Infinite Dimensional Analysis, Quantum Probability and Related Topics

THE STABILITY OF QUANTUM MARKOV FILTERS

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When are quantum filters asymptotically independent of the initial state? We show that this is the case for absolutely continuous initial states when the quantum stochastic model satisfies an observability condition. When the initial system is finite dimensional, this condition can be verified explicitly in terms of a rank condition on the coefficients of the associated quantum stochastic differential equation.

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1. Introduction

It is almost a tautology that laboratory measurements give rise to classical stochastic processes. For example, in quantum optics one usually detects, using a configuration of photodetectors, the light of a laser which is scattered off a cloud of atoms, and the resulting photocurrent is a classical stochastic process[2,4]. It is subsequently of interest to infer as well as possible the state of the atoms from the observed photocurrent, which is the purpose of quantum filtering theory. This theory has been extensively investigated both in the mathematical literature[3] (see Ref. 4 for a recent review) and in the physics literature, where it is known under the name of quantum trajectory theory or the theory of stochastic master equations[11].

In order to implement the quantum filter, however, the underlying quantum model is presumed to be known. It is not evident, *a priori*, that good estimates will be obtained in the presence of modelling errors which are inevitable in practice. Questions of robustness to modelling errors are particularly subtle on a long time interval, and have received much attention in the classical nonlinear filtering literature, see, e.g., Ref. 8 and the references therein. In particular, asymptotic stability of the filter—the independence of the filter, after a long time interval, of the initial estimate of the system—has been shown to hold in a wide range of classical nonlinear filtering models, and is the starting point for more general robustness questions.

The problem of asymptotic stability is related to the consistency of Bayes estimates and is of significant practical interest as it ensures optimal performance of the filter, after an initial transient, even under misspecification of the initial condition. To date, however, no such result is known in quantum filtering theory.

The goal of this paper is to develop a criterion which ensures asymptotic stability of quantum filters. This *observability* condition for stability is a natural one: it is the requirement that no two different initial states of the model give rise to an observation process with the same law. In the quantum optics example described above, this means that we must be able to determine precisely the initial state of the atoms if we have access to the full statistics of the photocurrent over the infinite time interval. If this is the case, then the filtered estimates of the atomic observables are insensitive to the initial state of the atoms after a long time interval, provided we restrict our attention to initial states that satisfy an absolute continuity condition.

The basic method of proof is based on the classical counterpart of this result, which has recently been developed by the author[21]. To extend this result to the quantum setting, it is most natural to work within an abstract quantum filtering setting which is a little more general than the usual setting in the quantum filtering literature[3,4]. We set up the problem in section 2 in the context of C^*-algebraic Markov process theory in the spirit of Accardi, Frigerio and Lewis[1]. The proof of the main result can be found in section 3. In section 4 we elaborate on the absolute continuity condition required by our main result, and provide a simpler sufficient condition. In the last section 5 we investigate a class of quantum filtering models, defined through the solution of a Hudson-Parthasarathy type quantum stochastic differential equation with a finite dimensional initial system, which have important applications, e.g., in quantum optics. In this setting one may find explicitly computable rank conditions for the model to be observable in terms of the coefficients of the quantum stochastic differential equation and the observation model.

Finally, let us note that the asymptotic stability of nonlinear filters is not only of interest by itself, but is also an important ingredient in the development of error bounds for filters under more general modelling errors or for approximate filters (see, e.g., Refs. 5, 6 in the classical setting). In that case, however, it is typically necessary to obtain more quantitative bounds on the rate of stability. Let us also mention that observability, though sufficient, is not a necessary condition for stability. One could conjecture that a natural counterpart of the *detectability* condition in Ref. 21 is necessary and sufficient for the stability of quantum filters in the finite dimensional setting of section 5, as it is in the classical case.

2. The quantum filtering model

We will consider quantum filtering theory in the abstract setting of Feller-type quantum Markov processes in the spirit of Accardi, Frigerio and Lewis[1,10]. One of the most important examples in practice is the quantum stochastic flow generated by a quantum stochastic differential equation with a finite-dimensional initial system;

this particular setting will be investigated in detail in section 5.

In this section, we introduce the quantum filtering model and fix the notation for the rest of the paper. Let us begin by defining the basic elements of the model.

- $\mathcal{A}$, the *initial system*, is a unital C^*-algebra with state space $\mathcal{S} \subset \mathcal{A}^*$;
- $\{P_t, t \geq 0\}$ is a one parameter semigroup of contractive and completely positive linear maps from $\mathcal{A}$ to itself, with $P_0[X] = X \ \forall X \in \mathcal{A}$ and $P_t[I] = I \ \forall t \geq 0$;
- $\mathfrak{M}$, the *universal algebra*, is a Von Neumann algebra;
- $\{\mathfrak{M}_{t]} : t \geq 0\}$ is a filtration of subalgebras of $\mathfrak{M}$ such that $(\bigcup_{t\geq 0} \mathfrak{M}_{t]})'' = \mathfrak{M}$ and $\mathfrak{M}_{0]} \simeq \mathcal{A}^{**}$ (i.e., $\mathfrak{M}_{0]}$ is the enveloping algebra of $\mathcal{A}$);
- $\{\Phi_\rho : \rho \in \mathcal{S}\}$ is a family of normal states on $\mathfrak{M}$ such that the conditional expectations $\Phi_\rho(\ \cdot\ |\mathfrak{M}_{t]}) : \mathfrak{M} \to \mathfrak{M}_{t]}$ exist for every $t \geq 0$ and $\rho \in \mathcal{S}$.

Remark 2.1. The requirement that $\mathcal{A}$ be unital is not overly restrictive; if $\mathcal{A}$ is not unital, we may always enlarge $\mathcal{A}$ by adjoining the identity without essentially changing the structure of the theory. When $\mathcal{A}$ is commutative, this corresponds to the one-point compactification of the spectrum (Ex. VII.8.5 in Ref. 7).

Before proceeding, we recall for the reader's convenience the definition of the conditional expectation in a Von Neumann algebra (see, e.g., Ref. 15).

Definition 2.1 (Conditional expectation). Let $\mathfrak{A}, \mathfrak{A}_0$ be Von Neumann algebras, $\mathfrak{A}_0 \subset \mathfrak{A}$ and let Φ be a normal state on $\mathfrak{A}$. Suppose there exists a linear map $\Phi(\ \cdot\ |\mathfrak{A}_0) : \mathfrak{A} \to \mathfrak{A}_0$ which satisfies the following properties:

- $\Phi(I|\mathfrak{A}_0) = I$;
- $\Phi(X^*X|\mathfrak{A}_0) \geq 0$ for all $X \in \mathfrak{A}$;
- $\Phi(X^*|\mathfrak{A}_0) = \Phi(X|\mathfrak{A}_0)^*$ for all $X \in \mathfrak{A}$;
- $\Phi(XYZ|\mathfrak{A}_0) = X\Phi(Y|\mathfrak{A}_0)Z$ for all $Y \in \mathfrak{A}$ and $X, Z \in \mathfrak{A}_0$;
- $\Phi(\Phi(X|\mathfrak{A}_0)) = \Phi(X)$ for all $X \in \mathfrak{A}$.

Then $\Phi(\ \cdot\ |\mathfrak{A}_0)$ is a *conditional expectation* from $\mathfrak{A}$ onto $\mathfrak{A}_0$ with respect to Φ.

It is not difficult to prove that any two maps $P, Q : \mathfrak{A} \to \mathfrak{A}_0$ which satisfy this definition are Φ-indistinguishable, i.e., $\Phi(|P(X) - Q(X)|^2) = 0$ (see, e.g., Thm. 3.16 in Ref. 4). Thus the conditional expectation, if it exists, is essentially unique. Existence, on the other hand, is not guaranteed in the noncommutative setting.

We now return to our filtering setup. We will presume that there is a family $\{j_t : t \geq 0\}$ of *-isomorphisms $j_t : \mathcal{A} \to \mathfrak{M}_{t]}$ such that the *Markov property* holds:

$$\Phi_\rho(j_{t+s}(X)|\mathfrak{M}_{s]}) = j_s(P_t[X]) \qquad \forall\, t, s \geq 0,\ X \in \mathcal{A},\ \rho \in \mathcal{S}.$$

Moreover, we presume that $j_0(\mathcal{A})'' = \mathfrak{M}_{0]}$ and that

$$\Phi_\rho(j_0(X)) = \rho(X) \qquad \forall\, X \in \mathcal{A},\ \rho \in \mathcal{S},$$

i.e., the state $\rho \in \mathcal{S}$ can be interpreted as the *initial state* of the quantum Markov process j_t. The latter plays the role of the signal process in classical filtering theory.

Remark 2.2. In order that $\Phi_\rho(j_0(X)) = \rho(X)$ for all ρ, it is necessary that every state $\rho \in \mathcal{S}$ extends to a normal state on $\mathfrak{M}_{0]}$. This forces us to work with the universal representation $\mathfrak{M}_{0]} \simeq \mathcal{A}^{**}$ as required above, see Thm. 1.17.2 in Ref. 18.

In addition, we must introduce the observations. To this end, we introduce the n-dimensional observation process $\{Y_t^k : t \geq 0,\ k = 1, \ldots, n\}$, where Y_t^k is a self-adjoint operator affiliated to $\mathfrak{M}_{t]}$ and $Y_0^k = 0$. Define $\mathfrak{Y}_{t]}$ to be the Von Neumann algebra generated by $\{Y_s^k : 0 \leq s \leq t,\ k = 1, \ldots, n\}$. We presume that

$$\mathfrak{Y}_{t]} \text{ is commutative,} \qquad j_t(X) \in \mathfrak{Y}_{t]}' \qquad \forall\, t \geq 0,\ X \in \mathcal{A}.$$

The first condition is known as the *self-nondemolition* property, and ensures that the process $\{Y_t\}$ can be represented as a classical stochastic process (as is befitting of laboratory observations). The second condition is the *nondemolition* property, and ensures that the conditional expectations $\pi_t^\rho(X) := \Phi_\rho(j_t(X)|\mathfrak{Y}_{t]})$ exist for every $X \in \mathcal{A}$ and $t \geq 0$ (see, e.g., Thm. 3.16 in Ref. 4). The goal of the *filtering problem* is to compute these conditional expectations. This problem can be solved explicitly in specific models, as is known since the work of Belavkin[3]; see Ref. 4 for an introduction and review. For the purpose of this paper, however, it will not be necessary to obtain explicit expressions for the filtered estimates $\pi_t^\rho(X)$.

Finally, we introduce the following Feller-type assumption. We presume that for any choice of $t_1, \ldots, t_k > 0$ and bounded continuous functions $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{R}$,

$$\Phi_\rho(f_1(Y_{t_1}) \cdots f_k(Y_{t_k})|\mathfrak{M}_{0]}) = j_0(Z(t_1, \ldots, t_k, f_1, \ldots, f_k))$$

for some $Z(t_1, \ldots, t_k, f_1, \ldots, f_k) \in \mathcal{A}$ independent of ρ, and moreover

$$\Phi_\rho(f_1(Y_{s+t_1} - Y_s) \cdots f_k(Y_{s+t_k} - Y_s)|\mathfrak{M}_{s]}) = j_s(Z(t_1, \ldots, t_k, f_1, \ldots, f_k))$$

for every $s \geq 0$. The latter assumption ensures, in a sense, that the observation process is time-homogeneous. An important example of a filtering model in which these constructions can be implemented is discussed in detail in section 5.

The goal of the remainder of the paper is to study the dependence of the filter $\pi_t^\rho(X) := \Phi_\rho(j_t(X)|\mathfrak{Y}_{t]})$ on the initial state $\rho \in \mathcal{S}$ as $t \to \infty$.

Definition 2.2 (Observability). Let $\mathfrak{Y} = (\bigcup_{t\geq 0} \mathfrak{Y}_{t]})''$. The model is *observable* if there do not exist $\rho_1, \rho_2 \in \mathcal{S}$ with $\rho_1 \neq \rho_2$ and $\Phi_{\rho_1}(Y) = \Phi_{\rho_2}(Y)$ for every $Y \in \mathfrak{Y}$.

We will prove the following result.

Theorem 2.1. *If the model is observable, then*

$$\Phi_{\rho_1}(|\pi_t^{\rho_1}(X) - \pi_t^{\rho_2}(X)|) \xrightarrow{t\to\infty} 0 \qquad \forall\, X \in \mathcal{A}$$

whenever the laws of the observations under Φ_{ρ_1} and Φ_{ρ_2} are absolutely continuous (i.e., if P is a projection in $\mathfrak{Y}$ and $\Phi_{\rho_2}(P) = 0$, then $\Phi_{\rho_1}(P) = 0$).

We can obtain a sufficient condition for the absolute continuity of the observation laws, as required in theorem 2.1, in terms of the initial states. This is developed in section 4. In the finite-dimensional setting, discussed in section 5, we will find explicitly computable conditions for the filtering model to be observable.

3. Observability and filter stability

The proof of the main result proceeds in two steps. First, we establish that

$$\Phi_{\rho_1}(|\pi_t^{\rho_1}(X) - \pi_t^{\rho_2}(X)|) \xrightarrow{t\to\infty} 0$$

for X of the form $Z(t_1,\ldots,t_k,f_1,\ldots,f_k)$. This holds without any further assumptions. Then, we show that the set of all such observables is total in $\mathcal{A}$ when the model is observable. A simple approximation argument then completes the proof.

3.1. *Stability of $Z(t_1,\ldots,t_k,f_1,\ldots,f_k)$*

We begin by proving a simple lemma. This result is almost trivial—it is just the tower property of the conditional expectation—but one should verify that the conditional expectations do in fact exist.

Lemma 3.1. *For any $\rho\in\mathcal{S}$, $s\ge 0$ and $t_1,\ldots,t_k,f_1,\ldots,f_k$,*

$$\pi_s^\rho(Z(t_1,\ldots,t_k,f_1,\ldots,f_k)) = \Phi_\rho(f_1(Y_{s+t_1}-Y_s)\cdots f_k(Y_{s+t_k}-Y_s)|\mathfrak{Y}_{s]})$$

up to Φ-indistinguishability.

Proof. First, note that by the nondemolition assumption

$$\Phi_\rho(f_1(Y_{s+t_1}-Y_s)\cdots f_k(Y_{s+t_k}-Y_s)|\mathfrak{M}_{s]}) = j_s(Z(t_1,\ldots,t_k,f_1,\ldots,f_k)) \in \mathfrak{Y}'_{s]}.$$

Hence the conditional expectation with respect to $\mathfrak{Y}_{s]}$ exists and

$$\pi_s^\rho(Z(t_1,\ldots,t_k,f_1,\ldots,f_k)) = \Phi_\rho(\Phi_\rho(f_1(Y_{s+t_1}-Y_s)\cdots f_k(Y_{s+t_k}-Y_s)|\mathfrak{M}_{s]})|\mathfrak{Y}_{s]}).$$

Moreover, the conditional expectation

$$\Phi_\rho(f_1(Y_{s+t_1}-Y_s)\cdots f_k(Y_{s+t_k}-Y_s)|\mathfrak{Y}_{s]})$$

exists as $f_1(Y_{s+t_1}-Y_s)\cdots f_k(Y_{s+t_k}-Y_s)\in\mathfrak{Y}'_{s]}$ (this follows directly as $\mathfrak{Y}$ is commutative). Finally, note that observables of the form $X\, f_1(Y_{s+t_1}-Y_s)\cdots f_k(Y_{s+t_k}-Y_s)$ with $X\in\mathfrak{Y}_{s]}$ are weak* total in $\mathfrak{Y}$. Hence the maps $\Phi_\rho(\,\cdot\,|\mathfrak{Y}_{s]}):\mathfrak{Y}\to\mathfrak{Y}_{s]}$ and $\Phi_\rho(\Phi_\rho(\,\cdot\,|\mathfrak{M}_{s]})|\mathfrak{Y}_{s]}):\mathfrak{Y}\to\mathfrak{Y}_{s]}$ are both well defined. It remains to note that both these maps satisfy the definition of the conditional expectation. □

We can now prove the stability of $Z(t_1,\ldots,t_k,f_1,\ldots,f_k)$. By virtue of the previous lemma the setting is essentially classical (as all the objects involved live in the commutative algebra $\mathfrak{Y}$), and we will exploit this fact explicitly in the proof.

Proposition 3.1. *Suppose that the law of the observations under Φ_{ρ_1} is absolutely continuous with respect to the law of the observations under Φ_{ρ_2}. Then*

$$\Phi_{\rho_1}(|\pi_t^{\rho_1}(Z(t_1,\ldots,t_k,f_1,\ldots,f_k)) - \pi_t^{\rho_2}(Z(t_1,\ldots,t_k,f_1,\ldots,f_k))|) \xrightarrow{t\to\infty} 0$$

for any $t_1,\ldots,t_k>0$ and bounded continuous functions $f_1,\ldots,f_k$.

Proof. We work exclusively on the commutative algebra $\mathfrak{Y}$. By the spectral theorem (Prop. 1.18.1 in Ref. 18), there exists a measure space $(\Omega, \mathscr{F}, \lambda)$ which admits a surjective *-isomorphism $\iota : \mathfrak{Y} \to L^\infty(\Omega, \mathscr{F}, \lambda)$, and every state Φ_φ induces a probability measure $\mathbf{P}_\varphi$ on Ω such that $\Phi_\varphi(X) = \mathbf{E}_\varphi(\iota(X))$ for all $X \in \mathfrak{Y}$. Moreover, there exists a classical stochastic process $\{y_t^k : t \geq 0,\ k = 1, \ldots, n\}$ on Ω such that

$$\iota(f(Y_{t_1}^{k_1}, \ldots, Y_{t_\ell}^{k_\ell})) = f(y_{t_1}^{k_1}, \ldots, y_{t_\ell}^{k_\ell}) \qquad \forall \text{ bounded measurable } f : \mathbb{R}^\ell \to \mathbb{R},$$

and it is straightforward to verify that

$$\begin{aligned}\iota(\Phi_\varphi(f_1(Y_{s+t_1} - Y_s) \cdots f_k(Y_{s+t_k} - Y_s)|\mathfrak{Y}_{s]})) \\ = \mathbf{E}_\varphi(f_1(y_{s+t_1} - y_s) \cdots f_k(y_{s+t_k} - y_s)|\mathscr{Y}_s)\end{aligned}$$

where $\mathscr{Y}_s = \sigma\{y_r : 0 \leq r \leq s\}$. Evidently it suffices to prove that

$$\mathbf{E}_{\rho_1}(|\mathbf{E}_{\rho_1}(\xi_t|\mathscr{Y}_t) - \mathbf{E}_{\rho_2}(\xi_t|\mathscr{Y}_t)|) \xrightarrow{t\to\infty} 0$$

whenever $\xi_s = f_1(y_{s+t_1} - y_s) \cdots f_k(y_{s+t_k} - y_s)$ for all $s \geq 0$ and $\rho_1, \rho_2 \in \mathcal{S}$ which give rise to absolutely continuous observation laws.

To proceed, note that by our absolute continuity assumption $\mathbf{P}_{\rho_1}|_{\mathscr{Y}_\infty} \ll \mathbf{P}_{\rho_2}|_{\mathscr{Y}_\infty}$. We can therefore apply the classical Bayes formula (Lem. 8.6.2 in Ref. 16):

$$\mathbf{E}_{\rho_2}(\Delta|\mathscr{Y}_t)\, \mathbf{E}_{\rho_1}(\xi_t|\mathscr{Y}_t) = \mathbf{E}_{\rho_2}(\Delta\, \xi_t|\mathscr{Y}_t) \qquad \mathbf{P}_{\rho_2}\text{-a.s.},$$

where $\Delta = d\mathbf{P}_{\rho_1}|_{\mathscr{Y}_\infty}/d\mathbf{P}_{\rho_2}|_{\mathscr{Y}_\infty}$ ($\mathscr{Y}_\infty = \bigvee_{t\geq 0} \mathscr{Y}_t$). Thus we find that

$$\mathbf{E}_{\rho_2}(\Delta|\mathscr{Y}_t)\, |\mathbf{E}_{\rho_1}(\xi_t|\mathscr{Y}_t) - \mathbf{E}_{\rho_2}(\xi_t|\mathscr{Y}_t)| = |\mathbf{E}_{\rho_2}((\Delta - \mathbf{E}_{\rho_2}(\Delta|\mathscr{Y}_t))\, \xi_t|\mathscr{Y}_t)| \qquad \mathbf{P}_{\rho_2}\text{-a.s.}$$

Taking the expectation with respect to $\mathbf{P}_{\rho_2}$, we obtain

$$\mathbf{E}_{\rho_1}(|\mathbf{E}_{\rho_1}(\xi_t|\mathscr{Y}_t) - \mathbf{E}_{\rho_2}(\xi_t|\mathscr{Y}_t)|) = \mathbf{E}_{\rho_2}(|\mathbf{E}_{\rho_2}((\Delta - \mathbf{E}_{\rho_2}(\Delta|\mathscr{Y}_t))\, \xi_t|\mathscr{Y}_t)|).$$

By Jensen's inequality

$$\mathbf{E}_{\rho_2}(|\mathbf{E}_{\rho_2}((\Delta - \mathbf{E}_{\rho_2}(\Delta|\mathscr{Y}_t))\, \xi_t|\mathscr{Y}_t)|) \leq K\, \mathbf{E}_{\rho_2}(|\Delta - \mathbf{E}_{\rho_2}(\Delta|\mathscr{Y}_t)|),$$

where $K = \|f_1\|_\infty \cdots \|f_k\|_\infty$. But note that ξ_t is measurable with respect to $\mathscr{Y}_\infty$, so by the martingale convergence theorem $\mathbf{E}_{\rho_2}(\Delta|\mathscr{Y}_t) \to \Delta$ in $L^1(\Omega, \mathscr{F}, \mathbf{P}_{\rho_2})$. Therefore $\mathbf{E}_{\rho_1}(|\mathbf{E}_{\rho_1}(\xi_t|\mathscr{Y}_t) - \mathbf{E}_{\rho_2}(\xi_t|\mathscr{Y}_t)|) \to 0$, and the proof is complete. □

Corollary 3.1. *Denote by $\mathcal{O}^0 \subset \mathcal{A}$ the linear span of $Z(t_1, \ldots, t_k, f_1, \ldots, f_k)$ for all $t_1, \ldots, t_k, f_1, \ldots, f_k$, and suppose that the law of the observations under Φ_{ρ_1} is absolutely continuous with respect to the law of the observations under Φ_{ρ_2}. Then*

$$\Phi_{\rho_1}(|\pi_t^{\rho_1}(Z) - \pi_t^{\rho_2}(Z)|) \xrightarrow{t\to\infty} 0 \qquad \forall\, Z \in \operatorname{cl} \mathcal{O}^0,$$

where $\operatorname{cl} \mathcal{O}^0$ denotes the (uniform) closure of $\mathcal{O}^0$ in $\mathcal{A}$.

Proof. Fix $Z \in \operatorname{cl} \mathcal{O}^0$ and a sequence $\{Z_n\} \subset \mathcal{O}^0$ such that $\|Z_n - Z\| \to 0$ as $n \to \infty$. For every $n < \infty$, we have $\Phi_{\rho_1}(|\pi_t^{\rho_1}(Z_n) - \pi_t^{\rho_2}(Z_n)|) \to 0$ as $t \to \infty$; to see this, it suffices to use the linearity of the conditional expectation and the fact that the triangle inequality holds for $|\cdot|$ when we restrict our attention to a commutative algebra (i.e., $|\sum_i X_i| \le \sum_i |X_i|$ provided that the X_i commute with each other and their adjoints). Reasoning in the same way, we find immediately that

$$\begin{aligned}\Phi_{\rho_1}(|\pi_t^{\rho_1}(Z) - \pi_t^{\rho_2}(Z)|) \\ \le \Phi_{\rho_1}(|\pi_t^{\rho_1}(Z - Z_n)|) + \Phi_{\rho_1}(|\pi_t^{\rho_1}(Z_n) - \pi_t^{\rho_2}(Z_n)|) + \Phi_{\rho_1}(|\pi_t^{\rho_2}(Z_n - Z)|).\end{aligned}$$

The first and the third term on the right are bounded above by $\|Z_n - Z\|$. Hence

$$\limsup_{t\to\infty} \Phi_{\rho_1}(|\pi_t^{\rho_1}(Z) - \pi_t^{\rho_2}(Z)|) \le 2\,\|Z_n - Z\|.$$

The result follows by letting $n \to \infty$. □

3.2. *Observability and approximation*

From the previous corollary, we see that a sufficient condition for the stability of the filter is that $\mathrm{cl}\,\mathcal{O}^0 = \mathcal{A}$. We will show that this is the case if and only if the model is observable. In fact, we will prove a more general result, from which this statement follows. We begin with the following definition.

Definition 3.1 (Observable space). For $\rho_1, \rho_2 \in \mathcal{S}$, we define the equivalence relation $\rho_1 \backsim \rho_2$ whenever $\Phi_{\rho_1}(Y) = \Phi_{\rho_2}(Y)$ for every $Y \in \mathfrak{Y}$. The Banach space

$$\mathcal{O} = \{X \in \mathcal{A} : \rho_1(X) = \rho_2(X) \text{ for all } \rho_1, \rho_2 \in \mathcal{S} \text{ such that } \rho_1 \backsim \rho_2\}$$

is called the observable space of the model.

The following result is key.

Proposition 3.2. *$\mathcal{O}^0$ is dense in $\mathcal{O}$.*

Proof. Suppose that $\mathcal{O}^0$ is not dense in $\mathcal{O}$. Then there must be an element X of $\mathcal{O}$ that is not in $\mathrm{cl}\,\mathcal{O}^0$. By the Hahn-Banach theorem, there exists an element $\varphi \in \mathcal{A}^*$ such that $\varphi(Z) = 0$ for all $Z \in \mathrm{cl}\,\mathcal{O}^0$ and $\varphi(X) \ne 0$. Then either $\varphi(X) + \varphi(X)^* \ne 0$, or $i(\varphi(X) - \varphi(X)^*) \ne 0$, so we may assume without loss of generality that φ is real-valued. In particular, we can write $\varphi = \varphi_1 - \varphi_2$ where φ_1, φ_2 are nonnegative (e.g., Prop. 1.17.1 in Ref. 18). But note that $I \in \mathcal{O}^0$, so $\varphi_1(I) = \varphi_2(I)$. We can thus define $\rho_1, \rho_2 \in \mathcal{S}$ by $\rho_1 = \varphi_1/\varphi_1(I)$ and $\rho_2 = \varphi_2/\varphi_2(I)$, and we find that $\rho_1(X) \ne \rho_2(X)$ and $\rho_1(Z) = \rho_2(Z)$ for all $Z \in \mathrm{cl}\,\mathcal{O}^0$. Now note that for any $\rho \in \mathcal{S}$

$$\rho(Z(t_1, \ldots, t_k, f_1, \ldots, f_k)) = \Phi_\rho(f_1(Y_{t_1}) \cdots f_k(Y_{t_k})).$$

Hence we find that

$$\Phi_{\rho_1}(f_1(Y_{t_1}) \cdots f_k(Y_{t_k})) = \Phi_{\rho_2}(f_1(Y_{t_1}) \cdots f_k(Y_{t_k}))$$

for all $t_1, \ldots, t_k, f_1, \ldots, f_k$. As the set of observables of the form $f_1(Y_{t_1}) \cdots f_k(Y_{t_k})$ is weak* total in $\mathfrak{Y}$, we conclude that $\Phi_{\rho_1}(Y) = \Phi_{\rho_2}(Y)$ for all $Y \in \mathfrak{Y}$. But then $\rho_1 \backsim \rho_2$, which implies $\rho_1(X) = \rho_2(X)$, and we have a contradiction. □

We immediately find the following corollary.

Corollary 3.2. *Suppose that the law of the observations under Φ_{ρ_1} is absolutely continuous with respect to the law of the observations under Φ_{ρ_2}. Then*

$$\Phi_{\rho_1}(|\pi_t^{\rho_1}(X) - \pi_t^{\rho_2}(X)|) \xrightarrow{t\to\infty} 0 \qquad \forall\, X \in \mathcal{O}.$$

Proof. Immediate from corollary 3.1 and $\mathrm{cl}\,\mathcal{O}^0 = \mathcal{O}$. □

We may finally complete the proof of theorem 2.1.

Proof. (Theorem 2.1). The model is observable, by definition, if $\rho_1 \backsim \rho_2$ implies $\rho_1 = \rho_2$. Clearly this is the case if and only if $\mathcal{O} = \mathcal{A}$. The result follows directly. □

Remark 3.1. The proof of proposition 3.2 clarifies why it is important to work in the C^*-algebraic setting, rather than starting off with an initial Von Neumann algebra. As the state space of a C^*-algebra is dual to the algebra itself, we may employ the Hahn-Banach theorem as in the proof of proposition 3.2 to characterize the observable space. For a Von Neumann algebra, however, the space of normal states is predual to the algebra. To employ the technique used in the proof of proposition 3.2, we would then have two options: we must either consider non-normal initial states, or prove density of $\mathcal{O}^0$ in $\mathcal{O}$ in the weak* topology on the initial Von Neumann algebra. The former is unphysical, while in the latter case corollary 3.1 can not be employed. It thus appears that the C^*-algebraic setting is the natural setting in which our results can be developed.

4. Absolute continuity and randomization

In our main result, theorem 2.1, we required that the initial state $\rho_1, \rho_2 \in \mathcal{S}$ are such that the law of the observations under Φ_{ρ_1} is absolutely continuous with respect to the law of the observations under Φ_{ρ_2}. One might expect that a sufficient condition would be that the initial states are themselves absolutely continuous in a suitable sense. The goal of this section is to develop this idea.

Before we turn to the filtering model of section 2, let us consider the general setting where $\mathcal{A}$ is any unital C^*-algebra. Given a state φ on $\mathcal{A}$, we denote by $(\pi_\varphi, \mathsf{H}_\varphi, \xi_\varphi)$ the cyclic representation of $\mathcal{A}$ induced by φ.

Let $\mathcal{S} \subset \mathcal{A}^*$ denote the state space of $\mathcal{A}$. We endow $\mathcal{A}^*$ with the weak* topology, and recall that this makes $\mathcal{S}$ a compact convex set. By a (finite) measure on $\mathcal{S}$ we mean a regular Borel measure on $\mathcal{S}$ or, equivalently, an element of $C(\mathcal{S})^*$ (see p. 232 in Ref. 19). A probability measure is a nonnegative measure with unit mass.

We now recall a basic construction in Choquet theory. Let μ be a probability measure on $\mathcal{S}$. Then (Lem. IV.6.3 in Ref. 19) there is a unique $\rho \in \mathcal{S}$ such that

$$F(\rho) = \int_{\mathcal{S}} F(\varphi)\,\mu(d\varphi) \qquad \forall\, F \in \mathcal{A}^{**}.$$

The state ρ is called the *barycenter* of the probability measure μ. The measure μ can be thought of as a *randomization* of the state ρ; indeed, we have replaced the state ρ by a random state, with law μ, which averages to ρ:

$$\rho(X) = \int_{\mathcal{S}} \varphi(X)\,\mu(d\varphi) \qquad \forall\, X \in \mathcal{A}.$$

The idea is now to seek randomizations which have desirable probabilistic properties. In particular, we will consider the following notion of absolute continuity.

Definition 4.1 (Absolute continuity). The state $\rho_1 \in \mathcal{S}$ is *absolutely continuous* with respect to $\rho_2 \in \mathcal{S}$, denoted as $\rho_1 \ll \rho_2$, if there exist probability measures μ_1, μ_2 on $\mathcal{S}$ such that ρ_1 is the barycenter of μ_1, ρ_2 is the barycenter of μ_2, and $\mu_1 \ll \mu_2$.

We now show that this natural definition of absolute continuity of ρ_1 with respect to ρ_2 is equivalent to the requirement that ρ_1 is *presque dominée* (almost dominated) by ρ_2 in the sense of Dixmier (Ch. I, §4, Ex. 8c in Ref. 9). Radon-Nikodym type results in this setting have been investigated by Naudts[13] and Gudder[12].

Proposition 4.1. *Let $\rho_1, \rho_2 \in \mathcal{S}$. Then the following are equivalent:*

(1) $\rho_1 \ll \rho_2$;
(2) For every sequence $\{X_n\} \subset \mathcal{A}$ such that $\lim_{m,n} \rho_1((X_m - X_n)^*(X_m - X_n)) = 0$, *we have* $\lim_n \rho_1(X_n^* X_n) = 0$ *whenever* $\lim_n \rho_2(X_n^* X_n) = 0$.
(3) There exists a positive self-adjoint operator T on H_{ρ_2}, affiliated to $\pi_{\rho_2}(\mathcal{A})'$, such that $\rho_1(X) = \langle T\xi_{\rho_2}, \pi_{\rho_2}(X) T\xi_{\rho_2}\rangle$ for all $X \in \mathcal{A}$.

Proof.

($1 \Rightarrow 2$) As $\rho_1 \ll \rho_2$, there are probability measures μ, ν on $\mathcal{S}$ with $\mu \ll \nu$ and

$$\rho_1(X) = \int_{\mathcal{S}} \varphi(X)\,\mu(d\varphi), \qquad \rho_2(X) = \int_{\mathcal{S}} \varphi(X)\,\nu(d\varphi), \qquad \forall\, X \in \mathcal{A}.$$

Let $\{X_n\}$ be such that $\lim_{m,n} \rho_1((X_m - X_n)^*(X_m - X_n)) = 0$, and define the random variables $\Phi_n : \mathcal{S} \to [0, \infty[$ by $\Phi_n(\varphi) = \varphi(X_n^* X_n)$. We begin by establishing that $\{\Phi_n\}$ is a Cauchy sequence in $L^1(\mathcal{S}, \mu)$. By Lem. 2.4 in Ref. 14

$$|\Phi_m(\varphi) - \Phi_n(\varphi)| \le \varphi((X_m - X_n)^*(X_m - X_n))^{1/2} \left[\varphi(X_m^* X_m)^{1/2} + \varphi(X_n^* X_n)^{1/2}\right].$$

Therefore, we find using the Cauchy-Schwarz inequality and $(a+b)^2 \le 2a^2 + 2b^2$

$$\left[\int_{\mathcal{S}} |\Phi_m - \Phi_n|\,d\mu\right]^2 \le 2\rho_1(X_m^* X_m + X_n^* X_n)\,\rho_1((X_m - X_n)^*(X_m - X_n)).$$

Thus $\{\Phi_n\}$ is Cauchy in $L^1(\mathcal{S}, \mu)$ provided that $\rho_1(X_n^* X_n)$ converges to a finite limit. To show this, define $\psi_n \in \mathsf{H}_{\rho_1}$ by $\psi_n = \pi_{\rho_1}(X_n)\xi_{\rho_1}$. Then $\rho_1(X_n^* X_n) = \|\psi_n\|^2$ and $\rho_1((X_m - X_n)^*(X_m - X_n)) = \|\psi_m - \psi_n\|^2$. As the latter converges to zero, we see that $\{\psi_n\}$ is a Cauchy sequence in H_{ρ_1} and thus $\rho_1(X_n^* X_n)$ has a finite limit.

Now suppose that, in addition, $\lim_n \rho_2(X_n^* X_n) = 0$. Then evidently $\Phi_n \to 0$ in $L^1(\mathcal{S}, \nu)$, so that in particular $\Phi_n \to 0$ in ν-probability as well as in μ-probability (as $\mu \ll \nu$). But as $\{\Phi_n\}$ is a Cauchy sequence in $L^1(\mathcal{S},\mu)$, it follows that $\Phi_n \to 0$ in $L^1(\mathcal{S},\mu)$. Thus $\lim_n \rho_1(X_n^* X_n) = 0$, which is what we set out to prove.

($2 \Leftrightarrow 3$) See Cor. 2 in Ref. 12.

($3 \Rightarrow 1$) Denote by $\mathfrak{C}$ the commutative Von Neumann algebra generated by T (i.e., this is the smallest Von Neumann subalgebra of $\mathfrak{B}(\mathsf{H}_{\rho_2})$ which contains the spectral projections of T). By §3.1 in Ref. 18, there is a unique probability measure ν on $\mathcal{S}$ with barycenter ρ_2 and surjective *-isomorphism $\Gamma : \mathfrak{C} \to L^\infty(\mathcal{S},\nu)$ so that

$$\langle \xi_{\rho_2}, \pi_{\rho_2}(X) V \xi_{\rho_2}\rangle = \int_{\mathcal{S}} \Gamma(V)(\varphi)\,\varphi(X)\,\nu(d\varphi) \qquad \forall\, V \in \mathfrak{C},\ X \in \mathcal{A}.$$

Now define $f_n(x) = nx/(n+x)$ and set $T_n = f_n(T)$. Then $T_n \in \mathfrak{C}$ is a bounded, self-adjoint operator and, writing the spectral measure of T as $E_T(d\lambda)$, we find that

$$\|T_n\xi_{\rho_2} - T\xi_{\rho_2}\|^2 = \int_{[0,\infty[} |f_n(\lambda)-\lambda|^2\,\langle \xi_{\rho_2}, E_T(d\lambda)\xi_{\rho_2}\rangle \xrightarrow{n\to\infty} 0$$

by dominated convergence (as $|f_n(\lambda)-\lambda|^2 \le 2f_n(\lambda)^2+2\lambda^2 \le 4\lambda^2$, and $\|T\xi_{\rho_2}\|^2 < \infty$ by construction). Consequently, we obtain

$$\int_{\mathcal{S}} \Gamma(T_n^2)(\varphi)\,\varphi(X)\,\nu(d\varphi) = \langle T_n\xi_{\rho_2}, \pi_{\rho_2}(X) T_n\xi_{\rho_2}\rangle \xrightarrow{n\to\infty} \rho_1(X).$$

But note that $\Gamma(T_n^2)(\varphi)$ is nonnegative, nondecreasing and

$$\int_{\mathcal{S}} \Gamma(T_n^2)(\varphi)\,\nu(d\varphi) \xrightarrow{n\to\infty} 1,$$

so by monotone convergence $\Gamma(T_n^2) \nearrow \Delta$ with $\Delta \in L^1(\mathcal{S},\nu)$. Thus

$$\rho_1(X) = \int_{\mathcal{S}} \Delta(\varphi)\,\varphi(X)\,\nu(d\varphi) \qquad \forall\, X \in \mathcal{A}$$

by dominated convergence. We now define $d\mu = \Delta\, d\nu$, and μ has barycenter ρ_1. $\square$

We now return to the filtering setting of section 2. The following result establishes that absolute continuity of the initial states is indeed a sufficient condition for absolute continuity of the observation laws. Absolute continuity of the initial states is often easily verified, e.g., in the finite dimensional setting of section 5.

Proposition 4.2. *Suppose that $\rho_1 \ll \rho_2$. Then the law of the observations under Φ_{ρ_1} is absolutely continuous with respect to the law of the observations under Φ_{ρ_2}.*

Proof. As in the proof of proposition 3.1, we begin by constructing a measure space $(\Omega,\mathscr{F},\lambda)$, a surjective *-isomorphism $\iota : \mathfrak{Y} \to L^\infty(\Omega,\mathscr{F},\lambda)$, and a family of probability measures $\mathbf{P}_\varphi$ on Ω such that $\Phi_\varphi(X) = \mathbf{E}_\varphi(\iota(X))$ for all $X \in \mathfrak{Y}$.

As $\rho_1 \ll \rho_2$, there exist two probability measures μ_1, μ_2 such that ρ_1 is the barycenter of μ_1, ρ_2 is the barycenter of μ_2, and $\mu_1 \ll \mu_2$. We will utilize these measures to construct randomizations of the classical probability measures $\mathbf{P}_{\rho_1}$ and

$\mathbf{P}_{\rho_2}$. To this end, let $\mathscr{B}$ be the Borel σ-algebra on the state space $\mathcal{S}$, and construct the enlarged probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbf{P}}_2)$ by setting $\tilde{\Omega} = \Omega \times \mathcal{S}$, $\tilde{\mathscr{F}} = \mathscr{F} \times \mathscr{B}$, and

$$\tilde{\mathbf{P}}_2(A) = \int_{\mathcal{S}} \left[\int_{\Omega} I_A(\omega, \varphi)\, \mathbf{P}_{\varphi}(d\omega) \right] \mu_2(d\varphi) \qquad \forall\, A \in \tilde{\mathscr{F}}.$$

Moreover, we define $d\tilde{\mathbf{P}}_1 = \Delta\, d\tilde{\mathbf{P}}_2$ where $\Delta(\omega, \varphi) = (d\mu_1/d\mu_2)(\varphi)$. Then

$$\int_{\tilde{\Omega}} F(\omega)\, \tilde{\mathbf{P}}_i(d\omega, d\varphi) = \int_{\Omega} F(\omega)\, \mathbf{P}_{\rho_i}(d\omega), \qquad i = 1, 2,$$

for all bounded measurable functions $F : \Omega \to \mathbb{R}$. As by construction $\tilde{\mathbf{P}}_1 \ll \tilde{\mathbf{P}}_2$, clearly the marginals satisfy $\mathbf{P}_{\rho_1} \ll \mathbf{P}_{\rho_2}$ also. The proof is complete. $\square$

5. The finite dimensional case

In this section, we consider a specific class of quantum filtering models which have important applications in quantum optics (see, e.g., Refs. 2, 4).

Fix $p, q \in \mathbb{N}$ and let $\mathsf{H} = \mathbb{C}^p \otimes \Gamma$, where $\Gamma = \Gamma_{\mathsf{s}}(L^2(\mathbb{R}_+) \otimes \mathbb{C}^q)$ is the symmetric Fock space of multiplicity q. Thus p is the dimension of the initial system, while q is the noise dimension. We set $\mathcal{A} = M_p$ (the *-algebra of $p \times p$ complex matrices), $\mathfrak{M} = \mathfrak{B}(\mathsf{H}) = \mathcal{A} \otimes \mathfrak{B}(\Gamma)$. Moreover, recalling that the Fock space admits the natural tensor product structure $\Gamma = \Gamma_{t]} \otimes \Gamma_{[t}$, we define the filtration of subalgebras

$$\mathfrak{M}_{t]} = \{X \otimes I : X \in \mathcal{A} \otimes \mathfrak{B}(\Gamma_{t]})\}.$$

Finally, we define the family of states $\Phi_\rho = \rho \otimes \Phi_V$ with $\Phi_V(X) = \langle \xi, X\xi \rangle$, where ξ is the vacuum vector in Γ. It is not difficult to verify that the conditional expectations $\Phi_\rho(\ \cdot\ |\mathfrak{M}_{t]})$ exist in this setting; in fact, they are given explicitly as follows:

$$H_{t]} : \mathbb{C}^p \otimes \Gamma_{t]} \to \mathbb{C}^p \otimes \Gamma, \quad H_{t]}\psi := \psi \otimes \xi_{[t}, \qquad \Phi_\rho(X|\mathfrak{M}_{t]}) = H_{t]}^* X H_{t]} \otimes I,$$

where $\xi_{[t}$ is the vacuum vector in $\Gamma_{[t}$. See, e.g., Ref. 17 for further details.

We now introduce, as usual, the canonical quantum noises $A_i(t), A_i^\dagger(t), \Lambda_{ij}(t)$, $i, j = 1, \ldots, q$ on Γ (we will denote their ampliations to H by the same notation), and consider the Hudson-Parthasarathy quantum stochastic differential equation

$$dU_t = \Bigg\{ \sum_{i,j=1}^{q_0} (S_{ij} - \delta_{ij})\, d\Lambda_{ij}(t) + \sum_{i=1}^{q_0} L_i\, dA_i^\dagger(t) \\ - \sum_{i,k=1}^{q_0} L_k^* S_{ki}\, dA_i(t) - \frac{1}{2} \sum_{k=1}^{q_0} L_k^* L_k\, dt - iH\, dt \Bigg\} U_t, \qquad U_0 = I,$$

where $q_0 \le q$ and $S_{ij}, L_i, H \in \mathcal{A}$, H is self-adjoint, and $\sum_{ij} S_{ij} \otimes e_i e_j^*$ is a unitary operator in $M_p \otimes M_q$ (e_i is the ith basis vector in $\mathbb{C}^q$). Then this equation has a unique solution $\{U_t : t \ge 0\}$ such that U_t is unitary for every $t \ge 0$ (Thm. 27.8 in

Ref. 17). Moreover, if we define $j_t : \mathcal{A} \to \mathfrak{M}_{t]}$ by $j_t(X) = U_t^*(X \otimes I)U_t$, then j_t satisfies the quantum Markov property for the semigroup $\{P_t : t \geq 0\}$ generated by

$$\mathscr{L}[X] = \lim_{t \searrow 0} \frac{P_t[X] - X}{t} = i[H, X] + \sum_{k=1}^{q_0} \left\{ L_k^* X L_k - \frac{1}{2} L_k^* L_k X - \frac{1}{2} X L_k^* L_k \right\},$$

see Cor. 27.10 in Ref. 17. As by construction $\Phi_\rho(j_0(X)) = \rho(X)$ for any $\rho \in \mathcal{S}$, this model satisfies the requirements of section 2.

It remains to introduce the observations. For sake of concreteness, we will consider in detail two common observation models: a one-dimensional homodyne detection model and a one-dimensional photon counting model. The generalization of these results to other observation models and to higher dimensional observations is straightforward. Before proceeding, however, we prove the following simple lemma.

Lemma 5.1. *Let ρ_1, ρ_2 be states on M_p which are defined by the density matrices ϱ_1, ϱ_2 (i.e., $\rho_i(X) = \mathrm{Tr}[\varrho_i X]$). Then $\rho_1 \ll \rho_2$ if and only if $\ker \varrho_1 \supset \ker \varrho_2$.*

Proof. We first prove that $\ker \varrho_1 \supset \ker \varrho_2$ implies $\rho_1 \ll \rho_2$. Let us restrict ϱ_1, ϱ_2 to the subspace $\mathsf{h} = (\ker \varrho_2)^\perp$. Note that $\varrho_2|_\mathsf{h}$ has full rank and hence is positive definite, so there is some $\varepsilon \in \,]0, 1[$ such that $\langle v, \varrho_2 v\rangle \geq \varepsilon \|v\|^2$ for all $v \in \mathsf{h}$. But the eigenvalues of ϱ_1 must be contained in $[0, 1]$, so that $\langle v, \varrho_1 v\rangle \leq \|v\|^2$ for any $v \in \mathsf{h}$. Thus we find that $\langle v, \varrho_2 v\rangle \geq \varepsilon \langle v, \varrho_1 v\rangle$ for all $v \in \mathsf{h}$, so evidently $\varrho_2 \geq \varepsilon \varrho_1$. But then $\varrho_1' = (\varrho_2 - \varepsilon \varrho_1)/(1 - \varepsilon)$ defines another state ρ_1' on M_p, and the measures $\mu_1 \ll \mu_2$ where $\mu_1 = \delta_{\{\rho_1\}}$ and $\mu_2 = \varepsilon \delta_{\{\rho_1\}} + (1 - \varepsilon)\delta_{\{\rho_1'\}}$ have barycenters ρ_1 and ρ_2.

It remains to prove that $\rho_1 \ll \rho_2$ implies $\ker \varrho_1 \supset \ker \varrho_2$. To this end, suppose there is a $v \in \ker \varrho_2$ such that $v \notin \ker \varrho_1$. Then $\rho_2(vv^*) = \langle v, \varrho_2 v\rangle = 0$ while $\rho_1(vv^*) = \langle v, \varrho_1 v\rangle = \|(\varrho_1)^{1/2} v\|^2 > 0$, contradicting $\rho_1 \ll \rho_2$ by proposition 4.1. □

Note that by proposition 4.2, this lemma makes the absolute continuity condition on the observation laws easy to verify explicitly in the finite-dimensional setting. In particular, the condition always holds if ϱ_2 has full rank. This is very convenient in practice: it means that if the model is observable, we can always obtain the correct filtered estimates as $t \to \infty$ even when the true initial state of the system is completely unknown by choosing an initial state for the filter of full rank.

5.1. *Homodyne detection*

For homodyne detection, we consider the observations

$$Y_t = U_t^* \{\sqrt{\eta}\,(A_1(t) + A_1^\dagger(t)) + \sqrt{1 - \eta}\,(A_q(t) + A_q^\dagger(t))\} U_t, \qquad \eta \in \,]0, 1], \; q_0 < q;$$

here η is the detection efficiency, and the qth quadrature plays the role of an independent corrupting noise (we allow $q_0 = q$ if $\eta = 1$). The operators Y_t are self-adjoint[a]

[a] The field quadrature $A_i(t) + A_i^\dagger(t)$ should be interpreted as the Stone generator of the appropriate Weyl operator[17]. This defines the correct domain for these operators on which they are self-adjoint.

and affiliated to $\mathfrak{M}_{t]}$. Before we can proceed, we must verify that the nondemolition and self-nondemolition properties hold, as well as the Feller property of section 2.

Lemma 5.2. *Denote by $\mathfrak{Z}_{t]}$ the Von Neumann algebra generated by*

$$\{Z_s := \sqrt{\eta}\,(A_1(s) + A_1^\dagger(s)) + \sqrt{1-\eta}\,(A_q(s) + A_q^\dagger(s)) : s \le t\}.$$

Then $U_T^*\mathfrak{Z}_{t]}U_T = U_t^*\mathfrak{Z}_{t]}U_t = \mathfrak{Y}_{t]}$ for every $0 \le t \le T$.

Proof. Denote by $U_{s,t}$ $(s \le t)$ the solution of the Hudson-Parthasarathy equation for U_t with the initial condition $U_s = I$. Then it is not difficult to verify that $U_{s,t}U_{r,s} = U_{r,t}$ for $r \le s \le t$, and that $U_{s,t}$ acts as the identity on $\Gamma_{s]}$ (Thm. 2.3 in Ref. 2). Thus $U_{s,t} \in (\mathfrak{Z}_{s]})'$, so that $U_T^*\mathfrak{Z}_{t]}U_T = U_t^*U_{t,T}^*\mathfrak{Z}_{t]}U_{t,T}U_t = U_t^*\mathfrak{Z}_{t]}U_t$. Finally, note that any spectral projection P of Y_s (with $s \le t$) can be written as $U_s^*QU_s$ where Q is a spectral projection of Z_s, so that $P = U_t^*QU_t$ also. But the set of all such Q generate $\mathfrak{Z}_{t]}$ and the set of all such P generate $\mathfrak{Y}_{t]}$, so $U_t^*\mathfrak{Z}_{t]}U_t = \mathfrak{Y}_{t]}$. $\square$

Corollary 5.1. *The self-nondemolition and nondemolition properties hold:*

$$\mathfrak{Y}_{t]} \text{ is commutative}, \qquad j_t(X) \in \mathfrak{Y}_{t]}' \qquad \forall\, t \ge 0,\ X \in \mathcal{A}.$$

Proof. As $\mathfrak{Z}_{t]}$ is a commutative algebra and $\mathfrak{Y}_{t]} = U_t^*\mathfrak{Z}_{t]}U_t$, evidently $\mathfrak{Y}_{t]}$ is commutative also. To prove the nondemolition property, fix $X \in \mathcal{A}$ and $P \in \mathfrak{Y}_{t]}$. Then $[j_t(X), P] = [U_t^*(X \otimes I)U_t, U_t^*QU_t] = U_t^*[X \otimes I, Q]U_t = 0$, as $[\mathfrak{M}_{0]}, \mathfrak{Z}_{t]}] = 0$. $\square$

Remark 5.1. By virtue of the nondemolition and self-nondemolition properties, the filtering problem is well-posed. In this setting, one can compute the filter explicitly as the solution of the following stochastic differential equation:

$$d\pi_t^\rho(X) = \pi_t^\rho(\mathscr{L}[X])\,dt + \sqrt{\eta}\,\{\pi_t^\rho(L_1^*X + XL_1) - \pi_t^\rho(L_1 + L_1^*)\,\pi_t^\rho(X)\}\,d\overline{W}_t^\rho,$$

where $d\overline{W}_t^\rho = dY_t - \sqrt{\eta}\,\pi_t^\rho(L_1 + L_1^*)\,dt$ and $\pi_0^\rho(X) = \rho(X)$, see, e.g., sec. 5.2.4 in Ref. 20. However, we do not need this representation of the filter in this paper.

We must still demonstrate the remaining requirement of section 2.

Lemma 5.3. *For any $t_1, \ldots, t_k > 0$ and bounded continuous $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{R}$,*

$$\Phi_\rho(f_1(Y_{t_1}) \cdots f_k(Y_{t_k})|\mathfrak{M}_{0]}) = j_0(Z(t_1, \ldots, t_k, f_1, \ldots, f_k))$$

for some $Z(t_1, \ldots, t_k, f_1, \ldots, f_k) \in \mathcal{A}$ independent of ρ, and moreover

$$\Phi_\rho(f_1(Y_{s+t_1} - Y_s) \cdots f_k(Y_{s+t_k} - Y_s)|\mathfrak{M}_{s]}) = j_s(Z(t_1, \ldots, t_k, f_1, \ldots, f_k))$$

for every $s \ge 0$.

Proof. The first assertion is trivial in the current setting, as the conditional expectation $\Phi_\rho(\,\cdot\,|\mathfrak{M}_{0]})$ does not depend on ρ and any element of $\mathfrak{M}_{0]}$ can be written as $j_0(X) = X \otimes I$ for some $X \in \mathcal{A}$. To prove the second assertion, note that

$$f_1(Y_{s+t_1} - Y_s) \cdots f_k(Y_{s+t_k} - Y_s) = U_T^* f_1(Z_{s+t_1} - Z_s) \cdots f_k(Z_{s+t_k} - Z_s) U_T,$$

where T is chosen to be greater than $\max\{s+t_\ell : \ell = 1,\ldots,k\}$. But as $U_s, U_s^* \in \mathfrak{M}_{s]}$, we find by the module property of the conditional expectation

$$\begin{aligned}\Phi_\rho(f_1(Y_{s+t_1}-Y_s)\cdots f_k(Y_{s+t_k}-Y_s)|\mathfrak{M}_{s]}) \\ = U_s^*\,\Phi_\rho(U_{s,T}^* f_1(Z_{s+t_1}-Z_s)\cdots f_k(Z_{s+t_k}-Z_s)U_{s,T}|\mathfrak{M}_{s]})\,U_s.\end{aligned}$$

Now note that for any pair of exponential vectors $e(f), e(g) \in \Gamma$ and $v, w \in \mathbb{C}^p$

$$\begin{aligned}&\langle v\otimes e(f), U_{s,T}^* f_1(Z_{s+t_1}-Z_s)\cdots f_k(Z_{s+t_k}-Z_s)U_{s,T}\ w\otimes e(g)\rangle \\ &\quad = \langle v\otimes e(\theta_s f), U_{T-s}^* f_1(Z_{t_1})\cdots f_k(Z_{t_k})U_{T-s}\ w\otimes e(\theta_s g)\rangle\ \langle e(f_{s]}), e(g_{s]})\rangle \\ &\quad = \langle v\otimes e(\theta_s f), f_1(Y_{t_1})\cdots f_k(Y_{t_k})\ w\otimes e(\theta_s g)\rangle\ \langle e(f_{s]}), e(g_{s]})\rangle,\end{aligned}$$

where $\theta_s f(t) = f(s+t)$ and $f_{s]}$ is the restriction of f to $[0,s]$. Hence

$$\begin{aligned}&\langle v\otimes e(f), \Phi_\rho(U_{s,T}^* f_1(Z_{s+t_1}-Z_s)\cdots f_k(Z_{s+t_k}-Z_s)U_{s,T}|\mathfrak{M}_{s]})\ w\otimes e(g)\rangle \\ &\quad = \langle v\otimes e(f_{s]})\otimes\xi_{[s}, U_{s,T}^* f_1(Z_{s+t_1}-Z_s) \\ &\qquad\qquad \cdots f_k(Z_{s+t_k}-Z_s)U_{s,T}\ w\otimes e(g_{s]})\otimes\xi_{[s}\rangle\ \langle e(f_{[s}), e(g_{[s})\rangle \\ &\quad = \langle v\otimes\xi, f_1(Y_{t_1})\cdots f_k(Y_{t_k})\ w\otimes\xi\rangle\ \langle e(f_{s]}), e(g_{s]})\rangle\ \langle e(f_{[s}), e(g_{[s})\rangle \\ &\quad = \langle v\otimes\xi, f_1(Y_{t_1})\cdots f_k(Y_{t_k})\ w\otimes\xi\rangle\ \langle e(f), e(g)\rangle \\ &\quad = \langle v\otimes e(f), \Phi_\rho(f_1(Y_{t_1})\cdots f_k(Y_{t_k})|\mathfrak{M}_{0]})\ w\otimes e(g)\rangle.\end{aligned}$$

The result now follows as the exponential vectors are total in Γ. □

We have now completed verifying that all the requirements of section 2 are met, and thus theorem 2.1 applies. The remainder of this section is devoted to the following problem: can one determine directly whether the model is observable on the basis of the coefficients S_{ij}, L_i, H? We will find that this is indeed the case, and we will give an explicit algorithm to test observability. Most of the work consists of the computation of the characteristic function of the finite-dimensional distributions of the observation process; we employ for this purpose a technique used by Barchielli[2].

Lemma 5.4. *For any $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_k$, we define*

$$\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k) = \Phi_\rho(e^{\sum_{\ell=1}^k\{i\lambda_\ell(Y_{t_\ell}-Y_{t_\ell-1})+\frac{1}{2}\lambda_\ell^2(t_\ell-t_{\ell-1})\}}|\mathfrak{M}_{0]}).$$

Then we can write

$$\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k) = e^{(\mathscr{L}+i\lambda_1\sqrt{\eta}\,\mathscr{K})t_1}\cdots e^{(\mathscr{L}+i\lambda_k\sqrt{\eta}\,\mathscr{K})(t_k-t_{k-1})}I,$$

where $\mathscr{K}[X] = L_1^*X + XL_1$.

Proof. Let $\kappa : [0,\infty[\,\to\mathbb{R}$ be locally bounded and measurable and define

$$\Xi_t(\kappa) = U_t^*\ \exp\left(i\int_0^t \kappa(s)\,dZ_s + \frac{1}{2}\int_0^t \kappa(s)^2\,ds\right)U_t.$$

Using the quantum Itô rules, we find that

$$d\Xi_t(\kappa) = i\kappa(t)\sqrt{\eta}\,\Xi_t(\kappa)\sum_{k=1}^{q_0}\{j_t(S_{1k})\,dA_k(t) + j_t(S_{1k}^*)\,dA_k^\dagger(t)\} \\ + i\kappa(t)\sqrt{\eta}\,j_t(L_1+L_1^*)\,\Xi_t(\kappa)\,dt + i\kappa(t)\sqrt{1-\eta}\,\Xi_t(\kappa)\,(dA_q(t)+dA_q^\dagger(t)).$$

Similarly, we find that

$$dj_t(X) = j_t(\mathscr{L}[X])\,dt + \sum_{i,j,k=1}^{q_0}(S_{ki}^*XS_{kj} - \delta_{ij}X)\,d\Lambda_{ij}(t) \\ + \sum_{i,k=1}^{q_0}\left\{j_t(S_{ki}^*[X,L_k])\,dA_i^\dagger(t) + j_t([L_k^*,X]S_{ki})\,dA_i(t)\right\}.$$

Using the quantum Itô rules once more and retaining only the time integrals,

$$j_t(X)\Xi_t(\kappa) = X + \int_0^t \{j_s(\mathscr{L}[X]) + i\kappa(s)\sqrt{\eta}\,j_s(\mathscr{K}[X])\}\,\Xi_s(\kappa)\,ds + \text{martingales}.$$

Thus evidently, if we define $\Upsilon_t(\kappa,X) = \Phi_\rho(j_t(X)\Xi_t(\kappa)|\mathfrak{M}_{0]})$, then

$$\frac{d}{dt}\,\Upsilon_t(\kappa,X) = \Upsilon_t(\kappa,\mathscr{L}[X] + i\kappa(t)\sqrt{\eta}\,\mathscr{K}[X]).$$

The result now follows directly by setting

$$\kappa(s) = \lambda_1 I_{[0,t_1[}(s) + \lambda_2 I_{[t_1,t_2[}(s) + \cdots + \lambda_k I_{[t_{k-1},t_k[}(s),$$

then solving the equation for $\Upsilon_t(\kappa,X)$ with $X=I$. □

Proposition 5.1. *The observable space $\mathcal{O}$ can be characterized as*

$$\mathcal{O} = \mathrm{span}\{\mathscr{L}^{c_1}\mathscr{K}^{d_1}\mathscr{L}^{c_2}\cdots\mathscr{L}^{c_k}\mathscr{K}^{d_k}I : k, c_i, d_i \geq 0\}.$$

In particular, $\mathcal{O}$ is the smallest linear subspace of $\mathcal{A}$ that contains I and is invariant under the action of $\mathscr{L}$ and $\mathscr{K}$. The model is observable if and only if $\dim\mathcal{O} = p^2$.

Proof. First, we claim that $\mathcal{O}$ coincides with the linear span of $\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k)$ for all $t_1,\ldots,t_k,\lambda_1,\ldots,\lambda_k$. To see this, note that the characteristic function of the joint distribution of $Y_{t_1},\ldots,Y_{t_k}$ under the state Φ_ρ is precisely $\rho(\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k))$ (up to a constant factor). As the finite dimensional distributions determine the law of the observations, we have $\rho_1 \backsim \rho_2$ if and only if

$$\rho_1(\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k)) = \rho_2(\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k)) \qquad \forall\, t_1,\ldots,t_k,\lambda_1,\ldots,\lambda_k.$$

Thus evidently every element of the linear span of $\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k)$ is in $\mathcal{O}$. Conversely, suppose that $X \in \mathcal{O}$ is not in the linear span of $\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k)$; then there must exist an $Y \in M_p$ such that $\mathrm{Tr}[Y^*X] \neq 0$, but $\mathrm{Tr}[Y^*Z] = 0$ for all Z in the linear span of $\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k)$. But writing Y as $\alpha(\varrho_a - \varrho_b) + i\beta(\varrho_c - \varrho_d)$ with $\alpha,\beta \in \mathbb{R}$ and $\varrho_a,\ldots,\varrho_d$ density matrices corresponding to states $\rho_a,\ldots,\rho_d$, we find that either $\rho_a(X) \neq \rho_b(X)$, or $\rho_c(X) \neq \rho_d(X)$, while nonetheless $\rho_a \backsim \rho_b$ and $\rho_c \backsim \rho_d$. We thus have a contradiction, and the claim is established.

We now claim that the linear span of $\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k)$ coincides with the linear span of $\mathscr{L}^{c_1}\mathscr{K}^{d_1}\cdots\mathscr{L}^{c_k}\mathscr{K}^{d_k}I$. First, note that any element of the latter form can be obtained from $\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k)$ by taking derivatives with respect to t_i and λ_i. This means, in particular, that any element of the latter form is in the closure of the linear span of $\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k)$. But we are working in finite dimensions, so the linear span is already closed. It remains to show that any $\Upsilon_{t_1,\ldots,t_k}(\lambda_1,\ldots,\lambda_k)$ is in the linear span of elements of the form $\mathscr{L}^{c_1}\mathscr{K}^{d_1}\cdots\mathscr{L}^{c_k}\mathscr{K}^{d_k}I$. This is an immediate consequence of the Cayley-Hamilton theorem, and the claim is established.

Finally, we must show that $\mathcal{O}$ is the smallest linear subspace of $\mathcal{A}$ that contains I and is invariant under the action of $\mathscr{L}$ and $\mathscr{K}$. Note that $\mathcal{O}$ is clearly invariant under $\mathscr{L}$ and $\mathscr{K}$ and contains I, so the smallest linear subspace such that this holds is contained in $\mathcal{O}$. Conversely, any element of $\mathcal{O}$ can be generated by applying $\mathscr{L}$ and $\mathscr{K}$ to I finitely many times and taking finitely many linear combinations, and the smallest subspace must contain at least these elements. This establishes the claim. Note that the model is observable if and only if $\mathcal{O} = \mathcal{A}$, which is clearly equivalent to $\dim \mathcal{O} = p^2$. The proof is complete. □

Using this characterization of $\mathcal{O}$ we can construct and explicit algorithm for verifying observability. To this end, we define the linear spaces $\mathcal{Z}_n \subset \mathcal{A}$ by

$$\mathcal{Z}_0 = \mathrm{span}\{I\}, \qquad \mathcal{Z}_n = \mathrm{span}\{\mathcal{Z}_{n-1},\ \mathscr{L}[\mathcal{Z}_{n-1}],\ \mathscr{K}[\mathcal{Z}_{n-1}]\}, \quad n \geq 1.$$

Clearly every element of $\mathcal{O}$ will be in $\mathcal{Z}_n$ for some finite n. Moreover, if $\mathcal{Z}_n = \mathcal{Z}_{n+1}$ for some $n = m$, then it is true for all $n > m$, and in particular $\mathcal{Z}_m = \mathcal{O}$. But this will always be the case for some finite n: after all, the linear spaces $\mathcal{Z}_n$ grow with n, but $\dim \mathcal{Z}_n$ can not exceed p^2. Hence this construction is guaranteed to yield $\mathcal{O}$ in a finite number of steps. To implement the procedure as a computational algorithm, one could start with $\{I\}$ in the first step, then apply the Gram-Schmidt procedure at every iteration n to obtain a basis for $\mathcal{Z}_n$.

5.2. *Photon counting*

In the photon counting case, we consider the observations

$$Y_t = U_t^*\{\eta\,\Lambda_{11}(t) + (1-\eta)\,\Lambda_{qq}(t) + \sqrt{\eta(1-\eta)}\,(\Lambda_{1q}(t) + \Lambda_{q1}(t))\}U_t,$$

where $\eta \in\,]0,1]$ is again the detection efficiency and $q_0 < q$. Once again Y_t is self-adjoint and affiliated to $\mathfrak{M}_{t]}$, and we must verify the various properties of section 2. The proofs of these properties are identical, however, to the homodyne case, so there is no need to repeat them. We only collect here the required facts.

Lemma 5.5. *Denote by $\mathfrak{N}_{t]}$ the Von Neumann algebra generated by*

$$\{N_s := \eta\,\Lambda_{11}(s) + (1-\eta)\,\Lambda_{qq}(s) + \sqrt{\eta(1-\eta)}\,(\Lambda_{1q}(s) + \Lambda_{q1}(s)) : s \leq t\}.$$

Then $U_T^*\mathfrak{N}_{t]}U_T = U_t^*\mathfrak{N}_{t]}U_t = \mathfrak{Y}_{t]}$ *for every* $0 \le t \le T$*. In particular, the self-nondemolition and nondemolition properties hold:*

$$\mathfrak{Y}_{t]} \text{ is commutative,} \qquad j_t(X) \in \mathfrak{Y}_{t]}' \qquad \forall\, t \ge 0,\ X \in \mathcal{A}.$$

Moreover, for any $t_1, \ldots, t_k > 0$ *and bounded continuous* $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{R}$*,*

$$\Phi_\rho(f_1(Y_{t_1}) \cdots f_k(Y_{t_k}) | \mathfrak{M}_{0]}) = j_0(Z(t_1, \ldots, t_k, f_1, \ldots, f_k))$$

for some $Z(t_1, \ldots, t_k, f_1, \ldots, f_k) \in \mathcal{A}$ *independent of* ρ*, and moreover*

$$\Phi_\rho(f_1(Y_{s+t_1} - Y_s) \cdots f_k(Y_{s+t_k} - Y_s) | \mathfrak{M}_{s]}) = j_s(Z(t_1, \ldots, t_k, f_1, \ldots, f_k))$$

for every $s \ge 0$*.*

Proof. The proofs of these facts are identical to the proofs of lemma 5.2, corollary 5.1, and lemma 5.3, and are thus omitted here. □

Remark 5.2. Also in this setting one can compute the filter explicitly as the solution of a stochastic differential equation driven by the observations:

$$d\pi_t^\rho(X) = \pi_t^\rho(\mathscr{L}[X])\, dt + \left[\frac{\pi_t^\rho(L_1^* X L_1)}{\pi_t^\rho(L_1^* L_1)} - \pi_t^\rho(X) \right] (dY_t - \eta\, \pi_t^\rho(L_1^* L_1)\, dt),$$

where $\pi_0^\rho(X) = \rho(X)$. We will not need this representation of the filter in this paper.

To proceed, we must adapt lemma 5.4 to the current setting.

Lemma 5.6. *For any* $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_k$*, we define*

$$\Upsilon_{t_1,\ldots,t_k}(\lambda_1, \ldots, \lambda_k) = \Phi_\rho(e^{\sum_{\ell=1}^k \{i\lambda_\ell(Y_{t_\ell} - Y_{t_\ell - 1})\}} | \mathfrak{M}_{0]}).$$

Then we can write

$$\Upsilon_{t_1,\ldots,t_k}(\lambda_1, \ldots, \lambda_k) = e^{(\mathscr{L} + (e^{i\lambda_1} - 1)\eta \mathscr{J})t_1} \cdots e^{(\mathscr{L} + (e^{i\lambda_k} - 1)\eta \mathscr{J})(t_k - t_{k-1})} I,$$

where $\mathscr{J}[X] = L_1^* X L_1$*.*

Proof. Let $\kappa : [0, \infty[\to \mathbb{R}$ be locally bounded and measurable and define

$$\Xi_t(\kappa) = U_t^*\, \exp\left(i \int_0^t \kappa(s)\, dN_s \right) U_t.$$

Using the quantum Itô rules, we find that

$$\begin{aligned}
d\Xi_t(\kappa) &= \eta\,(e^{i\kappa(t)}-1)\Xi_t(\kappa)\sum_{i,j=1}^{q_0} j_t(S_{1i}^*S_{1j})\,d\Lambda_{ij}(t) \\
&\quad + (1-\eta)\,(e^{i\kappa(t)}-1)\Xi_t(\kappa)\,d\Lambda_{qq}(t) \\
&\quad + \sqrt{\eta(1-\eta)}\,(e^{i\kappa(t)}-1)\Xi_t(\kappa)\sum_{i=1}^{q_0}\{j_t(S_{1i}^*)\,d\Lambda_{iq}(t)+j_t(S_{1i})\,d\Lambda_{qi}(t)\} \\
&\quad + \eta\,(e^{i\kappa(t)}-1)\Xi_t(\kappa)\sum_{i=1}^{q_0}\{j_t(S_{1i}^*L_1)\,dA_i^\dagger(t)+j_t(L_1^*S_{1i})\,dA_i(t)\} \\
&\quad + \sqrt{\eta(1-\eta)}\,(e^{i\kappa(t)}-1)\Xi_t(\kappa)\sum_{i=1}^{q_0}\{j_t(L_1^*)\,dA_q(t)+j_t(L_1)\,dA_q^\dagger(t)\} \\
&\quad + \eta\,(e^{i\kappa(t)}-1)\Xi_t(\kappa)\,j_t(L_1^*L_1)\,dt.
\end{aligned}$$

Using the quantum Itô rules once more and retaining only the time integrals,

$$j_t(X)\Xi_t(\kappa) = X + \int_0^t \{j_s(\mathscr{L}[X]) + (e^{i\kappa(s)}-1)\eta\, j_s(\mathscr{J}[X])\}\,\Xi_s(\kappa)\,ds + \text{martingales}.$$

Thus evidently, if we define $\Upsilon_t(\kappa,X)=\Phi_\rho(j_t(X)\Xi_t(\kappa)|\mathfrak{M}_{0]})$, then

$$\frac{d}{dt}\,\Upsilon_t(\kappa,X) = \Upsilon_t(\kappa,\mathscr{L}[X]+(e^{i\kappa(t)}-1)\eta\mathscr{J}[X]).$$

The result now follows directly by setting

$$\kappa(s) = \lambda_1 I_{[0,t_1[}(s) + \lambda_2 I_{[t_1,t_2[}(s) + \cdots + \lambda_k I_{[t_{k-1},t_k[}(s),$$

then solving the equation for $\Upsilon_t(\kappa,X)$ with $X=I$. □

The following result now follows precisely as before.

Proposition 5.2. *The observable space $\mathcal{O}$ can be characterized as*

$$\mathcal{O} = \mathrm{span}\{\mathscr{L}^{c_1}\mathscr{J}^{d_1}\mathscr{L}^{c_2}\cdots\mathscr{L}^{c_k}\mathscr{J}^{d_k}I : k, c_i, d_i \geq 0\}.$$

In particular, $\mathcal{O}$ is the smallest linear subspace of $\mathcal{A}$ that contains I and is invariant under the action of $\mathscr{L}$ and $\mathscr{J}$. The model is observable if and only if $\dim\mathcal{O}=p^2$.

Proof. The proof is identical to that of proposition 5.1. □

5.3. *Some remarks*

In this section, we have obtained precise characterizations of when a homodyne detection or photon counting model is observable (when the initial system is finite dimensional). This yields a simple algorithm to test observability, from which stability of the filter follows directly due to theorem 2.1. Even in the absence of observability, however, one can say something about the stability of certain observables using corollary 3.2. The simplest such result is the following.

Corollary 5.2. *For the homodyne detection model (resp. photon counting model), the observable $M = L_1 + L_1^*$ (resp. $L_1^* L_1$) is always stable in the sense that*

$$\Phi_{\rho_1}(|\pi_t^{\rho_1}(M) - \pi_t^{\rho_2}(M)|) \xrightarrow{t\to\infty} 0$$

whenever the law of the observations under Φ_{ρ_1} is absolutely continuous with respect to the law of the observations under Φ_{ρ_2}.

Proof. This is immediate from corollary 3.2 and the fact that $M = L_1 + L_1^* = \mathscr{K}[I]$ (resp. $M = L_1^* L_1 = \mathscr{J}[I]$) is clearly in $\mathcal{O}$. $\square$

In the physics literature, the observable M in this corollary is sometimes called the *measurement observable*. The fact that the measurement observable is always stable regardless of any other properties of the model was established for the homodyne detection case in §5.3.2 of Ref. 20 using a different method.

We conclude this section with an example that highlights the importance of the absolute continuity of the observations in our results.

Example 5.1. We consider the homodyne detection model, and let us choose $q_0 = 1$, $S_{11} = I$, $H = 0$, and $L_1 = F/2$ with $F = \mathrm{diag}\{1, 2, \ldots, p\}$. By the previous corollary, the measurement observable $M = F$ is stable in the sense that

$$\Phi_{\rho_1}(|\pi_t^{\rho_1}(F) - \pi_t^{\rho_2}(F)|) \xrightarrow{t\to\infty} 0$$

whenever the observations are absolutely continuous as required by theorem 2.1. It is easily verified, however, that any state ρ with density matrix of the form $\varrho = \mathrm{diag}\{0, \ldots, 0, 1, 0, \ldots, 0\}$ is a fixed point for the filtering equation in remark 5.1. Hence $\Phi_{\rho_1}(|\pi_t^{\rho_1}(F) - \pi_t^{\rho_2}(F)|) \not\to 0$ when ρ_1, ρ_2 are two different states of this form. Evidently the absolute continuity requirement is essential. We refer to Ref. 21 for a discussion of the connection between the weakening of the absolute continuity requirement and the notion of controllability in the classical setting.

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