# Tournament pathwidth and topological containment

Alexandra Fradkin and Paul Seymour[1]
Princeton University, Princeton, NJ 08544, USA

March 10, 2011

[1]Supported by NSF grant DMS-0901075 and ONR grant N00014-10-1-0680.

**Abstract**

We prove that for every set $\mathcal{S}$ of tournaments the following are equivalent:

- there exists $k$ such that every member of $\mathcal{S}$ has pathwidth at most $k$
- there is a digraph $H$ such that no subdivision of $H$ is a subdigraph of any member of $\mathcal{S}$
- there exists $k$ such that for each $T \in \mathcal{S}$, there do not exist $k$ vertices of $T$ that are pairwise $k$-connected.

As a consequence, we obtain a polynomial-time algorithm to test whether a tournament contains a subdivision of a fixed digraph $H$ as a subdigraph.

# 1 Introduction

In this paper, all digraphs are finite, and may have loops or parallel edges. A digraph is a *tournament* if it has no loops, and for every pair of distinct vertices $u, v$, there is exactly one edge with set of ends $\{u, v\}$. A digraph is *simple* if it has no loops, and for every pair of distinct vertices $u, v$ there is at most one edge with tail $u$ and head $v$. A digraph is *semi-complete* if it is simple and for every pair of vertices $u, v$ there is at least one edge with set of ends $\{u, v\}$.

Let $G$ be a digraph and let $V(G), E(G)$ denote the set of vertices and edges of $G$, respectively. Let $P, Q$ be directed paths of $G$ such that $P$ is from $u$ to $v$ and $Q$ is from $u'$ to $v'$. Then $P$ and $Q$ are *internally-disjoint* if $V(P) \cap V(Q) = \{u, v\} \cap \{u', v'\}$. Let $u, v \in V(G)$. By $\kappa_G(u, v)$ we denote the maximum number of internally-disjoint paths from $u$ to $v$ in $G$ (we often just write $\kappa(u, v)$ if it is clear which digraph we are dealing with). If $uv \in E(G)$ then $\kappa(u, v)$ is infinite. The vertices $u, v$ are *k-vertex-connected* (or just *k-connected*) if $\kappa(u, v) \geq k$ and $\kappa(v, u) \geq k$.

Let $X, Y \subseteq V(G)$ be disjoint. We say that $X$ is *complete to* $Y$ if $xy \in E(G)$ for all $x \in X$ and all $y \in Y$. We say that $X$ is *matched to* $Y$ if $|X| = |Y|$ and there exist edges $e_1, \ldots, e_{|X|}$ with $e_i = x_i y_i$ such that $x_1, \ldots, x_{|X|}, y_1, \ldots, y_{|X|}$ are all distinct. Let $k$ be an integer and let $A, B, C \subseteq V(G)$ be disjoint with $|A| = |B| = |C| = k$. We say that $(A, B, C)$ is a *k-triple* if $A$ is complete to $B$, $B$ is complete to $C$, and $C$ is matched to $A$.

Let $G, H$ be digraphs. Then $G$ is a *subdivision* of $H$ if it can be obtained from $H$ by repeatedly deleting an edge $uv$, adding a new vertex $w$, and adding two new edges $uw$ and $wv$. A digraph $G$ contains a subdivision of a digraph $H$ as a subdigraph if and only if there exists a map $\eta$, with domain $V(H) \cup E(H)$ that satisfies the following.

- $\eta(v) \in V(G)$ for each $v \in V(H)$
- $\eta(u) \neq \eta(v)$ for distinct $u, v \in V(H)$
- for each non-loop edge $e = uv$ of $H$, $\eta(e)$ is a path of $G$ from $\eta(u)$ to $\eta(v)$
- for each loop $e$ of $H$ incident with $v$, $\eta(e)$ is a directed cycle of $G$ passing through $\eta(v)$
- if $e, f \in E(H)$ are distinct with $e = uv$ and $f = xy$, then
$$V(\eta(e)) \cap V(\eta(f)) = \{\eta(u), \eta(v)\} \cap \{\eta(x), \eta(y)\}.$$

We call such a map an *expansion* of $H$ in $G$.

Given a digraph $D$, a sequence $W = [W_1, \ldots, W_r]$ of subsets of $V(D)$ is a *path decomposition* of $D$ if the following conditions are satisfied:

(i) $\bigcup_{i=1}^{r} W_i = V(D)$

(ii) $W_i \cap W_k \subseteq W_j$ for $1 \leq i < j < k \leq r$

(iii) for each edge $uv \in E(D)$, either $u, v \in W_i$ for some $i$ or $u \in W_i, v \in W_j$ for some $i > j$.

We call $W_1, \ldots, W_r$ the *terms* of the path decomposition. The *width* of a path decomposition is
$$\max_{1 \leq i \leq r} (|W_i| - 1).$$
The *pathwidth* of $D$ is the minimum width over all path decompositions of $D$.

The main result of this paper is the following.

**1.1** *For every set $\mathcal{S}$ of semi-complete digraphs, the following are equivalent:*

1. *there exists $k$ such that every member of $\mathcal{S}$ has pathwidth at most $k$*
2. *there is a digraph $H$ such that no subdivision of $H$ is a subdigraph of any member of $\mathcal{S}$*
3. *there exists $k$ such that for each $T \in \mathcal{S}$, there does not exist a $k$-triple in $T$*
4. *there exists $k$ such that for each $T \in \mathcal{S}$, there do not exist $k$ vertices of $T$ that are pairwise $k$-connected.*

The proof is given in the next section. We note that the equivalence of 1.1.3 and 1.1.4 is due to Maria Chudnovsky, Alex Scott, and the second author. We include the proof here (with the permission of its authors) because it naturally fits into this paper and does not appear anywhere else.

In [3] we proved a theorem of a similar flavor to 1.1 in which we showed that nine statements were equivalent for every set $\mathcal{S}$ of semi-complete digraphs (and in fact for a somewhat larger class of digraphs). One of the statements was

- there exists $k$ such that for each $T \in \mathcal{S}$, there do not exist $k$ vertices of $T$ that are pairwise $k$-edge-connected.

We also showed that the statements from [3] are not equivalent to the statements of 1.1.

As a consequence of the theorem in [3], we obtained a polynomial-time algorithm to test whether a semi-complete digraph $T$ contains an *immersion* of a fixed digraph $D$ (immersion is like expansion but with vertex-disjoint paths replaced by edge-disjoint paths; for a proper definition see [3]). Similarly, as a consequence of 1.1, we obtain a polynomial-time algorithm to test whether a semi-complete digraph $T$ contains an expansion of a fixed digraph $D$. This is the subject of Section 3. We note here that it is important that $D$ is semi-complete because the analogous problem for general digraphs is NP-complete.

We need a few more definitions. For $A, B \subseteq V(T)$, the pair $(A, B)$ is a *separation of order $l$* if

- $A \cup B = V(T)$,
- $|A \cap B| = l$, and
- there are no edges from $A \setminus B$ to $B \setminus A$.

Note that for a separation $(A, B)$ in a semi-complete digraph, the set $B \setminus A$ is complete to $A \setminus B$. We say that the set $A \cap B$ is the *cut* corresponding to the separation $(A, B)$. Two separations $(A, B)$ and $(C, D)$ *cross* unless one of the following holds:

(i) $C \subseteq A$ and $B \subseteq D$

(ii) $A \subseteq C$ and $D \subseteq B$.

A set of separations is *cross-free* if no two of its members cross.

**1.2** *Let $G$ be a digraph and $t \geq 0$ be a integer. Let $S$ be a cross-free set of separations with $|S| = t$. Then the members of $S$ can be ordered $(A_1, B_1), \dots, (A_t, B_t)$ such that $A_1 \subseteq \cdots \subseteq A_t$ and $B_t \subseteq \cdots \subseteq B_1$.*

**Proof.** For $t = 1$ there is nothing to prove. We proceed by induction on $t$. Let $(A_1, B_1)$ be the separation in $S$ with $|A_1| - |B_1|$ as small as possible. Then for all other separations $(A, B) \in S$, $A_1 \subseteq A$ and $B \subseteq B_1$. By induction, the members of $S \setminus (A_1, B_1)$ can be ordered $(A_2, B_2), \ldots, (A_t, B_t)$ such that $A_2 \subseteq \cdots \subseteq A_t$ and $B_t \subseteq \cdots \subseteq B_2$. Now $(A_1, B_1), \ldots, (A_t, B_t)$ is the desired ordering of the members of $S$. This proves 1.2. ■

## 2 The main proof

In this section we prove 1.1. We begin with some definitions and preliminary results.

Let $G$ be a digraph and let $U \subseteq V(G)$. We say that $U$ is a *$k$-jungle* in $G$ if $|U| = k$ and for all $u, v \in U$, $u$ and $v$ are $k$-connected. A separation $(A, B)$ of $G$ is an *$l$-separator* of $U$ if $|U \setminus A|, |U \setminus B| \geq l$. We say that $U$ is *$(k, l)$-separable* if there exists an $l$-separator of $U$ of order $\leq k$.

**2.1** *Let $T$ be a semi-complete digraph and let $(A, B), (C, D)$ be separations of $T$. Then either $A \cup D = V(T)$ or $B \cup C = V(T)$.*

**Proof.** Let $X = V(T) \setminus (B \cup C), Y = V(T) \setminus (A \cup D)$. Suppose that both $X$ and $Y$ are non-empty, and let $x \in X$ and $y \in Y$. Since $(A, B)$ is a separation, $yx \notin E(T)$ and since $(C, D)$ is a separation, $xy \notin E(T)$. This contradicts that $T$ is semi-complete. Therefore, one of $X$ and $Y$ is empty, and so either $A \cup D = V(T)$ or $B \cup C = V(T)$. This proves 2.1. ■

**2.2** *Let $G$ be a digraph and let $(A, B), (A', B')$, and $(C, D)$ be separations in $G$. Suppose that $(A, B)$ and $(A', B')$ do not cross $(C, D)$. Then $(A \cap A', B \cup B')$ and $(A \cup A', B \cap B')$ do not cross $(C, D)$.*

**Proof.** Let $(X, Y) = (A \cap A', B \cup B')$ and $(W, Z) = (A \cup A', B \cap B')$. If $A \cup A' \subseteq C$ and $D \subseteq B \cap B'$ then $X, W \subseteq C$ and $D \subseteq Y, Z$, and we are done. By symmetry we may assume that $A \not\subseteq C$ so since $(A, B)$ and $(C, D)$ do not cross, $C \subseteq A$ and $B \subseteq D$. If $C \subseteq A'$ and $B' \subseteq D$, then $C \subseteq X, W$ and $Y, Z \subseteq D$, and again we win. So we may assume that $A' \subseteq C$ and $B' \subseteq D$. But then $X \subseteq C \subseteq W$ and $Z \subseteq D \subseteq Y$, and this proves 2.2. ■

Next, we prove that there exists a function $f$ such that every semi-complete digraph that contains an $f(k)$-jungle also contains a $k$-triple. We begin with some preliminary results. For an integer $k$, let $R(k, k)$ denote the *Ramsey number*, that is the smallest integer such that every red-blue coloring of the edges of the complete graph on $R(k, k)$ vertices contains a monochromatic clique of size $k$ ($R(k, k)$ exists for all $k$ by Ramsey's theorem [4]).

**2.3** *Let $k$ be an integer and let $r = R(2k, 2k)$. Let $T$ be a semi-complete digraph, and let $A, B \subseteq V(T)$ be disjoint with $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$. Then there exists $X \subseteq \{1, \ldots, r\}$ with $|X| = k$ such that either $a_i b_j \in E(T)$ for all $i < j$ with $i, j \in X$ or $b_j a_i \in E(T)$ for all $i < j$ with $i, j \in X$.*

**Proof.** Let $H$ be the complete graph on $r$ vertices with $V(H) = \{v_1, \ldots, v_r\}$. For $i < j$, color the edge $v_i v_j$ red if $a_i b_j \in E(T)$ and blue otherwise. The result now follows from the definition of Ramsey number. ■

**2.4** *Let $T$ be a semi-complete digraph. Let $k \geq 0$ be an integer, let $r = R(2k, 2k)$ and $s = 2^r$. Suppose there exist disjoint $A, B, C \subseteq V(T)$ with $|A| = |B| = |C| = s$ such that $A$ is complete to $B$, $B$ is matched to $C$, and $C$ is matched to $A$. Then $T$ contains a $k$-triple.*

**Proof.** Let $A = \{a_1, \dots, a_s\}$, $B = \{b_1, \dots, b_s\}$, and $C = \{c_1, \dots, c_s\}$ such that $b_i$-$c_i$-$a_i$ is a two-edge path for all $1 \leq i \leq s$. The semi-complete digraph $G|B$ has $2^r$ vertices and therefore contains a subdigraph which is a transitive tournament on $r$ vertices. Hence, we may assume that $b_i b_j \in V(T)$ for all $1 \leq i < j \leq r$. By 2.3, and after possibly renumbering the vertices, we may assume that either $b_i c_j \in E(T)$ for all $1 \leq i < j \leq 2k$ or $c_j b_i \in E(T)$ for all $1 \leq i < j \leq 2k$. In the former case, let $A' = \{a_{k+1}, \dots, a_{2k}\}$, $B' = \{b_1, \dots, b_k\}$, and $C' = \{c_{k+1}, \dots, c_{2k}\}$; then $(A', B', C')$ is a $k$-triple in $T$. So we may assume that $c_j b_i \in E(T)$ for all $1 \leq i < j \leq 2k$. But then let $A' = \{b_1, \dots, b_k\}$, $B' = \{b_{k+1}, \dots, b_{2k}\}$, and $C' = \{c_{k+1}, \dots, c_{2k}\}$; it follows that $(C', A', B')$ is a $k$-triple. This proves 2.4. ■

**2.5** *Let $T$ be a semi-complete digraph. Let $k \geq 0$ be an integer and let $r = R(2k, 2k)$. Suppose there exist disjoint $A, B, C, D \subseteq V(T)$ with $|A| = |B| = |C| = |D| = r$ such that $A$ is matched to $B$, $B$ is complete to $C$, $C$ is matched to $D$, and $D$ is complete to $A$. Then $T$ contains a $k$-triple.*

**Proof.** By 2.3, there exists $A' \subseteq A$ and $C' \subseteq C$ with $|A'| = |C'| = k$ such that either $A'$ is complete to $C'$ or $C'$ is complete to $A'$. By symmetry, we may assume that $A'$ is complete to $C'$. Let $A' = \{a_1, \dots, a_k\}$, $C' = \{c_1, \dots, c_k\}$ and $D' = \{d_1, \dots, d_k\}$, where $D' \subseteq D$ and $c_i d_i$ is an edge for $1 \leq i \leq k$. Then $(D', A', C')$ is a $k$-triple. This proves 2.5. ■

**2.6** *For all integers $k \geq 0$, there exists an integer $m \geq 0$ such that every semi-complete digraph $T$ that contains an $m$-jungle contains a $k$-triple.*

**Proof.** Let $k \geq 0$ be an integer and let $s$ be as in the statement of 2.4. Let $m = 2^{12s}$. Let $T$ be a semi-complete digraph that contains an $m$-jungle, and suppose that $T$ does not contain a $k$-triple. Since every tournament with $m$ vertices contains a transitive tournament with at least $\log_2 m$ vertices, $T$ contains a subdigraph which is a transitive tournament whose vertex set, $X$ say, is a $12s$-jungle in $T$. Thus $X$ can be partitioned into $(X_1, X_2)$ such that $|X_1| = |X_2| = 6s$ and $X_1$ is complete to $X_2$. Since $X$ is a $12s$-jungle, there are $6s$ pairwise vertex-disjoint paths from $X_2$ to $X_1$ in $T$. Let $R$ be a minimal induced subdigraph of $T$ such that $X \subseteq V(R)$ and there are $6s$ vertex-disjoint paths from $X_2$ to $X_1$ in $R$. Thus, for every $v \in V(R)$ there is a separation $(A, B)$ of $R$ of order $6s$ such that $X_2 \subseteq A$, $X_1 \subseteq B$, and $v \in A \cap B$ (note that $(X_1, V(R))$ and $(V(R), X_2)$ are separations of order $6s$ in $R$). Let $S$ be a maximal cross-free collection of separations of order $6s$ such that $X_2 \subseteq A$ and $X_1 \subseteq B$ for each $(A, B) \in S$, and subject to that

$$\bigcup_{(A,B) \in S} A \cap B$$

is as large as possible. By 1.2, it follows that the members of $S$ can be ordered $(A_1, B_1), \dots, (A_t, B_t)$ such that $A_1 \subseteq \dots \subseteq A_t$ and $B_t \subseteq \dots \subseteq B_1$. For $1 \leq q \leq t$, let $C_q = A_q \cap B_q$.

(1) $C_1 \cup \dots \cup C_t = V(R)$.

Suppose there exists $v \in V(R)$ such that $v \notin C_1 \cup \cdots \cup C_t$. Let $(A, B)$ be a separation of $R$ of order $6s$ such that $v \in A \cap B$, $X_2 \subseteq A$, $X_1 \subseteq B$, and subject to that $(A, B)$ crosses as few members of $S$ as possible. Let $C = A \cap B$. Since $(A, B) \notin S$, it follows from the maximality of $S$ that $(A, B)$ crosses at least one member of $S$; say it crosses $(A_p, B_p)$. From 2.2 it follows that the separations $(A \cap A_p, B \cup B_p)$ and $(A \cup A_p, B \cap B_p)$ cross strictly fewer members of $S$ than $(A, B)$. Since they both separate $X_2$ from $X_1$ it follows that neither one has order less than $6s$; and since the sum of their orders is $12s$, each of them has order exactly $6s$. But $v$ is in the cut of at least one of the two separations, contrary to our choice of $(A, B)$. Therefore, there is no such $v$ and so $C_1 \cup \cdots \cup C_t = V(T)$. This proves (1).

Let $P_1, \ldots, P_{6s}$ be disjoint paths from $X_2$ to $X_1$. Let $\mathcal{P} = \{P_1, \ldots, P_{6s}\}$. For $1 \le i \le 6s$ let $a_i$ and $b_i$ be the first and second vertices of $P_i$, respectively, and let $y_i$ and $z_i$ be the second to last and last vertices of $P_i$, respectively. We call $V(P_i) \setminus \{a_i, z_i\}$ the set of *internal* vertices of $P_i$.

Let $v_1$-$\cdots$-$v_q$ be the vertices of $P_i$ in order. From the minimality of $R$ it follows that for all $1 \le j \le q$, the set $\{v_{j+1}, \ldots, v_q\}$ is complete to $\{v_1, \ldots, v_{j-1}\}$.

(2) *No $C_q$ contains internal vertices of at least $s$ members of $\mathcal{P}$.*

Suppose some $C_q$ contains internal vertices from $s$ members of $\mathcal{P}$, say $v_j \in V(P_j)$ for $1 \le j \le s$. For $1 \le j \le s$, let $u_j$ be the vertex of $P_j$ immediately preceding $v_j$ and let $w_j$ be the vertex immediately following $v_j$. Let $U = \{u_1, \ldots, u_s\}$, $V = \{v_1, \ldots, v_s\}$, and $W = \{w_1, \ldots, w_s\}$. Then $U$ is matched to $V$, $V$ is matched to $W$, and $W$ is complete to $U$. Thus, by 2.4, $R$ (and hence also $T$) contains a $k$-triple, a contradiction. This proves (2).

Choose $q$ maximum such that $|C_q \cap X_2| \ge 3s$. Note that $q \ne t$. It follows that $|C_{q+1} \cap X_2| < 3s$.

(3) $|(C_q \cap X_2) \setminus C_{q+1}| < s$.

Let $|(C_q \cap X_2) \setminus C_{q+1}| = l$. We may assume that $(C_q \cap X_2) \setminus C_{q+1} = \{a_1, \ldots, a_l\}$. Then from (1) it follows that $b_i \in C_{q+1}$ for $1 \le i \le l$. Now by (2) we have that $l < s$. This proves (3).

From (3) and the fact that $|C_{q+1} \cap X_2| < 3s$, it follows that $|C_q \cap X_2| < 4s$. So $C_q$ contains at most $4s$ vertices of $X_2$ and by (2) it contains at most $s$ internal vertices of members of $\mathcal{P}$. It follows that $|X_1 \cap C_q| \ge s$ vertices of $X_1$. Without loss of generality, we may assume that $a_1 \ldots, a_s \in C_q$ and that $z_{s+1}, \ldots, z_{2s} \in C_q$. Let $B' = \{b_1, \ldots, b_s\}$ and $Y' = \{y_{s+1}, \ldots, y_{2s}\}$; then since $(A_q, B_q)$ is a separation, $B'$ is complete to $Y'$. Now let $A' = \{a_1, \ldots, a_s\}$ and $Z' = \{z_{s+1}, \ldots, z_{2s}\}$. Note that $A'$ is matched to $B'$, $B'$ is complete to $Y'$, $Y'$ is matched to $Z'$, and $Z'$ is complete to $A'$. Consequently, by 2.5, $T$ contains a $k$-triple. This proves 2.6. ■

Next, we prove that if a semi-complete digraph $T$ does not have a $k$-jungle for some $k$, then $T$ has pathwidth bounded by a function of $k$. Once again, we begin with some lemmas and definitions.

**2.7** *Let $T$ be a semi-complete digraph and $W \subseteq V(T)$. Let $(A_1, B_1), \ldots, (A_t, B_t)$ be separations of $T$ of order $< k$ with $|(A_q \setminus B_q) \cap W| < l$ for $1 \le q \le t$. Then $|\bigcup_{q=1}^{t} (A_q \setminus B_q) \cap W| \le 2(l + k)$.*

**Proof.** Let $U = \bigcup_{q=1}^{t} (A_q \setminus B_q) \cap W$ and let $|U| = \alpha$. Then there exists $v \in U$ with $\delta^+_{T|U}(v) \geq \frac{\alpha-1}{2}$. Since $v \in U$, it follows that there exists $s$ such that $v \in (A_s \setminus B_s) \cap W$. Since the order of $(A_s, B_s)$ is less than $k$, we conclude that $\delta^+_{T|U}(v) < k + l$. Hence, $\alpha \leq 2(l + k)$. This proves 2.7. ■

*Remark*: We can switch the roles of $A_q$ and $B_q$ in the statement of 2.7.

**2.8** *Let $T$ be a semi-complete digraph that does not have a $k$-jungle, and let $W \subseteq V(T)$ with $|W| \geq 5k + 4l$. Then $W$ is $(k,l)$-separable.*

**Proof.** Suppose that $W$ is not $(k,l)$-separable. Let $(A_1, B_1), \ldots, (A_t, B_t)$ be the separations of $T$ of order $< k$ with $|(A_q \setminus B_q) \cap W| < l$ for all $1 \leq q \leq t$ and let $(C_1, D_1), \ldots, (C_s, D_s)$ be the separations of $T$ of order $< k$ with $|(D_q \setminus C_q) \cap W| < l$ for all $1 \leq q \leq s$. Note that these are all the separations of $T$ of order $< k$. Let $X = \bigcup_{q=1}^{t} (A_q \setminus B_q) \cap W$ and $Y = \bigcup_{q=1}^{s} (D_q \setminus C_q) \cap W$. By 2.7 and the remark, $|X| \leq 2(l + k)$ and $|Y| \leq 2(l + k)$. Let $Z = W \setminus (X \cup Y)$. Since $|W| \geq 5k + 4l$, it follows that $|Z| \geq k$ and for every separation $(A, B)$ of $T$ of order $< k$ either $Z \subseteq A$ or $Z \subseteq B$. It follows that $Z$ is a $k$-jungle, a contradiction. This proves 2.8. ■

Let $(A, B), (C, D)$ be separations of $T$ that do not cross and suppose they have orders $i, j$, respectively. Without loss of generality, suppose that $A \subseteq C$ and $D \subseteq B$. We say that $(A, B)$ and $(C, D)$ are *$\theta$-close* if $(B \setminus A) \cap (C \setminus D) < \theta|i - j|$.

For an integer $\theta > 0$, a *bundle* in $T$ of order $\theta$ is a cross-free set $\mathcal{B}$ of separations of $T$, each of order $< \theta$, such that

(i) no two members of $\mathcal{B}$ are $\theta$-close

(ii) if $(A, B)$ is a separation of $T$ of order $i < \theta$ then one of the following holds:

  (a) $(A, B) \in \mathcal{B}$

  (b) $(A, B)$ crosses some $(C, D) \in \mathcal{B}$ of order $\leq i$

  (c) $(A, B)$ is $\theta$-close to some $(C, D) \in \mathcal{B}$ of order $\leq i$.

We refer to these as the *first and second (bundle) axioms.*

Next, we prove some results about bundles.

**2.9** *Let $T$ be a semi-complete digraph with $V = V(T)$, and let $\mathcal{B}$ be a bundle in $T$. Then $(\emptyset, V), (V, \emptyset) \in \mathcal{B}$.*

**Proof.** Note that $(\emptyset, V)$ and $(V, \emptyset)$ are separations of order zero, and they do not cross and are not $\theta$-close to each other or any other separations of order 0. It now follows from the second bundle axiom that $(\emptyset, V), (V, \emptyset) \in \mathcal{B}$. ■

**2.10** *Let $T$ be a semi-complete digraph and let $\theta > 0$ be an integer. Then $T$ contains a bundle of order $\theta$.*

**Proof.** Let $(A_1, B_1), \ldots, (A_t, B_t)$ be a list of all the separations of $T$ of order $< \theta$ such that for all $1 \leq p < q \leq k$, the order of $(A_p, B_p)$ is at most that of $(A_q, B_q)$ and with $\{(A_1, B_1), (A_2, B_2)\} = \{(\emptyset, V(T)), (V(T), \emptyset)\}$.

We create a bundle in $T$ of order $\theta$ as follows. Let $\mathcal{B}_1 = \{(A_1, B_1)\}$. For $2 \leq q \leq k$, inductively let $\mathcal{B}_q = \mathcal{B}_{q-1} \cup \{(A_q, B_q)\}$ if $(A_q, B_q)$ does not cross and is not $\theta$-close to any members of $\mathcal{B}_{q-1}$, and let $\mathcal{B}_q = \mathcal{B}_{q-1}$ otherwise. Then it is clear that $\mathcal{B}_k$ is a bundle of order $\theta$ in $T$. This proves 2.10. ■

We now prove part of 1.1, the following.

**2.11** *If $T$ has pathwidth $\geq 4\theta^2 + 7\theta$ then $T$ has a $\theta$-jungle.*

**Proof.** Suppose that $T$ does not have a $\theta$-jungle. By 2.10, $T$ has a bundle $\mathcal{B}$ of order $\theta$. Let $(A_1, B_1), \ldots, (A_t, B_t)$ be all the members of $\mathcal{B}$, ordered such that $A_1 \subseteq \cdots \subseteq A_t$ and $B_t \subseteq \cdots \subseteq B_1$ (such an ordering exists by 1.2). Note that by 2.9, $(A_1, B_1) = (\emptyset, V)$ and $(A_t, B_t) = (V, \emptyset)$. For $1 \leq q \leq t$, let $C_q = A_q \cap B_q$. Next, for $1 \leq q \leq t-1$, let $X_q = (A_{q+1} \setminus B_{q+1}) \cap (B_q \setminus A_q)$ and let $W_q = X_q \cup C_q \cup C_{q+1}$. Then $W = [W_1, \ldots, W_{n-1}]$ is a path decomposition of $T$.

Since $T$ has pathwidth $\geq 4\theta^2 + 7\theta$ and $|C_q| < \theta$ for all $1 \leq q \leq t$, it follows that $|X_p| > 4\theta^2 + 5\theta$ for some $1 \leq p \leq t$. By 2.8, $X_p$ is $(\theta, \theta^2)$-separable. Hence, there exists a separation $(C, D)$ of $T$ of order $< \theta$ such that $|X_p \setminus C|, |X_p \setminus D| \geq \theta^2$, and let $(C, D)$ be such a separation that crosses as few members of $\mathcal{B}$ as possible. Note that $(C, D) \notin \mathcal{B}$ since $X_p \nsubseteq C$ and $X_p \nsubseteq D$. Let $d$ be the order of $(C, D)$. Then by the second bundle axiom, either $(C, D)$ crosses a member of $\mathcal{B}$ of order $\leq d$ or $(C, D)$ is $\theta$-close to a member of $\mathcal{B}$ of order $\leq d$.

(1) *$(C, D)$ crosses a member of $\mathcal{B}$ of order $\leq d$.*

Suppose not. Then $(C, D)$ is $\theta$-close to some $(A, B) \in \mathcal{B}$ of order $e \leq d$. Suppose that $A \subseteq C$ and $D \subseteq B$. Then $(B \setminus A) \cap (C \setminus D) < \theta(d - e) < \theta^2$. Since $(A, B) \in \mathcal{B}$, it follows that either $X_p \subseteq A \setminus B$ or $X_p \subseteq B \setminus A$. In the former case, it follows that $X_p \subseteq A \subseteq C$ contradicting the fact that $|X_p \setminus C| > 0$. In the latter case it follows that

$$|(B \setminus A) \cap (C \setminus D)| \geq |X_p \cap (C \setminus D) \geq \theta^2,$$

a contradiction. The case when $C \subseteq A$ and $B \subseteq D$ is analogous. This proves (1).

We may assume that $(C, D)$ crosses some $(A_q, B_q) \subseteq \mathcal{B}$ of order $\leq d$ with $q \leq p$ (the case where $q > p$ is analogous). Let $(A_r, B_r) \in \mathcal{B}$ be a separation of order $\leq d$ that crosses $(C, D)$ with $r$ as small as possible.

(2) *$B_r \cup C = V(T)$ and $|D \setminus B_r| < \theta$.*

By 2.1 either $A_r \cup D = V(T)$ or $B_r \cup C = V(T)$. Since $r < p$ it follows that $A_r \cap X_p = \emptyset$. Also, since $|(C \setminus D) \cap X_p| > 0$ it follows that $X_p \nsubseteq D$. Therefore, $X_p \nsubseteq A_r \cup D$ and so $A_r \cup D \neq V(T)$. Hence, $B_r \cup C = V(T)$, and so the first part of (2) holds. Consequently, $D \setminus B_r = (C \cap D) \setminus B_r$ and so $|D \setminus B_r| \leq |D \cap C| < \theta$. This proves (2).

Let $e$ be the order of $(A_r, B_r)$; then $e \leq d$. Then $(A_r \cup C, B_r \cap D)$ is a $\theta^2$-separator of $X_p$ and by 2.2 it crosses strictly fewer members of $\mathcal{B}$ of order $\leq d$ than $(C, D)$. Hence, from the way we chose $(C, D)$ we conclude that $(A_r \cup C, B_r \cap D)$ has order $> d$. It follows that $(C', D') = (C \cap A_r, D \cup B_r)$ has order $f < e$.

(3) *For all $(A_q, B_q) \in \mathcal{B}$ of order $\leq f$, the separations $(C', D')$ and $(A_q, B_q)$ do not cross.*

For $q \geq r$, we have $C' \subseteq A_r \subseteq A_q$ and $B_q \subseteq B_r \subseteq D'$, and hence $(C', D')$ and $(A_q, B_q)$ do not cross. For $q < r$, it follows from our choice of $(A_r, B_r)$ that $(A_q, B_q)$ and $(C, D)$ do not cross since the order of $(A_q, B_q)$ is $\leq d$. Hence, by 2.2, $(A_q, B_q)$ and $(C', D')$ do not cross. This proves (3).

(4) $(C', D') \notin \mathcal{B}$.

By (2), $C \cup B_r = V(T)$ and so $C' \cup B_r = V(T)$. It follows that $(D' \setminus C') \cap (A_r \setminus B_r) = \emptyset$ and so since $f < e$, $(C', D')$ is $\theta$-close to $(A_r, B_r)$. Therefore, $(C', D') \notin \mathcal{B}$. This proves (4).

From (3), (4) and the second bundle axiom, it follows that $(C', D')$ is $\theta$-close to some $(A_s, B_s) \in \mathcal{B}$ of order $g \leq f$.

(5) $s < r$.

First, we know that $s \neq r$ since the order of $(A_r, B_r)$ is $e > f$. Now suppose that $s > r$. Then $C' \subseteq A_r \subseteq A_s$. Let $Y = (D' \setminus C') \cap (A_s \setminus B_s)$. It follows from the definition of $\theta$-close that $|Y| < \theta(f - g)$. Let $Z = (B_r \setminus A_r) \cap (A_s \setminus B_s)$. Since $B_r \setminus A_r \subseteq D' \setminus C'$, we conclude that $Z \subseteq Y$. But then

$$|Z| \leq |Y| < \theta(f - g) < \theta(e - g),$$

implying that $(A_r, B_r)$ and $(A_s, B_s)$ are $\theta$-close. But $(A_r, B_r)$ and $(A_s, B_s)$ are both members of $\mathcal{B}$, a contradiction. This proves (5).

Next, we show that $A_s \subseteq C'$. Let $R = (B_s \setminus A_s) \cap (A_r \setminus B_r)$. Then $|R| > 0$ since $(A_s, B_s)$ and $(A_r, B_r)$ are not $\theta$-close. Since $B_r \cup C' = V(G)$, it follows that $A_r \setminus B_r \subseteq C'$ and so $R \subseteq C'$. But $R \not\subseteq A_s$ and so $C' \not\subseteq A_s$. Consequently, $A_s \subseteq C'$ since $(A_s, B_s)$ and $(C', D')$ do not cross.

Let $Q = (B_s \setminus A_s) \cap (C' \setminus D')$. Then since $(A_s, B_s)$ and $(C', D')$ are $\theta$-close, it follows that $|Q| < \theta(f - g)$; and since $(A_s, B_s)$ and $(A_r, B_s)$ are not $\theta$-close, it follows that $|R| \geq \theta(e - g)$. Hence, $|Q \setminus R| > \theta$. But

$$Q \setminus R \subseteq (A_r \setminus B_r) \setminus (C' \setminus D') \subseteq D' \setminus B_r = D \setminus B_r$$

and $|D \setminus B_r| < \theta$ by (2), a contradiction. We conclude that $T$ has a $\theta$-jungle and this proves 2.11. ■

We convert 2.11 to an algorithm as follows:

**2.12** *There is an algorithm with running time $O(n^{\theta+1})$, which, given as input a semi-complete digraph $T$ on $n$ vertices and an integer $\theta \geq 0$, outputs either a true statement that $T$ has a $\theta$-jungle or a path decomposition of $T$ with width $\leq 4\theta^2 + 7\theta$.*

**Proof.** We may assume that $\theta \geq 2$. Construct a list $(A_1, B_1), \ldots, (A_t, B_t)$ of all the separations of $T$ of order $< \theta$ such that for all $1 \leq p < q \leq t$, the order of $(A_p, B_p)$ is at most that of $(A_q, B_q)$. This can be done in time $O(n^{\theta+1})$ by listing all subsets $X$ of $V(T)$ of size at most $\theta - 1$ and then finding all separations of order zero of $T \setminus X$.

We construct a bundle in $T$ of order $\theta$ as in the proof of 2.10. Next, as in the proof of 2.11, for $1 \leq q \leq t-1$, let $X_q = (A_{q+1} \setminus B_{q+1}) \cap (B_q \setminus A_q)$ and let $W_q = X_q \cup C_q \cup C_{q+1}$. Then $W = [W_1, \ldots, W_{t-1}]$ is a path decomposition of $T$. If the width of $W$ is at most $4\theta^2 + 7\theta$, then output $W$. Otherwise, by 2.11, $T$ has a $\theta$-jungle, so output this statement. ■

**Proof of 1.1.** By 2.6 it follows that 1.1.3 implies 1.1.4 and by 2.11 it follows that 1.1.4 implies 1.1.1. Therefore, it suffices to prove that 1.1.1 implies 1.1.2 and that 1.1.2 implies 1.1.3.

(1) *For every digraph $H$ there is an integer $k \geq 0$ such that every semi-complete digraph that contains an $k$-triple also contains a subdivision of $H$ as a subdigraph. In particular, 1.1.2 implies 1.1.3.*

Let $V(H) = \{h_1, \ldots, h_r\}$ and $E(H) = \{e_1, \ldots, e_s\}$; let $k = \max(r, s)$. We claim that this value of $k$ satisfies (1). Let $T$ be a semi-complete digraph and let $(A, B, C)$ be an $k$-triple in $T$. Let $A = \{a_1, \ldots, a_k\}$, $B = \{b_1, \ldots, b_k\}$, and $C = \{c_1, \ldots, c_k\}$ such that $c_i a_i \in E(T)$. For $1 \leq i \leq r$ define $\eta(h_i) = b_i$. For $1 \leq p \leq s$, let $\eta(e_p)$ be the directed path

$$\eta(h_i) = b_i\text{-}c_p\text{-}a_p\text{-}b_j = \eta(h_j),$$

where $e_p = h_i h_j$. It is easy to check that $\eta$ is an expansion of $H$ in $G$.

It remains to show that 1.1.1 implies 1.1.2. Let $T$ be a semi-complete digraph and suppose that the pathwidth of $T$ is at most $k$. We will show that there exists a digraph $H$ such that $T$ does not contain a subdivision of $H$.

Let $T_1, T_2, T_3$ be transitive tournaments, each with $k+1$ vertices and let $H$ be obtained from $T_1 \cup T_2 \cup T_3$ by making $V(T_i)$ complete to $V(T_{i+1})$ for $i = 1, 2, 3$, where the subscripts are to be read modulo 3. Suppose that $T$ contains a subdivision of $H$ and let $\eta$ be an expansion of $H$ in $T$. Notice that $V(T_1)$ is a set of $k+1$ vertices of $H$ that are pairwise $(k+1)$-connected. Let $U = \eta(V(T_1))$. Then $U$ is a $(k+1)$-jungle in $T$. Let $[W_1, \ldots, W_r]$ be a path decomposition of $T$ of width at most $k$. It follows that there exist $1 \leq i, j \leq r$ and $u, v \in U$ such that $u \in W_i \setminus W_j$ and $v \in W_j \setminus W_i$. Without loss of generality, we may assume that $i < j$. Then there exists a set $X \subseteq V(T) \setminus \{u, v\}$ with $|X| \leq k$ such that every path from $u$ to $v$ contains a vertex of $X$. But $u$ and $v$ are $k+1$-connected, a contradiction. Therefore, $T$ does not contain a subdivision of $H$. This proves that 1.1.1 implies 1.1.2 and completes the proof of 1.1. ■

We note here that in [5], Thomassen proved the equivalence of 1.1.2 and 1.1.4 without mentioning 1.1.1 or 1.1.3.

# 3 Testing for a subdivision

In this section we use 2.12 to give a polynomial-time algorithm to test whether a fixed digraph $H$ is topologically contained in a given semi-complete digraph $T$. As mentioned in the introduction, it is important that $T$ is semi-complete, since for general digraphs the analogous problem is NP-complete.

We note that the algorithmic question of this section can also be solved by repeatedly running the algorithm in [1], which solves the vertex-disjoint paths problem in semi-complete digraphs. We present our algorithm because it is shorter and we believe it to be of interest in its own right.

The idea of our algorithm is as follows. Choose $k$ as in 1.1 such that every digraph with a $k$-jungle contains a subdivision of $H$. Now given as input a semi-complete digraph $T$, run 2.12 on $T$ with this value of $k$. If the output is a $k$-jungle, then we output that $T$ contains a subdivision of $H$ and we are done. Otherwise, we get a path decomposition of $T$ with width at most $4k^2 + 7k$; we can now use this decomposition to test for a subdivision of $H$ using dynamic programming. The remainder of this section is devoted to explaining the dynamic programming in more detail.

We begin with some notation and a consequence of a result from [1].

Let $G$ be a digraph and let $P_1, \dots, P_k$ be pairwise disjoint directed paths of $G$. If $P_i$ is from $a_i$ to $b_i$ for $1 \le i \le k$, then the set of pairs

$$\{(a_1, b_1), \dots, (a_k, b_k)\}$$

is *feasible* in $G$.

Now let $(v_1, \dots, v_n)$ be some enumeration of $V(G)$ and let $H = P_1 \cup \dots \cup P_k$. Then $(P_1, \dots, P_k)$ is *$t$-cohesive* (with respect to this enumeration) if $H|\{v_{i+1}, \dots, v_n\}$ is the disjoint union of at most $t$ directed paths for $0 \le i \le n$. The set of pairs $\{(a_1, b_1), \dots, (a_k, b_k)\}$ is then *$t$-cohesively feasible* with respect to the given enumeration. Note that if a set $T$ of pairs is $t$-cohesively feasible then $|T| \le t$. The following is a consequence of a result that appears in [1].

**3.1** *For all $t$ there is an algorithm as follows:*

- **Input:** *A digraph $G$ and an enumeration $(v_1, \dots, v_n)$ of its vertex set.*
- **Output:** *The set of all sets of pairs that are $t$-cohesively feasible in $G$ (with respect to the given enumeration).*
- **Running time:** *$O(n^{3t+4})$.*

A path decomposition $W = [W_1, \dots, W_r]$ is *nice* if $W_1 = W_r = \emptyset$ and for all $i$ with $2 \le i \le r$, $|(W_i \setminus W_{i-1}) \cup (W_{i-1} \setminus W_i)| = 1$. Note that for a digraph $G$ a nice path decomposition has exactly $2|V(G)| + 1$ terms. The following is immediate.

**3.2** *Let $G$ be a digraph with $n$ vertices. For every $k > 0$, there is an algorithm which given a path decomposition of $G$ that has width at most $k$ and such that no two consecutive terms are equal, finds a nice path decomposition of $G$ that has width at most $k$ in $O(n)$ time.*

We can associate a vertex ordering with a nice path decomposition as follows. Let $W = [W_1, \dots, W_r]$ be a nice path decomposition. For each $v \in V(G)$, there exists a unique $i$ such that $v \in W_i$ and $v \notin W_1 \cup \dots \cup W_{i-1}$. We say that $i$ is the *index* of $v$. It is clear that no two vertices have the same index and so there is a unique *increasing index enumeration* of $V(G)$ with respect to the decomposition $W$.

We need the following.

**3.3** *Let $G$ be a digraph and let $W$ be a nice path decomposition of $G$ that has width at most $k$. Let $(v_1, \dots, v_n)$ be the increasing index enumeration of $V(G)$ with respect to $W$, and let $P_1, \dots, P_m$ be vertex-disjoint paths in $G$. Then $(P_1, \dots, P_m)$ is $(k+m)$-cohesive with respect to the enumeration $(v_1, \dots, v_n)$.*

**Proof.** Let $A_i = \{v_{i+1}, \dots, v_n\}$ and let $G_i = G|A_i$. We need to show that for all $0 \le i \le n-1$, $H|A_i$ is the disjoint union of at most $k+m$ directed paths, where $H = P_1 \cup \dots \cup P_m$.

From the definition of nice path decomposition and increasing index enumeration, it follows that there exist at most $k$ vertices in $V(G) \setminus A_i$ with outneighbors in $A_i$. Let $X$ be the set of such vertices. For $1 \le j \le m$, let $|X \cap V(P_j)| = x_j$. Then $P_j|A_i$ is the disjoint union of at most $x_j + 1$ directed paths. Since

$$H|A_i = \bigcup_{j=1}^{m} P_j|A_i$$

it follows that $H|A_i$ is the disjoint union of at most $m + \sum_{j=1}^{m} x_j = m + k$ directed paths. This proves 3.3. ■

As a consequence of 3.3 we can solve in polynomial time the vertex-disjoint paths problem in digraphs with bounded pathwidth. We use this to obtain a polynomial-time algorithm to test for a subdivision of a fixed digraph, also in digraphs with bounded pathwidth.

Let $G, H$ be digraphs. Let $\eta$ be an expansion of $H$ in $G$. The *frame* of $\eta$ is a map $\phi$ such that

- $\phi(v) = \eta(v)$ for each $v \in V(H)$
- for each $e \in E(H)$ with $e = uv$, $\phi(e) = (x, y)$, where $x = \eta(u)$ and $y = \eta(v)$.

We now have:

**3.4** *For each digraph $H$ with $|V(H)| = r$ and $|E(H)| = s$ and every integer $k \ge 0$, there exists an algorithm as follows:*

- **Input:** *A digraph $G$ with $n$ vertices and a path decomposition $W$ of $G$ that has width at most $k$ and such that no two consecutive terms are equal.*
- **Output:** *The set of all frames of expansions of $H$ in $G$.*
- **Running time:** $O(n^{k+rs})$.

**Proof.** Let $V(H) = \{h_1, \dots, h_r\}$. For every injective map $\phi$ from $V(H)$ to $V(G)$ do the following: For $1 \le i \le r$, replace the vertex $\phi(h_i)$ by a set $X_i$ such that

- $|X_i|$ is the number of edges incident with $h_i$, counting loops twice
- $X_i$ is complete to all the outneighbors of $\phi(h_i)$ and complete from all the inneighbors of $\phi(h_i)$.

Let the resulting digraph be $G'$. Then we use $W$ to obtain a path decomposition of $G'$ of width at most $k + (r-1)s$, and we can run 3.2 on it to obtain a nice path decomposition $W'$. Compute the increasing index enumeration of $V(G')$ with respect to $W'$. Let this enumeration be $(v_1, \dots, v_m)$. Run 3.1 with $t = k + rs$ and with input $G'$ and this enumeration. For each set of pairs outputted by the algorithm, we determine whether in $G$ it corresponds to a frame of an expansion of $H$ in constant time (after identifying all the vertices in $X_i$ for each $1 \le i \le r$). Moreover, by 3.3, every frame of an expansion of $H$ will arise in such a way. This proves 3.4. ■

Finally, we have the main result of this section.

**3.5** *Let $H$ be a digraph with $|V(H)| = r$ and $|E(H)| = s$. Let $m = 2^{r(r+2)}, p = R(2m, 2m)$, and $\theta = 2^{12 \cdot 2^p}$. Then there is an algorithm with running time $O(n^{rs+4\theta^2+7\theta})$, which given as input a semi-complete digraph $T$ with $|V(T)| = n$ outputs yes or no, depending on whether there exists a subdivision of $H$ in $T$.*

**Proof.** Let $\theta$ be as in the statement of the theorem. Then every semi-complete digraph that contains a $\theta$-jungle contains a subdivision of $H$. Run 2.12 with input $T$ and $\theta$. If the output is a $\theta$-jungle, return that a subdivision exists. Otherwise, run 3.4 on the outputted pathwidth decomposition (of width at most $4\theta^2 + 7\theta$). If 3.4 returns at least one frame of a subdivision, return that a subdivision exists. Otherwise, return that a subdivision does not exist. ■

We note that we can modify the algorithm to actually return a subdivision if one exists, instead of just a yes answer. We omit the details.

## References


[1] M.Chudnovsky, A.Scott, P.Seymour, "Disjoint paths in tournaments", submitted for publication.

[2] S.Fortune, J.Hopcroft, J.Wyllie, "The directed subgraph homeomorphism problem", *Theor. Comp. Sci.* 10 (1980), 111-121.

[3] A. Fradkin and P.Seymour, "Edge-disjoint paths in digraphs with bounded independence number", submitted for publication.

[4] Ramsey, F. P. (1930), "On a Problem of Formal Logic", *Proc. London Math. Soc.* 30 (1930), 264-286.

[5] C. Thomassen, "Connectivity in tournaments", *Graph Theory and Combinatorics*, Academic Press, London (1984).