Wheel-free planar graphs

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Abstract

A *wheel* is a graph formed by a chordless cycle C and a vertex u not in C that has at least three neighbors in C. We prove that every 3-connected planar graph that does not contain a wheel as an induced subgraph is either a line graph or has a clique cutset. We prove that every planar graph that does not contain a wheel as an induced subgraph is 3-colorable.

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1 Introduction

All graphs in this paper are finite and simple. A graph G *contains* a graph F if an induced subgraph of G is isomorphic to F. A graph G is F*-free* if G does not contain F. For a set of graphs $\mathcal{F}$, G is $\mathcal{F}$*-free* if it is F-free for every $F \in \mathcal{F}$. An *element* of a graph is a vertex or an edge. When S is a set of elements of G, we denote by $G \setminus S$ the graph obtained from G by deleting all edges of S and all vertices of S.

A *wheel* is a graph formed by a chordless cycle C and a vertex u not in C that has at least three neighbors in C. Such a wheel is denoted by (u, C); u is the *center* of the wheel and C the *rim*. Observe that K_4 is a wheel (in some papers on the same subject, K_4 is not considered as a wheel). Wheels play an important role in the proof of several decomposition theorems. Little is known about wheel-free graphs. The only positive result is due to Chudnovsky (see [1] for a proof). It states that every non-null wheel-free graph contains a vertex whose neighborhood is made of disjoint cliques with no edges between them. No bound is known on the chromatic number of wheel-free graphs. No decomposition theorem is known for wheel-free graphs. However, several classes of wheel-free graphs were shown to have a structural description.

- Say that a graph is *unichord-free* if it does not contain a cycle with a unique chord as an induced subgraph. The class of $\{K_4$, unichord$\}$-free graphs is a subclass of wheel-free graphs (because every wheel contains a K_4 or a cycle with a unique chord as an induced subgraph), and unichord-free graphs have a complete structural description, see [10].

- It is easy to see that the class of graphs that do not contain a subdivision of wheel as an induced subgraph is the class of graphs that do not contain a wheel or a subdivision of K_4 as induced subgraphs. Here again, this subclass of wheel-free graphs has a complete structural description, see [7].

- The class of graphs that do not contain a wheel as a subgraph does not have a complete structural description so far. However, in [9] (see also [2]), several structural properties for this class are given.

- A *propeller* is a graph formed by a chordless cycle C and a vertex u not in C that has at least *two* neighbors in C. So, wheels are just special propellers, and the class of propeller-free graphs is a subclass of wheel-free graphs. In [3], a structural description of propeller-free graphs is given.

Interestingly, every graph that belongs to one of the four classes described above is 3-colorable (this is shown in cited papers). One might conjecture that every wheel-free graph is 3-colorable, but this is false as shown by the graph represented on Figure 1 (it is wheel-free and have chromatic number 4). Also, the four classes have polynomial time recognition algorithms, so one could conjecture that so does the class of wheel-free graphs. But it is proved in [5] that it is NP-hard to recognize them. All this suggest that possibly, no structural description of wheel-free graphs exists.

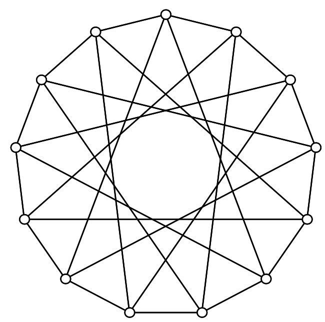

Figure 1: The Ramsey graph $R(3,5)$, that is the unique graph G satisfying $|V(G)| \geq 13$, $\alpha(G) = 4$ and $\omega(G) = 2$.

In this paper, we study planar wheel-free graphs. A *clique cutset* of a graph G is a clique K such that $G \setminus K$ is disconnected. When the clique has size three, it is referred to as a *K_3-cutset*. We prove the next theorems.

Theorem 1.1 *If G is a 3-connected wheel-free planar graph, then G either is a line graph or G has a clique cutset.*

We now give a complete description of 3-connected wheel-free planar graphs, but we first need some terminology. A graph is *basic* if it is the line graph of a graph H such that either H is $K_{2,3}$, or H can be obtained from a 3-connected cubic planar graph by subdividing every edge exactly once. We need to name four special graphs: the claw, the diamond, the butterfly and the paw, that are represented in Figure 2. Basic graphs have a simple characterization given below.

Theorem 1.2 *Let G be a graph. The following statements are equivalent.*

1. *G is basic.*

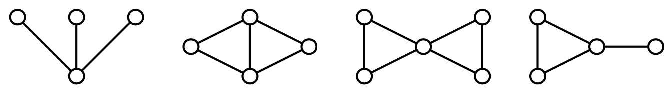

Figure 2: The claw, the diamond, the butterfly and the paw.

2. *G is a 3-connected wheel-free planar line graph.*

3. *G is 3-connected, planar and $\{K_4$, claw, diamond, butterfly$\}$-free.*

With Theorems 1.1 and 1.2, we may easily prove the complete description of 3-connected wheel-free planar graphs.

Theorem 1.3 *The class $\mathcal{C}$ of 3-connected wheel-free planar graphs is the class of graphs that can be constructed as follows: start with basic graphs and repeatedly glue previously constructed graphs along cliques of size three that are also face boundaries.*

PROOF — By Theorem 1.2, a basic graph is in $\mathcal{C}$. Also gluing along cliques of size three that are also face boundaries preserves being in $\mathcal{C}$ (in particular, it does not create wheels, because wheels have no clique cutset). It follows that the construction only constructs graphs in $\mathcal{C}$.

Conversely, let G be a graph in $\mathcal{C}$. We prove by induction on $|V(G)|$ that G can be constructed as we claim. We apply Theorem 1.1 to G. If G is a line graph, since it is also planar, 3-connected and wheel-free, by Theorem 1.2, it must be basic. So G has a clique cutset. Since G is 3-connected, this clique must be a triangle K. And since G is planar, for every connected component C of $G \setminus K$, K must be a face boundary of $G[C \cup K]$ (otherwise, a subdivision of $K_{3,3}$ exists in G, contradiction to the planarity of G). It follows by induction that G can be constructed from previously contructed graphs by gluing along a triangle that is also a face boundary. $\square$

A consequence of our description is the following.

Theorem 1.4 *Every wheel-free planar graph is 3-colorable.*

We have no conjecture (and no theorem) about the structure of wheel-free planar graphs in general (possibly not 3-connected). In Figure 3 several wheel-free planar graphs of connectivity 2 are represented. It can be checked that the three graphs belong to none of the four classes described above (each of them contains a cycle with a unique chord, an induced subdivision of K_4, a wheel as a subgraph and a propeller). So we do not understand them. We

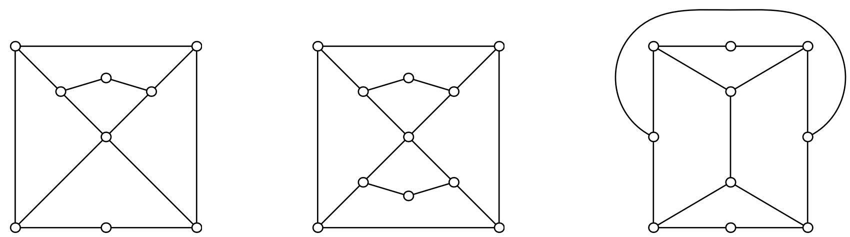

Figure 3: Some wheel-free planar graphs

leave the description of the most general wheel-free planar graph as an open question.

Section 2 gives the proof of Theorem 1.1, and in fact of a slight generalization that we need in Section 4. In Section 3, we prove Theorem 1.2. Theorem 1.4 is proved in Section 4.

Notation, definitions and preliminaries

We use notation and classical results from [4]. Let G be a graph, $X \subseteq V(G)$ and $u \in V(G)$. We denote by $G[X]$ the subgraph of G induced on X, by $N(u)$ the set of neighbors of u, and by $N(X)$ the set of vertices of $V(G) \setminus X$ adjacent to at least one vertex of X; and we define $N_X(u) = N(u) \cap X$. We sometimes write $G \setminus u$ instead of $G \setminus \{u\}$. When e is an edge of G, we denote by G/e the graph obtained from G by contracting e.

A *path* P is a graph with $k \geq 1$ vertices that can be numbered $p_1, \dots, p_k$, and with $k-1$ edges p_ip_{i+1} for $1 \leq i < k$. The vertices p_1 and p_k are the *end-vertices* of P, and $\{p_2, \dots, p_{k-1}\}$ is the *interior* of P. We also say that P is a p_1p_k*-path.* If P, Q are paths, disjoint except that they have one end-vertex v in common, then their union is a path and we often denote it by P-v-Q. If a, b are vertices of a path P, we denote the subpath of P with end-vertices a, b by a-P-b.

A *cycle* C is a graph with $k \geq 3$ vertices that can be numbered $p_1, \dots, p_k$, and with k edges p_ip_{i+1} for $1 \leq i \leq k$ (where $p_{k+1} = p_1$).

Let Q be a path or a cycle in a graph G. The *length* of Q is the number of its edges. An edge $e = xy$ of G is a *chord* of Q if $x, y \in V(Q)$, but xy is not an edge of Q. A chord is *short* if its ends are joined by a two-edge path in Q.

By the Jordan curve theorem, a simple closed curve C of the plane partitions its complement into a bounded open set and an unbounded open set. They are respectively the *interior* and the *exterior* of C.

We need the following.

Theorem 1.5 (Harary and Holzmann [6]) *A graph is the line graph of a triangle-free graph if and only if it is* {*diamond, claw*}*-free.*

2 Almost 3-connected wheel-free planar graphs

A graph G is *almost 3-connected* if it is 3-connected or if it can be obtained from a 3-connected graph by subdividing one edge exactly once. For a 2-connected graph drawn in the plane, the boundary of every face is a cycle. We need the following consequence.

Theorem 2.1 *Let G be an almost 3-connected graph drawn in the plane, and let x be a vertex of G such that all its neighbors have degree at least three. Let R be the face of $G \setminus \{x\}$ in which x is drawn. Then the boundary of R is a cycle C, and C goes through every vertex of $N(x)$.*

In this section, we prove the theorem below, which clearly implies Theorem 1.1. We prove the stronger statement below because we need it in the proof of Theorem 1.4.

Theorem 2.2 *If a graph G is an almost 3-connected wheel-free planar graph with no clique cutset, then G is a line graph.*

PROOF — The proof is by contradiction, so suppose that G is an almost 3-connected wheel-free planar graph that has no clique cutset and that is not a line graph.

(*1) Let $\{a, b, c\}$ be a clique of size three in G, and let P be a chordless path of $G \setminus \{b, c\}$ with one end a. Then at least one of b, c has no neighbor in $V(P) \setminus \{a\}$.*

Suppose b, c both have neighbors in $V(P) \setminus \{a\}$, and P' be the minimal subpath of P such that $a \in V(P')$, and both b and c have neighbors in $V(P') \setminus \{a\}$. We may assume that P' is from a to x, x is adjacent to b, and b has no neighbor in $V(P') \setminus \{a, x\}$. Then a-P'-x-b-a is an induced cycle, say C. Now since c is adjacent to a and b, and has a neighbor in $V(P') \setminus \{a\}$, it follows that (c, C) is a wheel, a contradiction. This proves (1).

(*2) G is diamond-free.*

Suppose that $\{a, x, b, y\}$ induces a diamond of G, and $xy \notin E(G)$. Since $\{a, b\}$ is not a cutset of G, there exists a chordless xy-path P in $G \setminus \{a, b\}$, contrary to (1). This proves (2).

A vertex e of G is a *corner* if e has degree two, and there exist four vertices a, b, c, d such that $E(G[\{a, b, c, d, e\}]) = \{ab, ac, bc, cd, de, eb\}$.

(3) No vertex of G is a corner.

Suppose that $e \in V(G)$ is a corner and let a, b, c, d be four vertices as in the definition. Since $\{b, c\}$ is not a cutset of G, there exists a chordless ad-path P in $G \setminus \{b, c\}$. But now the path a-P-d-e contradicts (1). This proves (3).

(4) G contains a claw.

Otherwise, by (2) and Theorem 1.5, G is a line graph, a contradiction. This proves (4).

The rest of the proof is in two steps. We first prove the existence of a special cutset, called an "I-cutset" (defined below). Then we use the I-cutset to obtain a contradiction.

Let $\{u, x, y\}$ be a cutset of size three of G. A component of $G \setminus \{u, x, y\}$ is said to be *degenerate* if it has only one vertex, or it has exactly two vertices a, b and $G[\{u, x, y, a, b\}]$ has the following edge-set: $\{xy, ax, ay, ab, bu\}$, and *nondegenerate* otherwise.

A cutset $\{u, x, y\}$ of size three of G is an *I-cutset* if $G[\{u, x, y\}]$ induces at least one edge and $G \setminus \{u, x, y\}$ has at least two connected components that are non-degenerate.

(5) G admits an I-cutset.

Fix a drawing of G in the plane. By (4), G contains a claw. Let u be the center of a claw. Let $u'_1, u_2, \dots, u_k$ ($k \geq 3$) be the neighbors of u, in cyclic order around u, where $u_2, \dots, u_k$ have degree at least three. If u'_1 has degree two, let u_1 be its neighbor different from u, and otherwise let $u_1 = u'_1$.

Deleting u, and also deleting u'_1 if u'_1 has degree two, yields a 2-connected graph, drawn in the plane, and therefore, the face R of this drawing in which u is drawn is bounded by a cycle C. By Theorem 2.1 $u_1, u_2, \dots, u_k$ all belong to C, and are in order in C. For $i = 1, \dots, k$, let $S_{u_i u_{i+1}}$ (subscripts are taken modulo k) be the unique $u_i u_{i+1}$-path included in C that contains none of $u_1, \dots, u_k$ except u_i and u_{i+1}.

Assume that xy is a chord of C. Vertices x and y edge-wise partition C into two xy-paths, say P' and P''. Since R is a face of $G \setminus \{u\}$ or of $G \setminus \{u, u_1'\}$, it follows that $\{u, x, y\}$ is a cutset of G that separates the interior of P' from the interior of P''. If xy is not a short chord, then both these interiors contain at least two vertices and therefore $\{u, x, y\}$ is an I-cutset of G. So we may assume that xy is short. If x, y both belong to $S_{u_i u_{i+1}}$ for some i, then $\{x, y\}$ is a clique-cutset of G, a contradiction. Thus we may assume that for every chord xy of C, there exists $i \in \{1, \dots, k\}$ such that $x \in S_{u_{i-1} u_i}$, $y \in S_{u_i u_{i+1}}$ and both xu_i and yu_i are edges.

Claim. u_1' exists, u is of degree three and is adjacent to u_1'.

To prove the claim, assume by way of contradiction that u has at least three neighbors of degree at least 3. Since G is wheel-free, C must have chords. Let xy be a chord, and choose $i \in \{1, \dots, k\}$ such that xu_i and yu_i are edges of C. Suppose first that we cannot choose xy and i such that u_i is adjacent to u. Consequently $i = 1$, and u_1' has degree two; moreover, the cycle obtained from C by replacing the edges xu_1 and $u_1 y$ by xy is induced. Since in this case $k \geq 4$, it follows that u has at least three neighbors in this cycle and so G contains a wheel, a contradiction.

We can therefore choose xy and i such that u_i is adjacent to u. It follows that u_{i+1}, u_{i-1} are not consecutive in C, since u is the center of a claw. We claim that there are no edges between $S_{u_i u_{i+1}} \setminus \{u_i\}$ and $S_{u_{i-1} u_i} \setminus \{u_i\}$, except xy. For suppose such an edge exists, say ab. Since u_{i+1}, u_{i-1} are not consecutive in C, it follows that ab is a chord of C. Since every chord of C is short, it follows that $\{a, b\} = \{u_{i+1}, u_{i-1}\}$ and that $u_{i+1} u_{i+2}$ and $u_{i+2} u_{i-1}$ are both edges of C. But now, $G[u, u_{i+1}, u_{i-1}, u_{i+2}]$ is a wheel or, in case one of $i-1$, $i+1$ or $i+2$ equals 1 and u_1' has degree 2, $G[u, u_{i+1}, u_{i-1}, u_{i+2}, u_1']$ is a wheel.

Hence there are no edges between $S_{u_i u_{i+1}} \setminus \{u_i\}$ and $S_{u_{i-1} u_i} \setminus \{u_i\}$ except xy and thus,

$$u\text{-}u_{i-1}\text{-}S_{u_{i-1}} u_i\text{-}x\text{-}y\text{-}S_{u_i u_{i+1}}\text{-}u_{i+1}\text{-}u$$

or, in the case where $i - 1 = 1$ and u_1' exists,

$$u\text{-}u_1'\text{-}u_1\text{-}S_{u_1} u_2\text{-}x\text{-}y\text{-}S_{u_1 u_2}\text{-}u_3\text{-}u$$

is an induced cycle containing three neighbors of u_i, a contradiction. This proves the claim.

Observe that the claim implies that every center of a claw in G has degree three and is adjacent to u_1'. In particular, $k = 3$.

Let x, y be the neighbors of u_2 in $S_{u_1u_2}$, $S_{u_2u_3}$ respectively. Note that possibly $x = u_1$. Observe that, since u is the center of a claw, $y \neq u_3$. Since every center of a claw is adjacent to u_1', it follows that u_2 is not the center of a claw and thus xy is an edge. Now,

$$x\text{-}y\text{-}S_{u_2u_3}\text{-}u_3\text{-}u\text{-}u_1'\text{-}u_1\text{-}S_{u_1u_2}\text{-}x$$

must admit a chord, for otherwise u_2 is the center of a wheel of G. Hence u_1u_3 is an edge. Let z be the neighbor of u_3 in $S_{u_2u_3}$. Since u_3 is not the center of claw, u_1z is an edge and thus u_1' is a corner, a contradiction to (3). This proves (5).

(6) Let $\{u, x, y\}$ be an I-cutset of G where xy is an edge and let C be a nondegenerate connected component of $G \setminus \{u, x, y\}$. Then there exist $v \in \{x, y\}$ and a path P of $G[C \cup \{u, x, y\}]$ from u to v, such that the vertex of $\{x, y\} \setminus \{v\}$ has no neighbor in $V(P) \setminus \{v\}$.

Since G does not admit a clique cutset, it follows that u is non-adjacent to at least one of x, y. If u is adjacent to exactly one vertex among x and y, then the claim holds. So we may assume that u is adjacent to neither x nor y.

Since G is {diamond, K_4}-free, at most one vertex of G is adjacent to both x and y. Let a be such a vertex, if it exists. Let $K = \{x, y, a\}$ if a exists, and let $K = \{x, y\}$ otherwise.

Since K is not a clique cutset in G, we deduce that u has a neighbor in every component of $C \setminus K$. Suppose first that there is a component C' of $C \setminus K$ containing a neighbor of one of x, y. Let P be a path with interior in C', one of whose ends is u, and the other one is in $\{x, y\}$, and subject to that as short as possible. Then only one of x, y has a neighbor in $V(P) \setminus \{x, y\}$, and (6) holds. So we may assume that no such component C' exists, and thus neither of x, y has neighbors in $V(C) \setminus K$.

Let $L = \{a, u\}$ if a exists, and otherwise let $L = \{u\}$. Then L is a cutset in G separating $C \setminus L$ from x, y. Since G is almost 3-connected, it follows that $L = \{a, u\}$, and $C \setminus L$ consists of a unique vertex of degree two, so C is degenerate, a contradiction. This proves (6).

For every I-cutset $\{u, x, y\}$, some nondegenerate component C_1 of $G \setminus \{u, x, y\}$ has no vertex with degree two in G; choose an I-cutset $\{u, x, y\}$ and C_1 such that $|V(C_1)|$ is minimum. We refer to this property as the *minimality of* C_1. Put $G_1 = G[C_1 \cup \{u, x, y\}]$, and $G_2 = G \setminus C_1$. Assume without loss of generality that xy is an edge, and let $C_2 \neq C_1$ be another nondegenerate component.

From (6) and the symmetry between x, y, we may assume without loss of generality that there is a chordless path Q of G_2 from u to x such that y has no neighbor in $V(Q) \setminus \{x\}$, and in particular u, y are non-adjacent. Also, since u, y both have neighbors in C_2, there is a chordless path R of G_2 between u, y not containing x. Since u, y both have neighbors in C_1, there is a chordless path P of G_1 between u, y not containing x. Consequently the union of P, Q and the edge xy is a cycle S. Let D be the disc bounded by S.

Suppose that some edge of G_1 incident with x is in the interior of D, and some other such edge is in the exterior of D. By adding these two edges to an appropriate path within $G[C_1]$, we obtain a cycle S_0 drawn in the plane, such that the path formed by the union of xy and P crosses it exactly once; and so one of y, u is in the interior of the disc bounded by S_0, and the other in the exterior. But this is impossible, because y, u are also joined by the path R, which is included in G_2 and thus is disjoint from $V(S_0)$. We deduce that we may arrange the drawing so that every edge of G_1 incident with x belongs to the interior of D. In addition we may arrange that the edge xy is incident with the infinite face.

Subject to this condition (and from now on with the drawing fixed), let us choose P so that D is minimum. Since $u, x, y \in V(S)$, every component of $G \setminus V(S)$ has vertex set either a subset of C_1 or disjoint from C_1. Suppose that some vertex c of C_1 is drawn in the interior of D, and let K be the component of $G \setminus V(S)$ containing it. From the choice of P, it follows that there do not exist two non-consecutive vertices of P both with neighbors in K and, since $|N(K)| \geq 3$ (because G is almost 3-connected and all vertices in C_1 are of degree at least three), and $N(K) \subseteq V(P) \cup \{x\}$, we deduce that $|N(K)| = 3$, and $N(K) = \{x, a, b\}$ say, where a, b are consecutive vertices of P. From the minimality of C_1, $\{x, a, b\}$ is not an I-cutset, and thus K is degenerate. Hence, since K has no vertex of degree 2, $|V(K)| = 1$, i.e. $V(K) = \{c\}$. Therefore c has degree three, with neighbors x, a, b. But then c has three neighbors in S, and so G contains a wheel, a contradiction.

Thus no vertex in C_1 is drawn in the interior of D. So, since all edges of G_1 incident with x belong to the interior of D, every neighbor of x in C_1 belongs to P. Since G is almost 3-connected and all vertices of C_1 are of degree at least 3, x has at least one neighbor in C_1. Since $P \cup R$ is a chordless cycle, it follows that x has at most two neighbors in P (counting y), and so only one neighbor in C_1. Let x_1 be the unique neighbor of x in C_1.

Since $|V(C_1)| \geq 2$, there is a vertex x_2 different from x_1 in C_1, and since G is almost 3-connected, there are two paths of G, from x_2 to u, y respectively, vertex-disjoint except for x_2, and not containing x_1. Consequently both these paths are paths of G_1, and so there is a path of G_1 between u, y, containing neither of x, x_1. We may therefore choose a chordless path P' of G_1 between u, y, containing neither of x, x_1. It follows that the union of P', Q and the edge xy is a chordless cycle S' say, bounding a disc D' say; choose P' such that D' is minimal. Since x_1 is in P and xy is incident with the infinite face, it follows that x_1 is in the interior of D'.

Let Z be the set of vertices in $C_1 \setminus \{x_1\}$ that are drawn in the interior of D'. We claim that every vertex in Z has degree three, and is adjacent to x_1 and to two consecutive vertices of P'. For let $c \in Z$, and let K be the component of $G \setminus (V(S') \cup \{x_1\})$ that contains c. From the choice of P', no two non-consecutive vertices of P' have neighbors in K, and so as before, $N(K) = \{a, b, x_1\}$, where a, b are consecutive vertices of P', and $|V(K)| = 1$. It follows that every vertex in Z has degree three and is adjacent to x_1 and to two consecutive vertices of P'.

Let x_1 have t neighbors in P'. Thus x_1 has at least $t+1$ neighbors in the chordless cycle S', and consequently $t \leq 1$ since G does not contain a wheel. The degree of x_1 equals $|Z| + t + 1$, and since x_1 has degree at least three and $t \leq 1$, we deduce that $Z \neq \emptyset$, and either $t = 1$, or $t = 0$ and $|Z| > 1$. Choose $z \in Z$, and let z be adjacent to a, b, x_1, where u, a, b, y are in order in P'.

We claim that x_1 is adjacent to neither u nor y. For suppose x_1 is adjacent to u or y. Since x_1 is the unique neighbor of x in C_1, $\{u, x_1, y\}$ is a cutset of G separating $C_1 \setminus \{x_1\}$ from the rest of the graph. So, since it is not an I-cutset and since all vertices in C_1 have degree at least 3, $|C_1 \setminus \{x_1\}| = 1$ and thus $C_1 \setminus \{x_1\} = \{z\}$. Since z has degree at least 3, z is adjacent to u, y and x_1 and, since $z \notin P'$, $P' = uy$. Hence $G[\{u, y, z, x_1\}]$ is a diamond, a contradiction to (2) or else x_1 is adjacent to both u and y and $G[\{u, y, z, x_1\}]$ is a wheel, a contradiction. So x_1 is adjacent to neither u nor y.

If x_1 has a neighbor in $P' \setminus \{u, y\}$ (a unique neighbor because $t \leq 1$) between u and a, say v, then z has three neighbors in the chordless cycle formed by the union of x_1v, the subpath of P' between v and y, and the edges yx and xx_1. On the other hand, if x_1 has a neighbor in P' between b and y, say v, then x_1 has three neighbors in the chordless cycle formed by the union of x_1v, the subpath of P' between v and u, the path Q and the edge xx_1. Thus x_1 has no neighbor in P', and so $t = 0$ and $|Z| \geq 2$. Let $z' \in Z \setminus \{z\}$, adjacent to x_1, a', b' say, where a', b' are consecutive vertices of P', and u, a', b', y are in order on P'. From planarity, $\{a, b\} \neq \{a', b'\}$, and so we may assume that u, a, a', y are in order on P'. But then z' has three neighbors in the chordless cycle formed by the path y-x-x_1-z-b and the subpath of P' between b and y, a contradiction. □

3 A characterization of basic graphs

We need the following.

Theorem 3.1 (Sedlaček [11]) *If H is a graph of maximum degree at most three, then $L(H)$ is planar if and only if H is planar.*

We now prove the following implications between the three statements of Theorem 1.2.

(1 ⇒ 2). Suppose that G is a basic graph. From the definition, G is a line graph of a planar graph R of maximum degree at most 3. Moreover, it is easy to see that G is 3-connected. By Theorem 3.1, G is planar. It remains to check that is wheel-free. If $R = K_{2,3}$, then G is obviously wheel-free. Otherwise, R is obtained from a 3-connected cubic planar graph by subdividing every edge exactly once. Suppose for a contradiction that (u, C) is a wheel in G. Since $G = L(R)$, u is an edge of R, and we set $u = xy$ where x has degree 3 and y has degree 2. Let x' be the other neighbor of y (so, x' has degree 3 in R). In R, there are two edges e and f different from xy and incident to x. And there are two edges e' and f' different from $x'y$ and incident to x'. Since u (seen as a vertex of G) has degree 3, the cycle C of G must go through e, f and yx' (also seen as vertices of G). But to go in and out from the vertex yx' of G, the only way is through e' and f' that are adjacent. It follows that C has a chord, a contradiction.

(2 ⇒ 3). Suppose that G is a 3-connected wheel-free planar line graph, say $G = L(R)$. Since G is a line graph, it is claw-free. Since G is wheel-free, it is K_4-free.

Suppose for a contradiction that G contains a diamond. It follows that R contains a paw (see Figure 2), say a triangle xyz and vertex t adjacent to x and to none of y or z. Since G is 3-connected, the removal of the edge xt in R keeps R connected. It follows that in R, there is path P from t to y or z, that does not use the edge tx. The edges of P, together with the edges tx, xy, yz and zx form a wheel in G, a contradiction.

Suppose finally that G contains a butterfly. The vertex of degree 4 in the butterfly is an edge xy in R, and both x and y have degree at least 3 (because of the butterfly), and in fact exactly 3 (because G contains no K_4). Since G is 3-connected, the removal of the vertex xy of G keeps G 2-connected. It follows that in R, there exists a cycle through x and y that does not go through the edge xy. Hence, the edges of this cycle form the rim of a wheel in G (the center is the vertex xy of G). This is a contradiction.

($\mathbf{3 \Rightarrow 1}$). By Theorem 1.5, G is the line graph of a triangle-free graph R. Since G is K_4-free, every vertex of R has degree at most 3. In particular, since G is planar, by Theorem 3.1, R must be planar. Also, if R has a cutvertex x, at least one pair of edges incident to x form a cutset (of vertices) in G, because G has at least four vertices since it is 3-connected. This is a contradiction to the 3-connectivity of G. It follows that R is 2-connected.

If two adjacent vertices of R have degree 3, then G contains a diamond or a butterfly, a contradiction. Hence, R is edge-wise partitioned into its branches, where a *branch* in a graph is a path of length at least 2, whose ends have degree at least 3 and whose internal vertices have degree 2. In fact, every branch of R has length exactly 2 because a branch of length at least 3 would yield a vertex of degree 2 in G, a contradiction to its 3-connectivity.

Suppose that there is a pair of vertices x, y of degree 3 in R such that at least two distinct branches P, Q have ends x and y. We denote by e (resp. f) the edge incident to x (resp. y) that does not belong to P or Q. Now, $G \setminus \{e, f\}$ is disconnected (contradicting G being 3-connected), unless e and f are the only edges of R that do not belong to P and Q. But in this case, $R = K_{2,3}$. So, from here on, we may assume that for all pair of vertices x, y from R, there is at most one branch of R with ends x and y.

It follows that by dissoluting all vertices of degree 2 of R, a cubic graph R' is obtained. Suppose that R' is not 3-connected. This means that $R' \setminus \{x, y\}$ is disconneted where x and y are vertices of R'. Since R' is cubic, for at least one component C_x of $R' \setminus \{x, y\}$, x has a unique neighbor x' in C_x. Also, y has a unique neighbor y' in some component C_y. Now, xx' and yy' are two edges of R' whose removal disconnects R'. These two edges are subdivided in R, but they still yield two edges whose removal disconnects R. This yields two vertices in G whose removal disconnects G, a contradiction to G being 3-connected. We proved that R is obtained from a 3-connected cubic graph (namely R') by subdividing once every edge.

4 Coloring wheel-free planar graphs

A *coloring* of G is a function $\pi : V(G) \to \mathcal{C}$ such that no two adjacent vertices receive the same color $c \in \mathcal{C}$. If $\mathcal{C} = \{1, 2, \ldots, k\}$, we say that π is a *k-coloring* of G. An *edge-coloring* of G is a function $\pi : E(G) \to \mathcal{C}$ such that no two adjacent edges receive the same color $c \in \mathcal{C}$. If $\mathcal{C} = \{1, 2, \dots, k\}$, we say that π is a *k-edge-coloring* of G. Observe that an edge-coloring of a graph H is also a coloring of $L(H)$.

A graph R is *chordless* if every cycle in R is chordless. A way to obtain a chordless graph is to take any graph and to subdivide all edges. It follows that basic graphs are in fact line graphs of chordless graphs. This is the property of basic graphs that we rely on in this section.

It is proved in [8] that for all $\Delta \geq 3$ and all chordless graphs G of maximum degree Δ, G is Δ-edge-colorable (for $\Delta = 3$, a simpler proof is given in [7]). Unfortunately, this result is not enough for our purpose and we reprove it for $\Delta = 3$ in a slightly more general form. A graph is *almost chordless* if at most one of its edges is the chord of a cycle.

Theorem 4.1 *If G is an almost chordless graph with maximum degree three, then G is 3-edge-colorable.*

PROOF — Let G be a counter-example with minimum number of edges. Let $X \subseteq V(G)$ be the set of vertices of degree three and $Y = V(G) \setminus X$ the set of vertices of degree at most two.

(1) Y is a stable set.

For suppose that there exists an edge uv such that u and v belong to Y. From the minimality of G there exists a 3-edge-coloring of $G \setminus uv$. Since $u, v \in Y$, it is easy to extend the 3-edge-coloring of $G \setminus uv$ to a 3-edge-coloring of G, a contradiction. This proves (1).

(2) G is 2-connected.

Otherwise G has a cut-vertex v, so $V(G) \setminus \{v\}$ partitions into two nonempty sets of vertices C_1 and C_2 with no edges between them. A 3-edge-coloring of G can be obtained easily from 3-edge-colorings of $G[C_1 \cup \{v\}]$ and $G[C_2 \cup \{v\}]$, a contradiction. This proves (2).

(3) If e, f are disjoint edges of G, then $G \setminus \{e, f\}$ is connected.

Suppose there exists two disjoint edges u_1u_2 and v_1v_2 such that $G \setminus \{u_1u_2, v_1v_2\}$ is not connected; then $G \setminus \{u_1u_2, v_1v_2\}$ partitions into two nonempty sets of vertices C_1 and C_2 with no edges between them. By (2) we may assume that $\{u_1, v_1\} \subseteq C_1$ and $\{u_2, v_2\} \subseteq C_2$. For $i = 1, 2$, let G_i be the graph obtained from $G[C_i]$ by adding a vertex m_i adjacent to both u_i and v_i. If G_1 contains a cycle C with a chord ab, then ab is a chord of a cycle of G (this is clear when C does not contain m_1, and when C contains m_1, the cycle is obtained by replacing m_1 by a u_2v_2-path included in C_2 that exists by (2)). It follows that G_1 and symmetrically G_2 are almost chordless. Moreover they both clearly have maximum degree at most three and, by (1), both C_1 and C_2 contain vertices of degree three, so G_1 and G_2 have fewer edges than G. Therefore G_1 and G_2 admit a 3-edge-coloring.

Let π_1 and π_2 be 3-edge-colorings of respectively G_1 and G_2. We may assume without loss of generality that $\pi_1(u_1m_1) = \pi_2(u_2m_2) = 1$ and $\pi_1(v_1m_1) = \pi_2(v_2m_2) = 2$. Now, the following coloring π is a 3-edge-coloring of G: $\pi(u_1v_1) = 1$, $\pi(u_2v_2) = 2$, $\pi(e) = \pi_1(e)$ if $e \in E(G_1)$ and $\pi(e) = \pi_2(e)$ if $e \in E(G_2)$, a contradiction. This proves (3).

(4) $G[X]$ has at most one edge, and if it has one, it is a chord of a cycle of G.

Suppose that xy is an edge of $G[X]$ such that $G \setminus xy$ is not 2-connected. Then, there exists a vertex w such that $G \setminus \{xy, w\}$ is disconnected. Let C_x and C_y be the two components of $G \setminus \{xy, w\}$, where $x \in C_x$ and $y \in C_y$. Since w is of degree at most three, w has a unique neighbor w' in one of C_x, C_y, say in C_x. If $w' = x$, then x is a cut-vertex of G (because $|C_x| > 1$ since x has degree three), a contradiction to (2). So $w' \neq x$ and hence xy, ww' are disjoint, a contradiction to (3).

Therefore, for every edge xy of $G[X]$, $G \setminus xy$ is 2-connected. So, if such an edge exists, by Menger's theorem there exists a cycle C going through both x and y in $G \setminus xy$, and thus xy is a chord of C. Since G is almost chordless, there is at most one such edge. This proves (4).

If G is chordless, then by (1) and (4), (X, Y) forms a bipartition of G, so by a classical theorem of Kőnig, G is 3-edge-colorable, a contradiction. So let xy be a chord of a cycle of G. Let x' and x'' be the two neighbors of x distinct from y and let y' and y'' be the two neighbors of y distinct from x. By (4), x', x'', y' and y'' are all of degree 2 and by (1), they induce a stable set. If $\{x', x''\} = \{y', y''\}$, then G is the diamond and thus is 3-edge-colorable. If $|\{x', x''\} \cap \{y', y''\}| = 1$, say $x' = y'$ and $x'' \neq y''$, then xx'', yy'' are disjoint and their deletion disconnects G, a contradiction to (3). Hence x', x'', y' and y'' are pairwise distinct.

Let x'_1 (resp. x''_1, y'_1, y''_1) be the unique neighbor of x' (resp. x'', y'_1, y''_1) distinct from x (resp. y). Let G' be the graph obtained from G by deleting the edge xy and contracting edges xx', xx'', yy' and yy''. We note x the vertex resulting from the contraction of xx' and xx'', and y the vertex resulting from the contraction of yy' and yy''. Since G' has maximum degree at most three and is bipartite by (1) and (4), it follows that G' has a 3-edge-coloring π' by Kőnig's theorem.

Assume without loss of generality that $\pi'(xx'_1) = 1$, $\pi'(xx''_1) = 2$, $\pi'(yy'_1) = a$ and $\pi'(yy''_1) = b$ where $\{a, b\} \subseteq \{1, 2, 3\}$. Since $\{a, b\} \cap \{1, 2\} \neq \emptyset$, we may assume without loss of generality that $a = 1$, so $b \neq 1$. Let us now extend this coloring to a 3-edge-coloring π of G. For any edge e of G such that its extremities are not both in $\{x, y, x', x'', y', y'', x'_1, x''_1, y'_1, y''_1\}$, set $\pi(e) = \pi'(e)$. Set $\pi'(x'x'_1) = 1$, $\pi'(x''x''_1) = 2$, $\pi'(y'y'_1) = 1$ and $\pi'(y''y''_1) = b$. Now we can set $\pi(xx') = 2$, $\pi(xx'') = 1$, $\pi(yy') = 2$, $\pi(yy'') = 1$ and $\pi(xy) = 3$. So π is a 3-edge-coloring of G. □

Note that in the next proof, we do not use planarity, except when we apply Theorem 2.2.

Proof of Theorem 1.4

We argue by induction on $|V(G)|$. Suppose first that G admits a clique cutset K. Let C_1 be the vertex set of a component of $G \setminus K$ and $C_2 = V(G) \setminus (K \cup C_1)$. By induction $G[C_1 \cup K]$ and $G[C_2 \cup K]$ are both 3-colorable and thus G is 3-colorable. So we may assume that G does not admit clique cutsets. If G has a vertex u of degree two, then we can 3-color $G \setminus \{u\}$ by induction and extend the coloring to a 3-coloring of G. So we may assume that every vertex of G has degree at least three.

Assume now that G is 3-connected. By Theorem 2.2, there exists a chordless graph H of maximum degree three such that $G = L(H)$. Hence, by Theorem 4.1, H is 3-edge-colorable and thus G is 3-colorable. So we may assume that the connectivity of G is two.

Let $\{a, b\} \subseteq V(G)$ be such that $G \setminus \{a, b\}$ is disconnected. We choose $\{a, b\}$ to minimize the smallest order of a component of $G \setminus \{a, b\}$, and let C be the vertex set of this component. If $|C| = 1$, then the vertex in C is of degree two in G, a contradiction. So $|C| \geq 2$. Let G'_C be the graph obtained from $G[C \cup \{a, b\}]$ by adding the edge ab (that did not exist since G has no clique cutset). Let us prove that G'_C is 3-connected. Since $|C| \geq 2$ and G'_C therefore has at least four vertices, we may assume by contradiction that G'_C admits a 2-cutset $\{x, y\}$. Let $C_1, \dots, C_k$ $(k \geq 2)$ be the vertex sets of the components of $G'_C \setminus \{x, y\}$. Since ab is an edge of G'_C, a and b are included in $G'_C[C_i \cup \{x, y\}]$ for some $i \leq k$, say $i = 2$. Hence $\{x, y\}$ is a cutset of G and C_1 is a component of $G \setminus \{x, y\}$ that is a proper subset of C, a contradiction to the minimality of C. So G'_C is 3-connected. (But it might not be wheel-free.)

Let G_C be the graph obtained from G'_C by subdividing ab once, and let m be the vertex of degree two of G_C. Since G'_C is 3-connected, G_C is almost 3-connected. Suppose that G_C admits a wheel (u, R). Since G is wheel-free, m must be a vertex of (u, R). Since m is of degree two, m is in R, and so a-m-b is a subpath of R. Since G is 2-connected, there exists a chordless ab-path P in $G \setminus C$. Hence by replacing a-m-b by P, we obtain a wheel in G, a contradiction. Therefore G_C is an almost 3-connected wheel-free planar graph.

By Theorem 2.2, there exists a chordless graph H of maximum degree three such that $L(H) = G_C$. We are now going to prove there exist two ways to 3-edge-color H, one giving the same color to a and b (that are edges of H), and the other giving distinct colors to a and b. This implies that there exist two ways to 3-color $G[C \cup \{a, b\}]$, one giving the same color to a and b and the other giving distinct colors to a and b. Since by the inductive hypothesis there exists a 3-coloring of $G \setminus C$, it follows that this 3-coloring can be extended to a 3-coloring of G.

We first prove that there exists a 3-edge-coloring π of H such that $\pi(a) \neq \pi(b)$. Observe that both ends of m are of degree two in H. Hence, H/m is also a chordless graph with maximum degree at most three. Therefore there exists a 3-edge-coloring π of H/m and clearly π statisfies $\pi(a) \neq \pi(b)$. It is easy to extend π to a 3-edge-coloring of H by giving a color distinct from $\pi(a)$ and $\pi(b)$ to m.

Let us now prove that there is a 3-edge-coloring of H such that $\pi(a) = \pi(b)$. Let $m = m_a m_b$, $a = m_a a_1$ and $b = m_b b_1$. We claim that $a_1 b_1$ is not an edge of H. For if $a_1 b_1$ is an edge of H, then there exists a vertex x in G_C adjacent to both a and b. Since G_C is almost 3-connected and m is the only vertex of degree 2 in G_C, $G_C \setminus \{x, m\}$ is connected, and thus there exists a path P between a and b avoiding x and m. Since a and b are not adjacent, P is of length at least 2. Naming u the vertex of P adjacent to a, u is adjacent to x, otherwise $G_C[\{a, x, u, m\}]$ is a claw of G_C, contradicting the fact that G_C is a line graph. Hence x has at least three neighbors in the chordless cycle formed by the path P and the edges am and bm, a contradiction to the fact that G_C is wheel-free. So $a_1 b_1$ is not an edge of H.

Let H' be the graph obtained from H by deleting the vertices m_a and m_b and adding the edge $a_1 b_1$. If an edge xy distinct from $a_1 b_1$ is the chord of a cycle Q, then since it is not a chord in H, Q must contain $a_1 b_1$. Then by replacing $a_1 b_1$ by a_1-m_a-m_b-b_1, we deduce that xy is also the chord of a cycle in H, a contradiction. Hence H' is almost chordless and thus, by Theorem 4.1, H' admits a 3-edge-coloring π'. Assume that $\pi'(a_1 b_1) = 1$. Then setting $\pi(a_1 m_a) = \pi(b_1 m_b) = 1$ and $\pi(m_a m_b) = 2$, we obtain a 3-

edge-coloring of H satisfying $\pi(a_1 m_a) = \pi(b_1 m_b)$. This completes the proof of Theorem 1.4. $\square$

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