A well-quasi-order for tournaments

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Abstract

A digraph H is *immersed* in a digraph G if the vertices of H are mapped to (distinct) vertices of G, and the edges of H are mapped to directed paths joining the corresponding pairs of vertices of G, in such a way that the paths are pairwise edge-disjoint. For graphs the same relation (using paths instead of directed paths) is a well-quasi-order; that is, in every infinite set of graphs some one of them is immersed in some other. The same is not true for digraphs in general; but we show it is true for tournaments (a *tournament* is a directed complete graph).

1 Introduction

In [6], Neil Robertson and the second author proved Wagner's conjecture, that in any infinite set of graphs, one of them is a minor of another; and in [7], the same authors proved a conjecture of Nash-Williams, that in any infinite set of graphs, one of them is weakly immersed in another (we define weak immersion below). It is tempting to try to extend these results to digraphs; but it not clear what we should mean by a "minor" of a digraph, and although digraph immersion makes sense, the statement analogous to Nash-Williams' conjecture is false.

Let us make this more precise. Let G, H be digraphs. (In this paper, all graphs and digraphs are finite, and may have multiples edges but not loops.) A *weak immersion* of H in G is a map η such that

- $\eta(v) \in V(G)$ for each $v \in V(H)$
- $\eta(u) \neq \eta(v)$ for distinct $u, v \in V(H)$
- for each edge $e = uv$ of H (this notation means that e is directed from u to v), $\eta(e)$ is a directed path of G from $\eta(u)$ to $\eta(v)$ (paths do not have "repeated" vertices)
- if $e, f \in E(H)$ are distinct, then $\eta(e), \eta(f)$ have no edges in common, although they may share vertices

If in addition we add the condition

- if $v \in V(H)$ and $e \in E(H)$, and e is not incident with v in H, then $\eta(v)$ is not a vertex of the path $\eta(e)$

we call the relation *strong immersion*. (For undirected graphs the definitions are the same except we use paths instead of directed paths.)

A *quasi-order* Q consists of a class $E(Q)$ and a transitive reflexive relation which we denote by $\leq$ or $\leq_Q$; and it is a *well-quasi-order* or *wqo* if for every infinite sequence q_i $(i = 1, 2 \ldots)$ of elements of $E(Q)$ there exist $j > i \geq 1$ such that $q_i \leq_Q q_j$. The result of [6] asserts that

1.1 *The class of all graphs is a wqo under the minor relation.*

At first sight this looks stronger than what we said before; but it is easy to show that a quasi-order is a wqo if and only if there is no infinite antichain and no infinite strictly descending chain, so 1.1 is not really stronger. Similarly, the theorem of [7] asserts:

1.2 *The class of all graphs is a wqo under weak immersion.*

It remains open whether the class of all graphs is a wqo under strong immersion (this is another conjecture of Nash-Williams); Robertson and the second author believe that at one time they had a proof, but it was extremely long and complicated, and was never written down.

What about directed graphs? Unfortunately weak immersion does not provide a wqo of the class of digraphs. To see this, let C_n be a cycle of length $2n$ and direct its edges alternately clockwise and counterclockwise; then no member of the set $\{C_i : i \geq 2\}$ is weakly immersed in another. Thor Johnson studied immersion for eulerian digraphs in his PhD thesis [3], and proved (although did not write down) that for any k, the class of all eulerian digraphs of maximum outdegree at most k is a wqo under weak immersion.

Immersion for another class of digraphs arose in our work on Rao's conjecture about degree sequences; we needed to prove that the class of all directed complete bipartite graphs is a wqo under strong immersion. (Moreover, we needed the immersion relation to respect the parts of the bipartition.) This we managed to do, and it led to a proof of Rao's conjecture, that we will publish in a separate paper [5].

This suggests what seems to be a more natural question; instead of directed complete bipartite graphs, what about using directed complete graphs (that is, tournaments)? We found that our proof also worked for tournaments, and in this context was much simpler; and since this seems to be of independent interest we decided to write up the tournament result separately. That is the content of this paper. Thus, the result of this paper asserts:

1.3 *The class of all tournaments is a wqo under strong immersion.*

2 Cutwidth

If $k \geq 0$ is an integer, an enumeration $(v_1, \ldots, v_n)$ of the vertex set of a digraph has *cutwidth* at most k if for all $j \in \{1, \ldots, n-1\}$, there are at most k edges uv such that $u \in \{v_1, \ldots, v_j\}$ and $v \in \{v_{j+1}, \ldots, v_n\}$; and a digraph has *cutwidth* at most k if there is an enumeration of its vertex set with cutwidth at most k. The following was proved in [1]:

2.1 *For every set $\mathcal{S}$ of tournaments, the following are equivalent:*

- *there exists k such that every member of $\mathcal{S}$ has cutwidth at most k*
- *there is a digraph H such that H cannot be strongly immersed in any member of $\mathcal{S}$.*

We will prove:

2.2 *For every integer $k \geq 0$, the class of all tournaments with cutwidth at most k is a wqo under strong immersion.*

Proof of 1.3, assuming 2.2. Suppose that the class of all tournaments is not a wqo under strong immersion. Then there is an infinite sequence T_i $(i = 1, 2, \ldots)$ such that for $1 \leq i < j$, there is no strong immersion of T_i in T_j. Let $\mathcal{S}$ be the set $\{T_2, T_3, \ldots\}$; then there is a digraph H such that H cannot be strongly immersed in any member of $\mathcal{S}$, namely T_1. By 2.1 there exists k such that every member of $\mathcal{S}$ has cutwidth at most k; but this is contrary to 2.2. This proves 1.3. ∎

The remainder of the paper is devoted to proving 2.2. The idea of the proof is roughly the following. Let T be a tournament of cutwidth at most k, and let $(v_1, \ldots, v_n)$ be an enumeration of $V(T)$ with cutwidth at most k. Let $1 \leq i \leq n$. Then there are at most k edges with tail in $\{v_1, \ldots, v_{i-1}\}$ and head in $\{v_i, \ldots, v_n\}$; and at most k edges with tail in $\{v_1, \ldots, v_i\}$ and head in $\{v_{i+1}, \ldots, v_n\}$. These two sets of edges may intersect; let us write a label on the vertex v_i consisting of the two sets (appropriately ordered). Thus we may regard $(v_1, \ldots, v_n)$ as a finite sequence of these labels, and it follows from Higman's theorem [2] that given infinitely many tournaments of cutwidth at most k, there are two such that the sequence of labels for the second tournament dominates that of the first. This is not sufficient to deduce that the first tournament is immersed in the second, however; we have to provide edge-disjoint directed paths of the second tournament linking the appropriate pairs of vertices. This is achieved by applying a standard technique from well-quasi-ordering, first making the enumerations "linked", and then applying a strengthened version of Higman's theorem with a gap condition.

3 Linked enumerations

Let G be a digraph, and let $\{v_1, \ldots, v_n\}$ be an enumeration of $V(G)$. For $1 \leq i < n$ let $B_i = \{v_1, \ldots, v_i\}$ and $A_i = \{v_{i+1}, \ldots, v_n\}$, and let F_i be the set of all edges from B_i to A_i. We say that the enumeration $\{v_1, \ldots, v_n\}$ is *linked* if for all h, j with $1 \leq h < j < n$, if $|F_h| = |F_j| = t$ say, and $|F_i| \geq t$ for all i with $h \leq i \leq j$, then there are t pairwise edge-disjoint directed paths of G from B_h to A_j. We need:

3.1 *Let G be a digraph and $k \geq 0$ an integer. If G has cutwidth at most k then there is a linked enumeration of G with cutwidth at most k.*

Proof. Let $\{v_1, \ldots, v_n\}$ be an enumeration of $V(G)$ with cutwidth at most k, chosen optimally in the following sense. For $1 \leq i < n$, let A_i, B_i, F_i be as before. For $0 \leq s \leq k$, let n_s be the number of values of $i \in \{1, \ldots, n-1\}$ with $|F_i| = s$. Let us choose the enumeration $\{v_1, \ldots, v_n\}$ such that n_0 is as large as possible; subject to that, n_1 is as large as possible; subject to that, n_2 is as large as possible, and so on. We claim that this enumeration is linked.

For let $1 \leq h < j < n$, and suppose that $|F_h| = |F_j| = t$ say, and $|F_i| \geq t$ for all i with $h \leq i \leq j$, and there do not exist t pairwise edge-disjoint directed paths of G from B_h to A_j. By Menger's theorem there is a partition (P, Q) of $V(G)$ with $B_h \subseteq P$ and $A_j \subseteq Q$, such that $|F| < t$, where F is the set of all edges of G with tail in P and head in Q. Choose such a partition (P, Q) with $|F|$ as small as possible. Let $P = \{x_1, \ldots, x_p\}$, and $Q = \{y_1, \ldots, y_q\}$, where both sets are enumerated in the order induced from the enumeration $\{v_1, \ldots, v_n\}$. Since $B_h \subseteq P$ it follows that $h \leq p$, and similarly $p \leq j$. Now $\{x_1, \ldots, x_p, y_1, \ldots, y_q\}$ is an enumeration of $V(G)$, say $\{v'_1, \ldots, v'_n\}$. For $1 \leq i < n$, let $B'_i = \{v'_1, \ldots, v'_i\}$ and $A'_i = \{v'_{i+1}, \ldots, v'_n\}$, and let F'_i be the set of all edges from B'_i to A'_i. Thus $B'_p = P$ and $A'_p = Q$, and $F'_p = F$. For a subset $Z \subseteq V(G)$, we denote by $\delta^+(Z)$ the set of edges of G with tail in Z and head in $V(G) \setminus Z$.

We claim that $|F'_1|, \ldots, |F'_{p-1}| \leq k$. For let $1 \leq r < p$, and choose $i < n$ such that $B'_r = B_i \cap P$ and $A'_r = A_i \cup Q$. Since $P \not\subseteq B_i$ (because $x_{r+1} \notin B'_r$), and $P \subseteq B_j$, it follows that $i < j$, and so $A_j \cap (B_i \cup P) = \emptyset$. Since $B_h \subseteq P \subseteq B_i \cup P$, the minimality of $|F|$ implies that $|\delta^+(B_i \cup P)| \geq |F|$. Now

$$|\delta^+(B_i)| + |\delta^+(P)| \geq |\delta^+(B_i \cap P)| + |\delta^+(B_i \cup P)|,$$

(this is easily seen by counting the contribution of each edge to both sides), and so

$$|F_i| + |F| \geq |F'_r| + |\delta^+(B_i \cup P)| \geq |F'_r| + |F|,$$

that is, $|F'_r| \leq |F_i|$. In particular, $|F'_1|, \ldots, |F'_{p-1}| \leq k$, and similarly $|F'_{p+1}|, \ldots, |F'_{n-1}| \leq k$, and since $F'_p = F$ and $|F| < t \leq k$, we see that the enumeration $\{v'_1, \ldots, v'_n\}$ has cutwidth at most k.

For $0 \leq s \leq k$, let n'_s be the number of values of $i \in \{1, \ldots, n-1\}$ with $|F'_i| = s$. We claim that $n'_s \geq n_s$ for $0 \leq s \leq t-1$. For let $0 \leq s \leq t-1$, and let $i \in \{1, \ldots, n-1\}$ with $|F_i| = s$. We claim that $|F'_i| = s$, and indeed $F'_i = F_i$. From the choice of h, j it follows that either $i < h$ or $i > j$, and from the symmetry we may assume the first. But then $B_i \subseteq P$, and so $B'_i = B_i$ and $A'_i = A_i$; and so $F'_i = F_i$. This proves that $n'_s \geq n_s$ for $0 \leq s \leq t-1$. From the choice of $\{v_1, \ldots, v_n\}$, we deduce that $n'_s = n_s$ for $0 \leq s \leq t-1$; and so for each $i \in \{1, \ldots, n-1\}$, if $|F'_i| < t$, then $i < h$ or $i > j$. But $|F'_p| = |F| < t$, and $h \leq p \leq j$, a contradiction. This proves that $\{v_1, \ldots, v_n\}$ is linked, and so proves 3.1. ∎

4 Codewords

Let Q be a quasi-order, and let $k \geq 0$ be an integer. A (Q,k)-*gap sequence* means a triple (P, f, a), where P is a directed path, f is a map from $V(P)$ into $E(Q)$, and a is a map from $E(P)$ into $\{0, \ldots, k\}$. We define a quasi-order on the class of all (Q,k)-gap sequences as follows. Let (P, f, a) and (R, g, b) be (Q,k)-gap sequences, and let P, R have vertices (in order) $p_1, \ldots, p_m$ and $r_1, \ldots, r_n$ respectively. We say the second *dominates* the first if there exist $s(1), \ldots, s(m)$ with $1 \leq s(1) < s(2) < \cdots < s(m) \leq n$, such that

- for $1 \leq i \leq m$, $f(p_i) \leq g(r_{s(i)})$
- for $1 \leq i < m$, let e be the edge $p_i p_{i+1}$ of P; then $a(e) \leq b(e')$ for every edge e' of the subpath of R between $r_{s(i)}$ and $r_{s(i+1)}$.

It is proved in [8, 4] that

4.1 *If Q is a wqo, then for all $k \geq 0$, domination defines a wqo of the class of all (Q,k)-gap sequences.*

A *march* is a finite sequence $x_1, \ldots, x_k$ of distinct elements, and k is the *length* of this march. If μ is a march $x_1, \ldots, x_k$, we define its *support* to be $\{x_1, \ldots, x_k\}$. If (μ_1, ν_1) and (μ_2, ν_2) are both pairs of marches, we say they are *equivalent* if

- μ_1 and μ_2 have the same length, say m
- ν_1 and ν_2 have the same length, say n
- for $1 \leq i \leq m$ and $1 \leq j \leq n$, the ith term of μ_1 equals the jth term of ν_1 if and only if the ith term of μ_2 equals the jth term of ν_2.

A *codeword of type k* is a pair (P, f), where P is a directed path and f is a map from $V(P)$ into the class of ordered pairs of marches both of length at most k, with the following properties:

- let P have vertices $p_1, \ldots, p_n$ in order; then for $1 \leq i < n$, the second term of the pair $f(p_i)$ and the first term of the pair $f(p_{i+1})$ have the same length
- the first term of the pair $f(p_1)$ and the second term of the pair $f(p_n)$ both have length zero.

For each edge $e = p_ip_{i+1}$ of P, let $a(e)$ be the common lengths of the second term of the pair $f(p_i)$ and the first term of the pair $f(p_{i+1})$. We call the function $a : E(P) \to \{0, 1, \ldots, k\}$ the *cutsize function* of the codeword.

We define a quasi-order $\mathcal{C}_k$ on the class of all codewords of type k as follows. Let (P, f) and (R, g) be codewords of type k, with cutsize functions a, b respectively. Thus (P, f, a) and (R, g, b) are (Q, k)-gap sequences, where Q is the class of all ordered pairs of marches both of length at most k, ordered by equivalence. We say that $(P, f) \le (R, g)$ if (R, g, b) dominates (P, f, a). Since Q is a wqo (since there are only finitely many equivalence classes), we have by 4.1 that:

4.2 *For each $k \ge 0$, the quasi-order $\mathcal{C}_k$ is a wqo.*

5 Encoding

We need the following lemma.

5.1 *Let G be a digraph, and let $\{v_1, \ldots, v_n\}$ be a linked enumeration of $V(G)$. For $1 \le i < n$ let $B_i = \{v_1, \ldots, v_i\}$ and $A_i = \{v_{i+1}, \ldots, v_n\}$, and let F_i be the set of all edges from B_i to A_i. Then for $1 \le i < n$ there is a march μ_i with support F_i, such that for all h, j with $1 \le h < j < n$, if $|F_h| = |F_j| = t$ say, and $|F_i| \ge t$ for all i with $h \le i \le j$, then there are t pairwise edge-disjoint directed paths $P_1, \ldots, P_t$ of G from B_h to A_j, such that for $1 \le s \le t$, the sth term of μ_h and the sth term of μ_j are both edges of P_s.*

Proof. Fix t such that $|F_i| = t$ for some i. Let $\{i(1), i(2), \ldots, i(m)\}$ be the set of all $i \in \{1, \ldots, n-1\}$ with $|F_i| = t$, where $i(1) < i(2) < \cdots < i(m)$. Choose a march $\mu_{i(1)}$ with support $F_{i(1)}$. Inductively, having defined a march $\mu_{i(j-1)}$ with support $F_{i(j-1)}$, with $j < m$, there are two cases:

- If there do not exist t directed paths of G from $B_{i(j-1)}$ to $A_{i(j)}$, pairwise edge-disjoint (that is, if there exists h with $i(j-1) < h < i(j)$ and with $|F_h| < t$), let $\mu_{j(i)}$ be some march with support $F_{j(i)}$, chosen arbitrarily.

- If there exist t directed paths of G from $B_{i(j-1)}$ to $A_{i(j)}$, pairwise edge-disjoint, choose some set of t such paths; we may number these paths as $P_1, \ldots, P_t$ in such a way that for $1 \le s \le t$, the sth term of $\mu_{i(j-1)}$ is an edge of P_s, and then choose $\mu_{j(i)}$ with support $F_{j(i)}$ in such a way that for $1 \le s \le t$, the sth term of $\mu_{i(j)}$ is an edge of P_s.

Then it follows easily that for all h, j with $1 \le h < j < n$, if $|F_h| = |F_j| = t$, and $|F_i| \ge t$ for all i with $h \le i \le j$, then there are t pairwise edge-disjoint directed paths $P_1, \ldots, P_t$ of G from B_h to A_j, such that for $1 \le s \le t$, the sth term of μ_h and the sth term of μ_j are both edges of P_s. By repeating this process for all values of t we obtain marches satisfying the theorem. This proves 5.1. ∎

Let G be a tournament of cutwidth at most k. We now define how to associate a codeword (not necessarily uniquely) with G. Choose a linked enumeration $\{v_1, \ldots, v_n\}$ of $V(G)$ of cutwidth at most k; this is possible by 3.1. For $1 \le i < n$, let A_i, B_i, F_i be as in 5.1, and choose a march μ_i as in 5.1. Define μ_0, μ_n to both be the march of length zero. Let P be a directed path with vertices $v_1, \ldots, v_n$ in order. (Note that P is not a path of G.) For $1 \le i \le n$, let $f(v_i) = (\mu_{i-1}, \mu_i)$. Then (P, f) is a codeword of type k, and we say this codeword is *associated* with G.

5.2 *Let G, H be tournaments of cutwidth at most k, with associated codewords (P, f) and (Q, g) respectively. Suppose that $(P, f) \le (Q, g)$ in $\mathcal{C}_k$. Then there is a strong immersion of G in H.*

Proof. Let $\{u_1, \ldots, u_m\}$ be a linked enumeration of $V(G)$ of cutwidth at most k giving rise to the codeword (P, f), and choose $\{v_1, \ldots, v_n\} = V(H)$ similarly. Thus P has vertices $u_1, \ldots, u_m$ in order. For $1 \le i < m$, let $B_i = \{u_1, \ldots, u_i\}$ and $A_i = \{u_{i+1}, \ldots, u_m\}$, and let E_i be the set of edges of G from B_i to A_i. For $1 \le j < n$, let $D_j = \{v_1, \ldots, v_j\}$ and $C_j = \{v_{j+1}, \ldots, v_n\}$, and let F_j be the set of edges of H from D_j to C_j. For $1 \le i < m$ let μ_i be the march with support E_i as in 5.1 used to obtain the codeword (P, f), and for $1 \le j < n$ let ν_j be a march with support F_j chosen similarly.

Since $(P, f) \le (Q, g)$ in $\mathcal{C}_k$, there exist $r(1), \ldots, r(m)$ with $1 \le r(1) < r(2) < \cdots < r(m) \le n$, such that

- for $1 \le i \le m$, $f(u_i)$ and $g(v_{r(i)})$ are equivalent pairs of marches
- for $1 \le i < m$, let e be the edge $u_i u_{i+1}$ of P; then $a(e) \le b(e')$ for every edge e' of the subpath of Q between $v_{r(i)}$ and $v_{r(i+1)}$, where a, b are the cutsize functions of (P, f) and (Q, g) respectively.

Thus we have

(1) *For $1 \le i \le m$, (μ_{i-1}, μ_i) and $(\nu_{r(i)-1}, \nu_{r(i)})$ are equivalent pairs of marches. In particular, $|E_{i-1}| = |F_{r(i)-1}|$, and $|E_i| = |F_{r(i)}|$, and $|E_{i-1} \cap E_i| = |F_{r(i)-1} \cap F_{r(i)}|$.*

This is just a reformulation of the first bullet statement above. Similarly the second bullet implies:

(2) *For $1 \le i < m$, $|F_{r(i)}| = |F_{r(i+1)-1}| = |E_i|$, and $|F_j| \ge |E_i|$ for all j with $r(i) \le j \le r(i+1) - 1$.*

(3) *Let $1 \le i < m$. For each edge $e \in E_i$ there is a directed path $P_i(e)$ of H with the following properties:*

- *the paths $P_i(e)$ ($e \in E_i$) are pairwise edge-disjoint*
- *the first edge of $P_i(e)$ is in $F_{r(i)}$, and has tail $v_{r(i)}$ if and only if e has tail u_i*
- *the last edge of $P_i(e)$ is in $F_{r(i+1)}$, and has head $v_{r(i+1)}$ if and only if e has head u_{i+1}*
- *all internal vertices of $P_i(e)$ belong to $\{v_{r(i)+1}, \ldots, v_{r(i+1)-1}\}$*
- *choose s such that e is the sth term of the march μ_i; then the first edge of $P_i(e)$ is the sth term of the march $\nu_{r(i)}$, and the last edge of $P_i(e)$ is the sth term of the march $\nu_{r(i+1)-1}$.*

For let $t = |E_i|$. By (2) and the choice of the marches ν_j, there are t pairwise edge-disjoint directed paths $Q_1, \ldots, Q_t$ of H from $D_{r(i)}$ to $C_{r(i+1)-1}$, such that for $1 \le s \le t$, the sth term of $\nu_{r(i)}$ and the sth term of $\nu_{r(i+1)-1}$ are both edges of Q_s. Since $Q_1, \ldots, Q_s$ are pairwise edge-disjoint and each contains an edge of $F_{r(i)}$, and $|F_{r(i)}| = t$, it follows that each Q_s has exactly one edge in $F_{r(i)}$, and similarly exactly one edge in $F_{r(i+1)-1}$. By choosing Q_s minimal we may assume that the sth term of $\nu_{r(i)}$ is the first edge of Q_s, and the sth term of $\mu_{r(i+1)-1}$ is the last edge of Q_s, for $1 \le s \le t$, and all internal vertices of Q_s belong to $\{v_{r(i)+1}, \ldots, v_{r(i+1)-1}\}$. Now let $1 \le s \le t$, and let e be the sth term of μ_i. We define $P_i(e) = Q_s$. This defines $P_i(e)$ for each $e \in E_i$, and we claim the five bullets above are all satisfied. We have already seen that the first, fourth and fifth bullet are satisfied; let us check the second. Certainly the first edge of $P_i(e)$ is in $F_{r(i)}$; let it be f say. We must show that f has tail $v_{r(i)}$ if and only if e has tail u_i. Now e has tail u_i if and only if $e \notin E_{i-1}$ or $i = 1$, that is, if and only if e does not belong to the support of μ_{i-1}. But since e is the sth term of μ_i and the pairs of marches (μ_{i-1}, μ_i) and $(\nu_{r(i)-1}, \nu_{r(i)})$ are equivalent, it follows that e is not in the support of μ_{i-1} if and only if the sth term of $\nu_{r(i)}$ is not in the support of $\nu_{r(i)-1}$, that is, if and only if f has tail $v_{r(i)}$. This proves the second bullet, and the third follows similarly. This proves (3).

(4) *For each edge $e \in E(G)$ with $e = u_h u_j$ say with $h < j$, there is a directed path $\eta(e)$ of H from $v_{r(h)}$ to $v_{r(j)}$, such that none of $v_{r(1)}, \ldots, v_{r(m)}$ is an internal vertex of $\eta(e)$, and the paths $\eta(e)$ $(e \in E(G))$ are pairwise edge-disjoint. Moreover, if e is the sth term of μ_h then the first edge of $\eta(e)$ is the sth term of $\nu_{r(h)}$, and if e is the tth term of μ_{j-1} then the last edge of $\eta(e)$ is the tth term of $\nu_{r(j)-1}$.*

For let $e = u_h u_j$ say with $h < j$. It follows that e belongs to each of the sets E_i for $h \le i < j$, and so the paths $P_h(e), P_{h+1}(e), \ldots, P_{j-1}(e)$ are all defined, as in (3). We claim that for $h+1 \le i \le j-1$, the last edge of $P_{i-1}(e)$ is the first edge of $P_i(e)$. For let e be the sth term of the march μ_{i-1} and the tth term of the march μ_i. Let f be the sth term of the march $\nu_{r(i)-1}$, and let g be the tth term of the march $\nu_{r(i)}$. Then by the last statement of (3), it follows that f is the last edge of $P_{i-1}(e)$, and g is the first edge of $P_i(e)$. But the sth term of μ_{i-1} equals the tth term of μ_i, and since the pairs (μ_{i-1}, μ_i) and $(\nu_{r(i)-1}, \nu_{r(i)})$ are equivalent, it follows that the sth term of $\nu_{r(i)-1}$ equals the tth term of $\nu_{r(i)}$, that is, $f = g$. This proves our claim that for $h+1 \le i \le j-1$, the last edge of $P_{i-1}(e)$ is the first edge of $P_i(e)$. Hence the union of the paths $P_h(e), P_{h+1}(e), \ldots, P_{j-1}(e)$ is a directed path $\eta(e)$ say from $v_{r(h)}$ to $v_{r(j)}$. If e is the sth term of μ_h, then from (3) the first edge of $P_h(e)$ is the sth term of $\nu_{r(h)}$, and hence the first edge of $\eta(e)$ is the sth term of $\nu_{r(h)}$; and similarly if e is the tth term of μ_{j-1} then the last edge of $\eta(e)$ is the tth term of $\nu_{r(j)-1}$. We see that the paths $\eta(e)$ $(e \in E(G))$ are pairwise edge-disjoint paths of H, and none of $v_{r(1)}, \ldots, v_{r(m)}$ is an internal vertex of any of the paths $\eta(e)$. This proves (4).

(5) *If $1 \le h < j \le m$ and $u_j u_h$ is ian edge of G, then $v_{r(j)} v_{r(h)}$ is an edge of H.*

For suppose not. Then $v_{r(h)} v_{r(j)}$ is an edge of H, say f. Thus $f \in F_{r(h)}$; let f be the sth term of $\nu_{r(h)}$. Let e be the sth term of μ_h; then by (4) f is an edge of $\eta(e)$. Thus both $v_{r(h)}, v_{r(j)}$ are vertices of $\eta(e)$, and since neither of them is an internal vertex of $\eta(e)$ by (4), we deduce that $\eta(e)$ is from $v_{r(h)}$ to $v_{r(j)}$. From the definition of $\eta(e)$ it follows that $e = u_h u_j$, a contradiction since u_j is adjacent to u_h in G by hypothesis, and G is a tournament, a contradiction. This proves (5).

From (5), if $e = u_j u_h$ is an edge of G with $h < j$, let us define $\eta(e)$ to be the path of H of length one from $v_{r(j)}$ to $v_{r(h)}$. (Thus these paths are pairwise edge-disjoint; and moreover, they are edge-disjoint from the paths $\eta(e)$ we defined in (4), since those paths have no internal vertex in $\{v_{r(1)}, \ldots, v_{r(m)}\}$.) Now for $1 \le i \le m$, let $\eta(u_i) = v_{r(i)}$; then η is a strong immersion of G in H. This proves 5.2. ∎

Proof of 2.2. Let G_i ($i = 1, 2\ldots$) be an infinite sequence of tournaments, all of cutwidth at most k. We must show that there exist $j > i \geq 1$ such that G_i is strongly immersed in G_j. For each i let (P_i, f_i) be a codeword of type k associated with G_i. By 4.2 there exist $j > i \geq 1$ such that $(P_i, f_i) \leq (P_j, f_j)$ in the wqo $\mathcal{C}_k$. By 5.2 it follows that there is a strong immersion of G_i in G_j. This proves 2.2, and hence completes the proof of 1.3. ∎

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