Tree-width and planar minors

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Abstract

Robertson and the second author [7] proved in 1986 that for all h there exists $f(h)$ such that for every h-vertex simple planar graph H, every graph with no H-minor has tree-width at most $f(h)$; but how small can we make $f(h)$? The original bound was an iterated exponential tower, but in 1994 with Thomas [9] it was improved to $2^{O(h^5)}$; and in 1999 Diestel, Gorbunov, Jensen, and Thomassen [3] proved a similar bound, with a much simpler proof. Here we show that $f(h) = 2^{O(h\log(h))}$ works. Since this paper was submitted for publication, Chekuri and Chuzhoy [2] have announced a proof that in fact $f(h)$ can be taken to be $O(h^{100})$.

1 Introduction

Graphs in this paper are finite, and may have loops or multiple edges. A *tree-decomposition* of a graph G is a pair (T, W), where T is a tree, $W = (W_t : t \in V(T))$ is a family of subsets of $V(G)$, and for each $t \in V(T)$, $W_t \subseteq V(G)$ satisfies the following:

- $\cup_{t \in V(T)} W_t = V(G)$, and for every edge uv of G there exists $t \in V(T)$ with $u, v \in W_t$
- if $r, s, t \in V(T)$, and s is on the path of T between r and t, then $W_r \cap W_t \subseteq W_s$.

A graph G has *tree-width* w if $w \geq 0$ is minimum such that G admits a tree-decomposition $(T, (W_t : t \in V(T)))$ satisfying $|W_t| \leq w + 1$ for each $t \in V(T)$.

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges, and if so, we say G has an *H-minor*. The following was proved in [7]:

1.1 *For every planar graph H there exists w such that every graph with no H-minor has tree-width at most w.*

This is of interest for two reasons. First, graphs of bounded tree-width are easier to handle (for algorithms, and for proving theorems) than general graphs, and 1.1 tells us that if we we have to give up this advantage, we have a large grid minor instead, which can also be useful (again for algorithms, and for proving theorems). Second, no non-planar graph H satisfies the conclusion of 1.1 (because if H is non-planar, then for any w a large enough grid has tree-width at least w and does not contain H as a minor).

In this paper we are concerned with the numerical dependence of w on H, and for this let us restrict ourselves to *simple* planar graphs H. In [9], it was shown that for every simple planar graph H with h vertices, every graph with no H-minor has tree-width at most 20^{64h^5}. Diestel, Gorbunov, Jensen, and Thomassen [3] proved a similar bound of $2^{O(h^5 \log(h))}$, with a much simpler proof. Here we prove that:

1.2 *For every simple planar graph H with h vertices, every graph with no H-minor has tree-width at most $2^{15h+8h\log(h)}$.*

(Logarithms have base two.) Independently, Kawarabayashi and Kobayashi [5] have proved a similar result. Since the present paper was submitted for publication in June 2012, Chandra Chekuri and Julia Chuzhoy [2] have announced a much stronger result, namely that the bound in 1.2 can be replaced by a polynomial in h (currently $O(h^{100})$).

Our proof uses the same approach as the proof of [3], but we implement some of the detailed arguments more efficiently. The work reported here is partially based on [6]. Let us sketch the proof.

- A "linkage" means a set of vertex-disjoint paths. Our main tool, which was also the main tool of [3], is the "linkage lemma", that if G has two sufficiently large linkages $\mathcal{P}, \mathcal{Q}$, where the paths in $\mathcal{P}$ are between two sets $A, B \subseteq V(G)$, then either G contains a large grid as a minor, or there is a path $Q \in \mathcal{Q}$ such that there is still a large linkage between A, B in G disjoint from Q, not as large as before, but still as large as we need. (This is more-or-less 3.1.)
- To prove the linkage lamma, we first prove (as did [3]) that if G contains a large grill, then G contains a large grid minor. A "grill" means a set of pairwise disjoint connected subgraphs,

and a set of vertex-disjoint paths, so that each path has one vertex from each subgraph, and always in the same order. We have found a better proof for this than that in [3], and this is the main place where we gain numerically. (This is 2.1.)

- To deduce the linkage lemma from the grill lemma, we choose the linkage $\mathcal{P}$ such that the union of its paths with the paths of the linkage $\mathcal{Q}$ is as small as possible; then for each edge e which belongs to a path in $\mathcal{P}$, if e also belongs to a member of $\mathcal{Q}$ we can contract it and win by induction, so we assume not; and the choice of $\mathcal{P}$ tells us that if we deleted e, there would be no linkage any more between A, B of cardinality $|\mathcal{P}|$. Consequently there is a cutset (consisting of the edge e and otherwise of vertices) that separates A from B, of order $|\mathcal{P}|$. This gives us many cutsets, one for each edge in each path of $\mathcal{P}$, and we can uncross them so they all line up linearly, and the paths in $\mathcal{P}$ cross each of them only once, and in the right order. Then we have something like a large grill, ready for the application of the grill lemma. (This is in section 3.)
- With this linkage lemma in hand, we turn to the main proof. If the tree-width is large, then any attempt to grow a tree-decomposition of small width must get stuck at some stage; and by "greedily" growing a tree-decomposition as far as we can, it is easy to obtain a separation (A, B) of large (but bounded) order, k say, such that every two subsets of $A \cap B$ of the same size are joined within $G|B$ by a linkage of that size. With more care, we can choose (A, B) so that in addition, it is possible to contract a path from $G|A$ onto the vertices in $A \cap B$. (This is 4.1.)
- Let us perform this contraction; so now we have a graph G' say with vertex set B, which is a minor of G, with a k-vertex path P say, with the property that any two subsets of $V(P)$ of the same size are joined by a linkage of that size. Partition P into many long subpaths, say $P_1, \dots, P_t$. Any two of these are joined by a large linkage in G'; and by repeated application of the linkage lemma, we can choose a path between every two of $P_1, \dots, P_t$, such that all these paths are pairwise disjoint. But then we have a large clique as a minor, and so we win.

This is the basic idea of the proof. There are some technical refinements that are not worth detailing here; for instance,

- we can use the part $G|A$ of the separation above to get more of the desired minor, by contracting a more complicated tree onto $A \cap B$ than just a path;
- it is wasteful to produce a large clique minor at the end, because all we need is a large grid minor, so we should just obtain paths between certain pairs of $P_1, \dots, P_t$ rather than between all of them;
- sometimes we don't even want a large grid minor, we want some particular planar graph as a minor, and it is wasteful to produce the large grid in its place; we get better numbers by going directly for the minor that we really want.

2 Linkages, grids and grills

If G is a graph, a *linkage* in G means a set $\mathcal{P}$ of paths of G, pairwise vertex-disjoint (a *path* has no "repeated" vertices); and if $A, B \subseteq V(G)$, an (A, B)-*linkage* means a linkage $\mathcal{P}$ such that each of its paths has one end in A and the other in B. If $\mathcal{P}$ is a linkage, $\cup\mathcal{P}$ means the subgraph formed by the union of the paths in $\mathcal{P}$.

For $g \geq 1$, the *$g \times g$-grid* is the graph with g^2 vertices $u_{i,j}$ $(1 \leq i, j \leq g)$ where $u_{i,j}$ and $u_{i',j'}$ are adjacent if $|i - i'| + |j - j'| = 1$.

For $a, b > 0$, an *(m,n)-grill* is a graph G with mn vertices

$$\{v_{i,j} : \; 1 \leq i \leq m, 1 \leq j \leq n\}$$

in which

- for $1 \leq p \leq m$ and $1 \leq j < n$, $v_{p,j}$ is adjacent to $v_{p,j+1}$, and so $v_{p,1}$-$v_{p,2}$-$\cdots$-$v_{p,n}$ are the vertices in order of a path, P_p say
- for $1 \leq j \leq n$, the subgraph T_j induced on $\{v_{1,j}, \ldots, v_{m,j}\}$ is connected.

The result of this section is the following. (This is the main improvement of our proof over the proof in [3]; they prove something analogous, but they require n to be exponentially large to get essentially the same conclusion.)

2.1 *Let $g \geq 1$ and $h \geq 3$ be integers, and let $m \geq (2g+1)(2h-5)+2$ and $n \geq h(2g+h-2)$ be integers. Let G be an (m,n)-grill. Then G contains either a $g \times g$-grid or the complete bipartite graph $K_{h,h}$ as a minor.*

The proof requires several lemmas.

2.2 *Let G be an (m,n)-grill, with the usual notation. Let $1 \leq h < j \leq n$, and let $X \subseteq V(T_h)$ and $Y \subseteq V(T_j)$, with $|X| = |Y| = k$ say. If $j - h \geq k+1$, there is an (X,Y)-linkage $\mathcal{P}$ of cardinality k in G, such that every vertex of $\cup\mathcal{P}$ belongs to $X \cup Y \cup \bigcup_{h<i<j} V(T_i)$.*

Proof. Let H be the subgraph of G induced on $X \cup Y \cup \bigcup_{h<i<j} V(T_i)$. We must show that in H there are k vertex-disjoint paths from X to Y. Suppose not; then there exists $Z \subseteq V(H)$ with $|Z| < k$ such that every connected subgraph of H containing a vertex of X and one of Y also contains a vertex of Z. Since $|X| > |Z|$, there exists $p \in \{1, \ldots, m\}$ such that $X \cap V(P_p) \neq \emptyset$ and $Z \cap V(P_p) = \emptyset$; and similarly there exists $q \in \{1, \ldots, m\}$ such that $Y \cap V(P_q) \neq \emptyset$ and $Z \cap V(P_q) = \emptyset$. Since $j - h > k$, and therefore there are at least $|Z|+1$ values of i with $h < i < j$, it follows that there exists i with $h < i < j$ such that $Z \cap V(T_i) = \emptyset$. But then

$$(P_p \cap H) \cup (P_q \cap H) \cup T_i$$

is a connected subgraph of H meeting both X, Y and not meeting Z, a contradiction. This proves 2.2. ∎

A *leaf* of a graph is a vertex of degree one. The following result is related to that of [4].

2.3 *If $r \geq 1$ and $h \geq 3$ are integers, and G is a connected simple graph with*

$$|V(G)| \geq (r+2)(2h-5)+2,$$

then either

- *G has a spanning tree T with at least h leaves, or*
- *there is a path of G with r vertices, such that all its internal vertices have degree two in G.*

Proof. Suppose that neither of these outcomes hold, for a contradiction. Since G is connected, it has a spanning tree; choose a spanning tree T with as many leaves as possible. (Note that $|V(G)| \geq 2$ and so no vertex of T has degree zero.) Now T has at most $h-1$ leaves, and hence (since $|V(G)| \geq 2$) has at most $2h-4$ vertices of degree different from two. Consequently T is a subdivision of a tree with at most $2h-4$ vertices and hence at most $2h-5$ edges. Thus T is the union of at most $2h-5$ paths, such that every internal vertex of each of these paths is in D, where D is the set of vertices that have degree two in T. Let the longest such path have vertices $v_1\text{-}\cdots\text{-}v_t$ say, in order. Thus $(t-2)(2h-5) \geq |D|$.

Let $4 \leq i \leq t-3$. We claim that v_i has degree two in G. For suppose that v_i is adjacent in G to some vertex $v \in V(G) = V(T)$ different from v_{i-1}, v_{i+1}. Let Q be the path of T between v_i and v. From the symmetry we may assume that $v_{i-1} \in V(Q)$. Let T' be the spanning tree of G obtained from T by adding the edge vv_i and deleting the edge $v_{i-2}v_{i-1}$. Now v_{i-1} is a leaf of T', and so is v_{i-2} unless $v = v_{i-2}$. It follows that T' has strictly more leaves than T, a contradiction. Thus v_i has degree two in G, for $4 \leq i \leq t-3$. Let P be the path with vertices $v_3\text{-}v_4\text{-}\cdots\text{-}v_{t-2}$ in order. Then every internal vertex of P has degree two in G. Since P has $t-4$ vertices, we may assume that $t-4 \leq r-1$, and since $(t-2)(2h-5) \geq |D|$, it follows that $(r+1)(2h-5) \geq |D|$. But $|V(G)| \leq 2h-4+|D|$, and so $|V(G)| \leq (r+2)(2h-5)+1$, a contradiction. This proves 2.3. ∎

If X is a subset of the vertex set of a graph G, we denote by $G \setminus X$ the graph obtained by deleting X (and we write $G \setminus v$ for $G \setminus \{v\}$.) We denote by $G|X$ the subgraph of G induced on X.

2.4 *Let $h \geq 1$ be an integer, and let $m \geq h+1$ and $n = h^2$. Let G be an (m,n)-grill, labeled as usual. Suppose that for $1 \leq j \leq h$, $T_{(h+1)j-h}$ has a spanning tree with at least h leaves. Then G contains $K_{h,h}$ as a minor.*

Proof. For $1 \leq j \leq h$, let $X_j \subseteq V(T_{(h+1)j-h})$ be a set of exactly h leaves of some spanning tree of $T_{(h+1)j-h}$; and let $T'_j = T_{(h+1)j-h} \setminus X_j$. Thus each T'_j is a connected subgraph, since $a > h$, and each vertex in X_j has a neighbour in $V(T'_j)$. By 2.2, for $1 \leq j < t$, there is an (X_j, X_{j+1})-linkage $\mathcal{P}_j$ with cardinality h such that every vertex of $\cup\mathcal{P}_j$ belongs to

$$X_j \cup X_{j+1} \cup \bigcup (V(T_i) \ : (h+1)j - h < i < (h+1)(j+1) - h).$$

The graph formed by the union of all the graphs $\cup\mathcal{P}_j$ $(1 \leq j < h)$ therefore has h components, each a path; and if we contract each of these components to a single vertex, and contract each T'_j $(1 \leq j \leq h)$ to a single vertex, we obtain $K_{h,h}$ as a minor. This proves 2.4. ∎

Proof of 2.1. We may assume that $n = h(2g+h-2)$. Let G be an (m,n)-grill, with notation as usual. Let $1 \leq i \leq n-2g+2$, and let H_i be the subgraph of G induced on the set $V(T_i \cup \cdots \cup T_{i+2g-2})$. Let J_i be the simple graph underlying the minor of H_i obtained by contracting all edges of $P_p \cap H_i$ for $1 \leq p \leq m$. For $1 \leq p \leq m$, let u_p be the vertex of J_i formed by contracting the edges of $P_p \cap H_i$.

(1) *If there exists i with $1 \leq i \leq n-2g+2$ such that J_i has no spanning tree with at least h leaves, then G contains a $g \times g$-grid as a minor.*

To prove this, we observe that if i satisfies the hypothesis of (1), then by 2.3, some path of J_i has $2g-1$ vertices, and all its internal vertices have degree two in J_i; and we may assume by renumbering that this path has vertices $u_1, \ldots, u_{2g-1}$ in order. Consequently for $2 \leq p \leq 2g-2$, and $i \leq j \leq i+2g-1$, the vertex $v_{p,j}$ has degree at most two in T_j, and has no neighbours in T_j except possibly $v_{p-1,j}$ and $v_{p+1,j}$. Since T_j is connected, it follows that one of

$$v_{1,j}\text{-}v_{2,j}\text{-}\cdots\text{-}v_{g,j},\ v_{g,j}\text{-}v_{g+1,j}\text{-}\cdots\text{-}v_{2g-1,j}$$

is (the sequence of vertices of) a path of T_j. From the symmetry, we may assume that $v_{1,j}\text{-}v_{2,j}\text{-}\cdots\text{-}v_{g,j}$ is a path of T_j for at least g of the $2g-1$ values of $j \in \{i, \ldots, i+2g-2\}$. But then these g paths, together with the paths $P_1 \cap H_i, \ldots, P_p \cap H_i$ (with some edges contracted appropriately), form a $g \times g$-grid. This proves (1).

From (1) we may assume that for $1 \leq i \leq n-2g+2$, J_i has a spanning tree with at least h leaves. Let $d = 2g+h-1$, and for $j = 1, \ldots, h$ and $1 \leq p \leq m$, let us contract the edges of $P_p \cap H_{jd-d+1}$. This yields an (m, h^2)-grill satisfying the hypotheses of 2.4, and so G contains $K_{h,h}$ as a minor. This proves 2.1. ∎

Let us say that for integers $m, n > 0$ and a real number ϵ with $0 \leq \epsilon \leq 1$, an (m, n, ϵ)-*pregrill* is a graph G such that there is an (A, B)-linkage $\{P_1, \ldots, P_m\}$ in G for some A, B, and there are vertex-disjoint connected subgraphs $T_1, \ldots, T_n$ of G, satisfying:

- for $1 \leq p \leq m$ and $1 \leq i < j \leq n$, every vertex of $P_p \cap T_i$ lies in P_p before every vertex of $P_p \cap T_j$, as P_p is traversed from A to B
- for $1 \leq j \leq n$, $P_p \cap T_j$ is null for at most ϵm values of $p \in \{1, \ldots, m\}$.

Thus if $\epsilon = 0$, then by contracting the edges of each P_p between the first and last vertex of each T_i, and some further contraction, we obtain an (m, n)-grill.

We need a small extension of 2.1:

2.5 *Let $g \geq 1$ and $h \geq 3$ be integers. Let $m \geq 2(2g+1)(h-2)$ and $n \geq 2h(2g+h-2)$ be integers, and let $\epsilon = (4(2g+1)(h-2))^{-1}$. Let G be an (m, n, ϵ)-pregrill, with $P_1, \ldots, P_m$ and $T_1, \ldots, T_n$ as above. Then G contains either a $g \times g$-grid or $K_{h,h}$ as a minor.*

Proof. Let $m' = 2(2g+1)(h-2)$. On average (over $1 \leq i \leq m$), P_i is disjoint from at most ϵn of $T_1, \ldots, T_n$; and so we can choose m' of $P_1, \ldots, P_m$, say $P_1, \ldots, P_{m'}$, such that at most $\epsilon m' n = n/2$ of $T_1, \ldots, T_n$ are disjoint from one of them. Consequently at least $\lceil n/2 \rceil$ of $T_1, \ldots, T_n$ meet all of $P_1, \ldots, P_{m'}$. But then we have an $(m', \lceil n/2 \rceil)$-grill as a minor, and the result follows from 2.1. ∎

3 Finding a path disjoint from a linkage

In this section we prove the following.

3.1 *Let $g \geq 1$, $h \geq 3$, $m \geq 2(2g+1)(h-2)$ and $n = 2h(2g+h-2)m$ be integers, and let $\epsilon = (4(2g+1)(h-2))^{-1}$. Suppose that*

- *G contains neither a $g \times g$-grid nor $K_{h,h}$ as a minor*
- *$A, B \subseteq V(G)$, and there is an (A,B)-linkage in G of cardinality m*
- *$\mathcal{Q}$ is a set of pairwise vertex-disjoint connected subgraphs of G, with $|\mathcal{Q}| \geq n$.*

Then for some $Q \in \mathcal{Q}$, there is an $(A \setminus V(Q), B \setminus V(Q))$-linkage of cardinality at least ϵm in $G \setminus V(Q)$.

We need the following lemma. A *separation* in a graph G is a pair (C, D) of subsets of $V(G)$, such that there is no edge of G between $C \setminus D$ and $D \setminus C$; and its *order* is $|C \cap D|$. The following is essentially theorem 12.1 of [8], but we sketch its proof for the reader's convenience.

3.2 *Let $\mathcal{P}$ be the only (A,B)-linkage of cardinality m in a graph G; and suppose that $V(\cup\mathcal{P}) = V(G)$. Let $|V(G)| = p$. Then there is a sequence (C_i, D_i) $(1 \leq i \leq p-m+1)$ of separations of G, each of order m, satisfying the following:*

- *$C_i \subseteq C_{i+1}$ and $D_{i+1} \subseteq D_i$, for $1 \leq i \leq p-m$;*
- *for $1 \leq i \leq p-m$ there is a unique vertex $u \in C_{i+1} \setminus C_i$, and a unique vertex $v \in D_i \setminus D_{i+1}$, and u, v are adjacent*
- *$C_1 = A$, $D_{p-m+1} = B$, and $D_1 = C_{p-m+1} = V(G)$.*

Proof. We proceed by induction on $|V(G)|$. If there is a separation (C, D) with order m and $A \subseteq C$ and $B \subseteq D$, and with $|C|, |D| > m$, then the result follows by the inductive hypothesis applied to $G|C$ (and the pair $A, C \cap D$) and to $G|D$ (and the pair $C \cap D, B$). We assume there is no such (C, D). If $\cup\mathcal{P}$ has no edges then the result is clear, so we assume that e is an edge of $\cup\mathcal{P}$. From the uniqueness of $\mathcal{P}$, there is no (A,B)-linkage of cardinality m in $G \setminus e$; and so there is a separation (C, D) of $G \setminus e$ of order $m-1$ with $A \subseteq C$, $B \subseteq D$. Thus e has ends u, v where $u \in C \setminus D$ and $v \in D \setminus C$. Both $(C, D \cup \{u\})$ and $C \cup \{v\}, D)$ are separations of G of order m, and since $D \cup \{u\} \neq B$ it follows that $C = A$, and similarly $D = B$. But then the result holds. This proves 3.2. ∎

Proof of 3.1. We proceed by induction on $|V(G)| + |E(G)|$. Let $\cup\mathcal{Q}$ denote the union of the members of $\mathcal{Q}$. Choose an (A,B)-linkage $\mathcal{P}$ in G of cardinality m. If $\mathcal{P}$ can be chosen such that some edge or vertex of G belongs to neither $\cup\mathcal{P}$ nor $\cup\mathcal{Q}$, we may delete it and apply the inductive hypothesis; and similarly if some edge belongs to both $\cup\mathcal{P}, \cup\mathcal{Q}$, we may contract it. Thus we assume that $E(\cup\mathcal{P}) = E(G) \setminus E(\cup\mathcal{Q})$ for every choice of $\mathcal{P}$. If some vertex does not belong to $\cup\mathcal{P}$, we may contract an edge incident with it and apply induction, if there is such an edge; and if there is no such edge then there is a one-vertex graph in $\mathcal{Q}$ disjoint from $\cup\mathcal{P}$ and this satisfies the theorem. So we may assume that $V(\cup\mathcal{P}) = V(G)$ for every choice of $\mathcal{P}$. In particular, $\mathcal{P}$ is the only (A,B)-linkage of cardinality m in G.

Thus $\mathcal{P}$ satisfies the hypotheses of 3.2; let (C_i, D_i) $(1 \le i \le |V(G)| - m + 1)$ be as in 3.2. For each $Q \in \mathcal{Q}$, let $I(Q)$ be the set of all $i \in \{1, \dots, |V(G)| - m + 1\}$ such that some vertex of Q belongs to $C_i \cap D_i$. Each set $I(Q)$ is non-empty, since Q has at least one vertex and every vertex belongs to $C_i \cap D_i$ for some choice of i. Moreover, each $I(Q)$ is an interval (of integers), since Q is connected. For each i, since $|C_i \cap D_i| = m$, there are at most m paths $Q \in \mathcal{Q}$ such that $i \in I(Q)$. It follows that no subset of $\{1, \dots, |V(G)| - m + 1\}$ of cardinality less than n/m has nonempty intersection with all the intervals $I(Q)$ $(Q \in \mathcal{Q})$, since $|\mathcal{Q}| \ge n$. Consequently, there are at least n/m members $Q \in \mathcal{Q}$ such that the corresponding intervals $I(Q)$ are pairwise disjoint, say $Q_1, \dots, Q_{n/m}$. Number them so that the corresponding intervals are in increasing order. Let $\mathcal{P} = \{P_1, \dots, P_m\}$ say. It follows that for $1 \le p \le m$, as P_p is traversed from A to B, for $j < j'$ every vertex of $P_p \cap Q_j$ is before every vertex of $P_p \cap Q_{j'}$. If each Q_j is disjoint from at most ϵm of $P_1, \dots, P_m$, 2.5 implies that G contains either a $g \times g$-grid or $K_{h,h}$ as a minor, a contradiction; and so some Q_j is disjoint from at least ϵm of $P_1, \dots, P_m$. This proves 3.1. ∎

Let $Z \subseteq V(G)$. A path P in G is *Z-proper* if its ends are in Z, but no internal vertex of P is in Z, and P does not have exactly one edge. (A path with no edges whose unique vertex is in Z counts as Z-proper.) A linkage is Z-proper if all its members are Z-proper. We deduce:

3.3 *Let $g \ge 1$, $h \ge 3$, $m \ge 2(2g+1)(h-2)$ and $n = 2h(2g+h-2)m$ be integers, and let $\epsilon = (4(2g+1)(h-2))^{-1}$. Suppose that*

- *G contains neither a $g \times g$-grid nor $K_{h,h}$ as a minor, and $Z \subseteq V(G)$*
- *$A, B \subseteq Z$, and there is a Z-proper (A, B)-linkage in G of cardinality m*
- *$\mathcal{Q}$ is a Z-proper linkage in G, with cardinality n.*

Then for some $Q \in \mathcal{Q}$ there is a $Z \setminus V(Q)$-proper $(A \setminus V(Q), B \setminus V(Q))$-linkage of cardinality at least ϵm in $G \setminus V(Q)$.

Proof. Let $\mathcal{P}$ be a Z-proper (A, B)-linkage with cardinality m. Let G' be the union of $\cup\mathcal{P}$ and $\cup\mathcal{Q}$. By 3.1 applied in G', for some $Q \in \mathcal{Q}$ there is an $(A \setminus V(Q), B \setminus V(Q))$-linkage $\mathcal{R}$ of cardinality at least ϵm in $G' \setminus V(Q)$. Choose $\mathcal{R}$ with $\cup\mathcal{R}$ minimal. We claim that $\mathcal{R}$ is $Z \setminus V(Q)$-proper. For certainly no member of $\mathcal{R}$ has exactly one edge, because no edge of G' has both ends in Z. Suppose for some $R \in \mathcal{R}$, some internal vertex v of R belongs to Z. Thus v has degree at least two in G'; and since $v \in Z$, it has degree at most one in $\cup\mathcal{P}$, and degree one only if v is an end of one of the paths in $\mathcal{P}$. The same holds for $\mathcal{Q}$; and we deduce that v is an end of a member of $\mathcal{P}$ and also an end of a member of $\mathcal{Q}$. In particular it belongs to $A \cup B$; but then some proper subpath of R is a path of G' from $A \setminus V(Q)$ to $B \setminus V(Q)$, contrary to the minimality of $\mathcal{R}$. This proves 3.3. ∎

3.4 *Let $g \ge 1$, $h \ge 3$ and $n \ge 1$ be integers, and let $\epsilon = (4(2g+1)(h-2))^{-1}$. Let $k_1, \dots, k_n \ge 2(2g+1)(h-2)$, where*

$$k_n \ge 2h(2g+h-2)(k_1 + \cdots + k_{n-1}).$$

Suppose that

- *G is a graph containing neither a $g \times g$-grid nor $K_{h,h}$ as a minor, and $Z \subseteq V(G)$, and*
- *for $1 \le i \le n$, there is a Z-proper (A_i, B_i)-linkage of cardinality k_i, where $A_i, B_i \subseteq Z$.*

Then there is a Z-proper path Q from A_n to B_n such that for $1 \le i \le n-1$ there is a $Z \setminus V(Q)$-proper $(A_i \setminus V(Q), B_i \setminus V(Q))$-linkage in $G \setminus V(Q)$ of cardinality at least ϵk_i.

Proof. Let $\mathcal{Q}$ be a Z-proper (A_n, B_n)-linkage of cardinality k_n. For $1 \le i \le n-1$, let $\mathcal{Q}_i$ be the set of all $Q \in \mathcal{Q}$ such that there is no $Z \setminus V(Q)$-proper $(A_i \setminus V(Q), B_i \setminus V(Q))$-linkage of cardinality at least ϵk_i in $G \setminus V(Q)$. By 3.3, $|\mathcal{Q}_i| < 2h(2g+h-2)k_i$, since $k_i \ge 2(2g+1)(h-2)$. Thus

$$|\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{n-1}| < 2h(2g+h-2)(k_1 + \cdots + k_{n-1}) \le k_n,$$

and so some member of $\mathcal{Q}$ belongs to none of $\mathcal{Q}_1, \dots, \mathcal{Q}_{n-1}$. This proves 3.4. ∎

3.5 *Let $g \ge 1$, $h \ge 3$ and $n \ge 1$ be integers, and let $\epsilon = (4(2g+1)(h-2))^{-1}$ and $d = 2h(2g+h-2)$. Let $k_1 = \epsilon^{1-n}$, and for $2 \le i \le n$, let $k_i = d(1+d)^{i-2}\epsilon^{1-n}$. Suppose that*

- *G is a graph containing neither a $g \times g$-grid nor $K_{h,h}$ as a minor, and $Z \subseteq V(G)$,*
- *for $1 \le i \le n$, there is a Z-proper (A_i, B_i)-linkage of cardinality k_i, where $A_i, B_i \subseteq Z$.*

Then there are Z-proper paths $P_1, \dots, P_n$ of G, pairwise vertex-disjoint, such that P_i is from A_i to B_i for $1 \le i \le n$.

Proof. We proceed by induction on n. If $n = 1$ the result is true, since $k_1 > 0$. If $n \ge 2$, note that $k_1, \dots, k_{n-1} \ge \epsilon^{-1} = 4(2g+1)(h-2)$; and

$$2h(2g+h-2)(k_1 + \cdots + k_{n-1}) = d\epsilon^{1-n}(1+d)^{n-2} = k_n.$$

Hence by 3.4, there is a path P between A_n and B_n such that for $1 \le i < n$, there is an $(A_i \setminus V(P), B_i \setminus V(P))$-linkage of cardinality at least ϵk_i in $G \setminus V(P)$. But then the result follows from the inductive hypothesis. This proves 3.5. ∎

4 Highly connected sets without making a mesh

Now we need to use 3.5 to give a bound on the tree-width of the graphs not containing these minors. Here we could just follow [3]; certainly their method is numerically as good as what we are about to present. But their argument at this point can be simplified, and this seems an appropriate place to explain how.

Let $Z \subseteq V(G)$. We say Z is *linked* in G if for every two subsets A, B of Z with $|A| = |B|$ (not necessarily disjoint) there is a Z-proper (A, B)-linkage in G of cardinality $|A|$. What we need to show now, is that if G has big tree-width, then it has a separation (A, B) such that $|A \cap B|$ is big (comparable with the tree-width), and $A \cap B$ is linked in $G|B$, and there are many disjoint connected subgraphs of $G|A$, each containing many members of $A \cap B$. (The first "many" here is the number of vertices in the minor we are excluding, and the second is the size of the k_i's in 3.5.) We will say all this more precisely later.

If we had such a thing, then 3.5 immediately gives our main result. In [3], they construct it by making what they call a "mesh", but this can be improved, in two ways:

- their proof actually constructs something better than a mesh
- a different proof gives something much better than a mesh.

Let us explain.

Let H be a simple graph with vertex set $\{v_1, \dots, v_h\}$. A *model* of H in G means a family $(C_1, \dots, C_h)$ of pairwise disjoint non-null subsets of $V(G)$, each inducing a connected subgraph of G, such that for each edge $v_i v_j$ of H, some vertex of C_i is adjacent in G to some vertex of C_j. If (A, B) is a separation of G, we say that (A, B) *left-contains* a model $(C_1, \dots, C_h)$ of H if $|A \cap B| = h$, and $C_1, \dots, C_h$ are all subgraphs of $G|A$, each containing exactly one vertex of $A \cap B$. If such a model exists we say that (A, B) left-contains H.

4.1 *Let $w \geq 1$ be an integer, and let G be a graph with tree-width at least $3w/2 - 1$. Then there is a separation (A, B) of G such that*

- $|A \cap B| = w$
- *$A \cap B$ is linked in $G|B$, and*
- *(A, B) left-contains a path.*

The proof is exactly the proof of [3], so there is little point in repeating it. In any case, this result is just a special case of the next. Not only can one persuade the separation to left-contain a path, one can make it left-contain any desired tree with the appropriate number of vertices. It seems the easiest proof uses the concept of a "bramble", so we begin with that. A *bramble of order k* in a graph G is a set $\mathcal{B}$ of non-null connected subgraphs of G, such that

- every two members $B_1, B_2 \in \mathcal{B}$ *touch*, that is, either $V(B_1 \cap B_2) \neq \emptyset$, or there is an edge of G with one end in $V(B_1)$ and the other in $V(B_2)$
- for every $X \subseteq V(G)$ with $|X| < k$, there exists $B \in \mathcal{B}$ with $X \cap V(B) = \emptyset$.

The following was proved in [10]:

4.2 *Let G be a graph and $k \geq 1$ an integer. Then G has tree-width at least $k - 1$ if and only if G admits a bramble of order k.*

We use this to show the following:

4.3 *Let $w \geq 1$ be an integer, let T be a tree with $|V(T)| = w$, and let G be a graph with tree-width at least $3w/2 - 1$. Then there is a separation (A, B) of G such that*

- $|A \cap B| = w$
- *$G|(B \setminus A)$ is connected, and every vertex in $A \cap B$ has a neighbour in $B \setminus A$*
- *$A \cap B$ is linked in $G|B$, and*
- *(A, B) left-contains T.*

Proof. The proof is that of [1], modified to use brambles instead of the "blockages" of that paper. Choose a vertex t_1 of T, and number the other vertices $t_2, \dots, t_w$ in such a way that for $2 \le i \le w$, t_i is adjacent to one of $t_1, \dots, t_{i-1}$. For $1 \le i \le w$, let T_i be the subtree of T induced on $\{t_1, \dots, t_i\}$. Now let G be a graph with tree-width at least $3w/2 - 1$; by 4.2 it has a bramble $\mathcal{B}$ of order at least $3w/2$. Choose $\mathcal{B}$ maximal; thus if C is a connected subgraph of G including a member of $\mathcal{B}$, then $C \in \mathcal{B}$ (because otherwise it could be added to $\mathcal{B}$, contrary to maximality). For each $X \subseteq V(G)$ with $|X| \le 3w/2 - 1$, there is therefore a unique component of $G \setminus X$ that belongs to $\mathcal{B}$; let its vertex set be $\beta(X)$.

Choose $v \in \beta(\emptyset)$; then $\beta(\{v\}) \subseteq \beta(\emptyset)$, and the separation $((V(G) \setminus \beta(\emptyset)) \cup \{v\}, \beta(\emptyset))$ left-contains T_1. Consequently we may choose a separation (A, B) of G with the following properties:

- (A, B) has order at least one, and at most w; say order k where $1 \le k \le w$;
- (A, B) left-contains T_k;
- $\beta(A \cap B) \subseteq B$;
- there is no separation (A', B') of G of order strictly less than k, with $A \subseteq A'$ and $B' \subseteq B$, and such that $\beta(A' \cap B') \subseteq B'$;
- subject to these conditions, $|A| - |B|$ is maximum.

(1) *There is no separation (A', B') of G of order k with $A \subseteq A'$ and $B' \subseteq B$ and $(A', B') \neq (A, B)$, such that $\beta(A' \cap B') \subseteq B'$.*

For suppose that there is such a separation (A', B'). From the optimality of (A, B), (A', B') does not left-contain T_k. Consequently there do not exist k vertex-disjoint paths of $G|(B \cap A')$ between $A \cap B$ and $A' \cap B'$; and so by Menger's theorem there is a separation (C, D) of order less than k, with $A \subseteq C$ and $B' \subseteq D$. Since $\beta(C \cap D)$ touches $\beta(A' \cap B')$, and $\beta(A' \cap B') \subseteq B' \subseteq D$, it follows that $\beta(C \cap D) \subseteq D$. But this contradicts the fourth condition above. This proves (1).

(2) *$G|(B \setminus A)$ is connected, and every vertex in $A \cap B$ has a neighbour in $B \setminus A$.*

Now $\beta(A \cap B) \subseteq B \setminus A$, and hence is the vertex set of a component of $G|(B \setminus A)$. Let $D = (A \cap B) \cup \beta(A \cap B)$, and let $C = V(G) \setminus \beta(A \cap B)$; then (C, D) is a separation of G satisfying the first four conditions above. From the optimality of (A, B) it follows that $(A, B) = (C, D)$, and in particular, $\beta(A \cap B) = B \setminus A$. This proves the first assertion. For the second, suppose some $v \in A \cap B$ has no neighbour in $B \setminus A$. Then $(A, B \setminus \{v\})$ is a separation, and since $\beta(A \cap (B \setminus \{v\}))$ touches $\beta(A \cap B)$, and hence is contained in $B \setminus \{v\}$, this contradicts the fourth condition above. This proves (2).

(3) $k = w$.

Suppose that $k < w$. Let $(C_1, \dots, C_k)$ be a model of T_k in $G|A$, where each C_i contains a unique vertex v_i say of $A \cap B$. Let t_{k+1} be adjacent in T to t_i. By (2), v_i has a neighbour in $B \setminus A$, say v_{k+1}. Let $A' = A \cup \{v_{k+1}\}$; then (A', B) is a separation of G, and it left-contains T_{k+1} (as we see by setting $C_{k+1} = \{v_{k+1}\}$). Moreover, since $\beta(A' \cap B)$ touches $\beta(A \cap B)$, it is a subset of B; and (A', B)

satisfies the fourth condition above because of (1). But this contradicts the optimality of (A, B), and hence proves (3).

(4) *$A \cap B$ is linked in $G|B$.*

For suppose not. Let $X, Y \subseteq A \cap B$, with $|X| = |Y|$, such that there is no $A \cap B$-proper (X, Y)-linkage in G of cardinality $|X|$. We may assume that $X \cap Y = \emptyset$ (by replacing X, Y by $X \setminus Y, Y \setminus X$). Let $Z = (A \cap B) \setminus (X \cup Y)$. Let F be the set of edges of G between X and Y, and let G' be obtained from $(G|B) \setminus F$ by deleting Z. Then there is a separation (C', D') of G', such that $|C' \cap D'| < |X|$. Let $C = C' \cup Z$ and $D = D' \cup Z$. Then (C, D) is a separation of $(G \setminus F)|B$ of order less than $|X| + |Z| \leq |C \cap A|, |C \cap B|$. It follows that $(A \cup C, D)$ is a separation of G of order

$$|C \cap D| + |(A \cap B) \setminus C| = |C \cap D| + |A \cap B| - |A \cap C| < |A \cap B| = w,$$

and so $\beta((A \cup C) \cap D)$ exists; and the fourth condition above implies that $\beta((A \cup C) \cap D)$ is not a subset of D. Consequently it is a subset of $A \cup C$. Similarly $\beta(((A \cup D) \cap C) \subseteq A \cup D$.

Let $X = (A \cap B) \cup (C \cap D)$. Since $A \cap B, (A \cup C) \cap D$, and $(A \cup D) \cap C$ are all subsets of X, and each vertex of X belongs to at least two of these three subsets, it follows that $2|X|$ is at most the sum of the cardinalities of these three subsets. Consequently $2|X| \leq w + (w - 1) + (w - 1) = 3w - 2$, and so $|X| \leq 3w/2 - 1$. Thus $\beta(X)$ exists. Since $\beta(X)$ touches $\beta(A \cap B)$, it follows that $\beta(X) \subseteq B$, and hence either $\beta(X) \subseteq C$ or $\beta(X) \subseteq D$. But $\beta(X)$ touches $\beta(((A \cup D) \cap C)$, and hence $\beta(X)$ is not a subset of C, and similarly it is not a subset of D, a contradiction. This proves (4).

From (2)–(4), this proves 4.3. ∎

We remark that 4.3 implies the following, which seems to be new.

4.4 *Let H be a simple graph with $|V(H)| \geq 2$ such that $H \setminus v$ is a forest for some vertex v. Then every graph with no H-minor has tree-width at most $3|V(H)|/2 - 3$.*

Proof. We may assume by adding edges that $H \setminus v$ is a tree T; let $w = |V(T)| \geq 1$. Now let G be a graph with tree-width at least $3w/2 - 1$. By 4.3, there is a separation (A, B) of G such that $|A \cap B| = w$, $G|(B \setminus A)$ is connected, every vertex in $A \cap B$ has a neighbour in $B \setminus A$, and (A, B) left-contains T. Let $(C_1, \ldots, C_w)$ be the corresponding model of T, and let $C_i \cap B = \{v_i\}$ for $1 \leq i \leq w$. Since each v_i has a neighbour in $B \setminus A$, and $G|(B \setminus A)$ is connected, by contracting $G|(B \setminus A)$ and each of $C_1, \ldots, C_w$ to a single vertex, we obtain an H-minor. This proves 4.4. ∎

5 Conclusion

Let us combine these lemmas to deduce our main result, the following.

5.1 *Let H be a connected simple graph, not a tree, with h vertices, and let $g \geq 1$ be an integer. Let G be a graph that G contains neither a $g \times g$-grid nor H as a minor. Then the tree-width of G is at most*

$$3(8h(h-2)(2g+h)(2g+1))^{|E(H)|-|V(H)|} + 3h/2.$$

Proof. Since H is not a tree, it follows that $h \geq 3$ and $m \geq 1$. Since H is a minor of $K_{h,h}$, G does not contain $K_{h,h}$ as a minor. Let $V(H) = \{t_1, \ldots, t_h\}$, and let $m = |E(H)| - |V(H)|$. Let T_0 be a spanning tree of H, let $f_1, \ldots, f_{m+1}$ be the edges of H not in $E(T)$, and for $1 \leq i \leq m+1$ let the ends of f_i be $t_{p(i)}, t_{q(i)}$. Let $\epsilon = (4(2g+1)(h-2))^{-1}$, and $d = 2h(2g+h-2)$. Let $k_1 = \epsilon^{-m}$, and for $2 \leq i \leq m+1$, let $k_i = d(1+d)^{i-2}\epsilon^{-m}$. For $1 \leq i \leq m+1$, take a set of $k_i - 1$ new vertices and make a tree T_i with vertex set consisting of these new vertices together with $t_{p(i)}$, where $t_{p(i)}$ is adjacent to all the new vertices. Also take another set of $k_i - 1$ new vertices and make a tree T_i' consisting of these new vertices and $t_{q(i)}$, where $t_{q(i)}$ is adjacent to all the new vertices. Altogether we need

$$2(k_1 + \cdots + k_{m+1}) - 2(m+1) = 2((d+1)/\epsilon)^m - 2(m+1)$$

new vertices. Let T be the tree consisting of the union of T_0 and all the trees T_i, T_i' $(1 \leq i \leq m+1)$. Thus T has at most $2((d+1)/\epsilon)^m + h$ vertices.

Let $w = |V(T)|$, and suppose that the tree-width of G is at least $3w/2 - 1$. By 4.3, there is a separation (A, B) of G such that $|A \cap B| = w$, $A \cap B$ is linked in $G|B$, and (A, B) left-contains T. It follows that there is a model $(C_1, \ldots, C_h)$ of T_0 in $G|A$, such that if $1 \leq i \leq m+1$ and $1 \leq j \leq h$ and f_i is incident with t_j then $|Z \cap V(C_j)| \geq k_i$, where $Z = A \cap B$. Thus for $1 \leq i \leq m+1$, there is a Z-proper $(Z \cap V(C_{p(i)}), Z \cap V(C_{q(i)}))$-linkage of cardinality k_i. By 3.5, there are Z-proper paths $P_1, \ldots, P_{m+1}$ of $G|B$, pairwise vertex-disjoint, such that for $1 \leq i \leq m+1$, P_i is between $Z \cap V(C_{p(i)})$ and $Z \cap V(C_{q(i)})$. But then contracting each C_i to a single vertex and contracting each P_i to an edge yields an H-minor, a contradiction. Thus the tree-width of G is less than $3w/2 - 1$, and hence at most $3((d+1)/\epsilon)^m + 3h/2$. But $(d+1)/\epsilon \leq 8h(h-2)(2g+h)(2g+1)$. This proves 5.1. ∎

We deduce:

5.2 *Let H be a simple connected planar graph, not a tree. Let $h = |V(H)|$ and $m = |E(H)| - |V(H)|$. Every graph with no H-minor has tree-width at most $3(160h^4)^m + 3h/2$.*

Proof. Let $g = 2h$. By theorems 1.3 and 1.4 of [9], H is a minor of the $g \times g$ grid. Let G have no H-minor. By 5.1, the tree-width of G is at most

$$3(8h(h-2)(4h+h)(4h+1))^m + 3h/2 \leq 3(160h^4)^m + 3h/2.$$

This proves 5.2. ∎

Consequently we have:

5.3 *Let H be a simple planar graph, with h vertices. Every graph with no H-minor has tree-width at most $2^{15h+8h\log(h)}$.*

Proof. We may assume that $h \geq 3$, and by adding edges we may assume that H is connected and not a tree. Let $m = |E(H)| - |V(H)|$. Since $m \leq 2h - 6$, 5.2 implies that if G has no H-minor, then the tree-width of G is at most

$$3(160h^4)^{2h-6} + 3h/2 \leq (160h^4)^{2h} \leq 2^{15h+8h\log(h)}.$$

This proves 5.3. ∎

We also have:

5.4 *Every graph not containing the $g \times g$ grid as a minor has tree-width at most g^{8g^2}.*

Proof. Let H be the $g \times g$ grid. We may assume that $g \geq 2$, and so H is not a tree. Since H has g^2 vertices and $2g(g-1)$ edges, 5.1 implies that if G does not contain H as a minor, then the tree-width of G is at most

$$3(8g^2(g^2-2)(2g+g^2)(2g+1))^{2g(g-1)-g^2} + 3g^2/2 \leq g^{8g^2}.$$

This proves 5.4. ∎

References

[1] D. Bienstock, N. Robertson, P.D. Seymour and R. Thomas, "Quickly excluding a forest", *J. Combin. Theory (Ser. B)*, 52 (1991), 274–283.

[2] C. Chekuri and J. Chuzhoy, "Polynomial bounds for the grid-minor theorem" [arXiv:1305.6577] (manuscript October 2013).

[3] R. Diestel, K.Yu. Gorbunov, T.R. Jensen and C. Thomassen, "Highly connected sets and the excluded grid theorem", *J. Combin. Theory (Ser. B)*, 75 (1999), 61-73.

[4] G. Ding, T. Johnson and P. Seymour, "Spanning trees with many leaves", *J. Graph Theory*, 37 (2001), 189-197.

[5] K. Kawarabayashi and Y. Kobayashi, "Linear min-max relation between the treewidth of H-minor-free graphs and its largest grid minor", *Symposium on Theoretical Aspects of Computer Science* 2012, 278–289.

[6] Alexander Leaf, *Improved Bounds for the Excluded Grid Theorem*, Senior Thesis (advisor Paul Seymour), Princeton University, 2012.

[7] N. Robertson and P.D. Seymour, "Graph minors. V. Excluding a planar graph", *J. Combinatorial Theory, Ser. B*, 41 (1986), 92–114.

[8] N. Robertson and P.D. Seymour, "Graph minors. XXI. Graphs with unique linkages", *J. Combinatorial Theory, Ser. B*, 99 (2009), 583–616.

[9] N. Robertson, P.D. Seymour and R. Thomas, "Quickly excluding a planar graph", *J. Combinatorial Theory, Ser. B*, 62 (1994), 323-348.

[10] P.D. Seymour and R. Thomas, "Graph searching, and a min-max theorem for tree-width", *J. Combinatorial Theory, Ser. B,* 58 (1993), 22–33.