Tree-chromatic number

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November 2, 2014; revised June 25, 2015

[1]Supported by ONR grant N00014-10-1-0680 and NSF grant DMS-1265563.

Abstract

Let us say a graph G has "tree-chromatic number" at most k if it admits a tree-decomposition $(T, (X_t : t \in V(T)))$ such that $G[X_t]$ has chromatic number at most k for each $t \in V(T)$. This seems to be a new concept, and this paper is a collection of observations on the topic. In particular we show that there are graphs with tree-chromatic number two and with arbitrarily large chromatic number; and for all $\ell \geq 4$, every graph with no triangle and with no induced cycle of length more than ℓ has tree-chromatic number at most $\ell - 2$.

1 Introduction

All graphs in this paper are finite, and have no loops or parallel edges. If G is a graph and $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced on X. The chromatic number of G is denoted by $\chi(G)$, and for $X \subseteq V(G)$, we write $\chi(X)$ for $\chi(G[X])$ when there is no danger of ambiguity.

A *tree-decomposition* of a graph G is a pair $(T, (X_t : t \in V(T)))$, where T is a tree and $(X_t : t \in V(T))$ is a family of subsets of $V(G)$, satisfying:

- for each $v \in V(G)$ there exists $t \in V(T)$ with $v \in X_t$; and for every edge uv of G there exists $t \in V(T)$ with $u, v \in X_t$
- for each $v \in V(G)$, if $v \in X_t \cap X_{t''}$ for some $t, t'' \in V(T)$, and t' belongs to the path of T between t, t'' then $v \in X_{t'}$.

The *width* of a tree-decomposition $(T, (X_t : t \in V(T)))$ is the maximum of $|X_t| - 1$ over all $t \in V(T)$, and the *tree-width* of G is the minimum width of a tree-decomposition of G. Tree-width was introduced in [4] (and independently discovered in [7]), and has been the subject of a great deal of study.

In this paper, we focus on a different aspect of tree-decompositions. Let us say the *chromatic number* of a tree-decomposition $(T, (X_t : t \in V(T)))$ is the maximum of $\chi(X_t)$ over all $t \in V(T)$; and G has *tree-chromatic number* at most k if it admits a tree-decomposition with chromatic number at most k. Let us denote the tree-chromatic number of G by $\Upsilon(G)$. This seems to be a new concept, and we begin with some easy observations.

Evidently $\Upsilon(G) \leq \chi(G)$, and if $\omega(G)$ denotes the size of the largest clique of G, then $\omega(G) \leq \Upsilon(G)$ (because if Z is a clique of G and $(T, (X_t : t \in V(T)))$ is a tree-decomposition of G, then there exists $t \in V(T)$ with $Z \subseteq X_t$, as is easily seen.) If H is an induced subgraph of G then $\Upsilon(H) \leq \Upsilon(G)$, but unlike tree-width, tree-chromatic number may increase when taking minors. For instance, let G be the graph obtained from the complete graph K_n by subdividing every edge once; then $\chi(G) = 2$, and so $\Upsilon(G) = 2$ (take the tree-decomposition using a one-vertex tree), and yet G contains K_n as a minor, and $\Upsilon(K_n) = n$.

For a graph G, how can we prove that $\Upsilon(G)$ is large? Here is one way. A *separation* of G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$ and there is no edge between $A \setminus B$ and $B \setminus A$.

1.1 *For every graph G, there is a separation (A, B) of G such that $\chi(A \cap B) \leq \Upsilon(G)$ and*

$$\chi(A \setminus B), \chi(B \setminus A) \geq \chi(G) - \Upsilon(G).$$

Proof. Let $(T, (X_t : t \in V(T)))$ be a tree-decomposition of G with chromatic number $\Upsilon(G)$. For any subtree T' of T we denote the union of the sets X_t ($t \in V(T')$) by $X(T')$. Let $t_0 \in V(T)$, let $t_1, \dots, t_k$ be the vertices of T adjacent to t_0, and let $T_1, \dots, T_k$ be the components of $T \setminus t_0$ containing $t_1, \dots, t_k$ respectively. For $1 \leq i \leq k$ let $Y_i = X(T_i) \setminus X_{t_0}$. Since there are no edges between Y_i and Y_j for $1 \leq i < j \leq k$, it follows that $\chi(Y_1 \cup \dots \cup Y_k)$ is the maximum of the numbers $\chi(Y_1), \dots, \chi(Y_k)$; and since $\chi(G) \leq \chi(X_{t_0}) + \chi(Y_1 \cup \dots \cup Y_k)$, we deduce that there exists i with $1 \leq i \leq k$ such that $\chi(Y_i) \geq \chi(G) - \chi(X_{t_0}) \geq \chi(G) - \Upsilon(G)$.

Suppose that there are two such values of i, say $i = 1$ and $i = 2$. Then $(Y_1 \cup X_{t_0}, Y_2 \cup \dots \cup Y_k \cup X_{t_0})$ is a separation of G satisfying the theorem. So we may assume that for each choice of $t_0 \in V(T)$

there is a unique component T' of $T \setminus t_0$ with $\chi(X(T') \setminus X_{t_0}) \geq \chi(G) - \Upsilon(G)$. For each t_0, let $f(t_0)$ be the neighbour of t_0 that belongs to the component T' of $T \setminus t_0$ just described. Since T has more vertices than edges, there exist adjacent $s, t \in V(T)$ such that $f(s) = t$ and $f(t) = s$. Let S', T' be the components of $T \setminus e$ (where e is the edge st). Then $(X(S'), X(T'))$ is a separation satisfying the theorem. This proves 1.1. ∎

It follows from 1.1 that the graphs from Erdős's random construction [2] of graphs with large chromatic number and large girth also have large tree-chromatic number (with high probability). It does not seem obvious that there is any graph with large chromatic number and small tree-chromatic number, but here is a construction to show that (apply it to a graph G with large chromatic number).

1.2 *Let G be a graph with vertex set $\{v_1, \dots, v_n\}$ say, and make a graph H as follows. The vertex set of H is $E(G)$, and an edge v_iv_j of G (where $i < j$) and an edge v_hv_k of G (where $h < k$) are adjacent in H if either $h = j$ or $i = k$. Then*

- *H is triangle-free;*
- *H admits a tree-decomposition $(T, (X_t : t \in V(T)))$ of chromatic number two, such that T is a path;*
- *$\chi(H) \geq \log(\chi(G))$; and*
- *$\binom{\chi(H)}{\lfloor \frac{\chi(H)}{2} \rfloor} \leq \chi(G)$, and so $\chi(H) \leq \log(\chi(G)) + \frac{1}{2}\log\log(\chi(G)) + \frac{1}{2}\log(\pi/2) + o(1)$.*

Proof. For the first claim, let v_av_b, v_cv_d, v_ev_f be edges of G, where $a < b$ and $c < d$ and $e < f$, and suppose that these three edges are pairwise adjacent vertices of H. We may assume that $a \leq c, e$, and so $a \neq d, f$; and since v_av_b is adjacent to v_cv_d in H, it follows that $c = b$, and similarly $e = b$. But then v_cv_d and v_ev_f are not adjacent in H. This proves the first claim.

For the second claim, let T be a path with vertices $t_1, \dots, t_n$ in order, and for $1 \leq i \leq n$ let X_i be the set of all edges v_av_b of G with $a \leq i \leq b$. We claim that $(T, (X_t : t \in V(T)))$ is a tree-decomposition of H. To see this, observe that if pq is an edge of H then there exist $a < b < c$ such that $p = v_av_b$ and $q = v_bv_c$ (or vice versa), and then $p, q \in X_b$. Also, if $h < i < j$ and v_av_b belongs to both X_h, X_j then $a \leq h \leq i$ and $i \leq j \leq b$, and so $v_av_b \in X_i$. Thus $(T, (X_t : t \in V(T)))$ is a tree-decomposition. For its chromatic number, let $1 \leq i \leq n$; then X_i is the union of two sets that are stable in H, namely $\{v_av_b : a < i \leq b\}$ and $\{v_av_b : a \leq i < b\}$, and so $\chi(X_i) \leq 2$. This proves the second claim.

For the third, let $k = \chi(H)$ and take a k-colouring ϕ of H; we must show that $\chi(G) \leq 2^k$. For each vertex v_i of G, there is no edge v_hv_i of G with $h < i$ which has the same colour as an edge v_iv_j of G with $j > i$ (since these two edges would be adjacent in H), and consequently there is a partition (A_i, B_i) of $\{1, \dots, k\}$ such that $\phi(v_hv_i) \in B_i$ for every edge v_hv_i with $h < i$, and $\phi(v_iv_j) \in A_i$ for every edge v_iv_j of G with $j > i$. For each $A \subseteq \{1, \dots, k\}$, let F_A be the set of all v_i with $1 \leq i \leq n$ such that $A_i = A$. It follows that each F_A is a stable set of G; because if $v_i, v_j \in F_A$ are adjacent in G and $i < j$, then $\phi(v_iv_j) \in A_i = A$ and $\phi(v_iv_j) \in B_j = \{1, \dots, k\} \setminus A$, a contradiction. This proves that $V(G)$ is the union of 2^k stable sets, and so $\chi(H) \geq \log(\chi(G))$.

For the fourth assertion (thanks to Alex Scott for this argument), let $k = \chi(G)$, take a k-colouring ϕ of G, and choose an integer s minimum such that

$$\binom{s}{\lfloor \frac{s}{2} \rfloor} \geq k.$$

Spencer [9] observed that

$$s = \log(k) + \frac{1}{2}\log\log(k) + \frac{1}{2}\log(\pi/2) + o(1),$$

and proved that there is a collection $(A_1, B_1), \dots, (A_s, B_s)$ of partitions of $\{1, \dots, k\}$ such that for all distinct $x, y \in \{1, \dots, k\}$, there exists i with $1 \leq i \leq s$ such that $x \in A_i$ and $y \in B_i$. For $1 \leq i \leq s$, let F_i be the set of all edges $v_a v_b$ of G with $a < b$ such that $\phi(v_a) \in A_i$ and $\phi(v_b) \in B_i$. Then $F_1 \cup \cdots \cup F_s = E(G)$, because for every edge $v_a v_b$ of G with $a < b$, $\phi(v_a) \neq \phi(v_b)$, and so there exists $i \in \{1, \dots, s\}$ with $\phi(v_a) \in A_i$ and $\phi(v_b) \in B_i$ and hence with $v_a v_b \in F_i$. Moreover each F_i is a stable set of H; because if $v_a v_b$ and $v_c v_d$ both belong to F_i, where $a < b$ and $c < d$, then $\phi(v_a), \phi(v_c) \in A_i$ and $\phi(v_b), \phi(v_d) \in B_i$, and so $a, c \neq b, d$, and consequently $v_a v_b$ and $v_c v_d$ are not adjacent in H. This proves that $\chi(H) \leq s$. This proves the fourth assertion, and so completes the proof of 1.2. ∎

A tree-decomposition $(T, (X_t : t \in V(T)))$ is a *path-decomposition* if T is a path. Let us say that G has *path-chromatic number* at most k if it admits a path-decomposition with chromatic number at most k. The construction of 1.2 yields a graph with large χ and with small path-chromatic number. To complete the picture, we should try to find an example with arbitrarily large path-chromatic number and bounded tree-chromatic number, but so far I have not been able to do this. Here is an example that I think works, but I am unable to prove it.

Take a uniform binary tree T of depth d, with root t_0. If $s, t \in V(T)$, s, t are *incomparable* if neither is an ancestor of the other. If $s, t \in V(T)$, the three paths of T between s and t, between s and t_0, and between t and t_0, have a unique common vertex, denoted by $\sup(s, t)$. Let H be the graph with vertex set all incomparable pairs (s, t) of vertices of T, and we say (s, t) and (p, q) are adjacent in H if either $\sup(s, t)$ is one of p, q, or $\sup(p, q)$ is one of s, t. It is easy to check that for d large, this graph H has large chromatic number, and tree-chromatic number two, and I suspect that it has large path-chromatic number, but have not found a proof. Indeed, in an earlier version of this paper I asked whether for all G the path-chromatic number and tree-chromatic number of G are equal; but this has now been disproved by Huynh and Kim [5].

2 Uncle trees

The remainder of the paper is directed to proving that graphs with no long induced cycle and no triangle have bounded tree-chromatic number, but for that we use a lemma that might be of interest in its own right. We prove the lemma in this section.

Let T be a tree, and let $t_0 \in V(T)$ be a distinguished vertex, called the *root*. If $s, t \in V(T)$, t is an *ancestor* of s if t lies in the path of T between s and t_0; and t is a *parent* of s if t is an ancestor of s and s, t are adjacent; and in this case, s is a *child* of t. Thus every vertex has a unique parent except t_0. For each vertex t of T, choose a linear order of its children; if s, s' are children of t, and s precedes s' in the selected linear order, we say that s is *older* than s'. We call T, together with t_0

and all the linear orders, an *ordered tree*. The *elder line* P of an ordered tree is the maximal path of T with one end t_0 with the property that if a vertex v of P has a child, then the eldest child of v also belongs to P. (In other words, we start with t_0, and keep choosing the eldest child until the process stops.) Given an ordered tree, and $u, v \in V(T)$, we say that u is an *uncle* of v if $u \neq t_0$, and there is a child u' of the parent of u that is older than u and that is an ancestor of v.

Now let G be a graph. An *uncle tree* in G is an ordered tree T, such that T is a spanning tree of G, and for every edge uv of G that is not an edge of T, one of u, v is an uncle of the other. Thus, if T is an uncle tree in G, then every path of T with one end t_0 is an induced path of G. We need:

2.1 *For every connected graph G and vertex t_0, there is an uncle tree in G with root t_0.*

Proof. For inductive purposes, it is helpful to prove a somewhat stronger statement: that for every induced path P of G with one end t_0, there is an uncle tree such that P is a subpath of its elder line. We prove this by induction on $2|V(G)| - |V(P)|$. Let P have vertices $p_1\text{-}\cdots\text{-}p_k$ say, where $p_1 = t_0$. If some neighbour v of p_k not in $V(P)$ is nonadjacent to $p_1, \dots, p_{k-1}$, then we add v to P, and the result follows from the inductive hypothesis applied to G and this longer path. Thus we may assume that:

(1) *Every neighbour of p_k not in $V(P)$ is adjacent to one of $p_1, \dots, p_{k-1}$.*

If $k = 1$ then (1) implies that t_0 has degree zero, and so $V(G) = \{t_0\}$ and the result is trivial. Thus we may assume that $k \geq 2$.

(2) *$G \setminus p_k$ is connected.*

For if not, let C_1, C_2 be distinct components of $G \setminus p_k$, where $p_1 \in V(C_1)$. It follows that $p_1, \dots, p_{k-1} \in V(C_1)$, and so by (1), every neighbour of p_k belongs to C_1. Since G is connected, p_k has a neighbour in C_2, a contradiction. This proves (2).

By the inductive hypothesis applied to $G \setminus p_k$ and the path $p_1, \dots, p_{k-1}$, there is an uncle tree T of $G \setminus p_k$ with root t_0 such that $p_1\text{-}\cdots\text{-}p_{k-1}$ is a subpath of its elder line. Let us add p_k to T, and the edge $p_{k-1}p_k$, and make p_k the eldest child of p_{k-1} (leaving the linear orders of the ordered tree otherwise unchanged). We thus obtain an ordered tree T', and P is a subpath of its elder line. We must check that it is an uncle tree of G. To do so it suffices to check that for every edge up_k of G with $u \neq p_{k-1}$, u is an uncle of p_k. Thus, let $up_k \in E(G)$, where $u \neq p_{k-1}$. It follows that $u \notin V(P)$ since P is induced. From (1), u is adjacent in G to some p_i where $i < k$. If the edge up_i is an edge of T then u is indeed an uncle of p_k as required, so we assume not; and since T is an uncle tree of $G \setminus p_k$, it follows that one of u, p_i is an uncle of the other. Suppose first that p_i is an uncle of u. Then $i \geq 2$, and there is a child q of p_{i-1}, older than p_i, such that q is an ancestor of u. But this is impossible since p_i is the eldest child of p_{i-1}. So u is an uncle of p_i. Hence the parent of u is one of $p_1, \dots, p_{i-1}$, and so u is also an uncle of p_k as required. This proves 2.1. ∎

Another proof, perhaps more intuitive, is as follows: start from t_0, and follow the procedure to grow a depth-first tree, subject to the condition that every path of the tree with one end t_0 is induced. Thus, we begin with a maximal induced path $p_1\text{-}\cdots\text{-}p_k$ say, where $p_1 = t_0$, and then back up the path to the largest value of i such that p_i has a neighbour v not in the path and which is nonadjacent to $p_1, \dots, p_{i-1}$, and add v and the edge vp_i to the tree. If v has a neighbour not yet in the tree and nonadjacent to $p_1, \dots, p_i$, we add the corresponding edge at v to the tree, and otherwise back down the tree again to the next vertex where growth is possible. And so on; the result is an uncle tree.

3 Long holes

A *hole* in a graph G is an induced subgraph which is a cycle of length at least four. In 1985, Gyárfás [3] made the conjecture that

3.1 Conjecture: *For every integer ℓ there exists n such that every graph with no hole of length $> \ell$ and no triangle has chromatic number at most n.*

(Since the paper was submitted for publication, we have proved this conjecture and stronger statements, in joint work with Maria Chudnovsky and Alex Scott [1, 8].) Here we prove the following. (Note that if G is triangle-free then we may set $d = 1$.)

3.2 *For all integers $d \geq 1$ and $\ell \geq 4$, if G is a graph with no hole of length $> \ell$, and such that for every vertex v, the subgraph induced on the set of neighbours of v has chromatic number at most d, then G has tree-chromatic number at most $d(\ell - 2)$.*

This follows immediately from the following. (A referee of this paper brought to my attention the paper [6] in which a very slightly weaker version of the same result was proved, independently.)

3.3 *For all integers $\ell \geq 4$, if G is a graph with no hole of length $> \ell$, then G admits a tree-decomposition $(T, (X_t : t \in V(T)))$ such that for each $t \in V(T)$, there is an induced path Q_t of $G[X_t]$ with at most $\ell - 2$ vertices, such that every vertex in X_t either belongs to Q_t or is adjacent to a vertex in Q_t.*

Proof. We may assume that G is connected. Choose a vertex t_0; by 2.1 there is an uncle tree T in G with root t_0. For each $t \in V(T)$, let P_t be the subpath of T between t and t_0, and let Q_t be the maximal subpath of P_t with one end t and with length at most $\ell - 3$. (Thus Q_t has length $\ell - 3$ unless P_t has length less than $l - 3$, and in that case $Q_t = P_t$.) If $s, t \in V(T)$, we say that s is *junior* to t if neither is an ancestor of the other, and there exists $w \in V(T)$, and distinct children s', t' of w, such that s' is an ancestor of s, and t' is an ancestor of t, and t' is older than s'. (It follows easily that for every two vertices s, t, if neither is an ancestor of the other then one is junior to the other.) For $t \in V(T)$, let X_t be the set of all vertices v of G such that either

- $v \in V(Q_t)$, or
- v is a child of t in T, or
- v is junior to t and is adjacent in G to a vertex in Q_t.

We claim that $(T, (X_t : t \in V(T)))$ is a tree-decomposition of G. To show this we must check several things. We start by verifying the first condition in the definition of "tree-decomposition".

(1) *For each* $v \in V(G)$ *there exists* $t \in V(T)$ *with* $v \in X_t$*; and for every edge* uv *of* G *there exists* $t \in V(T)$ *with* $u, v \in X_t$*.*

The first statement is clear, because $v \in X_v$. For the second, let uv be an edge of G. If $uv \in E(T)$, and u is a parent of v, then $u, v \in X_u$ as required, so we may assume that $uv \notin E(T)$; and hence we may assume that u is an uncle of v, and so is junior to v. Since uv is an edge it follows that $u \in X_v$ as required. This proves (1).

To verify the second condition in the definition of "tree-decomposition", it is easier to break it into two parts.

(2) *Let* $r, s, t \in V(T)$*, where* r *is an ancestor of* t *and* s *lies on the path of* T *between* r, t*; then* $X_r \cap X_t \subseteq X_s$*.*

We may assume that r, s, t are all different. Let $v \in X_r \cap X_t$. Suppose first that there is a path P of T with one end t_0 that contains all of r, s, t, v. Since $v \in X_r$, and is not junior to r (because $v \in P$), it follows that $v \in Q_r^+$, where Q_r^+ denotes the subpath of P consisting of Q_r together with the neighbour of r in P that is not in Q_r. Consequently v is not a child of t in T, and since $v \in X_t$ it follows that $v \in Q_t$; and so

$$v \in Q_r^+ \cap Q_t \subseteq Q_s \subseteq X_s$$

as required. Thus we may assume that there is no such path P. In particular, v does not belong to P_t, and is not adjacent in T to t, and so v is junior to t and has a neighbour in Q_t.

We claim that v is junior to s; for if v is junior to r then v is junior to s, and otherwise, since $v \in X_r$, it follows that v is a child of r in T, and therefore junior to s since v is junior to t. This proves that v is junior to s. Moreover, v has a neighbour in Q_r. Suppose that v has no neighbour in Q_s. Now Q_r, Q_s, Q_t are all subpaths of P_t, and v has a neighbour in Q_r and a neighbour in Q_t, and so has neighbours in $V(Q_r) \setminus V(Q_s)$ and in $V(Q_t) \setminus V(Q_s)$. Hence there is a subpath of P_t between two neighbours of v that includes Q_s; choose a minimal such subpath P' say. Since G has no hole of length $> \ell$, it follows that P' has length at most $\ell - 2$, and so Q_s has length at most $\ell - 4$, a contradiction. So v has a neighbour in Q_s and hence $v \in X_s$ as required. This proves (2).

(3) *Let* $r, s, t \in V(T)$*, where* s *lies on the path of* T *between* r, t*; then* $X_r \cap X_t \subseteq X_s$*.*

By (2) we may assume that neither of r, t is an ancestor of the other. Let $v \in X_r \cap X_t$. Choose $w \in V(T)$ with distinct children r', t' of w such that r' is an ancestor of r and t' is an ancestor of t. Then s belongs to either the path of T between r, w or the path of T between t, w, and so by (2), if $v \in X_w$ then $v \in X_s$; so we may assume that $v \notin X_w$, and hence we may assume that $s = w$. We may assume that t' is older than r' from the symmetry. If v belongs to P_t, then v is not junior to r, and so v belongs to Q_r, and hence $v \in Q_r \cap P_t \subseteq Q_s$ as required. We may assume then that $v \notin P_t$. Since $v \in X_r$, it follows that v is not a child of t in T, and so v is junior to t, and has a neighbour, say x, in Q_t. It follows that either v is adjacent in T to some vertex of Q_t, or v is junior to x, and in the latter case v is an uncle of x since T is an uncle tree. Thus both cases v is a child in T of some vertex y of P_t. Thus $v \in X_y$. Since $v \in X_r$, it follows that y belongs to P_s, and since $v \in X_y \cap X_t$, and s lies on the path of T between y, t, (2) implies that $v \in X_s$. This proves (3).

It follows that $(T, (X_t : t \in V(T)))$ is a tree-decomposition of G, and this completes the proof of 3.3. ∎

4 Acknowledgements

Thanks to Maria Chudnovsky and Alex Scott for helpful discussions on the topic of this paper.

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