Tournament minors

Ilhee Kim and Paul Seymour[1]
Princeton University, Princeton, NJ 08540

June 19, 2011; revised April 17, 2014

[1]Supported by ONR grant N00014-10-1-0608 and NSF grant DMS-0901075.

Abstract

We say a digraph G is a *minor* of a digraph H if G can be obtained from a subdigraph of H by repeatedly contracting a strongly-connected subdigraph to a vertex. Here, we show the class of all tournaments is a well-quasi-order under minor containment.

1 Introduction

The "minor" relation for graphs is well-established, but how it should be extended to digraphs is not clear. In digraphs, contracting an edge may yield a directed cycle, even starting from an acyclic digraph, and this seems undesirable for a theory of excluded minors. One way to avoid this is to permit the contraction only of certain special edges; for example, if an edge uv is the only edge with tail u or the only edge with head v, then contracting uv does not yield a new directed cycle (see for instance [5]). Another way, too complicated to explain here, is discussed for instance in [6].

A third way to extend minors of graphs to digraphs is as follows. For graphs, one can define contraction in terms of contracting edges, or in terms of contracting connected subgraphs, and it comes to the same thing. But for digraphs, contracting edges and contracting strongly-connected subdigraphs lead to different minor relations, and in this paper we study the second. (A digraph G is *strongly-connected* if G is non-null and there exists a directed path from u to v for every $u, v \in V(G)$.) We say a digraph H is a *minor* of a digraph G if H can be obtained from a subdigraph of G by repeatedly contracting a strongly-connected subdigraph to a vertex. (Note that we do not create "new" directed cycles after contracting a strongly-connected subdigraph.) Equivalently, a digraph H is a *minor* of a digraph G if there exists a mapping ϕ defined on $V(H)$ such that:

- for every $v \in V(H)$, $\phi(v)$ is a non-null strongly-connected subdigraph of G;
- if $u, v \in V(H)$ and $u \neq v$, then $\phi(u)$ and $\phi(v)$ are vertex-disjoint; and
- for every $u, v \in V(H)$ (not necessarily distinct), if there are k edges in H with tail u and head v, then there are at least k edges in G with head in $V(\phi(u))$ and tail in $V(\phi(v))$, and not contained in $E(\phi(x))$ for any $x \in V(H)$.

We call such a map ϕ a *model* of H in G.

We first give some definitions. Every digraph in this paper is finite. We say a digraph G is *simple* if it is loopless and there is at most one edge $uv \in E(G)$ for all distinct $u, v \in V(G)$. A simple digraph G is *semi-complete* if either $uv \in E(G)$ or $vu \in E(G)$ for all distinct $u, v \in V(G)$. A semi-complete digraph G is a *tournament* if exactly one of uv and vu is an edge of G for all distinct $u, v \in V(G)$.

An important property of minors for graphs is that they define a "well-quasi-order" [8]. A *quasi-order* $Q = (V(Q), \leq_Q)$ consists of a class $V(Q)$ and a reflexive transitive relation $\leq_Q$ on $V(Q)$. A quasi-order Q is called a *well-quasi-order* or *wqo* if for every infinite sequence $q_1, q_2, \ldots$ of elements of $V(Q)$, there exist $j > i \geq 1$ such that $q_i \leq_Q q_j$. Neil Robertson and the second author proved that the class of all graphs is a wqo under the minor relation in [8].

The analogous statement is not true for directed minors. For example, a directed cycle is not a minor of a bigger directed cycle, and so if we take an infinite set of digraphs, all directed cycles of different lengths, then this set is an infinite antichain under the minor order. However, what if we consider some subclass, say the class of all tournaments? (The subdigraph relation does not define a wqo even for the class of all tournaments. We leave finding a counterexample as an exercise for the reader.)

In this paper, we prove that minor containment defines a wqo for the class of all semi-complete digraphs, and therefore the same is true for the class of all tournaments. We also prove it is not a wqo for some closely-related classes.

1.1 *The class of all semi-complete digraphs is a wqo under minor containment.*

In [2], Maria Chudnovsky and the second author proved that the class of all semi-complete digraphs is a wqo under "immersion", by using a digraph parameter called "cut-width". Here, we prove the analogous statement for minors by using another parameter called *path-width*. Path-width for undirected graphs was introduced in [7], and it has a natural extension to digraphs, discussed for instance in [3].

For a digraph G, we say $P = (W_1, \dots, W_r)$ is a *path-decomposition* of G if:

- $r \geq 1$ and $\cup(W_i : 1 \leq i \leq r) = V(G)$,
- *(betweenness condition)* for $1 \leq h < i < j \leq r$, $W_h \cap W_j \subseteq W_i$, and
- *(cut condition)* if $uv \in E(G)$, then there exist i, j with $1 \leq i \leq j \leq r$ such that $u \in W_j$ and $v \in W_i$.

The betweenness condition implies that $\{i : v \in W_i\}$ is an integer interval for each $v \in V(G)$, and the cut condition implies that for $1 \leq i \leq r$ there are no edges from $\cup_{h<i} W_h$ to $\cup_{j>i} W_j$ in $G \setminus W_i$.

We define the *width* of a path-decomposition P to be $\max_{1 \leq i \leq r}(|W_i| - 1)$ and denote it by $pw(P)$. We say G has *path-width* k if $k \geq 0$ is minimum such that there exists some path-decomposition P of G with $pw(P) \leq k$, and we denote the path-width of G by $pw(G)$. For example, a loopless digraph G is acyclic if and only if $pw(G) = 0$.

For a path-decomposition $P = (W_1, \dots, W_r)$, we denote $\min_{1 \leq i \leq r} |W_i|$, $\max_{1 \leq i \leq r} |W_i|$, W_1, and W_r by $m(P)$, $M(P)$, $F(P)$, and $L(P)$, respectively. (In later sections, it is convenient to work with $M(P) = pw(P) + 1$ rather than with $pw(P)$.) We first prove that having bounded path-width is a minor-closed property.

1.2 *If a digraph has path-width at most k, then so do all its minors.*

Proof. Let $P = (W_1, \dots, W_r)$ be a path-decomposition of a digraph G. Then P is a path-decomposition of $G \setminus e$ for each edge $e \in E(G)$, and $(W_1 \setminus v, \dots, W_r \setminus v)$ is a path-decomposition of $G \setminus v$ for each vertex $v \in V(G)$. Therefore the path-width of a digraph G does not increase by deleting an edge or a vertex.

Thus, it remains to show that if G has path-width at most k, then so does G/H where H is a strongly-connected subdigraph of G. (G/H is the digraph obtained from G by contracting H to a single vertex w). Let $P = (W_1, ..., W_r)$ be a path-decomposition of G, and let $I_H = \{i : W_i \cap V(H) \neq \emptyset\}$.

(1) I_H *is an integer interval.*

Suppose that I_H is not an integer interval. Take indices $h < i < j$ such that $h, j \in I_H$ and $i \notin I_H$. Let $u \in W_h \cap V(H)$ and $v \in W_j \cap V(H)$. Since $\{t : u \in W_t\} \subseteq \{1, \dots, i-1\}$ and $\{t : v \in W_t\} \subseteq \{i+1, \dots, r\}$, the sets $\{t : u \in W_t\}$ and $\{t : v \in W_t\}$ do not intersect. Since H is strongly-connected and $V(H) \cap W_i = \emptyset$, there is a directed path from u to v in $G \setminus W_i$. However, this contradicts the cut condition since there are no edges from $\cup_{a<i} W_a$ to $\cup_{b>i} W_b$ in $G \setminus W_i$. This proves (1).

Let $P = (W_1, \dots, W_r)$ be a path-decomposition of G with $pw(P) \leq k$. Define W_i' by

$$W_i' = \begin{cases} (W_i \setminus V(H)) \cup \{w\} & \text{if } i \in I_H \\ W_i & \text{otherwise.} \end{cases}$$

We claim that $P' = (W'_1, \dots, W'_r)$ is a path-decomposition of G/H with $pw(P') \le k$. The betweenness condition follows from (1). For the cut condition, we only need to consider edges incident with w in G/H. For an edge $uw \in E(G/H)$, consider the corresponding edge $uv \in E(G)$. By the cut condition for P, there exist $i \le j$ such that $W_j \ni u$ and $W_i \ni v$. Therefore $W'_j \ni u$ and $W'_i \ni w$. The same argument applies for edges with tail w. Finally, $pw(P') \le pw(P) \le k$ from the definition of P'. This proves 1.2. ∎

We introduce a notion and a theorem from [3]. For a digraph G, let A, B and C be mutually disjoint subsets of $V(G)$. We say (A, B, C) is a *k-triple* if

- $|A| = |B| = |C| = k$,
- $ab \in E(G)$ for every $a \in A$ and $b \in B$,
- $bc \in E(G)$ for every $b \in B$ and $c \in C$, and
- A, C can be numbered as $\{a_1, \dots, a_k\}$ and $\{c_1, \dots, c_k\}$ respectively such that $c_i a_i \in E(G)$ for $i = 1, \dots, k$.

If G is a digraph and $A \subseteq V(G)$, $G[A]$ denotes the subdigraphi of G induced on A.

1.3 *Let (A, B, C) be a k-triple of a digraph G. Then G contains every semi-complete digraph with k vertices as a minor.*

Proof. Let $\{a_1, \dots, a_k\}$ and $\{c_1, \dots, c_k\}$ be numberings of A and C such that $c_i a_i \in E(G)$ for $i = 1, \dots, k$. Take an ordering $\{b_1, \dots, b_k\}$ of B. Then $G[\{a_i, b_i, c_i\}]$ is strongly-connected for each i. Let G' be the digraph obtained from $G[A \cup B \cup C]$ by contracting $G|\{a_i, b_i, c_i\}$ to a single vertex for each i. Then $|V(G')| = k$ and $uv \in E(G')$ for every distinct $u, v \in V(G')$. Therefore every semi-complete digraph with k vertices is a subdigraph of G' and hence, a minor of G. This proves 1.3. ∎

The following theorem says a semi-complete digraph has large path-width if and only if it has a large k-triple.

1.4 *For every set $\mathcal{S}$ of semi-complete digraphs, the following are equivalent:*

1. *There exists k such that every member of $\mathcal{S}$ has path-width at most k.*
2. *There is a digraph H such that no subdivision of H is a subdigraph of any member of $\mathcal{S}$.*
3. *There exists k such that for each $G \in \mathcal{S}$, there is no k-triple in G.*
4. *There exists k such that for each $G \in \mathcal{S}$, there do not exist k vertices of G that are pairwise k-connected.*
5. *There is a digraph H such that for each $G \in \mathcal{S}$, G does not contain H as a minor.*

Proof. The equivalence of the first four statements was proved by Alexandra Fradkin and the second author in [3]. Here, we prove $1 \Rightarrow 5 \Rightarrow 3$ to extend the theorem.

Suppose that 1 holds for k and $\mathcal{S}$ and let H be a digraph with $pw(H) > k$. Then 5 holds by 1.2. Now, suppose that 5 holds for H and $\mathcal{S}$. Let H' be a simple digraph containing H as a minor. Then 5 holds for H' and $\mathcal{S}$ as well. By 1.3, G has no $|V(H')|$-triple for each $G \in \mathcal{S}$. Therefore 3 holds. This proves 1.4. ∎

Thanks to 1.4, it is enough to show the following statement to prove 1.1.

1.5 *For all $k \geq 0$, the class of all semi-complete digraphs with path-width $\leq k$ is a wqo under minor containment.*

Proof of 1.1, assuming 1.5. Let $G_1, G_2, \ldots$ be an infinite sequence of semi-complete digraphs. We may assume that G_i does not contain G_1 as a minor for each $i \geq 2$. From 1.4, there exists k such that every member of $\mathcal{S} = \{G_2, G_3, \ldots\}$ has path-width at most k. From 1.5, there exist $j > i \geq 2$ such that G_i is a minor of G_j. This proves 1.1. ∎

Most of the remainder of this paper is devoted to proving 1.5. In section 2, we show the existence of a "linked" path-decomposition; and use it in sections 3 and 4 to prove a slightly more general version of 1.5. In section 5, we give counterexamples to disprove the extension of 1.1 to some larger classes of digraphs.

2 Linked path-decompositions

In this section, we show that every digraph of path-width k has a particularly nice path-decomposition of width k. It is designed so that when we break this path-decomposition into a sequence of small path-decompositions in the natural way, we will be able to apply Higman's sequence theorem to this sequence (in the proof of 4.4).

We first give some definitions. A directed path P in a semi-complete digraph G is *induced* if $v_iv_j \notin E(G)$ for $j - i \geq 2$ where $v_1, \ldots, v_n$ are the vertices of P in order. Note that $G[V(P)]$ is strongly-connected unless it is a one-edge directed path. For two sets A and B, denote by $A \Delta B$ the symmetric difference $(A \setminus B) \cup (B \setminus A)$. For a digraph G, we say (C, D) is a *separation of G of order s* if:

- $C \cup D = V(G)$,
- $|C \cap D| = s$, and
- there are no edges $uv \in E(G)$ such that $u \in C \setminus D$ and $v \in D \setminus C$.

If $A, B \subseteq V(G)$, a separation (C, D) of G *separates* A, B if $A \subseteq C$ and $B \subseteq D$. A path-decomposition $P = (W_1, \ldots, W_r)$ of a digraph G is called a *linked path-decomposition* if:

- *(increment condition)* $|W_i \ \Delta \ W_{i+1}| = 1$ for $i = 1, \ldots, r-1$,
- *(cardinality condition)* $|W_1| = |W_r| = m(P)$, and
- *(linked condition)* if $|W_i| \geq t$ for every i with $h \leq i \leq j$, then there exist t vertex-disjoint directed paths from W_h to W_j.

By Menger's theorem, the linked condition could be re-expressed as

- if $|W_i| \geq t$ for every i with $h \leq i \leq j$, then there is no separation (A, B) of order less than t with $W_h \subseteq A$ and $W_j \subseteq B$.

Observe that for every $v \notin W_1$, there exists a unique i with $1 \leq i \leq r-1$ such that $\{v\} = W_{i+1} \setminus W_i$, and for every $v \notin W_r$, there exists a unique j with $1 \leq j \leq r-1$ such that $\{v\} = W_j \setminus W_{j+1}$. Therefore the increment condition implies that $r-1 = 2\big(|V(G)| - m(P)\big)$. In particular, r is bounded above by $2|V(G)|+1$. We now prove the existence of a linked path-decomposition in a semi-complete digraph.

2.1 *Let G be a semi-complete digraph and $A, B \subseteq V(G)$ with $|A| = |B| = m \geq 0$. Suppose that there exist m vertex-disjoint directed paths from A to B in G, and there exists a path-decomposition (not necessarily linked) P of G such that $F(P) = A$, $L(P) = B$, and $M(P) \leq k$ for some k. Then there exists a linked path-decomposition P' such that $F(P') = A$, $L(P') = B$, and $M(P') \leq k$.*

In particular, every semi-complete digraph G with $pw(G) \leq k$ has a linked path-decomposition P with $pw(P) \leq k$ and $m(P) = 0$.

Proof. Observe that we can obtain a path-decomposition P' with $F(P') = A$, $L(P') = B$ and $M(P') \leq k$ satisfying the increment condition, by modifying P as follows: we remove (one of) any two consecutive sets that are equal, and insert appropriate sets between sets that differ by more than one vertex. The cardinality condition for P' is guaranteed by the existence of m vertex-disjoint paths from A to B.

Now, among all the path-decompositions $P = (W_1, \ldots, W_r)$ of G with $F(P) = A$, $L(P) = B$, and $M(P) \leq k$ satisfying the increment condition and the cardinality condition, we take one with $(n_0, n_1, \ldots, n_k)$ "lexicographically maximal", where $n_j = |\{i : |W_i| = j\}|$ for $j = 0, \ldots, k$. (More precisely, take one with n_0 as large as possible; subject to that, take n_1 as large as possible; and so on.) We can take such a path-decomposition since r is bounded above by $2|V(G)|+1$. Let P' be the path-decomposition we choose. We show P' satisfies the linked condition.

Let $P' = (W_1, \ldots, W_r)$. Suppose that $|W_i| \geq t$ for every i with $h \leq i \leq j$ and there do not exist t vertex-disjoint directed paths from W_h to W_j. Then from Menger's theorem, there is a separation (C, D) of order less than t that separates $\cup_{i \leq h} W_i$, $\cup_{i \geq j} W_i$. Take such a separation (C, D) with minimum order s. We claim that there exist two path-decompositions

$$P^C = (W_1^C, \ldots, W_j^C)$$

of $G[C]$ with $W_i^C = W_i$ for $1 \leq i \leq h$, $W_j^C = C \cap D$, and $M(P^C) \leq k$, and

$$P^D = (W_h^D, \ldots, W_r^D)$$

of $G[D]$ with $W_h^D = C \cap D$, $W_i^D = W_i$ for $j \leq i \leq r$, and $M(P^D) \leq k$. Then we will show that the "concatenation" of the two path-decompositions yields a path-decomposition lexicographically better than P', which contradicts our choice of P'.

We construct P^C as follows. Note that there exist s vertex-disjoint paths from W_h to W_j by the minimality of s. Take s vertex-disjoint directed paths $P_1, \ldots, P_s$ from $W_1 \cup \cdots \cup W_h$ to $W_j \cup \cdots \cup W_r$ with minimal union. For $1 \leq l \leq s$, the minimality of the union of $P_1, \ldots, P_s$ implies that P_l is induced and no vertex of P_l belongs to $W_1 \cup \cdots \cup W_h$ except its first vertex. Since there is no edge from $W_1 \cup \cdots \cup W_{h-1}$ to $W_{h+1} \cup \cdots \cup W_r$ in $G \setminus W_h$, it follows that the first vertex of P_l belongs to W_h. Similarly, the last vertex of P_l belongs to W_j, and no other vertex of P_l belongs to $W_j \cup \cdots \cup W_r$. Let $p_l \in V(P_l) \cap (C \cap D)$.

(1) *For $1 \leq l \leq s$, $\{i : W_i \cap (D \cap V(P_l)) \neq \emptyset,\ 1 \leq i \leq j\}$ is an integer interval containing j.*

If $G[D \cap V(P_l)]$ is strongly-connected, then (1) holds by the same argument as (1) in the proof of 1.2. Therefore we may assume that $G[D \cap V(P_l)]$ has exactly two vertices u, v with one edge uv, since P_l is induced. By the cut condition for the edge uv, there exist a, b with $1 \leq a \leq b \leq r$ such that $u \in W_b$ and $v \in W_a$. Since no vertex of P_l except its last is in $W_j \cup \cdots \cup W_r$, it follows that $b < j$. Since $v \in W_a \cap W_j$ and $a \leq b \leq j$, it follows that $v \in W_b$. In summary, there exists b with $1 \leq b < j$ such that $u, v \in W_b$.

On the other hand, $\{i : W_i \cap \{u\} \neq \emptyset,\ 1 \leq i \leq j\}$ and $\{i : W_i \cap \{v\} \neq \emptyset,\ 1 \leq i \leq j\}$ are both integer intervals by the betweenness condition. Since they intersect, the set in question is also an integer interval since it is the union of the two intersecting intervals. This proves (1).

For each i with $1 \leq i \leq j$, define

$$W_i^C = (W_i \cap C) \ \cup\ \{p_l : W_i \cap (D \cap V(P_l)) \neq \emptyset,\ 1 \leq l \leq s\}.$$

Let $P^C = (W_1^C, \ldots, W_j^C)$.

(2) *P^C is a path-decomposition of $G[C]$ with $W_i^C = W_i$ for $1 \leq i \leq h$, $W_j^C = C \cap D$, and $M(P^C) \leq k$.*

It is easy to check that $\cup_{i=1}^r W_i^C = C$, and the betweenness condition follows from (1). For the cut condition, we only need to consider edges incident with p_l in $G[C]$ and this is trivial since

$$\{i : p_l \in W_i, 1 \leq i \leq j\} \subseteq \{i : p_l \in W_i^C, 1 \leq i \leq j\}.$$

Therefore P^C is a path-decomposition of $G[C]$.

For $1 \leq i \leq h$, $W_i^C = W_i$ since $W_i \subseteq C$. And $W_j^C = C \cap D$ since $W_j \subseteq D$. Finally, $M(P_C) \leq M(P) \leq k$ since $|W_i^C| \leq |W_i|$ for every i with $1 \leq i \leq j$. This proves (2).

Similarly, let $P^D = (W_h^D, \ldots, W_r^D)$, where

$$W_i^D = (W_i \cap D) \ \cup\ \{p_l : W_i \cap C \cap V(P_l) \neq \emptyset,\ 1 \leq l \leq s\}$$

for each i with $h \leq i \leq r$, then it is a path-decomposition of $G[D]$ with $W_h^D = C \cap D$, $W_i^D = W_i$ for $j \leq i \leq r$, and $M(P_D) \leq k$.

Let P^* be the path-decomposition of G obtained by concatenating P^C and P^D and refining it to satisfy the increment condition. Then P^* is "lexicographically better" than P' since every W_a with $|W_a| \leq s$ is a term in the sequence P^* (because $W_i^C = W_i$ for $1 \leq i \leq h$ and $W_i^D = W_i$ for $j \leq i \leq r$, and $|W_i| > s$ for $h \leq i \leq j$), and there exists at least one more set of size s, namely $C \cap D$. This proves 2.1. ∎

3 Labeled minors

In this section, for a wqo Q and a semi-complete digraph G, we assign an element of $V(Q)$ to each vertex of G, and we fix a linked path-decomposition P of G together with $m(P)$ vertex-disjoint directed paths from $F(P)$ to $L(P)$. We define a minor relation for these slightly more general objects and prove a well-quasi-order theorem for them. Then 1.5 will follow as a corollary. Roughly speaking, we need this "Q-labeling" in order to handle the case when one of the induced directed paths has length one, and does not induce a strongly-connected subdigraph, and consequently we may not contract it.

For integers m, k with $k \geq m \geq 0$ and a well-quasi-order Q, we say $D = (G, P, R, l)$ is a (Q, m, k)-*digraph* if:

- G is a semi-complete digraph,
- P is a linked path-decomposition of G with $m(P) = m$ and $M(P) \leq k$,
- $R = (R_1, \dots, R_m)$ is a sequence of m vertex-disjoint induced directed paths from $F(P)$ to $L(P)$ in G, and
- l is a mapping from $V(G)$ to $V(Q)$.

Note that $|F(P) \cap V(R_i)| = |L(P) \cap V(R_i)| = 1$ for each $i = 1, \dots, m$. We say the vertex in $F(P) \cap V(R_i)$ is the *i-th source root* of D and the vertex in $L(P) \cap V(R_i)$ is the *i-th terminal root* of D. We denote the collection of all (Q, m, k)-digraphs by $\mathcal{G}_m^k(Q)$. We say D is *trivial* if $r = 1$ where $P = (W_1, \dots, W_r)$. Note that $|V(G)| = m$ if D is trivial.

Now we define a minor relation on $\mathcal{G}_m^k(Q)$. Let $D = (G, P, R, l)$, $D' = (G', P', R', l') \in \mathcal{G}_m^k(Q)$. Let a_i and a'_i be the i-th source roots of D and D', respectively, and similarly let b_i and b'_i be the i-th terminal roots of D and D', respectively. We say D is a *minor* of D' if there exists a model ϕ of G in G' such that:

- $a'_i \in V(\phi(a_i))$ and $b'_i \in V(\phi(b_i))$ for $i = 1, \dots, m$, and
- for every $v \in V(G)$, $l(v) \leq_Q l'(u)$ for some $u \in V(\phi(v))$.

Again, we call ϕ a *model* of D in D'.

Next, we define a "decomposition" of a (Q, m, k)-digraph. Let $D = (G, P, R, l)$ be a (Q, m, k)-digraph with $P = (W_1, \dots, W_r)$, $R = (R_1, \dots, R_m)$ and suppose that $|W_s| = m$ for some s with $1 < s < r$. Let $A = \cup_{i \leq s} W_i$ and define $D[A] = (G_A, P_A, R_A, l_A)$ by:

- $G_A = G[A]$,
- $P_A = (W_1, \dots, W_s)$,
- $R_A = (R_1[A], \dots, R_m[A])$, and
- $l_A = l|A$.

Then $D[A]$ is a (Q, m, k)-digraph and similarly, $D[B]$ is a (Q, m, k)-digraph where $B = \cup_{i \geq s} W_i$. We write $D = D_A \oplus D_B$ and say D is *decomposable*. We denote the class of all non-trivial non-decomposable (Q, m, k)-digraphs by $\mathcal{ND}_m^k(Q)$. More precisely, we say $D = (G, P, R, l) \in \mathcal{G}_m^k(Q)$ is in $\mathcal{ND}_m^k(Q)$ if:

- $r \geq 3$, and $|W_i| > m$ for every i with $2 \leq i \leq r - 1$ where $P = (W_1, \dots, W_r)$.

Note that $P' = (W_2, \dots, W_{r-1})$ is a linked path-decomposition of G with $m(P') = m+1$, $F(P') = W_2 \supseteq W_1$, and $L(P') = W_{r-1} \supseteq W_r$. Let R' be a sequence of $m+1$ vertex-disjoint induced directed paths from W_2 to W_{r-1}. Then we see that each $D = (G, P, R, l) \in \mathcal{ND}_m^k(Q)$ yields at least one member $D' = (G, P', R', l) \in \mathcal{G}_{m+1}^k$. (Notice that it could be the case that some path in R' joins the i-th source root of D to the j-th terminal root of D for some $j \neq i$.)

3.1 *Let m, k be integers with $k > m \geq 0$. Suppose that $\mathcal{G}_{m+1}^k(Q)$ is a wqo under minor containment for every wqo Q. Then $\mathcal{ND}_m^k(Q)$ is a wqo under minor containment for every wqo Q as well.*

Proof. Let Q be a wqo and $D_1, D_2, \dots$ be an infinite sequence in $\mathcal{ND}_m^k(Q)$. For each $D_i = (G_i, P_i, R_i, l_i)$, let $D_i' = (G_i, P_i', R_i', l_i) \in \mathcal{G}_{m+1}^k(Q)$ as described earlier. Recall that every source root of D_i is also a source root of D_i', and every terminal root of D_i is also a terminal root of D_i'.

For each $i \geq 1$, let $\sigma_i, \tau_i : \{1, \dots, m\} \to \{1, \dots, m+1\}$ be injections defined by

- the t-th source root of D_i equals the $\sigma_i(t)$-th source root of D_i'.
- the t-th terminal root of D_i equals the $\tau_i(t)$-th terminal root of D_i'.

Since there are only finitely many pairs (σ_i, τ_i), there exists some (σ, τ) such that $(\sigma_i, \tau_i) = (\sigma, \tau)$ for infinitely many i. Therefore we may assume that $\sigma_i = \sigma$ and $\tau_i = \tau$ for every $i \geq 1$.

Since $\mathcal{G}_{m+1}^k(Q)$ is a wqo under minor containment, D_i' is a minor of D_j' for some $i < j$. Let ϕ be a model of D_i' in D_j'. Then ϕ is also a model of D_i in D_j. This proves 3.1. ∎

We say $D = (G, P, R, l) \in \mathcal{G}_m^k(Q)$ is *contractible* if

- $G[V(R_j)]$ is strongly-connected for every $j \in \{1, \dots, m\}$ where $R = (R_1, \dots, R_m)$.

We denote the set of all non-contractible (Q, m, k)-digraphs by $\mathcal{NC}_m^k(Q)$.

Suppose that $G \in \mathcal{NC}_m^k(Q)$. In other words, $G[V(R_j)]$ is not strongly-connected for some j. Then $G[V(R_j)]$ must be a digraph with two vertices, namely the j-th source root u and the j-th terminal root v, and one edge uv. Note that every W_i contains either u or v where $P = (W_1, \dots, W_r)$. Let $G' = G \setminus \{u, v\}$, $\hat{W}_i = W_i \setminus \{u, v\}$ for $i = 1, \dots, r$, and $\hat{P} = (\hat{W}_1, \dots, \hat{W}_r)$. Then $\hat{P}$ is a path-decomposition (not necessarily linked) of G' with $M(\hat{P}) \leq k-1$. Note that we still have $m-1$ vertex-disjoint paths from $F(P) \setminus u$ to $L(P) \setminus v$. From 2.1, there exists a linked path-decomposition P' of G' with $F(P') = F(P) \setminus u$, $L(P') = L(P) \setminus v$, $m(P') = m-1$, and $M(P') \leq k-1$. Also, the sequence R' obtained from R by omitting R_j is a sequence of $m-1$ vertex-disjoint induced directed paths from $F(P')$ to $L(P')$. For labels, let Q' be a well-quasi-order defined by

- $V(Q') = V(Q) \times \{0, 1, 2\} \times \{0, 1, 2\}$, and
- $(q, x, y) \leq_{Q'} (q', x', y')$ if and only if $q \leq_Q q'$ and $x = x'$ and $y = y'$.

For each $w \in V(G) \setminus \{u, v\}$, let

$$x(w) = \begin{cases} 0 & \text{if } wu \in E(G) \text{ and } uw \notin E(G) \\ 1 & \text{if } wu \notin E(G) \text{ and } uw \in E(G) \\ 2 & \text{if } wu \in E(G) \text{ and } uw \in E(G) \end{cases}$$

and

$$y(w) = \begin{cases} 0 & \text{if } wv \in E(G) \text{ and } vw \notin E(G) \\ 1 & \text{if } wv \notin E(G) \text{ and } vw \in E(G) \\ 2 & \text{if } wv \in E(G) \text{ and } vw \in E(G). \end{cases}$$

Let l' be a mapping from $V(G')$ to $V(Q')$ defined by

- $l'(w) = (l(w), x(w), y(w))$ for each $w \in V(G) \setminus \{u, v\}$.

Then $D' = (G', P', R', l') \in \mathcal{G}_{m-1}^{k-1}(Q')$ and we see that each $D \in \mathcal{ND}_m^k(Q)$ yields at least one member D' in $\mathcal{G}_{m-1}^{k-1}(Q')$.

3.2 *Let m, k be integers with $k \geq m \geq 1$. Suppose that $\mathcal{G}_{m-1}^{k-1}(Q)$ is a wqo under minor containment for every wqo Q. Then $\mathcal{NC}_m^k(Q)$ is a wqo under minor containment for every wqo Q as well.*

Proof. Let Q be a wqo and $D_1, D_2, \ldots$ be an infinite sequence in $\mathcal{NC}_m^k(Q)$. For each $D_i = (G_i, P_i, R_i, l_i)$, let $D_i' = (G_i', P_i', R_i', l_i') \in \mathcal{G}_{m-1}^{k-1}(Q')$ as described earlier and let u_i and v_i be the source root and the terminal root of (G_i, P_i, R_i, l_i) such that $G_i' = G_i \setminus \{u_i, v_i\}$. Since Q is a wqo, we may assume that

- $l_i(u_i) \leq_Q l_j(u_j)$, and $l_i(v_i) \leq_Q l_j(v_j)$ for every $i < j$.

Since $\mathcal{G}_{m-1}^{k-1}(Q')$ is a wqo under minor containment, there exist i, j with $1 \leq i < j$ such that there is a model ϕ' of D_i' in D_j'. Define ϕ from D_i to D_j by:

$$\phi(w) = \begin{cases} (\{u_j\}, \emptyset) & \text{if } w = u_i \\ (\{v_j\}, \emptyset) & \text{if } w = v_i \\ \phi'(w) & \text{if } w \in V(G_i'). \end{cases}$$

Then it is easy to check that ϕ is a model of D_i in D_j, by the definition of Q'-labels. This proves 3.2. ∎

For two subclasses $\mathcal{A}, \mathcal{B}$ of $\mathcal{G}_m^k(Q)$, denote by $\mathcal{A} \oplus \mathcal{B}$ the class of all (Q, m, k)-digraphs D which are decomposable as $D_A \oplus D_B$ where $D_A \in \mathcal{A}$ and $D_B \in \mathcal{B}$.

3.3 *If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{G}_m^k(Q)$ are both wqo under minor containment, then $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \oplus \mathcal{B}$ are both wqo under minor containment.*

Proof. $\mathcal{A} \cup \mathcal{B}$ is a wqo under minor containment because every infinite sequence in $\mathcal{A} \cup \mathcal{B}$ contains either an infinite subsequence in $\mathcal{A}$ or an infinite subsequence in $\mathcal{B}$.

For $\mathcal{A} \oplus \mathcal{B}$, let $D_1, D_2, \ldots$ be an infinite sequence in $\mathcal{A} \oplus \mathcal{B}$. Let $D_i = D_i^a \oplus D_i^b$ where $D_i^a = (G_i^a, P_i^a, R_i^a, l_i^a) \in \mathcal{A}$ and $D_i^b = (G_i^b, P_i^b, R_i^b, l_i^b) \in \mathcal{B}$ for each $i \geq 1$. Then there exist $i < j$ such that:

- there is a model ϕ_a of D_i^a in D_j^a, and

- there is a model ϕ_b of D_i^b in D_j^b.

Define a mapping ϕ from D_i to D_j by

$$\phi(w) = \begin{cases} \phi_a(w) & \text{if } w \in V(G_i^a) \setminus V(G_i^b) \\ \phi_b(w) & \text{if } w \in V(G_i^b) \setminus V(G_i^a) \\ \phi_a(w) \cup \phi_b(w) & \text{if } w \in V(G_i^a) \cap V(G_i^b). \end{cases}$$

Since the union of two strongly-connected subdigraphs with non-empty intersection is also strongly-connected, $\phi_a(w) \cup \phi_b(w)$ is strongly-connected in G_j for each $w \in V(G_i^a) \cap V(G_i^b)$. Then it is easy to check that ϕ is a model of D_i in D_j. This proves 3.3. ∎

4 Links, and the main proof

We say a contractible (Q, m, k)-digraph D ($\notin \mathcal{NC}_m^k(Q)$) is a *link* if:

- $D \in \mathcal{ND}_m^k(Q) \ \cup \ (\mathcal{NC}_m^k(Q) \oplus \mathcal{ND}_m^k(Q))$.

We denote the collection of all links in $\mathcal{G}_m^k(Q)$ by $\mathcal{L}_m^k(Q)$. The following is an easy corollary of 3.3.

4.1 *Suppose that $\mathcal{NC}_m^k(Q)$ and $\mathcal{ND}_m^k(Q)$ are wqo under minor containment for some wqo Q. Then $\mathcal{L}_m^k(Q)$ is a wqo under minor containment as well.*

Now, we decompose $D \in \mathcal{G}_m^k(Q)$ into links (possibly except the last term) to apply Higman's sequence theorem.

4.2 *Let $D = (G, P, R, l)$ be a non-trivial (Q, m, k)-digraph. Then $D = D_1 \oplus \ldots \oplus D_t$ (perhaps with $t = 1$) such that:*

- *$D_i \in \mathcal{L}_m^k(Q)$ for $i \in \{1, \ldots, t-1\}$, and*
- *$D_t \in \mathcal{L}_m^k(Q) \ \cup \ \mathcal{NC}_m^k(Q)$.*

Proof. We may assume that D is contractible since otherwise $D \in \mathcal{NC}_m^k(Q)$ and the result holds with $t = 1$. Let $P = (W_1, \ldots, W_r)$. We proceed by induction on r. For the base case $r = 3$, D belongs to $\mathcal{ND}_m^k(Q)$ and hence, D itself is a link.

Let $1 = n_1 < \cdots < n_s = r$ be the indices such that

$$|W_{n_1}| = \cdots = |W_{n_s}| = m$$

Let $j > 1$ be the smallest index such that the initial segment $D[\cup_{i=1}^{n_j} W_i]$ is contractible (such j exists since D is contractible).

If $j = 2$, then the initial segment $D[\cup_{i=1}^{n_2} W_i]$ belongs to $\mathcal{ND}_m^k(Q)$, and hence it is a link. If $j > 2$, then

$$\begin{aligned} D[\cup_{i=1}^{n_j} W_i] &= D[\cup_{i=1}^{n_{j-1}} W_i] \ \oplus \ D[\cup_{i=n_{j-1}}^{n_j} W_i] \\ &\in \mathcal{NC}_m^k(Q) \ \oplus \ \mathcal{ND}_m^k(Q). \end{aligned}$$

Therefore, in either case, $D[\cup_{i=1}^{n_j} W_i]$ is a link. If $j = s$, then D is a link, and we are done. Otherwise, $D[\cup_{i=n_j}^{r} W_i]$ is non-trivial and satisfies the statement by the induction hypothesis. Therefore

$$D = D[\cup_{i=1}^{n_j} W_i] \ \oplus \ D[\cup_{i=n_j}^{r} W_i]$$

satisfies the statement as well. This proves 4.2. ∎

Let Q be a quasi-order. We define a quasi-order $Q^{<\omega}$ on the set of all finite sequences of elements of $V(Q)$. Let $p = (p_1, \ldots, p_a)$ and $q = (q_1, \ldots, q_b)$ be sequences of elements of $V(Q)$. Then $p \leq_{Q^{<\omega}} q$ if and only if:

- $a \leq b$, and
- there exist $1 \leq \alpha_1 < \ldots < \alpha_a \leq b$ such that $p_i \leq_Q q_{\alpha_i}$ for every $i = 1, \ldots, a$.

Higman [4] proved the following, called *Higman's sequence theorem*:

4.3 *If Q is a wqo, then so is $Q^{<\omega}$.*

The following is an easy corollary of 4.3.

4.4 *Let Q_1, Q_2 and Q_3 be wqos. Let Q be a quasi-order with $V(Q)$ the set of all finite sequences $(p_1, \ldots, p_a)$ with $a \geq 2$ such that $p_1 \in V(Q_1)$, $p_2, \ldots, p_{a-1} \in V(Q_2)$ and $p_a \in V(Q_3)$. For $p = (p_1, \ldots, p_a)$ and $q = (q_1, \ldots, q_b) \in V(Q)$, let $p \leq_Q q$ if and only if:*

- *$a \leq b$, and*
- *there exist $\alpha_1, \ldots, \alpha_a$ with $1 = \alpha_1 < \ldots < \alpha_a = b$ such that $p_1 \leq_{Q_1} q_1$, $p_a \leq_{Q_3} q_b$, and $p_i \leq_{Q_2} q_{\alpha_i}$ for all i with $2 \leq i \leq a-1$.*

Then Q is a wqo.

Next, we prove a key lemma for 1.5.

4.5 *Let m, k be integers with $k > m \geq 1$. Suppose that $\mathcal{G}^{k-1}_{m-1}(Q)$ and $\mathcal{G}^{k}_{m+1}(Q)$ are both wqo under minor containment for every wqo Q. Then $\mathcal{G}^k_m(Q)$ is a wqo under minor containment for every wqo Q.*

Proof. Let $D_1, D_2, \ldots$ be an infinite sequence of members of $\mathcal{G}^k_m(Q)$. We may assume that D_i is non-trivial for every $i \geq 1$ because every trivial D_i has m vertices. Decompose each D_i as

$$D_i = D_i^1 \oplus \ldots \oplus D_i^{t_i}$$

as in 4.2. By 3.1, 3.2, and 4.1, $\mathcal{L}^k_m(Q)$ and $\mathcal{NC}^k_m(Q)$ are both well-quasi-ordered under minor containment. Therefore we may assume that $t_i \geq 3$ for every $i \geq 1$. We apply 4.4 for $Q_1 = Q_2 = \mathcal{L}^k_m(Q)$, $Q_3 = \mathcal{L}^k_m(Q) \cup \mathcal{NC}^k_m(Q)$. Then there exist $i < j$ with $t_i \leq t_j$ and $1 = \alpha_1 < \ldots < \alpha_{t_i} = t_j$ such that:

- there is a model ϕ_p of D_i^p in $D_j^{\alpha_p}$, for every $1 \leq p \leq t_i$.

For each $w \in V(G_i)$, let $w_l \leq w_r$ be the indices such that $w_l = \min\{l : w \in V(G_i^l)\}$, and $w_r = \max\{r : w \in V(G_i^r)\}$. If $w_l < w_r$, then w must be on one of the m vertex-disjoint induced directed paths of D_i; let R_w be the path.

Now, define ϕ from D_i to D_j by

$$\phi(w) = \bigcup_{p \in \{w_l, \ldots, w_r\}} \phi_p(w) \bigcup_{\alpha_{w_l} < \alpha < \alpha_{w_r}} G_j^{\alpha}[V(R_w)]$$

Note that $G_j^{\alpha}[V(R_w)]$ is strongly-connected since D_j^{α} is a link. Thus ϕ is a model of D_i in D_j. This proves 4.5. ∎

4.6 *Let m, k be integers with $k \geq m \geq 0$. Then $\mathcal{G}^k_m(Q)$ is a wqo under minor containment for every wqo Q.*

Proof. We proceed by induction on k. For fixed k, we use induction on $k - m$. For the base case $k = m$, every member $D \in \mathcal{G}^k_m(Q)$ is trivial and hence the statement holds. For the inductive step, since $\mathcal{G}^{k-1}_{m-1}(Q)$ and $\mathcal{G}^k_{m+1}(Q)$ are wqo by the inductive hypotheses, the statement follows from 4.5. This proves 4.6. ∎

Proof of 1.5. Let $G_1, G_2, \ldots$ be an infinite sequence of semi-complete digraphs all with path-width at most k. Let Q be a wqo with $V(Q) = \{0\}$. For each $i \geq 1$, by 2.1 there is a linked path-decomposition P_i of G_i with $m(P_i) = 0$ and $pw(P_i) \leq k$. Let $R_i = ()$ be the empty sequence, let l_i be the constant mapping from $V(G_i)$ to $\{0\}$, and let $D_i = (G_i, P_i, R_i, l_i)$. Since $D_i \in \mathcal{G}^{k+1}_0(Q)$ for each $i \geq 1$, there exist $j > i \geq 1$ such that D_i is a minor of D_j by 4.6. Therefore G_i is a minor of G_j. This proves 1.5. ∎

5 Counterexamples

We have shown then that the class of semi-complete digraphs forms a wqo under minor containment. In this section, we show that two plausible enlargements of this class are not wqo's, the first allowing digraphs with slightly more edges, and the second allowing digraphs with slightly fewer.

Let us say the *parallelness* of a digraph G is the maximum size of a set of mutually parallel edges; and its *stability number* is the maximum size of a stable set in the underlying undirected graph of G. A loopless digraph is semi-complete if and only if its parallelness and its stability number are both at most one, and so the class of all loopless digraphs with parallelness one and stability number one is a wqo under minor containment. We will show

- the class of all loopless digraphs with parallelness at most two and stability number one is not a wqo under minor containment; and
- the class of all loopless digraphs with parallelness one and stability number at most two is not a wqo under minor containment.

For the first claim, for $i \geq 3$ let T_i be a transitive tournament with i vertices $v_1, \ldots, v_i$ such that $v_a v_b \in E(T_i)$ if and only if $a < b$. Let G_i be a digraph obtained from T_i by doubling the following i edges:

$$v_1v_2,\ v_2v_3,\ \ldots,\ v_{i-1}, v_i,\ v_1v_i.$$

("Doubling" means adding a new edge with the same head and tail as the given edge.) We claim that G_i is not a minor of G_j for $j > i \geq 3$. First, we cannot contract anything from G_j because G_j has no directed cycles. Therefore G_j must have G_i as a subdigraph in order to contain it as a minor. However, note that the underlying undirected graph of G_i has a cycle of length i with all edges doubled, while G_j does not. Therefore G_i is not a subdigraph of G_j, and hence not a minor of G_j. This shows the first claim.

For the second claim, for $i \geq 2$ let $A_i = \{a_1, a_2, a_3\}$, $B_i = \{b_1, b_2, b_3\}$, $C_i = \{c_1, \ldots, c_i\}$, and $D_i = \{d_1, \ldots, d_i\}$. Let G_i be a simple digraph with stability number two defined as follows. (See figure 1.)

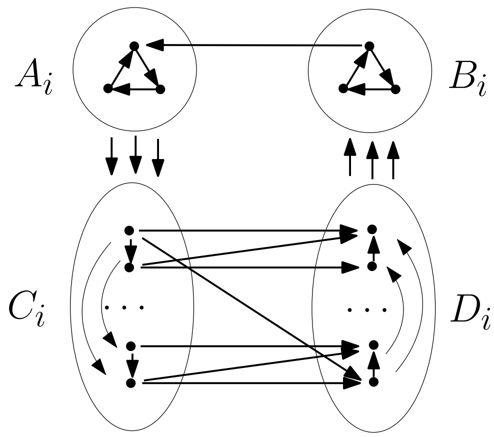

Figure 1: G_i

- $V(G_i)$ is the disjoint union of A_i, B_i, C_i and D_i,
- $G_i[A_i]$, $G_i[B_i]$ are directed triangles,
- $G_i[C_i]$, $G_i[D_i]$ are transitive tournaments,
- A_i is complete to C_i,
- D_i is complete to B_i,
- b_1a_1 is the only edge between A_i and B_i.
- Each edge between C_i and D_i goes from C_i to D_i, and the bipartite graph underlying $(C_i \cup D_i, \delta^+(C_i, D_i))$ is a Hamiltonian cycle.
- There are no other edges between $A \cup C_i$ and $B \cup D_i$.

We claim that there do not exist $j > i \geq 2$ such that G_i is a minor of G_j. For suppose that ϕ is a model of G_i in G_j. First, observe the following fact.

- If H is a strongly-connected subdigraph of G_j with $|V(H)| \geq 2$, then either $A_j \subseteq V(H)$ or $B_j \subseteq V(H)$ or $b_1a_1 \in E(H)$.

Therefore once we contract a non-trivial strongly-connected subdigraph H of G_j, there do not exist two disjoint directed cycles. That means we cannot contract anything in G_j if we hope to obtain G_i as a minor. Therefore ϕ must be a subdigraph mapping and $\phi(A_i) = G_j[A_j]$ and $\phi(B_i) = G_j[B_j]$ in order to preserve the existence of two disjoint directed cycles with an edge between them. Then $\phi(C_i)$ and $\phi(D_i)$ are subdigraphs of $G_j[C_j]$ and $G_j[D_j]$, respectively. However, the underlying bipartite graph of $(C_j \cup D_j, \delta^+(C_j, D_j))$ is a cycle of length $2j$, and hence does not contain a cycle of length $2i$ as a subgraph, a contradiction. This proves our claim.

References

[1] Maria Chudnovsky, Alexandra Fradkin, and Paul Seymour, "Tournament immersion and cutwidth", *J. Combinatorial Theory, Ser. B*, 102 (2012), 93–101.

[2] Maria Chudnovsky and Paul Seymour, "A well-quasi-order for tournaments", *J. Combinatorial Theory, Ser. B*, 101 (2011), 47–53.

[3] Alexandra Fradkin and Paul Seymour, "Tournament pathwidth and topological containment", *J. Combinatorial Theory, Ser. B*, 103 (2013), 374–384.

[4] G. Higman, "Ordering by divisibility in abstract algebras", *Proc. London Math. Soc., 3rd series*, 2 (1952), 326–336.

[5] T. Johnson, N. Robertson, P. Seymour, and R. Thomas, "Directed treewidth", *J. Combinatorial Theory, Ser. B*, 82 (2001), 128–154.

[6] S. Kreutzer, and S. Tazari, "Directed nowhere dense classes of graphs", *Proc. 23rd ACM-SIAM Symposium on Discrete Algorithms (SODA'12)*, to appear (2012).

[7] N. Robertson and P. D. Seymour, "Graph minors. I. Excluding a forest", *J. Combinatorial Theory, Ser. B*, 35 (1983), 39–61.

[8] N. Robertson and P. D. Seymour, "Graph minors. XX. Wagner's conjecture", *J. Combinatorial Theory, Ser. B*, 92 (2004), 325–357.