leq c\frac{1}{u'^2}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\left(\int_{\underline{u}_0}^{\underline{u}}\frac{1}{\tau_+}|\tau_-^2\,{}^{(T)}p_3|^2_{p=2,S}\right)^{\frac{1}{2}} \leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{u'^2} \qquad (6.2.9)
\end{aligned}
$$

where we have used Proposition 6.1.6 to estimate ${}^{(T)}p_3$ and the estimate for $\alpha(W)$ in 3.7.1, proved in subsection 5.1.6. The estimates of the remaining integrals in 6.2.8 are easier and we do not report them here.

To control the third integral of 6.2.7, recalling, see 6.1.23, that

$$
\begin{aligned}
\Theta^{(3)}(T,W) \;\;=\;\; &\mathrm{Qr}\left[\alpha\,;\,\underline{K}({}^{(T)}q)\right] + \mathrm{Qr}\left[\alpha\,;\,\underline{\Lambda}({}^{(T)}q)\right] + \mathrm{Qr}\left[\beta\,;\,(I,\underline{I})({}^{(T)}q)\right]\\
&+\mathrm{Qr}\left[(\rho,\sigma)\,;\,\Theta({}^{(T)}q)\right]
\end{aligned}
$$

we have to estimate the integrals:

$$
\begin{aligned}
&\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|(\underline{K}({}^{(T)}q),\underline{\Lambda}({}^{(T)}q))|^2|\alpha(W)|^2\\
&\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|(I({}^{(T)}q),\underline{I}({}^{(T)}q))|^2|\beta(W)|^2 \qquad (6.2.10)\\
&\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|\Theta({}^{(T)}q)|^2|(\rho,\sigma)(W)|^2
\end{aligned}
$$

Again the worst asymptotic behaviour is due to ${}^{(T)}p_3$ which is present in the explicit expression of $\underline{\Lambda}({}^{(T)}q)$, see 6.1.28. We write

$$
\underline{\Lambda}({}^{(T)}q) = \underline{\tilde{\Lambda}}({}^{(T)}q) + \frac{2}{3}{}^{(T)}p_3
$$

and observe that, for $p\in[2,\infty]$, $\sup_{\mathcal{K}}|r^{2-\frac{2}{p}}\tau_-\underline{\tilde{\Lambda}}({}^{(T)}q)|_{p,S}\leq c\epsilon_0$. Then, easily,

$$
\begin{aligned}
&\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|(\underline{K}({}^{(T)}q),\underline{\tilde{\Lambda}}({}^{(T)}q))|^2|\alpha(W)|^2\right)^{\frac{1}{2}}\\
&\leq c\frac{1}{u'^2}\left(\sup_{\mathcal{K}}|\tau_+^{\frac{7}{2}}\alpha(W)|\right)\left(\sup_{\mathcal{K}}|r^{2-\frac{2}{p}}\tau_-(K({}^{(T)}q),\underline{\tilde{\Lambda}}({}^{(T)}q))|_{p=2,S}\right)\\
&\leq c\epsilon_0^2\mathcal{Q}_{\mathcal{K}}\frac{1}{u'^2} \qquad (6.2.11)
\end{aligned}
$$

The remaining part $\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|^{(T)}p_3|^2|\alpha(W)|^2$ has already been estimated, see 6.2.9. The estimates of the second and third integrals are the same as the first one and will be omitted.

The estimate of the second integral of Proposition 6.2.1 proceeds in a similar way. We have, for the part associated to J^1,

$$\begin{aligned}
&\left|\int_{V_{(u,\underline{u})}}\tau_+^6\beta(\hat{\mathcal{L}}_T W)\cdot\Xi^{(1)}(T,W)\right|\\
&\leq c\int_{u_0}^{u}du'\frac{1}{|u'|}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4u'^2|\beta(\hat{\mathcal{L}}_T W)|^2\right)^{\frac12}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^8|\Xi^{(1)}(T,W)|^2\right)^{\frac12}\\
&\leq c\mathcal{Q}_{\mathcal{K}}^{\frac12}\int_{u_0}^{u}du'\frac{1}{|u'|}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^8|\Xi^{(1)}(T,W)|^2\right)^{\frac12} \qquad (6.2.12)
\end{aligned}$$

where the first factor in the integrand is bounded by $\mathcal{Q}_{\mathcal{K}}^{\frac12}$, according to 6.0.2 and the definition of $\mathcal{R}$. For the integrals associated to J^2, J^3 we proceed in a different way, using the second coarea formula in 6.2.5.

$$\begin{aligned}
&\left|\sum_{i=2}^{3}\int_{V_{(u,\underline{u})}}\tau_+^6\beta(\hat{\mathcal{L}}_T W)\cdot\Xi^{(i)}(T,W)\right|\\
&\leq c\int_{\underline{u}_0}^{\underline{u}}d\underline{u}'\left(\int_{\underline{C}(\underline{u}';[u_0,u])}\underline{u}'^6|\beta(\hat{\mathcal{L}}_T W)|^2\right)^{\frac12}\left(\sum_{i=2}^{3}\int_{\underline{C}(\underline{u}';[u_0,u])}\underline{u}'^6|\Xi^{(i)}(T,W)|^2\right)^{\frac12}\\
&\leq c\mathcal{Q}_{\mathcal{K}}^{\frac12}\sum_{i=2}^{3}\int_{\underline{u}_0}^{\underline{u}}d\underline{u}'\left(\int_{\underline{C}(\underline{u}';[u_0,u])}\underline{u}'^6|\Xi^{(i)}(T,W)|^2\right)^{\frac12} \qquad (6.2.13)
\end{aligned}$$

where, again, the first factor in the integrand has been bounded by $\mathcal{Q}_{\mathcal{K}}^{\frac12}$. The various terms in the right hand side of 6.2.12 and 6.2.13, associated to the currents J^1, J^2, J^3, are estimated separately. The result is formulated in the next lemma,

Lemma 6.2.2 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\begin{aligned}
\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^8|\Xi^{(1)}(T,W)|^2\right)^{\frac12} &\leq c\epsilon_0\,\mathcal{Q}_{\mathcal{K}}^{\frac12}\frac{1}{|u'|}\\
\left(\int_{\underline{C}(\underline{u}';[u_0,u])}\underline{u}'^6|\Xi^{(2)}(T,W)|^2\right)^{\frac12} &\leq c\epsilon_0\,\mathcal{Q}_{\mathcal{K}}^{\frac12}\frac{1}{|\underline{u}'|^{\frac32}} \qquad (6.2.14)
\end{aligned}$$

$$\left(\int_{\underline{C}(\underline{u}';[u_0,u])}\underline{u}'^6|\Xi^{(3)}(T,W)|^2\right)^{\frac{1}{2}}\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|\underline{u}'|^{\gamma}}$$

with $1<\gamma<\frac{3}{2}$.

Proof: All the various terms appearing in the decomposition of $\Xi^{(1)}$, see eq. 6.1.13, produce the same $|u'|^{-1}$ dependance. Let us consider, among them, the following one

$$\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^8|^{(T)}\mathbf{m}|^2|\beta_3(W)|^2$$

We proceed as in Lemma 6.2.1, using the Corollary 6.1.1 for ${}^{(T)}\mathbf{m}$ and 6.0.2,

$$\begin{aligned}&\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^8|^{(T)}\mathbf{m}|^2|\beta_3(W)|^2\leq c\left(\sup_{\mathcal{K}}|r^{2\,(T)}\mathbf{m}|^2\right)\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\beta_3|^2\\&\leq c\frac{1}{|u'|^2}\left(\sup_{\mathcal{K}}|r^{2\,(T)}\mathbf{m}|^2\right)\mathcal{Q}_1\leq c\epsilon_0^2\mathcal{Q}_{\mathcal{K}}\frac{1}{|u'|^2}\end{aligned}\tag{6.2.15}$$

To estimate the second integral of 6.2.14, we recall, see 6.1.18,

$$\Xi^{(2)}(T,W)=\mathrm{Qr}\left[{}^{(X)}\not{p};\,\alpha\right]+\mathrm{Qr}\left[{}^{(X)}p_4\,;\,\beta\right]\ ,$$

we are led to examine the integrals

$$\begin{aligned}&\int_{\underline{C}(\underline{u}';[u_0,u])}\underline{u}'^6|^{(T)}\not{p}|^2|\alpha(W)|^2\\&\int_{\underline{C}(\underline{u}';[u_0,u])}\underline{u}'^6|^{(T)}p_4|^2|\beta(W)|^2\end{aligned}\tag{6.2.16}$$

The first integral is estimated as follows, with the help of Proposition 6.1.4,

$$\begin{aligned}&\int_{\underline{C}(\underline{u}';[u_0,u])}\underline{u}'^6|^{(T)}\not{p}|^2|\alpha(W)|^2\leq\frac{1}{\underline{u}'^3}\left(\sup_{\mathcal{K}}|\tau_+^{\frac{7}{2}}\alpha(W)|^2\right)\int_{\underline{C}(\underline{u}';[u_0,u])}\frac{1}{u'^2}|r\tau_-{}^{(T)}\not{p}|^2\\&\leq c\mathcal{Q}_{\mathcal{K}}\frac{1}{\underline{u}'^3}\left(\sup_{\mathcal{K}}|r^{2-\frac{2}{p}}\tau_-{}^{(T)}\not{p}|^2_{p=2,S}\right)\int_{u_0}^{u}du'\frac{1}{u'^2}\leq c\epsilon_0^2\mathcal{Q}_{\mathcal{K}}\frac{1}{\underline{u}'^3}\end{aligned}\tag{6.2.17}$$

The second integral in 6.2.16 is estimates exactly as the previous one substituting $|\tau_+^{\frac{7}{2}}\alpha(W)|$ with $|\tau_+^{\frac{7}{2}}\beta(W)|$ and $|r^{2-\frac{2}{p}}\tau_-{}^{(T)}\not{p}|_{p,S}$ with $|r^{2-\frac{2}{p}}\tau_-{}^{(T)}p_4|_{p,S}$.

To estimate the third integral of 6.2.14, as, see 6.1.23,

$$\Xi^{(3)}(T,W) = \mathrm{Qr}\left[\alpha\,;\,(I,\underline{I})({}^{(X)}q)\right] + \mathrm{Qr}\left[\beta\,;\,(K,\Lambda,\Theta)({}^{(X)}q)\right] + \mathrm{Qr}\left[(\rho,\sigma)\,;\,\Xi({}^{(X)}q)\right]\ ,$$

we have to control the following integrals:

$$\begin{aligned}&\int_{\underline{C}(\underline{u}';[u_0,u])}\tau_+^6|\left(I({}^{(T)}q),\underline{I}({}^{(T)}q)\right)|^2|\alpha(W)|^2\\&\int_{\underline{C}(\underline{u}';[u_0,u])}\tau_+^6|\left(K({}^{(T)}q),\Lambda({}^{(T)}q),\Theta({}^{(T)}q)\right)|^2|\beta(W)|^2\\&\int_{\underline{C}(\underline{u}';[u_0,u])}\tau_+^6|\Xi({}^{(T)}q)|^2|(\rho,\sigma)(W)|^2\end{aligned}\tag{6.2.18}$$

For the first integral we observe that the $|\cdot|_{p,S}$ norms with which we bound $I({}^{(T)}q)$ and $\underline{I}({}^{(T)}q)$ are the same as those used to control ${}^{(T)}p_4$ or ${}^{(T)}\not{p}$, see 6.1.51, 6.1.52. Therefore we proceed as in 6.2.17.
Similarily the second integral can be bounded in the same way as the integral $\int_{V_{(u,\underline{u})}}\tau_+^6|{}^{(T)}p_4|^2|\beta(W)|^2$, see 6.2.16.

The third integral is controlled using the estimate of $\Xi({}^{(T)}q)$ in Proposition 6.1.8. We obtain, for any $\gamma<\frac{3}{2}$,

$$\begin{aligned}&\int_{\underline{C}(\underline{u}';[u_0,u])}\tau_+^6|\Xi({}^{(T)}q)|^2|(\rho,\sigma)(W)|^2\\&\leq\left(\sup_{\mathcal{K}}|r^3(\rho,\sigma)(W)|^2\right)\int_{\underline{C}(\underline{u}';[u_0,u])}\tau_+^6\frac{1}{r^{10}}|r^2\Xi({}^{(T)}q)|^2\\&\leq c\frac{1}{\underline{u}'^{2\gamma}}\left(\sup_{\mathcal{K}}|r^{3-\frac{2}{p}}\Xi({}^{(T)}q)|^2_{p=2,S}\right)\mathcal{Q}_{\mathcal{K}}\int_{u_0}^{u}du'\frac{1}{u'^{4-2\gamma}}\leq c\epsilon_0^2\mathcal{Q}_{\mathcal{K}}\frac{1}{\underline{u}'^{2\gamma}}\end{aligned}$$

Estimate of the B_2 integrals

The integrals in the B_2 group, see eq. 6.2.2, while similar to those we have analysed in the previous subsections, have a lower τ_+ weight and, therefore, are a little simpler. We shortly indicate how they are estimated. However we shall only analyze the integrals related to J^2, J^3, which we have shown in the previous discussion to be more delicate. Their estimates are collected in the following propositions,

Proposition 6.2.2 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\begin{aligned}
&\int_{V_{(u,\underline{u})}} \tau_+^4\tau_-^2\rho(\hat{\mathcal{L}}_T W)\Lambda^{(2)}(T,W) \leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}\\
&\int_{V_{(u,\underline{u})}} \tau_+^4\tau_-^2\sigma(\hat{\mathcal{L}}_T W)K^{(2)}(T,W) \leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}\\
&\int_{V_{(u,\underline{u})}} \tau_+^4\tau_-^2\beta(\hat{\mathcal{L}}_T W)\cdot I^{(2)}(T,W) \leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}
\end{aligned}\tag{6.2.19}$$

Proposition 6.2.3 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\begin{aligned}
&\int_{V_{(u,\underline{u})}} \tau_+^4\tau_-^2\rho(\hat{\mathcal{L}}_T W)\Lambda^{(3)}(T,W) \leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}\\
&\int_{V_{(u,\underline{u})}} \tau_+^4\tau_-^2\sigma(\hat{\mathcal{L}}_T W)K^{(3)}(T,W) \leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}\\
&\int_{V_{(u,\underline{u})}} \tau_+^4\tau_-^2\beta(\hat{\mathcal{L}}_T W)\cdot I^{(3)}(T,W) \leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}
\end{aligned}\tag{6.2.20}$$

Proof of Proposition 6.2.2 The first and the second integrals of 6.2.19 have the same structure, see 6.1.18, therefore we estimate only the first one. This decomposes in, see 6.1.18,

$$\begin{aligned}
&\int_{V_{(u,\underline{u})}} \tau_+^4\tau_-^2\rho(\hat{\mathcal{L}}_T W)^{(T)}\not{p}\beta(W)\\
&\int_{V_{(u,\underline{u})}} \tau_+^4\tau_-^2\rho(\hat{\mathcal{L}}_T W)^{(T)}p_4(\rho,\sigma)(W)
\end{aligned}\tag{6.2.21}$$

We estimate the first one as follows

$$\begin{aligned}
&\left|\int_{V_{(u,\underline{u})}} \tau_+^4\tau_-^2\rho(\hat{\mathcal{L}}_T W)({}^{(T)}\not{p})\beta(W)\right| \leq c\int_{V_{(u,\underline{u})}} \frac{1}{\underline{u}'^{\frac{5}{2}}}|\tau_+^2\tau_-\rho(\hat{\mathcal{L}}_T W)||r\tau_-{}^{(T)}\not{p}||\tau_+^{\frac{7}{2}}\beta(W)|\\
&\leq\ c\left(\sup_{\mathcal{K}}|\tau_+^{\frac{7}{2}}\beta|\right)\left(\sup_{\mathcal{K}}\int_{\underline{C}(\underline{u}';[u_0,u])}\tau_+^4\tau_-^2|\rho(\hat{\mathcal{L}}_T W)|^2\right)^{\frac{1}{2}}\int_{\underline{u}_0}^{\underline{u}}\frac{d\underline{u}'}{\underline{u}'^{\frac{3}{2}}}\left(\int_{u_0}^{u}\frac{1}{u'^2}|r^{2-\frac{2}{p}}\tau_-{}^{(T)}\not{p}|^2_{p=2,S}\right)^{\frac{1}{2}}\\
&\leq\ c\left(\sup_{\mathcal{K}}|r^{2-\frac{2}{p}}\tau_-{}^{(T)}\not{p}|^2_{p=2,S}\right)^{\frac{1}{2}}\left(\sup_{\mathcal{K}}|\tau_+^{\frac{7}{2}}\beta|\right)\underline{\mathcal{Q}}_1^{\frac{1}{2}}\int_{\underline{u}_0}^{\underline{u}}\frac{d\underline{u}'}{\underline{u}'^{\frac{3}{2}}}\left(\int_{u_0}^{u}\frac{du'}{u'^2}\right)^{\frac{1}{2}} \leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}
\end{aligned}\tag{6.2.22}$$

The estimate of the second integral in 6.2.21 is done exactly in the same way and we do not report here.

The most sensitive term in the third integral of 6.2.19 is [13] of $\int_{V_{(u,\underline{u})}}\tau_+^4\tau_-^2\beta(\hat{\mathcal{L}}_TW)({}^{(T)}p_3)\beta(W)$. We obtain

$$
\begin{aligned}
&\left|\int_{V_{(u,\underline{u})}}\tau_+^4\tau_-^2\beta(\hat{\mathcal{L}}_TW)^{(T)}p_3\beta(W)\right|\\
&\leq c\int d\underline{u}'\int_{\underline{C}(\underline{u}';[u_0,u])}|\tau_+^3\beta(\hat{\mathcal{L}}_TW)|\frac{1}{\tau_+^{\frac{5}{2}}}|\tau_-^{2\,(T)}p_3||\tau_+^{\frac{7}{2}}\beta(W)|\\
&\leq c\left(\sup_{\mathcal{K}}|\tau_+^{\frac{7}{2}}\beta(W)|\right)\left(\sup_{\mathcal{K}}\int_{\underline{C}(\underline{u}';[u_0,u])}\tau_+^6|\beta(\hat{\mathcal{L}}_TW)|^2\right)^{\frac{1}{2}}\int\frac{d\underline{u}'}{\underline{u}'^{\frac{5}{2}}}\left(\int_{u_0}^{u}du'|r^{1-\frac{2}{p}}\tau_-^{2\,(T)}p_3|^2_{p=2,S}\right)^{\frac{1}{2}}\\
&\leq c\left(\sup_{\mathcal{K}}|r^{1-\frac{2}{p}}\tau_-^{2\,(T)}p_3|^2_{p=2,S}\right)^{\frac{1}{2}}\left(\sup_{\mathcal{K}}|\tau_+^{\frac{7}{2}}\beta(W)|\right)\underline{\mathcal{Q}}_1^{\frac{1}{2}}\int\frac{d\underline{u}'}{\underline{u}'^{\frac{5}{2}}}\left(\int_{u_0}^{u}du'\right)^{\frac{1}{2}}\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}} \qquad (6.2.23)
\end{aligned}
$$

Proof of Proposition 6.2.3: The first and the second integral of 6.2.20 are similar as ρ and σ behave in the same way and $\Lambda^{(3)}(T,W)$ and $K^{(3)}(T,W)$ have the same structure, see 6.1.22. Therefore it is enough to estimate the first one. This amounts to control the three integrals

$$
\begin{aligned}
&\int_{V_{(u,\underline{u})}}\tau_+^4\tau_-^2\rho(\hat{\mathcal{L}}_TW)\underline{\Theta}({}^{(T)}q)\alpha(W)\\
&\int_{V_{(u,\underline{u})}}\tau_+^4\tau_-^2\rho(\hat{\mathcal{L}}_TW)\left(K({}^{(T)}q),\Lambda({}^{(T)}q)\right)(\rho,\sigma)(W) \qquad (6.2.24)\\
&\int_{V_{(u,\underline{u})}}\tau_+^4\tau_-^2\rho(\hat{\mathcal{L}}_TW)\Xi({}^{(T)}q)\underline{\beta}(W)
\end{aligned}
$$

The first integral is estimated in the following way

$$
\begin{aligned}
&\left|\int_{V_{(u,\underline{u})}}\tau_+^4\tau_-^2\rho(\hat{\mathcal{L}}_TW)(\underline{\Theta}({}^{(T)}q))\alpha(W)\right|\\
&\leq c\int_{\underline{u}_0}^{\underline{u}}d\underline{u}'\int_{\underline{C}(\underline{u}';[u_0,u])}|\tau_+^2\tau_-\rho(\hat{\mathcal{L}}_TW)|\frac{1}{\tau_-\tau_+^{\frac{3}{2}}}|\tau_-^2\underline{\Theta}({}^{(T)}q)||\tau_+^{\frac{7}{2}}\alpha(W)|\\
&\leq c\left(\sup_{\mathcal{K}}|r^{1-\frac{2}{p}}\tau_-^2\underline{\Theta}({}^{(T)}q)|^2_{p=2,S}\right)^{\frac{1}{2}}\left(\sup_{\mathcal{K}}|\tau_+^{\frac{7}{2}}\alpha|\right)\underline{\mathcal{Q}}_1^{\frac{1}{2}}\int_{\underline{u}_0}^{\underline{u}}\frac{d\underline{u}'}{\underline{u}'^{\frac{3}{2}}}\left(\int_{u_0}^{u}\frac{du'}{u'^2}\right)^{\frac{1}{2}}\\
&\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}} \qquad (6.2.25)
\end{aligned}
$$

The norm estimates of $K({}^{(T)}q),\Lambda({}^{(T)}q)$ are at least as good as that of ${}^{(T)}p_4$, see 6.1.51. Therefore the second integral in 6.2.24 is estimated as the second

[13] The other integral appearing in its decomposition, see 6.1.18, is easier to treat.

integral of 6.2.21, in the previous proposition. The estimate of the third integral proceeds as the previous one.
The estimate of the third integral of Proposition 6.2.3 does not differ from the previous ones and we do not report it here.

Estimate of the B_3, B_4 integrals

We recall the expressions of B_3 integrals:

$$\int_{V_{(u,\underline{u})}} \tau_+^2 \tau_-^4 \rho(\hat{\mathcal{L}}_T W)\underline{\Lambda}(T,W)$$
$$\int_{V_{(u,\underline{u})}} \tau_+^2 \tau_-^4 \sigma(\hat{\mathcal{L}}_T W)\underline{K}(T,W)$$
$$\int_{V_{(u,\underline{u})}} \tau_+^2 \tau_-^4 \underline{\beta}(\hat{\mathcal{L}}_T W)\cdot\underline{I}(T,W)$$

and those of the B_4 ones:

$$\int_{V_{(u,\underline{u})}} \tau_-^6 \underline{\alpha}(\hat{\mathcal{L}}_T W)\cdot\underline{\Theta}(T,W)$$
$$\int_{V_{(u,\underline{u})}} \tau_-^6 \underline{\beta}(\hat{\mathcal{L}}_T W)\cdot\underline{\Xi}(T,W)$$

The estimates of the various terms in which these integrals decompose are similar but easier than those for the integrals of groups B_2 and B_1, respectively. They are obtained with the obvious substitutions of the underlined quantities with the non underlined ones and viceversa.

The greater simplicity is due to the fact that, now, τ_-, although smaller than τ_+, plays an analogous role on the $\underline{C}$ null hypersurfaces. Moreover the factor $\mathbf{D}_3 \log\Omega$ with the slowest decay is now substituted by the better behaving factor $\mathbf{D}_4 \log\Omega$. Therefore we just collect the final results in the next proposition.

Proposition 6.2.4 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\int_{V_{(u,\underline{u})}} \tau_+^2 \tau_-^4 \rho(\hat{\mathcal{L}}_T W)\underline{\Lambda}(T,W) \leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$
$$\int_{V_{(u,\underline{u})}} \tau_+^2 \tau_-^4 \sigma(\hat{\mathcal{L}}_T W)\underline{K}(T,W) \leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$
$$\int_{V_{(u,\underline{u})}} \tau_+^2 \tau_-^4 \underline{\beta}(\hat{\mathcal{L}}_T W)\cdot\underline{I}(T,W) \leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$
$$\int_{V_{(u,\underline{u})}} \tau_-^6 \underline{\alpha}(\hat{\mathcal{L}}_T W)\cdot\underline{\Theta}(T,W) \leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$

$$\int_{V_{(u,\underline{u})}} \tau_-^6 \underline{\beta}(\hat{\mathcal{L}}_T W) \cdot \underline{\Xi}(T,W) \leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$

6.2.2 Estimate of $\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta)|$.

Proposition 6.2.5 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta)| \leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}} \tag{6.2.26}$$

Proof We write the explicit expression of the integrand

$$\begin{aligned}
&{}^{(\bar{K})}\pi^{\alpha\beta} Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}\bar{K}^\gamma \bar{K}^\delta = \\
&{}^{(\bar{K})}\pi^{\alpha\beta} \left\{ Q(\hat{\mathcal{L}}_T W)_{\alpha\beta 44}\tau_+^4 + 2Q(\hat{\mathcal{L}}_T W)_{\alpha\beta 43}\tau_+^2\tau_-^2 + Q(\hat{\mathcal{L}}_T W)_{\alpha\beta 33}\tau_-^4 \right\}
\end{aligned}$$

where, see 3.4.9,

$$\begin{aligned}
&{}^{(\bar{K})}\pi^{\alpha\beta} Q(\hat{\mathcal{L}}_T W)_{\alpha\beta 44}\tau_+^4 = \\
&\frac{1}{16}\tau_+^4 \Big\{ 2|\alpha(\hat{\mathcal{L}}_T W)|^{2\,(\bar{K})}\underline{\mathbf{n}} + 4(|\rho(\hat{\mathcal{L}}_T W)|^2 + |\sigma(\hat{\mathcal{L}}_T W)|^2)^{(\bar{K})}\mathbf{n} + |\beta(\hat{\mathcal{L}}_T W)|^{2\,(\bar{K})}\mathbf{j} \\
&- 8\alpha(\hat{\mathcal{L}}_T W)\cdot\beta(\hat{\mathcal{L}}_T W)\cdot{}^{(\bar{K})}\underline{\mathbf{m}} - 8\rho(\hat{\mathcal{L}}_T W)\beta(\hat{\mathcal{L}}_T W)\cdot{}^{(\bar{K})}\mathbf{m} \\
&+ 8\sigma(\hat{\mathcal{L}}_T W)^*\beta(\hat{\mathcal{L}}_T W)\cdot{}^{(\bar{K})}\mathbf{m} + 8(|\beta(\hat{\mathcal{L}}_T W)|^2)tr^{(\bar{K})}\mathbf{i} \\
&+ 8\rho(\hat{\mathcal{L}}_T W)\alpha(\hat{\mathcal{L}}_T W)\cdot{}^{(\bar{K})}\mathbf{i} - 8\sigma(\hat{\mathcal{L}}_T W)^*\alpha(\hat{\mathcal{L}}_T W)\cdot{}^{(\bar{K})}\mathbf{i} \Big\}
\end{aligned} \tag{6.2.27}$$

$$\begin{aligned}
&{}^{(\bar{K})}\pi^{\alpha\beta} Q(\hat{\mathcal{L}}_T W)_{\alpha\beta 43}\tau_+^2\tau_-^2 = \\
&\frac{1}{16}\tau_+^2\tau_-^2 \Big\{ 4|\beta(\hat{\mathcal{L}}_T W)|^{2\,(\bar{K})}\underline{\mathbf{n}} + 4|\underline{\beta}(\hat{\mathcal{L}}_T W)|^{2\,(\bar{K})}\mathbf{n} + 4(|\rho(\hat{\mathcal{L}}_T W)|^2 + |\sigma(\hat{\mathcal{L}}_T W)|^2)^{(\bar{K})}\mathbf{j} \\
&- 4\rho(\hat{\mathcal{L}}_T W)\beta(\hat{\mathcal{L}}_T W)\cdot{}^{(\bar{K})}\underline{\mathbf{m}} + 4\sigma(\hat{\mathcal{L}}_T W)^*\beta(\hat{\mathcal{L}}_T W)\cdot{}^{(\bar{K})}\underline{\mathbf{m}} \\
&+ 4\rho(\hat{\mathcal{L}}_T W)\underline{\beta}(\hat{\mathcal{L}}_T W)\cdot{}^{(\bar{K})}\mathbf{m} + 4\sigma(\hat{\mathcal{L}}_T W)^*\underline{\beta}(\hat{\mathcal{L}}_T W)\cdot{}^{(\bar{K})}\mathbf{m} \\
&+ 2(|\rho(\hat{\mathcal{L}}_T W)|^2 + \sigma(\hat{\mathcal{L}}_T W)|^2)tr^{(\bar{K})}\mathbf{i} - 2(\beta(\hat{\mathcal{L}}_T W)\widehat{\otimes}\underline{\beta}(\hat{\mathcal{L}}_T W))\cdot{}^{(\bar{K})}\mathbf{i} \Big\}
\end{aligned} \tag{6.2.28}$$

$$\begin{aligned}
&{}^{(\bar{K})}\pi^{\alpha\beta} Q(\hat{\mathcal{L}}_T W)_{\alpha\beta 33}\tau_-^4 = \\
&\frac{1}{16}\tau_-^4 \Big\{ 4(\rho(\hat{\mathcal{L}}_T W)|^2 + \sigma(\hat{\mathcal{L}}_T W)|^2)^{(\bar{K})}\underline{\mathbf{n}} + 2|\underline{\alpha}(\hat{\mathcal{L}}_T W)|^{2\,(\bar{K})}\mathbf{n} + 4|\underline{\beta}(\hat{\mathcal{L}}_T W)|^{2\,(\bar{K})}\mathbf{j} \\
&+ 8(\underline{\alpha}(\hat{\mathcal{L}}_T W)\cdot\underline{\beta}(\hat{\mathcal{L}}_T W))\cdot{}^{(\bar{K})}\mathbf{m} - 8\rho(\hat{\mathcal{L}}_T W)\underline{\beta}(\hat{\mathcal{L}}_T W)\cdot{}^{(\bar{K})}\underline{\mathbf{m}}
\end{aligned}$$

$$
\begin{aligned}
&- 8\sigma(\hat{\mathcal{L}}_T W)^* \underline{\beta}(\hat{\mathcal{L}}_T W) \cdot {}^{(\bar{K})}\underline{\mathbf{m}} + 8(|\underline{\beta}(\hat{\mathcal{L}}_T W)|^2) tr^{(\bar{K})}\mathbf{i} \\
&+ 8\rho(\hat{\mathcal{L}}_T W)\underline{\alpha}(\hat{\mathcal{L}}_T W) \cdot {}^{(\bar{K})}\mathbf{i} + 8\sigma(\hat{\mathcal{L}}_T W)^* \underline{\alpha}(\hat{\mathcal{L}}_T W) \cdot {}^{(\bar{K})}\mathbf{i} \Big\}
\end{aligned} \tag{6.2.29}
$$

All factors have the same structure. They are cubic terms, quadratic in the null components of $\hat{\mathcal{L}}_T W$ and linear in the deformation tensor of $\bar{K}$. Therefore they are all estimated in the same way. Let us discuss explicitely the integral relative to the term $\tau_+^4 {}^{(\bar{K})}\underline{\mathbf{n}}|\alpha(\hat{\mathcal{L}}_T W)|^2$ and the one relative to the term $\tau_+^4 {}^{(\bar{K})}\mathbf{n}|\rho(\hat{\mathcal{L}}_T W)|^2$, see 6.2.27. For the first integral we obtain, using Corollary 6.1.3,

$$
\begin{aligned}
&\int_{V_{(u,\underline{u})}} \tau_+^4 |\alpha(\hat{\mathcal{L}}_T W)|^2 |{}^{(\bar{K})}\underline{\mathbf{n}}| \leq c \int_{u_0}^{u} du' \int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^6 |\alpha(\hat{\mathcal{L}}_T W)|^2 \frac{1}{r^2} |{}^{(\bar{K})}\underline{\mathbf{n}}| \\
&\leq c \left(\sup_{\mathcal{K}} |\frac{\tau_-}{r} {}^{(\bar{K})}\underline{\mathbf{n}}| \right) \left(\sup_{\mathcal{K}} \int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^6 |\alpha(\hat{\mathcal{L}}_T W)|^2 \right) \int_{u_0}^{u} du' \frac{1}{ru'} \\
&\leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}}
\end{aligned}
$$

The estimate of the second integral proceeds exactly as before:

$$
\begin{aligned}
&\int_{V_{(u,\underline{u})}} \tau_+^4 |{}^{(\bar{K})}\mathbf{n}||\rho(\hat{\mathcal{L}}_T W)|^2 \leq c \int_{u_0}^{u} du' \frac{1}{u'^2} \int_{C(u';[\underline{u}_0,\underline{u}])} |\frac{r^2}{\tau_-^2} {}^{(\bar{K})}\mathbf{n}| \tau_+^2 \tau_-^4 |\rho(\hat{\mathcal{L}}_T W)|^2 \\
&\leq c \left(\sup_{\mathcal{K}} |\frac{r^2}{\tau_-^2} {}^{(\bar{K})}\mathbf{n}| \right) \left(\sup_{\mathcal{K}} \int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^2 \tau_-^4 |\rho(\hat{\mathcal{L}}_T W)|^2 \right) \int_{u_0}^{u} du' \frac{1}{u'^2} \\
&\leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}}
\end{aligned}
$$

6.2.3 Estimate of $\int_{V_{(u,\underline{u})}} |DivQ(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma T^\delta)|$

We have to control the following integrals, see 6.1.1, 6.1.2, 6.1.3,

$$
\begin{aligned}
&\int_{V_{(u,\underline{u})}} \tau_+^4 D(O,W)_{444} \ \ , \ \ \int_{V_{(u,\underline{u})}} \tau_+^4 D(O,W)_{344} \\
&\int_{V_{(u,\underline{u})}} \tau_+^2 \tau_-^2 D(O,W)_{344} \ , \ \int_{V_{(u,\underline{u})}} \tau_+^2 \tau_-^2 D(O,W)_{334} \\
&\int_{V_{(u,\underline{u})}} \tau_-^4 D(O,W)_{334} \ \ , \ \ \int_{V_{(u,\underline{u})}} \tau_-^4 D(O,W)_{333}
\end{aligned} \tag{6.2.30}
$$

The most sensitive terms are clearly the first two integrals containing the weight τ_+^4. We estimate explicitely the first integral in the first line of 6.2.30, whose explicit expression is

$$\int_{V_{(u,\underline{u})}} \tau_+^4 D(O,W)_{444} = \frac{1}{2}\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O W)\cdot\Theta(O,W) - \int_{V_{(u,\underline{u})}} \tau_+^4 \beta(\hat{\mathcal{L}}_O W)\cdot\Xi(O,W) \qquad (6.2.31)$$

Proposition 6.2.6 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\left|\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O W)\cdot\Theta(O,W)\right| \le c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$
$$\left|\int_{V_{(u,\underline{u})}} \tau_+^4 \beta(\hat{\mathcal{L}}_O W)\cdot\Xi(O,W)\right| \le c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$

where $\Theta(O,W) \equiv \sum_{i=1}^3 \Theta(J^i(O,W))$, $\Xi(O,W) \equiv \sum_{i=1}^3 \Xi(J^i(O,W))$.

Proof: Proceeding as in Subsection 6.2.1 we write

$$\begin{aligned}
&\left|\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O W)\cdot\Theta(O,W)\right| \\
&\le \left(\sup_{\mathcal{K}} \int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4 |\alpha(\hat{\mathcal{L}}_O W)|^2\right)^{\frac{1}{2}} \int_{u_0}^{u} du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4 |\Theta(O,W)|^2\right)^{\frac{1}{2}} \\
&\le c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}} \int_{u_0}^{u} du' \sum_{i=1}^{3} \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4 |\Theta^{(i)}(O,W)|^2\right)^{\frac{1}{2}} \qquad (6.2.32)
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{V_{(u,\underline{u})}} \underline{u}'^4 \beta(\hat{\mathcal{L}}_O W)\cdot\Xi(O,W)\right| \\
&\le \left(\sup_{\mathcal{K}} \int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4 |\beta(\hat{\mathcal{L}}_O W)|^2\right)^{\frac{1}{2}} \int_{u_0}^{u} du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4 |\Xi(O,W)|^2\right)^{\frac{1}{2}} \\
&\le c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}} \int_{u_0}^{u} du' \sum_{i=1}^{3} \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4 |\Xi^{i}(O,W)|^2\right)^{\frac{1}{2}} \qquad (6.2.33)
\end{aligned}$$

The result is obtained proving the next lemma.

Lemma 6.2.3 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\begin{aligned}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\Theta^{(1)}(O,W)|^2\right)^{\frac{1}{2}}&\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^2}\\ \left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\Theta^{(2)}(O,W)|^2\right)^{\frac{1}{2}}&\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^{\frac{3}{2}}}\\ \left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\Theta^{(3)}(O,W)|^2\right)^{\frac{1}{2}}&\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^{\frac{3}{2}}}\end{aligned}\tag{6.2.34}$$

$$\begin{aligned}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\Xi^{(1)}(O,W)|^2\right)^{\frac{1}{2}}&\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^2}\\ \left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\Xi^{(2)}(O,W)|^2\right)^{\frac{1}{2}}&\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^{\frac{3}{2}}}\\ \left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\Xi^{(3)}(O,W)|^2\right)^{\frac{1}{2}}&\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^{\frac{3}{2}}}\end{aligned}\tag{6.2.35}$$

Proof: We start estimating the integral in the first line, connected to the J^1 of the current. From Corollary 4.7.1 and 6.1.14, we have

$$\begin{aligned}|\Theta^{(1)}(O,W)|^2\quad\leq\quad& c\left(\sup_{\mathcal{K}}|r({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|\right)^2\cdot\frac{1}{r^2}\left[\left(|\nabla\!\!\!/\alpha|^2+|\alpha_3|^2+|\nabla\!\!\!/\beta|^2\right.\right.\\&\left.\left.+|\beta_4|^2+|(\rho_4,\sigma_4)|^2\right)+\frac{1}{r^2}\left(|\beta|^2+|(\rho,\sigma)|^2+|\alpha|^2\right)\right]+(\text{l.o.t})\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\Theta^{(1)}(O,W)|^2\right)^{\frac{1}{2}}\leq c\left(\sup_{\mathcal{K}}\left|r({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})\right|\right)\cdot\\&\frac{1}{|u'|}\left[\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^2u'^2\left(|\nabla\!\!\!/\alpha|^2+|\alpha_3|^2+|\nabla\!\!\!/\beta|^2+|\beta_4|^2+|(\rho_4,\sigma_4)|^2\right)\right.\\&\left.+u'^2\left(|\beta|^2+|(\rho,\sigma)|^2+|\alpha|^2\right)\right]^{\frac{1}{2}}\leq c\left(\sup_{\mathcal{K}}\left|r({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})\right|\right)\cdot\\&\frac{1}{|u'|^2}\left[\left(\sup_{\mathcal{K}}\int_{C(u';[\underline{u}_0,\underline{u}])}Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_4)\right)^{\frac{1}{2}}+\right.\end{aligned}$$

$$\left(\sup_{\mathcal{K}}\int_{\underline{C}(\underline{u}';[u_0,u])}Q(\hat{\mathcal{L}}_OW)(\bar{K},\bar{K},T,e_3)\right)^{\frac{1}{2}}+\sup_{\mathcal{K}\cap\Sigma_0}|r^3(\overline{\rho},\overline{\sigma})|\Bigg]$$
$$\leq\ c\frac{1}{u'^2}\left(\sup_{\mathcal{K}}|r({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|\right)\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\leq c\epsilon_0\,\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{u'^2} \tag{6.2.36}$$

where we used the results of Proposition 5.1.3. The fourth integral of Lemma 6.2.3 is estimated recalling that [14], see 6.1.13,

$$|\Xi^1(O,W)|^2\ \leq\ c\frac{1}{r^2}\left(\sup_{\mathcal{K}}|r({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|\right)^2\cdot$$
$$\left[|\nabla\!\!\!/\alpha|^2+|\alpha_4|^2+|\beta_4|^2+\frac{1}{r^2}(|\alpha|^2+|\beta|^2)\right]+(\text{l.o.t})$$

We obtain

$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\Xi^1(O,W)|^2\right)^{\frac{1}{2}}\leq c\left(\sup_{\mathcal{K}}|r({}^{(O)}\mathbf{i},{}^{(O)}\mathbf{j},{}^{(O)}\underline{\mathbf{m}})|\right)\cdot$$
$$\frac{1}{|u'|}\left[\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4\left(|\nabla\!\!\!/\alpha|^2+|\alpha_4|^2+|\beta_4|^2+\frac{1}{r^2}(|\alpha|^2+|\beta|^2)\right)\right]^{\frac{1}{2}}$$

The estimates of these terms proceed as in the previous case, using the results of Proposition 5.1.3.
We estimate the J^2 part only for the Θ integral in the second line of 6.2.34, as the estimate for the corresponding Ξ integral in the second line of 6.2.35, is done in the same way. Recalling the decomposition, see 6.1.18,

$$\Theta^{(2)}(O,W)=\text{Qr}\left[{}^{(O)}p_3\,;\,\alpha\right]+\text{Qr}\left[{}^{(O)}p\!\!\!/\,;\,\beta\right]+\text{Qr}\left[{}^{(O)}p_4\,;\,(\rho,\sigma)\right]$$

we write

$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\Theta^{(2)}(O,W)|^2\right)^{\frac{1}{2}}\leq \tag{6.2.37}$$
$$c\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4(|{}^{(O)}p_3|^2|\alpha(W)|^2+|{}^{(O)}p\!\!\!/|^2|\beta(W)|^2+|{}^{(O)}p_4|^2|(\rho,\sigma)(W)|^2)\right)^{\frac{1}{2}}$$

To estimate these integrals we have to control the $|\cdot|_{p,S}$ norms of ${}^{(O)}p_3$, ${}^{(O)}p\!\!\!/$ and ${}^{(O)}p_4$. Using the estimates of Proposition 6.1.10, for any $p\in[2,4]$, we

[14] In the following estimates we sistematically neglect the (l.o.t) terms.

obtain

$$\begin{aligned}&\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|^{(O)}p_3|^2|\alpha(W)|^2\right)^{\frac{1}{2}}\\&\leq c\left(\sup_{\mathcal{K}}|\tau_+^{\frac{7}{2}}\alpha(W)|\right)\left(\int_{\underline{u}_0}^{\underline{u}}d\underline{u}'\frac{1}{\underline{u}'^3u'^2}|r^{1-\frac{2}{p}}u'^{(O)}p_3|^2_{p=2,S}\right)^{\frac{1}{2}}\\&\leq c\left(\sup_{\mathcal{K}}|r^{1-\frac{2}{p}}u'^{(O)}p_3|_{p=2,S}\right)\left(\sup_{\mathcal{K}}|\tau_+^{\frac{7}{2}}\alpha(W)|\right)\frac{1}{|u'|}\left(\int_{\underline{u}_0}^{\underline{u}}d\underline{u}'\frac{1}{\underline{u}'^3}\right)^{\frac{1}{2}}\\&\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^2}\end{aligned}\tag{6.2.38}$$

The other integrals in 6.2.37 are estimated in the same way and we do not discuss them here
The part associated to the J^3 current in the third line of 6.2.34 is estimated starting from the decomposition, see 6.1.23,

$$\begin{aligned}\Theta^{(3)}(O,W)\;&=\;\mathrm{Qr}\left[\alpha\,;\,\underline{K}(^{(O)}q)\right]+\mathrm{Qr}\left[\alpha\,;\,\underline{\Lambda}(^{(O)}q)\right]+\mathrm{Qr}\left[\beta\,;\,(I,\underline{I})(^{(O)}q)\right]\\&+\mathrm{Qr}\left[(\rho,\sigma)\,;\,\Theta(^{(O)}q)\right]\;,\end{aligned}$$

obtaining the inequality

$$\begin{aligned}&\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\Theta^{(3)}(O,W)|^2\right)^{\frac{1}{2}}\leq c\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4(|\underline{K}(^{(O)}q)|^2|\alpha(W)|^2\right.\\&\left.+|\underline{\Lambda}(^{(O)}q)|^2|\alpha(W)|^2|+(I,\underline{I})(^{(O)}q)|^2|\beta(W)|^2+|\Theta(^{(O)}q)|^2|(\rho,\sigma)(W)|^2)\right)^{\frac{1}{2}}\end{aligned}$$

To estimate these integrals we use the norm estimates of Proposition 6.1.12 for $\underline{K}(^{(O)}q),\ \underline{\Lambda}(^{(O)}q),\ (I,\underline{I})(^{(O)}q),\ \Theta(^{(O)}q)$, for $p\in[2,4]$, we recall here

$$\begin{aligned}&\sup_{\mathcal{K}}|r^{2-\frac{2}{p}}(I(^{(O)}q),\underline{I}(^{(O)}q))|_{p,S}\leq c\epsilon_0\quad,\quad\sup_{\mathcal{K}}|r^{2-\frac{2}{p}}\Theta(^{(O)}q)|_{p,S}\leq c\epsilon_0\\&\sup_{\mathcal{K}}|r^{1-\frac{2}{p}}\tau_-\underline{\Lambda}(^{(O)}q)|_{p,S}\leq c\epsilon_0\quad,\quad\sup_{\mathcal{K}}|r^{2-\frac{2}{p}}\underline{K}(^{(O)}q)|_{p,S}\leq c\epsilon_0\end{aligned}$$

The estimates of all the integrals can be done in the same way, but the factor $|u'|^{-\frac{3}{2}}$ is due to the part depending on ρ, while the other ones produce the better factor $|u'|^{-2}$. We just report this term,

$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|\Theta(^{(O)}q)|^2|\rho(W)|^2\right)^{\frac{1}{2}}$$

$$\leq c\left(\sup_{\mathcal{K}}|r^3\rho(W)|\right)\left(\int_{\underline{u}_0}^{\underline{u}}d\underline{u}'\frac{1}{\underline{u}'^4}|r^{2-\frac{2}{p}}\Theta({}^{(O)}q)|^2_{p=2,S}\right)^{\frac{1}{2}}$$
$$\leq c\left(\sup_{\mathcal{K}}|r^{2-\frac{2}{p}}\Theta({}^{(O)}q)|_{p=2,S}\right)\left(\sup_{\mathcal{K}}|r^3\rho(W)|\right)\left(\int_{\underline{u}_0}^{\underline{u}}d\underline{u}'\frac{1}{\underline{u}'^4}\right)^{\frac{1}{2}}\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\frac{1}{|u'|^{\frac{3}{2}}}$$

6.2.4 Estimate of $\int_{V_{(u,\underline{u})}}|Q(\hat{\mathcal{L}}_OW)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)|$

Proposition 6.2.7 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\int_{V_{(u,\underline{u})}}|Q(\hat{\mathcal{L}}_OW)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)|\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}} \tag{6.2.39}$$

Proof: We write explicitely the various terms of the integrand

$$\begin{aligned}{}^{(\bar{K})}\pi^{\alpha\beta}Q(\hat{\mathcal{L}}_OW)_{\alpha\beta\gamma\delta}\bar{K}^\gamma T^\delta &= {}^{(\bar{K})}\pi^{\alpha\beta}\Big\{Q(\hat{\mathcal{L}}_OW)_{\alpha\beta44}\tau_+^2+Q(\hat{\mathcal{L}}_OW)_{\alpha\beta43}\tau_+^2\\ &+Q(\hat{\mathcal{L}}_OW)_{\alpha\beta34}\tau_-^2+Q(\hat{\mathcal{L}}_OW)_{\alpha\beta33}\tau_-^2\Big\}\end{aligned}$$

where

$$\begin{aligned}&{}^{(\bar{K})}\pi^{\alpha\beta}Q(\hat{\mathcal{L}}_OW)_{\alpha\beta44}\tau_+^2=\\ &\frac{1}{16}\tau_+^2\Big\{2|\alpha(\hat{\mathcal{L}}_OW)|^{2(\bar{K})}\underline{\mathbf{n}}+4(|\rho(\hat{\mathcal{L}}_OW)|^2+|\sigma(\hat{\mathcal{L}}_OW)|^2)^{(\bar{K})}\mathbf{n}+|\beta(\hat{\mathcal{L}}_OW)|^{2(\bar{K})}\mathbf{j}\\ &-8(\alpha(\hat{\mathcal{L}}_OW)\cdot\beta(\hat{\mathcal{L}}_OW))\cdot{}^{(\bar{K})}\underline{\mathbf{m}}-8\rho(\hat{\mathcal{L}}_OW)\beta(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar{K})}\mathbf{m}\\ &+8\sigma(\hat{\mathcal{L}}_OW)^*\beta(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar{K})}\mathbf{m}+8(|\beta(\hat{\mathcal{L}}_OW)|^2)tr^{(\bar{K})}\mathbf{i}\\ &+8\rho(\hat{\mathcal{L}}_OW)\alpha(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar{K})}\mathbf{i}-8\sigma(\hat{\mathcal{L}}_OW)^*\alpha(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar{K})}\mathbf{i}\Big\}\end{aligned} \tag{6.2.40}$$

$$\begin{aligned}&{}^{(\bar{K})}\pi^{\alpha\beta}Q(\hat{\mathcal{L}}_OW)_{\alpha\beta43}\tau_+^2=\\ &\frac{1}{16}\tau_+^2\Big\{4|\beta(\hat{\mathcal{L}}_OW)|^{2(\bar{K})}\underline{\mathbf{n}}+4|\underline{\beta}(\hat{\mathcal{L}}_OW)|^{2(\bar{K})}\mathbf{n}+4(|\rho(\hat{\mathcal{L}}_OW)|^2+|\sigma(\hat{\mathcal{L}}_OW)|^2)^{(\bar{K})}\mathbf{j}\\ &-4\rho(\hat{\mathcal{L}}_OW)\beta(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar{K})}\underline{\mathbf{m}}+4\sigma(\hat{\mathcal{L}}_OW)^*\beta(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar{K})}\underline{\mathbf{m}}\\ &+4\rho(\hat{\mathcal{L}}_OW)\underline{\beta}(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar{K})}\mathbf{m}+4\sigma(\hat{\mathcal{L}}_OW)^*\underline{\beta}(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar{K})}\mathbf{m}\\ &+2(|\rho(\hat{\mathcal{L}}_OW)|^2+\sigma(\hat{\mathcal{L}}_OW)|^2)tr^{(\bar{K})}\mathbf{i}-2(\beta(\hat{\mathcal{L}}_OW)\widehat{\otimes}\underline{\beta}(\hat{\mathcal{L}}_OW))\cdot{}^{(\bar{K})}\mathbf{i}\Big\}\end{aligned} \tag{6.2.41}$$

$${}^{(\bar{K})}\pi^{\alpha\beta}Q(\hat{\mathcal{L}}_OW)_{\alpha\beta34}\tau_-^2=$$

$$\frac{1}{16}\tau_-^2\Big\{4|\beta(\hat{\mathcal{L}}_OW)|^{2\,(\bar K)}\underline{\mathbf{n}}+4|\underline{\beta}(\hat{\mathcal{L}}_OW)|^{2\,(\bar K)}\mathbf{n}+4(|\rho(\hat{\mathcal{L}}_OW)|^2+|\sigma(\hat{\mathcal{L}}_OW)|^2)^{(\bar K)}\mathbf{j}$$
$$-4\rho(\hat{\mathcal{L}}_OW)\beta(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar K)}\underline{\mathbf{m}}+4\sigma(\hat{\mathcal{L}}_OW)^*\beta(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar K)}\underline{\mathbf{m}}$$
$$+4\rho(\hat{\mathcal{L}}_OW)\underline{\beta}(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar K)}\mathbf{m}+4\sigma(\hat{\mathcal{L}}_OW)^*\underline{\beta}(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar K)}\mathbf{m}$$
$$+2(|\rho(\hat{\mathcal{L}}_OW)|^2+\sigma(\hat{\mathcal{L}}_OW)|^2)tr^{(\bar K)}\mathbf{i}-2(\beta(\hat{\mathcal{L}}_OW)\widehat{\otimes}\underline{\beta}(\hat{\mathcal{L}}_OW))\cdot{}^{(\bar K)}\mathbf{i}\Big\}\qquad(6.2.42)$$

$$^{(\bar K)}\pi^{\alpha\beta}Q(\hat{\mathcal{L}}_OW)_{\alpha\beta33}\tau_-^2=$$
$$\frac{1}{16}\tau_-^2\Big\{4(\rho(\hat{\mathcal{L}}_OW)|^2+\sigma(\hat{\mathcal{L}}_OW)|^2)^{(\bar K)}\underline{\mathbf{n}}+2|\underline{\alpha}(\hat{\mathcal{L}}_OW)|^{2\,(\bar K)}\mathbf{n}+4|\underline{\beta}(\hat{\mathcal{L}}_OW)|^{2\,(\bar K)}\mathbf{j}$$
$$+8(\underline{\alpha}(\hat{\mathcal{L}}_OW)\cdot\underline{\beta}(\hat{\mathcal{L}}_OW))\cdot{}^{(\bar K)}\mathbf{m}-8\rho(\hat{\mathcal{L}}_OW)\underline{\beta}(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar K)}\underline{\mathbf{m}}$$
$$-8\sigma(\hat{\mathcal{L}}_OW)^*\underline{\beta}(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar K)}\underline{\mathbf{m}}+8(|\underline{\beta}(\hat{\mathcal{L}}_OW)|^2)tr^{(\bar K)}\mathbf{i}$$
$$+8\rho(\hat{\mathcal{L}}_OW)\underline{\alpha}(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar K)}\mathbf{i}+8\sigma(\hat{\mathcal{L}}_OW)^*\underline{\alpha}(\hat{\mathcal{L}}_OW)\cdot{}^{(\bar K)}\mathbf{i}\Big\}\qquad(6.2.43)$$

All the integrals appearing in the decomposition of 6.2.39 can be treated in the same way. We discuss explicitely the integral associated to the term $\frac{1}{8}\tau_+^2|\alpha(\hat{\mathcal{L}}_OW)|^{2\,(\bar K)}\underline{\mathbf{n}}$, see 6.2.40.

$$\int_{V_{(u,\underline{u})}}\tau_+^2|\alpha(\hat{\mathcal{L}}_OW)|^2|^{(\bar K)}\underline{\mathbf{n}}|\le c\int_{u_0}^u du'\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^4\frac{1}{ru'}|\alpha(\hat{\mathcal{L}}_OW)|^2|\frac{\tau_-}{r}{}^{(\bar K)}\underline{\mathbf{n}}|$$
$$\le c\left(\sup_{\mathcal{K}}|\tau_-r^{-1(\bar K)}\underline{\mathbf{n}}|\right)\left(\sup_{\mathcal{K}}\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^4|\alpha(\hat{\mathcal{L}}_OW)|^2\right)\left(\int_{u_0}^u du'\frac{1}{ru'}\right)\le c\epsilon_0\mathcal{Q}_{\mathcal{K}}$$

6.2.5 Estimate of $\int_{V_{(u,\underline{u})}}|Q(\hat{\mathcal{L}}_OW)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar K^\gamma\bar K^\delta)|$

Proposition 6.2.8 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold* [15]

$$\int_{V_{(u,\underline{u})}}|Q(\hat{\mathcal{L}}_OW)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar K^\gamma\bar K^\delta)|\le c\epsilon_0\mathcal{Q}_{\mathcal{K}}\qquad(6.2.44)$$

Proof: We write explicitely the various terms of

$$^{(T)}\pi^{\alpha\beta}Q(\hat{\mathcal{L}}_OW)_{\alpha\beta\gamma\delta}\bar K^\gamma\bar K^\delta=$$
$$^{(T)}\pi^{\alpha\beta}\Big\{Q(\hat{\mathcal{L}}_OW)_{\alpha\beta44}\tau_+^4+2Q(\hat{\mathcal{L}}_OW)_{\alpha\beta43}\tau_+^2\tau_-^2+Q(\hat{\mathcal{L}}_OW)_{\alpha\beta33}\tau_-^4\Big\}\ .$$

[15] $\mathcal{Q}_{\mathcal{K}}$ can be replaced by $\mathcal{Q}_1+\underline{\mathcal{Q}}_1$.

$$
\begin{aligned}
&{}^{(T)}\pi^{\alpha\beta}Q(\hat{\mathcal{L}}_O W)_{\alpha\beta 44}\tau_+^4 = \\
&\frac{1}{16}\tau_+^4\Big\{2|\alpha(\hat{\mathcal{L}}_O W)|^{2\,(T)}\underline{\mathbf{n}} + 4(|\rho(\hat{\mathcal{L}}_O W)|^2 + |\sigma(\hat{\mathcal{L}}_O W)|^2)^{(T)}\mathbf{n} + |\beta(\hat{\mathcal{L}}_O W)|^{2\,(T)}\mathbf{j} \\
&- 8(\alpha(\hat{\mathcal{L}}_O W)\cdot\beta(\hat{\mathcal{L}}_O W))\cdot{}^{(T)}\underline{\mathbf{m}} - 8\rho(\hat{\mathcal{L}}_O W)\beta(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\mathbf{m} \\
&+ 8\sigma(\hat{\mathcal{L}}_O W)^*\beta(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\mathbf{m} + 8(|\beta(\hat{\mathcal{L}}_O W)|^2)tr^{(T)}\mathbf{i} \\
&+ 8\rho(\hat{\mathcal{L}}_O W)\alpha(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\mathbf{i} - 8\sigma(\hat{\mathcal{L}}_O W)^*\alpha(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\mathbf{i}\Big\} \qquad (6.2.45)
\end{aligned}
$$

$$
\begin{aligned}
&{}^{(T)}\pi^{\alpha\beta}Q(\hat{\mathcal{L}}_O W)_{\alpha\beta 43}\tau_+^2\tau_-^2 = \\
&\frac{1}{16}\tau_+^2\tau_-^2\Big\{4|\beta(\hat{\mathcal{L}}_O W)|^{2\,(T)}\underline{\mathbf{n}} + 4|\underline{\beta}(\hat{\mathcal{L}}_O W)|^{2\,(T)}\mathbf{n} + 4(|\rho(\hat{\mathcal{L}}_O W)|^2 + |\sigma(\hat{\mathcal{L}}_O W)|^2)^{(T)}\mathbf{j} \\
&- 4\rho(\hat{\mathcal{L}}_O W)\beta(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\underline{\mathbf{m}} + 4\sigma(\hat{\mathcal{L}}_O W)^*\beta(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\underline{\mathbf{m}} \\
&+ 4\rho(\hat{\mathcal{L}}_O W)\underline{\beta}(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\mathbf{m} + 4\sigma(\hat{\mathcal{L}}_O W)^*\underline{\beta}(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\mathbf{m} \\
&+ 2(|\rho(\hat{\mathcal{L}}_O W)|^2 + \sigma(\hat{\mathcal{L}}_O W)|^2)tr^{(T)}\mathbf{i} - 2(\beta(\hat{\mathcal{L}}_O W)\widehat{\otimes}\underline{\beta}(\hat{\mathcal{L}}_O W))\cdot{}^{(T)}\mathbf{i}\Big\} \qquad (6.2.46)
\end{aligned}
$$

$$
\begin{aligned}
&{}^{(T)}\pi^{\alpha\beta}Q(\hat{\mathcal{L}}_O W)_{\alpha\beta 33}\tau_-^4 = \\
&\frac{1}{16}\tau_-^4\Big\{4(\rho(\hat{\mathcal{L}}_O W)|^2 + \sigma(\hat{\mathcal{L}}_O W)|^2)^{(T)}\underline{\mathbf{n}} + 2|\underline{\alpha}(\hat{\mathcal{L}}_O W)|^{2\,(T)}\mathbf{n} + 4|\underline{\beta}(\hat{\mathcal{L}}_O W)|^{2\,(T)}\mathbf{j} \\
&+ 8(\underline{\alpha}(\hat{\mathcal{L}}_O W)\cdot\underline{\beta}(\hat{\mathcal{L}}_O W))\cdot{}^{(T)}\mathbf{m} - 8\rho(\hat{\mathcal{L}}_O W)\underline{\beta}(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\underline{\mathbf{m}} \\
&- 8\sigma(\hat{\mathcal{L}}_O W)^*\underline{\beta}(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\underline{\mathbf{m}} + 8(|\underline{\beta}(\hat{\mathcal{L}}_O W)|^2)tr^{(T)}\mathbf{i} \\
&+ 8\rho(\hat{\mathcal{L}}_O W)\underline{\alpha}(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\mathbf{i} + 8\sigma(\hat{\mathcal{L}}_O W)^*\underline{\alpha}(\hat{\mathcal{L}}_O W)\cdot{}^{(T)}\mathbf{i}\Big\} \qquad (6.2.47)
\end{aligned}
$$

We use the estimates of Proposition 6.1.1 for the deformation tensor ${}^{(T)}\pi$, observe that all the integrals composing $\int_{V_{(u,\underline{u})}}|Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)|$ are estimated in same way and that the estimates are exactly of the same type as those for $\int_{V_{(u,\underline{u})}}|Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)$, therefore we do not report them here.

****************18/9/2001****************

6.3 The error terms $\mathcal{E}_2$

The remarks at the beginning of section 6.2 applies also to the estimates of $\mathcal{E}_2(u,\underline{u})$ and we do not repeat them here.

$\mathcal{E}_2(u,\underline{u})$ collects the error terms associated to the integrals of $\mathcal{Q}_2$ and $\underline{\mathcal{Q}}_2$,

$$\begin{aligned}
\mathcal{E}_2(u,\underline{u}) &= \int_{V_{(u,\underline{u})}} |DivQ(\hat{\mathcal{L}}_O^2 W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma T^\delta)| + \int_{V_{(u,\underline{u})}} |DivQ(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta)| \\
&+ \int_{V_{(u,\underline{u})}} |DivQ(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta)| + \int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta} \bar{K}^\gamma T^\delta)| \\
&+ \frac{1}{2}\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta)| + \frac{3}{2}\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta)| \\
&+ \frac{3}{2}\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta)|
\end{aligned}$$

Decomposing these integrals as we have done before leads to a very large number of terms. Nevertheless it is easy to see, with the experience accumulated in the previous sections, how to estimate all of them.

Remark: As far as asymptotic behaviour is concerned most terms can be treated as the corresponding ones in the previous section. The main new complication arising here is that of the presence of higher order derivatives. In particularit is for the reason of controlling the second order derivatives of ${}^{(O)}\pi$ that we had to work hard to ensure that they can be estimated in terms of only two derivatives of the curvature, rather than three derivatives which seem needed at first analysis. To stress this fact we shall concentrate here mainly on the terms involving the highest derivatives.

We start controlling the integrals

$$\begin{aligned}
&\int_{V_{(u,\underline{u})}} |DivQ(\hat{\mathcal{L}}_O^2 W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma T^\delta)| \\
&\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta} \bar{K}^\gamma T^\delta)| \\
&\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta)| \ .
\end{aligned}$$

They are needed to prove the boundedness of

$$\int_C Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4) \ , \ \int_{\underline{C}} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3)$$

6.3.1 Estimate of $\int_{V_{(u,\underline{u})}} |DivQ(\hat{\mathcal{L}}_O^2 W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma T^\delta)|$

By a straightforward calculation [16]

$$
\begin{aligned}
DivQ(\hat{\mathcal{L}}_O^2 W)_{\beta\gamma\delta} &= (\hat{\mathcal{L}}_O^2 W)_{\beta\ \delta}^{\ \mu\ \nu} J(O,O;W)_{\mu\gamma\nu} + (\hat{\mathcal{L}}_O^2 W)_{\beta\ \gamma}^{\ \mu\ \nu} J(O,O;W)_{\mu\delta\nu} \\
&+ {}^*(\hat{\mathcal{L}}_O^2 W)_{\beta\ \delta}^{\ \mu\ \nu} J(O,O;W)^*{}_{\mu\gamma\nu} + {}^*(\hat{\mathcal{L}}_O^2 W)_{\beta\ \gamma}^{\ \mu\ \nu} J(O,O;W)^*{}_{\mu\delta\nu}
\end{aligned}
$$

where [17]

$$
J(O,O;W) = J^0(O,O;W) + \frac{1}{2}\Big(J^1(O,O;W) + J^2(O,O;W) + J^3(O,O;W)\Big) \tag{6.3.1}
$$

and

$$
\begin{aligned}
J^0(O,O;W) &= \hat{\mathcal{L}}_O J(O;W) \\
J^1(O,O;W) &= J^1(O;\hat{\mathcal{L}}_O W) \\
J^2(O,O;W) &= J^2(O;\hat{\mathcal{L}}_O W) \\
J^3(O,O;W) &= J^3(O;\hat{\mathcal{L}}_O W)
\end{aligned} \tag{6.3.2}
$$

Recalling the equation, see 6.1.1,

$$
\begin{aligned}
DivQ(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T) &= \frac{1}{8}\tau_+^4 (D(O,O;W)_{444} + D(O,O;W)_{344}) \\
&+ \frac{1}{4}\tau_+^2 \tau_-^2 (D(O,O;W)_{344} + D(O,O;W)_{334}) \\
&+ \frac{1}{8}\tau_-^4 (D(O,O;W)_{334} + D(O,O;W)_{333})
\end{aligned}
$$

where, see 6.1.3,

$$
\begin{aligned}
D(O,O;W)_{444} &= 4\alpha(\hat{\mathcal{L}}_O^2 W)\cdot\Theta(O,O;W) - 8\beta(\hat{\mathcal{L}}_O^2 W)\cdot\Xi(O,O;W) \\
D(O,O;W)_{443} &= 8\rho(\hat{\mathcal{L}}_O^2 W)\Lambda(O,O;W) + 8\sigma(\hat{\mathcal{L}}_O^2 W)K(O,O;W) \\
&+ 8\beta(\hat{\mathcal{L}}_O^2 W)\cdot I(O,O;W) \\
D(O,O;W)_{334} &= 8\rho(\hat{\mathcal{L}}_O^2 W)\underline{\Lambda}(O,O;W) - 8\sigma(\hat{\mathcal{L}}_O^2 W)\underline{K}(O,O;W) \\
&- 8\underline{\beta}(\hat{\mathcal{L}}_O^2 W)\cdot\underline{I}(O,O;W) \\
D(O,O;W)_{333} &= 4\underline{\alpha}(\hat{\mathcal{L}}_O^2 W)\cdot\underline{\Theta}(O,O;W) + 8\underline{\beta}(\hat{\mathcal{L}}_O^2 W)\cdot\underline{\Xi}(O,O;W)
\end{aligned} \tag{6.3.3}
$$

[16] see [Ch-Kl], Propositions 7.1.1, 7.1.2.
[17] See also [Ch-Kl], equation (8.1.2d).

The terms $\Theta(O,O;W)$, ..., $\underline{\Xi}(O,O;W)$ [18] have the following structure:

$$X(O,O;W) = X^0(O,O;W) + \frac{1}{2}\Big(X^1(O,O;W) + X^2(O,O;W) + X^3(O,O;W)\Big)$$

and the upper indices $0,1,2,3$ refer to the various parts of the current $J(O,O;W)$, see 6.3.2. Therefore

$$X^i(O,O;W) = X^i(O,\hat{\mathcal{L}}_O W) \tag{6.3.4}$$

and

$$X^0(O,O;W) = \frac{1}{2}\Big[\hat{\mathcal{L}}_O X^1(O;W) + \hat{\mathcal{L}}_O X^2(O;W) + \hat{\mathcal{L}}_O X^3(O;W)\Big] + \text{l.o.t.} \tag{6.3.5}$$

We have to control the integrals

$$\begin{aligned} &\int_{V_{(u,\underline{u})}} \tau_+^4 D(O,O;W)_{444}\ ,\ \int_{V_{(u,\underline{u})}} \tau_+^4 D(O,O;W)_{443} \\ &\int_{V_{(u,\underline{u})}} \tau_+^2\tau_-^2 D(O,O;W)_{334}\ ,\ \int_{V_{(u,\underline{u})}} \tau_-^4 D(O,O;W)_{333} \end{aligned} \tag{6.3.6}$$

The more sensitive terms are the first two containing the highest weight in τ_+. We look at the estimate of the first one. the second one is potentially more dangerous [19]

$$\begin{aligned} \int_{V_{(u,\underline{u})}} \tau_+^4 D(O,O;W)_{444} &= \frac{1}{2}\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\cdot\Theta(O,O;W) \\ &\quad - \int_{V_{(u,\underline{u})}} \tau_+^4 \beta(\hat{\mathcal{L}}_O^2 W)\cdot\Xi(O,O;W) \end{aligned}$$

the other terms are simpler and can be treated similarly. The two integrals on the right hand side are estimated proving the next proposition, analogous to Proposition 6.2.6,

Proposition 6.3.1 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\left|\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\cdot\Theta(O,O;W)\right| \le c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$

$$\left|\int_{V_{(u,\underline{u})}} \tau_+^4 \beta(\hat{\mathcal{L}}_O^2 W)\cdot\Xi(O,O;W)\right| \le c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$

[18] They are the null components of $J(O,O;W)$.

[19] Both terms have the same weight, but $D(O,O;W)_{443}$ has lower signature, see 3.1.23.

Proof: We start looking at the first integral; equation 6.3.4 implies that the terms in the integrand associated to $J^1(O,O;W), J^2(O,O;W), J^3(O,O;W)$ are estimated exactly as the corresponding term of Proposition 6.2.6, substituting W with $\hat{\mathcal{L}}_O W$ and observing that, in the estimates analogous to those of Lemma 6.2.3, $\mathcal{Q}_1^{\frac{1}{2}}$ and $\underline{\mathcal{Q}}_1^{\frac{1}{2}}$ are replaced by $\mathcal{Q}_2^{\frac{1}{2}}$ and $\underline{\mathcal{Q}}_2^{\frac{1}{2}}$.
We have still to control

$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\cdot\Theta^0(O,O;W)\ .$$

For this integral we have to estimate

$$\Theta^0(O,O;W)=\frac{1}{2}\left[\hat{\mathcal{L}}_O\Theta^1(O;W)+\hat{\mathcal{L}}_O\Theta^2(O;W)+\hat{\mathcal{L}}_O\Theta^3(O;W)\right]+l.o.t.$$

The more delicate parts, for the regularity, are those associated to $\hat{\mathcal{L}}_O\Theta^2(O;W)$ and $\hat{\mathcal{L}}_O\Theta^3(O;W)$; therefore we start considering

$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\cdot\hat{\mathcal{L}}_O\Theta^2(O;W) \tag{6.3.7}$$

Recalling the expression of $\Theta^2(O;W)$, see 6.1.18,

$$\Theta^2(O;W)=\mathrm{Qr}\left[{}^{(O)}p_3\,;\,\alpha\right]+\mathrm{Qr}\left[{}^{(O)}\not{p};\,\beta\right]+\mathrm{Qr}\left[{}^{(O)}p_4\,;\,(\rho,\sigma)\right]$$

it follows that $\hat{\mathcal{L}}_O\Theta^2(O;W)$ has the following expression

$$\begin{aligned}\hat{\mathcal{L}}_O\Theta^2(O;W)&=\mathrm{Qr}\left[\hat{\mathcal{L}}_O{}^{(O)}p_3\,;\,\alpha(W)\right]+\mathrm{Qr}\left[\hat{\mathcal{L}}_O{}^{(O)}\not{p};\,\beta(W)\right]+\mathrm{Qr}\left[\hat{\mathcal{L}}_O{}^{(O)}p_4\,;\,(\rho,\sigma)\right]\\&+\mathrm{Qr}\left[{}^{(O)}p_3\,;\,\hat{\mathcal{L}}_O\alpha(W)\right]+\mathrm{Qr}\left[{}^{(O)}\not{p};\,\hat{\mathcal{L}}_O\beta(W)\right]+\mathrm{Qr}\left[{}^{(O)}p_4\,;\,\hat{\mathcal{L}}_O(\rho,\sigma)\right]\end{aligned} \tag{6.3.8}$$

Recalling the remark after Proposition 6.1.13 and the discussion in the appendix to this Chapter, the dependence on the third derivatives of the connection coefficients appears in $\hat{\mathcal{L}}_O{}^{(O)}p_3$ and $\hat{\mathcal{L}}_O{}^{(O)}\not{p}$ [20]. Therefore we concentrate our attention to

$$\mathrm{Qr}\left[\hat{\mathcal{L}}_O{}^{(O)}p_3\,;\,\alpha(W)\right]+\mathrm{Qr}\left[\hat{\mathcal{L}}_O{}^{(O)}\not{p};\,\beta(W)\right]$$

and check that the corresponding integrals in 6.3.7

$$\begin{aligned}&\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\mathrm{Qr}\left[\hat{\mathcal{L}}_O{}^{(O)}\not{p};\,\beta(W)\right]\\&\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\mathrm{Qr}\left[\hat{\mathcal{L}}_O{}^{(O)}p_3\,;\,\alpha(W)\right]\end{aligned} \tag{6.3.9}$$

verify,

[20] From the discussion in the appendix to this Chapter, see subsection 6.4.1, it follows that $\hat{\mathcal{L}}_O{}^{(O)}p_4$ does not depend on the third derivatives of connection coefficients.

Lemma 6.3.1 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold,*

$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W) Qr\left[\hat{\mathcal{L}}_O{}^{(O)}\not{p}\,;\,\beta(W)\right] \le c\mathcal{Q}_2\left(\sup_{\mathcal{K}}\int_{\underline{C}(\underline{u};[u_0(\underline{u}),u])}|\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2\right)^{\frac{1}{2}}$$
$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W) Qr\left[\hat{\mathcal{L}}_O{}^{(O)}p_3\,;\,\alpha(W)\right] \le c\mathcal{Q}_2\left(\sup_{\mathcal{K}}\int_{\underline{C}(\underline{u};[u_0(\underline{u}),u])}|\hat{\mathcal{L}}_O{}^{(O)}p_3|^2\right)^{\frac{1}{2}}$$

We postpone the proof of this lemma to subsection 6.3.2.

With the help of this lemma and Proposition 6.1.11, we obtain

$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\mathrm{Qr}\left[\hat{\mathcal{L}}_O{}^{(O)}\not{p}\,;\,\beta(W)\right] \le c\epsilon_0\mathcal{Q}_{\mathcal{K}}$$
$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\mathrm{Qr}\left[\hat{\mathcal{L}}_O{}^{(O)}p_3\,;\,\alpha(W)\right] \le c\epsilon_0\mathcal{Q}_{\mathcal{K}} \qquad (6.3.10)$$

which proves Proposition 6.3.1 for these particular terms. The contribution to the integral 6.3.7 due to $\mathrm{Qr}\left[\hat{\mathcal{L}}_O{}^{(O)}p_4\,;\,(\rho,\sigma)\right]$ is easier to treat and we do not discuss it, here.

The contributions to the integral 6.3.7 due to the terms present in the second line of 6.3.8, are

$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\mathrm{Qr}\left[{}^{(O)}\not{p}\,;\,\hat{\mathcal{L}}_O\beta(W)\right]$$
$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\mathrm{Qr}\left[{}^{(O)}p_3\,;\,\hat{\mathcal{L}}_O\alpha(W)\right] \qquad (6.3.11)$$
$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\mathrm{Qr}\left[{}^{(O)}p_4\,;\,\hat{\mathcal{L}}_O(\rho,\sigma)(W)\right]$$

and turn out to be easier to control. A short discussion on their estimates is given at the end of this section.

The estimate for the integral

$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\cdot\hat{\mathcal{L}}_O\Theta^3(O;W) \qquad (6.3.12)$$

is done precisely as for the integral 6.3.7. Again it follows, see 6.1.23, that the contributions to $\Theta^3(O;W)\equiv\Theta(J^3(O;W))$ which depend on the second derivatives of the connection coefficients are

$$\mathrm{Qr}\left[\alpha\,;\,\underline{K}({}^{(O)}q)\right]\ ,\ \mathrm{Qr}\left[\alpha\,;\,\underline{\Lambda}({}^{(O)}q)\right]\ ,\ \mathrm{Qr}\left[\beta\,;\,(I,\underline{I})({}^{(O)}q)\right]$$

and, therefore, the more delicate integrals to control are

$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\mathrm{Qr}\left[\hat{\mathcal{L}}_O \underline{K}({}^{(O)}q)\,;\,\alpha(W)\right]$$
$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\mathrm{Qr}\left[\hat{\mathcal{L}}_O \underline{\Lambda}({}^{(O)}q)\,;\,\alpha(W)\right]$$
$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\mathrm{Qr}\left[\hat{\mathcal{L}}_O \underline{I}({}^{(O)}q)\,;\,\alpha(W)\right]$$

These are estimated exactly as those in 6.3.9 [21] and we do not report it here.

The previous estimates have shown that the more delicate terms to estimate are those depending on the third order derivatives of the connection coefficients. So far we have discussed the estimates for the terms which depend on $\nabla\!\!\!/^3\eta$, $\nabla\!\!\!/^3\underline{\eta}$ and $\nabla\!\!\!/^3\chi$. Next we examine the terms which depend on $\nabla\!\!\!/^3\underline{\omega}$. Therefore we consider explicitely only those parts of the integrals 6.3.6 which depend on it [22]. The factor $\nabla\!\!\!/^2\underline{\omega}$ appears in the expression

$$\underline{\Xi}({}^{(O)}q)_a = \frac{1}{2}\mathbf{D}\!\!\!\!/_3{}^{(O)}\underline{\mathbf{m}}_a - \frac{1}{2}(\mathbf{D}_3\log\Omega)^{(O)}\underline{\mathbf{m}}_a + \frac{1}{2}tr\underline{\chi}^{(O)}\underline{\mathbf{m}}_a + \underline{\hat{\chi}}_{ac}{}^{(O)}\underline{\mathbf{m}}_c$$

through $\mathbf{D}\!\!\!\!/_3{}^{(O)}\underline{\mathbf{m}}$. The terms of $J^3(O;W)$ which depend on $\underline{\Xi}({}^{(O)}q)$ are, see 6.1.19, 6.1.21,

$$\begin{aligned}
\underline{\Xi}^3(O;W) &= \mathrm{Qr}\left[(\rho,\sigma)\,;\,\underline{\Xi}({}^{(O)}q)\right] + \cdots\\
\underline{\Lambda}^3(O;W) &= \mathrm{Qr}\left[\beta\,;\,\underline{\Xi}({}^{(O)}q)\right] + \cdots\\
\underline{K}^3(O;W) &= \mathrm{Qr}\left[\beta\,;\,\underline{\Xi}({}^{(O)}q)\right] + \cdots\\
I^3(O;W) &= \mathrm{Qr}\left[\alpha\,;\,\underline{\Xi}({}^{(O)}q)\right] + \cdots
\end{aligned} \tag{6.3.13}$$

These $J^3(O;W)$ components are present in, see 6.1.3,

$$\begin{aligned}
D(O,W)_{443} &= 8\beta(\hat{\mathcal{L}}_O W)\cdot I^3(O,W) + \cdots\\
D(O,W)_{334} &= 8\rho(\hat{\mathcal{L}}_O W)\underline{\Lambda}^3(O,W) - 8\sigma(\hat{\mathcal{L}}_O W)\underline{K}^3(O,W) + \cdots\\
D(O,W)_{333} &= 8\underline{\beta}(\hat{\mathcal{L}}_O W)\cdot\underline{\Xi}^3(O,W) + \cdots
\end{aligned} \tag{6.3.14}$$

[21] See the discussion in the appendix to this Chapter, subsection 6.4.1, about the dependance on the connection coefficients of $\underline{K}({}^{(O)}q), \underline{\Lambda}({}^{(O)}q), \underline{I}({}^{(O)}q)$.

[22] The term $\nabla\!\!\!/^3\underline{\omega}$ does not appear in the integrals examinated up to now.

In the error term $\int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_O^2 W)_{\beta\gamma\delta}\bar{K}^\beta\bar{K}^\gamma T^\delta$ we have to consider the terms $D(O,O;W)_{443}$, $D(O,O;W)_{334}$, $D(O,O;W)_{333}$. The contributions to these terms coming from $\hat{\mathcal{L}}_O J^3(O;W)$ [23] are:

$$\begin{aligned}
D(O,O;W)_{443} &= 8\beta(\hat{\mathcal{L}}_O^2 W)\cdot I(O,O;W)+\cdots\\
D(O,O;W)_{334} &= 8\rho(\hat{\mathcal{L}}_O^2 W)\underline{\Lambda}(O,O;W)-8\sigma(\hat{\mathcal{L}}_O^2 W)\underline{K}(O,O;W)+\cdots\\
D(O,O;W)_{333} &= 8\underline{\beta}(\hat{\mathcal{L}}_O^2 W)\cdot\underline{\Xi}(O,O;W)+\cdots
\end{aligned}$$

and from $I(O,O;W)$, $\underline{\Lambda}(O,O;W)$, $\underline{K}(O,O;W)$, $\underline{\Xi}(O,O;W)$ we select the parts containing $\hat{\mathcal{L}}_O\underline{\Xi}({}^{(O)}q)$,

$$\begin{aligned}
\underline{\Xi}(O,O;W) &= \mathrm{Qr}\left[(\rho,\sigma)(W)\,;\,\hat{\mathcal{L}}_O\underline{\Xi}({}^{(O)}q)\right]+\cdots\\
\underline{\Lambda}(O,O;W) &= \mathrm{Qr}\left[\beta(W)\,;\,\hat{\mathcal{L}}_O\underline{\Xi}({}^{(O)}q)\right]+\cdots\\
\underline{K}(O,O;W) &= \mathrm{Qr}\left[\beta(W)\,;\,\hat{\mathcal{L}}_O\underline{\Xi}({}^{(O)}q)\right]+\cdots\\
I(O,O;W) &= \mathrm{Qr}\left[\alpha(W)\,;\,\hat{\mathcal{L}}_O\underline{\Xi}({}^{(O)}q)\right]+\cdots
\end{aligned}$$

In conclusion the terms which depend on $\nabla\!\!\!/^3\underline{\omega}$ are

$$\begin{aligned}
D(O,O;W)_{443} &= 8\beta(\hat{\mathcal{L}}_O^2 W)\cdot\mathrm{Qr}\left[\alpha(W)\,;\,\hat{\mathcal{L}}_O\underline{\Xi}({}^{(O)}q)\right]+\cdots\\
D(O,O;W)_{334} &= 8\rho(\hat{\mathcal{L}}_O^2 W)\cdot\mathrm{Qr}\left[\beta(W)\,;\,\hat{\mathcal{L}}_O\underline{\Xi}({}^{(O)}q)\right]+\cdots\\
D(O,O;W)_{333} &= 8\underline{\beta}(\hat{\mathcal{L}}_O^2 W)\cdot\mathrm{Qr}\left[(\rho,\sigma)(W)\,;\,\hat{\mathcal{L}}_O\underline{\Xi}({}^{(O)}q)\right]+\cdots
\end{aligned}$$

They appear in the following integrals

$$\begin{aligned}
\int_{V_{(u,\underline{u})}}\tau_+^4(D(O,O;W)_{443} &= \int_{V_{(u,\underline{u})}}\tau_+^4\beta(\hat{\mathcal{L}}_O^2 W)\cdot\mathrm{Qr}\left[\alpha(W)\,;\,\hat{\mathcal{L}}_O\underline{\Xi}({}^{(O)}q)\right]+\ldots.\\
\int_{V_{(u,\underline{u})}}\tau_+^2\tau_-^2(D(O,O;W)_{334} &= \int_{V_{(u,\underline{u})}}\tau_+^2\tau_-^2\rho(\hat{\mathcal{L}}_O^2 W)\cdot\mathrm{Qr}\left[\beta(W)\,;\,\hat{\mathcal{L}}_O\underline{\Xi}({}^{(O)}q)\right]+\ldots.\\
\int_{V_{(u,\underline{u})}}\tau_-^4(D(O,O;W)_{333} &= \int_{V_{(u,\underline{u})}}\tau_-^4\underline{\beta}(\hat{\mathcal{L}}_O^2 W)\cdot\mathrm{Qr}\left[(\rho,\sigma)(W)\,;\,\hat{\mathcal{L}}_O\underline{\Xi}({}^{(O)}q)\right]+\ldots.
\end{aligned} \tag{6.3.15}$$

The neglected terms or have either been already estimated in Proposition 6.3.1 or are easier to control. The integrals in the right hand side are estimated in the following proposition

[23] $\hat{\mathcal{L}}_O J^3(O;W)$ appears in the current $J^0(O,O;W)=\hat{\mathcal{L}}_O J^3(O;W)+\cdots$, see 6.3.1.

Proposition 6.3.2 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\int_{V_{(u,\underline{u})}} \tau_+^4 \beta(\hat{\mathcal{L}}_O^2 W)\cdot Qr\Big[\alpha(W)\,;\,\hat{\mathcal{L}}_O \underline{\Xi}({}^{(O)}q)\Big] \le c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$
$$\int_{V_{(u,\underline{u})}} \tau_+^2\tau_-^2 \rho(\hat{\mathcal{L}}_O^2 W)\, Qr\Big[\beta(W)\,;\,\hat{\mathcal{L}}_O \underline{\Xi}({}^{(O)}q)\Big] \le c\epsilon_0 \mathcal{Q}_{\mathcal{K}} \qquad (6.3.16)$$
$$\int_{V_{(u,\underline{u})}} \tau_-^4 \underline{\beta}(\hat{\mathcal{L}}_O^2 W)\cdot Qr\Big[(\rho,\sigma)(W)\,;\,\hat{\mathcal{L}}_O \underline{\Xi}({}^{(O)}q)\Big] \le c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$

Proof: The proof is based on the following lemma, whose proof is postponed to the next subsection,

Lemma 6.3.2 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold, with* $\epsilon > 0$,

$$\int_{V_{(u,\underline{u})}} \tau_+^4 \beta(\hat{\mathcal{L}}_O^2 W)\cdot Qr\Big[\alpha(W)\,;\,\hat{\mathcal{L}}_O \underline{\Xi}({}^{(O)}q)\Big] \le$$
$$c\mathcal{Q}_{\mathcal{K}}\left(\sup_{\mathcal{K}}\int_{\underline{C}(\underline{u};[u_0(\underline{u}),u])}\frac{1}{r^{1-2\epsilon}}|\hat{\mathcal{L}}_O \underline{\Xi}({}^{(O)}q)|^2\right)^{\frac{1}{2}}$$
$$\int_{V_{(u,\underline{u})}} \tau_+^2\tau_-^2 \rho(\hat{\mathcal{L}}_O^2 W)\, Qr\Big[\beta(W)\,;\,\hat{\mathcal{L}}_O \underline{\Xi}({}^{(O)}q)\Big] \le$$
$$c\mathcal{Q}_{\mathcal{K}}\left(\sup_{\mathcal{K}}\int_{\underline{C}(\underline{u};[u_0(\underline{u}),u])}\frac{1}{r^{1-2\epsilon}}|\hat{\mathcal{L}}_O \underline{\Xi}({}^{(O)}q)|^2\right)^{\frac{1}{2}}$$
$$\int_{V_{(u,\underline{u})}} \tau_-^4 \underline{\beta}(\hat{\mathcal{L}}_O^2 W)\cdot Qr\Big[(\rho,\sigma)(W)\,;\,\hat{\mathcal{L}}_O \underline{\Xi}({}^{(O)}q)\Big] \le$$
$$c\mathcal{Q}_{\mathcal{K}}\left(\sup_{\mathcal{K}}\int_{\underline{C}(\underline{u};[u_0(\underline{u}),u])}\frac{1}{r^{1-2\epsilon}}|\hat{\mathcal{L}}_O \underline{\Xi}({}^{(O)}q)|^2\right)^{\frac{1}{2}}$$

Proposition 6.3.2 is an immediate consequence of Proposition 6.1.13 and Lemma 6.3.2.

6.3.2 Proof of Lemma 6.3.1 and Lemma 6.3.2

To prove Lemma 6.3.1 we estimate the first integral of 6.3.9,

$$\int_{V_{(u,\underline{u})}} \tau_+^4 \alpha(\hat{\mathcal{L}}_O^2 W)\mathrm{Qr}\left[\hat{\mathcal{L}}_O{}^{(O)}\not{p};\,\beta(W)\right] \le \int_{V_{(u,\underline{u})}} \tau_+^4|\alpha(\hat{\mathcal{L}}_O^2 W)||\hat{\mathcal{L}}_O{}^{(O)}\not{p}||\beta(W)|$$
$$\le \left(\int_{V_{(u,\underline{u})}} \tau_+^{2\gamma}|\alpha(\hat{\mathcal{L}}_O^2 W)|^2\right)^{\frac{1}{2}}\left(\int_{V_{(u,\underline{u})}} \tau_+^{2\sigma}|\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2|\beta(W)|^2\right)^{\frac{1}{2}} \qquad (6.3.17)$$

with $\gamma+\sigma=4$. The first factor satisfies the inequality

$$\int_{V_{(u,\underline{u})}} \tau_+^{2\gamma}|\alpha(\hat{\mathcal{L}}_O^2 W)|^2 \leq c\mathcal{Q}_2 \tag{6.3.18}$$

with $2\gamma<3$ which implies $2\sigma>5$. In fact, from the results of Chapter 5, see **??**,

Osservazione 6.3.1 *C'e' un riferimento mancante, correggere.*

$$\begin{aligned}
\int_{V_{(u,\underline{u})}} \tau_+^{2\gamma}|\alpha(\hat{\mathcal{L}}_O^2 W)|^2 &= \int_{u_0(\underline{u})}^{u} du' \int_{C(u';[\underline{u}_0(u'),\underline{u}])} \tau_+^{2\gamma}|\alpha(\hat{\mathcal{L}}_O^2 W)|^2 \\
&\leq \int_{u_0(\underline{u})}^{u} du' \frac{1}{u'^{4-2\gamma}} \left(\sup_{V(u,\underline{u})} \int_{C(u';[\underline{u}_0(u'),\underline{u}])} \tau_+^{4}|\alpha(\hat{\mathcal{L}}_O^2 W)|^2\right) \\
&\leq c\left(\sup_{\mathcal{K}} \int_{C(u')} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4)\right) \leq c\mathcal{Q}_2
\end{aligned}$$

The second factor satisfies

$$\begin{aligned}
\int_{V_{(u,\underline{u})}} \tau_+^{2\sigma}|\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2|\beta(W)|^2 &\leq \left(\sup|r^{\frac{7}{2}}\beta(W)|\right)^2 \int_{V_{(u,\underline{u})}} \tau_+^{-(7-2\sigma)}|\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2 \\
&\leq c\mathcal{Q}_{\mathcal{K}} \int_{V_{(u,\underline{u})}} \tau_+^{-(7-2\sigma)}|\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2
\end{aligned} \tag{6.3.19}$$

We decompose the integral in the following way, writing $7-2\sigma=1+\tau$ with $\tau\in(0,1)$,

$$\begin{aligned}
\int_{V_{(u,\underline{u})}} \tau_+^{-(7-2\sigma)}|\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2 &\leq c\int_{\underline{u}_0(u)}^{\underline{u}} d\underline{u}' \int_{\underline{C}(\underline{u}';[u_0(\underline{u}'),u])} \underline{u}'^{-(1+\tau)}|\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2 \\
&\leq c\int_{\underline{u}_0(u)}^{\underline{u}} d\underline{u}' \frac{1}{\underline{u}'^{(1+\tau)}} \left(\int_{\underline{C}(\underline{u}';[u_0(\underline{u}'),u])} |\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2\right) \\
&\leq c\left(\sup_{\mathcal{K}} \int_{\underline{C}(\underline{u}';[u_0(\underline{u}'),u])} |\hat{\mathcal{L}}_O{}^{(O)}\not{p}|^2\right) \leq c\epsilon_0
\end{aligned} \tag{6.3.20}$$

where we have used, in the last inequality, the second estimate of Proposition 6.1.11. Inequalities 6.3.20 and 6.3.18, toghether, prove the first estimate of the lemma. The estimate of the second line of 6.3.9 proceeds in the same way, but relying on the first inequality of Proposition 6.1.11.

To prove Lemma 6.3.2 we prove only the first inequality of the lemma, the others are proved in the same way. The proof is similar, but easier, to that of Lemma 6.3.1,

$$
\begin{aligned}
&\int_{V_{(u,\underline{u})}} \tau_+^4 \beta(\hat{\mathcal{L}}_O^2 W)\cdot \mathrm{Qr}\left[\alpha(W)\,;\,\hat{\mathcal{L}}_O \Xi({}^{(O)}q)\right]\\
&\leq \left(\sup_{V(u,\underline{u})}\int_{\underline{C}(\underline{u};[u_0,u])}\underline{u}'^4|\beta(\hat{\mathcal{L}}_O^2 W)|^2\right)^{\frac{1}{2}}\int_{\underline{u}_0}^{\underline{u}} d\underline{u}'\left(\int_{\underline{C}(\underline{u}';[u_0,u])}\underline{u}'^4|\alpha(W)|^2||\hat{\mathcal{L}}_O\Xi({}^{(O)}q)|^2\right)^{\frac{1}{2}}\\
&\leq c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\left(\sup_{\mathcal{K}}|r^{\frac{7}{2}}\alpha(W)|\right)\int_{\underline{u}_0}^{\underline{u}} d\underline{u}'\left(\int_{\underline{C}(\underline{u}';[u_0,u])}\frac{1}{\underline{u}'^3}|\hat{\mathcal{L}}_O\Xi({}^{(O)}q)|^2\right)^{\frac{1}{2}}\\
&\leq c\mathcal{Q}_{\mathcal{K}}\int_{\underline{u}_0}^{\underline{u}} d\underline{u}'\frac{1}{\underline{u}'^{1+\epsilon}}\left(\int_{\underline{C}(\underline{u};[u_0,u])}\frac{1}{r^{1-2\epsilon}}|\hat{\mathcal{L}}_O\Xi({}^{(O)}q)|^2\right)^{\frac{1}{2}} \qquad (6.3.21)
\end{aligned}
$$

with $\epsilon > 0$, where in the last line we used Proposition 6.1.13.

We shall now provide a sketch of the estimate of the second integral of 6.3.11 [24]. We use the estimates 6.1.56, valid for any $p \in [2,4]$,

$$
\sup_{\mathcal{K}}|r^{1-\frac{2}{p}}\tau_-{}^{(O)}p_3|_{p,S} \leq c\epsilon_0 \quad , \quad \sup_{\mathcal{K}}|r^{2-\frac{2}{p}}({}^{(O)}p_4, {}^{(O)}\not{p})|_{p,S} \leq c\epsilon_0 \ ,
$$

and obtain

$$
\begin{aligned}
&\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|{}^{(O)}p_3|^2|\hat{\mathcal{L}}_O\alpha(W)|^2 \leq c\int_{\underline{u}_0}^{\underline{u}} d\underline{u}'\underline{u}'^4\int_{S(u',\underline{u}')}|{}^{(O)}p_3|^2|\hat{\mathcal{L}}_O\alpha(W)|^2\\
&\leq c\int_{\underline{u}_0}^{\underline{u}} d\underline{u}'\underline{u}'^4\frac{1}{r^2\tau_-^2}|r^{1-\frac{2}{p}}\tau_-{}^{(O)}p_3|^2_{p=4,S}|r^{1-\frac{2}{p}}\hat{\mathcal{L}}_O\alpha(W)|^2_{p=4,S}\\
&\leq c\left(\sup_{\mathcal{K}}|r^{1-\frac{2}{p}}\tau_-{}^{(O)}p_3|_{p=4,S}\right)^2\int_{\underline{u}_0}^{\underline{u}} d\underline{u}'\underline{u}'^4\frac{1}{r^2\tau_-^2}|r^{1-\frac{2}{p}}\hat{\mathcal{L}}_O\alpha(W)|^2_{p=4,S}\\
&\leq c\epsilon_0^2\int_{\underline{u}_0}^{\underline{u}} d\underline{u}'\underline{u}'^4\frac{1}{r^7\tau_-^2}|r^{\frac{7}{2}-\frac{2}{p}}\hat{\mathcal{L}}_O\alpha(W)|^2_{p=4,S} \leq c\epsilon_0^2\mathcal{Q}_{\mathcal{K}}\left(\int_{\underline{u}_0}^{\underline{u}} d\underline{u}'\underline{u}'^4\frac{1}{r^7\tau_-^2}\right)\\
&\leq c\epsilon_0^2\mathcal{Q}_{\mathcal{K}}\frac{1}{u'^4} \leq c\epsilon_0^2\mathcal{Q}_{\mathcal{K}} \qquad (6.3.22)
\end{aligned}
$$

Exactly the same argument can be used for the other integrals in 6.3.11 and for those associated to the current $J^3(O;W)$.

[24] Compare the way this integral is estimated with the estimate of the second integral of eq. 6.2.34 in Lemma 6.2.3, see inequality 6.2.38.

6.3.3 Estimate of $\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)|$

Proposition 6.3.3 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)| \leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}} \tag{6.3.23}$$

Proof: These estimates are exactly of the same type as those obtained for $\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)|$ with the obvious substitution in the first and in the second factor of $\mathcal{Q}_1^{\frac{1}{2}}$ or $\underline{\mathcal{Q}}_1^{\frac{1}{2}}$ with $\mathcal{Q}_2^{\frac{1}{2}}$ or $\underline{\mathcal{Q}}_2^{\frac{1}{2}}$.

6.3.4 Estimate of $\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta)|$

Proposition 6.3.4 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta}\bar{K}^\gamma \bar{K}^\delta)| \leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}} \tag{6.3.24}$$

Proof: These estimates are exactly of the same type as those obtained for $\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)|$ with the obvious substitution in the first and in the second factor of $\mathcal{Q}_1^{\frac{1}{2}}$ or $\underline{\mathcal{Q}}_1^{\frac{1}{2}}$ with $\mathcal{Q}_2^{\frac{1}{2}}$ or $\underline{\mathcal{Q}}_2^{\frac{1}{2}}$.

This completes the control of the error terms associated to the integrals

$$\int_C Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4)\ ,\ \int_{\underline{C}} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3)\ .$$

6.3.5 Estimate of $\int_{V_{(u,\underline{u})}} |DivQ(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta|$

We recall the equation, see [Ch-Kl], Propositions 7.1.1, 7.1.2,

$$\begin{aligned}DivQ(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\beta\gamma\delta} &= (\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\beta\ \delta}^{\ \mu\ \nu}J(T,O;W)_{\mu\gamma\nu} + (\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\beta\ \gamma}^{\ \mu\ \nu}J(T,O;W)_{\mu\delta\nu}\\ &+{}^*(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\beta\ \delta}^{\ \mu\ \nu}J(T,O;W)^*{}_{\mu\gamma\nu} + {}^*(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\beta\ \gamma}^{\ \mu\ \nu}J(T,O;W)^*{}_{\mu\delta\nu}\end{aligned}$$

where

$$J(T,O;W) = J^0(T,O;W) + \frac{1}{2}\left(J^1(T,O;W) + J^2(T,O;W) + J^3(T,O;W)\right)$$

and

$$J^0(T,O;W) = \hat{\mathcal{L}}_O J(T;W)$$
$$J^i(T,O;W) = J^i(O;\hat{\mathcal{L}}_T W) \quad , i \in \{1,2,3\}.$$

As in the case of $\int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_O^2 W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma T^\delta)$ we see that the terms associated to this current $J^1(T,O;W) = J^1(O;\hat{\mathcal{L}}_T W)$ are the same as those of

$$\int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma T^\delta)$$

with $\hat{\mathcal{L}}_O W$ substituted by $\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W$ and $J^1(O;W)$ replaced by $J^1(O;\hat{\mathcal{L}}_T W)$. Considering the equation, see 6.1.3,

$$D^1(T,O;W)_{444} = 4\alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)\cdot\Theta^1(O,\hat{\mathcal{L}}_T W) - 8\beta(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)\cdot\Xi^1(O,\hat{\mathcal{L}}_T W)\ ,$$

it follows that we have to control the integrals

$$\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)\cdot\Theta^1(O,\hat{\mathcal{L}}_T W)$$
$$\int_{V_{(u,\underline{u})}} \tau_+^6 \beta(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)\cdot\Xi^1(O,\hat{\mathcal{L}}_T W) \tag{6.3.25}$$

It is immediate to realize that the main difference with respect to Proposition 6.2.6 is that the Riemann components present in the terms $\Theta^1(O,\hat{\mathcal{L}}_T W)$ and $\Xi^1(O,\hat{\mathcal{L}}_T W)$, see 6.1.13, 6.1.14, appear as first derivatives along the tangential and the null directions of the null components of $\hat{\mathcal{L}}_T W$. We conclude that terms like $\mathbf{D}\!\!\!/_3\mathbf{D}\!\!\!/_3 W$, $\mathbf{D}\!\!\!/_3\mathbf{D}\!\!\!/_4 W$, $\mathbf{D}\!\!\!/_4\mathbf{D}\!\!\!/_3 W$, $\mathbf{D}\!\!\!/_4\mathbf{D}\!\!\!/_4 W$ are present [25]. In particular we have to control the following integrals

$$\int \frac{du'}{u'}\left(\int_{C(u')} r^6 |\mathbf{D}\!\!\!/_T \alpha(\hat{\mathcal{L}}_T W)|^2\right)^{\frac{1}{2}}$$
$$\int \frac{d\underline{u}'}{\underline{u}'}\left(\int_{\underline{C}(\underline{u}')} r^6 |\mathbf{D}\!\!\!/_T \underline{\alpha}(\hat{\mathcal{L}}_T W)|^2\right)^{\frac{1}{2}}$$

[25] Observe that the terms in $\Theta^1(O,\hat{\mathcal{L}}_T W)$, $\Xi^1(O,\hat{\mathcal{L}}_T W)$ involve, in particular $\mathbf{D}\!\!\!/_3\mathbf{D}\!\!\!/_3\underline{\alpha}$ and $\mathbf{D}\!\!\!/_4\mathbf{D}\!\!\!/_4\alpha$ which do not appear in the Bianchi equations. These terms have been treated in Chapter 5 by expressing them in terms of $\mathbf{D}\!\!\!/_T\alpha(\hat{\mathcal{L}}_T W)$ or $\mathbf{D}\!\!\!/_T\underline{\alpha}(\hat{\mathcal{L}}_T W)+$ [easier terms].

The terms associated to $J^2(T,O;W)$ and $J^3(T,O;W)$ are treated as the corresponding terms of $\int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma T^\delta)$ with the obvious modifications.
Finally to control the terms associated to the current $J^0(T,O;W)=\hat{\mathcal{L}}_O J(T;W)$ we have to look, carefully, at the following integrals,

$$\int_{V_{(u,\underline{u})}} \tau_+^6\alpha(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O{}^{(T)}p_3)\alpha(W)$$
$$\int_{V_{(u,\underline{u})}} \tau_+^6\alpha(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O{}^{(T)}\not{p})\beta(W)$$
$$\int_{V_{(u,\underline{u})}} \tau_+^6\alpha(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O{}^{(T)}p_4)(\rho,\sigma)(W)$$

their estimates are summarized in the following proposition:

Proposition 6.3.5 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\begin{aligned}
&\int_{V_{(u,\underline{u})}} \tau_+^6\alpha(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O{}^{(T)}p_3)\alpha(W)\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\\
&\int_{V_{(u,\underline{u})}} \tau_+^6\alpha(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O{}^{(T)}\not{p})\beta(W)\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\\
&\int_{V_{(u,\underline{u})}} \tau_+^6\alpha(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)(\hat{\mathcal{L}}_O{}^{(T)}p_4)(\rho,\sigma)(W)\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}
\end{aligned}\tag{6.3.26}$$

Proof: The first term is the more delicate one. It has to be treated as the term $\int_{V_{(u,\underline{u})}}\tau_+^6\alpha(\hat{\mathcal{L}}_T W)({}^{(T)}p_3)\alpha(W)$, see 6.2.9, using, in this case, the following bound of Proposition 6.1.6,

$$\left(\sup_{u\in\mathcal{K}}\int_{\underline{u}_0}^{\underline{u}}\frac{1}{\tau_+(u,\underline{u}')}|r\tau_-^2\nabla\!\!\!\!/\,{}^{(T)}p_3|^2_{p=2,S}(u,\underline{u}')d\underline{u}'\right)^{\frac{1}{2}}\leq c\epsilon_0\ .$$

The second and third integrals in 6.3.26 are treated in the same way, but, in this case, it is enough to use only the estimates of Proposition 6.1.5 [26].

6.3.6 Estimate of $\int_{V_{(u,\underline{u})}}|Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)|$

Proposition 6.3.6 *Under the assumptions 6.0.1 and 6.0.2 the following inequalities hold*

$$\int_{V_{(u,\underline{u})}}|Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)|\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}\tag{6.3.27}$$

[26]This is due to the better asymptotic behaviour of ${}^{(T)}p_4$, ${}^{(T)}\not{p}$ with respect to ${}^{(T)}p_3$, see the remark after Proposition 6.1.5.

Proof: The estimate of this term proceeds exactly as the estimate of $\int_{V_{(u,\underline{u})}} |Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)|$. The final result is the same with the obvious substitutions of the factors $\mathcal{Q}_1^{\frac{1}{2}}$ or $\underline{\mathcal{Q}}_1^{\frac{1}{2}}$ with $\mathcal{Q}_2^{\frac{1}{2}}$ or $\underline{\mathcal{Q}}_2^{\frac{1}{2}}$.

6.3.7 Estimate of $\int_{V_{(u,\underline{u})}} |DivQ(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta)|$

We recall the equation[27], see [Ch-Kl], Propositions 7.1.1, 7.1.2,

$$\begin{aligned} DivQ(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\beta\gamma\delta} &= (\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\beta\ \delta}^{\ \mu\ \nu} J(T,S;W)_{\mu\gamma\nu} + (\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\beta\ \gamma}^{\ \mu\ \nu} J(T,S;W)_{\mu\delta\nu} \\ &+ {}^*(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\beta\ \delta}^{\ \mu\ \nu} J(T,S;W)^*{}_{\mu\gamma\nu} + {}^*(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\beta\ \gamma}^{\ \mu\ \nu} J(T,S;W)^*{}_{\mu\delta\nu} \end{aligned}$$

where

$$J(T,S;W) = J^0(T,S;W) + \frac{1}{2}\Big(J^1(T,S;W) + J^2(T,S;W) + J^3(T,S;W)\Big)$$

and

$$\begin{aligned} &J^0(T,S;W) = \hat{\mathcal{L}}_S J(T;W)\ ,\ J^1(T,S;W) = J^1(S;\hat{\mathcal{L}}_T W) \\ &J^2(T,S;W) = J^2(S;\hat{\mathcal{L}}_T W)\ ,\ J^3(T,S;W) = J^3(S;\hat{\mathcal{L}}_T W) \end{aligned}$$

Proceeding as in the subsection 6.3.5 we have to analyze the integrals

$$\begin{aligned} &\int_{V_{(u,\underline{u})}} \tau_+^6 D(T,S;W)_{444}\ ,\ \int_{V_{(u,\underline{u})}} \tau_+^4\tau_-^2 D(T,S;W)_{344} \\ &\int_{V_{(u,\underline{u})}} \tau_+^2\tau_-^4 D(T,S;W)_{334}\ ,\ \int_{V_{(u,\underline{u})}} \tau_-^6 D(T,S;W)_{333} \end{aligned}$$

We examine the first of these integrals as the other ones are controlled in the same way and are, in fact, even easier. we recall, see 6.1.3, the equation

$$D(T,S;W)_{444} = 4\alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)\cdot\Theta(T,S;W) - 8\beta(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)\cdot\Xi(T,S;W) \quad (6.3.28)$$

where [28]

$$\Theta(T,S;W) \ = \ \Theta^0(T,S;W) + \frac{1}{2}\Big(\Theta^1(T,S;W) + \Theta^2(T,S;W) + \Theta^3(T,S;W)\Big)$$

and the indices $0,1,2,3$ refer to the various parts of the current $J(T,S;W)$. Therefore, for $i\in\{1,2,3\}$,

$$\Theta^i(T,S;W) = \Theta^i(S,\hat{\mathcal{L}}_T W)$$

[27] The estimate of this term is similar to that of $\int_{V_{(u,\underline{u})}} |DivQ(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta)|$.

[28] The corresponding expressions hold for $\Xi(T,S;W)$.

We consider first the terms with $i = 1, 2, 3$ and in particular the first term of 6.3.28,

$$\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W) \cdot \Theta^i(T, S; W) \ .$$

Proceeding as in the case of $\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_T W) \cdot \Theta^i(T, W)$, see 6.2.6, we have

$$\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W) \cdot \Theta^i(S, \hat{\mathcal{L}}_T W)$$
$$\leq c \int_{u_0}^{u} du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6 |\alpha(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)|^2 \right)^{\frac{1}{2}} \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6 |\Theta^i(S, \hat{\mathcal{L}}_T W)|^2 \right)^{\frac{1}{2}}$$

The first integral factor is estimated by, see 3.5.1,

$$\sup_{\mathcal{K}} \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6 |\alpha(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)|^2 \right)^{\frac{1}{2}} \leq c \mathcal{Q}_2^{\frac{1}{2}} \tag{6.3.29}$$

To estimate

$$\int_{u_0}^{u} du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6 |\Theta^i(S, \hat{\mathcal{L}}_T W)|^2 \right)^{\frac{1}{2}}$$

we compare it with the integral

$$\int_{u_0}^{u} du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6 |\Theta^i(T, W)|^2 \right)^{\frac{1}{2}}$$

which has been estimated in subsection 6.2.1 [29]. Proceeding in the same way we obtain

$$\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W) \cdot \Theta^1(T, S; W) \leq c\epsilon_0 \mathcal{Q}_2^{\frac{1}{2}} (\mathcal{Q}_1 + \mathcal{Q}_2)^{\frac{1}{2}} \leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$

Operating in an analogous way for the second term we derive

$$\int_{V_{(u,\underline{u})}} \tau_+^6 \beta(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W) \cdot \Xi^1(S, T; W) \leq c\epsilon_0 \mathcal{Q}_2^{\frac{1}{2}} (\mathcal{Q}_1 + \mathcal{Q}_2)^{\frac{1}{2}} \leq c\epsilon_0 \mathcal{Q}_{\mathcal{K}}$$

[29] The main difference is that the deformation tensors or their derivatives refer now to the S vector field instead of T and that the Riemann null components or their derivatives are relative to $\hat{\mathcal{L}}_T W$ instead of W. As the estimates for the deformation tensor relative to $S : {}^{(S)}\hat{\pi}$ are worst, relatively to those for the deformation tensor ${}^{(T)}\hat{\pi}$, by a factor r or τ_- and, at the same time, the estimates for the null Riemann components of $\hat{\mathcal{L}}_T W$ are better than those relative to the null components of W by a factor r, we easily conclude that the the term $\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W) \cdot \Theta^1(T, S; W)$ is under control.

In the case of the currents $J^2(T,S;W)$ and $J^3(T,S;W)$, we have to proceed somewhat differently than in the corresponding proof of subsection 6.2.1. This is due to the fact that we have to deal with the null components of $\hat{\mathcal{L}}_T W$ which cannot be estimated in the sup norm. We shall sketch here the estimates for $\Theta^{(2)}(S,T;W)$, those for $\Theta^{(3)}(S,T;W)$ are obtained in the same way. From

$$\begin{aligned}\Theta^{(2)}(S,T;W) &= \mathrm{Qr}\left[{}^{(S)}p_3\,;\,\alpha(\hat{\mathcal{L}}_T W)\right] + \mathrm{Qr}\left[{}^{(S)}\not{p}\,;\,\beta(\hat{\mathcal{L}}_T W)\right]\\ &+ \mathrm{Qr}\left[{}^{(S)}p_4\,;\,(\rho,\sigma)(\hat{\mathcal{L}}_T W)\right]\end{aligned}$$

the following integrals have to be estimated

$$\begin{aligned}&\int_{V_{(u,\underline{u})}} \tau_+^6\alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)({}^{(S)}p_3)\alpha(\hat{\mathcal{L}}_T W)\\ &\int_{V_{(u,\underline{u})}} \tau_+^6\alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)({}^{(S)}\not{p})\beta(\hat{\mathcal{L}}_T W)\\ &\int_{V_{(u,\underline{u})}} \tau_+^6\alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)({}^{(S)}p_4)(\rho,\sigma)(\hat{\mathcal{L}}_T W)\end{aligned} \tag{6.3.30}$$

To estimate the first integral of 6.3.30 we write

$$\begin{aligned}&\int_{V_{(u,\underline{u})}} \tau_+^6\alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)^{(S)}p_3\,\alpha(\hat{\mathcal{L}}_T W)\\ &\le \int\frac{du'}{u'}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|\alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)|^2\right)^{\frac12}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}|\tau_-{}^{(S)}p_3|^2\tau_+^6|\alpha(\hat{\mathcal{L}}_T W)|^2\right)^{\frac12}\\ &\le\left(\sup_{\mathcal{K}}\int_{C(u';[\underline{u}_0,\underline{u}])}Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4)\right)^{\frac12}\int\frac{du'}{u'}\left[\int_{\underline{u}_0}^{\underline{u}}d\underline{u}'\left(\int_{S(u',\underline{u}')}|\tau_+^{\frac12}\tau_-{}^{(S)}p_3|^4\right)^{\frac12}\cdot\right.\\ &\left.\cdot\left(\int_{S(u',\underline{u}')}\tau_+^{10}|\alpha(\hat{\mathcal{L}}_T W)|^4\right)^{\frac12}\right]^{\frac12}\\ &\le c\mathcal{Q}_2^{\frac12}\int\frac{du'}{u'}\left[\int_{\underline{u}_0}^{\underline{u}}d\underline{u}'|r^{1-\frac{2}{p}}\tau_-{}^{(S)}p_3|^2_{p=4,S}\left(\int_{S(u',\underline{u}')}\tau_+^{10}|\alpha(\hat{\mathcal{L}}_T W)|^4\right)^{\frac12}\right]^{\frac12}\end{aligned} \tag{6.3.31}$$

Using Proposition 6.1.7, see 6.1.50, and the estimate, proved in Chapter 5,

$$\left(\int_{S(u',\underline{u}')}\tau_+^{16}|\alpha(\hat{\mathcal{L}}_T W)|^4\right)^{\frac14} \le c\left(\int_{C(u')} r^6|\alpha(\hat{\mathcal{L}}_T W)|^2 + r^8|\nabla\!\!\!/\,\alpha(\hat{\mathcal{L}}_T W)|^2\right.$$

$$+ \quad r^8|\not{D}_4\alpha(\hat{\mathcal{L}}_T W)|^2 \Big)^{\frac{1}{2}} \leq c\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}} \tag{6.3.32}$$

we have

$$\begin{aligned} &\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)({}^{(S)}p_3)\alpha(\hat{\mathcal{L}}_T W) \\ &\leq c\left(\sup|r^{1-\frac{2}{p}}\tau_-{}^{(S)}p_3|_{p=4,S}\right)\mathcal{Q}_2^{\frac{1}{2}}\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\int\frac{du'}{u'}\left(\int_{\underline{u}_0}^{\underline{u}} d\underline{u}'\frac{1}{\underline{u}'^3}\right)^{\frac{1}{2}} \\ &\leq c\left(\sup|r^{1-\frac{2}{p}}\tau_-{}^{(S)}p_3|_{p=4,S}\right)\mathcal{Q}_{\mathcal{K}} \leq c\epsilon_0\mathcal{Q}_{\mathcal{K}} \end{aligned} \tag{6.3.33}$$

completing the estimate. The remaining integrals in 6.3.30 are estimated in the same way. We are left with the estimate of the part associated to $J^0(T,S;W) = \hat{\mathcal{L}}_S J(T;W)$. From

$$\Theta^0(T,S;W) = \frac{1}{2}\left(\hat{\mathcal{L}}_S\Theta^1(T;W) + \hat{\mathcal{L}}_S\Theta^2(T;W) + \hat{\mathcal{L}}_S\Theta^3(T;W)\right)$$

we have to control the integrals

$$\begin{aligned} &\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)\hat{\mathcal{L}}_S\Theta^1(T;W) \\ &\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)\hat{\mathcal{L}}_S\Theta^2(T;W) \\ &\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)\hat{\mathcal{L}}_S\Theta^3(T;W) \end{aligned} \tag{6.3.34}$$

We write, using the estimates proven in Chapter 6,

$$\begin{aligned} &\int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)\hat{\mathcal{L}}_S\Theta^i(T;W) \\ &\leq \int du'\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|\alpha(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)|^2\right)^{\frac{1}{2}}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|\hat{\mathcal{L}}_S\Theta^i(T;W)|^2\right)^{\frac{1}{2}} \\ &\leq c\mathcal{Q}_2^{\frac{1}{2}}\int du'\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|\hat{\mathcal{L}}_S\Theta^i(T;W)|^2\right)^{\frac{1}{2}} \end{aligned} \tag{6.3.35}$$

Recall that $\Theta^i(T;W)$ consists of a sum of terms which are products between a deformation tensor, or its first derivatives, and a null Riemann components, or its first derivatives, see eqs. 6.1.8...6.1.23. Let us consider first

$$\begin{aligned} \Theta(J^1) &= \mathrm{Qr}\left[{}^{(T)}\underline{\mathbf{m}}\,;\,\not{\nabla}\alpha\right] + \mathrm{Qr}\left[{}^{(T)}\underline{\mathbf{n}}\,;\,\alpha_4\right] + \mathrm{Qr}\left[{}^{(T)}\mathbf{j}\,;\,\alpha_3\right] \\ &+ \mathrm{Qr}\left[{}^{(T)}\mathbf{i}\,;\,\not{\nabla}\beta\right] + \mathrm{Qr}\left[{}^{(T)}\underline{\mathbf{m}}\,;\,\beta_4\right] + \mathrm{Qr}\left[{}^{(T)}\mathbf{m}\,;\,\beta_3\right] \\ &+ \ \ldots \end{aligned}$$

The Lie derivative $\hat{\mathcal{L}}_S$ can operate on both factors. When it operates on the null Riemann components it adds a derivative [30]. Therefore in $\hat{\mathcal{L}}_S\Theta(J^1)$ the null Riemann components can appear derived twice, but multiplied by non derived deformation tensors. In this case the estimates are done taking the sup norms for the deformation tensors and the $L^2(C)$ or $L^2(\underline{C})$ norms for the twice derived Riemann components.
In the case of

$$\Theta(J^2) = \mathrm{Qr}\left[{}^{(X)}p_3\,;\,\alpha\right] + \mathrm{Qr}\left[{}^{(X)}\not{p};\,\beta\right] + \mathrm{Qr}\left[{}^{(X)}p_4\,;\,(\rho,\sigma)\right]$$

when $\hat{\mathcal{L}}_S$ operate on the Riemann components we estimate the integrals taking the $L^p(S)$ norms both for the first derivatives of the Riemann components and for the ${}^{(T)}p_3, {}^{(T)}p_3, {}^{(T)}\not{p}$ terms which depend on the first derivatives of the deformation tensors. A similar procedure applies in the case of $\Theta(J^3)$. Therefore the estimate of this error term, when $\hat{\mathcal{L}}_S$ operates on the Riemann components, does not produce complications as the asymptotic behaviours are not changed and the derivatives involved are only the zero and first derivatives for the deformation tensors and at most the first derivatives for the Riemann components.

We consider now the case where $\hat{\mathcal{L}}_S$ operates on the deformation tensors. This is simple for the $\Theta(J^1)$ part as, in this case, the only effect is that the deformation tensor is substituted by its first derivatives which depend on the first derivatives of the connection coefficients we already know how to control.
When $\hat{\mathcal{L}}_S$ operates on the deformation tensor parts of $\Theta(J^2)$ and $\Theta(J^3)$ the situation is more delicate. In this case the Riemann tensor components are not derived and can be estimated with the sup norms, but $\hat{\mathcal{L}}_S$ operating on ${}^{(T)}p_3, {}^{(T)}p_3, {}^{(T)}\not{p}$ produce terms which depend on the second derivatives of the Ricci null coefficients and this requires a careful control of their norms. The same happens if $\hat{\mathcal{L}}_S$ operates on the ${}^{(T)}q$ factors of $\Theta(J^3)$.
Let us consider, as an example, the integral of 6.3.35, for $i = 2$. This amounts to control the terms

$$\begin{aligned}
&\int du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^6 |\hat{\mathcal{L}}_S{}^{(T)}p_3|^2 |\alpha(W)|^2\right)^{\frac{1}{2}}\\
&\int du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^6 |\hat{\mathcal{L}}_S{}^{(T)}\not{p}|^2 |\beta(W)|^2\right)^{\frac{1}{2}} \qquad (6.3.36)
\end{aligned}$$

[30] Without changing the asymptotic behaviour of the component.

$$\int du'\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|\hat{\mathcal{L}}_S{}^{(T)}p_4|^2|(\rho,\sigma)(W)|^2\right)^{\frac{1}{2}}$$

To estimate the first term we write

$$\begin{aligned}
&\int du'\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|\hat{\mathcal{L}}_S{}^{(T)}p_3|^2|\alpha(W)|^2\right)^{\frac{1}{2}} &(6.3.37)\\
&\leq c\left(\sup|\tau_+^{\frac{7}{2}}\alpha(W)|\right)\int du'\frac{1}{\tau_-^2}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\frac{1}{\tau_+}|\tau_-^2\hat{\mathcal{L}}_S{}^{(T)}p_3|^2\right)^{\frac{1}{2}}\\
&\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}
\end{aligned}$$

with the last inequality using Proposition 6.1.6. The estimates of the remaining two integrals is easier due to the better asymptotic behaviour of ${}^{(T)}\not{p}$ and ${}^{(T)}p_4$ compared to ${}^{(T)}p_3$. In this case it is enough to use Proposition 6.1.5. In fact we have, for the last integral,

$$\begin{aligned}
&\int du'\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\tau_+^6|\hat{\mathcal{L}}_S{}^{(T)}p_4|^2|(\rho(W),\sigma(W))|^2\right)^{\frac{1}{2}} &(6.3.38)\\
&\leq c\left(\sup|\tau_+^3(\rho(W),\sigma(W))|\right)\int du'\frac{1}{\tau_-}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\frac{1}{\tau_+^2}|\tau_+\tau_-\hat{\mathcal{L}}_S{}^{(T)}p_4|^2\right)^{\frac{1}{2}}\\
&\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}\int du'\frac{1}{\tau_-}\left(\int d\underline{u}'\frac{1}{\tau_+^2}\right)^{\frac{1}{2}}\leq c\epsilon_0\mathcal{Q}_{\mathcal{K}}^{\frac{1}{2}}
\end{aligned}$$

6.3.8 Estimate of $\int_{V_{(u,\underline{u})}}|Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)|$

The estimate of this integral is done exactly in the same way as the estimate of $\int_{V_{(u,\underline{u})}}|Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)|$ with the obvious substitutions of $\mathcal{Q}_1$ with $\mathcal{Q}_2$.

6.4 Appendix to Chapter 6

6.4.1 The third order derivatives of the connection coefficients

From the evolution equations, in Section 4.6, of ${}^{(i)}F, {}^{(i)}H_{ab}, {}^{(i)}Z_a$ and from the explicit expressions of the rotation deformation tensors, see 4.6.10, 4.6.11, it follows that ${}^{(O)}\underline{\mathbf{m}}_a$ depends on $\nabla\!\!\!/\eta$ and $\nabla\!\!\!/\underline{\eta}$, ${}^{(O)}\mathbf{i}_{ab}$ depends on $\nabla\!\!\!/\chi$ and ${}^{(O)}\mathbf{j}$ depends on $\nabla\!\!\!/\log\Omega$. From the expression, see 6.1.24,

$$\begin{aligned}{}^{(O)}p_3 &= \operatorname{d\!\!\!/iv}{}^{(O)}\underline{\mathbf{m}} - \frac{1}{2}\mathbf{D\!\!\!/}_3{}^{(O)}\mathbf{j} + (2\underline{\eta}+\eta-\zeta)\cdot{}^{(O)}\underline{\mathbf{m}} - \hat{\chi}\cdot{}^{(O)}\mathbf{i} \\ &\quad -\frac{1}{2}tr\chi(tr{}^{(O)}\mathbf{i} + {}^{(O)}\mathbf{j})\end{aligned}$$

it follows that ${}^{(O)}p_3$ depends on the second derivatives of the connection coefficients, $\nabla\!\!\!/^2\eta$ and $\nabla\!\!\!/^2\underline{\eta}$ through $\operatorname{d\!\!\!/iv}{}^{(O)}\underline{\mathbf{m}}$ [31] which at their turn depend on the first derivatives of the Riemann tensor [32]. Therefore it follows immediately that $\hat{\mathcal{L}}_O{}^{(O)}p_3$ will depend on $\nabla\!\!\!/^3\eta$, $\nabla\!\!\!/^3\underline{\eta}$ and through it on the second derivatives of the Riemann tensor. From

$$\begin{aligned}{}^{(O)}p\!\!\!/ &= \nabla\!\!\!/_c{}^{(O)}\mathbf{i} - \frac{1}{2}\mathbf{D\!\!\!/}_4{}^{(O)}\underline{\mathbf{m}} - \frac{1}{2}(\mathbf{D}_4\log\Omega){}^{(O)}\underline{\mathbf{m}} + \frac{1}{2}{}^{(O)}\mathbf{j}(\eta+\underline{\eta}) \\ &\quad + {}^{(O)}\mathbf{i}\cdot(\eta+\underline{\eta}) - \frac{3}{4}tr\chi{}^{(O)}\underline{\mathbf{m}} - \frac{1}{2}\hat{\chi}\cdot{}^{(O)}\underline{\mathbf{m}}\end{aligned}$$

we see that ${}^{(O)}p\!\!\!/$ depends, through the term $\nabla\!\!\!/_c{}^{(O)}\mathbf{i}$, on $\nabla\!\!\!/^2\chi$ [33]. This implies that $\hat{\mathcal{L}}_O{}^{(O)}p\!\!\!/$ will depend on the third derivative $\nabla\!\!\!/^3\chi$ and through it on the second derivatives of the Riemann tensor. From

$${}^{(O)}p_4 = -\frac{1}{2}\mathbf{D\!\!\!/}_4{}^{(O)}\mathbf{j} - \underline{\hat{\chi}}\cdot{}^{(O)}\mathbf{i} - \frac{1}{2}tr\underline{\chi}(tr{}^{(O)}\mathbf{i} + {}^{(O)}\mathbf{j}) - \frac{1}{2}tr\chi{}^{(O)}\underline{\mathbf{n}} - (\mathbf{D}_4\log\Omega){}^{(O)}\underline{\mathbf{n}}$$

we see that ${}^{(O)}p_4$ does not depend on second order derivatives of connection coefficients.

An analogous argument holds for the derivatives of the various components of ${}^{(O)}q$. From the explicit expressions 6.1.27 and 6.1.28, in the case $X = O$, we obtain:

a) $\Lambda({}^{(O)}q) = -\frac{1}{4}\mathbf{D}_4{}^{(O)}\mathbf{j} + \frac{2}{3}{}^{(O)}p_4$ does not depend on second derivatives of connection coefficients.

---

[31] As $\nabla\!\!\!/Z$ depends also on $\nabla\!\!\!/H$, it follows that through the dependance on $\operatorname{d\!\!\!/iv}{}^{(O)}\underline{\mathbf{m}}$ we have also a dependance on $\nabla\!\!\!/^2\chi$ and finally $\hat{\mathcal{L}}_O{}^{(O)}p_3$ depend on $\nabla\!\!\!/^3(\eta,\underline{\eta},\chi)$.

[32] In fact through $\mathbf{D\!\!\!/}_3{}^{(O)}\mathbf{j}$ there is also a dependence on $\nabla\!\!\!/\underline{\omega}$. This is, nevertheless, not harmful as, when we consider the tangential derivatives of ${}^{(O)}p_3$, $\nabla\!\!\!/^3\eta$ and $\nabla\!\!\!/^3\underline{\eta}$ will depend on the second derivatives of the Riemann tensor, while $\nabla\!\!\!/^2\underline{\omega}$ still depends only on the first derivative as follows from Proposition 4.4.1.

[33] Apparently, through the term $\mathbf{D\!\!\!/}_4{}^{(O)}\underline{\mathbf{m}}$, it seems that ${}^{(O)}p\!\!\!/$ depends also on $\nabla\!\!\!/^2\omega$. This is not true as $\mathbf{D\!\!\!/}_4{}^{(O)}\underline{\mathbf{m}}$ is proportional to $\frac{d}{d\underline{u}}(\Omega{}^{(i)}Z_a)$ which, see Proposition 4.6.2, is proportional to $\nabla\!\!\!/_a{}^{(i)}F$, ${}^{(i)}H_{ab}$ and $L_{{}^{(i)}O}(\zeta+\eta)$ and therefore depends on $\nabla\!\!\!/\eta$ $\nabla\!\!\!/\underline{\eta}$ and $\nabla\!\!\!/\chi$, all first derivatives of the connection coefficients.

b) $K({}^{(O)}q)_{ab} = -\frac{1}{2}\left(\hat{\chi}_{ac}{}^{(O)}\mathbf{i}_{cb} - \hat{\chi}_{bc}{}^{(O)}\mathbf{i}_{ca}\right)$ does not depend on second derivatives of connection coefficients.

c)
$$I({}^{(O)}q)_a = \frac{1}{2}\mathbf{D\!\!\!/}_4{}^{(O)}\underline{\mathbf{m}}_a - \frac{1}{2}\nabla\!\!\!\!/_a{}^{(O)}\mathbf{j} - \frac{1}{2}(\mathbf{D}_4\log\Omega)^{(O)}\underline{\mathbf{m}}_a + \frac{1}{4}tr\chi^{(O)}\underline{\mathbf{m}}_a + \frac{1}{2}\hat{\chi}_{ac}{}^{(O)}\underline{\mathbf{m}}_c - \frac{1}{2}\underline{\eta}_c{}^{(O)}\mathbf{i}_{ca} + \frac{3}{2}{}^{(O)}p\!\!\!/_a$$

depends, through ${}^{(O)}p\!\!\!/$, on the second derivatives of Ricci coefficient $\nabla\!\!\!\!/^2\chi$.

d)
$$\Theta({}^{(O)}q)_{ab} = 2\left(\mathbf{D\!\!\!/}_4{}^{(O)}\mathbf{i}_{ab} - \frac{1}{2}\delta_{ab}\mathrm{tr}(\mathbf{D\!\!\!/}_4{}^{(O)}\mathbf{i})\right) + \mathrm{tr}\chi^{(\hat{O})}\mathbf{i}_{ab} + \hat{\chi}_{ab}\mathrm{tr}^{(O)}\mathbf{i} + \hat{\chi}_{ab}{}^{(O)}\mathbf{j}$$

does not depend on the second derivatives of the connection coefficients [34].

e) $\underline{\Lambda}({}^{(O)}q) = -\underline{\eta}\cdot{}^{(O)}\underline{\mathbf{m}} - \frac{1}{4}\left(\mathbf{D}_3{}^{(O)}\mathbf{j} - 2\eta\cdot{}^{(O)}\underline{\mathbf{m}}\right) + \frac{2}{3}{}^{(O)}p_3$ depends, through ${}^{(O)}p_3$, on $\nabla\!\!\!\!/^2\eta$ and $\nabla\!\!\!\!/^2\underline{\eta}$ [35].

f)
$$\underline{K}({}^{(O)}q)_{ab} = \frac{1}{2}\left(\nabla\!\!\!\!/_a{}^{(O)}\underline{\mathbf{m}}_b - \nabla\!\!\!\!/_b{}^{(O)}\underline{\mathbf{m}}_a\right) - \frac{1}{2}\left(\zeta_a{}^{(O)}\underline{\mathbf{m}}_b - \zeta_b{}^{(O)}\underline{\mathbf{m}}_a\right) - \frac{1}{2}\left(\underline{\hat{\chi}}_{ac}{}^{(O)}\mathbf{i}_{cb} - \underline{\hat{\chi}}_{bc}{}^{(O)}\mathbf{i}_{ca}\right)$$

depends, through $\nabla\!\!\!\!/_a{}^{(O)}\underline{\mathbf{m}}$, on $\nabla\!\!\!\!/^2\eta$ and $\nabla\!\!\!\!/^2\underline{\eta}$.

h) $\underline{\Xi}({}^{(O)}q)_a = \frac{1}{2}\mathbf{D\!\!\!/}_3{}^{(O)}\underline{\mathbf{m}}_a - \frac{1}{2}(\mathbf{D}_3\log\Omega)^{(O)}\underline{\mathbf{m}}_a + \frac{1}{2}tr\underline{\chi}^{(O)}\underline{\mathbf{m}}_a + \underline{\hat{\chi}}_{ac}{}^{(O)}\underline{\mathbf{m}}_c$
depends, through $\mathbf{D\!\!\!/}_3{}^{(O)}\underline{\mathbf{m}}$, on $\nabla\!\!\!\!/^2\underline{\omega}$.

i) $\underline{I}({}^{(O)}q)_a = -\frac{1}{2}\nabla\!\!\!\!/_a{}^{(X)}\mathbf{j} + \frac{1}{4}tr\chi^{(O)}\underline{\mathbf{m}}_a + \frac{1}{2}\hat{\chi}_{ac}{}^{(O)}\underline{\mathbf{m}}_c - \frac{1}{2}\eta_c{}^{(O)}\mathbf{i}_{ca} + \frac{3}{2}{}^{(O)}p\!\!\!/_a$
depends, through ${}^{(O)}p\!\!\!/$, on $\nabla\!\!\!\!/^2\chi$.

l)
$$\underline{\Theta}({}^{(O)}q)_{ab} = 2\left(\mathbf{D\!\!\!/}_3{}^{(O)}\mathbf{i}_{ab} - \frac{1}{2}\delta_{ab}tr(\mathbf{D\!\!\!/}_3{}^{(O)}\mathbf{i})\right) - \left(\nabla\!\!\!\!/_a{}^{(O)}\underline{\mathbf{m}}_b + \nabla\!\!\!\!/_b{}^{(O)}\underline{\mathbf{m}}_a - \delta_{ab}\nabla\!\!\!\!/_c{}^{(O)}\underline{\mathbf{m}}_c\right)$$
$$-2\left(\eta_a{}^{(O)}\underline{\mathbf{m}}_b + \eta_b{}^{(O)}\underline{\mathbf{m}}_a - \delta_{ab}\eta_c{}^{(O)}\underline{\mathbf{m}}_c\right) + \left(\zeta_a{}^{(O)}\underline{\mathbf{m}}_b + \zeta_b{}^{(O)}\underline{\mathbf{m}}_a - \delta_{ab}\zeta_c{}^{(O)}\underline{\mathbf{m}}_c\right)$$
$$+ tr\underline{\chi}^{(O)}\mathbf{i}_{ab} + \underline{\hat{\chi}}_{ab}tr^{(O)}\mathbf{i} + \underline{\hat{\chi}}_{ab}{}^{(O)}\mathbf{j}$$

depends, through $\nabla\!\!\!\!/^{(O)}\underline{\mathbf{m}}$, on $\nabla\!\!\!\!/^2\eta$ and $\nabla\!\!\!\!/^2\underline{\eta}$.

[34] The term $\mathbf{D\!\!\!/}_4{}^{(O)}\mathbf{i}$ can be expressed, using the structure equations, in terms of first derivatives of connection coefficients.

[35] The term $\mathbf{D}_3{}^{(O)}\mathbf{j}$ gives a dependence on $\nabla\!\!\!\!/\underline{\omega}$ which, as discussed before, is harmless.

Chapter 7

The initial hypersurface and the last slice

7.1 Initial hypersurface foliations

7.1.1 Some general properties of a foliation of Σ_0

Let the function $w(p)$ define a foliation on Σ_0. Its leaves are

$$S_0(\nu) = \{p \in \Sigma_0 | w(p) = \nu\}$$

We define on $\Sigma_0 \backslash K$ a moving orthonormal frame $\{\tilde{N}, e_A\}$[1] adapted to this foliation, where $A \in \{1,2\}$ and $\tilde{N}^i = \frac{1}{|\partial w|} g^{ij}\partial_j w$, is the unit vector field, defined on Σ_0, normal to each $S_0(\nu)$. The metric on Σ_0 can be written, in adapted coordinates $\{w, \phi^a\}$, as

$$g(\cdot,\cdot) = a^2 dw^2 + \gamma_{ab} d\phi^a d\phi^b \tag{7.1.1}$$

and, in these coordinates, $\tilde{N} = \frac{1}{a}\frac{\partial}{\partial w}$, $a^{-2} = |\partial w|^2$.

The second fundamental form associated to the leaves of this foliation is

$$\theta_{ij} = \Pi_i^l \Pi_j^s \nabla_l \tilde{N}_s = \nabla_i \tilde{N}_j - \tilde{N}_i \nabla_{\tilde{N}} \tilde{N}_j$$

where $\Pi_s^l = (\delta_s^l - \tilde{N}^l \tilde{N}_s)$, is the projection over TS_0 and ∇ the covariant derivative relative to Σ_0. A simple computation shows that, in adapted coordinates,

$$\theta_{ab} = \frac{1}{2a}\partial_w \gamma_{ab} \ .$$

[1]During this chapter we use, differently from the rest of the book, the capital letters $A, B, C, ...$ for an orthonormal basis tangent to S, $e_A = e_A^b \frac{\partial}{\partial \phi_b}$. We use the small ones $a, b, c, ...$, as coordinate indices.

Moreover the adapted moving frame satisfies the following equations, where $\nabla\!\!\!/_A \equiv \nabla\!\!\!/_{e_A}$,

$$\begin{aligned}
&\nabla_{\tilde{N}}\tilde{N} = (a^{-1}\nabla\!\!\!/_A a)e_A\\
&\nabla_A\tilde{N} = \theta_{AB}e_B\\
&\nabla_{\tilde{N}}e_A = \nabla\!\!\!/_{\tilde{N}}e_A + (a^{-1}\nabla\!\!\!/_A a)\tilde{N}\\
&\nabla_B e_A = \nabla\!\!\!/_B e_A - \theta_{AB}\tilde{N}
\end{aligned} \tag{7.1.2}$$

7.1.2 The structure equations on Σ_0

We start writing the Gauss and the Codazzi-Mainardi equations [2]

$$\begin{aligned}
{}^{(3)}R_{abcd} &= {}^{(2)}R_{abcd} - \theta_{ac}\theta_{bd} + \theta_{ad}\theta_{bc}\\
{}^{(3)}R_{wabc} &= -a\left(\nabla\!\!\!/_b\theta_{ca} - \nabla\!\!\!/_c\theta_{ab}\right)
\end{aligned} \tag{7.1.3}$$

Contracting the Gauss equation, with respect to the indices b, d, we obtain[3] the

Contracted Gauss equation

$${}^{(3)}R_{ac} = {}^{(2)}R_{ac} + {}^{(3)}R^{w}{}_{awc} - (\mathrm{tr}\theta)\theta_{ac} + \theta_{ad}\theta^{d}{}_{c} \tag{7.1.4}$$

where ${}^{(3)}R_{ac}$ is the Ricci tensor of Σ_0 and ${}^{(2)}R_{ac}$ the one associated to $S_0(\nu)$. The explicit computation of ${}^{(3)}R^{w}{}_{awc}$ gives

$${}^{(3)}R^{w}{}_{awc} = -a^{-1}\nabla\!\!\!/_a\nabla\!\!\!/_c a - \nabla\!\!\!/_{\tilde{N}}\theta_{ac} - \theta_{ad}\theta^{d}{}_{c} \tag{7.1.5}$$

which, substituted in 7.1.4, gives

$${}^{(3)}R_{ac} = {}^{(2)}R_{ac} - a^{-1}\nabla\!\!\!/_a\nabla\!\!\!/_c a - \nabla\!\!\!/_{\tilde{N}}\theta_{ac} - \mathrm{tr}\theta\theta_{ac} \tag{7.1.6}$$

Contracting 7.1.5 with respect to the indices a, c, we obtain

$${}^{(3)}R_{\tilde{N}\tilde{N}} = -a^{-1}\triangle\!\!\!/ a - \nabla\!\!\!/_{\tilde{N}}\mathrm{tr}\theta - |\theta|^2 \tag{7.1.7}$$

[2] Here the index w denotes the corresponding coordinate w and a, b are associated to ϕ^a, ϕ^b. In arbitrary coordinates equations 7.1.3 become

$$\begin{aligned}
{}^{(3)}R_{rspq}\Pi_i^r\Pi_j^s\Pi_k^p\Pi_l^q &= {}^{(2)}R_{ijkl} - (\theta_{ik}\theta_{jl} - \theta_{il}\theta_{jk})\\
{}^{(3)}R_{srpq}\tilde{N}^s\Pi_k^r\Pi_i^p\Pi_j^q &= -\left(\nabla\!\!\!/_i\theta_{jk} - \nabla\!\!\!/_j\theta_{ki}\right)
\end{aligned}$$

[3] In arbitrary coordinates it has the form

$${}^{(3)}R_{rp}\Pi_i^r\Pi_k^p = {}^{(2)}R_{ik} + {}^{(3)}R_{iskq}\tilde{N}^s\tilde{N}^q - (\mathrm{tr}\theta)\theta_{ik} + \theta_{is}\theta^{s}{}_{k}$$

Moreover contracting 7.1.4 with respect to the indices a, c, we obtain

$$ {}^{(3)}R - 2{}^{(3)}R_{\tilde{N}\tilde{N}} = {}^{(2)}R - (\mathrm{tr}\theta)^2 + |\theta|^2 \tag{7.1.8} $$

where ${}^{(3)}R$ is the scalar curvature of Σ_0 and ${}^{(2)}R$ the one of S_0. Finally contracting the Codazzi-Mainardi equation, with respect to the a, c indices, we obtain the

Contracted Codazzi-Mainardi equation:

$$ {}^{(3)}R_{\tilde{N}b} = -\nabla\!\!\!/_b \mathrm{tr}\theta + \nabla\!\!\!/^c \theta_{cb} \tag{7.1.9} $$

Decomposing θ in its traceless and trace parts, $\theta = \hat{\theta} + \frac{1}{2}\gamma\mathrm{tr}\theta$, and using equations 7.1.6, 7.1.7 and 7.1.8 we obtain the following evolution equations for $\mathrm{tr}\theta$ and $\hat{\theta}$,

Evolution equations:

$$ \begin{aligned} \nabla\!\!\!/_{\tilde{N}}\hat{\theta}_{ac} + \mathrm{tr}\theta\ \hat{\theta}_{ac} &= -a^{-1}\widehat{\nabla\!\!\!/_a\nabla\!\!\!/_c} a - \left[{}^{(3)}R_{ac} + 2^{-1}\gamma_{ac}({}^{(3)}R_{\tilde{N}\tilde{N}} - {}^{(3)}R)\right] \\ \nabla_{\tilde{N}}\mathrm{tr}\theta + \frac{1}{2}\mathrm{tr}\theta^2 &= -a^{-1}\Delta\!\!\!/ a - |\hat{\theta}|^2 - {}^{(3)}R_{\tilde{N}\tilde{N}} \end{aligned} \tag{7.1.10} $$

where $\widehat{\nabla\!\!\!/_a\nabla\!\!\!/_c} = \nabla\!\!\!/_a\nabla\!\!\!/_c - \frac{1}{2}\gamma_{ac}\Delta\!\!\!/$.

Definition 7.1.1 *We denote $\widehat{{}^{(3)}R}$ the traceless part of the Ricci tensor ${}^{(3)}R$, with respect to the metric of Σ_0, g_{ij},*

$$ \widehat{{}^{(3)}R} = {}^{(3)}R - \frac{1}{3}g_{ij}{}^{(3)}R \tag{7.1.11} $$

where ${}^{(3)}R$ is the scalar curvature of Σ_0. The various components of the traceless part of the Ricci tensor, in the frame adapted to the foliation, are defined as [4]

$$ S_{AB} \equiv \widehat{{}^{(3)}R}_{AB}\ ,\ P_A \equiv \widehat{{}^{(3)}R}_{A\tilde{N}}\ ,\ Q \equiv \widehat{{}^{(3)}R}_{\tilde{N}\tilde{N}} \tag{7.1.12} $$

It follows easily that the traceless part of S, $\hat{S}_{AB} = S_{AB} - \frac{1}{2}\delta_{AB}\mathrm{tr}S$, satisfies

$$ \hat{S}_{AB} = {}^{(3)}R_{AB} + \frac{1}{2}\delta_{AB}\left({}^{(3)}R_{\tilde{N}\tilde{N}} - {}^{(3)}R\right) \tag{7.1.13} $$

which allows us to rewrite the evolution equation for $\hat{\theta}$, 7.1.10, in the following way

$$ \nabla\!\!\!/_{\tilde{N}}\hat{\theta}_{ac} + \mathrm{tr}\theta\ \hat{\theta}_{ac} = -a^{-1}\widehat{\nabla\!\!\!/_a\nabla\!\!\!/_c} a - \hat{S}_{ac} \tag{7.1.14} $$

[4] In arbitrary coordinates we have $S_{ij} = \Pi_i^l\Pi_j^t\widehat{{}^{(3)}R}_{lt}\ ,\ P_i = \Pi_i^l\widehat{{}^{(3)}R}_{l\tilde{N}}\ ,\ Q = \widehat{{}^{(3)}R}_{\tilde{N}\tilde{N}}$.

The second Bianchi identities and the definition [5]

$$B_{ij} = (\text{curl}\ \widehat{^{(3)}R})_{ij} \tag{7.1.15}$$

imply that the components of $\widehat{^{(3)}R}$, S, P, Q, satisfy the following equations, see [Ch-Kl], Chapter 5, assuming the adapted frame with $a = 1$ and Fermi trasported [6].

$$\begin{aligned}
\text{d\!\!/iv}\, P &= \frac{1}{6}\nabla_N R - \nabla_N Q - \frac{3}{2}\text{tr}\theta Q + \hat{S}\cdot\hat{\theta} \\
\text{cu\!\!/rl}\, P &= B_{NN} + \hat{\theta}\wedge\hat{S} \\
\nabla\!\!\!/_N P + \text{tr}\theta P &= \frac{1}{12}\nabla\!\!\!/_a R + {}^*B\!\!\!/_N + \nabla\!\!\!/ Q - 2\hat{\theta}\cdot P \\
\text{d\!\!/iv}\,\hat{S} &= \left(\frac{1}{12}\nabla\!\!\!/ R - {}^*B\!\!\!/_N\right) - \frac{1}{12}\nabla\!\!\!/ Q + (\hat{\theta}\cdot P) - \frac{1}{2}\text{tr}\theta P \\
\nabla\!\!\!/_N\hat{S} + \frac{1}{2}\text{tr}\theta\hat{S} &= {}^*B\!\!\!/ + \frac{1}{2}\nabla\!\!\!/\hat{\otimes}P + \frac{3}{2}\hat{\theta}Q
\end{aligned} \tag{7.1.16}$$

where $\nabla\!\!\!/_N X$ is the projection of $\nabla_N X$ on TS_0, $B\!\!\!/$ is the S-tangent symmetric two tensor $B\!\!\!/_{AB} = B_{AB}$ and ${}^*B\!\!\!/_{ab} = \epsilon_a{}^c\epsilon_b{}^d B\!\!\!/_{cd}$.

7.1.3 The construction of the background foliation of Σ_0

We start with the following

Theorem 7.1.1 *Assume that, given $\varepsilon > 0$, the initial data are such that $J_K(\Sigma_0, g, k) \le \varepsilon^2$ is bounded. There exists a global geodesic foliation on $\Sigma_0\backslash K$ with lapse function $a = 1$, such that the following inequalities [7] hold,*

$$\begin{aligned}
&\inf_{\Sigma_0\backslash K} r\,tr\theta \le c\varepsilon,\ \sup_{\Sigma_0\backslash K} r\,tr\theta \le c\varepsilon,\ \inf_{\Sigma_0\backslash K} r^2\tilde{K} \le c\varepsilon,\ \sup_{\Sigma_0\backslash K} r^2\tilde{K} \le c\varepsilon \\
&\sup_{\Sigma_0\backslash K}\frac{r^2}{1+|\log r|}|\hat{\theta}| \le c\varepsilon,\ \sup_{\Sigma_0\backslash K} r^2(tr\theta - \overline{tr\theta}) \le c\varepsilon
\end{aligned} \tag{7.1.17}$$

Also

$$\begin{aligned}
&|r^{\frac{7}{2}-\frac{2}{p}}\hat{S}|_{p,S_0} + |r^{\frac{9}{2}-\frac{2}{p}}\nabla\!\!\!/\hat{S}|_{p,S_0} \le c\varepsilon \\
&|r^{\frac{7}{2}-\frac{2}{p}}P|_{p,S_0} + |r^{\frac{9}{2}-\frac{2}{p}}\nabla\!\!\!/ P|_{p,S_0} \le c\varepsilon \\
&|r^{\frac{7}{2}-\frac{2}{p}}(Q-\overline{Q})|_{p,S_0} + |r^{\frac{9}{2}-\frac{2}{p}}\nabla\!\!\!/ Q|_{p,S_0} \le c\varepsilon
\end{aligned} \tag{7.1.18}$$

[5] $(\text{curl}\ \widehat{^{(3)}R})_{ij} \equiv \epsilon_j{}^{ls}\nabla_l\left({}^{(3)}R_{is} - \frac{1}{3}g_{is}{}^{(3)}R\right)$.

[6] $\nabla\!\!\!/_N e_A = 0$.

[7] We denote, here, $\tilde{K}$ the Gauss curvature of $S_0(\nu)$, to distinguish it from the compact set K.

The proof of Theorem 7.1.1 follows by a simple adaptation of the proof of Proposition 5.0.1 in [Ch-Kl], Chapter 5. Observe that one can also prove additional estimates for the derivatives of $\mathrm{tr}\theta$ and $\hat\theta$ as well of $\hat S$, P and $(Q-\overline{Q})$, up to second order, see also 7.1.18.

Remark: We can choose K such that ∂K coincides with a leave of the background foliation.

[Added the formulas previously in the note explaining how to get the estimates for the four dimensional Riemann tensor on σ_0 once we know S, P, Q and the global smallness conditions]

The results of Theorem 7.1.1 and assumption $J_K(\Sigma_0,g,k)\leq\varepsilon^2$ allow to control on Σ_0 both the connection coefficients and the various components of the four dimensional Riemann tensor. To achieve this result we use the following relationships, for the four dimensional Riemann tensor,

$$
\begin{aligned}
{}^{(3)}R_{\tilde N\tilde N} &= \rho' + k^2_{\tilde N\tilde N} + \sum_A |k_{e_A\tilde N}|^2\\
{}^{(3)}R_{e_A\tilde N} &= -\frac12(\beta'_A+\underline{\beta}'_A) - \frac12(\chi'+\underline{\chi}')_{AC}\zeta'_C + (\omega'+\underline{\omega}')\zeta'_A\\
{}^{(3)}R_{e_Ae_B} &= \frac14(\alpha'_{AB}+\underline{\alpha}'_{AB}) - \frac12\delta_{AB}\rho'_B + \frac14(\chi'+\underline{\chi}')_{AC}(\chi'+\underline{\chi}')_{CB} + \zeta'_A\zeta'_B\\
0 &= \frac14(\alpha'_{AB}-\underline{\alpha}'_{AB}) + \frac12\epsilon_{AB}\sigma' + \nabla_{\tilde N}k_{BA} - \nabla_B k_{\tilde N A} \qquad (7.1.19)\\
0 &= \frac14(\beta'_A-\underline{\beta}'_A) + \nabla_{\tilde N}k_{A\tilde N} - \nabla_A k_{\tilde N\tilde N}
\end{aligned}
$$

the first three relations coming from the Gauss equation,

$$
{}^{(3)}R^{\mu}{}_{\nu\rho\sigma} = {}^{(4)}R^{\tau}{}_{\gamma\lambda\zeta}\Pi^{\mu}_{\tau}\Pi^{\gamma}_{\nu}\Pi^{\lambda}_{\rho}\Pi^{\zeta}_{\sigma} - k^{\mu}{}_{\rho}k_{\nu\sigma} + k^{\mu}{}_{\sigma}k_{\nu\rho} \qquad (7.1.20)
$$

where $\Pi^{\mu}_{\nu}=(\delta^{\mu}_{\nu}-T^{\mu}_0T_{0\nu})$ is the projection on $T\Sigma_0$, and the remaining two from the Codazzi-Mainardi equation,

$$
{}^{(4)}R_{T_0\gamma\lambda\zeta} = -\nabla_\lambda k_{\zeta\gamma} + \nabla_\zeta k_{\lambda\gamma} \qquad (7.1.21)
$$

For the connection coefficients the relations involved are

$$
\begin{aligned}
\zeta'_A &= \frac12 g(\mathbf{D}_Ae_4,e_3) = k_{A\tilde N}\\
\chi'_{AB} &= g(\mathbf{D}_Ae_4,e_B) = -k_{AB}+\theta_{AB}\\
\underline{\chi}'_{AB} &= g(\mathbf{D}_Ae_3,e_B) = -k_{AB}-\theta_{AB}\\
\omega'+\underline{\omega}' &= k_{\tilde N\tilde N} \qquad (7.1.22)\\
2\omega' &= -\mathbf{D}_4\log a\ ,\ 2\underline{\omega}' = -\mathbf{D}_3\log a\\
2\Omega' &= a
\end{aligned}
$$

7.1.4 The construction of the canonical foliation of Σ_0

We start giving the motivation for the introduction of the canonical foliation on $\Sigma_0 \backslash K$. Let us consider the evolution equation for $\mathrm{tr}\theta$ on Σ_0, see 7.1.10,

$$\nabla_{\tilde{N}} \mathrm{tr}\theta + \frac{1}{2}\mathrm{tr}\theta^2 = -a^{-1} \not{\triangle} a - |\hat{\theta}|^2 - {}^{(3)}R_{\tilde{N}\tilde{N}} \tag{7.1.23}$$

Expressing the spacetime curvature tensor $\mathbf{R}_{\alpha\beta\gamma\delta}$ relative [8] to the null pair $\{e_4 = T_0 + \tilde{N} \ , \ e_3 = T_0 - \tilde{N}\}$ and using the Gauss equation, see 7.1.20,

$${}^{(3)}R_{\tilde{N}\tilde{N}} = \rho + k^2_{\tilde{N}\tilde{N}} + \sum_A |k_{e_A \tilde{N}}|^2 \tag{7.1.24}$$

Using this equation we rewrite the evolution equation 7.1.23 as

$$\nabla_{\tilde{N}} \mathrm{tr}\theta + \frac{1}{2}(\mathrm{tr}\theta)^2 = -(\not{\triangle} \log a + \rho) + \left[-|\not{\nabla} \log a|^2 - |\hat{\theta}|^2 + g(k)\right] \tag{7.1.25}$$

where $g(k) \equiv k^2_{\tilde{N}\tilde{N}} + \sum_a |k_{e_A \tilde{N}}|^2$. Observe that the right hand side of 7.1.25 depends through ρ on the second derivatives of the metric g which implies that we can estimate, at most, two angular derivatives of $\mathrm{tr}\theta$. To do better we have to modify the $(\not{\triangle} \log a + \rho)$ term. This leads to the following definition,

Definition 3.3.1: *We say that a foliation is canonical on $\Sigma_0 \backslash K$ if it is defined by a function $\underline{u}_{(0)}(p)$ solution of the following problem, "The initial slice problem",*

$$\begin{aligned} |\nabla \underline{u}_{(0)}| = a^{-1} \ , \ \underline{u}_{(0)}|_{\partial K} = \nu_0 \\ \not{\triangle} \log a = -(\rho - \overline{\rho}) \ , \ \overline{\log a} = 0 \end{aligned} \tag{7.1.26}$$

The leaves of the canonical foliation are denoted

$$S_{(0)}(\nu) = \{p \in \Sigma_0 | \underline{u}_{(0)}(p) = \nu\}$$

Moreover the initial leave $S_0(\nu_0) = \partial K$ of the background foliation is also the initial leave, $S_{(0)}(\nu_0)$, of the canonical foliation.

7.1.5 Proof of Theorem M3:

Let us recall, now, the statement of Theorem **M3**:

[8] T_0 is the vector field defined on Σ_0 orthonormal to Σ_0.

Theorem 3.3.1 (Theorem M3) *Consider an initial data set which satisfies the exterior global smallness condition $J_K(\Sigma_0, g, k) \leq \varepsilon^2$, with ε sufficiently small. There exists a canonical foliation on $\Sigma_0\backslash K$, such that the following estimates hold*

$$\mathcal{O}_{[3]}(\Sigma_0\backslash K) \leq c\varepsilon \ , \ \underline{\mathcal{O}}_{[3]}(\Sigma_0\backslash K) \leq c\varepsilon \tag{7.1.27}$$

Proof of Theorem M3, part I: The proof consists of a local existence argument followed by a continuation argument. The local existence part is of the same type, but easier than the local part of the proof of the "last slice problem" as presented in[Ch-Kl], Chapter 6. The global extension argument is far simpler than the corresponding global argument given in Chapter 14 of [Ch-Kl]. Both the local existence and the extension argument are based on Theorem 7.1.1

7.2 The initial hypersurface connection estimates

In this section we continue the proof Theorem **M3**. We assume that the initial hypersurface $\Sigma_0\backslash K$ is endowed with a canonical foliation and prove that we can estimate the $\Sigma_0\backslash K$ norms of the connection coefficients and their derivatives up to third order, in terms of the initial data norm $J_K(\Sigma_0, g, k)$. We shall in fact prove slightly stronger estimates expressed in the following definitions of the connection coefficients [9] norms $\mathcal{O}'(\Sigma_0\backslash K)$ and $\underline{\mathcal{O}}'(\Sigma_0\backslash K)$.

[Some corrections in the definitions of the norms; recall that here both the norm for ω and $\underline{\omega}$ are given as in the initial layer region all the evolution equations start from Σ_0.]

$$\begin{aligned}
\mathcal{O}'^{\infty}_{[0]}(\Sigma_0\backslash K) &\equiv \mathcal{O}'^{\infty}_0(\Sigma_0\backslash K) + \sup_{\Sigma_0\backslash K}|r^2\tau_-^{\frac{1}{2}}(\overline{\mathrm{tr}\chi'} - \frac{2}{r})| + \sup_{\Sigma_0\backslash K}|r(\Omega' - \frac{1}{2})| \\
\underline{\mathcal{O}}'^{\infty}_{[0]}(\Sigma_0\backslash K) &\equiv \underline{\mathcal{O}}'^{\infty}_0(\Sigma_0\backslash K) + \sup_{\Sigma_0\backslash K}|r\tau_-^{\frac{3}{2}}(\overline{\mathrm{tr}\underline{\chi}'} + \frac{2}{r})| \\
\mathcal{O}'_{[1]}(\Sigma_0\backslash K) &\equiv \left[\mathcal{O}'_1(\Sigma_0\backslash K) + \sup_{p\in[2,4]} \mathcal{O}'^{p,S}_0(\Sigma_0\backslash K)(\mathbf{D}'_3\underline{\omega}')\right] + \mathcal{O}'^{\infty}_{[0]}(\Sigma_0\backslash K) \\
\underline{\mathcal{O}}'_{[1]}(\Sigma_0\backslash K) &\equiv \left[\underline{\mathcal{O}}'_1(\Sigma_0\backslash K) + \sup_{p\in[2,4]} \mathcal{O}'^{p,S}_0(\Sigma_0\backslash K)(\mathbf{D}'_4\omega')\right] + \underline{\mathcal{O}}'^{\infty}_{[0]}(\Sigma_0\backslash K) \qquad (7.2.1) \\
\mathcal{O}'_{[2]}(\Sigma_0\backslash K) &\equiv \left[\mathcal{O}'_2(\Sigma_0\backslash K) + \sup_{p\in[2,4]} \left(\mathcal{O}'^{p,S}_1(\Sigma_0\backslash K)(\mathbf{D}'_3\underline{\omega}') + \mathcal{O}'^{p,S}_0(\Sigma_0\backslash K)(\mathbf{D}'^2_3\underline{\omega}')\right)\right] \\
&\quad + \mathcal{O}'_{[1]}(\Sigma_0\backslash K)
\end{aligned}$$

[9]Observe that here all the connection coefficients $\chi', \underline{\chi}'$... are primed as they are the restrictions on Σ_0 of the connection coefficients relative to the *initial layer foliation*, see subsection 3.3.4.

$$\underline{\mathcal{O}'}_{[2]}(\Sigma_0\backslash K) \equiv \Big[\underline{\mathcal{O}'}_2(\Sigma_0\backslash K) + \sup_{p\in[2,4]}\Big(\mathcal{O}'^{p,S}_1(\Sigma_0\backslash K)(\mathbf{D}'_4\omega') + \mathcal{O}'^{p,S}_0(\Sigma_0\backslash K)(\mathbf{D}'^2_4\omega')\Big)\Big]$$
$$+\underline{\mathcal{O}'}_{[1]}(\Sigma_0\backslash K)$$
$$\mathcal{O}'_{[3]}(\Sigma_0\backslash K) \equiv \mathcal{O}'_3(\Sigma_0\backslash K) + \mathcal{O}'_{[2]}(\Sigma_0\backslash K)$$

where, for $q \leq 2$,

$$\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(X) \equiv \sup_{\Sigma_0\backslash K}\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(X)(\nu) \tag{7.2.2}$$

The $\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(X)(\nu)$ norms, present in 7.2.1 [10] are listed below,

$$\begin{aligned}
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(\hat{\chi}')(\nu) &= |r^{(\frac{5}{2}+q-\frac{2}{p})}\nabla\!\!\!/'^q\hat{\chi}'|_{p,S_{(0)}(\nu)}\\
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(\hat{\underline{\chi}}')(\nu) &= |r^{(\frac{5}{2}+q-\frac{2}{p})}\nabla\!\!\!/'^q\hat{\underline{\chi}}'|_{p,S_{(0)}(\nu)}\\
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(tr\chi')(\nu) &= |r^{(\frac{5}{2}+q-\frac{2}{p})}\nabla\!\!\!/'^q(\mathrm{tr}\chi' - \overline{tr\chi'})|_{p,S_{(0)}(\nu)}\\
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(tr\underline{\chi}')(\nu) &= |r^{(\frac{5}{2}+q-\frac{2}{p})}\nabla\!\!\!/'^q(\mathrm{tr}\underline{\chi}' - \overline{tr\underline{\chi}'})|_{p,S_{(0)}(\nu)}\\
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(\eta')(\nu) &= |r^{(\frac{5}{2}+q-\frac{2}{p})}\nabla\!\!\!/'^q\eta'|_{p,S_{(0)}(\nu)}\\
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(\underline{\eta}')(\nu) &= |r^{(\frac{5}{2}+q-\frac{2}{p})}\nabla\!\!\!/'^q\underline{\eta}'|_{p,S_{(0)}(\nu)}\\
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(\underline{\omega}')(\nu) &= |r^{(\frac{5}{2}+q-\frac{2}{p})}\nabla\!\!\!/'^q\underline{\omega}'|_{p,S_{(0)}(\nu)}\\
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(\mathbf{D}'_3\underline{\omega}')(\nu) &= |r^{(\frac{7}{2}+q-\frac{2}{p})}\nabla\!\!\!/'^q\mathbf{D}'_3\underline{\omega}'|_{p,S_{(0)}(\nu)}\\
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(\mathbf{D}'^2_3\underline{\omega}')(\nu) &= |r^{(\frac{9}{2}+q-\frac{2}{p})}\nabla\!\!\!/'^q\mathbf{D}'^2_3\underline{\omega}'|_{p,S_{(0)}(\nu)}\\
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(\omega')(\nu) &= |r^{(\frac{5}{2}+q-\frac{2}{p})}\nabla\!\!\!/'^q\omega'|_{p,S_{(0)}(\nu)}\\
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(\mathbf{D}'_4\omega')(\nu) &= |r^{(\frac{7}{2}+q-\frac{2}{p})}\nabla\!\!\!/^q\mathbf{D}'_3\omega'|_{p,S_{(0)}(\nu)}\\
\mathcal{O}'^{p,S}_q(\Sigma_0\backslash K)(\mathbf{D}'^2_4\omega')(\nu) &= |r^{(\frac{9}{2}+q-\frac{2}{p})}\nabla\!\!\!/'^q\mathbf{D}'^2_4\omega'|_{p,S_{(0)}(\nu)}
\end{aligned} \tag{7.2.3}$$

Finally we introduce the $\mathcal{O}'_3(\Sigma_0\backslash K)$ and the $\underline{\mathcal{O}'}_3(\Sigma_0\backslash K)$ norms on the initial slice. They are defined in the following way:

$$\begin{aligned}
\mathcal{O}'_3(\Sigma_0\backslash K) &= \mathcal{O}'_3(\Sigma_0\backslash K)(\mathrm{tr}\chi') + \mathcal{O}'_3(\Sigma_0\backslash K)(\underline{\omega}') \\
\underline{\mathcal{O}'}_3(\Sigma_0\backslash K) &= \mathcal{O}'_3(\Sigma_0\backslash K)(\mathrm{tr}\underline{\chi}') + \mathcal{O}'_3(\Sigma_0\backslash K)(\omega')
\end{aligned} \tag{7.2.4}$$

[10]Observe that all these norms are different from those defined on the whole $\mathcal{K}$, see 3.5.22,....., 3.5.28, for a factor $r^{\frac{1}{2}}$.

where

$$
\begin{aligned}
\mathcal{O}_3'(\mathrm{tr}\underline{\chi}') &= \sup_{\nu\in[\nu_0,\nu_*]}\left(r^{\frac{1}{2}}(\nu)||r^3\nabla\!\!\!\!/'^3\mathrm{tr}\underline{\chi}'||_{L^2(\Sigma_0\setminus K)}\right)\\
\mathcal{O}_3'(\omega') &= \sup_{\nu\in[\nu_0,\nu_*]}\left(r^{\frac{1}{2}}(\nu)||r^3\nabla\!\!\!\!/'^3\omega'||_{L^2(\Sigma_0\setminus K)}\right)\\
\mathcal{O}_3'(\mathrm{tr}\chi') &= \sup_{\nu\in[\nu_0,\nu_*]}\left(r^{\frac{1}{2}}(\nu)||r^3\nabla\!\!\!\!/'^3\mathrm{tr}\chi'||_{L^2(\Sigma_0\setminus K)}\right) \qquad (7.2.5)\\
\mathcal{O}_3'(\omega') &= \sup_{\nu\in[\nu_0,\nu_*]}\left(r^{\frac{1}{2}}(\nu)||r^3\nabla\!\!\!\!/'^3\underline{\omega}'||_{L^2(\Sigma_0\setminus K)}\right)
\end{aligned}
$$

Proof of Theorem M3, part II:

We sketch below the proof of the theorem.

1) We recall the evolution equation satisfied by $\mathrm{tr}\theta$, the Codazzi-Mainardi equation for $\hat{\theta}$, 7.1.9, and the elliptic equation satisfied by a, relative to the canonical foliation:

$$
\begin{aligned}
&\nabla_{\tilde{N}}\mathrm{tr}\theta+\frac{1}{2}(\mathrm{tr}\theta)^2=-\overline{\rho}+\left[-|\nabla\!\!\!\!/\log a|^2-|\hat{\theta}|^2+g(k)\right]\\
&\nabla\!\!\!\!/^c\hat{\theta}_{cb}=\frac{1}{2}\nabla\!\!\!\!/_b\mathrm{tr}\theta+P_b \qquad (7.2.6)\\
&\Delta\!\!\!\!/\log a=-(\rho-\overline{\rho})\ ,\ \overline{\log a}=0
\end{aligned}
$$

Observe that the quantities $\hat{S},P,(Q-\overline{Q})$, see 7.1.12, which decompose the Ricci tensor relative to $\Sigma_0\setminus K$, are expressed relative to the canonical foliation.

2) Using equations 7.2.6 we proceed precisely as in the proof of Theorem 7.1.1 and deduce the following results

$$
\begin{aligned}
&|r^{\frac{7}{2}-\frac{2}{p}}\hat{S}|_{p,S_{(0)}}+|r^{\frac{9}{2}-\frac{2}{p}}\nabla\!\!\!\!/\hat{S}|_{p,S_{(0)}}+|r^{\frac{9}{2}-\frac{2}{p}}\nabla_{\tilde{N}}\hat{S}|_{p,S_{(0)}}\leq c\varepsilon\\
&|r^{\frac{7}{2}-\frac{2}{p}}P|_{p,S_{(0)}}+|r^{\frac{9}{2}-\frac{2}{p}}\nabla\!\!\!\!/P|_{p,S_{(0)}}+|r^{\frac{9}{2}-\frac{2}{p}}\nabla_{\tilde{N}}P|_{p,S_{(0)}}\leq c\varepsilon \qquad (7.2.7)\\
&|r^{\frac{7}{2}-\frac{2}{p}}(Q-\overline{Q})|_{p,S_{(0)}}+|r^{\frac{9}{2}-\frac{2}{p}}\nabla\!\!\!\!/Q|_{p,S_{(0)}}+|r^{\frac{9}{2}-\frac{2}{p}}Q_{\tilde{N}}|_{p,S_{(0)}}\leq c\varepsilon\\
&||r^4\nabla\!\!\!\!/\nabla_{\tilde{N}}\hat{S}||_{L^2(\Sigma_0\setminus K)}+||r^4\nabla^2_{\tilde{N}}\hat{S}||_{L^2(\Sigma_0\setminus K)}\leq c\varepsilon\\
&||r^4\nabla\!\!\!\!/\nabla_{\tilde{N}}P||_{L^2(\Sigma_0\setminus K)}+||r^4\nabla^2_{\tilde{N}}P||_{L^2(\Sigma_0\setminus K)}\leq c\varepsilon\\
&||r^4\nabla_{\tilde{N}}Q_{\tilde{N}}||_{L^2(\Sigma_0\setminus K)}+||r^4\nabla\!\!\!\!/Q_{\tilde{N}}||_{L^2(\Sigma_0\setminus K)}+||r^4\nabla\!\!\!\!/^2Q||_{L^2(\Sigma_0\setminus K)}\leq c\varepsilon
\end{aligned}
$$

Remark: In the process of proving this result we prove also the same results for $\hat{\theta}$, $\mathrm{tr}\theta$ and their derivatives up to second order as in Theorem 7.1.1, see 7.1.17 and [Ch-Kl], Chapter 5. In fact, we will do better in what follows.

3) Using the curvature estimates established above we use the equations 7.2.6 to obtain the estimates 7.1.27. We sketch the main ideas of the proof below.
a) The first important observation is that the curvature term, $\overline{\rho}$, in the third equation of 7.2.6 is constant on the $S_{(0)}$ surfaces, therefore we can eliminate it taking tangential derivatives. This leads to a coupled system between the evolution equation for $\nabla\!\!\!\!/ \mathrm{tr}\theta$ and the Codazzi equation for $\hat{\theta}$. This system is similar but simpler than the system for $\nabla\!\!\!\!/ \mathrm{tr}\chi$ and $\nabla\!\!\!\!/ \hat{\chi}$ studied in Chapter 4 subsection 4.3.1, and can be treated as in that case. Therefore we derive the following estimates

$$\begin{aligned}
&|r^{\frac{5}{2}-\frac{2}{p}}\hat{\theta}|_{p,S_{(0)}} \leq c\varepsilon \ , \ |r^{\frac{7}{2}-\frac{2}{p}}\nabla\!\!\!\!/ \hat{\theta}|_{p,S_{(0)}} \leq c\varepsilon \\
&|r^{\frac{5}{2}-\frac{2}{p}}(\mathrm{tr}\theta-\overline{\mathrm{tr}\theta})|_{p,S_{(0)}} \leq c\varepsilon \ , \ |r^{2-\frac{2}{p}}(\overline{\mathrm{tr}\theta}-\frac{2}{r})|_{p,S_{(0)}} \leq c\varepsilon \ , \\
&|r^{\frac{7}{2}-\frac{2}{p}}\nabla\!\!\!\!/ \mathrm{tr}\theta|_{p,S_{(0)}} \leq c\varepsilon
\end{aligned} \tag{7.2.8}$$

b) It is easy to see that one can also obtain the estimates up to two more derivatives for these quantities obtaining, for $p \in [2,4]$,

$$\begin{aligned}
&|r^{\frac{9}{2}-\frac{2}{p}}\nabla\!\!\!\!/^2\mathrm{tr}\theta|_{p,S_{(0)}} \leq c\varepsilon \ , \ |r^{\frac{9}{2}-\frac{2}{p}}\nabla\!\!\!\!/^2\hat{\theta}|_{p,S_{(0)}} \leq c\varepsilon \\
&r^{\frac{1}{2}}||r^3\nabla\!\!\!\!/^3\mathrm{tr}\theta||_{L^2(\Sigma_0\setminus K)} \leq c\varepsilon
\end{aligned} \tag{7.2.9}$$

and the estimate, for the scalar function a, for $q=0,1,2$,

$$|r^{(\frac{3}{2}+q)-\frac{2}{p}}\nabla\!\!\!\!/^q \log a|_{p,S_{(0)}} \leq c\varepsilon \tag{7.2.10}$$

due to the elliptic character of the equation satisfied by $\log a$.

4) The final step consists in using these results to estimate the various connection coefficients χ', $\underline{\chi}'$, ζ', ω' and $\underline{\omega}'$. We consider the future directed unit vector normal to Σ_0, T_0, and the null frame, adapted to the canopnical foliation, we denote $\{e_4', e_3', e_A'\}$ where $e_4' = T_0 + \tilde{N}$, $e_3' = T_0 - \tilde{N}$. The connection coefficients we consider here are those computed with respect to this null frame [11]. The following expressions hold, see also 7.1.22,

$$\begin{aligned}
&2\Omega' = a \\
&\zeta_A' = \frac{1}{2}g(\mathbf{D}_{A'}e_4', e_3') = -g(\mathbf{D}_{e_A'}T_0, \tilde{N}) = k(e_A', \tilde{N}) \\
&\chi_{AB}' = g(\mathbf{D}_A' e_4', e_B') = -k_{AB} + \theta_{AB} \\
&\underline{\chi}_{AB}' = g(\mathbf{D}_A' e_3', e_B') = -k_{AB} - \theta_{AB} \\
&2\omega' = -\mathbf{D}_{e_4'}\log a \ , \ 2\underline{\omega}' = -\mathbf{D}_{e_3'}\log a
\end{aligned} \tag{7.2.11}$$

[11] Observe that, as discussed in Chapters 3 and 4, this frame is the restriction to Σ_0 of the null frame relative to the *initial layer foliation*.

In view of the estimates for k, implicit in the assumption, $J_K(\Sigma_0, \mathbf{g}, k) \le \varepsilon^2$, we obtain the following result

$$\mathcal{O}'_{[3]}(\Sigma_0 \backslash K) \le c\varepsilon \ , \ \underline{\mathcal{O}}'_{[3]}(\Sigma_0 \backslash K) \le c\varepsilon \tag{7.2.12}$$

7.2.1 Proof of Lemma 3.7.1

We recall the statement of the lemma:

Lemma 3.7.1: *Assuming $J_K(\Sigma_0, g, k)$ sufficiently small, the following inequality holds*

$$\mathcal{Q}_{\Sigma_0 \cap \mathcal{K}} \le cJ_K(\Sigma_0, g, k) \ .$$

The proof of this Lemma is straightforward. We sketch here only the main steps. Looking at the definitions 3.5.9, 3.5.10, 3.5.12, it follows that we have to estimate on Σ_0 the L^2 norms of the various components of the Riemann tensors $\mathbf{R}$, $\hat{\mathcal{L}}_0\mathbf{R}$, $\hat{\mathcal{L}}_T\mathbf{R}$, $\hat{\mathcal{L}}_0^2\mathbf{R}$, $\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\mathbf{R}$. In 7.1.19 these Riemann components are written in terms of the Ricci tensor of Σ_0 and of quadratic expressions in k. Using the previous Theorem **M3**, see 7.2.7, and the conditions on k imposed by the smallness of J_K, we can easily estimate these quantities. The main simplification is that, on Σ_0, all these Riemann components have the same asymptotic behaviour as $r \to \infty$. More precisely we have $O(r^{-\frac{7}{2}})$ for the terms[12] associated to $\mathbf{R}$, $\hat{\mathcal{L}}_0\mathbf{R}$, $\hat{\mathcal{L}}_0^2\mathbf{R}$ and stronger for $\hat{\mathcal{L}}_T\mathbf{R}$ and $\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\mathbf{R}$.

7.3 The last slice foliation

7.3.1 Construction of the canonical foliation of $\underline{C}_*$

The construction of the canonical foliation on $\underline{C}_*$ is based on the proof of Theorem **M6** whose statement we recall:

Theorem M6: *Assume given on $\underline{C}_*$ a radial foliation, not necessarily canonical, whose connection coefficients and null curvature components satisfy* [13] *the inequalities*

$$\begin{aligned}\mathcal{R}'(\underline{C}_*) &\equiv \mathcal{R}_{[2]}{}'(\underline{C}_*) + \underline{\mathcal{R}}_{[2]}{}'(\underline{C}_*) \le \epsilon_0' \\ \mathcal{O}'(\underline{C}_*) &\equiv \underline{\mathcal{O}}_{[2]}{}'(\underline{C}_*) + \mathcal{O}_{[2]}{}'(\underline{C}_*) \le \epsilon_0' \end{aligned} \tag{7.3.1}$$

[12]With the only exception of Q, see 7.1.12, where we have to subtract its average $\overline{Q}$.

[13]These are the "appropriate smallness assumptions" of the first statement of the theorem, see Theorem 3.3.2.

where ${\mathcal{R}_{[2]}}'(\underline{C}_*)$, ${\underline{\mathcal{R}}_{[2]}}'(\underline{C}_*)$, ${\underline{\mathcal{O}}_{[2]}}'(\underline{C}_*), {\mathcal{O}_{[2]}}'(\underline{C}_*)$ *are the norms introduced in Chapter 3 section 3.5, restricted to* $\underline{C}_*$*, relative to the radial foliation. Then there exists a canonical foliation, on* $\underline{C}_*$ *relative to which we have*

$$\begin{aligned}\mathcal{R}(\underline{C}_*) &\equiv \mathcal{R}_{[2]}(\underline{C}_*) + \underline{\mathcal{R}}_{[2]}(\underline{C}_*) \leq c\epsilon'_0 \\ \mathcal{O}(\underline{C}_*) &\equiv \underline{\mathcal{O}}_{[2]}(\underline{C}_*) + \mathcal{O}_{[3]}(\underline{C}_*) \leq c\epsilon'_0\end{aligned} \tag{7.3.2}$$

In addition it can be shown that these two foliations remain close to each other in a sense which can be made precise.

Proof: We divide the proof into four parts.

a) We prove that the canonical foliation exists locally on $\underline{C}_*$ close to $S_*(\lambda_1)$.

b) Given a canonical foliation on $\underline{C}_*$, such that $\mathcal{R}(\underline{C}_*)$ is sufficiently small, we show that the following inequality holds

$$\mathcal{O}(\underline{C}_*) \leq c\,(\mathcal{I}_0 + \mathcal{R}(\underline{C}_*)) \tag{7.3.3}$$

This is the content of Theorem **M4** whose proof is given in section 7.4.

c) We compare the norms $\mathcal{R}(\underline{C}_*)$ and the norms $\mathcal{R}'(\underline{C}_*)$ and establish the following inequality which we write, schematically, as

$$\mathcal{R}(\underline{C}_*) \leq \mathcal{R}'(\underline{C}_*) + c(\mathcal{O}(\underline{C}_*) + \mathcal{O}'(\underline{C}_*))\mathcal{R}'(\underline{C}_*) \tag{7.3.4}$$

see the proof in the appendix.

d) Combining b) and c) and using assumptions 7.3.1 we deduce

$$\begin{aligned}\mathcal{O}(\underline{C}_*) &\leq c\Big(\mathcal{I}_0 + \mathcal{R}'(\underline{C}_*) + c(\mathcal{O}(\underline{C}_*) + \mathcal{O}'(\underline{C}_*))\mathcal{R}'(\underline{C}_*)\Big) \\ &\leq c\left(\mathcal{I}_0 + \mathcal{R}'(\underline{C}_*) + \mathcal{O}(\underline{C}_*)\epsilon'_0 + O(\epsilon_0'^2)\right)\end{aligned} \tag{7.3.5}$$

Therefore, if ϵ'_0 is sufficiently small we obtain the result of the theorem for $\mathcal{O}(\underline{C}_*)$ and using again inequality 7.3.5 in 7.3.4 we derive also the estimate for $\mathcal{R}(\underline{C}_*)$.

The proof of the local existence result, a), is sketched in the appendix to this chapter, see subsection 7.7.2, and proved in detail elsewhere, see [Ni]. The continuation argument, b), to extend the foliation to the whole $\underline{C}_*$, is based on the a priori estimates 7.3.3, proved in the next section.
Next corollary specifies in which sense the two foliations on $\underline{C}_*$ are near.

Corollary 7.3.1 *Under the assumptions of Theorem* **M6**, *let S' be a leave on $\underline{C}_*$ of the radial background foliation* [14], *let $u(p)=u_*(p)$ the function, defined on $\underline{C}_*$, whose level surfaces are the leaves of the canonical foliation of $\underline{C}_*$, then*

$$\sup_{(p,p')\in S'}|u(p)-u(p')|\leq c\epsilon'_0 \tag{7.3.6}$$

Proof: It is an immediate consequence of the proof of Theorem **M6** and we do not report it here.

7.4 The last slice connection estimates

In this section we give the proof of Theorem **M4**. We shall in fact prove slightly stronger estimates which are encompassed in the following definitions of the connection coefficients norms $\mathcal{O}^*$ and $\underline{\mathcal{O}}^*$ on $\underline{C}_*$.

7.4.1 $\mathcal{O}$ norms on the last slice

The norms we introduce here are slightly different from the corresponding ones defined in $\mathcal{K}$, in fact they are a little stronger. It is possible to prove their boundedness due to the fact that the $\underline{C}_*$ is endowed with a canonical foliation. Their expressions are

[Some corrections in the definitions of the norms; in particular those for $\overline{\mathrm{tr}\chi}-\frac{2}{r}$ and for $\overline{\mathrm{tr}\underline{\chi}}+\frac{2}{r}$ where an extra factor $\tau_-^{\frac{1}{2}}$ is wrongly added.]

$$\begin{aligned}
\mathcal{O}^{*\infty}_{[0]} &\equiv \mathcal{O}^{*\infty}_{0}+\sup_{\underline{C}_*}|r^2(\overline{\mathrm{tr}\chi}-\frac{2}{r})|+\sup_{\underline{C}_*}|r(\Omega-\frac{1}{2})|\\
\underline{\mathcal{O}}^{*\infty}_{[0]} &\equiv \underline{\mathcal{O}}^{*\infty}_{0}+\sup_{\underline{C}_*}|r\tau_-(\overline{\mathrm{tr}\underline{\chi}}+\frac{2}{r})|\\
\mathcal{O}^{*}_{[1]} &\equiv \left[\mathcal{O}^{*}_{1}+\sup_{p\in[2,4]}\mathcal{O}^{*p,S}_{0}(\mathbf{D}_3\underline{\omega})\right]+\mathcal{O}^{*\infty}_{[0]}\\
\underline{\mathcal{O}}^{*}_{[1]} &\equiv \underline{\mathcal{O}}^{*}_{1}+\underline{\mathcal{O}}^{*\infty}_{[0]}\\
\mathcal{O}^{*}_{[2]} &= \left[\mathcal{O}^{*}_{2}+\sup_{p\in[2,4]}\left(\mathcal{O}^{*p,S}_{1}(\mathbf{D}_3\underline{\omega})+\mathcal{O}^{*p,S}_{0}(\mathbf{D}_3^2\underline{\omega})\right)\right]+\mathcal{O}^{*}_{[1]}\\
\underline{\mathcal{O}}^{*}_{[2]} &= \underline{\mathcal{O}}^{*}_{2}+\underline{\mathcal{O}}^{*}_{[1]}\ \ ,\quad \mathcal{O}^{*}_{[3]}=\mathcal{O}^{*}_{3}+\mathcal{O}^{*}_{[2]}
\end{aligned} \tag{7.4.1}$$

where, for $q\leq 2$,

$$\mathcal{O}^{*p,S}_{q}(X)\equiv\sup_{\underline{C}_*}\mathcal{O}^{*p,S}_{q}(X)(\lambda,\nu) \tag{7.4.2}$$

[14] In the Oscillation Lemma this result is needed only for a small distance from $\underline{C}_*\cap\Sigma_0$.

The ${\mathcal O^*}_q^{p,S}$ norms, present in 7.4.1 [15] are listed below,

$$\begin{aligned}
{\mathcal O^*}_q^{p,S}(\hat{\chi}) &= |r^{(2+q-\frac{2}{p})}\tau_-^{\frac{1}{2}}\nabla\!\!\!\!/^q\hat{\chi}|_{p,S}\\
{\mathcal O^*}_q^{p,S}(\underline{\hat{\chi}}) &= |r^{(1+q-\frac{2}{p})}\tau_-^{\frac{3}{2}}\nabla\!\!\!\!/^q\underline{\hat{\chi}}|_{p,S}\\
{\mathcal O^*}_q^{p,S}(tr\chi) &= |r^{(2+q-\frac{2}{p})}\tau_-^{\frac{1}{2}}\nabla\!\!\!\!/^q(\mathrm{tr}\chi-\overline{tr\chi})|_{p,S}\\
{\mathcal O^*}_q^{p,S}(tr\underline{\chi}) &= |r^{(2+q-\frac{2}{p})}\tau_-^{\frac{1}{2}}\nabla\!\!\!\!/^q(\mathrm{tr}\underline{\chi}-\overline{tr\underline{\chi}})|_{p,S}\\
{\mathcal O^*}_q^{p,S}(\eta) &= |r^{(2+q-\frac{2}{p})}\tau_-^{\frac{1}{2}}\nabla\!\!\!\!/^q\eta|_{p,S}\\
{\mathcal O^*}_q^{p,S}(\underline{\eta}) &= |r^{(2+q-\frac{2}{p})}\tau_-^{\frac{1}{2}}\nabla\!\!\!\!/^q\underline{\eta}|_{p,S}\\
{\mathcal O^*}_q^{p,S}(\underline{\omega}) &= |r^{(1+q-\frac{2}{p})}\tau_-^{\frac{3}{2}}\nabla\!\!\!\!/^q\underline{\omega}|_{p,S}\\
{\mathcal O^*}_q^{p,S}(\mathbf{D}_3\underline{\omega}) &= |r^{(1+q-\frac{2}{p})}\tau_-^{\frac{5}{2}}\nabla\!\!\!\!/^q\mathbf{D}_3\underline{\omega}|_{p,S}\\
{\mathcal O^*}_q^{p,S}(\mathbf{D}_3^2\underline{\omega}) &= |r^{(1+q-\frac{2}{p})}\tau_-^{\frac{7}{2}}\nabla\!\!\!\!/^q\mathbf{D}_3^2\underline{\omega}|_{p,S}
\end{aligned}\tag{7.4.3}$$

Finally we introduce the $\mathcal{O}_3^*$ norm on the final slice,

$$\mathcal{O}_3^* = \mathcal{O}_3^*(\mathrm{tr}\chi) + \mathcal{O}_3^*(\underline{\omega})\tag{7.4.4}$$

where

$$\begin{aligned}
\mathcal{O}_3^*(\mathrm{tr}\chi) &= \sup_{\lambda\in[\lambda_0,\lambda_1]}\left(r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!\!/^3\mathrm{tr}\chi||_{L^2(\underline{C}_*(\nu_*,[\lambda_1,\lambda]))}\right)\\
\mathcal{O}_3^*(\underline{\omega}) &= \sup_{\lambda\in[\lambda_0,\lambda_1]}\left(r^{\frac{1}{2}}(\lambda,\nu)||r^3\nabla\!\!\!\!/^3\underline{\omega}||_{L^2(\underline{C}_*(\nu_*,[\lambda_1,\lambda]))}\right)
\end{aligned}\tag{7.4.5}$$

Using these norms the improved version of Theorem **M4** is the following:

Theorem M4: *Given a canonical foliation on $\underline{C}_*$, relative to which*

$$\mathcal{R}_{[2]} + \underline{\mathcal{R}}_{[2]} \leq \Delta\ .$$

Moreover assume

$$\mathcal{O}_{[2]}(\underline{C}_*\cap\Sigma_0) + \mathcal{O}_3(\Sigma_0) + \underline{\mathcal{O}}_{[3]}(\Sigma_0) \leq \mathcal{I}_0\ .$$

If Δ, $\mathcal{I}_0$ are sufficiently small, then the following estimate holds

$$\underline{\mathcal{O}}^*_{[2]} + \mathcal{O}^*_{[3]} \leq c(\mathcal{I}_0 + \Delta)\ .$$

[15] Observe that all these norms are different from those defined on the whole $\mathcal{K}$, see 3.5.22,....., 3.5.28, for a factor $\tau_-^{\frac{1}{2}}$. Moreover we do not need, on $\underline{C}_*$, norms relative to ω.

The proof of this theorem is divided in Proposition 7.4.1 and Proposition 7.4.2.

Proposition 7.4.1 *Assume that, relative to a canonical foliation on the last slice,* $\underline{C}_*$ *we have*

$$\begin{aligned}&\mathcal{R}_0^\infty+\underline{\mathcal{R}}_0^\infty\leq\Delta_0\\&\mathcal{R}_1^S+\underline{\mathcal{R}}_1^S\leq\Delta_1\end{aligned}\tag{7.4.6}$$

Moreover assume that

$$\underline{\mathcal{O}}^*_{[1]}(\underline{C}_*\cap\Sigma_0)+\mathcal{O}^*_{[1]}(\underline{C}_*\cap\Sigma_0)\leq\mathcal{I}_0\tag{7.4.7}$$

If Δ_0, Δ_1 *and* $\mathcal{I}_0$ *are sufficiently small, the following inequality holds*

$$\underline{\mathcal{O}}^*_{[1]}+\mathcal{O}^*_{[1]}\leq c\,(\mathcal{I}_0+\Delta_0+\Delta_1)\tag{7.4.8}$$

Proposition 7.4.2 *Assume that, relative to a canonical foliation on the last slice,* $\underline{C}_*$, *we have*

$$\begin{aligned}&\mathcal{R}_0^\infty+\underline{\mathcal{R}}_0^\infty\leq\Delta_0\\&\mathcal{R}_1^S+\underline{\mathcal{R}}_1^S\leq\Delta_1\\&\mathcal{R}_2+\underline{\mathcal{R}}_2\leq\Delta_2\end{aligned}\tag{7.4.9}$$

Moreover assume [16]

$$\mathcal{O}^*_{[2]}(\underline{C}_*\cap\Sigma_0)+\underline{\mathcal{O}}^*_{[2]}(\underline{C}_*\cap\Sigma_0)\leq\mathcal{I}_0\tag{7.4.10}$$

then if $\Delta_0,\Delta_1,\Delta_2$ *and* $\mathcal{I}_0$ *are sufficiently small the following inequalities hold*

$$\begin{aligned}&\underline{\mathcal{O}}^*_{[2]}+\mathcal{O}^*_{[2]}\leq c\,(\mathcal{I}_0+\Delta_0+\Delta_1)\\&\mathcal{O}^*_3\leq c\,(\mathcal{I}_0+\Delta_0+\Delta_1+\Delta_2)\end{aligned}\tag{7.4.11}$$

Proof of Proposition 7.4.1: The proposition is proved by a bootstrap argument similar to the one of Theorem 4.2.1.
i) We prove that, assuming

$$\mathcal{O}^{*\infty}_{[0]}\leq\Gamma_0\ ,\ \underline{\mathcal{O}}^{*\infty}_{[0]}\leq\Gamma_0\tag{7.4.12}$$

[16]Observe that the assumptions 7.4.10 follow from the "global initial data conditions" 3.6.3, see Theorem 3.3.1.

with Γ_0 sufficiently small [17], the following inequalities hold

$$\underline{\mathcal{O}}_0^* + \mathcal{O}_0^* \leq c\,(\mathcal{I}_0 + \Delta_0)$$
$$\underline{\mathcal{O}}_1^* + \mathcal{O}_1^* \leq c\,(\mathcal{I}_0 + \Delta_0) \tag{7.4.13}$$

Also,

$$\sup_{p\in[2,4]}\left(\sup_{\underline{C}_*}|r^{1-\frac{2}{p}}\tau_-(\overline{\mathrm{tr}\underline{\chi}}+\frac{2}{r})|_{p,S}\right) \leq c\,(\mathcal{I}_0 + \Delta_0)$$
$$\sup_{p\in[2,4]}\left(\sup_{\underline{C}_*}|r^{2-\frac{2}{p}}(\overline{\mathrm{tr}\chi}-\frac{2}{r})|_{p,S}\right) \leq c\,(\mathcal{I}_0 + \Delta_0) \tag{7.4.14}$$
$$\sup_{p\in[2,4]}\left(\sup_{\underline{C}_*}|r^{1-\frac{2}{p}}(\Omega-\frac{1}{2})|_{p,S} + \mathcal{O}^{*p,S}_{\ 0}(\underline{\omega})\right) \leq c\,(\mathcal{I}_0 + \Delta_0)$$
$$\sup_{p\in[2,4]}\left(\mathcal{O}^{*p,S}_{\ 1}(\underline{\omega}) + \mathcal{O}^{*p,S}_{\ 0}(\mathbf{D}_3\underline{\omega})\right) \leq c\,(\mathcal{I}_0 + \Delta_0 + \Delta_1)$$

ii) From the estimates 7.4.13, 7.4.14, using the Sobolev inequality of Lemma 4.1.3, it is possible to control $\underline{\mathcal{O}}^{*\infty}_{[0]}$ and $\mathcal{O}^{*\infty}_{[0]}$, obtaining

$$\underline{\mathcal{O}}^{*\infty}_{[0]}(\underline{C}_*) + \mathcal{O}^{*\infty}_{[0]}(\underline{C}_*) \leq c\,(\mathcal{I}_0 + \Delta_0) \tag{7.4.15}$$

iii) To complete the bootstrap argument we consider the portion of $\underline{C}_*$,

$$\underline{C}_*([\lambda_1, \lambda_2)) = \{p \in \underline{C}_* | u(p) \in [\lambda_1, \lambda_2)\}\ ,$$

where the following inequality holds [18]

$$\underline{\mathcal{O}}^{*\infty}_{[0]}(\underline{C}_*([\lambda_1,\lambda_2))) + \mathcal{O}^{*\infty}_{[0]}(\underline{C}_*([\lambda_1,\lambda_2))) \leq \Gamma_0$$

Using the result in ii), assuming $\mathcal{I}_0 + \Delta_0$ sufficiently small, it follows that in this portion of $\underline{C}_*$ a better inequality holds, namely

$$\underline{\mathcal{O}}^{*\infty}_{[0]}(\underline{C}_*([\lambda_1,\lambda_2))) + \mathcal{O}^{*\infty}_{[0]}(\underline{C}_*([\lambda_1,\lambda_2))) \leq \frac{\Gamma_0}{2}$$

so that the region can be made larger. To avoid a contradiction one has to conclude that it coincides with the whole $\underline{C}_*$.
Repeating step i) again, we have also proved that inequalities 7.4.13 and 7.4.14 hold on the whole $\underline{C}_*$, and, therefore, the inequality,

$$\underline{\mathcal{O}}^*_{[1]} + \mathcal{O}^*_{[1]} \leq c\,(\mathcal{I}_0 + \Delta_0 + \Delta_1)$$

completing the proposition.

[17] Γ_0 must be such that $\Gamma_0^2 < (\Delta_0 + \mathcal{I}_0) < \Gamma_0$.

[18] That the interval $[\lambda_1, \lambda_2)$ is not empty is guaranteed from the assumptions 7.4.7.

Corollary 7.4.1 *The previous result implies the following inequality, see also Corollary 4.3.1,*

$$\sup_{\underline{C}_*} |r\tau_-(tr\chi + tr\underline{\chi})| \leq c(\mathcal{I}_0 + \mathcal{I}_* + \Delta_0) \tag{7.4.16}$$

7.4.2 Implementation of Proposition 7.4.1

Part i): Under the bootstrap assumption 7.4.12 we prove the first estimate of 7.4.13.

We start controlling the norms $\mathcal{O}_q^{p,S}(\underline{C}_*)(\eta)$ for $q = 0, 1$, $p \in [2, 4]$, using the Hodge system

$$\begin{aligned}
\mathrm{d\!/\!iv}\,\eta &= \frac{1}{2}(\hat{\chi}\cdot\hat{\underline{\chi}} - \overline{\hat{\chi}\cdot\hat{\underline{\chi}}}) - (\rho - \overline{\rho}) \\
\mathrm{c\!/\!url}\,\eta &= \frac{1}{2}\hat{\underline{\chi}}\wedge\hat{\chi} + \sigma
\end{aligned} \tag{7.4.17}$$

Observe that this Hodge system is derived from the Hodge system 4.3.34, with the help of the relation $\mu = \overline{\mu}$ defining the canonical foliation, see 3.3.9. We use the estimate of $\hat{\chi}$ and $\hat{\underline{\chi}}$ contained in the bootstrap assumption 7.4.12 and the assumptions on $\rho - \overline{\rho}$ and σ contained in the first line of 7.4.6. All this implies the inequality:

$$\mathcal{O}_0^{*p,S}(\eta) + \mathcal{O}_1^{*p,S}(\eta) \leq c(\Delta_0 + \Gamma_0^2) \leq c(\mathcal{I}_0 + \Delta_0) \tag{7.4.18}$$

We estimate next the norms $\mathcal{O}_0^{*p,S}(\mathrm{tr}\chi)$ and $\mathcal{O}_0^{*p,S}(\hat{\chi})$. For them we use the evolution equations for $\mathrm{tr}\chi$ and $\hat{\chi}$, along $\underline{C}_*$, see 3.1.45,

$$\begin{aligned}
\frac{d}{du}(\Omega\mathrm{tr}\chi) + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}(\Omega\mathrm{tr}\chi) &= 2\Omega^2(|\eta|^2 - \frac{1}{2}\overline{\hat{\chi}\hat{\underline{\chi}}} + \overline{\rho}) \\
\frac{d}{du}(\Omega\hat{\chi}_{ab}) + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}(\Omega\hat{\chi}_{ab}) &= -\frac{1}{2}\Omega\hat{\underline{\chi}}_{ab}(\Omega\mathrm{tr}\chi) + \Omega^2(\nabla\!\!\!/\widehat{\otimes}\eta - \eta\widehat{\otimes}\eta)_{ab}
\end{aligned} \tag{7.4.19}$$

Using the bootstrap assumption 7.4.12 for χ and for $\underline{\chi}$, the estimate for $\overline{\rho}$ and the previous result for η and $\nabla\!\!\!/\eta$, we estimate the right hand side of the equations 7.4.19. Applying the evolution Lemma 4.1.5 we obtain immediately [19]

$$\begin{aligned}
|r^{1-\frac{2}{p}}\mathrm{tr}\chi|_{p,S_*} &\leq |r^{1-\frac{2}{p}}\mathrm{tr}\chi|_{p,\underline{C}_*\cap\Sigma_0} + \frac{1}{r}c\left(\Gamma_0\left(\mathcal{I}_0 + \Delta_0\right) + \Gamma_0^2\right) \\
|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\hat{\chi}|_{p,S_*} &\leq |r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\hat{\chi}|_{p,\underline{C}_*\cap\Sigma_0} + c\left(\Gamma_0\left(\mathcal{I}_0 + \Delta_0\right) + \Gamma_0^2\right)
\end{aligned}$$

[19]The estimate for $\hat{\chi}$ is better on $\underline{C}_*$ than on the whole $\mathcal{K}$ due to the fact that, on $\underline{C}_*$, the estimates for η are stronger.

and, using the assumption 7.4.10 for the connection coefficients on Σ_0, we obtain

$$|r^{1-\frac{2}{p}}\mathrm{tr}\chi|_{p,S_*} \leq c\left((\mathcal{I}_0+\Delta_0)+\Gamma_0\,(\mathcal{I}_0+\Delta_0)+\Gamma_0^2\right) \leq c\,(\mathcal{I}_0+\Delta_0)$$
$$|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\hat{\chi}|_{p,S_*} \leq c\left((\mathcal{I}_0+\Delta_0)+\Gamma_0\,(\mathcal{I}_0+\Delta_0)+\Gamma_0^2\right) \leq c\,(\mathcal{I}_0+\Delta_0) \quad (7.4.20)$$

To control the norms relative to $\underline{\eta}$, $\mathrm{tr}\chi-\overline{\mathrm{tr}\chi}$ and $\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\underline{\chi}}$, we need first to estimate $(\mathbf{D}_3\log\Omega-\overline{\mathbf{D}_3\log\Omega})$ and $\nabla\!\!\!/\mathbf{D}_3\log\Omega$. This is achieved in the following way:
The function $\log\Omega$ satisfies, on the last slice, see 3.3.12, the elliptic equation

$$\triangle\!\!\!/\log\Omega=\frac{1}{2}\left[\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\eta}+\frac{1}{2}(\hat{\chi}\cdot\underline{\hat{\chi}}-\overline{\hat{\chi}\cdot\underline{\hat{\chi}}})-(\rho-\overline{\rho})\right] .$$

Differentiating this equation with respect to $\mathbf{D}_3$ we infer that $\mathbf{D}_3\log\Omega$ satisfies the following elliptic equation, see Lemma 7.7.4,

$$\triangle\!\!\!/(\Omega\mathbf{D}_3\log\Omega)=\mathrm{d}\!\!\!/\mathrm{iv}\,F_1+G_1-\overline{G_1} \quad (7.4.21)$$

$$\text{where}\quad F_1=\Omega\underline{\beta}+\tilde{F}_1\quad,\quad \tilde{F}_1=\left(\frac{3}{2}\Omega\eta\cdot\underline{\hat{\chi}}+\frac{1}{4}\Omega\eta\mathrm{tr}\underline{\chi}\right)$$
$$G_1=H+\frac{1}{4}\Omega\mathbf{D}_3(\hat{\chi}\cdot\underline{\hat{\chi}})-\frac{1}{2}(\Omega tr\underline{\chi})(\rho-\overline{\rho})+\frac{1}{4}(\Omega\mathrm{tr}\underline{\chi})(\hat{\chi}\cdot\underline{\hat{\chi}}-\overline{\hat{\chi}\cdot\underline{\hat{\chi}}}) .$$

Remark: The term in $\tilde{F}_1$

$$\frac{1}{4}\Omega\eta\mathrm{tr}\underline{\chi}=\frac{1}{2r}\Omega\eta+\frac{1}{4}\Omega\eta\left(\mathrm{tr}\underline{\chi}-\frac{2}{r}\right)$$

contains a linear term; observe, nevertheless, that $\mathrm{d}\!\!\!/\mathrm{iv}\,(\Omega\eta\mathrm{tr}\underline{\chi})=\Omega\mathrm{tr}\chi\mathrm{d}\!\!\!/\mathrm{iv}\,\eta$ plus quadratic terms which can be estimated by Γ_0^2. Thus using again the relation $\mu=\overline{\mu}$,

$$\mathrm{d}\!\!\!/\mathrm{iv}\,(\Omega\eta\mathrm{tr}\underline{\chi})=(\Omega\mathrm{tr}\chi)\left(\frac{1}{2}(\hat{\chi}\cdot\underline{\hat{\chi}}-\overline{\hat{\chi}\cdot\underline{\hat{\chi}}})-(\rho-\overline{\rho})\right)+O(\Gamma_0^2)\leq c(\Delta_0+\Gamma_0^2) .$$

Therefore applying Proposition 4.1.3 to 7.4.21 we obtain, for $p\in[2,4]$,

$$|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}(\mathbf{D}\!\!\!/_3\log\Omega-\overline{\mathbf{D}\!\!\!/_3\log\Omega})|_{p,S}\leq c(\Delta_0+\Gamma_0^2)$$
$$|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\nabla\!\!\!/\mathbf{D}\!\!\!/_3\log\Omega|_{p,S}\leq c(\Delta_0+\Gamma_0^2) \quad (7.4.22)$$

To estimate ${\mathcal O}^{*p,S}_0(\mbox{tr}\chi)$, see 7.4.3, we have to control $\mbox{tr}\chi-\mbox{tr}\underline{\chi}$. It is a simple calculation to show that its evolution equation is, see also Proposition 4.3.14, denoting $\tilde{W}=\mbox{tr}\chi-\overline{\mbox{tr}\chi}$, $\underline{\tilde{W}}=\mbox{tr}\underline{\chi}-\overline{\mbox{tr}\underline{\chi}}$,

$$\frac{d}{du}\tilde{W}+\frac{1}{2}\Omega\mbox{tr}\underline{\chi}\tilde{W}=-\frac{1}{2}\underline{\tilde{W}}(\overline{\Omega\mbox{tr}\chi})-\frac{1}{2}\overline{\underline{\tilde{W}}(\Omega\mbox{tr}\chi)}+F-\overline{F} \qquad (7.4.23)$$

where $F=2\Omega^2(|\eta|^2-\frac{1}{2}\overline{\hat{\chi}\hat{\underline{\chi}}})$. To use this evolution equation we need an estimate on $\underline{C}_*$ of $\underline{\tilde{W}}$. We observe that $\underline{V}=\Omega\mbox{tr}\underline{\chi}-\overline{\Omega\mbox{tr}\underline{\chi}}$ satisfies the following evolution equation, see 4.3.113 and Proposition 4.3.15 [20],

$$\begin{aligned}\frac{d}{du}\underline{V}+\Omega\mbox{tr}\chi\underline{V} &= \frac{1}{2}\underline{V}^2-\overline{\underline{V}^2}+2(\Omega\mathbf{D}_3\log\Omega)\underline{V}-\left[\Omega^2|\hat{\underline{\chi}}|^2-\overline{\Omega^2|\hat{\underline{\chi}}|^2}\right]\\ &+2(\overline{\Omega\mbox{tr}\underline{\chi}})\left(\Omega\mathbf{D}_3\log\Omega-\overline{\Omega\mathbf{D}_3\log\Omega}\right)\\ &-2\overline{(\Omega\mbox{tr}\underline{\chi})\left(\Omega\mathbf{D}_3\log\Omega-\overline{\Omega\mathbf{D}_3\log\Omega}\right)}\end{aligned}$$

Applying to it the evolution Lemma 4.1.5 and Gronwall's lemma, using the estimate for $\hat{\underline{\chi}}$ in the bootstrap assumption 7.4.12 and the estimate, see 7.4.22,

$$|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}(\Omega\mathbf{D}_3\log\Omega-\overline{\Omega\mathbf{D}_3\log\Omega})|_{p,S_*}\leq c\,({\mathcal I}_0+\Delta_0)$$

and, finally, using the assumptions for the connection coefficients on Σ_0 [21], see 7.4.7, we obtain the following estimate for $\underline{\tilde{W}}$,

$$|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}\underline{\tilde{W}}|_{p,S_*}\leq c\left(({\mathcal I}_0+\Delta_0)+\Gamma_0\,({\mathcal I}_0+\Delta_0)+\Gamma_0^2\right)\leq c\,({\mathcal I}_0+\Delta_0) \quad (7.4.24)$$

Therefore the right hand side of 7.4.23 can be bounded in the $|\cdot|_{p,S}$ norm as

$$\left|\frac{1}{2}\underline{\tilde{W}}(\overline{\Omega\mbox{tr}\chi})+\frac{1}{2}\overline{\underline{\tilde{W}}(\Omega\mbox{tr}\chi)}-(F-\overline{F})\right|_{p,S_*}\leq\frac{1}{r^2\tau_-^{\frac{3}{2}}}c(\Gamma_0)\,({\mathcal I}_0+\Delta_0)\ .$$

Applying the evolution Lemma 4.1.5 and Gronwall's lemma to the evolution equation 7.4.23 for $\tilde{W}=\mbox{tr}\chi-\overline{\mbox{tr}\chi}$, using assumptions 7.4.7 relative to Σ_0, we conclude that

$$\begin{aligned}|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\mbox{tr}\chi-\overline{\mbox{tr}\chi})|_{p,S_*} &\leq |r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\mbox{tr}\chi-\overline{\mbox{tr}\chi})|_{p,\underline{C}_*\cap\Sigma_0}+c(\Gamma_0)\,({\mathcal I}_0+\Delta_0)\\ &\leq c\,({\mathcal I}_0+\Delta_0) \qquad (7.4.25)\end{aligned}$$

[20] In Proposition 4.3.15 the estimate of $\underline{\tilde{W}}$ depends also on ${\mathcal I}_*$, due to the factor $\underline{\omega}-\overline{\underline{\omega}}$. Here the dependance on ${\mathcal I}_*$ is absente as, on $\underline{C}_*$, $\underline{\omega}-\overline{\underline{\omega}}$ is estimated differently, see the proof of Proposition 7.4.4.

[21] $\underline{\mathcal O}^{\infty}_{[1]}(\Sigma_0\backslash K)(\Sigma_0)\leq{\mathcal I}_0$, ${\mathcal O}^{\infty}_{[1]}(\Sigma_0)\leq{\mathcal I}_0$

which, together with the inequalities 7.4.20, completes the proof of the first inequality of 7.4.13 for the non underlined connection coefficients. Inequality 7.4.24 allows also to conclude that

$$\begin{aligned} |r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}(\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\underline{\chi}})|_{p,S_*} &\leq |r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}(\mathrm{tr}\underline{\chi}-\overline{\mathrm{tr}\underline{\chi}})|_{p,\underline{C}_*\cap\Sigma_0} + c(\Gamma_0)\,(\mathcal{I}_0+\Delta_0) \\ &\leq c\,(\mathcal{I}_0+\Delta_0) \end{aligned} \tag{7.4.26}$$

We shall now estimate the angular derivatives $\nabla\!\!\!/\mathrm{tr}\chi$, $\nabla\!\!\!/\hat{\chi}$ and $\nabla\!\!\!/\eta$. The last term has been already estimated, see 7.4.18. To estimate $\nabla\!\!\!/\mathrm{tr}\chi$ we take the tangential derivative of the first equation of 7.4.19. The estimate then proceeds in the same way as the one for $\mathrm{tr}\chi$. Observe that, as the foliation on the last slice is canonical, the dependance on the Riemann components disappear, as $\nabla\!\!\!/\overline{\rho}=0$. We obtain the result, for $p\in[2,4]$,

$$|r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!/\mathrm{tr}\chi|_{p,S_*}\leq c(\mathcal{I}_0+\Delta_0) \tag{7.4.27}$$

To estimate $\nabla\!\!\!/\hat{\chi}$ we can proceed in the same way, differentiating tangentially the evolution equation of $\hat{\chi}$ along $\underline{C}_*$, see 7.4.19. Observe that the right hand side of this evolution equation for $\nabla\!\!\!/\hat{\chi}$ does not contain curvature terms. We obtain the estimate

$$|r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!/\hat{\chi}|_{p,S_*}\leq c(\mathcal{I}_0+\Delta_0) \tag{7.4.28}$$

Remark: It is interesting to remark that the estimate of $\nabla\!\!\!/\hat{\chi}$ could also be obtained from the Hodge system,

$$\mathrm{d}\!\!\!/\mathrm{iv}\,\hat{\chi}=\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi-\zeta\cdot\chi-\beta\ ,$$

see 4.3.13, applying to it Proposition 4.1.3. To achieve this result, however, we need a better estimate for ζ. This is done below.

To estimate ζ we consider its evolution equation along the incoming null hypersurfaces, see 3.1.45,

$$\mathbf{D}\!\!\!\!/_3\zeta+2\underline{\chi}\cdot\zeta-\mathbf{D}_3\nabla\!\!\!/\log\Omega=-\underline{\beta}$$

Due to the "better" estimate for $\nabla\!\!\!/\mathbf{D}_3\log\Omega$, see 7.4.22, it follows immediately that, on $\underline{C}_*$, ζ satisfies the inequality

$$|r^{2-2/p}\tau_-^{\frac{1}{2}}\zeta|_{p,S_*}\leq c(\mathcal{I}_0+\Delta_0) \tag{7.4.29}$$

This estimate for ζ and the analogous one for its tangential derivative, obtained exactly in the same way, together with the estimates for η and $\nabla\!\!\!/\eta$, see 7.4.18, allows also to conclude that $\underline{\eta}$ and $\nabla\!\!\!/\underline{\eta}$ satisfy on $\underline{C}_*$ the same estimate as η and $\nabla\!\!\!/\eta$,

$$\mathcal{O}_0^{p,S}(\underline{C}_*)(\underline{\eta})+\mathcal{O}_1^{p,S}(\underline{C}_*)(\underline{\eta})\leq c(\Delta_0+\Gamma_0^2)\leq c(\mathcal{I}_0+\Delta_0) \tag{7.4.30}$$

To control $\nabla\!\!\!/\mathrm{tr}\underline{\chi}$ we derive the evolution equation for $\mathrm{tr}\underline{\chi}$, along the incoming "cones", and using again the bootstrap assumption 7.4.12 for $\hat{\underline{\chi}}$ and $\nabla\!\!\!/\hat{\underline{\chi}}$ and the estimate 7.4.47 for $\nabla\!\!\!/\underline{\omega}$, we obtain

$$\begin{aligned}|r^{3-\frac{2}{p}}\tau_-^{\frac{1}{2}}\nabla\!\!\!/\mathrm{tr}\underline{\chi}|_{p,S_*} &\leq |r^{3-\frac{2}{p}}\tau_-^{\frac{1}{2}}\nabla\!\!\!/\mathrm{tr}\underline{\chi}|_{p,\underline{C}_*\cap\Sigma_0}+c\Big(\Gamma_0(\mathcal{I}_0+\Delta_0)+\Gamma_0^2\Big) \\ &\leq c\Big((\mathcal{I}_0+\Delta_0)+\Gamma_0(\mathcal{I}_0+\Delta_0)+\Gamma_0^2\Big)\leq c\,(\mathcal{I}_0+\Delta_0)\end{aligned} \tag{7.4.31}$$

To complete the proof of the inequalities in 7.4.13 we are left with estimating, on $\underline{C}_*$, $\hat{\underline{\chi}}$ and $\nabla\!\!\!/\hat{\underline{\chi}}$. These estimates are obtained applying Proposition 4.1.3 to the Hodge system

$$\mathrm{d}\!\!\!/\mathrm{iv}\,\hat{\underline{\chi}}=\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\underline{\chi}+\zeta\cdot\underline{\chi}-\underline{\beta}\ ,$$

obtaining

$$\begin{aligned}&|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}\hat{\underline{\chi}}|_{p,S_*}\leq c\,((\mathcal{I}_0+\Delta_0)+\Gamma_0(\mathcal{I}_0+\Delta_0))\leq c\,(\mathcal{I}_0+\Delta_0) \\ &|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\tau_-^{\frac{3}{2}}\nabla\!\!\!/\hat{\underline{\chi}}|_{p,S_*}\leq c\,((\mathcal{I}_0+\Delta_0)+\Gamma_0(\mathcal{I}_0+\Delta_0))\leq c\,(\mathcal{I}_0+\Delta_0)\end{aligned} \tag{7.4.32}$$

To complete the proof of part i) we have still to prove the inequalities 7.4.14. We obtain the estimates for $\overline{\mathrm{tr}\chi}$ and $\overline{\mathrm{tr}\underline{\chi}}$, $p\in[2,4]$,

$$\begin{aligned}&|r^{2-\frac{2}{p}}(\overline{\mathrm{tr}\chi}-\frac{2}{r})|_{p,S_*}\leq c(\Gamma_0)\left((\mathcal{I}_0+\Delta_0)+\Gamma_0^2\right) \\ &|r^{1-\frac{2}{p}}\tau_-(\overline{\mathrm{tr}\underline{\chi}}+\frac{2}{r})|_{p,S_*}\leq c(\Gamma_0)\left((\mathcal{I}_0+\Delta_0)+\Gamma_0^2\right)\end{aligned} \tag{7.4.33}$$

starting again from the evolution equations for $\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$ along $\underline{C}_*$ and using the estimate 7.2.8 for $\left(\overline{\mathrm{tr}\theta}-\frac{2}{r}\right)$ at $\Sigma_0\cap\underline{C}_*$, proved in Theorem **M3**.

The remaining estimates of 7.4.14 are obtained in the following way. We control $\Delta\!\!\!\!/\log\Omega$ from the estimates, we already have, of $\mathrm{d}\!\!\!/\mathrm{iv}\,\eta$ and $\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\eta}$. From it we control $\log\Omega$, recalling that on $\underline{C}_*$, $\overline{\log\Omega}=0$, and its first and second tangential derivatives. This result is collected in the following proposition,

Proposition 7.4.3 *Under the bootstrap assumptions 7.4.12 and the assumptions 7.4.6 for the Riemann components, we have, for any* $p \geq 2$*,*

$$\begin{aligned}
&|r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!/^2\log\Omega|_{p,S_*} \leq c|r^{3-2/p}\tau_-^{\frac{1}{2}}\triangle\!\!\!/\log\Omega|_{p,S_*} \leq c(\Gamma_0)(\mathcal{I}_0+\Delta_0)\\
&|r^{2-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!/\log\Omega|_{p,S_*} \leq c|r^{3-2/p}\tau_-^{\frac{1}{2}}\triangle\!\!\!/\log\Omega|_{p,S_*} \leq c(\Gamma_0)(\mathcal{I}_0+\Delta_0) \quad (7.4.34)\\
&|r^{1-2/p}\tau_-^{\frac{1}{2}}\log 2\Omega|_{p,S_*} \leq c|r^{3-2/p}\tau_-^{\frac{1}{2}}\triangle\!\!\!/\log\Omega|_{p,S_*} \leq c(\Gamma_0)(\mathcal{I}_0+\Delta_0)
\end{aligned}$$

To complete part i) we have still to control $\mathcal{O}_0^{p,S}(\mathbf{D}_3\underline{\omega})(\underline{C}_*)$. This is discussed in the next subsection together with the proof of Proposition 7.4.2.

Proof of Corollary 7.4.1: The estimate of $(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})$ is a easy consequence of the previous estimates 7.4.25, 7.4.26 and 7.4.33, the final result is

$$\sup_{\underline{C}_*}|r\tau_-(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})| \leq c\,(\mathcal{I}_0+\Delta)\ .$$

7.4.3 Implementation of Proposition 7.4.2

The proof of Proposition 7.4.2 is similar to that one of Proposition 7.4.1. We use, in fact, for the norms associated to $\chi,\underline{\chi}$ the evolution equations obtained by differentiating the previous ones.
The estimates for the second tangential derivatives $\nabla\!\!\!/^2\eta$ are obtained differentiating the Hodge system 7.4.17 and those for $\nabla\!\!\!/^2\underline{\eta}$ differentiating twice the evolution equation for ζ and taking into account the estimate for $\nabla\!\!\!/^2\mathbf{D}\!\!\!\!/_3\log\Omega$.
The control of the norms $\mathcal{O}_0^*(\mathbf{D}\!\!\!\!/_3^2\underline{\omega})$, $\mathcal{O}_1^*(\mathbf{D}\!\!\!\!/_3\underline{\omega})$ and $\mathcal{O}_2^*(\underline{\omega})$ is obtained, via elliptic estimates, in Propositions 7.4.4, 7.4.5. These propositions allow to control also the norm $\mathcal{O}_0^{*p,S}(\mathbf{D}_3\underline{\omega})$ completing, therefore, the proof of Proposition 7.4.1.
We omit the details concerning the estimates for the second angular derivatives of $\mathrm{tr}\chi,\hat{\chi},\mathrm{tr}\underline{\chi},\underline{\hat{\chi}},\eta,\underline{\eta}$ and we sketch below the estimate for the third derivatives of $\mathrm{tr}\chi$ and $\underline{\omega}$.

Estimate for $\mathcal{O}_3^*(\mathrm{tr}\chi)$:

The estimate of $||r^3\nabla\!\!\!/^3\mathrm{tr}\chi||_{L^2(\underline{C}_*\cap V(u,\underline{u}_*))}$ procceds in a way similar to the estimate of $\nabla\!\!\!/\underline{\psi\!\!\!/}$ in Proposition 4.5.1 where $\underline{\psi\!\!\!/} \equiv \mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\psi\!\!\!/}+\Omega^{-1}\mathrm{tr}\chi\rho$, but is simpler due to the canonical foliation of $\underline{C}_*$. We start with the evolution equation for $\nabla\!\!\!/^3\mathrm{tr}\chi$ obtained deriving three times tangentially the evolution equation, along $\underline{C}_*$, for $\mathrm{tr}\chi$,

$$\frac{d}{du}(\mathrm{tr}\chi)+\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}(\mathrm{tr}\chi) = \mathcal{X}_0 \qquad (7.4.35)$$

where

$$\mathcal{X}_0 = -\Omega(\mathbf{D}_3 \log \Omega)\mathrm{tr}\chi + 2\Omega|\eta|^2 + \Omega\left(2\overline{\rho} - \overline{\hat{\chi}\underline{\hat{\chi}}}\right) \tag{7.4.36}$$

satisfies, using the results of Proposition 7.4.1

$$|\mathcal{X}_0| = O(r^{-3}) + O(r^{-2}\tau_-^{-\frac{3}{2}}) \tag{7.4.37}$$

The evolution equation for $\nabla\!\!\!/\mathrm{tr}\chi$ can be written as

$$\frac{d}{du}(\nabla\!\!\!/\mathrm{tr}\chi) + \Omega\mathrm{tr}\underline{\chi}(\nabla\!\!\!/\mathrm{tr}\chi) + (\mathbf{D}_3 \log \Omega)(\nabla\!\!\!/\mathrm{tr}\chi) = -\underline{\hat{\chi}}(\nabla\!\!\!/\mathrm{tr}\chi) + \mathcal{X}_1 \tag{7.4.38}$$

where

$$\mathcal{X}_1 = \left[-\frac{1}{2}(\nabla\!\!\!/\mathrm{tr}\underline{\chi})\mathrm{tr}\chi - (\nabla\!\!\!/\mathbf{D}_3 \log \Omega)\mathrm{tr}\chi + 4\eta \cdot \nabla\!\!\!/\eta + \nabla\!\!\!/ \log \Omega(\mathbf{D}_3 \log \Omega)\right] \tag{7.4.39}$$

and

$$|\mathcal{X}_1| = O(r^{-4}\tau_-^{-\frac{1}{2}}) + O(r^{-3}\tau_-^{-\frac{3}{2}}) \tag{7.4.40}$$

Iterating the procedure we obtain for $\nabla\!\!\!/^3\mathrm{tr}\chi$ the evolution equation

$$\frac{d}{du}(\nabla\!\!\!/^3\mathrm{tr}\chi) + 2\Omega\mathrm{tr}\underline{\chi}(\nabla\!\!\!/^3\mathrm{tr}\chi) + (\mathbf{D}_3 \log \Omega)(\nabla\!\!\!/^3\mathrm{tr}\chi) = -\underline{\hat{\chi}}(\nabla\!\!\!/^3\mathrm{tr}\chi) + \mathcal{X}_3 \tag{7.4.41}$$

where $\mathcal{X}_3$ is a term, up to fourth order in the connection coefficients up to second tangential derivatives and linear in the Riemann components up to first derivatives. Its asymptotic behaviour is

$$|r^{-\frac{2}{p}}\mathcal{X}_3|_{p,S} = O(r^{-6}\tau_-^{-\frac{1}{2}}) + O(r^{-5}\tau_-^{-\frac{3}{2}}) \tag{7.4.42}$$

Therefore applying the evolution Lemma 4.1.5 and the Gronwall Lemma we obtain

$$\begin{aligned}
|r^{4-\frac{2}{p}}\nabla\!\!\!/^3\mathrm{tr}\chi|_{p=2,S}(u,\underline{u}_*) &\leq c|r^{4-\frac{2}{p}}\nabla\!\!\!/^3\mathrm{tr}\chi|_{p=2,\underline{C}_*\cap\Sigma_0} + \int_{u_0(\underline{u}_*)}^{u} du'|r^{4-\frac{2}{p}}\mathcal{X}_3|_{p=2,S}\\
&\leq c|r^{4-\frac{2}{p}}\nabla\!\!\!/^3\mathrm{tr}\chi|_{p=2,\underline{C}_*\cap\Sigma_0} + c\frac{(\mathcal{I}_0+\Delta)}{r(u,\underline{u}_*)u^{\frac{1}{2}}}
\end{aligned} \tag{7.4.43}$$

From this expression we obtain, with $\epsilon > 0$,

$$\begin{aligned}
&||r^3\nabla\!\!\!/^3\mathrm{tr}\chi||_{L^2(\underline{C}_*\cap V(u,\underline{u}_*))} \qquad (7.4.44)\\
&\leq c\left(\int_{u_0(\underline{u}_*)}^{u}|r^3\nabla\!\!\!/^3\mathrm{tr}\chi|^2_{p=2,\underline{C}_*\cap\Sigma_0}\right)^{\frac{1}{2}} + c(\mathcal{I}_0+\Delta)\left(\int_{u_0(\underline{u}_*)}^{u}\frac{1}{r(u,\underline{u}_*)^2u}\right)^{\frac{1}{2}}\\
&\leq c|u_0(\underline{u}_*)-u|^{\frac{1}{2}}\left(|r^3\nabla\!\!\!/^3\mathrm{tr}\chi|_{p=2,\underline{C}_*\cap\Sigma_0}\right) + c(\mathcal{I}_0+\Delta)\frac{1}{r^{1-\epsilon}(u,\underline{u}_*)u^{\epsilon}}
\end{aligned}$$

The estimate of the first term is connected to the estimate of $\nabla\!\!\!/^3\text{tr}\theta$ on Σ_0. In fact, on Σ_0, $\text{tr}\chi = \text{tr}\theta - k_{\tilde{N},\tilde{N}}$ and the second fundamental form is controlled from the assumption on J_K. Therefore we are left with estimating $|r^3\nabla\!\!\!/^3\text{tr}\theta|_{p=2,\underline{C}_*\cap\Sigma_0}$. Observing that $\underline{C}_*\cap\Sigma_0$ is, by construction, a leave of the canonical foliation on Σ_0 we can integrate the evolution equation for $\text{tr}\theta$, 7.1.25, after deriving it tangentially three times,

$$\nabla_{\tilde{N}}(\nabla\!\!\!/^3\text{tr}\theta) + \frac{5}{2}\text{tr}\theta(\nabla\!\!\!/^3\text{tr}\theta) = -2\hat{\theta}\cdot\nabla\!\!\!/^3\hat{\theta} + \mathcal{P}_3 \tag{7.4.45}$$

wher $\mathcal{P}_3$ depends on the second fundamental form up to third derivatives and to terms with less tangential derivative. Taking into account that, due to the *strong asymptotic flatness*, $\lim_{r\to\infty}|r^{\frac{11}{2}}\nabla\!\!\!/^3\text{tr}\theta| = 0$, and the estimate 7.2.9, we conclude that

$$|r^{\frac{11}{2}-\frac{2}{p}}\nabla\!\!\!/^3\text{tr}\theta|_{p=2,C_*\cap\Sigma_0} \leq c(\mathcal{I}_0+\Delta)$$

This implies

$$|r^3\nabla\!\!\!/^3\text{tr}\theta|_{p=2,C_*\cap\Sigma_0} \leq c\frac{(\mathcal{I}_0+\Delta)}{r^{\frac{3}{2}}(u_0,\underline{u}_*)} \tag{7.4.46}$$

which, substituted in 7.4.44, completes the estimate of $r^{\frac{1}{2}}||r^3\nabla\!\!\!/^3\text{tr}\chi||_{L^2(\underline{C}_*\cap V(u,\underline{u}_*))}$.

Estimate for $\mathcal{O}_3^*(\underline{\omega})$: This estimate is a corollary of the elliptic estimates proved in the next subsection.

The elliptic estimates on the last slice

Osservazione 7.4.1 *Si osservi che come detto nella nota 23 su $\underline{C}_*$ il risultato per $\Omega\mathbf{D}_3\log\Omega$ è migliore di quello che si ottiene in $\mathcal{K}$ anche se in $\mathcal{K}$ si usa la foliazione canonica. Questo non e‘ vero per le derivate tangenziali prime e seconde che hanno la stessa stima in $\mathcal{K}$ se si usa la foliazione canonica. Si veda il corollario 4.4.1. Questa osservazione è importante perchè è connessa al fatto che $\Omega\mathbf{D}_3\log\Omega$ è diverso da zero anche nel caso di Schwarzschild . Piu‘ precisamente la ragione è la seguente: su $\underline{C}_*$, $\overline{\log\Omega}=0$ mentre questo non è vero all'interno di $\mathcal{K}(\lambda_0,\nu_*)$. Pertanto la stima su $\underline{C}_*$ è come se fosse una stima di $\log\Omega-\overline{\log\Omega}$ è quindi migliore perchè è $\overline{\log\Omega}$ che si "ricorda" di $\overline{\rho}$.*

Proposition 7.4.4 *Under the same assumptions* [22] *as in Proposition 7.4.1 there exists a generic constant c such that* [23], *for* $p \in [2,4]$, *on* $\underline{C}_*$,

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}\Omega\mathbf{D}_3\log\Omega|_{p,S}(u,\underline{u}_*) \leq c(\mathcal{I}_0+\Delta_0)\\
&|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\nabla\!\!\!\!/(\Omega\mathbf{D}_3\log\Omega)|_{p,S}(u,\underline{u}_*) \leq c_0(\mathcal{I}_0+\Delta_0) \qquad (7.4.47)\\
&|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}\nabla\!\!\!\!/^2(\Omega\mathbf{D}_3\log\Omega)|_{p,S}(u,\underline{u}_*) \leq c_0(\mathcal{I}_0+\Delta_0+\Delta_1)
\end{aligned}
$$

Proposition 7.4.5 *Under the same assumptions as in Proposition 7.4.2 there exists a generic constant c such that*

$$
\begin{aligned}
&|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}(\Omega\mathbf{D}_3)^2\log\Omega|_{p,S}(u,\underline{u}_*) \leq c(\mathcal{I}_0+\Delta_0)\\
&|r^{2-\frac{2}{p}}\tau_-^{\frac{5}{2}}\nabla\!\!\!\!/(\Omega\mathbf{D}_3)^2\log\Omega)|_{p,S}(u,\underline{u}_*) \leq c(\mathcal{I}_0+\Delta_0+\Delta_1)\\
&|r^{1-\frac{2}{p}}\tau_-^{\frac{7}{2}}(\Omega\mathbf{D}_3)^3\log\Omega|_{p,S_*}(u,\underline{u}_*) \leq c(\mathcal{I}_0+\Delta_0+\Delta_1) \qquad (7.4.48)
\end{aligned}
$$

The proofs of these propositions are in the appendix to this chapter.

7.5 The last slice rotation deformation estimates

We prove in this section Theorem **M5** which we recall here,

Theorem M5: *Assume that, relative to a canonical foliation on* $\underline{C}_*$, *we have*

$$\mathcal{R}_{[2]}+\underline{\mathcal{R}}_{[2]} \leq \Delta$$

$$\mathcal{O}_{[2]}(\underline{C}_*\cap\Sigma_0)+\mathcal{O}_3(\Sigma_0\backslash K)+\underline{\mathcal{O}}_{[3]}(\Sigma_0\backslash K) \leq c\mathcal{I}_0\ .$$

If Δ, $\mathcal{I}_0$ *are sufficiently small, the following estimate holds*

$$\mathcal{D}(\underline{C}_*) \leq c(\mathcal{I}_0+\Delta)\ .$$

The proof of this theorem is divided in two propositions.

[22] We stress here that the proof of this proposition does not require the completion of Proposition 7.4.1.

[23] Observe that this result for $\Omega\mathbf{D}_3\log\Omega$ on $\underline{C}_*$ is stronger than the one which holds on $\mathcal{K}$, see Propositions 4.3.4.

Proposition 7.5.1 *Under the same assumptions of Proposition 7.4.2, assuming also, that*

$$\mathcal{D}_{[1]}(\underline{C}_* \cap \Sigma_0) \leq c(\mathcal{I}_0 + \Delta) \tag{7.5.1}$$
$$\left(\int_{u_0(\underline{u}')}^{u} du' |r^{2-\frac{2}{p}} \nabla\!\!\!/^2 H|^2_{p=2,\underline{C}_*\cap\Sigma_0}\right)^{\frac{1}{2}} \leq c(\mathcal{I}_0 + \Delta)$$

then the following estimates hold, for $p \in [2, 4]$,

$$|r^{-1(i)}O|_{p,S_*} \leq c\,(\mathcal{I}_0 + \Delta) \quad , \quad |\nabla\!\!\!/^{(i)}O|_{p,S_*} \leq c\,(\mathcal{I}_0 + \Delta) \tag{7.5.2}$$
$$|r^{(i)}H_{ab}|_{p,S_*} \leq c\,(\mathcal{I}_0 + \Delta) \quad , \quad |r^{2-\frac{2}{p}}\nabla\!\!\!/^{(i)}H_{ab}|_{p,S_*} \leq c\,(\mathcal{I}_0 + \Delta)$$

and

$$||r\nabla\!\!\!/^2 H||_{L^2(\underline{C}_*\cap V(u,\underline{u}_*))} = \leq c\,(\mathcal{I}_0 + \Delta) \tag{7.5.3}$$

Proposition 7.5.2 *Assume that, relative to a canonical foliation on* $\Sigma_0 \backslash K$,

$$\mathcal{R}_{[2]} + \underline{\mathcal{R}}_{[2]} \leq \Delta$$
$$\textit{and} \qquad \mathcal{O}_{[2]}(\underline{C}_* \cap \Sigma_0) + \mathcal{O}_{[3]}(\Sigma_0 \backslash K) + \underline{\mathcal{O}}_{[3]}(\Sigma_0 \backslash K) \leq c\mathcal{I}_0$$

then, if Δ *and* $\mathcal{I}_0$ *are sufficiently small,*

$$\mathcal{D}_{[1]}(\underline{C}_* \cap \Sigma_0) \leq c(\mathcal{I}_0 + \Delta) \tag{7.5.4}$$
$$\left(\int_{u_0(\underline{u}')}^{u} du' |r^{2-\frac{2}{p}} \nabla\!\!\!/^2 H|^2_{p=2,\underline{C}_*\cap\Sigma_0}\right)^{\frac{1}{2}} \leq c(\mathcal{I}_0 + \Delta)$$

Proof of Proposition 7.5.1: We first recall the construction of the rotation vector fields on $\underline{C}_*$. We start with the vector fields defined in $S_*(\lambda_1) = \underline{C}_* \cap \Sigma_0$ where we have already defined the rotation group, see [Ch-Kl], Chapter 3, and the proof of next proposition. This is achieved in a similar way as we did for the extension discussed in Chapter 4, subsection 4.6.1. Let $q \in S_*(\lambda)$ be an arbitrary point of $\underline{C}_*$. As $S_*(\lambda)$ is diffeomorphic, via $\underline{\phi}_\Delta$, to $S_*(\lambda_1)$, with $\Delta = \lambda - \lambda_1$, there exists a point $p \in S_*(\lambda_1)$ such that $q = \underline{\phi}_\Delta(p)$. We define the element O_* of the rotation group operating on $q \in \underline{C}_*$ in the following way[24]:

$$(O_*; q) \equiv \underline{\phi}_\Delta(O; p)$$

[24]At the differential level the extension is defined through: ${}^{(i)}O_* \equiv \underline{\phi}_{\Delta *}{}^{(i)}O$.

where $(O_*; q)$ is a point of $S_*(\lambda)$ and $(O; p)$ is the point of $S_*(\lambda_1)$ obtained applying O [25] to the point p. This extension of the action of the rotation group to the whole $\underline{C}_*$ satisfies

$$O_* = \underline{\phi}_t^{-1} O_* \underline{\phi}_t \ .$$

This implies that the generators, ${}^{(i)}O_*$, satisfy

$$[\underline{N}, {}^{(i)}O_*] = 0$$

From the previous definitions we easily check that

$$[{}^{(i)}O_*, {}^{(j)}O_*] = \epsilon_{ijk} {}^{(k)}O_* \ , \ i, j, k \in \{1, 2, 3\}$$

In conclusion the generators ${}^{(i)}O_*$, defined on the whole $\underline{C}_*$, tangent to $S_*(\lambda)$ at each point, satisfy

$$\begin{aligned} &[{}^{(i)}O_*, {}^{(j)}O_*] = \epsilon_{ijk} {}^{(k)}O_* \\ &[\underline{N}, {}^{(i)}O_*] = 0 \\ &g({}^{(i)}O_*, e_3) = g({}^{(i)}O_*, e_4) = 0 \end{aligned} \tag{7.5.5}$$

Moreover as $\underline{N} = \Omega \hat{\underline{N}} = \Omega e_3$, it follows

$$[{}^{(i)}O_*, e_3] = {}^{(i)}F e_3 \tag{7.5.6}$$

where ${}^{(i)}F \equiv -{}^{(i)}O_{*c}(\nabla\!\!\!/_c \log \Omega)$, ${}^{(i)}O_{*c} \equiv g({}^{(i)}O_*, e_c)$.

Proposition 7.5.3 *the quantities* ${}^{(i)}O_{*a}$ *and* $(\nabla\!\!\!/ {}^{(i)}O_*)_{ab}$ *satisfy the following evolution equations*

$$\frac{d}{du} {}^{(i)}O_{*b} = \Omega \underline{\chi}_{bc} {}^{(i)}O_{*c} \tag{7.5.7}$$

$$\begin{aligned} \frac{d}{du}(\nabla\!\!\!/ {}^{(i)}O_*)_{ab} = \Omega \Big[&\hat{\underline{\chi}}_{bc}(\nabla\!\!\!/ {}^{(i)}O_*)_{ac} - \hat{\underline{\chi}}_{ac}(\nabla\!\!\!/ {}^{(i)}O_*)_{cb} + {}^{(i)}O_{*c}(\underline{\chi}_{cb}\eta_a - \underline{\chi}_{ca}\eta_b) \\ &+ {}^{(i)}O_{*c}R_{3abc} - {}^{(i)}O_{*c}\underline{\chi}_{cb}\zeta_a + \underline{\chi}_{ab}({}^{(i)}O_{*c}\eta_c) + {}^{(i)}O_{*c}(\nabla\!\!\!/_a \underline{\chi})_{cb} \Big] \end{aligned} \tag{7.5.8}$$

Proof: From $[\underline{N}, {}^{(i)}O_*] = 0$ we infer that

$$\Omega \mathbf{D}_3 {}^{(i)}O_* = {}^{(i)}O_{*c}(\nabla\!\!\!/_c \log \Omega)\underline{N} + \Omega \mathbf{D}_{{}^{(i)}O_*} e_3 \tag{7.5.9}$$

[25] The rotation group operating on $S_*(\lambda_1) = \underline{C}_* \cap \Sigma_0$ is the one obtained extending on Σ_0 the rotation group defined, asymptotically, at spatial infinity.

and, choosing a moving frame satisfying $\not{D}_3 e_b = 0$, we obtain 7.5.7.
To obtain an evolution equation for $\not\nabla_a {}^{(i)}O_{*b} \equiv (\not\nabla^{(i)}O_*)_{ab}$ we start from equation 7.5.9, which we rewrite as

$$\mathbf{D}_3{}^{(i)}O_* = {}^{(i)}O_{*c}\underline{\chi}_{cb}e_b + {}^{(i)}O_{*c}\eta_c e_3$$

Using the commutation relations proved in the appendix to Chapter 4, see Proposition 4.8.1, we obtain

$$\begin{aligned}\frac{d}{du}(\not\nabla^{(i)}O_*)_{ab} &= \Omega\Big[\hat{\underline{\chi}}_{bc}(\not\nabla^{(i)}O_*)_{ac} - \hat{\underline{\chi}}_{ac}(\not\nabla^{(i)}O_*)_{cb} + {}^{(i)}O_{*c}(\underline{\chi}_{cb}\eta_a - \underline{\chi}_{ca}\eta_b)\\ &\quad + {}^{(i)}O_{*c}R_{3abc} - {}^{(i)}O_{*c}\underline{\chi}_{cb}\zeta_a + \underline{\chi}_{ab}({}^{(i)}O_{*c}\eta_c) + {}^{(i)}O_{*c}(\not\nabla_a\underline{\chi})_{cb}\Big]\end{aligned}$$

Using the evolution equations 7.5.7 and 7.5.8, we obtain immediately the estimates 7.5.2 for ${}^{(i)}O_*$. To estimate ${}^{(i)}Z_a$ and ${}^{(i)}H_{ab}$ we recall their last slice expressions, see 4.6.11,

$$\begin{aligned}{}^{(i)}Z_a &\equiv \frac{1}{4}\Big(g(\mathbf{D}_a{}^{(i)}O_*, e_3) + g(\mathbf{D}_3{}^{(i)}O_*, e_a)\Big) = \frac{1}{4}{}^{(i)}\pi_{3a}\\ {}^{(i)}H_{ab} &\equiv \frac{1}{2}\Big(g(\not\nabla_a{}^{(i)}O_*, e_b) + g(\not\nabla_b{}^{(i)}O_*, e_a)\Big) = \frac{1}{2}{}^{(i)}\pi_{ab}\end{aligned}$$

An elementary calculation shows that on $\underline{C}_*$, ${}^{(i)}Z_a = 0$. To prove the remaining estimates of 7.5.2 and 7.5.3 we need the evolution equation of ${}^{(i)}H_{ab}$ and of its tangential derivatives along $\underline{C}_*$, up to second order. These evolution equations can be obtained as in Chapter 4, subsection 4.8.4, with the obvious modifications. The final result is

$$\begin{aligned}\frac{d}{du}{}^{(i)}H_{ab} &= -\Omega\left(\hat{\underline{\chi}}_{ac}{}^{(i)}H_{cb} + \hat{\underline{\chi}}_{bc}{}^{(i)}H_{ca}\right) + \Omega^{(i)}O_c(\not\nabla_c\underline{\chi})_{ab} + \Omega\underline{\chi}_{ab}(\not\nabla_c\log\Omega)^{(i)}O_c\\ &\quad + \Omega\left(\hat{\underline{\chi}}_{bc}\not\nabla_a{}^{(i)}O_c + \hat{\underline{\chi}}_{ac}\not\nabla_b{}^{(i)}O_c\right)\\ \frac{d}{du}(\not\nabla_c H_{ab}) + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}(\not\nabla_c H_{ab}) &= -\Omega\hat{\underline{\chi}}_{ad}(\not\nabla H)_{cdb} + \hat{\underline{\chi}}_{ad}\left[(\not\nabla H)_{bdc} - (\not\nabla H)_{dbc}\right]\\ &\quad + \hat{\underline{\chi}}_{bd}\left[(\not\nabla H)_{adc} - (\not\nabla H)_{dac}\right] + \underline{\mathcal{H}}_1 \end{aligned} \tag{7.5.10}$$

From these evolution equations the estimates in the last line of 7.5.2 follow immediately.

To prove inequality 7.5.3, which involves up to third derivatives for the connection coefficients, we need the evolution equation for the second tangential derivative of ${}^{(i)}H$ along the incoming null hypersurface. Proceeding as in Chapter 4, with the obvious modifications we obtain an evolution equation of the form

$$\frac{d}{du}(\not\nabla^2 H) + \Omega\mathrm{tr}\underline{\chi}(\not\nabla^2 H) = \hat{\underline{\chi}}(\not\nabla^2 H) + (L_O\not\nabla^2\underline{\chi}) + \underline{\mathcal{H}}_2 \tag{7.5.11}$$

where $\underline{\mathcal{H}}_2$ collects all the quadratic error terms which do not depend on third order derivatives of the connection coefficients. From this equation it is immediate to prove that

$$|r^{2-\frac{2}{p}}\nabla\!\!\!/^2 H|_{p=2,S}(u',\underline{u}_*) \leq c\left(|r^{2-\frac{2}{p}}\nabla\!\!\!/^2 H|_{p=2,\underline{C}_*\cap\Sigma_0} + \frac{1}{r(u',\underline{u}_*)u'}\left(\mathcal{I}_0+\Delta\right)\right)$$

where the second factor comes from the integration along $\underline{C}_*$ and Proposition 7.4.2. Substituting this estimate in the integral in 7.5.4 we obtain the expected result, once we prove the inequality

$$\int_{u_0(\underline{u}*)}^{u} du'|r^{2-\frac{2}{p}}\nabla\!\!\!/^2 H|^2_{p=2,\underline{C}_*\cap\Sigma_0} \leq c\left(\mathcal{I}_0+\Delta\right)^2 \tag{7.5.12}$$

Proof of Proposition 7.5.2:

To control the left hand side of 7.5.12 let us observe that

$$\begin{aligned}\int_{u_0(\underline{u}')}^{u} du'|r^{2-\frac{2}{p}}\nabla\!\!\!/^2 H|^2_{p=2,\underline{C}_*\cap\Sigma_0} &= |u_0(\underline{u}')-u||r^{2-\frac{2}{p}}\nabla\!\!\!/^2 H|^2_{p=2,\underline{C}_*\cap\Sigma_0}\\ &\leq cr_0(\lambda_1)|r^{2-\frac{2}{p}}\nabla\!\!\!/^2 H|^2_{p=2,\underline{C}_*\cap\Sigma_0}\end{aligned} \tag{7.5.13}$$

where $r_0(\lambda_1)\equiv\left[\frac{1}{4\pi}|S_*(\lambda_1)|\right]^{\frac{1}{2}}$ is the radius of of $S_*(\lambda_1)=S_{(0)}(\nu_*)=\underline{C}_*\cap\Sigma_0$, see definition 3.3.1 and definition 3.3.8. To control $|r^{2-\frac{2}{p}}\nabla\!\!\!/^2 H|_{p=2,\underline{C}_*\cap\Sigma_0}$ we have, therefore, to control $|r^{2-\frac{2}{p}}\nabla\!\!\!/^2 H|_{p=2,S_{(0)}(\nu)}$ on Σ_0. ${}^{(i)}H_{ab}$ on Σ_0 has the following expression

$${}^{(i)}H_{ab}\equiv\frac{1}{2}\left(g(\nabla\!\!\!/_a{}^{(i)}O,e_b)+g(\nabla\!\!\!/_b{}^{(i)}O,e_a)\right) \tag{7.5.14}$$

where ${}^{(i)}O={}^{(i)}O_{\Sigma_0}$ are the generators of the rotation group defined on Σ_0 and $g=\mathbf{g}|_{\Sigma_0}$ is the metric restricted to Σ_0. To define the rotation group on Σ_0, we use exactly the same procedure used in [Ch-Kl], Chapter 3, where the function $u(p)$, defining the foliation, is $u(p)=u_{(0)}(p)$, see definition 3.3.1. The strategy consists in using the "global initial data conditions, to define the rotation group at the spacelike infinity of Σ_0 and extend it to the whole Σ_0 using the diffeomorphism generated by the vector field $N\equiv\frac{\partial}{\partial u}$. The final result is that the ${}^{(i)}O$'s satisfy the following relations [26]:

[26] The relation between $N=\frac{\partial}{\partial u}$ and $\tilde{N}=\frac{1}{a}\frac{\partial}{\partial u}$ on Σ_0 is of the same type as the relation between N and e_3 on $\underline{C}_*$.

[Observe that the relation between $N=\frac{\partial}{\partial u}$ and $\tilde{N}=\frac{1}{a}\frac{\partial}{\partial u}$ on Σ_0 is of the same type as the relation between N and e_3 on $\underline{C}_*$.]

$$
\begin{aligned}
&[{}^{(i)}O, {}^{(j)}O] = \epsilon_{ijk}{}^{(k)}O \\
&[N, {}^{(i)}O] = 0 \\
&g({}^{(i)}O, N) = 0
\end{aligned}
\tag{7.5.15}
$$

On Σ_0, in the coordinates $\{u, \theta, \phi\}$, the metric is written as

$$
g(\cdot,\cdot) = a^2 du^2 + \gamma_{ab} d\phi^a d\phi^b
$$

and $\{\phi^a\} = \{\theta, \phi\}$. The adapted moving frame, Fermi propagated along Σ_0, is denoted $\{\tilde{N}, e_A\}$, with $A \in \{1,2\}$ and $\tilde{N} = \frac{1}{a}\frac{\partial}{\partial u}$.
From the commutation relation $[N, {}^{(i)}O] = 0$ and the properties of the adapted moving frame we obtain $(\nabla_N {}^{(i)}O)_A = \theta_{AB}{}^{(i)}O_B$ which can be rewritten as

$$
\frac{d}{du}{}^{(i)}O_A = a\theta_{AB}{}^{(i)}O_B \tag{7.5.16}
$$

To obtain the evolution equation for $\nabla_A {}^{(i)}O_B \equiv (\nabla {}^{(i)}O)_{AB}$, we derive tangentially the equation $(\nabla_N {}^{(i)}O)_A = \theta_{AB}{}^{(i)}O_B$ and use the commutation relations $[\nabla\!\!\!/_{\tilde{N}}, \nabla\!\!\!/]$, which have the following expression, see the appendix to Chapter 4,

$$
\begin{aligned}
([\nabla\!\!\!/_{\tilde{N}}, \nabla\!\!\!/]V)_{AB} \;=\;& -\theta_{AC}(\nabla\!\!\!/ V)_{CB} - (a^{-1}\nabla\!\!\!/_A a)\theta_{AC}V_C + \theta_{AB}(a^{-1}\nabla\!\!\!/_C a)V_C \\
& + (a^{-1}\nabla\!\!\!/_A a)(\nabla\!\!\!/_{\tilde{N}} V)_B + \tilde{N}^j e^s_A e^t_B [\nabla_j, \nabla_s] V_t
\end{aligned}
$$

where V is a vector field tangent to $S_{(0)}$. Choosing $V = {}^{(i)}O$ we obtain

$$
\begin{aligned}
\frac{d}{du}(\nabla_A {}^{(i)}O)_B \;=\;& a\Big[\hat{\theta}_{BC}(\nabla_A {}^{(i)}O)_C - \hat{\theta}_{AC}(\nabla_C {}^{(i)}O)_B + {}^{(i)}O_C\Big(\theta_{CB}(a^{-1}\nabla\!\!\!/_A a) \\
& - \theta_{CA}(a^{-1}\nabla\!\!\!/_B a)\Big) + {}^{(i)}O_C(\nabla\!\!\!/_A \theta)_{BC} + \theta_{AB}(a^{-1}\nabla\!\!\!/_C a){}^{(i)}O_C \\
& + \tilde{N}^j e^s_A [\nabla_j, \nabla_s]{}^{(i)}O_B\Big]
\end{aligned}
\tag{7.5.17}
$$

and from it, immediately, the evolution equation for ${}^{(i)}H$ on Σ_0,

$$
\frac{d}{du}{}^{(i)}H_{AB} = a\left(\hat{\theta}_{BC}{}^{(i)}H_{CA} - \hat{\theta}_{AC}{}^{(i)}H_{CB}\right) + \mathcal{H}_0 \tag{7.5.18}
$$

where

$$
\begin{aligned}
\mathcal{H}_0 \;=\;& a\Big[-\left(\hat{\theta}_{BC}(\nabla_C {}^{(i)}O)_A + \hat{\theta}_{AC}(\nabla_C {}^{(i)}O_B)\right) + \frac{1}{2}{}^{(i)}O_C\,((\nabla\!\!\!/_A\theta)_{BC} \\
& + (\nabla\!\!\!/_B\theta)_{AC}) + \theta_{AB}(a^{-1}\nabla\!\!\!/_C a){}^{(i)}O)_C + \frac{1}{2}\left({}^{(3)}\mathbf{R}_{BC\tilde{N}A} + {}^{(3)}\mathbf{R}_{AC\tilde{N}B}\right){}^{(i)}O_C\Big]
\end{aligned}
\tag{7.5.19}
$$

From the results of section 7.2, the estimates for ${}^{(i)}O$ and $\nabla\!\!\!/^{(i)}O$ obtained using the evolution equations 7.5.16, 7.5.17 and the properties of the rotation group at the spacelike infinity of Σ_0 it is immediate to infer that $\mathcal{H}_0 = O(\frac{1}{r^2})$. Differentiating twice tangentially this evolution equation and using the commutation relations of $[\nabla\!\!\!/_{\tilde{N}}, \nabla\!\!\!/]$ we obtain, the following evolution equation for $(\nabla\!\!\!/^2_A{}^{(i)}H)$,

$$\frac{d}{du}(\nabla\!\!\!/^{2\,(i)}H)_{AB} + a\mathrm{tr}\theta(\nabla\!\!\!/^{2\,(i)}H)_{AB} = a\left(\hat{\theta}_{BC}(\nabla\!\!\!/^{2\,(i)}H)_{CA} - \hat{\theta}_{AC}(\nabla\!\!\!/^{2\,(i)}H)_{CB}\right) + \mathcal{H}_{2AB}$$

where $\mathcal{H}_2$ depends on the second derivatives of the Riemann components. Applying the evolution Lemma 4.1.5 and Gronwall Lemma to $|r^{2-\frac{2}{p}}\nabla\!\!\!/^{2\,(i)}H|_{p=2,S_{(0)}}$ we obtain

$$\begin{aligned} |r^{2-\frac{2}{p}}\nabla\!\!\!/^{2\,(i)}H|_{p=2,S_{(0)}}(\underline{u}_*) &\leq c\left(|r^{2-\frac{2}{p}}\nabla\!\!\!/^{2\,(i)}H|_{p=2,S_{(0)}}(\infty) + \int_{\underline{u}_*}^{\infty}|r^{2-\frac{2}{p}}\mathcal{H}_2|_{p=2,S_{(0)}}\right) \\ &\leq c\int_{\underline{u}_*}^{\infty}|r^{2-\frac{2}{p}}\mathcal{H}_2|_{p=2,S_{(0)}} \end{aligned} \tag{7.5.20}$$

recalling that $\lim_{\underline{u}\to\infty}|r^{2-\frac{2}{p}}\nabla\!\!\!/^{(i)}H|_{p=2,S}(\underline{u}) = 0$. The integral on the right hand side can be estimated, using the explicit expression of $\mathcal{H}_2$ and the estimates of section 7.2, as $r_0(\lambda_1)^{-\frac{1}{2}}(\mathcal{I}_0 + \Delta)$, which substituted in 7.5.13, gives

$$\int_{u_0(\underline{u}')}^{u} du'|r^{2-\frac{2}{p}}\nabla\!\!\!/^2 H|^2_{p=2,\underline{C}_*\cap\Sigma_0} \leq c(\mathcal{I}_0 + \Delta)^2 \tag{7.5.21}$$

completing the proof of Proposition 7.5.2.

7.6 The extension argument

In this section we present the proof of Theorem **M9** which we recall below

Theorem M9: *Consider the spacetime $\mathcal{K}(\lambda_0, \nu_*)$ together with its double null (canonical) foliation given by the functions u and $\underline{u}$ such that*

1) The norms $\mathcal{Q}, \mathcal{O}, \mathcal{R}$ are sufficiently small

$$\mathcal{Q} \leq \epsilon_0'\ ,\ \mathcal{O} \leq \epsilon_0'\ ,\ \mathcal{R} \leq \epsilon_0'\ .$$

2) The initial conditions on Σ_0 are such that

$$\mathcal{O}(\Sigma_0[\nu_*, \nu_* + \delta]) \leq \epsilon_0'\quad ,$$

where $\Sigma_0[\nu_*, \nu_* + \delta] \equiv \{p \in \Sigma_0 | u_{(0)}(p) \in [\nu_*, \nu_* + \delta]\}$.

Then we can extend the spacetime $\mathcal{K}(\lambda_0, \nu_*)$ *and the double null foliation* $\{u, \underline{u}\}$ *to a larger spacetime* $\mathcal{K}(\lambda_0, \nu_* + \delta)$*, with* δ *sufficiently small, such that the extended norms, denoted* $\mathcal{O}', \mathcal{R}'$ *satisfy*

$$\mathcal{O}' \leq c\epsilon_0' \ , \ \mathcal{R}' \leq c\epsilon_0' \ .$$

Proof:

1) By an adapted version of the classical local existence theorem, starting with initial data in the annulus $\mathcal{A}_0 = \left\{p \in \Sigma_0 | u_{(0)}(p) \in [\nu_*, \nu_* + \delta]\right\}$ we can construct, provided δ is sufficiently small, a solution of the Einstein equations in its future domain of dependance, we denote $\mathcal{K}_0^{(\delta)}$,

$$\mathcal{K}_0^{(\delta)} \equiv \mathcal{K}(\lambda_1, \nu_* + \delta) \ .$$

The boundary of $\mathcal{K}_0^{(\delta)}$ consists of the annulus $\mathcal{A}_0 \subset \Sigma_0$ and of two null hypersurfaces, one incoming given by the portion [27],

[A mistake here has been corrected; in fact the correct expression is $\underline{C}_{**}(\lambda_1 - \sigma) \equiv \underline{C}(\nu_* + \delta; [\lambda_1 - \sigma, \lambda_1])$]

$$\underline{C}_{**}(\lambda_1 - \sigma) \equiv \underline{C}(\nu_* + \delta; [\lambda_1 - \sigma, \lambda_1])$$

of the null hypersurface $\underline{C}_{**} \equiv \underline{C}(\nu_* + \delta)$, where

$$\lambda_1 - \sigma \equiv u|_{\underline{C}(\nu_* + \delta) \cap \Sigma_0} \ , \ \lambda_1 = u|_{\underline{C}(\nu_*) \cap \Sigma_0} \ ,$$

and one outgoing given by the portion of the null outgoing hypersurface initiating at $S_*(\lambda_1)$ and contained in $\mathcal{K}(\lambda_0, \nu_* + \delta)$, which we denote $C^*(\delta)$. We can endow the region $\mathcal{K}_0^{(\delta)}$ with a double null foliation $\{u, \underline{u}\}$, where u and $\underline{u}$ are incoming and outgoing solutions of the eikonal equation with, as initial data, the function $u_{(0)}$ restricted to $\mathcal{A}_0$.

It is trivial to see that, provided δ is sufficiently small, relative to this double null foliation we have

$$\mathcal{O}' \leq \frac{3}{2}\epsilon_0' \ , \ \mathcal{R}' \leq \frac{3}{2}\epsilon_0' \tag{7.6.1}$$

2) Using a non standard version of the local existence theorem (see the discussion in the Remark 3 below) we can extend the spacetime $\mathcal{K}_0^{(\delta)} \cup \mathcal{K}(\lambda_1 + \sigma, \nu_*)$ to the future domain of dependance of $C^*(\delta) \cup \underline{C}_*(\lambda_1 + \sigma)$, where

$$\underline{C}_*(\lambda_1 + \sigma) \equiv \underline{C}(\nu_*; [\lambda_1, \lambda_1 + \sigma]) \tag{7.6.2}$$

provided σ is sufficiently small. Moreover starting with the foliation induced on $C^*(\delta)$ and on $\underline{C}_*(\lambda_1+\sigma)$ we can extend the double null foliation in this region in such a way that $\mathcal{O}'$ and $\mathcal{R}'$ satisfy

stake here has been cor- ; in fact the correct ex- on is $\mathcal{O}'$ and $\mathcal{R}'$ satisfy ead of $\mathcal{O}$ and $\mathcal{R}$ satisfy...]

$$\mathcal{O}' \le 2\epsilon'_0 \ , \ \mathcal{R}' \le 2\epsilon'_0 \tag{7.6.3}$$

3) Denote $\overline{\sigma}$, the supremum of all the values of σ for which this extension can be done in such a way that $\mathcal{O}'$ and $\mathcal{R}'$ satisfy

$$\mathcal{O}' \le c_0\epsilon'_0 \ , \ \mathcal{R}' \le c_0\epsilon'_0 \tag{7.6.4}$$

where the constant c_0 will be specified later on. If $\overline{\sigma}=\lambda_0-\lambda_1$ the proof is completed. Otherwise let us consider the spacetime $\mathcal{K}(\lambda_1+\overline{\sigma},\nu_*+\delta)$ where $\mathcal{O}'$ and $\mathcal{R}'$ satisfy 7.6.4.
By using a somewhat simplified [28] version of the apriori estimates developed in Chapters 4,5,6, we show that, in fact, $\mathcal{O}'$ and $\mathcal{R}'$ are strictly less than $c_0\epsilon'_0$ and thus reach a contradiction, if we choose c_0 sufficiently large.

[A mistake here has been corrected; in fact the correct expression is "strictly less than $c_0\epsilon'_0$..." instead of "strictly less than $2\epsilon'_0$..."]

This is accomplished in the following steps:

a) using a variant of the methods of Chapter 4 we show that inside the region [29]

$$\tilde{\Delta}(\lambda_1+\sigma,\nu_*+\delta) \equiv \mathcal{K}(\lambda_1+\sigma,\nu_*+\delta)\setminus\{\mathcal{K}_0^{(\delta)}\cup\mathcal{K}(\lambda_1+\sigma,\nu_*)\} \tag{7.6.5}$$

$\mathcal{O}'$ can be bounded as

$$\mathcal{O}'|_{\tilde{\Delta}} \le c\left(\mathcal{O}'|_{C^*(\delta)}+\mathcal{O}'|_{\underline{C}_*(\lambda_1+\sigma)}+\mathcal{R}'|_{\tilde{\Delta}}\right) \le c(3\epsilon'_0+\mathcal{R}') \tag{7.6.6}$$

b) By the comparison argument of Chapter 5 we know that $\mathcal{R}'$ can be bounded by $c\mathcal{Q}^{\frac{1}{2}}$ where $\mathcal{Q}$ is the quantity defined in Chapter 3, subsection 3.5.1, relative to the vector fields S,T,K_0 and the rotation vector fields ${}^{(i)}O$, defined in $\tilde{\Delta}(\lambda_1+\sigma,\nu_*+\delta)$.

Remark 1: The vector fields S,T,K_0 are defined, as before, with the help of the extended functions u and $\underline{u}$ defined in $\tilde{\Delta}(\lambda_1+\sigma,\nu_*+\delta)$. The rotation vector fields ${}^{(i)}O$ are defined in the same way as in Chapter 4 by an extension argument starting from $\underline{C}_*$. With the help of the diffeomorphism ϕ_t, along $C(u)$, we are extending in the future direction.

c) To complete the argument it remains to apply again, but in a slightly different situation, Theorem **M8**. Therefore we prove that, in the region

[27] Recall that $u<0$, therefore λ varies inside $\mathcal{K}\equiv\mathcal{K}(\lambda_0,\nu_*)$ in the interval $[\lambda_1,\lambda_0]$
[28] Simplified with respect to the length of the interval in which u varies.
[29] Recalling definition 3.7.23, we have $\Delta(\lambda_1+\sigma,\nu_*+\delta)=\tilde{\Delta}(\lambda_1+\sigma,\nu_*+\delta)\cup\mathcal{K}_0^{(\delta)}$

$\tilde{\Delta}(\lambda_1+\sigma,\nu_*+\delta)$, $\mathcal{Q}$ is bounded by a constant multiple of its restriction to $C^*(\delta)\cup\underline{C}_*(\lambda_1+\sigma)$. As, on the other side, $\mathcal{Q}^{\frac{1}{2}}|_{C^*(\delta)\cup\underline{C}_*(\lambda_1+\sigma)}$ is bounded by

$$\mathcal{Q}^{\frac{1}{2}}|_{C^*(\delta)\cup\underline{C}_*(\lambda_1+\sigma)}=\mathcal{Q}^{\frac{1}{2}}|_{C^*(\delta)}+\mathcal{Q}^{\frac{1}{2}}|_{\underline{C}_*(\lambda_1+\sigma)}\leq 4\epsilon_0' \tag{7.6.7}$$

Therefore $\mathcal{Q}|^{\frac{1}{2}}_{\tilde{\Delta}}\leq 4c\epsilon_0'$ and $\mathcal{R}|'_{\Delta}\leq 4c^2\epsilon_0'$ which implies, from 7.6.6, that

$$\mathcal{O}'|_{\tilde{\Delta}}\leq c\left(\mathcal{O}'|_{C^*(\delta)}+\mathcal{O}'|_{\underline{C}_*(\lambda_1+\sigma)}+\mathcal{R}'|_{\tilde{\Delta}}\right)\leq(3c\epsilon_0'+4c^3\epsilon_0') \tag{7.6.8}$$

Therefore, choosing $c_0>8c^3$, this completes the proof of the theorem.

Remark 2: The only argument which is somewhat different with respect to those developed in Chapters 4,5,6 is the one relative to the $\mathcal{O}$ norms. The estimates needed above to control the $\mathcal{O}$ and $\mathcal{D}$ norms in $\tilde{\Delta}$ are somewhat different from those of Chapter 4. Indeed in Chapter 4 the not underlined quantities $\mathrm{tr}\chi$, $\hat{\chi}$ and η were estimated using the evolution equations along $C(u)$ moving backward in time starting from $\underline{C}_*$. Now, on the other hand we start from $\underline{C}_*$ and move forward in time for a very short interval of size δ. The smallness of the interval makes this procedure straightforward.

Remark 3: The non standard local existence theorem which we have introduced above does not seem to exist in the literature. Although we do not prove it here we sketch below a possible approach to the proof.

Using the Einstein equations written relatively to a double null foliation, see subsection 3.1.7, we can first prove an adapted version of the Cauchy-Kowaleski theorem assuming that the spacetimes $\mathcal{K}_0^{(\delta)}$ and $\mathcal{K}(\lambda_1+\sigma,\nu_*)$ are real analitic. Once we have that, in order to get rid of the analiticity assumption, we propose the approach outlined in [Kl-Ni] which is based on a priori estimates similar, but far simpler than the ones described in Chapters 4,5,6.

An alternative approach would be to make use of Rendall's solution, [Ren], of the Characteristic Cauchy problem. More precisely we will need an adaptation of his approach to the H^k category, see also **??** and **??**. Starting with $\hat{\chi}$ on $C^*(\delta)$ and $\underline{\hat{\chi}}$ on $\underline{C}_*(\lambda_1+\sigma)$ and assuming that they are sufficiently differentiable, say $\hat{\chi}\in H^k(C^*(\delta))$ and $\underline{\hat{\chi}}\in H^k(C^*(\underline{C}_*(\lambda_1+\sigma))$ for k sufficiently large, this H^k variant of Rendall's result should allow us to construct a spacetime $\tilde{\Delta}(\lambda_1+\tilde{\sigma},\nu_*+\tilde{\delta})$, with $\tilde{\sigma}$ and $\tilde{\delta}$ depending on the H^k norms of $\hat{\chi}$ and $\underline{\hat{\chi}}$. Due to the loss of derivatives inherent in the characteristic Cauchy problem to apply this result we need a degree of smoothness for $\hat{\chi}$ and $\underline{\hat{\chi}}$ incompatible with our setup. This loss of derivatives can be attributed to the fact that in the characteristic Cauchy problem one treats the data $\hat{\chi},\underline{\hat{\chi}}$ (or

$\hat{\gamma}$ and Ω, see discussion in subsection **??**) as arbitrary. In reality, however, our $\hat{\chi}, \underline{\hat{\chi}}$, as they are induced by the spacetimes $\mathcal{K}_0^{(\delta)}$ and $\mathcal{K}(\lambda_1+\sigma,\nu_*)$ satisfy additional equations. In particular this means that we do not just know $\hat{\chi}, \underline{\hat{\chi}}$ on $C^*(\delta)$, $\underline{C}_*(\lambda_1+\sigma)$, but also their derivatives $\mathbf{D}\!\!\!/_3\hat{\chi}, \mathbf{D}\!\!\!/_4\underline{\hat{\chi}}$. Of course we cannot in general prescribe both $\hat{\chi}, \underline{\hat{\chi}}$ and $\mathbf{D}\!\!\!/_3\hat{\chi}, \mathbf{D}\!\!\!/_4\underline{\hat{\chi}}$; in our case, however, these quantities satisfy on $C^*(\delta)$, $\underline{C}_*(\lambda_1+\sigma)$ compatibility relations induced by the structure equations.

To avoid the loss of derivatives in the solution of the characteristic Cauchy problem one needs to appropriately approximate our $\hat{\chi}, \underline{\hat{\chi}}$, taking into account also the compatibility relations mentioned above, induced by the spacetimes $\mathcal{K}_0^{(\delta)}$ and $\mathcal{K}(\lambda_1+\sigma,\nu_*)$, by a smooth sequence $\hat{\chi}_n, \underline{\hat{\chi}}_n$ and associated with them the spacetimes $E_n=\tilde{\Delta}(\lambda_1+\tilde{\sigma}_n,\nu_*+\tilde{\delta}_n)$ constructed by the variant of Rendall's result mentioned above. Once this is done we can apply a vastly simplified version of the apriori estimates described in chapters from 3 to 7 to show that these spacetimes can be extended to values of $\tilde{\sigma},\tilde{\delta}$ independent of n and then pass to the limit. As the details of this argument are not very relevant to this book we plan to present them in a separate publication.

7.7 Appendix to Chapter 7

Osservazione 7.7.1 *Si osservi che in questo paragone tra le due foliazioni nella parte iniziale della regione estesa* $\Delta\equiv\tilde{\Delta}\cup\mathcal{K}_0^{\delta}$, *le due foliazioni sono entrambe nella* Initial layer region *o almeno parte di* Δ *certamente lo è. Ora le notazioni possono trarre in inganno e far sembrare che si sta paragonando la* Null canonical foliation *con la* Initial layer foliation *come viene fatto estesamente nell'Oscillation Lemma. Qui la situazione è diversa. Entrambe le foliazioni sono nella* initial layer region *estesa o nella regione sopra a* $\tilde{\Sigma}_0$. *La dimostrazione la possiamo considerare fatta solo sopra a* $\tilde{\Sigma}_0$. *Nella initial layer region la vicinanza delle due foliazioni e' conseguenza immediata della piccolezza di questa. Qui si dovrebbe aprire una parentesi per dire che nel Main Theorem, si deve assumere che* $\mathcal{K}$ *sia foliato dalla* Null canonical foliation *sopra a* $\tilde{\Sigma}_0$ *estesa e dall* initial layer foliation *nella* Initial layer region. *Il teorema mostra che* $\mathcal{K}$ *esteso cioè* $\mathcal{K}(\lambda_0,\nu_*+\delta)$ *ha le stesse proprietà. La seconda osservazione è che tale comparison si può fare anche guardando l'equazione di evoluzione di* $\mathbf{g}(L,L')$ *lungo una generica* $\underline{C}(\nu_*+\gamma)$ *di* Δ *come è fatto dopo su* $\underline{C}_{**}$.

7.7.1 Comparison between different foliations

We discuss here how to compare different foliations associated to different solutions of the eikonal equation and how to prove that, under appropriate conditions, they stay near one to each other.
Consider two double null foliations $\{u',\underline{u}\}$ and $\{u,\underline{u}\}$ with common incoming null hypersurfaces $\underline{C}(\nu)$. We denote by $C'(\lambda)$, $C(\lambda)$ the null outgoing hypersurfaces $u'=\lambda$, $u=\lambda$.
In the application of this result to the proof of the *Main Theorem*, in particular Step 6, we need to assume that the $\{u',\underline{u}\}$ foliation is globally defined and small, that is $\mathcal{O}'\leq\epsilon'_0$ and that the foliation $\{u,\underline{u}\}$ is defined in a neighbourhood Δ of $\underline{C}_{**}$, see 3.7.23. We can assume also that $\mathcal{O}$ is sufficiently small in Δ.

We want to establish a quantitative relationship between the two foliations in Δ. Associated to the null hypersurfaces of these foliations we introduce the null geodesics vector fields

$$
\begin{aligned}
&L'=-g^{\mu\nu}\partial_\nu u'\frac{\partial}{\partial x^\mu}\ ,\ L=-g^{\mu\nu}\partial_\nu u\frac{\partial}{\partial x^\mu}\\
&\underline{L}=-g^{\mu\nu}\partial_\nu\underline{u}\frac{\partial}{\partial x^\mu}
\end{aligned}
\tag{7.7.1}
$$

and the corresponding *"spacetime lapse functions"* Ω and Ω', see definition 3.1.12,

$$
g^{\mu\nu}\partial_\mu u'\partial_\nu\underline{u}=-(2\Omega'^2)^{-1}\ ,\ g^{\mu\nu}\partial_\mu u\partial_\nu\underline{u}=-(2\Omega^2)^{-1}
\tag{7.7.2}
$$

Associated to the two double null foliations we have two different double null integrable S-foliations whose leaves are

$$
S'(\lambda,\nu)=C'(\lambda)\cap\underline{C}(\nu)\ ,\ S(\lambda,\nu)=C(\lambda)\cap\underline{C}(\nu)\ .
$$

Starting from the geodesic vector fields we associate to these foliations two adapted null frames [30] $\{\hat{e}'_4,\hat{e}'_3,e'_a\}$, and $\{\hat{e}_4,\hat{e}_3,e_a\}$ in the following way:

$$
\begin{aligned}
&\hat{e}'_4=4\Omega'^2L'\ ,\ \hat{e}'_3=\underline{L}\ ,\ \ e'_a\text{ tangent to }S'(\lambda,\nu)\\
&\hat{e}_4=4\Omega^2L'\ ,\ \hat{e}_3=\underline{L}\ ,\ \ e_a\text{ tangent to }S(\lambda,\nu)
\end{aligned}
\tag{7.7.3}
$$

[30] The null vector fields chosen here are not normalized null pairs in the sense of definition 3.1.13. In fact we have $\hat{e}_4=2\Omega\hat{N}=2N$ and $\hat{e}_3=\underline{L}=(2\Omega)^{-1}\underline{\hat{N}}$ and the same for the primed ones.

The two null frames are related in the following way

$$
\begin{aligned}
\hat{e}_4 &= \hat{e}'_4 + \left[4\Omega^2\Omega'^2(-2\mathbf{g}(L,L'))\right]\hat{e}'_3 + 2\left[4\Omega^2\Omega'^2(-2\mathbf{g}(L,L'))\right]^{\frac{1}{2}}\hat{\sigma}_a e'_a \\
\hat{e}_3 &= \hat{e}'_3 \\
e_a &= e'_a + \left[4\Omega^2\Omega'^2(-2\mathbf{g}(L,L'))\right]^{\frac{1}{2}}\hat{\sigma}_a\hat{e}'_3
\end{aligned}
\tag{7.7.4}
$$

where $|\hat{\sigma}|^2 = 1$. Moreover

$$
\begin{aligned}
\hat{e}'_4 &= \hat{e}_4 + \left[4\Omega^2\Omega'^2(-2\mathbf{g}(L,L'))\right]\hat{e}_3 - 2\left[4\Omega^2\Omega'^2(-2\mathbf{g}(L,L'))\right]^{\frac{1}{2}}\hat{\sigma}_a e_a \\
\hat{e}'_3 &= \hat{e}_3 \\
e'_a &= e_a - \left[4\Omega^2\Omega'^2(-2\mathbf{g}(L,L'))\right]^{\frac{1}{2}}\hat{\sigma}_a\hat{e}_3
\end{aligned}
\tag{7.7.5}
$$

These formulas follow immediately from the fact that both frames are null frames and from the relation $\mathbf{g}(\hat{e}_4,\hat{e}'_4) = 16\Omega^2\Omega'^2\mathbf{g}(L,L')$.
How much the foliations are , one to each other, is controlled by the term

$$
\Theta \equiv \left[4\Omega^2\Omega'^2(-2\mathbf{g}(L,L'))\right] \tag{7.7.6}
$$

To estimate $\mathbf{g}(L,L')$ we start from its expression

$$
\mathbf{g}(L,L') = g^{\mu\nu}\partial_\mu u\partial_\nu u' = \frac{1}{2}g^{\mu\nu}\partial_\mu(u-u')\partial_\nu(u'-u) \tag{7.7.7}
$$

and express the right hand side of 7.7.7 using a specific choice of coordinates. We choose $\{v,\underline{u},\omega^a\}$ as coordinates, where v is the affine parameter of the null incoming geodesic curves along the hypersurfaces $\underline{C}(\underline{u})$. It is an easy computation to write the explicit expression of the metric,

$$
\mathbf{g}(\cdot,\cdot) = X^2 d\underline{u}^2 - (dvd\underline{u} + d\underline{u}dv) - X_a\left(d\underline{u}d\omega^a + d\omega^a d\underline{u}\right) + \gamma_{ab}d\omega^a d\omega^b
$$

where, analogously to what was done in subsection 3.1.6, see equation 3.1.61, $N = \frac{\partial}{\partial\underline{u}} + X$ and

$$
\frac{\partial}{\partial v}X^a = Z^a = 2\gamma^{ab}\zeta_b \ .
$$

The components of the inverse metric are

$$
g^{vv} = 0\ ,\ g^{\underline{u}\,\underline{u}} = 0\ ,\ g^{v\underline{u}} = -1\ ,\ g^{vd} = X^d\ ,\ g^{\underline{u}d} = 0\ ,\ g^{ab} = \gamma^{ab}\ .
$$

Using thes coordinates we observe that, along the $\underline{C}(\nu)$ null hypersurfaces, common to both foliations, u and u' satisfy

$$
\begin{aligned}
u(p) = u(v,\underline{u},\omega) &= \int^v \frac{du}{dv}dv' = \int^v (2\Omega^2)^{-1}(v',\underline{u},\omega)dv' \\
u'(p) = u'(v,\underline{u},\omega) &= \int^v \frac{du'}{dv}dv' = \int^v (2\Omega'^2)^{-1}(v',\underline{u},\omega)dv' \qquad (7.7.8)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
2\mathbf{g}(L,L') &= g^{v\underline{u}}\left[\partial_v(u-u')\partial_{\underline{u}}(u'-u) + \partial_{\underline{u}}(u-u')\partial_v(u'-u)\right] \\
&\quad + g^{vd}\left[\partial_v(u-u')\partial_d(u'-u) + \partial_d(u-u')\partial_v(u'-u)\right] \\
&\quad + \gamma^{ab}\partial_a(u-u')\partial_b(u'-u) \qquad (7.7.9) \\
&= -\left[\partial_v(u-u')\partial_{\underline{u}}(u'-u) + \partial_{\underline{u}}(u-u')\partial_v(u'-u)\right] \\
&\quad - X^d\left[\partial_v(u-u')\partial_d(u'-u) + \partial_d(u-u')\partial_v(u'-u)\right] \\
&\quad + \gamma^{ab}\partial_a(u-u')\partial_b(u'-u)
\end{aligned}
$$

In the chosen coordinates the right hand side of 7.7.9 becomes

$$
\begin{aligned}
2\mathbf{g}(L,L') &= -\Bigg[\partial_v\left(\int^v\left[(2\Omega^2)^{-1}-(2\Omega'^2)^{-1}\right](v',\underline{u},\omega)dv'\right)\partial_{\underline{u}}\left(\int^v\left[(2\Omega'^2)^{-1}-(2\Omega^2)^{-1}\right](v',\underline{u},\omega)d\right. \\
&\quad + \partial_{\underline{u}}\left(\int^v\left[(2\Omega^2)^{-1}-(2\Omega'^2)^{-1}\right](v',\underline{u},\omega)dv'\right)\partial_v\left(\int^v\left[(2\Omega'^2)^{-1}-(2\Omega^2)^{-1}\right](v',\underline{u},\omega)dv'\right. \\
&\quad - X^d\Bigg[\partial_v\left(\int^v\left[(2\Omega^2)^{-1}-(2\Omega'^2)^{-1}\right](v',\underline{u},\omega)dv'\right)\partial_d\left(\int^v\left[(2\Omega'^2)^{-1}-(2\Omega^2)^{-1}\right](v',\underline{u},\right. \\
&\quad + \partial_d\left(\int^v\left[(2\Omega^2)^{-1}-(2\Omega'^2)^{-1}\right](v',\underline{u},\omega)dv'\right)\partial_v\left(\int^v\left[(2\Omega'^2)^{-1}-(2\Omega^2)^{-1}\right](v',\underline{u},\omega)dv'\right. \\
&\quad \gamma^{ab}\partial_a\left(\int^v\left[(2\Omega^2)^{-1}-(2\Omega'^2)^{-1}\right](v',\underline{u},\omega)dv'\right)\partial_b\left(\int^v\left[(2\Omega'^2)^{-1}-(2\Omega^2)^{-1}\right](v',\underline{u},\omega)d\right.
\end{aligned}
$$

Computing these terms explicitely we obtain

$$
\begin{aligned}
\mathbf{g}(L,L') &= -\frac{(\Omega'^2-\Omega^2)}{2\Omega^2\Omega'^2}\Bigg\{\int_0^v\left(\frac{1}{\Omega^2}\partial_{\underline{u}}\log\Omega - \frac{1}{\Omega'^2}\partial_{\underline{u}}\log\Omega'\right) + X^d\int_0^v\left(\frac{1}{\Omega^2}\partial_d\log\Omega - \frac{1}{\Omega'^2}\partial_d\log\Omega\right. \\
&\quad - \gamma^{ab}\int_0^v\left(\frac{1}{\Omega^2}\partial_a\log\Omega - \frac{1}{\Omega'^2}\partial_a\log\Omega'\right)\left(\frac{1}{\Omega^2}\partial_b\log\Omega - \frac{1}{\Omega'^2}\partial_b\log\Omega'\right)\Bigg\} \\
&\equiv [I]+[II]+[III] \qquad (7.7.1
\end{aligned}
$$

Writing this expression in terms of the null frames 7.7.4, 7.7.5, we obtain the following expressions

$$[I]+[II] =$$

$$= -\frac{(\Omega'^2-\Omega^2)}{2\Omega^2\Omega'^2}\left\{\int_0^v\left(\frac{1}{2\Omega^2}\partial_{\hat{e}_4}\log\Omega-\frac{1}{2\Omega'^2}\partial_{\hat{e}'_4}\log\Omega'\right)+\int_0^v\left(\frac{\Theta}{2\Omega'^2}\partial_{\hat{e}'_3}\log\Omega'+\frac{\Theta^{\frac12}}{\Omega'^2}\hat{\sigma}_a\partial_{e'_a}\log\Omega'\right)\right\}$$
$$+\frac{(\Omega'^2-\Omega^2)}{2\Omega^2\Omega'^2}\left[\int_0^v\left(\frac{1}{\Omega^2}X^d\partial_d\log\Omega-\frac{1}{\Omega'^2}X^d\partial_d\log\Omega'\right)-X^d\int_0^v\left(\frac{1}{\Omega^2}\partial_d\log\Omega-\frac{1}{\Omega'^2}\partial_d\log\Omega'\right)\right]$$
$$= -\frac{(\Omega'^2-\Omega^2)}{2\Omega^2\Omega'^2}\left\{\int_0^v\left(\frac{1}{2\Omega^2}\partial_{\hat{e}_4}\log\Omega-\frac{1}{2\Omega'^2}\partial_{\hat{e}'_4}\log\Omega'\right)+\int_0^v\left(\frac{\Theta}{2\Omega'^2}\partial_{\hat{e}'_3}\log\Omega'+\frac{\Theta^{\frac12}}{\Omega'^2}\hat{\sigma}_a\partial_{e'_a}\log\Omega'\right)\right\}$$
$$+\frac{(\Omega'^2-\Omega^2)}{2\Omega^2\Omega'^2}\left[\int_0^v\left(\frac{1}{\Omega^2}\theta^d(X)\partial_{e_d}\log\Omega-\frac{1}{\Omega'^2}\theta^d(X)\partial_{e'_d}\log\Omega'-\frac{1}{\Omega'^2}\Theta^{\frac12}(\theta^d(X)\hat{\sigma}_d)\partial_{\hat{e}'_3}\log\Omega'\right)\right.$$
$$\left.-X^c\int_0^v\left(\frac{1}{\Omega^2}\theta^d_c\partial_{e_d}\log\Omega-\frac{1}{\Omega'^2}\theta^d_c\partial_{e'_d}\log\Omega'-\frac{1}{\Omega'^2}\Theta^{\frac12}(\theta^d_c\hat{\sigma}_d)\partial_{\hat{e}'_3}\log\Omega'\right)\right] \quad (7.7.11)$$

and

$$[III] = -\frac{(\Omega'^2-\Omega^2)}{2\Omega^2\Omega'^2}\gamma^{ab}\int_0^v\frac{1}{\Omega^2}\theta^d_a\partial_{e_d}\log\Omega\int_0^v\frac{1}{\Omega'^2}\theta^d_b\left(\partial_{e'_d}\log\Omega'+\Theta^{\frac12}(\theta^d_b\hat{\sigma}_d)\partial_{\hat{e}'_3}\log\Omega'\right) \quad (7.7.12)$$

where θ^d is the one form ossociated to the vector fields e_d, $d\in\{1,2\}$. Putting together 7.7.10 and 7.7.12 we obtain

$$\Theta(v,\underline{u},\omega) =$$
$$= 4(\Omega'^2-\Omega^2)\left\{\int_0^v\left(\frac{1}{2\Omega^2}\partial_{\hat{e}_4}\log\Omega-\frac{1}{2\Omega'^2}\partial_{\hat{e}'_4}\log\Omega'\right)+\int_0^v\left(\frac{\Theta}{2\Omega'^2}\partial_{\hat{e}'_3}\log\Omega'+\frac{\Theta^{\frac12}}{\Omega'^2}\hat{\sigma}_a\partial_{e'_a}\log\Omega'\right)\right.$$
$$-\left[\int_0^v\left(\frac{1}{\Omega^2}\theta^d(X)\partial_{e_d}\log\Omega-\frac{1}{\Omega'^2}\theta^d(X)\partial_{e'_d}\log\Omega'-\frac{1}{\Omega'^2}\Theta^{\frac12}(\theta^d(X)\hat{\sigma}_d)\partial_{\hat{e}'_3}\log\Omega'\right)\right.$$
$$\left.-X^c\int_0^v\left(\frac{1}{\Omega^2}\theta^d_c\partial_{e_d}\log\Omega-\frac{1}{\Omega'^2}\theta^d_c\partial_{e'_d}\log\Omega'-\frac{1}{\Omega'^2}\Theta^{\frac12}(\theta^d_c\hat{\sigma}_d)\partial_{\hat{e}'_3}\log\Omega'\right)\right]$$
$$\left.+\gamma^{ab}\int_0^v\frac{1}{\Omega^2}\theta^d_a\partial_{e_d}\log\Omega\int_0^v\frac{1}{\Omega'^2}\theta^d_b\left(\partial_{e'_d}\log\Omega'+\Theta^{\frac12}(\theta^d_b\hat{\sigma}_d)\partial_{\hat{e}'_3}\log\Omega'\right)\right\} \quad (7.7.13)$$

and from this expression we have immediately

[There is a modification and a correction in the statement of lemma 7.7.1]

Lemma 7.7.1 *Assume the spacetime lapse function Ω bounded, then, if $\mathcal{O}'_{[0]}$ is sufficiently small, we have* [31]

$$|r^2\Theta| \le c(\mathcal{O}_{[0]}+\mathcal{O}'_{[0]})(1+\mathcal{O}'_{[0]}) \quad (7.7.14)$$

Proof: Taking the sup of Θ along $\underline{C}(\nu)$ and using that $\mathcal{O}'_{[0]}$ is small and that the connection coefficients ω' and $\eta',\underline{\eta}'$ have the appropriate decay, the result follows immediately. The next corollary is an immediate consequence of this lemma.

[31] The decay is stronger on $\underline{C}_*$.

Osservazione 7.7.2 *mi sembra che la formula del Lemma dovrebbe essere*

$$\Theta \leq c\mathcal{O}_{[0]}(1+\mathcal{O}'_{[0]})$$

[There is a modification and a correction in the statement of corollary 7.7.1]

Corollary 7.7.1 *Assume the spacetime lapse function Ω bounded, then if $\mathcal{O}'_{[0]}$ is sufficiently small we have*

$$\mathcal{R} \leq \mathcal{R}' + c\mathcal{O}_{[2]}(1+\mathcal{O}'_{[2]})\mathcal{R}' \tag{7.7.15}$$

Osservazione 7.7.3 *mi sembra che la formula del Corollario dovrebbe essere*

$$\mathcal{R} \leq \mathcal{R}' + c\mathcal{O}_{[2]}(1+\mathcal{O}'_{[2]})\mathcal{R}'$$

qui le norme $\mathcal{O}_{[2]}$ e $\mathcal{O}'_{[2]}$ intervengono poiche' in $\mathcal{R}$ ci sono le derivate seconde di Riemann.

Osservazione 7.7.4 *Nel lemma precedente sembra che in realtà basti chiedere meno per le norme accentate, ma questo è da verificare con attenzione. Sembra, infatti, che basta chiedere, per il Lemma 7.7.1, solamente che $\mathcal{O}'_{[0]}$ sia piccolo. Questo sarebbe soddisfacente anche sotto un altro aspetto. Infatti si ricordi che nella prova del Teorema* **M6**, *che sostanzialmente assicura l'esistenza della foliazione canonica su $\underline{C}_*$ e delle stime per i corrispondenti coefficienti di connessione, le norme "primed" entrano solo nella prova della esistenza locale.*
Nella dimostrazione della esistenza locale della foliazione canonica non sembra ci sia bisogno di controllare $\mathcal{O}'_{[2]}$, ma solo $\mathcal{O}'_{[1]}$, (questo quando si deve controllare il "mapping" per ${}^{(0)}\nabla\!\!\!/'^3 W$, altrimenti basterebbe solo $\mathcal{O}'_{[0]}$). Invece non sembra si debba mai controllare $\nabla\!\!\!/'^2\chi'$. Ora si ricordi che su $\underline{C}_*$, ma in generale su ogni $\underline{C}$, se la foliazione non è canonica si ha che $\nabla\!\!\!/'^2\chi'$ dipende da $\nabla\!\!\!/'^2\rho$ e quindi non ammette stime $L^p(S')$.
Tuttavia si deve anche ricordare che l'esistenza della foliazione canonica si usa per costruire tale foliazione su $\underline{C}_{**}$ avendolo assunta esistente su $\underline{C}_*$. La foliazione di "background" su $\underline{C}_{**}$ è costruita estendendo la foliazione canonica su $\underline{C}_*$ a $\underline{C}_{**}$. Proprio la vicinanza tra $\underline{C}_{**}$ e $\underline{C}_*$ permette di conservare il controllo sulla norma $\mathcal{O}'_{[2]}$. Rigorosamente questa estensione viene fatta facendo evolvere tutti i coefficienti di connessione "forward", cioè verso il futuro, e quindi avendo la possibilità di controllare, ad esempio, $\nabla\!\!\!/'^2\chi'$ su $\underline{C}_{**}$.

Si ricordi anche che l'esistenza della foliazione canonica viene usata due volte, la prima volta per dimostrare che esiste una regione $\mathcal{K}$ con tutte le proprieta' richieste dal bootstrap. Ora in questo caso poichè la regione può essere piccolissima di nuovo $\nabla\!\!\!/'^2\chi$ può essere stimato muovendosi sembre dal basso, "forward", e sfruttando che in questo caso la dipendenza da Riemann è migliore. Questa considerazione tuttavia non serve se si è certi che per la costruzione della foliazione canonica il controllo di $\mathcal{O}'_{[2]}$ non serve. Questo sara' esaminato in dettaglio in [Ni]. Supponendo infine di non volere usare mai $\mathcal{O}'_{[2]}$ si deve osservare che l'equazione 7.3.4,

$$\mathcal{R}(\underline{C}_*) \leq \mathcal{R}'(\underline{C}_*) + c\left(\mathcal{O}(\underline{C}_*) + \left[\mathcal{O}_{[2]}'(\underline{C}_*) + \underline{\mathcal{O}}_{[2]}'(\underline{C}_*)\right]\right)\mathcal{R}'(\underline{C}_*) \ ,$$

andrebbe riscritta anche se tutto funzionerebbe ancora, infatti in questo caso le seconde derivate di Riemann non primed sono stimate dalle $L^2(\underline{C}_*)$ norme delle stesse quantità primed e da $L^2(\underline{C}_*)$ norme per alcuni coefficienti di connessione che nel caso canonico ammettono anche $L^p(S)$ norme.
In conclusione penso che si possa concludere che

a) Le norme $\mathcal{O}'_{[2]}$ possono essere utilizzate.

b) Nella costruzione della foliazione canonica non c'e' bisogno di queste stime, ma solo di controllare $\mathcal{O}'_{[1]}$. (Invece per $\underline{\mathcal{O}}'_{[2]}$ tutto va bene.)

7.7.2 Proof of the local existence part of Theorem 3.3.2

We recall the equations which define the canonical foliation on the last slice, see 3.3.12,

$$\begin{aligned}
&\frac{du_*}{dv} = (2\Omega^2)^{-1}; \quad u_*|_{\underline{C}_*\cap\Sigma_0} = \lambda_1 \\
&\triangle\!\!\!/ \log\Omega = \frac{1}{2}\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\eta} + \frac{1}{2}\left(\mathbf{K} - \overline{\mathbf{K}} + \frac{1}{4}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \overline{\mathrm{tr}\chi\mathrm{tr}\underline{\chi}})\right) \\
&\overline{\log 2\Omega} = 0
\end{aligned} \tag{7.7.16}$$

We rewrite these equations, with respect to the following null frame

$$\underline{L} = \frac{1}{2\Omega}\underline{\hat{N}} \ , \ L^* = 2\Omega\hat{N} \ , \ e''_A = e_A \ ,$$

where $\{\hat{N}, \underline{\hat{N}}\}$ is the normalized null pair associated to the canonical foliation. The quantities which refer to the null frame $\{\underline{L}, L^*, e''_A\}$ will be denoted with a double prime, for instance $\chi'', \zeta'', \rho''...$, and the following relations hold between primed and unprimed quantities

$$\zeta''_A = \zeta_A - \nabla\!\!\!/_A \log\Omega \ \ , \ \ \underline{\eta}''_A = \underline{\eta}_A = -\zeta'' \ \ , \ \ \eta''_A = \eta_A$$
$$\underline{\chi}''_{AB} = \frac{1}{2\Omega}\underline{\chi}_{AB} \ \ , \ \ \chi''_{AB} = 2\Omega\chi_{AB} \ \ , \ \ \rho'' = \rho \qquad (7.7.17)$$

the equations 7.7.16 become

$$\frac{du_*}{dv} = (2\Omega^2)^{-1}; \quad u_*|_{S'_*(0)} = \lambda_1$$
$$\Delta\!\!\!/ \log 2\Omega = \frac{1}{2}\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\eta}'' + \frac{1}{2}\left[(\frac{1}{2}\hat{\chi}''\hat{\underline{\chi}}'' - \frac{1}{2}\overline{\hat{\chi}''\hat{\underline{\chi}}''}) - (\rho'' - \overline{\rho''})\right]$$
$$\overline{\log 2\Omega} = 0 \qquad (7.7.18)$$

where we write $S_0 = S_*(\lambda_1) = \underline{C}_* \cap \Sigma_0$.

To solve this system of equations we make the following preliminary steps:

a: We observe that we can replace the given background foliation on $\underline{C}_*$ by the geodesic foliation which we define below. This can be easily done locally near S_0. The geodesic foliation is defined by the level surfaces of the affine parameter v,

$$S'(\tau) = \{p \in \underline{C}_* | v(p) = \tau \in [0, v_1]\} \ ,$$

where $S'(0) = S_0$ and v_1 is defined later on.

b: Associated to this new "background foliation" we define a null frame adapted to it,

$$\{\underline{L}, N', e'_A\} \ ,$$

with the e'_A vector fields, Fermi transported along $\underline{C}_*$.

c: We choose (v, ω) as coordinates of a point $p \in \underline{C}_*$ where $\omega = (\theta, \phi)$ are the angular coordinates of S_0 [32]. The vector fields e'_A can be expressed in the form

$$e'_A|_p = {e'_A}^a \frac{\partial}{\partial\omega^a}\Big|_{\omega(p)}$$

We denote $\gamma(v)$ the restriction of the metric g on the two dimensional surfaces $S'(\tau) \subset \underline{C}_*$,

$$\gamma(v, \omega)(\cdot, \cdot) = \mathbf{g}(p)|_{S'(\tau)}(\cdot, \cdot) \ .$$

[32] Let p be a point $\in \underline{C}_*$, there exists a null geodesic λ starting at $p_0 \in S_0$ such that $p = \lambda(\overline{v}, p_0)$. Then $(v(p), \omega(p)) = (\overline{v}, \theta(p_0), \phi(p_0))$

The null geodesics on $\underline{C}_*$ define a family of maps $\{\psi_v\}$ between $S'(0)$ and $S'(v)$. In our adapted coordinates they are given by

$$S'(0) : p_0 \equiv (0, \omega) \to p = \psi_v(p_0) = (v, \omega) \in S'(v) \ ,$$

therefore the metrics $\{\gamma(v, \cdot)\}$ can be thought as a family of metrics on $S'(0)$,

$$\gamma(v, \omega) = \gamma_{ab}(v; \omega) d\omega^a d\omega^b \ .$$

d: We consider the class of foliations, defined through the functions $W(\lambda, \omega)$[33],

$$\begin{aligned} &{}^{(W)}F : [\lambda_1, \lambda_2] \times S_0 \to \underline{C}_* \\ &{}^{(W)}F(\lambda, \omega) = (W(\lambda, \omega), \omega) \ , \ W(\lambda_1, \omega) = 0 \end{aligned} \tag{7.7.19}$$

The leaves of the ${}^{(W)}F$ foliations are the two dimensional surfaces

$$S_{(W)}(\lambda) \equiv \{p \in \underline{C}_* | (v(p), \omega(p)) = (W(\lambda, \omega), \omega)\} \tag{7.7.20}$$

and $S_{(W)}(\lambda_1) = S'(0)$. Observe that the background geodesic foliation corresponds to $W_0(\lambda, \omega) = \lambda - \lambda_1$.

Once we have introduced this space of foliations, we define an appropriate norm on it and construct a transformation such that its fixed point will be the solution of the system 7.7.18. This is achieved through the following steps:

A: Observe that the vector fields

$$\frac{\partial}{\partial \omega^a} + \frac{\partial W}{\partial \omega^a} \frac{\partial}{\partial v} \tag{7.7.21}$$

are tangent to $S_{(W)}(\lambda)$ for every λ. Using them we define the orthonormal frame $\{{}^{(W)}e_A\}$, adapted to the $W(\lambda, \cdot)$ foliation, as

$${}^{(W)}e_A = e'^{\,a}_A \left(\frac{\partial}{\partial \omega^a} + \frac{\partial W}{\partial \omega^a} \frac{\partial}{\partial v} \right) \equiv e'_A + (\partial'_A W) \underline{L} \tag{7.7.22}$$

B: We construct a null frame adapted to $S_{(W)}(\lambda)$

$$\{\underline{L}, {}^{(W)}N, {}^{(W)}e_A\} \ .$$

[33]The function $W(\lambda, \omega)$ must have some appropriate properties to define a foliation. In particular one has to require that W has no critical points and that, for any fixed λ, the level surfaces of $W(\lambda, \omega)$ are diffeomorphic to S^2.

The relation of this null frame with the background one, $\{\underline{L}, N', e'_A\}$, is given by [34]

$$\begin{aligned} {}^{(W)}N &= N' + (\partial' W)^2\underline{L} + 2(\partial'_B W)e'_B \\ {}^{(W)}e_A &= e'_A + (\partial'_A W)\underline{L} \end{aligned} \tag{7.7.23}$$

where $\qquad (\partial' W)^2 = \sum_A (\partial'_A W)^2 \equiv \sum_A (\partial_{e'_A} W)^2.$

The connection coefficients and the relevant null curvature component relative to this $S_{(W)}$ foliation can be expressed in terms of those relative to the background foliation [35]:

$$\begin{aligned} &{}^{(W)}\underline{\chi}_{AB} = \underline{\chi}'_{AB} \\ &{}^{(W)}\zeta_A = \zeta'_A - (\partial'_C W)\underline{\chi}'_{C,A} \\ &{}^{(W)}\underline{\eta}_A = \underline{\eta}'_A + (\partial'_C W)\underline{\chi}'_{C,A} = -{}^{(W)}\zeta_A \\ &{}^{(W)}\chi_{AB} = \chi'_{AB} + (\partial' W)^2\underline{\chi}'_{AB} + 2\left[(\partial_{e'_B} W)\zeta'_A + (\partial_{e'_A} W)\zeta'_B\right] \\ &\qquad -2(\partial'_C W)\left[(\partial'_B W)\underline{\chi}'_{CA} + (\partial'_A W)\underline{\chi}'_{CB}\right] + 2\left(\nabla\!\!\!\!/'(\nabla\!\!\!\!/' W)|_{S_W(\lambda)}\right)_{AB} \\ &{}^{(W)}\rho = \rho' - \frac{1}{2}(\partial'_C W)\underline{\beta}'_C + \frac{1}{4}(\partial'_B W)(\partial'_C W)\underline{\alpha}'_{BC} \end{aligned} \tag{7.7.24}$$

C: We introduce the nonlinear map $\mathcal{A}$ whose fixed point will be the solution of the system 7.7.18 in the following way: we denote with $||G||_{L^p(S_0)}$ the following norm:

$$||G||_{L^p(S_0)} = \left(\int_{S_0} |G|^p d\mu_0\right)^{\frac{1}{p}} \tag{7.7.25}$$

where $d\mu_0$ is the measure on S_0. On the space function $C^0(I; L^p(S_0))$, with[36] $I = [0, \lambda_2]$, we introduce the nonlinear map

$$\mathcal{A} : W(\lambda, \omega) \to \widetilde{W}(\lambda, \omega) \equiv \mathcal{A}(W)(\lambda, \omega)$$

defined through the following steps:

---

[34]These relations are the same as used in subsection 7.7.1. Here, nevertheless, we have a more refined control over W.

[35]Repeated capital indices mean sum over them. We use the following notations $\chi'_{AB} \equiv \chi'(e'_A, e'_B)$, ${}^{(W)}\chi_{AB} \equiv {}^{(W)}\chi({}^{(W)}e_A, {}^{(W)}e_B)$ for all the connection coefficients.

[36]Here we use $\lambda \equiv \lambda - \lambda_1$.

C$_1$: We consider the portion of the $\underline{C}_*$ null hypersurface

$$\underline{C}_*(I;W) = \{p \in \underline{C}_* | p \in S_{(W)}(\lambda); \lambda \in I\} \tag{7.7.26}$$

C$_2$: Given W we consider on $\underline{C}_*(I;W)$ the null frame

$$\{\underline{L}, {}^{(W)}N, {}^{(W)}e_A\} \ ,$$

the associated connection coefficients ${}^{(W)}\zeta, {}^{(W)}\underline{\eta}, {}^{(W)}\chi, {}^{(W)}\underline{\chi}$ and the curvature component ${}^{(W)}\rho - \overline{{}^{(W)}\rho}$ where the average is done with respect to S_W.

C$_3$: On the two dimensional surfaces $S_{(W)}(\lambda)$ of $\underline{C}_*(I;W)$ we solve the elliptic equation

$$\begin{aligned} {}^{(W)}\triangle\!\!\!/ \log 2{}^{(W)}\Omega &= \frac{1}{2}{}^{(W)}\text{d}\!\!\!/\text{iv}\,{}^{(W)}\underline{\eta} + \frac{1}{2}\left[(\frac{1}{2}{}^{(W)}\hat{\chi}{}^{(W)}\underline{\hat{\chi}} - \frac{1}{2}\overline{{}^{(W)}\hat{\chi}{}^{(W)}\underline{\hat{\chi}}}) - ({}^{(W)}\rho - \overline{{}^{(W)}\rho})\right] \\ \overline{\log 2{}^{(W)}\Omega} &= 0 \end{aligned} \tag{7.7.27}$$

where, for any given λ, ${}^{(W)}\triangle\!\!\!/ = {}^{(W)}\nabla\!\!\!\!/_A{}^{(W)}\nabla\!\!\!\!/_A$ is the intrinsic Laplacian relative to the surface $S_{(W)}(\lambda)$.

C$_4$: We define the non linear map

$$\mathcal{A}(W)(\lambda,\omega) = \int_0^\lambda 2\left({}^{(W)}\Omega|_{S_W(\lambda')}(\omega)\right)^2 d\lambda' \tag{7.7.28}$$

To prove the local existence of a canonical foliation we have to show that the trasformation $\mathcal{A}(W) = \widetilde{W}$ has a fixed point W_*,

$$\mathcal{A}(W_*) = W_* \ .$$

Indeed given W_* we can define implicitely $u_* = u_*(v,\omega)$ according to [37]:

$$u_*(W_*(\lambda,\omega),\omega) = \lambda \tag{7.7.29}$$

Then,

$$1 = \frac{du_*}{d\lambda} = \frac{du_*}{dv}\frac{dW_*}{d\lambda} = \frac{du_*}{dv}2({}^{(W_*)}\Omega)^2$$

[37] We remark that, as the "background foliation" is equivariant with respect to the vector field $\underline{L}$, the (local) foliation ${}^{\mathcal{A}(W)}S(\lambda) = \{p \in \underline{C}_* | \mathcal{A}(W)(p) = \lambda\}$ is equivariant with respect to the vector field ${}^{(\mathcal{A}(W))}\underline{N} \equiv 2({}^{(W)}\Omega)^2\underline{L}$.

which implies

$$\frac{du_*}{dv} = \frac{1}{2({}^{(W_*)}\Omega)^2}$$

as desired. The portion of $\underline{C}_*$ endowed with this canonical foliation is

$$\underline{C}_*(I; W_*) = \{p \in \underline{C}_* | p \in S_{(W_*)}(\lambda); \lambda \in I\} \ .$$

In the following we consider a simplified version of the map $\mathcal{A}(W)$ close to the previous one. To obtain it we rewrite 7.7.28 in the following way:

$$\mathcal{A}(W)(\lambda,\omega) = \frac{\lambda}{2} + \int_0^\lambda \left(2\left({}^{(W)}\Omega|_{S_{(W)}(\lambda')}(\omega)\right)^2 - \frac{1}{2}\right) d\lambda'$$

Assuming $\left|2\left({}^{(W)}\Omega|_{S_{(W)}(\lambda')}(\omega)\right)^2 - \frac{1}{2}\right|$ small for any function W and any $\lambda \in I$, we expand $2\left({}^{(W)}\Omega|_{S_{(W)}(\lambda')}(\omega)\right)^2 - \frac{1}{2}$ in terms of $\log {}^{(W)}\Omega$ and consider only the lowest order term of the expansion [38] in $\log 2{}^{(W)}\Omega$,

$$\begin{aligned}\mathcal{A}(W)(\lambda,\omega) &= \frac{\lambda}{2} + \int_0^\lambda \left(\log 2{}^{(W)}\Omega|_{S_{(W)}(\lambda')}\right)(\omega) d\lambda' \qquad (7.7.30)\\ &= \frac{\lambda}{2} + \int_0^\lambda \left(\log 2{}^{(W)}\Omega\right)(W(\lambda',\omega),\omega) d\lambda'\end{aligned}$$

where the last line follows since by definition

$${}^{(W)}\Omega|_{S_W(\lambda)}(\omega) = {}^{(W)}\Omega(W(\lambda,\omega),\omega) \ .$$

$\mathbf{C}_5$: We look for a fixed point of the map 7.7.30 in the space

$$\mathcal{E} \equiv \cap_{p=2}^4 C^1(I; W_2^p(S_0)) \qquad (7.7.31)$$

with the Sobolev norms $||\cdot||_{W_k^p(S_0)}$ defined by

$$||G||_{W_k^p(S_0)} \equiv \left(\sum_{l=0}^k \int_{S_0} |{}^{(0)}\nabla\!\!\!\!/^{\,l} G|^p d\mu_0\right)^{\frac{1}{p}} < \infty \qquad (7.7.32)$$

where ${}^{(0)}\nabla\!\!\!\!/$ denotes the covariant derivative with respect to S_0.

---

[38] We consider only the first term of the expansion

$$2\Omega^2 - \frac{1}{2} = \log 2\Omega + \frac{1}{2}\sum_{k=2}^\infty \frac{2^k}{k!}(\log 2\Omega)^k \ .$$

It will be clear during the proof that the result obtained using this "approximate" map can be immediately extended to the exact map if the portion of the null hypersurface, $\underline{C}_*(I)$, does not differ too much from a portion of a null cone in the Minkowski spacetime.

Definition 7.7.1 *In $\mathcal{E}$ we define the closed set $\mathcal{K}_{\delta_0,\sigma_1}$ as follows:*
$W \in \mathcal{K}_{\delta_0,\sigma_1}$ if, for any $p \in [2,4]$ [39]

$$\begin{cases} W(\lambda=0,\omega)=0 \\ \sup_{\lambda\in I} ||{}^{(0)}\!\nabla\!\!\!/^{0,1,2}\left(W(\lambda,\cdot)-\frac{\lambda}{2}\right)||_{L^p(S_0)} \leq \delta_0 \\ \sup_{\lambda\in I} ||\partial_\lambda {}^{(0)}\!\nabla\!\!\!/^{0,1,2}\left(W(\lambda,\cdot)-\frac{\lambda}{2}\right)||_{L^p(S_0)} \leq \sigma_1 \end{cases} \tag{7.7.33}$$

We can finally state a precise form of the local existence part of Theorem **M6**

Theorem 7.7.2 *Assume that*

$$\mathcal{R}_{[2]}{}'(\underline{C}_*) + \underline{\mathcal{R}}_{[2]}{}'(\underline{C}_*) \leq \epsilon_0'$$
$$\underline{\mathcal{O}}_{[2]}{}'(\underline{C}_*) + \mathcal{O}_{[2]}{}'(\underline{C}_*) \leq \epsilon_0'$$

relative to the background foliation. There exist positive constants $|I|, \delta_0, \sigma_1$ such that $\mathcal{A}$ has a fixed point in the set $\mathcal{K}_{\delta_0,\sigma_1}$.

Proof: We need to show that:

i) $\mathcal{A}$ maps $\mathcal{K}_{\delta_0,\sigma_1}$, subset of $\cap_{p=2}^4 C^1(I; W_2^p(S_0))$, into itself.
ii) $\mathcal{A}$ is a contraction on $\mathcal{K}_{\delta_0,\sigma_1}$.

We shall omit the proof here, see [Ni].

We restrict now the comparison between different foliations discussed in subsection 7.7.1 to the specific case where the two foliations are the background and the canonical foliation on $\underline{C}_*$. We recall that the existence of the canonical foliation has been proved in Theorem **M6**. As discussed in subsection 7.7.1 and required in the proof of the "Oscillation Lemma", see also Lemma 4.8.2, we have to control the quantity $(-2\mathbf{g}(L',L))$ in the "initial" portion of $\underline{C}_*$. The proof we present here is different from the one discussed in subsection 7.7.1 [40]. The result is expressed in the following lemma.

Osservazione 7.7.5 *Nel lemma 7.7.2 seguente si dimostra la stima su $\underline{C}_*$ di $\mathbf{g}(L,L')$ che viene utilizzata come ipotesi nella prova dell'"Oscillation*

[39] Conditions 7.7.33 imply also that the following norms are bounded

$$\sup_{\lambda\in I}|{}^{(0)}\!\nabla\!\!\!/^{0,1}W|_{L^\infty(S_0)}\ ,\ \sup_{\lambda\in I}|\partial_\lambda {}^{(0)}\!\nabla\!\!\!/^{0,1}W|_{L^\infty(S_0)}\ .$$

[40] The proof used here can be easily adapted to prove Lemma 7.7.1.

Lemma" nel Capitolo 4 e in particolare nella prova del comportamento "asintotico" di $\mathbf{g}(L,L')$ *su* Σ'_{δ_0}*, Lemma 4.8.2. In questa dimostrazione la stima di* $\mathbf{g}(L,L')$ *su* $\underline{C}_*$ *è ottenuta usando le proprietà dei coefficienti di connessione rispetto alla foliazione canonica su* $\underline{C}_*$ *e quindi come assunzione su* $\underline{C}_*$ *l'ipotesi* $\mathcal{O}(\underline{C}_*) \leq \epsilon'_0$ *va sostituita da* $\mathcal{O}^*(\underline{C}_*) \leq \epsilon'_0$*. Inoltre poichè si userà anche del fatto che* $|r'^2\tau_-^{\frac{1}{2}}\eta'| \leq c\epsilon'_0$ *perchè ciò possa avvenire occorre assumere* $|r'^{\frac{5}{2}}\eta'|_{\Sigma_0} \leq \epsilon'_0$ *perchè questo implica, applicando l'equazione di evoluzione a* η' *nella "Initial layer region", il comportamento desiderato. Inoltre questo lemma ci interessa ed è provato solo nella regione* $\underline{C}_*(I)$ *dove la stima di* $\mathbf{g}(L,L')$ *è necessaria.*

Lemma 7.7.2 *Assume on* $\underline{C}_*(I)$ [41]

$$\mathcal{O}'(\underline{C}_*(I)) \leq \epsilon'_0 \ , \ \mathcal{O}^*(\underline{C}_*(I)) \leq \epsilon'_0 \tag{7.7.34}$$

Assume also that

$$|r'^{\frac{5}{2}}\eta'|_{\Sigma_0} \leq \epsilon'_0 \quad , \quad |\mathbf{g}(L',L)|_{\underline{C}_*\cap\Sigma_0} = 0 \tag{7.7.35}$$

then, on $\underline{C}_*(I)$*, the following inequality holds*

$$|r^2\tau_-\mathbf{g}(L',L)| \leq c\epsilon'_0 \tag{7.7.36}$$

Proof: The proof is similar, but easier to the one for the estimate of $\mathbf{g}(L',L)$ on Σ'_{δ_0}. The evolution equation along $\underline{C}_*$ for $\mathbf{g}(L',L)$ is

$$\frac{d}{du'}\mathbf{g}(L',L) = 4\frac{\Omega'}{\Omega}(\Omega\underline{\omega}'+\Omega'\underline{\omega})\mathbf{g}(L',L)+\frac{\Omega'}{\Omega}(-2\mathbf{g}(L',L))^{\frac{1}{2}}\hat{\sigma}\cdot(\eta'-\eta) \tag{7.7.37}$$

which is obtained by a direct computation. Applying now Gronwall's Lemma we obtain

$$|\mathbf{g}(L',L)|_{\underline{C}_*}(u',\underline{u}_*) \leq c\int_{u'_0}^{u'} du''\left(|\mathbf{g}(L',L)|^{\frac{1}{2}}|\eta'-\eta|\right)(u'',\underline{u}_*) \tag{7.7.38}$$

As, due to the assumptions of the lemma, $|\eta'-\eta| = O(r'^{-2}\tau_-^{-\frac{1}{2}})$, the thesis follows mimicking the argument used to complete the proof of Lemma 4.8.2.

[41] $\mathcal{O}^*(\underline{C}_*)$ has the same expression of $\mathcal{O}(\underline{C}_*)$, with all the connection coefficients norms substituted by those defined in 7.4.1, 7.4.3, 7.4.4.

7.7.3 Proof of Propositions 7.4.4, 7.4.5

Proof: The proof of these propositions is a consequence of the next two lemma.

Lemma 7.7.3 *Let us consider on* S [42] *a solution in the sense of distributions of the equation*

$$\not{\Delta} u = G$$

where G *satisfies the condition:* $\overline{G} = 0$. *Moreover let us assume that*

$$G \in W^{-2,p}(S)$$

where an element of $W^{-2,p}(S)$ *is a bounded linear functional on the space of the test functions* $\mathcal{C}^\infty(S)$ *such that the following inequality holds*

$$| < G, \phi > | \leq C \left(|\not{\nabla}^2 \phi|_{L^q(S)} + r^{-1} |\not{\nabla} \phi|_{L^q(S)} + r^{-2} |\phi|_{L^q(S)} \right)$$

with q *the number conjugate to* p *and a constant* C *independent from* ϕ. *Introducing the norm* $|G|_{W^{-2,p}(S)}$ *as the infimum of the possible constants* C, *the solution* u *is an element of* $L^p(S)$ *and the following inequality holds*

$$|u - \overline{u}| \leq c |G|_{W^{-2,p}(S)}$$

with a constant c *depending on* ${k_m}^{-1}$ *and* k_M.

Lemma 7.7.4 *Assuming the last slice endowed with the canonical foliation, then* $\Omega \mathbf{D}_3 \log \Omega$, $(\Omega \mathbf{D}_3)^2 \log \Omega$, $(\Omega \mathbf{D}_3)^3 \log \Omega$ *satisfy the following elliptic equations*

$$\not{\Delta}(\Omega \mathbf{D}_3 \log \Omega) = \not{d}iv F_1 + G_1 - \overline{G_1} \tag{7.7.39}$$

$$\not{\Delta}((\Omega \mathbf{D}_3)^2 \log \Omega) = \not{d}iv F_2 + G_2 - \overline{G_2} \tag{7.7.40}$$

$$\not{\Delta}((\Omega \mathbf{D}_3)^3 \log \Omega) = \not{d}iv F_3 + G_3 - \overline{G_3} \tag{7.7.41}$$

where

$$F_1 = \Omega \underline{\beta} + \tilde{F}_1 \quad , \quad \tilde{F}_1 = \left(\frac{3}{2} \Omega \eta \cdot \hat{\underline{\chi}} + \frac{1}{4} \Omega \eta tr \underline{\chi} \right)$$

[42] Here S is a two dimensional compact surface such that $k_m > 0$, where $k_m = \min_S r^2 K$, $k_M = \max_S r^2 K$ and K is its Gauss curvature.

$$G_1 = H + \frac{1}{4}\Omega\mathbf{D}_3(\hat{\chi}\cdot\hat{\underline{\chi}}) - \frac{1}{2}(\Omega tr\underline{\chi})(\rho-\overline{\rho}) + \frac{1}{4}(\Omega tr\underline{\chi})(\hat{\chi}\cdot\hat{\underline{\chi}} - \overline{\hat{\chi}\cdot\hat{\underline{\chi}}}) \tag{7.7.42}$$
$$H = \frac{\Omega}{2}(\frac{3}{2}tr\underline{\chi}\rho + \frac{1}{2}\hat{\chi}\cdot\underline{\alpha} + \eta\cdot\underline{\beta})$$

$$F_2 = \Omega \not{d}iv\underline{\alpha} + \widetilde{F_2}$$
$$\widetilde{F_2} = (\mathbf{D}_3\Omega\underline{\beta} - \Omega\not{d}iv\underline{\alpha}) + \Omega\left(\mathbf{D}_3\widetilde{F_1} + 2\not\nabla(\Omega\mathbf{D}_3\log\Omega)\cdot\hat{\underline{\chi}} - \hat{\underline{\chi}}\cdot F_1 + \frac{1}{2}tr\underline{\chi}F_1\right)$$
$$G_2 = \Omega\mathbf{D}_3 G_1 + (\Omega tr\underline{\chi})(G_1 - \overline{G_1}) \tag{7.7.43}$$

$$F_3 = \Omega\not{d}iv\mathbf{D}_3\underline{\alpha} + \widetilde{F_3}$$
$$\widetilde{F_3} = (\mathbf{D}_3\Omega\not{d}iv\underline{\alpha} - \Omega\not{d}iv\mathbf{D}_3\underline{\alpha}) + \Omega\left(\mathbf{D}_3\widetilde{F_2} + 2\not\nabla[(\Omega D_3)^2\log\Omega]\cdot\hat{\underline{\chi}} - \hat{\underline{\chi}}\cdot F_2 + \frac{1}{2}tr\underline{\chi}F_2\right)$$
$$G_3 = \Omega\mathbf{D}_3 G_2 + (\Omega tr\underline{\chi})(G_2 - \overline{G_2}) \tag{7.7.44}$$

Proof: The proof of this lemma is postponed to the end of the appendix.

Proof of Lemma 7.7.3: Let ϕ be a function $\in \mathcal{C}^\infty(S)$, let ψ be a solution of $\not\Delta\psi = \phi$ then it is easy to prove that ψ satisfies the following bound for any $q > 0$

$$|\not\nabla^2\psi|_{L^q(S)} + r^{-1}|\not\nabla\psi|_{L^q(S)} + r^{-2}|\psi - \overline{\psi}|_{L^q(S)} \leq c|\phi|_{L^q(S)}$$

therefore

$$\begin{aligned} |<u-\overline{u},\phi>| &= |<u-\overline{u},\not\Delta\psi>| = |<\not\Delta u,\psi>| \\ &= |<G,\psi>| = |<G,\psi-\overline{\psi}>| \\ &\leq |G|_{W^{-2,p}(S)}\left(|\not\nabla^2\psi|_{L^q(S)} + r^{-1}|\not\nabla\psi|_{L^q(S)} + r^{-2}|\psi-\overline{\psi}|_{L^q(S)}\right) \\ &\leq c|G|_{W^{-2,p}(S)}|\phi|_{L^q(S)} \end{aligned} \tag{7.7.45}$$

so that
$$|<u-\overline{u},\phi>| \leq c|G|_{W^{-2,p}(S)}|\phi|_{L^q(S)}$$
for any $\phi \in \mathcal{C}^\infty(S)$ and, from it,
$$|u-\overline{u}|_{L^p(S)} \leq c|G|_{W^{-2,p}(S)}\ .$$

Proof of Proposition 7.4.4

We apply Lemma 7.7.3 to equation 7.7.39 with

$$u = \Omega\mathbf{D}_3\log\Omega$$

$$\begin{aligned} G &= \mathrm{d\!\!/iv}\, F_1 + (G_1 - \overline{G_1}) = \mathrm{d\!\!/iv}\, \Omega\underline{\beta} + \mathrm{d\!\!/iv}\, \tilde{F}_1 + (G_1 - \overline{G_1}) \\ &= \mathrm{d\!\!/iv}\, \Omega\underline{\beta} + \mathrm{d\!\!/iv} \left(\frac{3}{2}\Omega\eta \cdot \hat{\underline{\chi}} + \frac{1}{4}\Omega\eta tr\underline{\chi} \right) + (G_1 - \overline{G_1}) \end{aligned}$$

The only derivative of the Riemann component, is $\mathrm{d\!\!/iv}\, \Omega\underline{\beta}$, all the other terms of G are controlled from the sup norms estimates of the non derived Riemann components and the estimates of the connection coefficients and their first derivatives on the last slice. In particular, assuming $\mathcal{I}_0$, Δ_0 sufficiently small,

$$\begin{aligned} &|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}(G_1 - \overline{G_1})|_{p,S} \leq c\,(\mathcal{I}_0 + \Delta_0) \ \ , \ \ \forall p \geq 2 \\ &|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}\mathrm{d\!\!/iv}\, \Omega\underline{\beta}|_{p,S} \leq c\Delta_1 \ , \ \ \forall p \in [2,4] \end{aligned} \tag{7.7.46}$$

and

$$|r^{4-\frac{2}{p}}\tau_-^{\frac{1}{2}}\mathrm{d\!\!/iv}\, \tilde{F}_1|_{p,S} \leq c\,(\mathcal{I}_0 + \Delta_0) \ \ , \ \ \forall p \geq 2 \tag{7.7.47}$$

The last estimate is true only on the last slice [43] due to its canonical foliation. In fact from 7.7.42 it follows

$$\tilde{F}_1 = \left(\frac{3}{2}\Omega\eta \cdot \hat{\underline{\chi}} + \frac{1}{4}\Omega\eta\mathrm{tr}\underline{\chi} \right)$$

and, to control $|r^{4-\frac{2}{p}}\tau_-^{\frac{1}{2}}\mathrm{d\!\!/iv}\, \tilde{F}_1|_{p,S}$, recalling the estimate for $\hat{\underline{\chi}}$, we have to control $|r^{4-\frac{2}{p}}\tau_-^{\frac{1}{2}}\Omega\nabla\!\!\!/\,\eta\mathrm{tr}\underline{\chi}|_{p,S}$. Differently from what happens in $\mathcal{K}$, $\mathrm{d\!\!/iv}\,\eta$ has a better asymptotic behaviour on $\underline{C}_*$. In fact from 3.3.12 we obtain

$$\begin{aligned} \frac{1}{2}\mathrm{d\!\!/iv}\,\eta = \frac{1}{2}\mathrm{d\!\!/iv}\,\zeta + \frac{1}{2}\Delta\!\!\!/ \log\Omega &= \frac{1}{2}\left(K - \overline{K} + \frac{1}{4}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \overline{\mathrm{tr}\chi\mathrm{tr}\underline{\chi}}) \right) \\ &= \frac{1}{2}\left(\frac{1}{2}\hat{\chi}\hat{\underline{\chi}} - \frac{1}{2}\overline{\hat{\chi}\hat{\underline{\chi}}} - (\rho - \overline{\rho}) \right) \end{aligned} \tag{7.7.48}$$

and from this expression the required estimate for $\mathrm{d\!\!/iv}\,\eta$ follows. Therefore

$$\begin{aligned} |<G,\psi>| &\leq \left| \int_S (\mathrm{d\!\!/iv}\, \Omega\underline{\beta})\psi \right| + \left| \int_S (\mathrm{d\!\!/iv}\, \tilde{F})\psi \right| + \left| \int_S (G_1 - \overline{G_1})\psi \right| \\ &\leq \int_S |\Omega\underline{\beta}||\nabla\!\!\!/\,\psi| + \int_S |\mathrm{d\!\!/iv}\, \tilde{F}_1||\psi| + \int_S |G_1 - \overline{G_1}||\psi| \\ &\leq \frac{1}{|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}|}|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\Omega\underline{\beta}|_{p,S}|\nabla\!\!\!/\,\psi|_{q,S} + \frac{1}{|r^{4-\frac{2}{p}}\tau_-^{\frac{1}{2}}|}|r^{4-\frac{2}{p}}\tau_-^{\frac{1}{2}}\mathrm{d\!\!/iv}\, \tilde{F}_1|_{p,S}|\psi|_{q,S} \end{aligned}$$

[43] In $\mathcal{K}$ the analogous of the estimate 7.7.47 holds without the factor $\tau_-^{\frac{1}{2}}$.

$$
\begin{aligned}
&+ \quad \frac{1}{|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}|}|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}(G_1-\overline{G_1})|_{p,S}|\psi|_{q,S}\\
&\leq \quad c\frac{1}{|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}|}\sup_{\underline{C}_*}\Bigg[\bigg(|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\Omega\underline{\beta}|_{p,S}+|r^{4-\frac{2}{p}}\tau_-^{\frac{1}{2}}\nabla\!\!\!\!/\tilde{F}_1|_{p,S}\bigg)\\
&+ \quad |r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}(G_1-\overline{G_1})|_{p,S}\Bigg]\left(\frac{1}{r}|\nabla\!\!\!\!/\psi|_{q,S}+\frac{1}{r^2}|\psi|_{q,S}\right) \qquad (7.7.49)
\end{aligned}
$$

which implies

$$
\begin{aligned}
|G|_{W^{-2,p}(S)} \quad &\leq \quad c_0\frac{1}{|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}|}\sup_{\underline{C}_*}\Bigg[\bigg(|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\Omega\underline{\beta}|_{p,S}+|r^{4-\frac{2}{p}}\tau_-^{\frac{1}{2}}\tilde{F}_1|_{p,S}\bigg)\\
&+ \quad |r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}(G_1-\overline{G_1})|_{p,S}\Bigg] \leq c_0\frac{1}{|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}|}(\mathcal{I}_0+\Delta_0)\ .
\end{aligned}
$$

From it we obtain, for any $p \geq 2$,

$$
|\Omega\mathbf{D}_3\log\Omega-\overline{\Omega\mathbf{D}_3\log\Omega}|_{p,S} \leq c_0\frac{1}{|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}|}(\mathcal{I}_0+\Delta_0) \qquad (7.7.50)
$$

Considering $\nabla\!\!\!\!/(\Omega\mathbf{D}_3\log\Omega)$ and $\nabla\!\!\!\!/^2(\Omega\mathbf{D}_3\log\Omega)$ and proceeding exactly in the same way we easily conclude that, for $p \geq 2$, [44]

$$
|\nabla\!\!\!\!/(\Omega\mathbf{D}_3\log\Omega)|_{p,S} \leq c\frac{1}{|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}|}(\mathcal{I}_0+\Delta_0) \qquad (7.7.51)
$$

and, for $p \in [2,4]$,

$$
|\nabla\!\!\!\!/^2(\Omega\mathbf{D}_3\log\Omega)|_{p,S} \leq c_0\frac{1}{|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}|}(\mathcal{I}_0+\Delta_0+\Delta_1) \qquad (7.7.52)
$$

Finally, as on the last slice $\overline{\log 2\Omega}=0$, it follows

$$
\begin{aligned}
0=\frac{\Omega}{2}\mathbf{D}_3\overline{\log 2\Omega} \quad &= \quad \frac{1}{2}\frac{d}{du}\frac{1}{|S|}\int_S\log 2\Omega=-\frac{1}{2}\frac{1}{|S|^2}(\frac{d|S|}{du})\int_S\log 2\Omega+\frac{1}{2}\frac{1}{|S|}\frac{d}{du}\int_S\log 2\Omega\\
&= \quad \frac{1}{2}\left[-\frac{1}{|S|}(\frac{d|S|}{du})\overline{\log 2\Omega}+\overline{(\Omega tr\underline{\chi})\log 2\Omega}+\overline{\Omega\mathbf{D}_3\log 2\Omega}\right]\\
&= \quad \overline{(\Omega tr\underline{\chi})\log 2\Omega}+\overline{\Omega\mathbf{D}_3\log 2\Omega}\quad .
\end{aligned}
$$

[44]From it $\sup_{\underline{C}_*}|r\tau_-^{\frac{3}{2}}\Omega\mathbf{D}_3\log\Omega| \leq c(\mathcal{I}_0+\Delta_0)$ follows.

Therefore, recalling the inequality [45]

$$|\overline{f}|_{p,S} \leq |f|_{p,S} \tag{7.7.53}$$

and Proposition 7.4.3 we have

$$|\overline{\Omega \mathbf{D}_3 \log 2\Omega}|_{p,S_*} \leq |\overline{(\Omega tr\underline{\chi}) \log 2\Omega}|_{p,S_*} \leq c\frac{1}{r}|(\log 2\Omega - \overline{\log 2\Omega})|_{p,S_*} \tag{7.7.54}$$

and

$$|r^{1-\frac{2}{p}}\tau_- \overline{\Omega \mathbf{D}_3 \log 2\Omega}|_{p,S_*} \leq c\frac{\tau_-^{\frac{1}{2}}}{r}|r^{1-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\log 2\Omega - \overline{\log 2\Omega})|_{p,S_*} \leq c\frac{\tau_-^{\frac{1}{2}}}{r}(\mathcal{I}_0 + \Delta_0)$$

so that, finally,

$$|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}} \overline{\Omega \mathbf{D}_3 \log 2\Omega}|_{p,S_*} \leq c(\mathcal{I}_0 + \Delta_0) \tag{7.7.55}$$

completing the proof of Proposition 7.4.4.

Proof of Proposition 7.4.5

To apply Lemma 7.7.3 to this case we use the following definitions,

$$\begin{aligned}
u &= \Omega \mathbf{D}_3(\Omega \mathbf{D}_3 \log \Omega) \\
G &= \mathrm{d\!\!/iv}\left(\Omega \mathbf{D}_3 F_1 + 2\Omega(\nabla\!\!\!\!/(\Omega \mathbf{D}_3 \log \Omega)) \cdot \hat{\underline{\chi}} - \Omega \hat{\underline{\chi}} \cdot F_1 + \frac{1}{2}\Omega tr\underline{\chi} F_1\right) + (G_2 - \overline{G_2}) \\
&\equiv \mathrm{d\!\!/iv}\, \mathrm{d\!\!/iv}\, \underline{\alpha} + \mathrm{d\!\!/iv}\, \tilde{F}_2 + (G_2 - \overline{G_2})
\end{aligned}$$

Therefore we have to control the integrals

$$\textstyle\int_S \mathrm{d\!\!/iv}\, \mathrm{d\!\!/iv}\, \underline{\alpha}\psi \ , \ \int_S \mathrm{d\!\!/iv}\, \tilde{F}_2\psi \ , \ \int_S (G_2 - \overline{G_2})\psi \ .$$

The following estimates hold, for any $p \geq 2$,

$$\begin{aligned}
\left|\int_S \mathrm{d\!\!/iv}\, \mathrm{d\!\!/iv}\, \underline{\alpha}\psi\right| &\leq c\int_S |\underline{\alpha}||\nabla\!\!\!\!/^2\psi| \leq c\frac{1}{|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}|}\left(|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}\underline{\alpha}|_{p,S}\right)|\nabla\!\!\!\!/^2\psi|_{q,S} \\
\left|\int_S \mathrm{d\!\!/iv}\, \tilde{F}_2\psi\right| &\leq c\int_S |\tilde{F}_2||\nabla\!\!\!\!/\psi| \leq c\frac{1}{|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}|}|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}\tilde{F}_2|_{p,S}|\nabla\!\!\!\!/\psi|_{q,S} \\
&\leq c\frac{1}{|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}|}|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}\tilde{F}_2|_{p,S}\left(\frac{1}{r}|\nabla\!\!\!\!/\psi|_{q,S}\right)
\end{aligned} \tag{7.7.56}$$

[45] It follows immediately from the Holder inequality. In fact

$$|\overline{f}|_{p,S} = \left(\int_S \frac{1}{|S|^p}\left(\int_S f\right)^p\right)^{\frac{1}{p}} = |S|^{\frac{1}{p}-1}\int_S f \leq |S|^{\frac{1}{p}-1}|1|_{q,S}|f|_{p,S} = |S|^{\frac{1}{p}+\frac{1}{q}-1}|f|_{p,S} = |f|_{p,S} \ .$$

Clearly the last estimate is the appropriate one as a long but simple analysis of all the terms composing $\tilde{F}_2$ allows to conclude that, for any $p \geq 2$,

$$|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}\tilde{F}_2|_{p,S} \leq c(\mathcal{I}_* + \Delta_0) \tag{7.7.57}$$

To estimate the third integral $\int_S(G_2-\overline{G_2})\psi$ we have to examine the structure of the term $G_2 = \Omega\mathbf{D}_3G_1 + (\Omega tr\underline{\chi})(G_1-\overline{G_1})$. We already have the estimates of $(G_1 - \overline{G_1})$ therefore we have only to investigate the term $\Omega\mathbf{D}_3G_1$. Extracting the terms where $\mathbf{D}_3$ operates on the Riemann components we can write

$$\mathbf{D}_3G_1 = O(\frac{1}{r})\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta} + O(\frac{1}{r^2})\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha} + O(\frac{1}{r^2})\rho + O(\frac{1}{r^3})(\underline{\beta}+\underline{\alpha}) + \widetilde{\mathbf{D}_3G_1}$$

where $\widetilde{\mathbf{D}_3G_1}$ is the part of $\mathbf{D}_3G_1$ which does not depend explicitely from the Riemann components and for which it is easy to prove that the following bounds hold, for any $p \geq 2$,

$$|r^{3-\frac{2}{p}}\tau_-^{\frac{5}{2}}\widetilde{\mathbf{D}_3G_1}|_{p,S} \leq c(\mathcal{I}_* + \Delta_0)\ .$$

Therefore we write

$$\begin{aligned}
G_2 &= \Omega\mathbf{D}_3G_1 + (\Omega tr\underline{\chi})(G_1-\overline{G_1}) = \left(O(\frac{1}{r})\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta} + O(\frac{1}{r^2})\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha}\right)\\
&+ \left[\left(O(\frac{1}{r^2})\rho + O(\frac{1}{r^3})(\underline{\beta}+\underline{\alpha})\right) + \widetilde{\mathbf{D}_3G_1} + (\Omega tr\underline{\chi})(G_1-\overline{G_1})\right]\\
&\equiv \left(O(\frac{1}{r})\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta} + O(\frac{1}{r^2})\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha}\right) + \tilde{G}_2
\end{aligned}$$

and it is easy to prove, collecting all the previous estimates,

$$|r^{3-\frac{2}{p}}\tau_-^{\frac{5}{2}}\tilde{G}_2|_{p,S} \leq c(\mathcal{I}_* + \Delta_0)$$

for any $p \geq 2$. Collecting all these results together we obtain

$$\begin{aligned}
|\int_S(G_2-\overline{G_2})\psi| &\leq c\int_S|\left(O(\frac{1}{r})\underline{\beta}\right)||\nabla\!\!\!\!/\,\psi| + c\int_S|\left(O(\frac{1}{r^2})\underline{\alpha}\right)||\nabla\!\!\!\!/\,\psi| + c\int_S|\tilde{G}_2-\overline{\tilde{G}_2}||\psi|\\
&\leq c\frac{1}{|r^{3-\frac{2}{p}}\tau_-^{\frac{5}{2}}|}|r^{3-\frac{2}{p}}\tau_-^{\frac{5}{2}}(\tilde{G}_2-\overline{\tilde{G}_2})|_{p,S}|\psi|_{q,S} + c\frac{1}{|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}|}|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\underline{\beta}|_{p,S}|\nabla\!\!\!\!/\,\psi|_{q,S}\\
&\quad + c\frac{1}{|r^{3-\frac{2}{p}}\tau_-^{\frac{5}{2}}|}|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}\underline{\alpha}|_{p,S}|\nabla\!\!\!\!/\,\psi|_{q,S}\\
&\leq c\frac{1}{|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}|}\left[|r^{3-\frac{2}{p}}\tau_-^{\frac{5}{2}}(\tilde{G}_2-\overline{\tilde{G}_2})|_{p,S} + \frac{1}{r}\left(|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\underline{\beta}|_{p,S} + |r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}\underline{\alpha}|_{p,S}\right)\right]\\
&\quad \cdot\left(\frac{1}{r}|\nabla\!\!\!\!/\,\psi|_{q,S} + \frac{1}{r^2}|\psi|_{q,S}\right)
\end{aligned} \tag{7.7.58}$$

Using Lemma 7.7.3 and the analogous of 7.7.54 we conclude

$$|(\Omega D_3(\Omega D_3 \log\Omega))|_{p,S} \leq c\frac{1}{|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}|}\left\{\left[|r^{3-\frac{2}{p}}\tau_-^{\frac{5}{2}}(\tilde{G}_2-\overline{\tilde{G}_2})|_{p,S}\right.\right.$$
$$\left.+\frac{1}{r}\left(|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\underline{\beta}|_{p,S}+|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}\underline{\alpha}|_{p,S}\right)\right]+\frac{1}{r}\left(|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}\tilde{F}_2|_{p,S}\right) \quad (7.7.59)$$
$$\left.+\frac{1}{\tau_-^{\frac{1}{2}}}\left(|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}\underline{\alpha}|_{p,S}\right)\right\}\cdot\left(\frac{1}{r^2}|\nabla\!\!\!/^2\psi|_{q,S}+\frac{1}{r}|\nabla\!\!\!/\psi|_{q,S}+\frac{1}{r^2}|\psi|_{q,S}\right)$$

Therefore

$$|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}(\Omega D_3(\Omega D_3\log\Omega))|_{p,S}\leq c|G|_{W^{-p,S}} \quad (7.7.60)$$

and, for any $p \geq 2$,

$$|G|_{W^{-p,S}}\leq\left\{\left[|r^{3-\frac{2}{p}}\tau_-^{\frac{5}{2}}(\tilde{G}_2-\overline{\tilde{G}_2})|_{p,S}+\frac{1}{r}\left(|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}\underline{\beta}|_{p,S}+|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}\underline{\alpha}|_{p,S}\right)\right]\right.$$
$$\left.+\frac{1}{r}\left(|r^{3-\frac{2}{p}}\tau_-^{\frac{3}{2}}\tilde{F}_2|_{p,S}\right)+\frac{1}{\tau_-^{\frac{1}{2}}}\left(|r^{1-\frac{2}{p}}\tau_-^{\frac{5}{2}}\underline{\alpha}|_{p,S}\right)\right\}\leq c(\mathcal{I}_*+\Delta_0) \quad (7.7.61)$$

Proceeding in a similar way we obtain, for any $p\in[2,4]$,

$$|r^{2-\frac{2}{p}}\tau_-^{\frac{5}{2}}\nabla\!\!\!/(\Omega\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega))|_{p,S}\leq c(\mathcal{I}_*+\Delta_0+\Delta_1)\ .$$

Estimate for $(\Omega\mathbf{D}_3)^3\log\Omega$

The proof goes basically as in the case of $(\Omega\mathbf{D}_3)^2\log\Omega$, we stress only the main differences. From the explicit expressions of F_3 and G_3 we have to examine the dependance on the various Riemann components. In a symbolic way we can write

$$\begin{aligned}
F_3 &= \mathrm{d}\!\!/\mathrm{iv}\,\mathbf{D}_3\underline{\alpha}+O(\frac{1}{r})\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha}+O(\frac{1}{r^2})\mathbf{D}_3\underline{\alpha}\\
&= \mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha}(\hat{\mathcal{L}}_TW)+\mathrm{d}\!\!/\mathrm{iv}\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta}+O(\frac{1}{r})\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha}+O(\frac{1}{r^2})(\underline{\alpha}(\hat{\mathcal{L}}_TW)+\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta})\\
G_3 &= O(\frac{1}{r})\mathrm{d}\!\!/\mathrm{iv}\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha}+O(\frac{1}{r^2})\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha}(\hat{\mathcal{L}}_TW)+O(\frac{1}{r^2})\mathrm{d}\!\!/\mathrm{iv}\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta} \qquad (7.7.62)
\end{aligned}$$

where we reported just the dependance on the various Riemann components associated to the Weyl tensor W except when explicitely indicated. The factors $O(\frac{1}{r})$, $O(\frac{1}{r^2})$ in front to some of them just remind the asymptotic behaviour of the factors multiplying the various components.

It is now easy to apply to this case Lemma 7.7.3 and obtain the thesis of the Lemma [46].

Proof of equation 7.7.39 of Lemma 7.7.4

We write

$$\Delta\!\!\!/\,\mathbf{D}_3\log\Omega=\mathbf{D}_3\Delta\!\!\!/\log\Omega+[\Delta\!\!\!/,\mathbf{D}_3]\log\Omega$$

Using Proposition 4.8.1 for $[\Delta\!\!\!/,\mathbf{D}_3]\log\Omega$ we obtain

$$\begin{aligned}\Delta\!\!\!/\,\mathbf{D}_3\log\Omega &= \mathbf{D}_3\Delta\!\!\!/\log\Omega\\ &-\Big\{-\eta_a\underline{\chi}_{ab}\nabla\!\!\!\!/_b\log\Omega+tr\underline{\chi}\eta_b\nabla\!\!\!\!/_b\log\Omega-2\underline{\chi}_{ab}(\nabla\!\!\!\!/\nabla\!\!\!\!/\log\Omega)_{ab}\\ &\quad-\zeta_a\underline{\chi}_{ab}\nabla\!\!\!\!/_b\log\Omega-(\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\chi})_b\nabla\!\!\!\!/_b\log\Omega-\underline{\beta}_b\nabla\!\!\!\!/_b\log\Omega\Big\}\\ &-\Big[(\nabla\!\!\!\!/_a\log\Omega)(\mathbf{D}_3\nabla\!\!\!\!/\log\Omega)_a-\zeta_a(\mathbf{D}_3\nabla\!\!\!\!/\log\Omega)_a+\eta_a\nabla\!\!\!\!/_a\mathbf{D}_3\log\Omega\\ &\quad+(\Delta\!\!\!/\log\Omega)\mathbf{D}_3\log\Omega+\zeta_a(\nabla\!\!\!\!/_a\log\Omega)\mathbf{D}_3\log\Omega\Big]\end{aligned}\qquad(7.7.63)$$

A long but easy computation allows to rewrite this equation in the following way:

$$\begin{aligned}\Delta\!\!\!/(\Omega\mathbf{D}_3\log\Omega) &= \Omega\mathbf{D}_3(\Delta\!\!\!/\log\Omega)+\Omega\Big[2\underline{\chi}_{ab}(\nabla\!\!\!\!/\nabla\!\!\!\!/\log\Omega)_{ab}+(\nabla\!\!\!\!/_b\log\Omega)(\nabla\!\!\!\!/_a\log\Omega)\underline{\chi}_{ab}\\ &+(\nabla\!\!\!\!/_b\log\Omega)(\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\chi})_b+(\nabla\!\!\!\!/_b\log\Omega)\left(\eta_a\underline{\chi}_{ab}-\eta_b tr\underline{\chi}+\underline{\beta}_b\right)\Big]\end{aligned}\qquad(7.7.64)$$

Recalling the structure equation, see 3.1.46,

$$\nabla\!\!\!\!/ tr\underline{\chi}-\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\chi}+\zeta\cdot\underline{\chi}-\zeta tr\underline{\chi}+\underline{\beta}=0$$

we obtain

$$(\eta_a\underline{\chi}_{ab}-\eta_b tr\underline{\chi}+\underline{\beta}_b)=(\nabla\!\!\!\!/_a\log\Omega)\underline{\chi}_{ab}-(\nabla\!\!\!\!/_b\log\Omega)\mathrm{tr}\underline{\chi}+(\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\chi})_b-\nabla\!\!\!\!/_b\mathrm{tr}\underline{\chi}$$

so that the [ ] part of the previous equation, 7.7.64, becomes:

$$\Omega[\quad]=2\mathrm{d}\!\!\!/\mathrm{iv}\,(\Omega\nabla\!\!\!\!/\log\Omega\cdot\underline{\chi})-(\nabla\!\!\!\!/_b\log\Omega)(\nabla\!\!\!\!/_b\Omega\mathrm{tr}\underline{\chi})$$

[46] Remark that the control of $|r^{\frac{3}{2}-\frac{2}{p}}\tau_-^{\frac{7}{2}}\hat{\mathcal{L}}_T\underline{\alpha}|_{p,S}$ required in the assumptions of the Lemma is provided by the control of

$$\int_{\underline{C}(\underline{u})}Q(\hat{\mathcal{L}}_{\mathcal{O}}\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)\ and\ \int_{\underline{C}(\underline{u})}Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3)$$

while we do not have, proceeding in the same way, the analogous bound for $|\nabla\!\!\!\!/\hat{\mathcal{L}}_T\underline{\alpha}|_{p,S}$.

and, therefore,

$$\Delta\!\!\!/(\Omega\mathbf{D}_3\log\Omega)=\Omega\mathbf{D}_3(\Delta\!\!\!/\log\Omega)+\left[2\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\nabla\!\!\!\!/\log\Omega\cdot\underline{\chi})-(\nabla\!\!\!\!/_b\log\Omega)(\nabla\!\!\!\!/_b\Omega\mathrm{tr}\underline{\chi})\right]$$

As the foliation we choose on the last slice $\underline{C}_*$ is canonical, $\log\Omega$ satisfies the elliptic equation, see 3.3.12,

$$\Delta\!\!\!/\log\Omega=\frac{1}{2}\left[\mathrm{d}\!\!/\mathrm{iv}\,\underline{\eta}+\frac{1}{2}(\hat{\chi}\cdot\underline{\hat{\chi}}-\overline{\hat{\chi}\cdot\underline{\hat{\chi}}})-(\rho-\overline{\rho})\right]$$

so that

$$\Omega\mathbf{D}_3(\Delta\!\!\!/\log\Omega)=\frac{\Omega}{2}\left[\mathbf{D}_3(\mathrm{d}\!\!/\mathrm{iv}\,\underline{\eta})-(\mathbf{D}_3\rho-\mathbf{D}_3\overline{\rho})+\frac{1}{2}\left(\mathbf{D}_3(\hat{\chi}\cdot\underline{\hat{\chi}})-\mathbf{D}_3\overline{(\hat{\chi}\cdot\underline{\hat{\chi}})}\right)\right]$$

Recalling the evolution equation of $\underline{\eta}$ along the $\underline{C}$ null hypersurfaces

$$\mathbf{D}\!\!\!\!/_3\underline{\eta}=(\eta-\underline{\eta})\cdot\underline{\chi}+\underline{\beta}$$

a long but simple computation [47] gives

$$\begin{aligned}\frac{\Omega}{2}\mathbf{D}_3(\mathrm{d}\!\!/\mathrm{iv}\,\underline{\eta}) &= \frac{1}{2}\left[\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\underline{\beta}+\Omega(\eta-\underline{\eta})\cdot\underline{\chi})-(\mathrm{d}\!\!/\mathrm{iv}\,\Omega\underline{\eta}\cdot\underline{\chi})+(\nabla\!\!\!\!/_b\Omega tr\underline{\chi})\underline{\eta}_b\right]\\ &= \frac{1}{2}\left[\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\underline{\beta}+\Omega\eta\cdot\underline{\chi}-2\Omega\underline{\eta}\cdot\underline{\chi})\right]+\frac{1}{2}(\nabla\!\!\!\!/_b\Omega tr\underline{\chi})\underline{\eta}_b \quad (7.7.65)\end{aligned}$$

where we used again the structure equation

$$\nabla\!\!\!\!/ tr\underline{\chi}-\mathrm{d}\!\!/\mathrm{iv}\,\underline{\chi}+\zeta\cdot\underline{\chi}-\zeta tr\underline{\chi}+\underline{\beta}=0\ .$$

From it

$$\begin{aligned}\Omega\mathbf{D}_3(\Delta\!\!\!/\log\Omega) &= \frac{1}{2}\left[\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\underline{\beta}+\Omega\eta\cdot\underline{\chi}-2\Omega\underline{\eta}\cdot\underline{\chi})\right]+\frac{1}{2}(\nabla\!\!\!\!/_b\Omega tr\underline{\chi})\underline{\eta}_b\\ &+ \left[-\frac{\Omega}{2}\mathbf{D}_3(\rho-\overline{\rho})+\frac{\Omega}{4}(\mathbf{D}_3(\hat{\chi}\cdot\underline{\hat{\chi}})-\mathbf{D}_3\overline{(\hat{\chi}\cdot\underline{\hat{\chi}})})\right] (7.7.66)\end{aligned}$$

---

[47] More explicitely

$$\begin{aligned}&\mathbf{D}_3(\mathrm{d}\!\!/\mathrm{iv}\,\underline{\eta})=\mathrm{d}\!\!/\mathrm{iv}\,(\mathbf{D}_3\underline{\eta})+(\nabla\!\!\!\!/_a\log\Omega)(\mathbf{D}_3\underline{\eta})_a-(\eta_a\underline{\chi}_{ab}-\eta_b tr\underline{\chi}+\underline{\beta}_b)\underline{\eta}_b-\underline{\chi}_{ab}(\nabla\!\!\!\!/_b\underline{\eta})_a\\ &=\mathrm{d}\!\!/\mathrm{iv}\,(\underline{\beta}+(\eta-\underline{\eta})\cdot\underline{\chi})+(\nabla\!\!\!\!/_a\log\Omega)(\underline{\beta}+(\eta-\underline{\eta})\cdot\underline{\chi})_a-(\eta_a\underline{\chi}_{ab}-\eta_b tr\underline{\chi}+\underline{\beta}_b)\underline{\eta}_b-\underline{\chi}_{ab}(\nabla\!\!\!\!/_b\underline{\eta})_a\\ &=\frac{1}{\Omega}\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\underline{\beta}+\Omega(\eta-\underline{\eta})\cdot\underline{\chi})-\left((\nabla\!\!\!\!/_a\log\Omega)\underline{\chi}_{ab}+(\mathrm{d}\!\!/\mathrm{iv}\,\underline{\chi})_b-(\nabla\!\!\!\!/_b\log\Omega)tr\underline{\chi}-\nabla\!\!\!\!/_b tr\underline{\chi}\right)\underline{\eta}_b-\underline{\chi}_{ab}(\nabla\!\!\!\!/_b\underline{\eta})_a\\ &=\frac{1}{\Omega}\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\underline{\beta}+\Omega(\eta-\underline{\eta})\cdot\underline{\chi})-\frac{1}{\Omega}(\mathrm{d}\!\!/\mathrm{iv}\,\Omega\underline{\chi})_b\underline{\eta}_b-\underline{\chi}_{ab}(\nabla\!\!\!\!/_a\underline{\eta})_b+\frac{1}{\Omega}(\nabla\!\!\!\!/(\Omega tr\underline{\chi}))\underline{\eta}_b\\ &=\frac{1}{\Omega}\left[\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\underline{\beta}+\Omega(\eta-\underline{\eta})\cdot\underline{\chi})-(\mathrm{d}\!\!/\mathrm{iv}\,\Omega\underline{\eta}\cdot\underline{\chi})+(\nabla\!\!\!\!/_b\Omega tr\underline{\chi})\underline{\eta}_b\right]\end{aligned}$$

so that finally

$$\begin{aligned} \not\Delta(\Omega\mathbf{D}_3\log\Omega) &= \not{d}\mathrm{iv}\left(\frac{1}{2}\Omega\underline{\beta}+\frac{3}{2}\Omega\eta\cdot\underline{\chi}\right)-\frac{1}{2}\eta_b(\not\nabla_b\Omega tr\underline{\chi}) \\ &+ \left[-\frac{\Omega}{2}\mathbf{D}_3(\rho-\overline{\rho})+\frac{\Omega}{4}(\mathbf{D}_3(\hat{\chi}\cdot\hat{\underline{\chi}})-\mathbf{D}_3\overline{(\hat{\chi}\cdot\hat{\underline{\chi}})})\right] \end{aligned} \tag{7.7.67}$$

Recalling that, on the last slice, see 3.3.10,

$$\not{d}\mathrm{iv}\,\eta=-\not{d}\mathrm{iv}\,\underline{\eta}+2\not\Delta\log\Omega=\left[\tfrac{1}{2}(\hat{\chi}\cdot\hat{\underline{\chi}}-\overline{\hat{\chi}\cdot\hat{\underline{\chi}}})-(\rho-\overline{\rho})\right]$$

we have

$$\begin{aligned} -\frac{1}{2}\eta_b(\not\nabla_b\Omega tr\underline{\chi}) &= -\frac{1}{2}\not{d}\mathrm{iv}\,(\Omega\mathrm{tr}\underline{\chi}\eta)+\frac{1}{2}(\not{d}\mathrm{iv}\,\eta)\Omega\mathrm{tr}\underline{\chi} \\ &= -\frac{1}{2}\not{d}\mathrm{iv}\,(\Omega\mathrm{tr}\underline{\chi}\eta)+\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\left[\frac{1}{2}(\hat{\chi}\cdot\hat{\underline{\chi}}-\overline{\hat{\chi}\cdot\hat{\underline{\chi}}})-(\rho-\overline{\rho})\right] \end{aligned} \tag{7.7.68}$$

Moreover from

$$\begin{aligned} \frac{\Omega}{2}\mathbf{D}_3\overline{\rho} &= \frac{1}{2}\frac{d}{du}\left(\frac{1}{|S|}\int_S\rho d\mu_\gamma\right)=-\frac{1}{2}\frac{1}{|S|^2}(\frac{d|S|}{du})\int_S\rho d\mu_\gamma+\frac{1}{2}\frac{1}{|S|}\frac{d}{du}\int_S\rho d\mu_\gamma \\ &= \frac{1}{2}\left\{-\frac{1}{|S|}(\frac{d|S|}{du})\overline{\rho}+\overline{(\Omega tr\underline{\chi})\rho}+\overline{\Omega\mathbf{D}_3\rho}\right\} \end{aligned}$$

we have

$$\frac{\Omega}{2}\mathbf{D}_3(\rho-\overline{\rho})=\frac{1}{2}\left(\Omega\mathbf{D}_3\rho-\overline{\Omega\mathbf{D}_3\rho}\right)-\frac{1}{2}\left(\overline{(\Omega tr\underline{\chi})\rho}-\overline{(\Omega tr\underline{\chi})}\overline{\rho}\right) \tag{7.7.69}$$

Using the Bianchi equation for ρ, see eq. 3.2.8,

$$\mathbf{D}_3\rho=-(\frac{3}{2}tr\underline{\chi}\rho+\frac{1}{2}\hat{\chi}\cdot\underline{\alpha}+\eta\cdot\underline{\beta}+\not\nabla\log\Omega\cdot\underline{\beta})-\not{d}\mathrm{iv}\,\underline{\beta}$$

we obtain

$$-\frac{\Omega}{2}\mathbf{D}_3(\rho-\overline{\rho})=\frac{1}{2}\not{d}\mathrm{iv}\,(\Omega\underline{\beta})+(H-\overline{H}) \tag{7.7.70}$$

where

$$H\equiv\frac{\Omega}{2}(\frac{3}{2}tr\underline{\chi}\rho+\frac{1}{2}\hat{\chi}\cdot\underline{\alpha}+\eta\cdot\underline{\beta}) \tag{7.7.71}$$

In the same way

$$\begin{aligned} \frac{\Omega}{4}\left(\mathbf{D}_3(\hat{\chi}\cdot\hat{\underline{\chi}})-\mathbf{D}_3\overline{(\hat{\chi}\cdot\hat{\underline{\chi}})}\right) &= \frac{1}{4}\left(\Omega\mathbf{D}_3(\hat{\chi}\cdot\hat{\underline{\chi}})-\overline{\Omega\mathbf{D}_3(\hat{\chi}\cdot\hat{\underline{\chi}})}\right) \\ &- \frac{1}{4}\left(\overline{(\Omega tr\underline{\chi})(\hat{\chi}\cdot\hat{\underline{\chi}})}-(\overline{\Omega tr\underline{\chi}})\overline{(\hat{\chi}\cdot\hat{\underline{\chi}})}\right) \end{aligned} \tag{7.7.72}$$

and, finally, collecting all these equations togheter,

$$\triangle\!\!\!/(\Omega\mathbf{D}_3\log\Omega) = \mathrm{d}\!\!/\mathrm{iv}\left(\Omega\underline{\beta}+\frac{3}{2}\Omega\eta\cdot\underline{\hat{\chi}}+\frac{1}{4}\Omega\eta\mathrm{tr}\underline{\chi}\right)+G_1-\overline{G_1} \quad (7.7.73)$$

where [48]

$$\begin{aligned} G_1 &= H+\frac{1}{4}\Omega\mathbf{D}_3(\hat{\chi}\cdot\underline{\hat{\chi}})-\frac{1}{2}(\Omega tr\underline{\chi})(\rho-\overline{\rho}) \\ &+ \frac{1}{4}(\Omega tr\underline{\chi})(\hat{\chi}\cdot\underline{\hat{\chi}}-\overline{\hat{\chi}\cdot\underline{\hat{\chi}}}) \end{aligned} \quad (7.7.74)$$

Denoting

$$\mathrm{d}\!\!/\mathrm{iv}\,F_1 \equiv \mathrm{d}\!\!/\mathrm{iv}\left(\Omega\underline{\beta}+\frac{3}{2}\Omega\eta\cdot\underline{\hat{\chi}}+\frac{1}{4}\Omega\eta tr\underline{\chi}\right) \quad (7.7.75)$$

the final equation is

$$\triangle\!\!\!/(\Omega\mathbf{D}_3\log\Omega) = \mathrm{d}\!\!/\mathrm{iv}\,F_1+G_1-\overline{G_1} \quad (7.7.76)$$

Proof of equation 7.7.40 of Lemma 7.7.4

From

$$\begin{aligned} \triangle\!\!\!/(\Omega\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)) &= \Omega(\triangle\!\!\!/\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega))+(\triangle\!\!\!/\Omega)\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega) \\ &+ 2\Omega(\nabla\!\!\!/\log\Omega)\cdot\nabla\!\!\!/(\mathbf{D}_3\Omega\mathbf{D}_3\log\Omega) \end{aligned} \quad (7.7.77)$$

it follows

$$\begin{aligned} \frac{1}{\Omega}\triangle\!\!\!/(\Omega\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)) &= \triangle\!\!\!/\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)+\left[\frac{\triangle\!\!\!/\Omega}{\Omega}\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)\right. \\ &\left.+\,2(\nabla\!\!\!/\log\Omega)\nabla\!\!\!/\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)\right] \end{aligned} \quad (7.7.78)$$

From the previous lemma, see 7.7.63, we have

$$\triangle\!\!\!/\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega) = \mathbf{D}_3\triangle\!\!\!/(\Omega\mathbf{D}_3\log\Omega)-[\cdot]_{(\Omega D_3\log\Omega)}-\{\cdot\}_{(\Omega D_3\log\Omega)} \quad (7.7.79)$$

where $\{\cdot\}_{(\Omega D_3\log\Omega)}$ and $[\cdot]_{(\Omega D_3\log\Omega)}$ have the same expressions as in 7.7.63 with $\log\Omega$ substituted with $\Omega\mathbf{D}_3\log\Omega$,

$$\begin{aligned} \{\cdot\}_{(\Omega D_3\log\Omega)} &= \Big\{-\eta_a\underline{\chi}_{ab}\nabla\!\!\!/_b(\Omega\mathbf{D}_3\log\Omega)+tr\underline{\chi}\eta_b\nabla\!\!\!/_b(\Omega\mathbf{D}_3\log\Omega) \\ &-2\underline{\chi}_{ab}(\nabla\!\!\!/\nabla\!\!\!/(\Omega\mathbf{D}_3\log\Omega))_{ab}-\zeta_a\underline{\chi}_{ab}\nabla\!\!\!/_b(\Omega\mathbf{D}_3\log\Omega) \\ &-(\mathrm{d}\!\!/\mathrm{iv}\,\underline{\chi})_b\nabla\!\!\!/_b(\Omega\mathbf{D}_3\log\Omega)-\underline{\beta}_b\nabla\!\!\!/_b(\Omega\mathbf{D}_3\log\Omega)\Big\} \end{aligned} \quad (7.7.80)$$

[48] Looking at the expression of H it seems that G_1 depends on $\underline{\alpha}$, but in fact the $\underline{\alpha}$ present in H is cancelled from the $\underline{\alpha}$ with the opposite sign appearing in $\frac{1}{4}\Omega\mathbf{D}_3(\hat{\chi}\cdot\underline{\hat{\chi}})$.

and

$$
\begin{aligned}
[\cdot]_{(\Omega D_3 \log\Omega)} &= [(\not\nabla_a \log\Omega)(\mathbf{D}_3 \not\nabla_a(\Omega\mathbf{D}_3\log\Omega)) - \zeta_a(\mathbf{D}_3\not\nabla_a(\Omega\mathbf{D}_3\log\Omega)) \\
&+ \eta_a \not\nabla_a \mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega) + (\not\Delta\log\Omega)\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega) \\
&+ \zeta_a(\not\nabla_a\log\Omega)\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)] \qquad (7.7.81)
\end{aligned}
$$

Observe that a simple use of commutation relations allows to rewrite $[\cdot]_{(\Omega D_3\log\Omega)}$ in the following way, which will be subsequently used,

$$
\begin{aligned}
[\cdot]_{(\Omega D_3 \log\Omega)} &= \Big[2(\not\nabla_a\log\Omega)(\not\nabla_a\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)) + \frac{\not\Delta\Omega}{\Omega}\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega) \\
&- \underline{\chi}_{ab}(\not\nabla_a\log\Omega)\not\nabla_b(\Omega\mathbf{D}_3\log\Omega) + \zeta_a\underline{\chi}_{ab}\not\nabla_b(\Omega\mathbf{D}_3\log\Omega)\Big] \qquad (7.7.82)
\end{aligned}
$$

Substituting the expression for $\not\Delta\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)$, see 7.7.79, in 7.7.78 we obtain

$$
\begin{aligned}
&\frac{1}{\Omega}\not\Delta(\Omega\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)) = \mathbf{D}_3\not\Delta(\Omega\mathbf{D}_3\log\Omega) + \Big[\frac{\not\Delta\Omega}{\Omega}\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega) \\
&+ 2(\not\nabla_a\log\Omega)\not\nabla_a\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega) - [\cdot]_{(\Omega D_3\log\Omega)}\Big] - \{\cdot\}_{(\Omega D_3\log\Omega)} (7.7.83)
\end{aligned}
$$

A long but easy computation exactly on the same lines as in the previous case allows to rewrite the last equation as

$$
\begin{aligned}
\not\Delta(\Omega\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)) &= \Omega\mathbf{D}_3\not\Delta(\Omega\mathbf{D}_3\log\Omega) + \Omega\Big[2\underline{\chi}_{ab}(\not\nabla\not\nabla(\Omega\mathbf{D}_3\log\Omega))_{ab} \\
&+ (\not\nabla_b\log\Omega)\not\nabla_a(\Omega\mathbf{D}_3\log\Omega)\underline{\chi}_{ab} + (\not\nabla_b(\Omega\mathbf{D}_3\log\Omega))(\not{d}\mathrm{iv}\,\underline{\chi})_b \\
&+ (\not\nabla_b(\Omega\mathbf{D}_3\log\Omega))\left(\eta_a\underline{\chi}_{ab} - \eta_b tr\underline{\chi} + \underline{\beta}_b\right)\Big] \\
&\equiv \Omega\mathbf{D}_3\not\Delta(\Omega\mathbf{D}_3\log\Omega) + \Omega[IV]_{(\Omega D_3\log\Omega)} \qquad (7.7.84)
\end{aligned}
$$

where [49]

$$
\begin{aligned}
&\Omega[IV]_{(\Omega D_3\log\Omega)} = \Omega\Big[2\underline{\chi}_{ab}(\not\nabla\not\nabla(\Omega\mathbf{D}_3\log\Omega))_{ab} + (\not\nabla_b\log\Omega)\not\nabla_a(\Omega\mathbf{D}_3\log\Omega)\underline{\chi}_{ab} \\
&\quad + (\not\nabla_b(\Omega\mathbf{D}_3\log\Omega))(\not{d}\mathrm{iv}\,\underline{\chi})_b + (\not\nabla_b(\Omega\mathbf{D}_3\log\Omega))\left(\eta_a\underline{\chi}_{ab} - \eta_b tr\underline{\chi} + \underline{\beta}_b\right)\Big] \\
&= \Omega\Big[2\underline{\chi}_{ab}(\not\nabla\not\nabla(\Omega\mathbf{D}_3\log\Omega))_{ab} + 2(\not\nabla_b\log\Omega)\not\nabla_a(\Omega\mathbf{D}_3\log\Omega)\underline{\chi}_{ab} \\
&\quad + 2(\not\nabla_b(\Omega\mathbf{D}_3\log\Omega))(\not{d}\mathrm{iv}\,\underline{\chi})_b - (\not\nabla_b(\Omega\mathbf{D}_3\log\Omega))\left((\not\nabla_b\log\Omega)tr\underline{\chi} + \not\nabla_b tr\underline{\chi}\right)\Big] \\
&= 2\not{d}\mathrm{iv}\,(\Omega(\not\nabla(\Omega\mathbf{D}_3\log\Omega))\cdot\underline{\chi}) - (\not\nabla_b(\Omega\mathbf{D}_3\log\Omega))(\not\nabla_b\Omega tr\underline{\chi}) \qquad (7.7.85)
\end{aligned}
$$

[49]We used the structure equation

$$
\eta_a\underline{\chi}_{ab} - \eta_b tr\underline{\chi} + \underline{\beta}_b = (\not\nabla_a\log\Omega)\underline{\chi}_{ab} - (\not\nabla_b\log\Omega)tr\underline{\chi} + (\not{d}\mathrm{iv}\,\underline{\chi})_b - \not\nabla_b tr\underline{\chi}\ .
$$

Finally we obtain

$$
\begin{aligned}
\triangle\!\!\!/(\Omega\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)) &= \Omega\mathbf{D}_3\left[\mathrm{d}\!\!/\mathrm{iv}\,\Omega\underline{\beta}+\mathrm{d}\!\!/\mathrm{iv}\,\tilde{F}_1+(G_1-\overline{G_1})\right]\\
&+ \mathrm{d}\!\!/\mathrm{iv}\,(2\Omega(\nabla\!\!\!\!/\,(\Omega\mathbf{D}_3\log\Omega))\cdot\underline{\chi})-(\nabla\!\!\!\!/_b(\Omega\mathbf{D}_3\log\Omega))(\nabla\!\!\!\!/_b\Omega tr\underline{\chi})\\
&= \Omega\mathbf{D}_3\left[\mathrm{d}\!\!/\mathrm{iv}\,F_1+(G_1-\overline{G_1})\right]\\
&+ \mathrm{d}\!\!/\mathrm{iv}\,(2\Omega(\nabla\!\!\!\!/\,(\Omega\mathbf{D}_3\log\Omega))\cdot\underline{\chi})-(\nabla\!\!\!\!/_b(\Omega\mathbf{D}_3\log\Omega))(\nabla\!\!\!\!/_b\Omega tr\underline{\chi})
\end{aligned}
\tag{7.7.86}
$$

where $\mathrm{d}\!\!/\mathrm{iv}\,F_1\equiv\mathrm{d}\!\!/\mathrm{iv}\,\Omega\underline{\beta}+\mathrm{d}\!\!/\mathrm{iv}\,\tilde{F}_1$ and $\tilde{F}_1=\left(\frac{3}{2}\Omega\eta\cdot\hat{\underline{\chi}}+\frac{1}{4}\Omega\eta tr\underline{\chi}\right)$. Repeating the previous computation, we have

$$
\begin{aligned}
\mathbf{D}_3\mathrm{d}\!\!/\mathrm{iv}\,F_1 &= \mathrm{d}\!\!/\mathrm{iv}\,(\mathbf{D}_3F_1)+(\nabla\!\!\!\!/_a\log\Omega)(\mathbf{D}_3F_1)_a+(-\eta_a\underline{\chi}_{ab}+\eta_b\mathrm{tr}\underline{\chi}-\underline{\beta}_b)F_{1b}-\underline{\chi}_{ab}(\nabla\!\!\!\!/_bF_1)_a\\
&= \frac{1}{\Omega}\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\mathbf{D}_3F_1)+\left[(-\eta_a\hat{\underline{\chi}}_{ab}+\frac{1}{2}\eta_b\mathrm{tr}\underline{\chi}-\underline{\beta}_b)F_{1b}-\underline{\chi}_{ab}(\nabla\!\!\!\!/_bF_1)_a\right]
\end{aligned}
$$

and using again the previous structure equation we obtain

$$
\Omega\mathbf{D}_3\mathrm{d}\!\!/\mathrm{iv}\,F_1=\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\mathbf{D}_3F_1-\Omega\underline{\chi}\cdot F_1)+(\nabla\!\!\!\!/_b\Omega tr\underline{\chi})F_{1b}
\tag{7.7.87}
$$

Inserting this expression in the equation for $\triangle\!\!\!/(\Omega\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega))$, see 7.7.86, it is easy to conclude that

$$
\begin{aligned}
\triangle\!\!\!/(\Omega\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)) &= \mathrm{d}\!\!/\mathrm{iv}\,(2\Omega\nabla\!\!\!\!/\,(\Omega\mathbf{D}_3\log\Omega)\cdot\underline{\chi})+\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\mathbf{D}_3F_1-\Omega\underline{\chi}\cdot F_1)\\
&+ (F_{1b}-\nabla\!\!\!\!/_b(\Omega\mathbf{D}_3\log\Omega))\,(\nabla\!\!\!\!/_b\Omega\mathrm{tr}\underline{\chi})+\Omega\mathbf{D}_3(G_1-\overline{G_1})
\end{aligned}
\tag{7.7.88}
$$

Observing that

$$
\begin{aligned}
&(F_{1b}-\nabla\!\!\!\!/_b(\Omega\mathbf{D}_3\log\Omega))\,(\nabla\!\!\!\!/_b\Omega tr\underline{\chi})=\\
&=\mathrm{d}\!\!/\mathrm{iv}\,\Big((F_{1b}-\nabla\!\!\!\!/_b(\Omega\mathbf{D}_3\log\Omega))(\Omega tr\underline{\chi})\Big)-\Omega\mathrm{tr}\underline{\chi}\,(\mathrm{d}\!\!/\mathrm{iv}\,F_1-\triangle\!\!\!/(\Omega\mathbf{D}_3\log\Omega))\\
&=\mathrm{d}\!\!/\mathrm{iv}\,\Big((F_{1b}-\nabla\!\!\!\!/_b(\Omega\mathbf{D}_3\log\Omega))(\Omega tr\underline{\chi})\Big)+(\Omega tr\underline{\chi})(G_1-\overline{G_1})
\end{aligned}
$$

the final result is

$$
\begin{aligned}
\triangle\!\!\!/(\Omega\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)) &= \mathrm{d}\!\!/\mathrm{iv}\,\Big[(\Omega\mathbf{D}_3F_1-\Omega\underline{\chi}\cdot F_1)+2\Omega(\nabla\!\!\!\!/\,(\Omega\mathbf{D}_3\log\Omega))\cdot\underline{\chi}\\
&\quad+F_1\Omega\mathrm{tr}\underline{\chi}-(\nabla\!\!\!\!/\,(\Omega\mathbf{D}_3\log\Omega))\Omega\mathrm{tr}\underline{\chi}\Big]\\
&\quad+\Big\{\Omega\mathbf{D}_3(G_1-\overline{G_1})+(\Omega tr\underline{\chi})(G_1-\overline{G_1})\Big\}\\
&\equiv \mathrm{d}\!\!/\mathrm{iv}\,F_2+G_2-\overline{G_2}
\end{aligned}
\tag{7.7.89}
$$

where

$$F_2 = \Omega\mathbf{D}_3 F_1 + 2\Omega(\nabla\!\!\!/(\Omega\mathbf{D}_3\log\Omega))\cdot\hat{\underline{\chi}} - \Omega\hat{\underline{\chi}}\cdot F_1 + \frac{1}{2}\Omega tr\underline{\chi}F_1$$
$$G_2 = \Omega\mathbf{D}_3 G_1 + (\Omega tr\underline{\chi})(G_1 - \overline{G_1})$$

Therefore the final equation has the same structure of the one for $\Omega\mathbf{D}_3\log\Omega$:

$$\triangle\!\!\!/(\Omega\mathbf{D}_3(\Omega\mathbf{D}_3\log\Omega)) = d\!\!\!/iv\, F_2 + G_2 - \overline{G_2} \tag{7.7.90}$$

and obviously $\overline{d\!\!\!/iv\, F_2 + G_2 - \overline{G_2}} = 0$.

Proof of equation 7.7.41 of Lemma 7.7.4

Proceeding as in the previous case we have the following expressions

$$\begin{aligned}\triangle\!\!\!/((\Omega\mathbf{D}_3)^3\log\Omega)) &= \Omega(\triangle\!\!\!/\mathbf{D}_3[(\Omega\mathbf{D}_3)^2\log\Omega] + (\triangle\!\!\!/\Omega)\mathbf{D}_3[(\Omega\mathbf{D}_3)^2\log\Omega] \\ &+ 2\Omega(\nabla\!\!\!/\log\Omega)\cdot\nabla\!\!\!/\mathbf{D}_3[(\Omega\mathbf{D}_3)^2\log\Omega]\end{aligned} \tag{7.7.91}$$

As before

$$\begin{aligned}\triangle\!\!\!/\mathbf{D}_3[(\Omega\mathbf{D}_3)^2\log\Omega] &= \mathbf{D}_3\triangle\!\!\!/[(\Omega\mathbf{D}_3)^2\log\Omega] - [\cdot]_{[(\Omega D_3)^2\log\Omega]} \\ &-\{\cdot\}_{[(\Omega D_3)^2\log\Omega]}\end{aligned} \tag{7.7.92}$$

where, see 7.7.82,

$$\begin{aligned}[\cdot]_{[(\Omega D_3)^2\log\Omega]} &= \Big[2(\nabla\!\!\!/\log\Omega)\nabla\!\!\!/\mathbf{D}_3[(\Omega\mathbf{D}_3)^2\log\Omega] + (\frac{\triangle\!\!\!/\Omega}{\Omega})\mathbf{D}_3[(\Omega\mathbf{D}_3)^2\log\Omega] \\ &- \underline{\chi}_{ab}(\nabla\!\!\!/_a\log\Omega)\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega] + \zeta_a\underline{\chi}_{ab}\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega]\Big]\end{aligned}$$

and

$$\begin{aligned}\{\cdot\}_{[(\Omega D_3)^2\log\Omega]} &= \Big\{-\eta_a\underline{\chi}_{ab}\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega] + tr\underline{\chi}\eta_b\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega] \\ &- 2\underline{\chi}_{ab}(\nabla\!\!\!/\nabla\!\!\!/[(\Omega\mathbf{D}_3)^2\log\Omega])_{ab} - \zeta_a\underline{\chi}_{ab}\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega] \\ &- (d\!\!\!/iv\,\underline{\chi})_b\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega] - bb_b\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega]\Big\}\end{aligned}$$

Substituting the expression for $\triangle\!\!\!/\mathbf{D}_3[(\Omega\mathbf{D}_3)^2\log\Omega]$ in 7.7.91 we obtain

$$\begin{aligned}\frac{1}{\Omega}\triangle\!\!\!/((\Omega\mathbf{D}_3)^3\log\Omega) &= \mathbf{D}_3\triangle\!\!\!/[(\Omega\mathbf{D}_3)^2\log\Omega] + \Big[\underline{\chi}_{ab}(\nabla\!\!\!/_a\log\Omega)\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega] \\ &- \zeta_a\underline{\chi}_{ab}\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega]\Big] - \{\cdot\}_{[(\Omega D_3)^2\log\Omega]}\end{aligned} \tag{7.7.93}$$

A long but easy computation on the same lines as for the previous case allows to rewrite the equation as

$$\begin{aligned}\not\triangle((\Omega\mathbf{D}_3)^3\log\Omega)) &= \Omega\mathbf{D}_3\not\triangle[(\Omega\mathbf{D}_3)^2\log\Omega] + \Omega\Big[\underline{\chi}_{ab}(\nabla\!\!\!/_a\log\Omega)\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega]\\ &\quad + 2\underline{\chi}_{ab}(\nabla\!\!\!/\nabla\!\!\!/[(\Omega\mathbf{D}_3)^2\log\Omega])_{ab} + (\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\chi})_b(\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega])\\ &\quad + (\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega])\left(\eta_a\underline{\chi}_{ab} - \eta_b tr\underline{\chi} + \underline{\beta}_b\right)\Big]\\ &\equiv \Omega\mathbf{D}_3\not\triangle[(\Omega\mathbf{D}_3)^2\log\Omega] + \Omega[IV]_{[(\Omega\mathbf{D}_3)^2\log\Omega]} \qquad (7.7.94)\end{aligned}$$

After some easy, but long computations, where we used again the structure equation: $\eta_a\underline{\chi}_{ab} - \eta_b tr\underline{\chi} + \underline{\beta}_b = (\nabla\!\!\!/_a\log\Omega)\underline{\chi}_{ab} - (\nabla\!\!\!/_b\log\Omega)tr\underline{\chi} + (\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\chi})_b - \nabla\!\!\!/_b tr\underline{\chi}$, we obtain, see 7.7.85,

$$\Omega[IV]_{[(\Omega\mathbf{D}_3)^2\log\Omega]} = \mathrm{d}\!\!\!/\mathrm{iv}\,(2\Omega[(\Omega\mathbf{D}_3)^2\log\Omega])\cdot\underline{\chi}) - (\nabla\!\!\!/_b[(\Omega\mathbf{D}_3)^2\log\Omega])\nabla\!\!\!/_b(\Omega tr\underline{\chi})$$

and finally, repeating the previous steps, we conclude

$$\not\triangle((\Omega\mathbf{D}_3)^3\log\Omega)) = \mathrm{d}\!\!\!/\mathrm{iv}\,F_3 + G_3 - \overline{G_3} \qquad (7.7.95)$$

where

$$\begin{aligned}F_3 &= \Omega\mathbf{D}_3F_2 + 2\Omega(\nabla\!\!\!/[(\Omega\mathbf{D}_3)^2\log\Omega])\cdot\underline{\hat{\chi}} - \Omega\underline{\hat{\chi}}\cdot F_2 + \frac{1}{2}\Omega tr\underline{\chi}F_2\\ G_3 &= \Omega\mathbf{D}_3G_2 + (\Omega tr\underline{\chi})(G_2 - \overline{G_2}) \qquad (7.7.96)\end{aligned}$$

and obviously $\overline{\mathrm{d}\!\!\!/\mathrm{iv}\,F_3 + G_3 - \overline{G_3}} = 0$.

Chapter 8

Conclusions

In this chapter we derive the most important consequences of our *Main Theorem*, in particular we give a rigorous derivation of the Bondi mass law and the precise asymptotic formula for the optical function u expressed in terms of t and r. Due to the construction of our spacetime based on the double null foliation which allows us, in particular, to give a straightforward definition of the outgoing null infinity, the derivation of these results is simpler and more intuitive than the one in [Ch-Kl]. In addition our approach allows us to give a simple derivation of the connection between the Bondi mass and the ADM mass.
Before embarking on the main topic of this chapter it helps to summarize some of the main relevant features of our proof of the *Main Theorem*.

We recall that our spacetime has been constructed together with a double null foliation generated by the level hypersurfaces, $C(\lambda)$, $\underline{C}(\nu)$ of the optical functions $u, \underline{u}$.
Associated to these null hypersurfaces we have the two dimensional surfaces $S(\lambda,\nu)=C(\lambda)\cap\underline{C}(\nu)$, the adapted null frames, see definition 3.1.13, and the connection coefficients $\chi, \underline{\chi}$, $\eta, \underline{\eta}$,$\omega, \underline{\omega}$ which satisfy the structure equations 3.1.45, 3.1.46 and 3.1.47.
We have also decomposed the curvature tensor, relative to the adapted null frames, into its null components α, β, ρ, σ, $\underline{\beta}, \underline{\alpha}$, see 3.1.19. The boundedness of the $\mathcal{R}$ norms implies in particular the uniform decay of these components,

$$\begin{aligned}
&\sup_{\mathcal{K}} r^{7/2}|\alpha| \leq C_0 \ , \ \sup_{\mathcal{K}} r|\tau_-|^{\frac{5}{2}}|\underline{\alpha}| \leq C_0 \\
&\sup_{\mathcal{K}} r^{7/2}|\beta| \leq C_0 \ , \ \sup_{\mathcal{K}} r^2|\tau_-|^{\frac{3}{2}}|\underline{\beta}| \leq C_0 \qquad (8.0.1)\\
&\sup_{\mathcal{K}} r^3|\rho| \leq C_0 \ , \ \sup_{\mathcal{K}} r^3|\tau_-|^{\frac{1}{2}}|(\rho-\overline{\rho},\sigma)| \leq C_0
\end{aligned}$$

see 3.7.1. Observe that in Theorem 8.5.2 of section 8.5, we show that the $\overline{\rho}$ component, whose decay is $O(r^{-3})$, is intimately tied to the ADM mass. In fact the following relation is proved

$$\overline{\rho} = -2M\frac{1}{r^3} + O\left(\frac{1}{r^3\lambda^2}\right) .$$

The boundedness of the $\mathcal{O}$ norms implies pointwise estimates for the connection coefficients and their first derivatives. Based on the properties of the canonical foliation we were in fact able to prove somewhat stronger estimates for the connection coefficients in Chapters 4. These stronger estimates were not relevant [1] in the proof of the *Main Theorem*, but play a fundamental role in this chapter. We repeat here those estimates which will be used in the sequel and we refer to Chapter 4, see in particular subsection 4.3.16, for a more extended discussion,

$$\begin{aligned}
&|r^{2-2/p}\tau_-^{\frac{1}{2}}\hat{\chi}|_{p,S} \leq C_0 \quad , \quad |r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!\!/\,\hat{\chi}|_{p,S} \leq C_0\\
&|r^{1-2/p}\tau_-^{\frac{3}{2}}\underline{\hat{\chi}}|_{p,S} \leq C_0 \quad , \quad |r^{2-2/p}\tau_-^{\frac{3}{2}}\nabla\!\!\!\!/\,\underline{\hat{\chi}}|_{p,S} \leq C_0\\
&|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\left(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}\right)|_{p,S} \leq C_0 \ , \ |r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}\left(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}}\right)|_{p,S} \leq C_0\\
&|r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!\!/\,\mathrm{tr}\chi|_{p,S} \leq C_0 \quad , \quad |r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!\!/\,\mathrm{tr}\underline{\chi}|_{p,S} \leq C_0\\
&|r^{2-2/p}\tau_-^{\frac{1}{2}}\eta|_{p,S} \leq C_0 \quad , \quad |r^{2-2/p}\tau_-^{\frac{1}{2}}\underline{\eta}|_{p,S} \leq C_0 \qquad\qquad (8.0.2)\\
&|r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!\!/\,\eta|_{p,S} \leq C_0 \quad , \quad |r^{3-2/p}\tau_-^{\frac{1}{2}}\nabla\!\!\!\!/\,\underline{\eta}|_{p,S} \leq C_0\\
&|r^{2-\frac{2}{p}}\tau_-^{\frac{3}{2}}(\Omega\nabla\!\!\!\!/\,\mathbf{D}_3\log\Omega)|_{p,S} \leq C_0 \ , \ |r^{3-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\Omega\nabla\!\!\!\!/\,\mathbf{D}_4\log\Omega)|_{p,S} \leq C_0\\
&|r^{1-\frac{2}{p}}\tau_-^{\frac{3}{2}}(\Omega\mathbf{D}_3\log\Omega - \overline{\Omega\mathbf{D}_3\log\Omega})|_{p,S} \leq C_0\\
&|r^{2-\frac{2}{p}}\tau_-^{\frac{1}{2}}(\Omega\mathbf{D}_4\log\Omega - \overline{\Omega\mathbf{D}_4\log\Omega})|_{p,S} \leq C_0
\end{aligned}$$

with C_0 a constant depending on the initial data.

Observe that the terms ω, $\underline{\omega}$, $\mathbf{D}_4\omega$, $\mathbf{D}_3\underline{\omega}$, $(\overline{\Omega\mathrm{tr}\chi} - \frac{1}{r})$ and $(\overline{\Omega\mathrm{tr}\underline{\chi}} + \frac{1}{r})$ do not have the $\tau_-^{\frac{1}{2}}$ improvement manifest in all the other terms. This is due to their relation (in the structure equations) to the ρ component of the curvature tensor, more precisely to its $\overline{\rho}$ part, which is tied to the ADM mass as explained above.

[1] But, of course, the fact that the double null foliation is canonical was.

Observe that these connection coefficients are the only non trivial ones in a Schwarzschild spacetime. From this perspective our main result can be interpreted as a stability of the external region of the Schwarzschild spacetime. One of the results proved in this chapter further justifies this conclusion. We show in Proposition 8.6.1 that on any null outgoing hypersurface $C(\lambda)$ we have

$$\frac{dr}{dt} = -2M\frac{1}{r} + O\left(\frac{1}{r^2}\right) \tag{8.0.3}$$

where t is the time function $t(p) = \frac{1}{2}(\underline{u}(p) + u(p))$ introduced [2] in Chapter 3, see Proposition 3.3.1 . This proves that the null outgoing hypersurfaces converge asymptotically to the null outgoing Schwarzschild cones.

As mentioned above our double null foliation approach allows us to define the outgoing null infinity by simply taking the limit of the incoming null hypersurfaces $\underline{C}(\nu)$ as $\nu \to \infty$. This approach not only simplifies significantly the derivation of the main conclusions in [Ch-Kl], but also allows us to connect the outgoing null infinity $\mathcal{J}^+$ to the spacelike infinity i_0. In particular we are able to connect [3] the Bondi mass to the ADM mass. In fact defining the Bondi mass as... **put the definition** we have

$$\lim_{\lambda \to -\infty} M_B(\lambda) = M_{ADM}$$

8.1 The spacetime null infinity

8.1.1 The existence of a global optical function

The *Main Theorem* provides us with a family of optical functions ${}^{(\nu_*)}u(p)$ which are outgoing solutions of the eikonal equation with initial data on $\underline{C}(\nu_*)$. The following corollary allows us to conclude the existence of a global optical function u with initial data at "null infinity".

[2] Remark that, although we do not use the maximal spacelike foliation, nevertheless we have a spacelike foliation at our disposal and, therefore, a global time function. Both are in fact provided from Proposition 3.3.1 and, in this case, the spacelike hypersurfaces are such that each surface $S(\lambda,\nu)$ is immersed in the hypersurface $\tilde{\Sigma}_t = \{p \in \mathcal{M} | t(p) = t\}$.

[3] Analogously, using these null incoming hypersurfaces, the derivation of the asymptotic rotational symmetry of the spacetime is more straightforward than in [Ch-Kl]. In fact, as described in subsection 3.4.1, the angular momentum vector fields on $\mathcal{M}$ are defined starting from the last slice null hypersurface $\underline{C}(\nu_*)$. On the other hand the angular momentum vector fields on the last slice are defined starting from $\underline{C}(\nu_*) \cap \Sigma_0$; so that the connection between the limits is easily established as $\nu_* \to \infty$.

Corollary 8.1.1 *Under all the assumptions of the "Main Theorem" the following limit holds in $\mathcal{M}$,*

$$u(p)=\lim_{\nu_*\to\infty}{}^{(\nu_*)}u(p) \tag{8.1.1}$$

Proof: We shall show that $\{{}^{(\nu_*)}u(p)\}$ form a Cauchy sequence. We first prove that for $\tilde{\epsilon}$ arbitrary small, it is possible to choose $\overline{\nu}_*$ such that, given $\nu_{*,2}>\nu_{*,1}>\overline{\nu}_*$, we have $|{}^{(\nu_{*,2})}u-{}^{(\nu_{*,1})}u|_{\underline{C}_*(\nu_{*,1})}\leq\tilde{\epsilon}$ or, in view of our definition ${}^{(\nu_{*,1})}u(p)|_{\underline{C}_*(\nu_{*,1})}={}^{(\nu_{*,1})}u_*(p)$,

$$\sup_{p\in\underline{C}_*(\nu_{*,1})}\left|{}^{(\nu_{*,2})}u(p)-{}^{(\nu_{*,1})}u_*(p)\right|\leq\tilde{\epsilon} \tag{8.1.2}$$

where ${}^{(\nu_{*,1})}u_*(p)$ is the solution of the last slice problem on $\underline{C}_*(\nu_{*,1})$, see **??**. Once 8.1.2 is proved we can conclude that in any spacetime region $\mathcal{K}(\lambda_0,\nu_1)\subset\mathcal{M}$ the difference between ${}^{(\nu_{*,2})}u(p)$ and ${}^{(\nu_{*,1})}u(p)$ tends to zero as $\nu_{*,1}\to\infty$. This implies the convergence of ${}^{(\nu_*)}u(p)$ to $u(p)$ proving the result.

To prove 8.1.2 we choose $\overline{\nu}_*$ sufficiently large such that the initial data norm **write the explicit expresssion.** outside $\underline{C}_*(\nu_{*,1})\cap\Sigma_0$ is of order $O(\tilde{\epsilon})$. The restriction of ${}^{(\nu_{*,2})}u$ induces a foliation on $\underline{C}_*(\nu_{*,1})$. Though strictly speaking this is not a background foliation [4] we can nevertheless deform it, in the neighbourhood of $\underline{C}_*(\nu_{*,1})\cap\Sigma_0$, generating only errors of order $O(\tilde{\epsilon})$, such that it becomes one. More precisely we can assume that ${}^{(\nu_{*,2})}u|_{\underline{C}_*(\nu_{*,1})}=v$ induces a background foliation on $\underline{C}_*(\nu_{*,1})$ verifying the assumptions of Theorem 3.3.2 . As ${}^{(\nu_{*,1})}u_*$ is the canonical foliation of $\underline{C}_*(\nu_{*,1})$ we infer from 3.3.12 that

$$\frac{d\,{}^{(\nu_{*,1})}u_*}{dv}(p)=(4({}^{(\nu_{*,1})}\Omega)^2)^{-1} \tag{8.1.3}$$

where ${}^{(\nu_{*,1})}\Omega$ satisfies the following elliptic equation

$$\begin{aligned}\not{\Delta}\log 2\,{}^{(\nu_{*,1})}\Omega&=\frac{1}{2}\not{d}\text{iv}\,{}^{(\nu_{*,1})}\underline{\eta}+\left[\frac{1}{2}{}^{(\nu_{*,1})}\hat{\chi}\,{}^{(\nu_{*,1})}\underline{\hat{\chi}}-\overline{{}^{(\nu_{*,1})}\hat{\chi}\,{}^{(\nu_{*,1})}\underline{\hat{\chi}}})-({}^{(\nu_{*,1})}\rho-\overline{{}^{(\nu_{*,1})}\rho})\right]\\ \overline{\log 2\,{}^{(\nu_{*,1})}\Omega}&=0\end{aligned} \tag{8.1.4}$$

[4] This requires some extra work. Between the leaves defined on $\underline{C}_*(\nu_{*,1})$ by the restriction of ${}^{(\nu_{*,2})}u(p)$ there exists one which is "near" of order $\tilde{\epsilon}$ to $\underline{C}_*(\nu_{*,1})\cap\Sigma_0$. The proof of this requires the use of the *Oscillation Lemma*, see**??** and has to take into account that $\overline{\nu}_*$ is sufficiently large.

Integrating equation 8.1.3 along $\underline{C}_*(\nu_{*,1})$ we obtain, apart from higher order corrections,

$$ {}^{(\nu_{*,1})}u_*(p) - {}^{(\nu_{*,2})}u(p)|_{\underline{C}_*(\nu_{*,1})} = \int_{\lambda_1}^{\lambda(p)} \left(\frac{1}{4({}^{(\nu_{*,1})}\Omega)^2} - 1 \right) \tag{8.1.5} $$

and from it we infer that

$$ \left| {}^{(\nu_{*,1})}u_*(p) - {}^{(\nu_{*,2})}u(p)|_{\underline{C}_*(\nu_{*,1})} \right| \le 4 \int_0^{\lambda(p)} \left| \log({}^{(\nu_{*,1})}\Omega) \right| \tag{8.1.6} $$

In view of the elliptic equation 8.1.4 satisfied by $\log({}^{(\nu_{*,1})}\Omega)$, it follows immediately, using the estimates for the connection coefficients on the last slice proved in Chapter 7, **put the references of this chapter** that

$$ \begin{aligned} |\log({}^{(\nu_{*,1})}\Omega)| &\le r^2 \left(|\mathrm{d}\!\!/\mathrm{iv}\, {}^{(\nu_{*,1})}\underline{\eta}| + |{}^{(\nu_{*,1})}\hat{\chi}||{}^{(\nu_{*,1})}\underline{\hat{\chi}}| + |({}^{(\nu_{*,1})}\rho - \overline{{}^{(\nu_{*,1})}\rho})| \right) \\ &= O\left(\frac{1}{r\lambda^{\frac{1}{2}}} \right) \end{aligned} \tag{8.1.7} $$

Therefore, integrating the right hand side of 8.1.6 we obtain,

$$ \left| {}^{(\nu_{*,1})}u_*(p) - {}^{(\nu_{*,2})}u(p)|_{\underline{C}_*(\nu_{*,1})} \right| \le c\frac{1}{r^{\frac{1}{2}}} \le \tilde{\epsilon} \tag{8.1.8} $$

as $r = r(\lambda, \overline{\nu_*})$ can be made sufficiently large provided we choose $\overline{\nu_*}$ appropriately large.

8.1.2 The outgoing null infinity limit $\mathcal{J}^+$

To define the outgoing null infinity limit $\mathcal{J}^+$, we start by defining, in the global spacetime $\mathcal{M}$ constructed by the *Main Theorem*, a family of diffeomorphisms $\psi(\lambda, \nu)$ such that

$$ \psi(\lambda, \nu) : S^2 \to S(\lambda, \nu) \tag{8.1.9} $$

These diffeomorphisms are associated, in a way we will make precise, to the diffeomorphisms ϕ_ν and $\underline{\phi}_\lambda$ generated by the null equivariant vector field $N, \underline{N}$ introduced in subsection 3.1.4, Lemma 3.1.1. We will construct the diffeomorphisms $\psi(\lambda, \nu)$ as follows:

According to the results of the previous subsection we already have the two optical functions $u(p) = \lim_{\nu_* \to \infty} {}^{(\nu_*)}u(p)$ and $\underline{u}(p)$ defining a double null canonical foliation in $\mathcal{M}$. Associated to it we define the function Ω, see subsection 3.1.4,

$$2\Omega^2 = -(g^{\rho\sigma}\partial_\rho u \partial_\sigma \underline{u})^{-1} \tag{8.1.10}$$

and the null geodesic vector fields,

$$L^\rho \equiv -g^{\rho\mu}\partial_\mu u \ \text{ and } \ \underline{L}^\rho \equiv -g^{\rho\mu}\partial_\mu \underline{u} \tag{8.1.11}$$

as well as, see Lemma 3.1.2, the outgoing null vector field $N = 2\Omega^2 L$ and the incoming null vector field $\underline{N} = 2\Omega^2 \underline{L}$, equivariant relative to the double null integral S-foliation $\{S(\lambda,\nu) = C(\lambda) \cap \underline{C}(\nu)\}$. Using N and $\underline{N}$ we consider the diffeomorphisms ϕ_t and $\underline{\phi}_s$ generated by them, see **??**, and recall that they map the leaves of the double null integral S-foliation $\{S(\lambda,\nu)\}$ into themselves.
Consider now the diffeomorphisms $\psi(\lambda,\nu) : S^2 \to S(\lambda,\nu)$ defined by

$$\psi(\lambda,\nu) = \underline{\phi}_{(\lambda-\lambda_0)} \circ \phi_{(\nu-\nu_0)} \tag{8.1.12}$$

where $\nu_0 = \underline{u}|_{C(\lambda_0)\cap\Sigma_0}$, $\lambda_0 = u|_{C(\lambda_0)\cap\Sigma_0}$. Here we have identified we have identified $S_{(0)}(\nu_0)$ with the topological sphere S^2 through a diffeomorphism.

Given the diffeomorphism $\psi(\lambda,\nu)$ we can map the $S(\lambda,\nu)$-tangent tensor fields defined on $\mathcal{M}$ to tensor fields defined on S^2 with the help of the pull back map $\psi^*(\lambda,\nu)$. Therefore, given an $S(\lambda,\nu)$-tangent p-covariant tensor field ω, we define the p-covariant tensor field $\tilde{\omega}$ on S^2 by the relation

$$\tilde{\omega}(\lambda,\nu) \equiv \psi^*(\lambda,\nu)(r^{-p}\omega) \tag{8.1.13}$$

This allows us to introduce a precise definition of null outgoing infinite limit of ω,

Definition 8.1.1 *We say that the $S(\lambda,\nu)$-tangent p-covariant tensor field ω has W as its null outgoing infinite limit along $C(\lambda)$,*

$$\lim_{C(\lambda),\nu\to\infty} \omega = W$$

if the following limit exists,

$$W(\lambda) = \lim_{\nu\to\infty} \tilde{\omega}(\lambda,\nu) = \lim_{\nu\to\infty} \psi^*(\lambda,\nu)(r^{-p}\omega) \tag{8.1.14}$$

In this case W is a p-covariant tensor field on S^2.

As we are interested to study the null outgoing infinite limit of some of the structure equations we need also an explicit expression for $\frac{\partial}{\partial\lambda}W$. To obtain it we need the following lemma,

Lemma 8.1.1 *the following relations hold*

$$\begin{aligned}\frac{\partial}{\partial\lambda}\psi^*(\lambda,\nu)\omega &= \psi^*(\lambda,\nu)(\mathcal{L}_{\underline{N}}\omega)\\ \frac{\partial}{\partial\nu}\psi^*(\lambda,\nu)\omega &= \psi^*(\lambda,\nu)(\mathcal{L}_V\omega)\end{aligned} \tag{8.1.15}$$

where $V = \underline{\phi}^{-1}_{*(\lambda-\lambda_0)}N$ *is the vector field generating the one parameter diffeomorphisms* $\underline{\phi}^{-1}_{(\lambda-\lambda_0)}\circ\phi_h\circ\underline{\phi}_{(\lambda-\lambda_0)}$.

Proof: Let $p\in S(\lambda,\nu)$ and $p_0=\psi^{-1}(\lambda,\nu)(p)\in S^2$,

$$(\frac{\partial}{\partial\lambda}\psi^*(\lambda,\nu)\omega)|_{p_0} = \lim_{h\to 0}\frac{1}{h}\left[(\psi^*(\lambda+h,\nu)\omega)_{p_0} - (\psi^*(\lambda,\nu)\omega)_{p_0}\right] \tag{8.1.16}$$

As

$$(\psi^*(\lambda+h,\nu)\omega)_{p_0} = \phi^*_{(\nu-\nu_0)}\underline{\phi}^*_{(\lambda+h-\lambda_0)}\omega = \psi^*(\lambda,\nu)(\underline{\phi}^*_h\omega)_p \tag{8.1.17}$$

we have

$$\begin{aligned}(\frac{\partial}{\partial\lambda}\psi^*(\lambda,\nu)\omega)|_{p_0} &= \lim_{h\to 0}\frac{1}{h}\left[\left(\psi^*(\lambda,\nu)\underline{\phi}^*_h\omega\right)_{p_0} - (\psi^*(\lambda,\nu)\omega)_{p_0}\right] \quad (8.1.18)\\ &= \psi^*(\lambda,\nu)\left(\lim_{h\to 0}\frac{1}{h}\left[(\underline{\phi}^*_h\omega)_p-\omega_p\right]\right) = \psi^*(\lambda,\nu)\left(\mathcal{L}_{\underline{N}}\omega_p\right)\end{aligned}$$

To prove the second relation we write

$$(\frac{\partial}{\partial\nu}\psi^*(\lambda,\nu)\omega)|_{p_0} = \lim_{h\to 0}\frac{1}{h}\left[(\psi^*(\lambda,\nu+h)\omega)_{p_0} - (\psi^*(\lambda,\nu)\omega)_{p_0}\right] \tag{8.1.19}$$

As

$$(\psi^*(\lambda,\nu+h)\omega)_{p_0} = \psi^*(\lambda,\nu)\left((\underline{\phi}^*_{(\lambda-\lambda_0)})^{-1}\phi^*_h\underline{\phi}^*_{(\lambda-\lambda_0)}\omega\right)_p \tag{8.1.20}$$

we obtain

$$\begin{aligned}(\frac{\partial}{\partial\nu}\psi^*(\lambda,\nu)\omega)|_{p_0} &= \psi^*(\lambda,\nu)\lim_{h\to 0}\frac{1}{h}\left[\left((\underline{\phi}^*_{(\lambda-\lambda_0)})^{-1}\phi^*_h\underline{\phi}^*_{(\lambda-\lambda_0)}\omega\right)_p - \omega_p\right]\\ &= \psi^*(\lambda,\nu)(\mathcal{L}_V\omega|_p)\end{aligned} \tag{8.1.21}$$

where V is the vector field generating the one parameter diffeomorphisms $\underline{\phi}^{-1}_{(\lambda-\lambda_0)}\phi_h\underline{\phi}_{(\lambda-\lambda_0)}$.

From Lemma 8.1.1 it follows that

$$\frac{\partial}{\partial\lambda}W(\lambda) = \lim_{C(\lambda),\nu\to\infty}\mathcal{L}_{\underline{N}}\omega = \lim_{\nu\to\infty}\psi^*(\lambda,\nu)(r^{-p}\mathcal{L}_{\underline{N}}\omega) \tag{8.1.22}$$

8.1.3 The null outgoing limit of the metric

Let $\gamma = g|_{S(\lambda,\nu)}$ be the induced metric on $S(\lambda,\nu)$. Define $\tilde{\gamma}(\lambda,\nu)$ the Riemannian metric on S^2,

$$\tilde{\gamma}(\lambda,\nu) = \psi^*(\lambda,\nu)(r^{-2}\gamma) \tag{8.1.23}$$

Lemma 8.1.2 *$\tilde{\gamma}$ satisfies the following equation on S^2,*

$$\frac{\partial\tilde{\gamma}}{\partial\lambda} = \left(\Omega tr\underline{\chi} - \overline{\Omega tr\underline{\chi}}\right)\tilde{\gamma} + 2\Omega\tilde{\underline{\hat{\chi}}} \tag{8.1.24}$$

Proof:

$$\begin{aligned}\frac{\partial\tilde{\gamma}}{\partial\lambda} &= \psi^*(\lambda,\nu)(\mathcal{L}_{\underline{N}}r^{-2}\gamma) = \psi^*(\lambda,\nu)\left\{-2r^{-1}\underline{N}(r)r^{-2}\gamma + r^{-2}\mathcal{L}_{\underline{N}}\gamma\right\}\\ &= -(\overline{\Omega\mathrm{tr}\underline{\chi}})\tilde{\gamma} + r^{-2}\psi^*(\lambda,\nu)(2\Omega\underline{\chi}) = -(\overline{\Omega\mathrm{tr}\underline{\chi}})\tilde{\gamma} + r^{-2}\psi^*(\lambda,\nu)(\gamma\Omega\mathrm{tr}\underline{\chi} + 2\Omega\underline{\hat{\chi}})\\ &= \left(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}}\right)\tilde{\gamma} + 2\Omega r^{-2}\psi^*(\lambda,\nu)(\underline{\hat{\chi}}) = \left(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}}\right)\tilde{\gamma} + 2\Omega\tilde{\underline{\hat{\chi}}}\ .\end{aligned}$$

Using this Lemma the following proposition holds,

Proposition 8.1.1 *The metric $\tilde{\gamma}_{(\lambda,\nu)}$ converges as $\nu\to\infty$ to a metric $\tilde{\gamma}_\infty$ on S^2,*

$$\lim_{\nu\to\infty}\tilde{\gamma}_{(\lambda,\nu)} = \tilde{\gamma}_\infty \tag{8.1.25}$$

Moreover $\tilde{\gamma}_\infty$ has Gauss curvature 1, is independent from λ and can be considered as the standard metric on S^2.

Proof: We start to compare $\tilde{\gamma}_{(\lambda,\nu)}$ to $\tilde{\gamma}_{(\lambda_0(\nu),\nu)}$. To do it we choose an orthonormal basis $\{E_A\}$ on $(S^2,\tilde{\gamma}_{(\lambda_0(\nu),\nu)})$ such that the matrix $\tilde{\gamma}_{(\lambda,\nu)}(E_A,E_B)$ is diagonal, with smallest eigenvalue $\Lambda_-(\lambda,\nu)$ and highest eigenvalue $\Lambda_+(\lambda,\nu)$. Proceeding as in section 3.3 of [Ch-Kl] we denote,

$$\mu_{\tilde{\gamma}}(\lambda,\nu) \equiv \sqrt{\Lambda_-\Lambda_+}\ \ ,\ \ \nu_{\tilde{\gamma}}(\lambda,\nu) \equiv \sqrt{\frac{\Lambda_+}{\Lambda_-}}$$

and we prove, under appropriate conditions on the connection coefficients (satisfied in view of the *Main Theorem*) listed below, that $\mu_{\tilde{\gamma}}(\lambda,\nu)$ and $\nu_{\tilde{\gamma}}(\lambda,\nu)$ can be bounded by their values at $(\lambda,\nu)=(\lambda_0(\nu),\nu)$ plus lower order corrections going to zero as $\nu\to\infty$. Therefore

$$\begin{aligned}\lim_{\nu\to\infty}\mu_{\tilde{\gamma}}(\lambda,\nu) &= \lim_{\nu\to\infty}\mu_{\tilde{\gamma}}(\lambda_0(\nu),\nu)\\ \lim_{\nu\to\infty}\nu_{\tilde{\gamma}}(\lambda,\nu) &= \lim_{\nu\to\infty}\nu_{\tilde{\gamma}}(\lambda_0(\nu),\nu)\end{aligned} \tag{8.1.26}$$

This means that, for $\nu \to \infty$, the metric $\tilde{\gamma}_{(\lambda,\nu)}$ converges to the metric $\tilde{\gamma}_{(\lambda_0(\nu),\nu)}$ in the sense that its eigenvalues converge to $\lim_{\nu\to\infty} \Lambda_-(\lambda_0(\nu),\nu)$ and $\lim_{\nu\to\infty} \Lambda_+(\lambda_0(\nu),\nu)$, respectively. Moreover the connection $\Gamma(\lambda,\nu)$ of the metric $\tilde{\gamma}_{(\lambda,\nu)}$ is bounded by the connection of the metric $\tilde{\gamma}_{(\lambda_0(\nu),\nu)}$ plus correction terms, see the remark below. Thus in the $\nu \to \infty$ limit we have

$$\lim_{\nu\to\infty} |\Gamma(\lambda,\nu) - \Gamma(\lambda_0(\nu),\nu)| = 0 \tag{8.1.27}$$

with the pointwise norm taken relative to the metric $\tilde{\gamma}_{(\lambda_0(\nu),\nu)}$.
Similarily, using the diffeomorphism on Σ_0 generated by the gradient flow of the canonical function $u_{(0)}$, see definition 3.3.1, we can connect $\Lambda_-(\lambda_0(\nu),\nu)$ and $\Lambda_+(\lambda_0(\nu),\nu)$ to the corresponding eigenvalues of the rescaled metric $r^{-2}\gamma_{(\lambda_0(\nu),\nu)}$. In view of our the initial conditions, see definition 3.6.1 it is easily seen that this rescaled metric tends to the standard metric at spacelike infinity as $\nu \to \infty$ and therefore

$$\lim_{\nu\to\infty} \Lambda_-(\lambda_0(\nu),\nu) = \lim_{\nu\to\infty} \Lambda_+(\lambda_0(\nu),\nu) = 1 \tag{8.1.28}$$

We have, therefore proved that the limit $\lim_{\nu\to\infty} \tilde{\gamma}_{(\lambda,\nu)} = \tilde{\gamma}_\infty$ exists. Moreover, in view of the boundedness of the pointwise norms

$$|r^2 \left(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}}\right)|, |r\tau_-\hat{\underline{\chi}}| \tag{8.1.29}$$

we deduce from 8.1.24 that

$$\lim_{\nu\to\infty} \left|\frac{\partial\tilde{\gamma}}{\partial\lambda}\right|_{\tilde{\gamma}_\infty} = 0 \tag{8.1.30}$$

Thus $\tilde{\gamma}_\infty$ does not depend on λ.
Remark: To prove the limit of the connection in 8.1.27 we need the boundedness of $|r^3\nabla\!\!\!/\mathrm{tr}\underline{\chi}|, |r^2\tau_-\hat{\underline{\chi}}|,\ |r^2\eta|,\ |r^2\underline{\eta}|$ and the fact that $r^2K(\lambda,\nu)$ tends as $\nu \to \infty$ to the Gauss curvature of the spacelike infinity surface $lim_{\nu\to\infty}S(\lambda_0(\nu),\nu)$.

8.1.4 The null outgoing infinite limit of the $S(\lambda,\nu)$ orthonormal frame

Let e_a be an orthonormal basis for the tangent space to $S(\lambda,\nu)$. Using the diffeomorphism $\psi(\lambda,\nu)$ introduced above, we define a basis on S^2 as follows

$$\tilde{E}_a|_{p_0} = {\psi_*}^{-1}(\lambda,\nu)(re_a|_p) \tag{8.1.31}$$

where $p = \psi(\lambda,\nu)(p_0)$, and p_0 is a point on S^2.

Lemma 8.1.3 *The frame $\{\tilde{E}_a(\lambda,\nu)\}$ converges as $\nu\to\infty$ to a frame orthonormal with respect to $\tilde{\gamma}_\infty$.*

Proof: The frame $\{\tilde{E}_a = \tilde{E}_a(\lambda,\nu)\}$ is orthonormal with respect to the metric $\tilde{\gamma}(\lambda,\nu)=\psi^*(\lambda,\nu)(r^{-2}\gamma)$. Deriving $\{\tilde{E}_a\}$ with repect to ν we obtain

$$\begin{aligned}\frac{\partial \tilde{E}_a|_{p_0}}{\partial\nu} &= \lim_{h\to 0}\frac{1}{h}\left[{\psi_*}^{-1}(\lambda,\nu+h)(re_a)-{\psi_*}^{-1}(\lambda,\nu)(re_a)\right]\Big|_{p_0} \quad (8.1.32)\\ &= {\psi_*}^{-1}(\lambda,\nu)\lim_{h\to 0}\frac{1}{h}\left[\Big((\underline{\phi}_{\lambda-\lambda_0}\circ\phi_h\circ\underline{\phi}^{-1}_{\lambda-\lambda_0})(re_a)\Big)|_p-(re_a)|_p\right]\\ &= {\psi_*}^{-1}(\lambda,\nu)(\mathcal{L}_V(re_a)|_p)\end{aligned}$$

where $V=\underline{\phi}^{-1}_{*(\lambda-\lambda_0)}N$. Using the definition of the $|\cdot|_{\tilde{\gamma}_\infty}$ and the definition of the Lie derivative it is immediate to infer that

$$\left|\frac{\partial\tilde{E}_a}{\partial\nu}\right|_{\tilde{\gamma}_\infty}\le\frac{1}{r}|\mathcal{L}_N(re_a)|_\gamma+o(\frac{1}{r^2}) \quad (8.1.33)$$

To proceed we can further assume that the frame $\{e_a\}$ is Fermi transported along $C(\lambda)$, that is $\not{\mathcal{L}}_N e_a=-\Omega\chi_{ab}e_b$. An explicit calculation gives

$$\not{\mathcal{L}}_N(re_a)=-\frac{1}{2}(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})re_a-\Omega\hat{\chi}_{AB}re_B \quad (8.1.34)$$

from which immediately $|\mathcal{L}_N(re_a)|_\gamma=|\not{\mathcal{L}}_N(re_a)|_\gamma\le cr^{-1}$ and finally

$$\left|\frac{\partial\tilde{E}_a}{\partial\nu}\right|_{\tilde{\gamma}_\infty}=O(r^{-2}) \quad (8.1.35)$$

This implies that $\{\tilde{E}_a(\lambda,\nu)\}$ converges as $\nu\to\infty$ to a frame orthonormal with respect to $\tilde{\gamma}_\infty$.
With this definition of the $\{\tilde{E}_a(\lambda,\nu)\}$ frame, given ω a $S(\lambda,\nu)$ tangent, p-covariant tensor field on $\mathcal{M}$ and $\tilde{\omega}$ its rescaled pull back on S^2, the following relation holds

$$\tilde{\omega}(\tilde{E}_{a_1},...\tilde{E}_{a_p})=\omega(e_{a_1}....e_{a_p}) \quad (8.1.36)$$

Therefore we have proved the following lemma,

Lemma 8.1.4 *An $S(\lambda,\nu)$ tangent, p-covariant tensor field ω has a null outgoing limit, in the sense of definition 8.1.1, if and only if the limit $\lim_{\nu\to\infty}\omega|_{S(\lambda,\nu)}(e_{a_1}....e_{a_p})$ exists.*

8.2 The behaviour of the curvature tensor at the spacetime null infinity

To examine the behaviour of the various components of the Riemann tensor moving toward the future infinity along an outgoing null hypersurface $C(\lambda)$ we recall that, as discussed in section 8.1, a covariant p-tensor w defined on $S(\lambda,\nu)$ has a null infinity limit along $C(\lambda)$ if the following limit exists for any λ [5],

$$\lim_{C(\lambda);\nu\to\infty}\psi^*(\lambda,\nu)(r^{-p}w)=\lim_{C(\lambda);\nu\to\infty}\tilde{w}(\lambda,\nu)\equiv W(\lambda) \tag{8.2.1}$$

Moreover Lemma 8.1.4 shows that this is equivalent to proving that the following limit does exist

$$\lim_{C(\lambda);\nu\to\infty}w(p)(e_{a_1},....,e_{a_p}) \tag{8.2.2}$$

Using these results we have the following proposition

Proposition 8.2.1 *The null components of the Riemann tensor have the following future outgoing null infinity limits*

$$\lim_{C(\lambda);\nu\to\infty}r\underline{\alpha}=\underline{A}(\lambda,\omega)\quad,\quad\lim_{C(\lambda);\nu\to\infty}r^2\underline{\beta}=\underline{B}(\lambda,\omega) \tag{8.2.3}$$
$$\lim_{C(\lambda);\nu\to\infty}r^3\rho=P(\lambda,\omega)\quad,\quad\lim_{C(\lambda);\nu\to\infty}r^3\sigma=Q(\lambda,\omega)$$

and $\underline{A}(\lambda,\omega),\underline{B}(\lambda,\omega),P(\lambda,\omega),Q(\lambda,\omega)$ *satisfy the following estimates*

$$|\underline{A}(\lambda,\omega)|\leq c(1+|\lambda|)^{-\frac{5}{2}}\quad;\quad|\underline{B}(\lambda,\omega)|\leq c(1+|\lambda|)^{-\frac{3}{2}} \tag{8.2.4}$$
$$|(P-\overline{P})(\lambda,\omega)|\leq c(1+|\lambda|)^{-\frac{1}{2}}\ ;\ |(Q-\overline{Q})(\lambda,\omega)|\leq c(1+|\lambda|)^{-\frac{1}{2}}$$

Remark: The limits $\lim_{\lambda\to\lambda_0}\overline{P}$ and $\lim_{\lambda\to\lambda_0}\overline{Q}$ will be discussed in section 8.5.

Osservazione 8.2.1 *Le stime in* **??** *sono un risultato nuovo da dimostrare, è ragionevole pensare che siano collegate alla massa all'interno della regione compatta K definita su Σ_0.*

Proof: This result is, basically, the same as the one in **Conclusion 17.0.1** in [Ch-Kl]. We sketch its proof for completeness.

[5] Any λ means in fact any $\lambda\leq\lambda_0$.

Using the Bianchi equation for $\underline{\alpha}$, see 3.2.8, and assuming the frame $\{e_a\}$ Fermi transported along the null outgoing hypersurfaces we have the evolution equation for $\underline{\alpha}(e_a, e_b)$

$$\frac{\partial \underline{\alpha}_{ab}}{\partial \nu}+\frac{1}{2}\Omega\mathrm{tr}\chi\underline{\alpha}_{ab}=\Omega\left\{-(\nabla\widehat{\otimes}\underline{\beta})_{ab}+\left[4\omega\underline{\alpha}_{ab}-3(\underline{\hat{\chi}}_{ab}\rho-{}^\star\underline{\hat{\chi}}_{ab}\sigma)+\Big((\zeta-4\underline{\eta})\widehat{\otimes}\underline{\beta}\Big)_{ab}\right]\right\}$$

Recalling that $\frac{\partial}{\partial\nu}r = \frac{1}{2}r\overline{\Omega\mathrm{tr}\chi}$, see 4.1.30, from the previous equation we derive

$$\frac{\partial(r\underline{\alpha}_{ab})}{\partial\nu}=f(\nu,\lambda,\cdot)(r\underline{\alpha}_{ab})+F(\nu,\lambda,\cdot) \tag{8.2.5}$$

where

$$f(\nu,\lambda,\omega^a) = \left[-\frac{1}{2}(\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi})+4\Omega\omega\right] \tag{8.2.6}$$
$$F(\nu,\lambda,\omega^a) = r\Omega\left\{-(\nabla\widehat{\otimes}\underline{\beta})_{ab}+\left[-3(\underline{\hat{\chi}}_{ab}\rho-{}^\star\underline{\hat{\chi}}_{ab}\sigma)+\Big((\zeta-4\underline{\eta})\widehat{\otimes}\underline{\beta}\Big)_{ab}\right]\right\}$$

From 8.2.5 we easily obtain, omitting the dependance on the angular variables,

$$|r\underline{\alpha}_{ab}(\nu,\lambda)-r\underline{\alpha}_{ab}(\nu',\lambda)| \le \int_{\nu'}^{\nu}\left(|fr\underline{\alpha}|(\nu'',\lambda)+|F|(\nu'',\lambda)\right)d\nu'' \tag{8.2.7}$$

From the *Main Theorem*, see 8.0.1 and 8.0.2, $f(\nu,\lambda) = O(r^{-2}(\lambda,\nu))$ and $\sup_{\mathcal{K}} r|\lambda|^{\frac{5}{2}}|\underline{\alpha}| \le C_0$. Therefore

$$\int_{\nu'}^{\nu}|fr\underline{\alpha}|(\nu'',\lambda)d\nu'' \le C_0\frac{1}{r(\lambda,\nu')}\frac{1}{\lambda^{\frac{5}{2}}} \tag{8.2.8}$$

Moreover in view of the results of our *Main Theorem*, the $\mathcal{R}$ norms are uniformily bounded by a constant C_0 which implies immediately the following estimate for $\int_{\nu'}^{\nu}F(\nu',\lambda)d\nu'$, taking into account only the principal term of $F(\nu,\lambda,\omega^a)$, $r\Omega(\nabla\widehat{\otimes}\underline{\beta})_{ab}$,

$$\left|\int_{\nu'}^{\nu}F(\nu'',\lambda)d\nu''\right| \le c\int_{\nu'}^{\nu}|r(\nabla\widehat{\otimes}\underline{\beta})_{ab}|d\nu'' \tag{8.2.9}$$
$$\le c\left(\int_{\nu'}^{\nu}|r\nabla\underline{\beta}|^2r^2d\nu''\right)^{\frac{1}{2}}\left(\int_{\nu'}^{\nu}r^{-2}d\nu''\right)^{\frac{1}{2}} \le \frac{1}{r(\lambda,\nu')^{\frac{1}{2}}}\frac{1}{\lambda^2}$$

Choosing ν' sufficiently large we can make the right hand side of 8.2.8 and 8.2.9 arbitrarily small, proving the existence of the limit for $r\underline{\alpha}(\lambda,\nu)$. Then, using again 8.2.5, Gronwall lemma and the estimate for $\underline{\alpha}$ on Σ_0, see **??**, we obtain

$$\lim_{\nu\to\infty}|r\underline{\alpha}_{ab}|(\nu,\lambda)\leq c\left(|r\underline{\alpha}_{ab}|(\nu_0(\lambda))+\int_{\nu_0(\lambda)}^{\infty}F(\nu',\lambda)d\nu'\right)\leq c\frac{1}{\lambda^{\frac{5}{2}}} \quad (8.2.10)$$

proving our result.

The limits for $\underline{\beta},\rho,\sigma$ are obtained in the same way using the corresponding Bianchi equations, see 3.2.8, and we do not report them here.

According to the Penrose hypothesis of smooth conformal compactification, [Pe1], [Pe2], [**?**], α and β should also have outgoing null infinity limits. It is in fact known that smooth compactification implies the following

$$\lim_{C(\lambda);\nu\to\infty} r^5\alpha=A(\lambda,\omega) \quad , \quad \lim_{C(\lambda);\nu\to\infty} r^4\beta=B(\lambda,\omega) \qquad (8.2.11)$$

These results are, however out of reach with our methods. In fact in this work, as well as in [Ch-Kl], we have been only able to prove the boundedness of $r^{\frac{7}{2}}\alpha$ and $r^{\frac{7}{2}}\beta$. This, by itself, does not exclude [6] the possibility that α,β have better bounds under, possibly, more stringent conditions on the initial data.

The issue of smooth conformal compactification has drawn a lot of attention in the last twenty years. In particular one of the main promoter of the idea that there must exist an important class of data which lead to a smooth compactification is H.Friedrich, [Fr1], [**?**]. On the other hand a lot of evidence has been accumulated suggesting that one cannot expect the smoothness of the compactification for generic initial data [**?**] or for physically relevant ones, [**?**], [**?**]. In particular D.Christodoulou has shown, under some reasonable "physical" assumptions concerning [7] the past null infinity $\mathcal{J}^-$, that the following limit hold,

$$\begin{aligned}\alpha&=r^{-4}A_1(\theta,\phi)+r^{-5}\log rA_2(\lambda;\theta,\phi)+r^{-5}A_3(\lambda;\theta,\phi)\\ \beta&=r^{-4}\log rB_1(\theta,\phi)+r^{-4}B_2(\lambda;\theta,\phi)\end{aligned} \qquad (8.2.12)$$

[6] Observe that we cannot show that the quantities $r^{\frac{7}{2}}\alpha$, $r^{\frac{7}{2}}\beta$ have a null outgoing limit. Indeed if we were trying to implement the same startegy for α and β as for the other quantities we encounter the following difficulties: in the case of α the Bianchi equations, see 3.2.8, do not contain an evolution equation for α along the e_4 direction. On the other hand for β we have an evolution equation along $C(\lambda)$ which would suggest that $r^4\beta$ has a null outgoing limit. To prove this, however, we would need to control the quantity $\int_{\nu'}^{\nu}|r^4\nabla\!\!\!/\alpha|$ whose boundedness is not at our disposal.

[7] It is assumed there is no incoming radiation from $\mathcal{J}^-$ and the outgoing radiation had the structure suggested from the quadrupole approximation of the gravitational radiation produced by N point accelerated massses.

This is in agreement with the polyhomogeneity expansions suggested by the work of many authors, see **??** and the references within.

8.3 The behaviour of the connection coefficients at the spacetime null infinity

Proposition 8.3.1 *The following null outgoing infinity limits hold*

$$\lim_{C(\lambda);\nu\to\infty}\Omega=\frac{1}{2}$$
$$\lim_{C(\lambda);\nu\to\infty} r\,tr\chi=2\ ,\quad \lim_{C(\lambda);\nu\to\infty} r\,tr\underline{\chi}=-2 \tag{8.3.1}$$

Proof: From the estimates 8.0.2 proved in the *Main Theorem* it follows that, choosing $\nu_1\geq\nu_2\geq M>0$, $|\log 2\Omega(\nu_1,\lambda,\cdot)-\log 2\Omega(\nu_2,\lambda,\cdot)|$ can be made arbitrarily small. In fact

$$|\log 2\Omega(\nu_1,\lambda,\cdot)-\log 2\Omega(\nu_2,\lambda,\cdot)|\leq c\int_{\nu_2}^{\nu_1}|(\Omega\mathbf{D}_4\log\Omega)|(\nu',\lambda,\cdot)d\nu'$$
$$\leq c\frac{1}{\lambda^{\frac{1}{2}}}\left|\frac{1}{r(\lambda,\nu_1)}-\frac{1}{r(\lambda,\nu_2)}\right| \tag{8.3.2}$$

Therefore the limit $\lim_{C(\lambda);\nu\to\infty}\log 2\Omega(\nu,\lambda,\cdot)$ exists. To prove that it is equal to $\frac{1}{2}$ we observe first that $|\log\Omega-\overline{\log\Omega}|$ goes to zero as ν goes to infinity and subsequently we look at the evolution equation of $\log 2\Omega$ along the $\underline{C}(\nu)$ null hypersurfaces, similarily to the procedure in Lemma 4.3.4, and we obtain the final result estimating $\log 2\Omega(\lambda,\nu)$ in terms of $\log 2\Omega(\lambda,\nu)|_{\underline{C}(\nu)\cap\Sigma_0}=\log 2\Omega(\lambda_0(\nu),\nu)$ and correction terms which go to zero as $n\to\infty$ and recall that $\log 2\Omega(\lambda,\nu)|_{\underline{C}(\nu)\cap\Sigma_0}$ goes to zero as $\nu\to\infty$. The proof of the limits for $r\mathrm{tr}\chi$ and $r\mathrm{tr}\underline{\chi}$ proceeds exactly ion the same way. First by looking at their evolution equation along $C(\lambda)$ and using the estimates from the *Main Theorem* one proves that these limits do exist. Then using the corresponding evolution equations along the $\underline{C}(\nu)$ null hypersurfaces it is possible to connect these limits to the limits of the corresponding quantities on Σ_0 obtaining the result.

Proposition 8.3.2 *The connection coefficients $\hat{\chi}$ and $\hat{\underline{\chi}}$ have the null outgoing infinity limits*

$$\lim_{C(\lambda);\nu\to\infty} r^2\hat{\chi}=X(\lambda,\cdot)\quad ,\quad \lim_{C(\lambda);\nu\to\infty} r\hat{\underline{\chi}}=\underline{X}(\lambda,\cdot) \tag{8.3.3}$$

and

$$|X(\lambda,\cdot)| \leq c(1+|\lambda|)^{-\frac{1}{2}} \quad , \quad |\underline{X}(\lambda,\cdot)| \leq c(1+|\lambda|)^{-\frac{3}{2}} \tag{8.3.4}$$

Proof: We start looking at the null infinity limit of $\hat{\chi}$. As it was done for the underlined null components of the curvature tensor, we look at the evolution equation along the null outgoing hypersurfaces $C(\lambda)$ for $\hat{\chi}$, starting with initial data on Σ_0 and use [8] it for estimating the limit $\nu \to \infty$. The evolution equation for $\hat{\chi}$, see 3.1.45, can be written as

$$\frac{\partial}{\partial\nu}\hat{\chi}_{ab} + \Omega\mathrm{tr}\chi\hat{\chi}_{ab} + 2\Omega\omega\hat{\chi}_{ab} = -\alpha_{ab} \tag{8.3.5}$$

and also

$$\frac{\partial(r^2\hat{\chi}_{ab})}{\partial\nu} = h(\nu,\lambda,\cdot)(r^2\hat{\chi}_{ab}) - \Omega\alpha_{ab} \tag{8.3.6}$$

where

$$h(\nu,\lambda,\omega^a) = -(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}) + 2\Omega\omega \tag{8.3.7}$$

In view of the *Main Theorem* results, see 8.0.1, 8.0.2, we easily check the pointwise bounds,

$$|r^{\frac{7}{2}}\alpha| \leq C_0 \ , \ |r^2 h| \leq C_0 \ , \ |r^2\tau_-^{\frac{1}{2}}\hat{\chi}| \leq C_0 \tag{8.3.8}$$

thus by integration of 8.3.6, choosing $\nu_1 \geq \nu_2 \geq M > 0$, we have

$$\left|r^2\hat{\chi}_{ab}(\nu_1,\lambda) - r^2\hat{\chi}_{ab}(\nu_2,\lambda)\right| \leq c\left(\int_{\nu_2}^{\nu_1}|r^2\alpha_{ab}|(\lambda,\nu')d\nu' + \int_{\nu_2}^{\nu_1} O\left(\frac{1}{r^2\lambda^{\frac{1}{2}}}\right)\right)$$

As the right hand side can be made arbitrarily small as $M \to \infty$, we conclude that the following limit exists,

$$\lim_{C(\lambda);\nu\to\infty} r^2\hat{\chi} = X(\lambda,\cdot) \tag{8.3.9}$$

To obtain an estimate for the behaviour of $X(\lambda,\cdot)$ with respect to λ we use the evolution equation for $\hat{\chi}$ along the null incoming hypersurfaces $\underline{C}(\nu)$ written as

$$\frac{\partial}{\partial\lambda}\hat{\chi}_{ab} = -\left[\frac{1}{2}\Omega\mathrm{tr}\underline{\chi} + 2\Omega\underline{\omega}\right]\hat{\chi}_{ab} - \frac{1}{2}\Omega\mathrm{tr}\chi\underline{\hat{\chi}}_{ab} - \Omega(\nabla\!\!\!/\widehat{\otimes}\eta - \eta\widehat{\otimes}\eta) \tag{8.3.10}$$

[8]Observe that our procedure here differs from that of Chapter 4 where the integration took place with the initial conditions given on the last slice.

from which we derive

$$\frac{\partial}{\partial\lambda}(r\hat{\chi}_{ab}) = t(\nu,\lambda,\omega^a)(r\hat{\chi}_{ab}) + T(\nu,\lambda,\omega^a) \tag{8.3.11}$$

where

$$\begin{aligned}\underline{t}(\nu,\lambda,\omega^a) &= \left[-\frac{1}{2}(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}}) + 2\Omega\underline{\omega}\right]\\ \underline{T}(\nu,\lambda,\omega^a) &= -\frac{1}{2}\Omega\mathrm{tr}\chi(r\hat{\underline{\chi}}_{ab}) - \Omega r(\nabla\!\!\!/\widehat{\otimes}\eta - \eta\widehat{\otimes}\eta)\end{aligned} \tag{8.3.12}$$

Applying Gronwall lemma, and denoting $\lambda_0(\nu) = u|_{\Sigma_0\cap\underline{C}(\nu)}$, we obtain

$$\begin{aligned}|r^2\hat{\chi}_{ab}|(\nu,\lambda) &\leq c\left(|r^2\hat{\chi}_{ab}|(\lambda_0(\nu),\nu) + \int_{\lambda_0(\nu)}^{\lambda}|r^2(\nabla\!\!\!/\widehat{\otimes}\eta - \eta\widehat{\otimes}\eta)|(\lambda',\nu)d\lambda'\right)\\ &\leq c\int_{\lambda_0(\nu)}^{\lambda}(|r^2\nabla\!\!\!/\eta| + r^2|\eta|^2)(\lambda',\nu)d\lambda' \leq c\lambda^{-\frac{1}{2}}\end{aligned} \tag{8.3.13}$$

where the last inequality follows from the estimates 8.0.2. Taking the limit as ν goes to infinity and recalling the asymptotic behaviour of $|r^2\hat{\chi}_{ab}|$ on Σ_0, see 7.2.3, we infer that $|X(\lambda,\cdot)| \leq c(1+|\lambda|)^{-\frac{1}{2}}$.
We proceed similarily for $\hat{\underline{\chi}}$. Consider the evolution equation for $\hat{\underline{\chi}}$ along $C(\lambda)$, which we express in the following way

$$\frac{\partial}{\partial\nu}\hat{\underline{\chi}}_{ab} = -\left[\frac{1}{2}\Omega\mathrm{tr}\chi + 2\Omega\omega\right]\hat{\underline{\chi}}_{ab} - \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\hat{\chi}_{ab} - \Omega(\nabla\!\!\!/\widehat{\otimes}\underline{\eta} - \underline{\eta}\widehat{\otimes}\underline{\eta}) \tag{8.3.14}$$

which implies also

$$\frac{\partial}{\partial\nu}(r\hat{\underline{\chi}}_{ab}) = q(\nu,\lambda,\omega^a)(r\hat{\underline{\chi}}_{ab}) + Q(\nu,\lambda,\omega^a) \tag{8.3.15}$$

where

$$\begin{aligned}q(\nu,\lambda,\omega^a) &= \left[-\frac{1}{2}(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}) + 2\Omega\omega\right]\\ Q(\nu,\lambda,\omega^a) &= -\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}(r\hat{\chi}_{ab}) - \Omega r(\nabla\!\!\!/\widehat{\otimes}\underline{\eta} - \underline{\eta}\widehat{\otimes}\underline{\eta})\end{aligned} \tag{8.3.16}$$

In view of the *Main Theorem* results, see 8.0.1, 8.0.2, we easily check the pointwise bounds,

$$|r^2q| \leq C_0\ ,\ |r^3\tau_-^{\frac{1}{2}}(\nabla\!\!\!/\widehat{\otimes}\underline{\eta} - \underline{\eta}\widehat{\otimes}\underline{\eta})| \leq C_0\ ,\ |r\tau_-^{\frac{3}{2}}\hat{\underline{\chi}}| \leq C_0 \tag{8.3.17}$$

Proceeding as before, we obtain

$$|r\underline{\hat{\chi}}_{ab}(\nu_1,\lambda) - r\underline{\hat{\chi}}_{ab}(\nu_2,\lambda)| \leq c\int_{\nu_2}^{\nu_1} |r(\nabla\!\!\!\!/\,\widehat{\otimes}\underline{\eta} - \underline{\eta}\widehat{\otimes}\underline{\eta})|(\lambda,\nu')d\nu' + \int_{\nu_2}^{\nu_1} O\left(\frac{1}{r^2\lambda^{\frac{3}{2}}}\right) \quad (8.3.18)$$

Letting ν_1, ν_2 going to ∞, as before, we infer that $r\underline{\hat{\chi}}$ has a limit

$$\lim_{C(\lambda);\nu\to\infty} r\underline{\hat{\chi}} = \underline{X}(\lambda,\cdot)\ .$$

Moreover, recalling the asymptotic behaviour of $|r\underline{\hat{\chi}}_{ab}|$ on Σ_0, derived from the boundedness of the norms 7.2.3, it is immediate to prove, proceeding as in the case of $X(\lambda,\cdot)$, that $\underline{X}$ behaves as $O(\lambda^{-\frac{3}{2}})$. Therefore

$$|\underline{X}(\lambda,\cdot)| \leq c(1+|\lambda|)^{-\frac{3}{2}}\ .$$

8.4 The null outgoing infinity limit of the structure equations

We show in this section that some of the structure equations have some limit equations when $\nu\to\infty$, involving $X(\lambda,\cdot), \underline{X}(\lambda,\cdot)$ and the null infinity limit of the null Riemann tensor components,

Proposition 8.4.1 *The following equations are satisfied by the null outgoing infinity limit of the connection coefficients and of the null Riemann components,*

$$\widetilde{\mathrm{d}\!\!\!/iv}\,\underline{X} = \underline{B} \quad,\quad \frac{\partial}{\partial\lambda}\underline{X} = -\frac{1}{2}\underline{A} \quad,\quad \frac{\partial}{\partial\lambda}X = -\underline{X} \qquad (8.4.1)$$

Proof: Let us consider the structure equation, see 3.1.46,

$$\nabla\!\!\!\!/\,\mathrm{tr}\underline{\chi} - \mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\chi} + \zeta\cdot\underline{\chi} - \zeta\mathrm{tr}\underline{\chi} = -\underline{\beta}\ .$$

Multiplying it by r^2 we obtain

$$r\mathrm{d}\!\!\!/\mathrm{iv}\,(r\underline{\hat{\chi}})_a = r^2(\nabla\!\!\!\!/_a\mathrm{tr}\underline{\chi}) + r^2(\zeta\cdot\underline{\chi})_a - (\zeta\mathrm{tr}\underline{\chi})_a + r^2\underline{\beta}_a \qquad (8.4.2)$$

and taking the limit $\nu\to\infty$, recalling the estimates for the connection coefficients and for the Riemann null components provided by the *Main Theorem*, we obtain, denoting with $\widetilde{\mathrm{d}\!\!\!/\mathrm{iv}}$ the divergence on S^2 relative to the $\tilde{\gamma}_\infty$ metric,

$$\widetilde{\mathrm{d}\!\!\!/\mathrm{iv}}\,\underline{X} = \underline{B} \qquad (8.4.3)$$

Let us now consider the structure equation

$$\not{\mathbf{D}}_3\underline{\hat{\chi}}+\mathrm{tr}\underline{\chi}\underline{\hat{\chi}}-(\mathbf{D}_3\log\Omega)\hat{\chi}=-\underline{\alpha}$$

which we rewrite as

$$\frac{\partial}{\partial\lambda}\underline{\hat{\chi}}_{ab}+\Omega\mathrm{tr}\underline{\chi}\ \underline{\hat{\chi}}_{ab}+2\Omega\underline{\omega}\underline{\hat{\chi}}_{ab}=-\Omega\underline{\alpha}_{ab} \tag{8.4.4}$$

and also, multiplying by r and recalling the definition and asymptotic properties of h, see 8.3.7 and 8.0.2,

$$\frac{\partial(r\underline{\hat{\chi}}_{ab})}{\partial\lambda}=h(\nu,\lambda,\cdot)(r^2\underline{\hat{\chi}}_{ab})-\Omega(r\underline{\alpha}_{ab}) \tag{8.4.5}$$

Therefore, taking the $\nu\to\infty$ limit, we obtain

$$\frac{\partial}{\partial\lambda}\underline{X}=-\frac{1}{2}\underline{A} \tag{8.4.6}$$

Finally, from the structure equation

$$\frac{\partial}{\partial\lambda}\hat{\chi}_{ab}=-\left[\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}+2\Omega\underline{\omega}\right]\hat{\chi}_{ab}-\frac{1}{2}\Omega\mathrm{tr}\chi\underline{\hat{\chi}}_{ab}-\Omega(\not\nabla\widehat{\otimes}\eta-\eta\widehat{\otimes}\eta) \tag{8.4.7}$$

proceeding exactly in the same way, multiplying by r^2 and taking the limit $\nu\to\infty$ we obtain

$$\lim_{C(\lambda);\nu\to\infty}\frac{\partial}{\partial\lambda}(r^2\hat{\chi}_{ab})=-\lim_{C(\lambda);\nu\to\infty}(r\underline{\hat{\chi}}_{ab}) \tag{8.4.8}$$

which implies

$$\frac{\partial}{\partial\lambda}X=-\underline{X} \tag{8.4.9}$$

8.4.1 Other structure equations in the outgoing null infinity limit

We start proving the following lemma

Lemma 8.4.1 *the following limit holds*

$$\lim_{C(\lambda);\nu\to\infty}r^2\underline{\omega}=\frac{P}{8} \tag{8.4.10}$$

Proof: $\underline{\omega}$ satisfies the following evolution equation, see 4.3.58

$$\Omega\mathbf{D}_4(\Omega\underline{\omega}) = \Omega^2(\mathbf{D}_4 \log\Omega)\underline{\omega} + \Omega^2\mathbf{D}_4\underline{\omega} = -\frac{1}{2}\Omega(\underline{\hat{F}} - \Omega\rho) \qquad (8.4.11)$$

which we rewrite as

$$\Omega^2\mathbf{D}_4\underline{\omega} = -\Omega^2(\mathbf{D}_4 \log\Omega)\underline{\omega} - \frac{1}{2}\Omega(\underline{\hat{F}} - \Omega\rho) \qquad (8.4.12)$$

or

$$\frac{\partial}{\partial\nu}\underline{\omega} = -\Omega(\mathbf{D}_4 \log\Omega)\underline{\omega} - \frac{1}{2}(\underline{\hat{F}} - \Omega\rho) \qquad (8.4.13)$$

As $\lim_{C(\lambda);\nu\to\infty}\underline{\omega} = 0$, integrating 8.4.13 we obtain

$$\underline{\omega}(\lambda,\nu) = \int_\nu^\infty \Omega(\mathbf{D}_4 \log\Omega)\underline{\omega} + \frac{1}{2}\int_\nu^\infty \underline{\hat{F}} + \frac{1}{2}\int_\nu^\infty \Omega\rho \qquad (8.4.14)$$

and, multiplying both sides by r^2,

$$(r^2\underline{\omega})(\lambda,\nu) = r^2(\lambda,\nu)\int_\nu^\infty \Big[\Omega(\mathbf{D}_4 \log\Omega)\underline{\omega} + \underline{\hat{F}}\Big](\lambda,\nu') + r^2(\lambda,\nu)\int_\nu^\infty \Omega\rho \qquad (8.4.15)$$

From the previous estimates, see **??**, validated in the *Main Theorem*, it is immediate to see that

$$\int_\nu^\infty \Big[\Omega(\mathbf{D}_4 \log\Omega)\underline{\omega} + \underline{\hat{F}}\Big] = O(r^{-3}) \qquad (8.4.16)$$

Therefore, performing the limit $\nu \to \infty$, it follows

$$\lim_{\nu\to\infty}(r^2\underline{\omega})(\lambda,\nu) = \lim_{\nu\to\infty} r^2(\lambda,\nu)\int_\nu^\infty \Omega\rho \qquad (8.4.17)$$
$$= \lim_{\nu\to\infty}\left(r^2(\lambda,\nu)\int_\nu^\infty \frac{\Omega}{r^3}(r^3\rho - P)\right) + \frac{P}{2}\lim_{\nu\to\infty}\left(r^2(\lambda,\nu)\int_\nu^\infty \frac{\Omega}{r^3}\right)$$

From the result of Proposition 8.2.1 we have immediately that the first integral in the right hand side goes to zero and we are left with

$$\lim_{\nu\to\infty}(r^2\underline{\omega})(\lambda,\nu) = \frac{P}{8} \qquad (8.4.18)$$

proving the lemma.

Remark: Observe that the estimate of Lemma 8.4.1 is stronger than the estimate provided by the *Main Theorem*, see Proposition 4.3.4 and Proposition 7.4.4. Once this estimate is obtained one can obtain a better estimate also for $\nabla\!\!\!/\underline{\omega}$. In fact we prove

Lemma 8.4.2 *the following limit holds*

$$\lim_{C(\lambda);\nu\to\infty} r^2 \nabla\!\!\!/ \underline{\omega} = 0 \tag{8.4.19}$$

moreover we can also prove $r^2\nabla\!\!\!/\underline{\omega} = O(r^{-1}\tau_-^{-\frac{1}{2}})$.

Proof: The proof is similar to the one of the previous lemma. First we observe that $\lim_{C(\lambda);\nu\to\infty}(r\nabla\!\!\!/\underline{\omega}) = 0$ as follows from the $L^p(S)$ norm estimates for $\nabla\!\!\!/\underline{\omega}$ and $\nabla\!\!\!/^2\underline{\omega}$. Then we look at the evolution equation satisfied from $\nabla\!\!\!/\underline{\omega}$, 4.3.79

$$\begin{aligned}\mathbf{D}\!\!\!/_4(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega) + \frac{1}{2}tr\chi(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega) &= -\hat{\chi}(\Omega\nabla\!\!\!/\mathbf{D}_3\log\Omega) + \nabla\!\!\!/\rho \\ &-(\mathbf{D}_3\log\Omega)(\Omega\nabla\!\!\!/\mathbf{D}_4\log\Omega) - \underline{\hat{H}}\end{aligned}$$

where $\underline{\hat{H}}$ satisfies the following bound $|r^{-\frac{2}{p}}\underline{\hat{H}}|_{p,S} \leq C_0 r^{-5}$, see Proposition 4.3.8. This equation can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial\nu}(\nabla\!\!\!/\underline{\omega}) + \frac{\Omega\mathrm{tr}\chi}{2}(\nabla\!\!\!/\underline{\omega}) &= -\Omega\hat{\chi}(\nabla\!\!\!/\underline{\omega}) + 2\left(\Omega\omega\nabla\!\!\!/\underline{\omega} - \underline{\omega}\nabla\!\!\!/\omega\right) + \frac{1}{2}\underline{\hat{H}} - \frac{1}{2}\nabla\!\!\!/\rho \\ &= -\frac{1}{2}\nabla\!\!\!/\rho + O\left(\frac{1}{r^4\tau_-}\right)\end{aligned}$$

This allows to write

$$\frac{\partial}{\partial\nu}(r\nabla\!\!\!/\underline{\omega}) = -\frac{1}{2}\left(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}\right)(r\nabla\!\!\!/\underline{\omega}) - \frac{1}{2}r\nabla\!\!\!/\rho + O\left(\frac{1}{r^3\tau_-^2}\right) = -\frac{1}{2}r\nabla\!\!\!/\rho + O\left(\frac{1}{r^3\tau_-^2}\right)$$

and from it, using that $\lim_{C(\lambda);\nu\to\infty}(r\nabla\!\!\!/\underline{\omega}) = 0$,

$$(r^2\nabla\!\!\!/\underline{\omega})(\lambda,\nu) = -\frac{1}{2}r(\lambda,\nu)\int_\nu^\infty r\nabla\!\!\!/\rho(\lambda,\nu') - \frac{1}{2}r(\lambda,\nu)\int_\nu^\infty O\left(\frac{1}{r^3\tau_-^2}\right) \tag{8.4.20}$$

which implies that

$$(r^2\nabla\!\!\!/\underline{\omega})(\lambda,\nu) = O(r^{-1}\tau_-^{-\frac{1}{2}}) \tag{8.4.21}$$

As before we can estimate the limit of the right hand side as $\nu\to\infty$ and conclude that, in this case, it goes to zero. This is due to the fact that the integrand $\nabla\!\!\!/\rho$ in 8.4.20 goes to zero as $O(r^{-4}\tau_-^{-\frac{1}{2}})$, proving the lemma.

Remark: Observe that in this case we do not have a pointwise limit as $\nu\to\infty$ for $r^3\nabla\!\!\!/\underline{\omega}$ due to the fact that its evolution equation involves the first tangential derivatives of ρ which has not a pointwise limit.

Using the previous Lemma we can prove the following proposition

Proposition 8.4.2 *The quantities $\nabla\!\!\!/ tr\chi$ and ζ have the null outgoing infinity limits*

$$\lim_{C(\lambda);\nu\to\infty} r^2 \nabla\!\!\!/ r\, tr\chi = \tilde{\nabla\!\!\!/} H(\lambda,\cdot) \quad , \quad \lim_{C(\lambda);\nu\to\infty} r^2\zeta = Z(\lambda,\cdot) \tag{8.4.22}$$

Moreover the following "structure equation" holds at infinity

$$\widetilde{d\!\!/iv} X = \frac{1}{2}\tilde{\nabla\!\!\!/} H + Z \tag{8.4.23}$$

Proof: to be added from my notes.

8.5 The Bondi mass

Definition 8.5.1 *The Hawking mass enclosed by a two surface $S(\lambda,\nu)$ is given by the following expression, see[Ch-Kl], Chapter 17,*

$$m(\lambda,\nu) = \frac{r(\lambda,\nu)}{2}\left(1 + \frac{1}{16\pi}\int_{S(\lambda,\nu)} tr\chi tr\underline{\chi}\right) \tag{8.5.1}$$

Recalling the definition of the mass aspect function, see 3.3.6,

$$\underline{\mu}(\lambda,\nu) = \mathbf{K} + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - d\!\!/\mathrm{iv}\,\underline{\eta} = -d\!\!/\mathrm{iv}\,\underline{\eta} + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \rho \tag{8.5.2}$$

and using the Gauss Bonnet theorem, we can reexpress the Hawking mass in the form [9],

$$\begin{aligned} m(\lambda,\nu) &= \frac{r(\lambda,\nu)}{2}\left(1 + \frac{1}{4\pi}\int_{S(\lambda,\nu)} (\underline{\mu} - \mathbf{K} + d\!\!/\mathrm{iv}\,\underline{\eta})\right) \\ &= \frac{r(\lambda,\nu)}{8\pi}\int_{S(\lambda,\nu)} \underline{\mu} = \frac{r(\lambda,\nu)}{8\pi}\int_{S(\lambda,\nu)} \left(\frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \rho\right) \end{aligned} \tag{8.5.3}$$

Proposition 8.5.1 *the Hawking mass $m(\lambda,\nu)$ satisfies the following*

$$\frac{\partial}{\partial\nu} m(\lambda,\nu) = O(r^{-2}) \tag{8.5.4}$$

[9] In most of the equations we omit the dependance on the angular variables, except where strictly needed.

Proof: From definition 8.5.1 we have

$$\begin{aligned}
\frac{\partial}{\partial\nu}m(\lambda,\nu) &= \frac{1}{2}\frac{\partial r}{\partial\nu}\left(1+\frac{1}{16\pi}\int_{S(\lambda,\nu)}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\right)+\frac{r}{32\pi}\frac{\partial}{\partial\nu}\int_{S(\lambda,\nu)}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} \\
&= \frac{\partial r}{\partial\nu}\frac{1}{8\pi}\int_{S(\lambda,\nu)}\underline{\mu}+\frac{r}{32\pi}\frac{\partial}{\partial\nu}\int_{S(\lambda,\nu)}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} \\
&= \frac{\overline{\Omega\mathrm{tr}\chi}}{2}\frac{r}{8\pi}\int_{S(\lambda,\nu)}\underline{\mu}+\frac{r}{32\pi}\int_{S(\lambda,\nu)}\left(\frac{\partial}{\partial\nu}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi})+\Omega\mathrm{tr}\chi^2\mathrm{tr}\underline{\chi}\right) \\
&= \frac{r}{16\pi}\int_{S(\lambda,\nu)}\left(\overline{\Omega\mathrm{tr}\chi}\underline{\mu}+\frac{1}{2}\Omega\mathrm{tr}\chi^2\mathrm{tr}\underline{\chi}+\frac{1}{2}\frac{\partial}{\partial\nu}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi})\right) \qquad (8.5.5)
\end{aligned}$$

From the structure equations 3.1.45, it is easy to derive

$$\begin{aligned}
\frac{\partial}{\partial\nu}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} &= -\Omega(\mathrm{tr}\chi)^2\mathrm{tr}\underline{\chi}-\Omega|\hat{\chi}|^2\mathrm{tr}\underline{\chi}+2\Omega\mathrm{tr}\chi|\eta|^2+2\Omega\mathrm{tr}\chi\left(\mathrm{d\!\!/iv}\,\underline{\eta}-\frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}}+\rho\right) \\
&= -2\Omega\mathrm{tr}\chi\underline{\mu}-\Omega(\mathrm{tr}\chi)^2\mathrm{tr}\underline{\chi}-\Omega|\hat{\chi}|^2\mathrm{tr}\underline{\chi}+2\Omega\mathrm{tr}\chi|\eta|^2 \qquad (8.5.6)
\end{aligned}$$

Plugging this relation in 8.5.5 we obtain

$$\frac{\partial}{\partial\nu}m(\lambda,\nu)=\frac{r}{16\pi}\int_{S(\lambda,\nu)}\left[(\overline{\Omega\mathrm{tr}\chi}-\Omega\mathrm{tr}\chi)\underline{\mu}+\Omega\mathrm{tr}\chi|\eta|^2-\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}|\hat{\chi}|^2\right] \qquad (8.5.7)$$

and due to the estimates 8.0.1 and 8.0.2 the right hand side of 8.5.7 is $O(r^{-2})$, proving the proposition.

From the expression of the Hawking mass in equation 8.5.3 and the existence of the null outgoing infinity limit of $r^2\hat{\chi}$, $r\underline{\hat{\chi}}$ and $r^3\rho$ proved in propositions 8.2.1 and 8.3.2 it follows immediately that $m(\lambda,\nu)$ has a limit as $\nu\to\infty$, uniform in λ. We can therefore introduce the Bondi mass [10], see [**?**], in the following way,

Definition 8.5.2 *The Bondi mass relative to the null outgoing hypersurface $C(\lambda)$ is*

$$M_B(\lambda)=\lim_{\nu\to\infty}m(\lambda,\nu) \qquad (8.5.8)$$

As a corollary to Proposition 8.5.1 we have

Corollary 8.5.1 *On any $C(\lambda)$ the following relation holds*

$$m(\lambda,\nu)=M_B(\lambda)+O(r^{-1}) \qquad (8.5.9)$$

[10]For a discussion about the Bondi mass see [Wa2], Chapter 11.

Proof: It follows immediately by integrating the right hand side of 8.5.4.

Observe that, on any $\underline{C}(\nu)$, λ varies in the interval $[\lambda_0(\nu), \lambda_0]$, where $\lambda_0(\nu) = u|_{\underline{C}(\nu)\cap\Sigma_0}$. As $\lim_{\nu\to\infty}\lambda_0(\nu) = -\infty$, $M_B(\lambda)$ is defined in the interval $(-\infty, \lambda_0]$ and we have the following proposition

Theorem 8.5.2 *The Bondi mass has the following limit*

$$\lim_{\lambda\to-\infty} M_B(\lambda) = M \tag{8.5.10}$$

where M, defined in the global initial data conditions, see Definition 3.6.1, is the ADM energy on Σ_0.

Proof: From equation 8.5.3, and the definition of the Bondi mass it follows immediately that

$$M_B(\lambda) = \frac{1}{8\pi}\int_{S^2}(X\cdot\underline{X} - P)(\lambda,\cdot) \tag{8.5.11}$$

where the integration is relative to the standard volume element of S^2. The asymptotic behaviour in λ of $X(\lambda,\cdot)$ and $\underline{X}(\lambda,\cdot)$, proved in Proposition 8.3.2, implies that

$$M_B(\lambda) = -\frac{1}{8\pi}\int_{S^2}P(\lambda,\cdot) + O\left(\frac{1}{\lambda^2}\right) \tag{8.5.12}$$

and

$$M_B(-\infty) = -\frac{1}{2}\lim_{\lambda\to-\infty}\overline{P}(\lambda) = -\frac{1}{2}\lim_{\lambda\to-\infty}\left(\lim_{\nu\to\infty}(r^3\overline{\rho})(\lambda,\nu)\right) \tag{8.5.13}$$

We express $(r^3\overline{\rho})(\lambda,\nu)$ using its evolution equation along $\underline{C}(\nu)$, see subsection 5.1.6,

$$\begin{aligned}(r^3\overline{\rho})(\lambda,\nu) &= (r^3\overline{\rho})(\lambda_0(\nu),\nu) + \frac{1}{8\pi}\int_{\lambda_0(\nu)}^{\lambda}\frac{1}{8\pi}\int_{S(\lambda,\nu)} r(\overline{\Omega tr\underline{\chi}} - \Omega\mathrm{tr}\underline{\chi})(\rho-\overline{\rho})\\ &-\frac{1}{4\pi}\int_{\lambda_0(\nu)}^{\lambda}\int_{S(\lambda,\nu)}\Omega r\left((\frac{3}{2}\eta - \frac{1}{2}\underline{\eta})\cdot\underline{\beta} + \frac{1}{2}\hat{\chi}\cdot\underline{\alpha} - \zeta\cdot\underline{\beta}\right)\end{aligned} \tag{8.5.14}$$

and recalling Propositions 8.2.1 and 8.3.2, we can write

$$\begin{aligned}\overline{P}(\lambda) &= \lim_{\nu\to\infty}(r^3\overline{\rho})(\lambda_0(\nu),\nu) - \frac{1}{8\pi}\int_{-\infty}^{\lambda}\int_{S^2}X\underline{A}(\lambda,\cdot)\\ &= \lim_{\nu\to\infty}(r^3\overline{\rho})(\lambda_0(\nu),\nu) + O\left(\frac{1}{\lambda^2}\right)\end{aligned} \tag{8.5.15}$$

where the last integral in 8.5.15 has been estimated observing that $X = O\left(\lambda^{-\frac{1}{2}}\right)$ and $\underline{A} = O\left(\lambda^{-\frac{5}{2}}\right)$. Computing expicitely the asymptotic expression of ρ on Σ_0 we obtain, see subsection 7.1.3,

$$M_B(-\infty) = -\frac{1}{2}\lim_{\nu\to\infty}(r^3\overline{\rho})(\lambda_0(\nu),\nu) = E_{ADM} = M \tag{8.5.16}$$

proving the proposition.

We are now ready to give a rigorous derivation of the *Bondi mass formula*

Theorem 8.5.3 *The following equation is satisfied, in the null outgoing infinity limit,*

$$\frac{\partial M_B(\lambda)}{\partial\lambda} = -\frac{1}{32\pi}\int_{S^2}|\underline{X}(\lambda,\cdot)|^2 \tag{8.5.17}$$

Proof: To prove equation 8.5.17 we first differentiate with respect to λ the Hawking mass $m(\lambda,\nu)$ and subsequently take the limit $\nu\to\infty$. Proceeding as in the derivation of 8.5.5

$$\begin{aligned}\frac{\partial}{\partial\lambda}m(\lambda,\nu) &= \frac{\partial r}{\partial\lambda}\frac{1}{8\pi}\int_{S(\lambda,\nu)}\underline{\mu} + \frac{r}{32\pi}\frac{\partial}{\partial\lambda}\int_{S(\lambda,\nu)}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} \qquad (8.5.18)\\ &= \frac{r}{16\pi}\int_{S(\lambda,\nu)}\left(\overline{\Omega\mathrm{tr}\underline{\chi}}\,\underline{\mu} + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}^2\mathrm{tr}\chi + \frac{1}{2}\frac{\partial}{\partial\lambda}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi})\right)\end{aligned}$$

From the structure equations 3.1.45, it is easy to obtain

$$\begin{aligned}\frac{\partial}{\partial\lambda}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} &= -\Omega(\mathrm{tr}\underline{\chi})^2\mathrm{tr}\chi - \Omega|\hat{\underline{\chi}}|^2\mathrm{tr}\chi + 2\Omega\mathrm{tr}\underline{\chi}|\eta|^2 + 2\Omega\mathrm{tr}\underline{\chi}\left(\mathrm{d\!\!/iv}\,\eta - \frac{1}{2}\hat{\underline{\chi}}\cdot\hat{\chi} + \rho\right)\\ &= -2\Omega\mathrm{tr}\underline{\chi}\mu - \Omega(\mathrm{tr}\underline{\chi})^2\mathrm{tr}\chi - \Omega|\hat{\underline{\chi}}|^2\mathrm{tr}\chi + 2\Omega\mathrm{tr}\chi|\eta|^2 \qquad (8.5.19)\end{aligned}$$

and equation 8.5.18 can be rewritten as

$$\frac{\partial}{\partial\lambda}m(\lambda,\nu) = \frac{r}{16\pi}\int_{S(\lambda,\nu)}\left(\overline{\Omega\mathrm{tr}\underline{\chi}}\,\underline{\mu} - \Omega\mathrm{tr}\underline{\chi}\mu - \frac{1}{2}\Omega\mathrm{tr}\chi|\hat{\underline{\chi}}|^2 + \Omega\mathrm{tr}\chi|\eta|^2\right) \tag{8.5.20}$$

Using the previous results we see that the integrand of 8.5.20 admits a limit for $\nu\to\infty$, uniform in λ. Moreover the only term in the right hand side not converging to zero is $-(32\pi)^{-1}r\int_{S(\lambda,\nu)}\Omega\mathrm{tr}\chi|\hat{\underline{\chi}}|^2$. Thus we conclude

$$\frac{\partial}{\partial\lambda}M_B(\lambda) = -\frac{1}{32\pi}\int_{S^2}|\underline{X}(\lambda,\cdot)|^2$$

proving our result.

*****************************3/3/2002***************************

add some physical comment

Using equation 8.5.15 we can complete Proposition 8.2.1 proving the following lemma

Lemma 8.5.1 *The following limits hold*

$$\lim_{\lambda\to\lambda_0} \overline{P} = ? \quad , \quad \lim_{\lambda\to\lambda_0} \overline{Q} = 0 \tag{8.5.21}$$

Proof:

8.6 Asymptotic behaviour of null outgoing hypersurfaces

In this section we recover **conclusion 17.0.6** of [Ch-Kl]. We want to show that, as $\nu \to \infty$, the null outgoing hypersurfaces $C(\lambda)$ approach the null outgoing cones of the Schwarzschild spacetime with ADM mass $M = M_B(-\infty)$. In particular we show that they diverge logarithmically from the standard position of outgoing null cones in Minkowski spacetime.

Proposition 8.6.1 *On any null outgoing hypersurface $C(\lambda)$ the following relation holds*

$$\frac{dr}{dt} = -2M\frac{1}{r} + O\left(\frac{1}{r^2}\right) \tag{8.6.1}$$

Proof: We first recall the definition of the time function in the spacetime $\mathcal{K}$, see Proposition 3.3.1,

$$t(\lambda,\nu) = \frac{1}{2}(\lambda+\nu) \ .$$

Recall also that $r = r(\lambda,\nu)$ is defined by the formula

$$r(\lambda,\nu) = (4\pi)^{-\frac{1}{2}}|S(\lambda,\nu)|^{\frac{1}{2}} \ .$$

Computing $\frac{d}{dt}r$ on a null hypersurface $C(\lambda)$ we obtain, see 4.1.30,

$$\frac{d}{dt}r|_{C(\lambda)} = 2\frac{\partial}{\partial\nu}r = r\overline{\Omega\mathrm{tr}\chi} = 1 + r(\overline{\Omega\mathrm{tr}\chi} - \frac{1}{r}) \tag{8.6.2}$$

To obtain an explicit relation between $r(\overline{\Omega\mathrm{tr}\chi} - \frac{1}{r})$ and the Bondi mass, we express this quantity as an integral along the null incoming hypersurface $\underline{C}(\nu)$,

$$\frac{1}{4\pi r(\lambda,\nu)}\int_{S(\lambda,\nu)}(\Omega\mathrm{tr}\chi - \frac{1}{r}) = \frac{1}{4\pi r(\lambda_0(\nu),\nu)}\int_{S(\lambda_0(\nu),\nu)}(\Omega\mathrm{tr}\chi - \frac{1}{r})$$
$$+\frac{1}{4\pi}\int_{\lambda_0(\nu)}^{\lambda}\frac{\partial}{\partial\lambda}\left(\frac{1}{r}\int_S(\Omega\mathrm{tr}\chi - \frac{1}{r})\right)(\lambda',\nu) \tag{8.6.3}$$

Using Lemma 3.1.3 we have

$$\frac{1}{4\pi}\int_{\lambda_0(\nu)}^{\lambda}\frac{\partial}{\partial\lambda}\left(\frac{1}{r}\int_S(\Omega\mathrm{tr}\chi-\frac{1}{r})\right)=\frac{1}{4\pi}\int_{\lambda_0(\nu)}^{\lambda}\left\{-\frac{1}{r^2}\left(\frac{\partial}{\partial\lambda}r\right)\int_{S(\lambda,\nu)}(\Omega\mathrm{tr}\chi-\frac{1}{r})\right.$$
$$\left.+\frac{1}{r}\int_{S(\lambda,\nu)}\left(\frac{\partial}{\partial\lambda}(\Omega\mathrm{tr}\chi-\frac{1}{r})+\Omega\mathrm{tr}\underline{\chi}(\Omega\mathrm{tr}\chi-\frac{1}{r})\right)\right\}$$
$$=\frac{1}{4\pi}\int_{\lambda_0(\nu)}^{\lambda}\left\{\frac{1}{r}\int_{S(\lambda,\nu)}\left[\frac{\partial}{\partial\lambda}(\Omega\mathrm{tr}\chi-\frac{1}{r})+\frac{\Omega\mathrm{tr}\underline{\chi}}{2}(\Omega\mathrm{tr}\chi-\frac{1}{r})\right]\right.$$
$$\left.+\frac{1}{2r}\int_{S(\lambda,\nu)}(\Omega\mathrm{tr}\underline{\chi}-\overline{(\Omega\mathrm{tr}\underline{\chi})})(\Omega\mathrm{tr}\chi-\frac{1}{r})\right\} \tag{8.6.4}$$
$$=\frac{1}{4\pi}\int_{\lambda_0(\nu)}^{\lambda}\frac{1}{r}\int_{S(\lambda,\nu)}\left[\frac{\partial}{\partial\lambda}(\Omega\mathrm{tr}\chi-\frac{1}{r})+\frac{\Omega\mathrm{tr}\underline{\chi}}{2}(\Omega\mathrm{tr}\chi-\frac{1}{r})\right]+O\left(\frac{1}{r^3(\lambda,\nu)}\right)$$

where the estimate of the last term uses the boundedness of $r^2|\Omega\mathrm{tr}\underline{\chi}-\overline{(\Omega\mathrm{tr}\underline{\chi})}|$ and of $r^2|(\Omega\mathrm{tr}\chi-\frac{1}{r})|$ implicit in the bounds for the $\mathcal{O}$ norms proved in Theorem M1. Using the structure equations 3.1.45 we calculate

$$\left[\frac{\partial}{\partial\lambda}(\Omega\mathrm{tr}\chi-\frac{1}{r})+\frac{\Omega\mathrm{tr}\underline{\chi}}{2}(\Omega\mathrm{tr}\chi-\frac{1}{r})\right]=\Omega\left[\left(-\hat{\underline{\chi}}\cdot\hat{\chi}+2\rho\right)+2\mathrm{d}\!\!/\mathrm{iv}\,\zeta+2\triangle\!\!\!/\log\Omega\right.$$
$$\left.+2|\zeta|^2+4\zeta\cdot\nabla\!\!\!/\log\Omega+2|\nabla\!\!\!/\log\Omega|^2\right] \tag{8.6.5}$$

Using once more the estimates for the connection coefficients implicit in the bounds for the $\mathcal{O}$ norms provided by the *Main Theorem*, we write

$$\frac{\partial}{\partial\lambda}\left(\frac{1}{r}\int_{S(\lambda,\nu)}(\Omega\mathrm{tr}\chi-\frac{1}{r})\right)=\frac{2}{r}\int_{S(\lambda,\nu)}\left(-\frac{1}{2}\hat{\underline{\chi}}\cdot\hat{\chi}+\rho\right)+O\left(\frac{1}{r^3}\right) \tag{8.6.6}$$

Therefore, from 8.6.4 and using 8.5.3 we have,

$$\frac{1}{4\pi}\int_{\lambda_0(\nu)}^{\lambda}\frac{\partial}{\partial\lambda}\left(\frac{1}{r}\int_{S(\lambda,\nu)}(\Omega\mathrm{tr}\chi-\frac{1}{r})\right)=2\int_{\lambda_0(\nu)}^{\lambda}\frac{1}{r^2}\left(-\frac{r}{4\pi}\int_{S(\lambda',\nu)}\left(\frac{1}{2}\hat{\underline{\chi}}\cdot\hat{\chi}-\rho\right)\right)$$
$$=-2\int_{\lambda_0(\nu)}^{\lambda}\frac{1}{r^2(\lambda',\nu)}m(\lambda',\nu)+O\left(\frac{1}{r^2}\right) \tag{8.6.7}$$

Recalling that from Corollary 8.5.1, $m(\lambda,\nu)=M_B(\lambda)+O(r^{-1})$, we write

$$\frac{d}{dt}r=\frac{1}{4\pi r(\lambda_0(\nu),\nu)}\int_{S(\lambda_0(\nu),\nu)}(\Omega\mathrm{tr}\chi-\frac{1}{r})+\frac{1}{4\pi}\int_{\lambda_0(\nu)}^{\lambda}\frac{\partial}{\partial\lambda}\left(\frac{1}{r}\int_S(\Omega\mathrm{tr}\chi-\frac{1}{r})\right)$$
$$=\frac{1}{4\pi r(\lambda_0(\nu),\nu)}\int_{S(\lambda_0(\nu),\nu)}(\Omega\mathrm{tr}\chi-\frac{1}{r})-2\int_{\lambda_0(\nu)}^{\lambda}\frac{1}{r^2(\lambda',\nu)}M_B(\lambda')+O\left(\frac{1}{r^2}\right) \tag{8.6.8}$$

Using the Bondi masss formula, see **??**, we write

$$
\begin{aligned}
&-2\int_{\lambda_0(\nu)}^{\lambda}\frac{1}{r^2(\lambda',\nu)}M_B(\lambda') = -2\int_{\lambda_0(\nu)}^{\lambda}\frac{1}{r^2(\lambda',\nu)}\left(M_B(-\infty)+\int_{-\infty}^{\lambda'}\frac{\partial M_B}{\partial\lambda}(\lambda'')\right)\\
&= -2\int_{\lambda_0(\nu)}^{\lambda}\frac{1}{r^2(\lambda',\nu)}\left(M_B(-\infty)-\frac{1}{32\pi}\int_{-\infty}^{\lambda'}\int_{S^2}|\underline{X}(\lambda'',\cdot)|^2\right)\\
&= -2\int_{\lambda_0(\nu)}^{\lambda}\frac{1}{r^2(\lambda',\nu)}M_B(-\infty)+O\left(\frac{1}{r^2}\right) \qquad (8.6.9)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{d}{dt}r = \frac{1}{4\pi r(\lambda_0(\nu),\nu)}\int_{S(\lambda_0(\nu),\nu)}(\Omega\mathrm{tr}\chi-\frac{1}{r}) - 2M_B(-\infty)\int_{\lambda_0(\nu)}^{\lambda}\frac{1}{r^2(\lambda',\nu)}d\lambda'+O\left(\frac{1}{r^2}\right)\\
&= \frac{1}{4\pi r(\lambda_0(\nu),\nu)}\int_{S(\lambda_0(\nu),\nu)}(\Omega\mathrm{tr}\chi-\frac{1}{r}) - 2M_B(-\infty)\left(\frac{1}{r(\lambda,\nu)}-\frac{1}{r(\lambda_0(\nu),\nu)}\right)+O\left(\frac{1}{r^2}\right)
\end{aligned}
$$

where we have used the relation $d\lambda = -(1+\frac{c\varepsilon}{r})dr$, which follows from Lemma 4.1.8. To complete the proof we consider the first term in the last line of 8.6.8,

$$
\begin{aligned}
&\frac{1}{4\pi r(\lambda_0(\nu),\nu)}\int_{S(\lambda_0(\nu),\nu)}(\Omega\mathrm{tr}\chi-\frac{1}{r}) \qquad (8.6.10)\\
&= \int_{-\infty}^{\lambda_0(\nu)}\frac{\partial}{\partial\lambda}\left(\frac{1}{4\pi r(\lambda,\nu_0(\lambda))}\int_{S(\lambda,\nu_0(\lambda))}(a\mathrm{tr}\theta-\frac{1}{r})\right)d\lambda+O\left(\frac{1}{r^2}\right)
\end{aligned}
$$

where the $O\left(\frac{1}{r^2}\right)$ term originates from the integration of the terms due to $(\Omega\mathrm{tr}\chi-a\mathrm{tr}\theta)$, see subsections 3.3.1 and **??**. Recall that

$$\lambda = u|_{\Sigma_0\cap C(\lambda)} \quad , \quad \nu_0(\lambda)=\underline{u}|_{\Sigma_0\cap C(\lambda)} \ .$$

Repeating the computation done in 8.6.4 with $\tilde{N}$ the unit vector field along Σ_0 normal to the canonical foliation $\{S_0(\nu)\}$ and taking into account equation 7.2.6,

$$\nabla_{\tilde{N}}\mathrm{tr}\theta+\frac{1}{2}(\mathrm{tr}\theta)^2 = -\overline{\rho}+\left[-|\nabla\!\!\!/ \log a|^2-|\hat{\theta}|^2+g(k)\right]$$

and $\nabla_{\tilde{N}}r = \frac{\overline{a\mathrm{tr}\theta}}{2}$ as well as the estimates of the norms $\mathcal{O}(\Sigma_0\backslash K)$ we write **there is a factor between $\frac{\partial}{\partial\lambda}$ and $\tilde{N}$ to check!!!**

$$\frac{\partial}{\partial\lambda}\left(\frac{1}{4\pi r(\lambda,\nu_0(\lambda))}\int_{S(\lambda,\nu_0(\lambda))}(a\mathrm{tr}\theta-\frac{1}{r})\right) = -\tilde{N}\left(\frac{1}{4\pi r(\lambda,\nu_0(\lambda))}\int_{S(\lambda,\nu_0(\lambda))}(a\mathrm{tr}\theta-\frac{1}{r})\right)$$

$$= \frac{1}{4\pi r^2}(\nabla_{\tilde N} r)\int_{S(\lambda,\nu_0(\lambda))}(a\mathrm{tr}\theta - \frac{1}{r}) - \frac{1}{4\pi r}\int_{S(\lambda,\nu_0(\lambda))}\left(\nabla_{\tilde N}(a\mathrm{tr}\theta - \frac{1}{r}) + a\mathrm{tr}\theta(a\mathrm{tr}\theta - \frac{1}{r})\right)$$
$$= -\frac{1}{4\pi r}\int_{S(\lambda,\nu_0(\lambda))}\left[\nabla_{\tilde N}(a\mathrm{tr}\theta - \frac{1}{r}) + \frac{a\mathrm{tr}\theta}{2}(a\mathrm{tr}\theta - \frac{1}{r})\right] + \frac{1}{8\pi r}\int_{S(\lambda,\nu_0(\lambda))}(a\mathrm{tr}\theta - \overline{(a\mathrm{tr}\theta)})(a\mathrm{tr}\theta - \frac{1}{r})$$
$$= -\frac{1}{4\pi r}\int_{S(\lambda,\nu_0(\lambda))}\left[\nabla_{\tilde N}(a\mathrm{tr}\theta - \frac{1}{r}) + \frac{a\mathrm{tr}\theta}{2}(a\mathrm{tr}\theta - \frac{1}{r})\right] + O\left(\frac{1}{r^3(\lambda,\nu_0(\lambda))}\right)$$
$$= \frac{1}{4\pi r}\int_{S(\lambda,\nu_0(\lambda))}\overline{\rho} + O\left(\frac{1}{r^3(\lambda,\nu_0(\lambda))}\right) = r(\lambda,\nu_0(\lambda))\overline{\rho} + O\left(\frac{1}{r^3(\lambda,\nu_0(\lambda))}\right) \tag{8.6.11}$$

Plugging this result in 8.6.10 we obtain easily, recalling the *Global initial data conditions*, see definition 3.6.1, taking into account the equation, see ...,

$$\overline{\rho} = ...$$

$$\frac{1}{4\pi r(\lambda_0(\nu),\nu)}\int_{S(\lambda_0(\nu),\nu)}(\Omega\mathrm{tr}\chi - \frac{1}{r}) = \int_{-\infty}^{\lambda_0(\nu)}\left(\frac{1}{r^2}r^3\overline{\rho}\right)d\lambda + O\left(\frac{1}{r^2}\right)$$
$$= -\frac{2}{r(\lambda_0(\nu),\nu)}M + O\left(\frac{1}{r^2}\right) \tag{8.6.12}$$

The constant M above is the ADM mass associated to the initial data and concide, as proved in Proposition 8.5.2, with the Bondi mass for $\lambda \to -\infty$, $M = M_B(-\infty)$. Using this relation, equation 8.6.8 can be written as

$$\frac{d}{dt}r(\lambda,\nu) = -2M\frac{1}{r(\lambda,\nu)} + O\left(\frac{1}{r^2(\lambda,\nu)}\right) \tag{8.6.13}$$

completing the proof of Proposition 8.6.1.

Remarks:

Bibliography

[Ad] R.E. Adams *"Sobolev Spaces"*. NewYork: Academic.(1975).

[An-Mon] L.Andersson, V.Moncrief, *"On the global evolution problem in 3+1 gravity"*. J.Geom.Phys., under preparation.

[Ar-De-Mi] R.Arnowitt, S.Deser, C.Misner, *"Coordinate invariance and energy expressions in general relativity"*. Phys.Rev. 122, (1961), 997-1006.

[Ba] R.Bartnik, *"Existence of maximal surfaces in asymptotically flat space-times"*. Comm.Math.Phys. 94, (1984), 155-175.

[Bel] L.Bel, *"Introduction d'un tenseur du quatrieme ordre"*. C.R.Acad.Sci.Paris 247, (1959), 1094-1096.

[Bl-D] L.Blanchet, T.Damour, *"Hereditary effects in gravitational radiation"*. Phys.Rev. D46, (1992), 4304-4319.

[Br1] Y.Choquet-Bruhat, *"Théorème d' existence pour certain systèmes d'équations aux dériveés partielles nonlinéaires"*. Acta Matematica 88, (1952), 141-225.

[Br2] Y.Choquet-Bruhat, *"Solutions C^∞ d'équations hyperboliques non linéaires"*. C.R.Acad.Sci.Paris 272, (1968), 386-388.

[Br3] Y.Choquet-Bruhat, *"Un théorème d'instabilité pour certain équations hyperboliques nonlinéaires"*. C.R.Acad.Sci.Paris 276A, (1973), 281.

[Br-Ch2] Y.Choquet-Bruhat, D.Christodoulou, *"Elliptic systems in $H_{s,\delta}$ spaces on manifolds which are euclidean at infinity"* Acta Math. 145, (1981), 129-150.

[Br-Fo] Y.Choquet-Bruhat, Y.Fourès, *"Un théorème d'existence pour certain systèmes d'équations aux derivées partielles non linéaires"*. Acta Math. 88, (1952), 141-225.

[Br-Ge] Y.Choquet-Bruhat, R.P.Geroch, *"Global aspects of the Cauchy problem in General Relativity"*. Comm.Math.Phys. 14, (1969), 329-335.

[Br-Y] Y.Choquet-Bruhat, J.York, *"The Cauchy Problem"* in Held, A, ed. General Relativity and Gravitation volume 1, 99-172, Plenum , N.Y. (1980).

[Ch1] D.Christodoulou, *"Solutions globales des equations de champ de Yang-Mills"* C.R.Acad.Sci.Paris 293, series A, (1981), 39-42.

[Ch2] D.Christodoulou, *"Global solutions for nonlinear hyperbolic equations for small data"*. Comm.Pure appl.Math.39, (1986), 267-282.

[Ch3] D.Christodoulou, *"The formation of black holes and singularities in spherically symmetric gravitational collapse"* Comm. Pure and Appl. Math. XLIV, (1991),339-373.

[Ch4] D.Christodoulou, *"Examples of naked singularity formation in the gravitational collapse of a scalar field"* Ann.Math. 140, (1994), 607-653.

[Ch5] D.Christodoulou, *"The instability of naked singularity formation in the gravitational collapse of a scalar field"* Ann.Math. 149, (1999), 149-183.

[Ch6] D.Christodoulou, *"On the global inital value problem and the issue of singulairties"* Classical Quantum Gravity 13, (1999), A23-A35.

[Ch7] D.Christodoulou, *"The Stability of Minkowski Spacetime"* Proceedings of the 'international Congress of Mathematicians Kyoto 1990, (1990), 1114-1121.

[Ch-Kl1] D.Christodoulou, S.Klainerman, *"Asymptotic properties of linear field equations in Minkowski space"*. Comm.Pure Appl.Math. XLIII,(1990), 137-199.

[Ch-Kl] D.Christodoulou, S.Klainerman, *"The global non linear stability of the Minkowski space"*. Princeton Mathematical series, 41 (1993).

[Ch-Mu] D.Christodoulou, N.O' Murchadha, *"The boost problem in General Relativity"*. Comm.Math.Phys. 80, (1981), 271-300.

[Cha] I.Chavel *"Riemannian Geometry: A modern introduction"*. Cambridge University Press, 108(1993).

[Chr] P.T.Chrúsciel, *"On the uniqueness in the large of solutions to the Einstein equations ("strong cosmic censorship")"*. Mathematical aspects of classical field theory (Seattle,WA,1991),Amer.Math.Soc., Providence,RI,(1992), 235-273.

[Ea-Mon] D.Eardley, V.Moncrief, *"The global existence problem and cosmic censorship in genneral relativity"*. Gen.Rel.Gravit. 13, (1981), 887-892.

[F-Ms1] A.Fisher, J.E.Marsden, *"The Einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system.I"*. Comm.Math.Phys. 28, (1972), 1-38.

[F-Ms2] A.Fisher, J.E.Marsden, *"General relativity, partial differential equations and dynamical systems"*. AMS Proc.Symp.Pure Math. 23, (1973), 309-327.

[Fr1] H.Friedrich, *"Cauchy problems for the conformal vacuum field equations in General Relativity"*. Comm.Math.Phys. 91, (1983), 445-472.

[Fr2] H.Friedrich, *"Existence and structure of past asymptotically simple solutions of Einstein's field equations with positive cosmological constant"*. J.Geom.Phys. 3, (1986), 101-117.

[Fr3] H.Friedrich, *"On the global existence and the asymptotic behavior of solutions to the Einstein-Maxwell-Yang-Mills equations"*. J.Differential Geom. 34, no.2, (1991), 275-345.

[Fr4] H.Friedrich, *" Hyperbolic reductions for Einstein equations"* Classical Quantum Gravity 13, (1996), 1451-1469.

[Fr-Re] H.Friedrich, A.Rendall *" The Cauchy problem for the Einstein equations"* arXiv:gr-qc/0002074 (22 Feb 2000).

[Ge] R.P.Geroch, *"The domain of dependence"*. J.Math.Phys. 11, (1970), 437-439.

[Haw-El] S.W.Hawking, G.F.R.Hellis "The Large Scale Structure of Spacetime" *Cambridge Monographs on Mathematical Physics*, 1973.

[Ho] L.Hörmander, *"Lectures on Nonlinear Hyperbolic Equations"* Mathematics and Applications 26, Springer-Verlag (1987).

[Hu-Ka-Ms] T.J.R.Hughes, T.Kato, J.E.Marsden, *"Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity"*. Arch.Rational Mech.Anal. 63, (1976), no.3, 273-294.

[John1] F.John, *"Formation of Singularities in Elastic Waves"*. Lecture Notes in Physics Berlin: Springer-Verlag, (1984), pp. 190-214.

[John2] F.John, *"Nonlinear wave equations, formation of singularities"*. Amer. Math. Soc. Providence, (1990).

[Kl1] S. Klainerman, *"Long time behavior of solutions to nonlinear wave equations"*, Proceedings of the Intern. Congress of Mathematicians, Warsaw, (1982)

[Kl2] S.Klainerman, *"The null condition and global existence to nonlinear wave equations"*. Lect. Appl. Math. 23, (1986), 293-326.

[Kl3] S.Klainerman, *"Remarks on the global Sobolev inequalities in Minkowski Space"*. Comm.Pure appl.Math. 40, (1987), 111-117.

[Kl4] S.Klainerman, *"Uniform decay estimates and the Lorentz invariance of the classical wave equation"*. Comm.Pure appl.Math. 38, (1985), 321-332.

[Kl5] S.Klainerman, *A commuting vectorfield approach to Strichartz type inequalities and applications to quasilinear wave equations. IMRN* to appear.

[Kl-Ni] S.Klainerman, F.Nicolò "On local and global aspects of the Cauchy problem in General Relativity" Classical Quantum Gravity16,(1999), R73-R157.

[Le] J.Leray, *"Lectures on Hyperbolic Equations"*. Institute for Advanced Study, Notes (1953).

[Ne-Pe1] E.T.Newman, R.Penrose, *"New conservation laws for zero rest-mass fields in asymptotically flat space-time"*. J.Math.Phys. 3, (1962), 566.

[Ne-Pe2] E.T.Newman, R.Penrose, *"New conservation laws for zero rest-mass fields in asymptotically flat space-time"*. Proc.Roy.Soc.Lond. A305, (1968), 175-204.

[Ni] F.Nicolò "Canonical foliation on a null hypersurface". To appear.

[Pe1] R.Penrose, *"Conformal treatment of infinity"* Relativity, Groups, and Topology B. deWitt and C. deWitt, (eds), Gordon and Breach, (1963).

[Pe2] R.Penrose, *"Zero rest mass fields including gravitation: asymptotic behavior"*. Proc.Roy.Soc.Lond. A284, (1962), 159-203.

[Pe3] R.Penrose, *"Gravitational collapse and spacetime singularities"*. Phys.Rev.Lett. 14, (1965), 57-59.

[Ren] A.D.Rendall, *"Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations"*. Proc.Roy.Soc.Lond. A427, (1990), 221-239.

[Se] H.Seifert, *"Kausal Lorentzraume"*. Doctoral thesis (1968).

[Sc-Yau1] R.Schoen, S.T.Yau, *"Proof of the positive mass theorem I"*. Comm.Math.Phys. 65, (1979), 45-76.

[Sc-Yau2] R.Schoen, S.T.Yau, *"Proof of the positive mass theorem II"*. Comm.Math.Phys. 79, (1981), 231-260.

[Sp] M.Spivak, *"A comprehensive introduction to Differential Geometry"*. Wilmington: Publish or Perish, Inc. (1970).

[Wa1] R.Wald, *"Gravitational collapse and cosmic censorship"*. (1997), gr-qc/9710068.

[Wa2] R.Wald, *"General Relativity"*. University of Chicago Press. (1984).

[W] T.Wolff, *"Recent work connected with the Kakeya problem"*. Prospects in Mathematics. AMS (1998), Editor H. Rossi.