Domination in tournaments

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Abstract

We investigate the following conjecture of Hehui Wu: for every tournament S, the class of S-free tournaments has bounded domination number. We show that the conjecture is false in general, but true when S is 2-colourable (that is, its vertex set can be partitioned into two transitive sets); the latter follows by a direct application of VC-dimension. Our goal is to go beyond this; we give a non-2-colourable tournament S that satisfies the conjecture. The key ingredient here (perhaps more interesting than the result itself) is that we overcome the unboundedness of the VC-dimension by showing that the set of shattered sets is sparse.

1 Introduction

If there is an edge of a digraph G with head v and tail u, we say that "v is adjacent from u" and "u is adjacent to v". If T is a tournament and $X, Y \subseteq V(T)$, we say that X *dominates* Y if every vertex in $Y \setminus X$ is adjacent from some vertex in X. The *domination number* of T is the smallest cardinality of a set that dominates $V(T)$. A class $\mathcal{C}$ of tournaments has *bounded domination* if there exists c such that every tournament in $\mathcal{C}$ has domination number at most c. If S, T are tournaments, we say that T is *S-free* if no subtournament of T is isomorphic to S. A tournament S is a *rebel* if the class of all S-free tournaments has bounded domination. In this paper we investigate the following conjecture, recently proposed by Hehui Wu (private communication):

1.1 Conjecture: *Every tournament is a rebel.*

We will disprove this; and that leads to the question, which tournaments are rebels? We will give a partial answer:

- all 2-colourable tournaments are rebels, and so is at least one more;
- all rebels are poset tournaments.

This needs some definitions. A *k-colouring* of a tournament T is a partition of $V(T)$ into k transitive sets, and if T admits such a partition it is *k-colourable.* The *chromatic number* of a tournament T is the minimum k such that T is k-colourable. We will prove below that all 2-colourable tournaments are rebels, using VC-dimension; and since not all tournaments are rebels, one might anticipate the converse, that all rebels are 2-colourable. The main goal of this paper is to give a counterexample to this. The tournament on seven vertices, obtained by substituting a cyclic triangle for two of the three vertices of a cyclic triangle, is not 2-colourable, but we will show it is a rebel. This is proved in sections 5 and 6. Again the proof uses VC-dimension, using an extension of a theorem of Haussler and Welzl [5], proved in section 4, that permits large shattered sets provided they are sparse.

Let us say a tournament is a *poset* tournament if its vertex set can be ordered $\{v_1, \dots, v_n\}$ such that for all $i < j < k$, if v_j is adjacent from v_i and adjacent to v_k then v_i is adjacent to v_k; that is, the "forward" edges under this linear order form the comparability graph of a partial order. In section 2 we prove that every rebel is a poset tournament, and consequently disprove 1.1.

Domination in tournaments is an old and much-studied question [6]. For instance, let us say a tournament T is *k-majority* if there are $2k - 1$ linear orders on $V(T)$ such that for all distinct $u, v \in V(T)$, if u is adjacent to v then u is before v in at least k of the $2k-1$ orders. Alon, Brightwell, Kierstead, Kostochka and Winkler showed in [1] that k-majority tournaments have bounded domination number, and indeed this paper is where the idea of using VC-dimension for tournament domination was introduced. Their result follows from the fact that 2-colourable tournaments are rebels, since it is easy to see (by estimating the number of n-vertex tournaments in each class) that some 2-colourable tournament S is not k-majority; and since S is a rebel and the class of k-majority tournaments is S-free, the latter has bounded domination. But this proof of the result of [1] is basically the same as one of the proofs in [1], and indeed, 3.6 and its proof are only slight extensions of ideas in that paper.

A tournament is *k-transitive* if its edge set can be partitioned into k sets each of which is transitively oriented; and Gyárfás proposed the conjecture that k-transitive tournaments have bounded domination number (see [7] for a discussion). Not every tournament is k-transitive (for instance, by theorem 4 of [7]), so let S be a tournament that is not k-transitive; then no k-transitive tournament contains S, and so if we could show that some such S is a rebel, this would imply Gyárfás' conjecture, which remains open. We mention also the well-known conjecture of Sands, Sauer and Woodrow [8], that for all k there exists $f(k)$ such that if the edges of a tournament are coloured with k colours, then there is a set of at most $f(k)$ vertices such that every other vertex can be reached from some vertex in the set by a monochromatic path.

Conjecture 2.6 of [2] states that for all k there exists $f(k)$ such that for every tournament T, if for every vertex v the set of out-neighbours of v has chromatic number at most k, then T has chromatic number at most $f(k)$. If some rebel is not k-colourable, then this conjecture is true for that value of k. To see this, let S be a rebel that is not k-colourable, and let T be a tournament satisfying the condition above on out-neighbour sets. If T contains a copy of S with vertex set X say, then X is not a subset of the out-neighbour set of any vertex of T, from the hypothesis, and so X is dominating; and if not, then since S is a rebel, it follows that the domination number of T is bounded. Thus in either case there exists a dominating set $X \subseteq V(T)$ with $|X|$ at most some function of k. Since the set of out-neighbours of each vertex in X is k-colourable, it follows that T is $k|X|$-colourable. In particular, 5.1 implies that some rebel is not 2-colourable, and so conjecture 2.6 of [2] is true when $k = 2$. (This was previously open.)

2 Poset tournaments

In this section we show that not all tournaments are rebels, disproving 1.1. First, let us observe that not every tournament is a poset tournament. Let T be a poset tournament with n vertices. From Dilworth's theorem applied to the poset, there is a chain or antichain in the poset with cardinality at least $n^{1/2}$; and hence there is a transitively oriented subset of $V(T)$ with cardinality at least $n^{1/2}$. But in a random n-vertex tournament, the largest transitively oriented subset has cardinality $O(\log(n))$ with high probability; so if n is large enough then not every n-vertex tournament is a poset tournament. It follows that not every tournament is a rebel, because we will show that:

2.1 *Every rebel is a poset tournament.*

Proof. Let S be a rebel, and choose c such that every tournament not containing S has domination number at most c. Let $k = c + 1$.

Let V be a set defined as follows. Let V_0 be a set of cardinality k^k, and define $V_1, \dots, V_k$ inductively by: having defined V_{i-1} where $i \leq k$, let V_i be the set of all subsets $X \subseteq V_{i-1}$ with $|X| = k$. In particular, it follows that $|V_i| \geq k^k$ for all i. Let $V = V_0 \cup \dots \cup V_k$. Choose a linear order of V, say $\{v_1, \dots, v_n\}$, where for $0 \leq i < j \leq k$, if $v_s \in V_i$ and $v_t \in V_j$ then $s > t$.

Let H_1 be the digraph with vertex set V, where if $X \in V_i$ where $i > 0$, then X is adjacent in H_1 to every $Y \in V_{i-1}$ with $Y \in X$. Let H_2 be the transitive closure of H_1.

Let T be the tournament with vertex set V, in which for $1 \leq s < t \leq n$, v_s is adjacent to v_t in T if and only if v_s is adjacent to v_t in H_2. We see that T is a poset tournament.

Suppose that the domination number of T is at most c, and choose $X \subseteq V$ dominating T with $|X| \leq c$. Since $|X| \leq k - 1$, there exists i with $1 \leq i \leq k$ such that $X \cap V_i = \emptyset$. Let $Y = X \cap (V_0 \cup \dots \cup V_{i-1})$ and $Z = X \cap (V_{i+1} \cup \dots \cup V_k)$. For each $y \in Y$, choose $p(y) \in V_{i-1}$ such that there is a directed path of H_1 from $p(y)$ to y (possibly $p(y) = y$). Let $P = \{p(y) : y \in Y\}$.

Thus $|P| \le |Y| \le c$; choose a subset Q of V_{i-1} of cardinality $c = k-1$ including P. Now there are $|V_{i-1}|-k+1$ k-subsets of V_{i-1} including Q, and each is a member of V_i; and so there are $|V_{i-1}|-k+1$ members of V_i that are adjacent to every member of Q in H_1. These vertices are therefore adjacent in H_2 to every vertex in Y, and so not dominated in T by Y. Consequently they are all dominated by Z. But each vertex in Z dominates at most k^{k-1} vertices in V_i, since $k-i \le k-1$; and since $|Z| \le k-1$, it follows that $|V_{i-1}|-k+1 \le (k-1)k^{k-1}$, and so $|V_{i-1}| \le (k-1)(k^{k-1}+1) < k^k$, a contradiction.

This proves that the domination number of T is more than c, and from the definition of c, it follows that T contains S. Consequently S is a poset tournament. This proves 2.1. ∎

Is every poset tournament a rebel? As a first step, is the seven-vertex Paley tournament a rebel? We have not been able to answer this.

3 2-colourable tournaments and VC-dimension

In this section we prove that all 2-colourable tournaments are rebels. Let H be a hypergraph. (A *hypergraph* consists of a set $V(H)$ of *vertices* and a set $E(H)$ of subsets of $V(H)$ called *edges*.) We say that $X \subseteq V(H)$ is *shattered* by H if for every $Y \subseteq X$, there exists $A \in E(H)$ with $A \cap X = Y$. The largest cardinality of a shattered set is called the *Vapnik-Chervonenkis dimension* or *VC-dimension* of H, after [12].

If T is a tournament and $X \subseteq V(T)$, $T[X]$ denotes the subtournament induced on X. If $\{A, B\}$ is a 2-colouring of a tournament T, and A', B' are disjoint subsets of the vertex set of a tournament T', an isomorphism from T to a subtournament of T' mapping A to a subset of A' and B to a subset of B' is called an *embedding of* (T, A, B) *into* (T', A', B'); and if in addition the isomorphism maps T to T' (and hence A to A' and B to B') we call it an *isomorphism from* (T, A, B) *to* (T', A', B'). If S, T are tournaments and $V(S) \subseteq V(T)$, then $T \leftarrow S$ denotes the tournament obtained from T by replacing the edges of T with both ends in $V(S)$ by the edges of S.

3.1 *For every 2-colourable tournament S, there exists $d \ge 0$ with the following property. Let $\{C, D\}$ be a 2-colouring of S. Let T be a tournament, and let $A, B \subseteq V(T)$ be disjoint. For each $v \in B$, let $N(v)$ denote the set of all $u \in A$ adjacent to v. Let H be the hypergraph with vertex set A and edge set $\{N(v) : v \in B\}$. Let $X \subseteq A$ be shattered by H with $|X| \ge d$. Then there is an embedding of (S, C, D) into (T, X, B).*

Proof. By adding vertices to D, we may assume that no two vertices in C are adjacent to exactly the same subset of vertices in D. Let $|C| = m$ and $|D| = n$. We claim first that:

(1) *There is a tournament R with a 2-colouring $\{C, I\}$, with $|I| = m!n$, such that for every transitive tournament M with vertex set C, there is an embedding of (S, C, D) into $(R \leftarrow M, C, I)$. Moreover, no two vertices in C are adjacent to exactly the same vertices in I.*

There are $m!$ transitive tournaments $M_1, \dots, M_{m!}$ with vertex set C; one of them is $S[C]$. For each such tournament M_i, extend it to a tournament S'_i by adding a set D_i of n new vertices, in such a way that there is an isomorphism from (S, C, D) to (S'_i, C, D_i). For each i, let $S_i = S'_i \leftarrow S[C]$.

Thus the union of all the tournaments $S_1, \dots, S_{m!}$ is a digraph, although not a tournament; extend it to a tournament R by making the set I of all the $m!n$ new vertices transitive. This proves (1).

Let R, I be as in (1). Let $c = 2^{|I|}$ and $d = 2^{2^c}$; we claim that d satisfies the theorem. For let T, A, B, H, X be as in the theorem. By a theorem of [11], there is a subset $X_1 \subseteq X$ with $|X_1| = 2^c$ such that $T[X_1]$ is transitive. Since $|X_1| = 2^c$, we can number the members of X_1 as $X_1 = \{x_P : P \subseteq \{1, \dots, c\}\}$. Since H shatters X_1, for each $p \in \{1, \dots, c\}$ there exists $y_p \in B$ such that for each $P \subseteq \{1, \dots, c\}$, x_P is adjacent to y_p if and only if $p \in P$. Let $Y_1 = \{y_p : 1 \leq p \leq c\}$; then for every $Z \subseteq Y_1$, there exists $x \in X_1$ such that x is adjacent to every vertex in Z and adjacent from every vertex in $Y_1 \setminus Z$, namely the vertex $x = x_P$, where $P = \{p \in \{1, \dots, c\} : y_p \in Z\}$. By [11] again, since $c = 2^{|I|}$, there exists $Y_2 \subseteq Y_1$ with $|Y_2| = |I|$ such that $T[Y_2]$ is transitive. Since for every $Z \subseteq Y_2$, there exists $x \in X_1$ such that x is adjacent to every vertex in Z and adjacent from every vertex in $Y_2 \setminus Z$, and since no two vertices of R in C have the same out-neighbours in I, it follows that there is a subset $X_2 \subseteq X_1$ with $|X_2| = |C|$, and a transitive tournament M with $V(M) = C$, and an isomorphism from $(R \leftarrow M, C, I)$ to $(T[X_2 \cup Y_2], X_2, Y_2)$. From (1) there is an embedding of (S, C, D) into $(R \leftarrow M, C, I)$; and so there is an embedding of (S, C, D) into $(T[X_2 \cup Y_2], X_2, Y_2)$. This proves 3.1. ∎

Let T be a tournament, and for each vertex v let $N^-_T(v)$ denote the set of all vertices of T that are either adjacent to v or equal to v. Thus $\{N^-_T(v) : v \in V(T)\}$ is the edge set of a hypergraph with vertex set $V(T)$, called the *hypergraph of in-neighbourhoods* of T.

3.2 *For every 2-colourable tournament S, there is a number d such that for every S-free tournament T, its hypergraph of in-neighbourhoods has VC-dimension at most d.*

Proof. Let d' satisfy 3.1 with d replaced by d', and let d be an integer with $d > d' + \log_2(d')$. We claim that d satisfies the theorem. Let T be a tournament, and suppose that $X \subseteq V(T)$ is shattered by the hypergraph of in-neighbourhoods of T, where $|X| > d$. For each $Y \subseteq X$, there exists $v \in V(T)$ such that $N^-_T(v) \cap X = Y$; let us write $v_Y = v$. Choose $X' \subseteq X$ with $|X'| = d'$. Then for each $Y' \subseteq X'$, there are at least $d' + 1$ subsets Y of X with $Y \cap X' = Y'$, and so $v_Y \notin X'$ for at least one such set Y. But then 3.1 implies that T is not S-free. This proves 3.2. ∎

A similar, simpler proof (which we omit) shows:

3.3 *For every 2-colourable tournament S and every 2-colouring $\{C, D\}$ of S, there is a number d with the following property. Let T be a tournament, let $A \subseteq V(T)$, let H be the hypergraph with vertex set $V(T)$ and edge set $\{N^-_T(v) : v \in A\}$, and suppose that $X \subseteq V(T)$ is shattered by H, with $|X| \geq d$. Then there is an embedding of (S, C, D) into $(T, A \setminus X, X)$.*

If H is a hypergraph, τ_H denotes the minimum cardinality of a set which has nonempty intersection with every edge of H, and τ^*_H is a fractional relaxation of this: the minimum of $\sum_{v \in V(H)} f(v)$ over all functions f from $V(H)$ to the nonnegative real numbers such that $\sum_{v \in A} f(v) \geq 1$ for every edge A of H.

We need the following theorem of [3], a slight refinement of earlier work of Haussler and Welzl [5]. (Logarithms are to base two.)

3.4 *Let $d \geq 1$, and let H be a hypergraph with VC-dimension at most d. Then*

$$\tau_H \leq 2d\tau_H^* \log(11\tau_H^*).$$

3.5 *Let T be a tournament, let $d \geq 1$, and let the VC-dimension of its hypergraph of in-neighbourhoods be at most d. Then the domination number of T is at most $18d$.*

Proof. Let H be the hypergraph of in-neighbourhoods of T. Then $\tau_H^* \leq 2$ (by corollary 6 of [1]), so $\tau_H \leq 4d \log(22) \leq 18d$ by 3.4. This proves 3.5. ∎

Finally we deduce:

3.6 *Every 2-colourable tournament is a rebel.*

Proof. Let S be a 2-colourable tournament, let $\{C, D\}$ be a 2-colouring, and let d be as 3.2. If T is an S-free tournament, its hypergraph of in-neighbourhoods has VC-dimension at most d by 3.2, and so its domination number is at most $18d$ by 3.5. This proves 3.6. ∎

4 Sparse shattered sets

When we are excluding a tournament S that is not 2-colourable, we find that the S-free tournaments do not necessarily have hypergraphs of in-neighbourhoods with bounded VC-dimension; but for our application we can prove that large shattered sets are sparse, and this turns out to be enough to carry over the proof of 3.6. We will need the Sauer-Shelah lemma [9, 10], the following.

4.1 *Let H be a hypergraph, and let $X \subseteq V(H)$ with $|X| = n$, such that no $d+1$-subset of X is shattered by H. Then there are at most $\sum_{0 \leq i \leq d} \binom{n}{i}$ distinct sets $A \cap X$ where $A \in E(H)$.*

In this section we prove our main lemma, a version of 3.4 which permits large shattered sets provided they are sparse. Its proof is a modification of the proof of 3.4 in [3] and of the original proof of Haussler and Welzl [5].

4.2 *Let H be a hypergraph with n vertices, and let $0 < \epsilon \leq 1$, such that $|f| \geq \epsilon n$ for every edge f. Let $d \geq 1$ be an integer, and let $0 \leq c \leq 1$, such that at most $c\binom{n}{d}$ d-subsets of $V(H)$ are shattered by H. If $t \geq d$ is an integer such that $2^{-\epsilon t}(2t)^d(\frac{1}{5}(e/d)^d + c2^{2t}(2t)!) < 1/4$, then $\tau_H \leq t$.*

Proof. Suppose that $\tau_H > t$. Let $\mathcal{T}$ be the set of all sequences $(x_1, \ldots, x_{2t})$, where $x_1, \ldots, x_{2t} \in V(T)$, not necessarily distinct. For each $f \in E(H)$, let $\mathcal{A}_f$ be the set of all $(x_1, \ldots, x_{2t}) \in \mathcal{T}$ such that $x_1, \ldots, x_t \notin f$, and $x_i \in f$ for at least $\epsilon t - 1$ values of $i \in \{t+1, \ldots, 2t\}$. Let $\mathcal{A} = \cup_{f \in E(H)} \mathcal{A}_f$.

(1) *Let $x_1, \ldots, x_t \in V(H)$, and let $f \in E(H)$ with $x_1, \ldots, x_t \notin f$. Then there are at least $n^t/2$ sequences $(x_{t+1}, \ldots, x_{2t})$ such that $(x_1, \ldots, x_{2t}) \in \mathcal{A}_f$.*

Since $|f| \geq \epsilon t$, if we choose $x_{t+1}, \ldots, x_{2t}$ independently at random from $V(H)$, the expected number of $i \in \{t+1, \ldots, 2t\}$ with $x_i \in f$ is at least ϵt; and since the median of a binomial distribution is within 1 of its mean, it follows that at least half of all the choices of $(x_{t+1}, \ldots, x_{2t})$ have at least

$\epsilon t - 1$ terms in f. This proves (1).

(2) $|\mathcal{A}| \geq n^{2t}/2$.

There are n^t choices of $(x_1, \ldots, x_t)$, and for each one, there exists $f \in E(H)$ with $x_1, \ldots, x_t \notin f$, since $t < \tau_H$ from our assumption. Consequently, for each choice of $(x_1, \ldots, x_t)$, (1) implies that there are at least $n^t/2$ sequences $(x_{t+1}, \ldots, x_{2t})$ such that $(x_1, \ldots, x_{2t}) \in \mathcal{A}_f \subseteq \mathcal{A}$; and this proves (2).

For each $(x_1, \ldots, x_{2t}) \in \mathcal{T}$, its *support* is the function μ with domain $V(H)$ where $\mu(v)$ is the number of values of $i \in \{1, \ldots, 2t\}$ with $x_i = v$; and a function μ that is the support of some member of $\mathcal{T}$ is called a *supporter*. For the moment, let us fix some supporter μ. Let $\mathcal{S}^\mu$ be the set of all $(x_1, \ldots, x_{2t}) \in \mathcal{T}$ with support μ, and for each $f \in E(H)$, let $\mathcal{S}^\mu_f$ be the set of members of $\mathcal{A}_f$ with support μ, that is, $\mathcal{S}^\mu_f = \mathcal{S}^\mu \cap \mathcal{A}_f$.

(3) $|\mathcal{S}^\mu_f| \leq 2^{1-\epsilon t}|\mathcal{S}^\mu|$ *for each* $f \in E(H)$.

Let $f \in E(H)$ and let $k = \sum_{v \in f} \mu(v)$. If $k > t$ or $k < \epsilon t - 1$ then $\mathcal{S}^\mu_f = \emptyset$ and the claim holds; so we may assume that $\epsilon t - 1 \leq k \leq t$. It follows that a sequence $(x_1, \ldots, x_{2t}) \in \mathcal{S}^\mu$ belongs to $\mathcal{S}^\mu_f$ if and only if $x_1, \ldots, x_t \notin f$. Let $P(x_1, \ldots, x_{2t}) = \{i \in \{1, \ldots, 2t\} : x_i \in f\}$. Thus $|P(x_1, \ldots, x_{2t})| = k$; for each k-subset Q of $\{1, \ldots, 2t\}$ there is the same number of sequences $(x_1, \ldots, x_{2t}) \in \mathcal{S}^\mu$ with $P(x_1, \ldots, x_{2t}) = Q$; and for each such Q, either all these sequences belong to $\mathcal{S}^\mu_f$ or none do, depending whether $Q \subseteq \{t+1, \ldots, 2t\}$ or not. Thus $|\mathcal{S}^\mu_f|/|\mathcal{S}^\mu|$ equals the proportion of k-subsets of $\{1, \ldots, 2t\}$ that are included in $\{t+1, \ldots, 2t\}$, that is,

$$\frac{|\mathcal{S}^\mu_f|}{|\mathcal{S}^\mu|} = \frac{\binom{t}{k}}{\binom{2t}{k}} = \prod_{0 \leq i \leq k-1} \frac{t-i}{2t-i} \leq 2^{-k} \leq 2^{1-\epsilon t}.$$

This proves (3).

Let $V(\mu) = \{v : \mu(v) > 0\}$. Thus $|V(\mu)| \leq 2t$.

(4) *If* $V(\mu)$ *includes no* d*-set that is shattered by* H *then* $|\cup_{f \in E(H)} \mathcal{S}^\mu_f| \leq \frac{1}{5}(2et/d)^d 2^{1-\epsilon t}|\mathcal{S}^\mu|$.

Let there be r distinct sets of the form $f \cap V(\mu)$ where $f \in E(H)$. By 4.1,

$$r \leq \sum_{0 \leq i < d} \binom{2t}{i} \leq \frac{(2t)^{d-1}}{(d-1)!} \sum_{0 \leq i < d} x^i \leq \frac{(2t)^{d-1}}{(d-1)!}(1-x)^{-1},$$

where $x = (d-1)/(2t)$; and since $t \geq d$, $(1-x)^{-1} \leq \frac{2t}{d+1}$. Consequently,

$$r \leq \frac{(2t)^{d-1}}{(d-1)!} \frac{2t}{d+1} = \frac{d}{d+1} \frac{(2t)^d}{d!}.$$

By a form of Stirling's approximation,

$$d! \geq \sqrt{2\pi}\, d^{d+\frac{1}{2}} e^{-d} \geq \frac{5}{2}(d/e)^d d^{1/2} \geq 5(d/e)^d \frac{d}{d+1}.$$

It follows that

$$r \leq \frac{d}{d+1}\frac{(2t)^d}{d!} \leq \frac{1}{5}(2et/d)^d.$$

For $f, f' \in E(H)$, if $f \cap V(\mu) = f' \cap V(\mu)$ then $\mathcal{S}^\mu_f = \mathcal{S}^\mu_{f'}$; and so there are at most r distinct sets $\mathcal{S}^\mu_f$ ($f \in E(H)$). By (3),

$$|\cup_{f\in E(H)} \mathcal{S}^\mu_f| \leq \frac{1}{5}(2et/d)^d 2^{1-\epsilon t}|\mathcal{S}^\mu|.$$

This proves (4).

(5) *There are at most $c\binom{2t}{d}n^{2t}$ supporters μ for which $V(\mu)$ includes a d-subset that is shattered by H.*

If $(x_1, \dots, x_{2t}) \in \mathcal{T}$ and $Y \subseteq V(H)$, we say that $(x_1, \dots, x_{2t})$ *covers* Y if $Y \subseteq \{x_1, \dots, x_{2t}\}$. There are n^{2t} members of $\mathcal{T}$, and each covers at most $\binom{2t}{d}$ d-subsets of $V(H)$. Moreover, each d-subset X of $V(H)$ is covered by the same number of members of $\mathcal{T}$. It follows that each d-subset of $V(H)$ is covered by at most $n^{2t}\binom{2t}{d}/\binom{n}{d}$ members of $\mathcal{T}$. Since there are at most $c\binom{n}{d}$ d-subsets of $V(H)$ that are shattered by H, it follows that there are at most $cn^{2t}\binom{2t}{d}$ members of $\mathcal{T}$ that cover a d-set that is shattered by H. Consequently there are at most that many supporters that do so. This proves (5).

(6) $|\mathcal{A}| \leq n^{2t}2^{1-\epsilon t}(2t)^d(\frac{1}{5}(e/d)^d + c2^{2t}(2t)!)$.

Every member of $\mathcal{A}$ belongs to $\cup_{f\in E(H)}\mathcal{S}^\mu_f$ for some supporter μ. There are two kinds of supporters μ, those such that $V(\mu)$ includes no shattered d-set and those that do. We call these the "first" and "second" kinds. By (4), the union of all the sets $\cup_{f\in E(H)}\mathcal{S}^\mu_f$ over all μ of the first kind has cardinality at most $\frac{1}{5}(2et/d)^d 2^{1-\epsilon t}$ times the sum of the cardinalities of the sets $\mathcal{S}^\mu$ over all such μ, and since these sets $\mathcal{S}^\mu$ are pairwise disjoint subsets of $\mathcal{T}$, the sum of their cardinalities is at most $|\mathcal{T}| = n^{2t}$. Thus, the union of the sets $\cup_{f\in E(H)}\mathcal{S}^\mu_f$ over all supporters μ of the first kind has cardinality at most $\frac{1}{5}(2et/d)^d 2^{1-\epsilon t}n^{2t}$. For each supporter μ of the second kind, the set $\cup_{f\in E(H)}\mathcal{S}^\mu_f$ has cardinality at most $2^{2t}2^{1-\epsilon t}|\mathcal{S}^\mu|$, since there are only 2^{2t} distinct sets $\mathcal{S}^\mu_f$, and each has cardinality at most $2^{1-\epsilon t}|\mathcal{S}^\mu|$ by (3). Since $|\mathcal{S}^\mu| \leq (2t)!$, and there are at most $c(2t)^d n^{2t}$ such supporters μ by (5), it follows that the union of the sets $\cup_{f\in E(H)}\mathcal{S}^\mu_f$ over all μ of the second kind has cardinality at most

$$c(2t)^d n^{2t}2^{2t}2^{1-\epsilon t}(2t)!.$$

Adding, the result follows.

From (2) and (6), we deduce that $2^{-\epsilon t}(2t)^d(\frac{1}{5}(e/d)^d + c2^{2t}(2t)!) \geq 1/4$, contradicting the choice of t. Thus $\tau_H \leq t$. This proves 4.2. ∎

We deduce (logarithms are to base two):

4.3 *For all ϵ with $0 < \epsilon \leq 1$, and all integers $d \geq 1$, there exists $c > 0$ with the following property. Let H be a hypergraph with n vertices, such that $|f| \geq \epsilon n$ for every edge f, and such that at most $c\binom{n}{d}$ d-subsets of $V(H)$ are shattered by H. Then $\tau_H \leq \lceil 2d\epsilon^{-1}\log(6\epsilon^{-1})\rceil$.*

Proof. Let $t = \lceil 2d\epsilon^{-1}\log(6\epsilon^{-1})\rceil$, and choose $p \geq 6$ such that $t = 2d\epsilon^{-1}\log(p\epsilon^{-1})$. Then

$$2^{-\epsilon t}(2t)^d(e/d)^d \leq 2^{-2d\log(p\epsilon^{-1})}(2t)^d(e/d)^d = (2etp^{-2}\epsilon^2/d)^d.$$

Now

$$2etp^{-2}\epsilon^2/d = 4ep^{-2}\epsilon\log(p\epsilon^{-1});$$

but

$$\frac{\log(p\epsilon^{-1})}{p\epsilon^{-1}} < \frac{\log p}{p}$$

since $p \geq 6 > e$, and so

$$2etp^{-2}\epsilon^2/d \leq \frac{4e\log p}{p^2} < 1$$

since $p \geq 6$. Consequently $2^{-\epsilon t}(2t)^d(e/d)^d < 1$. Choose c satisfying $c2^{2t}(2t)! = \frac{1}{20}(e/d)^d$; then

$$2^{-\epsilon t}(2t)^d(\frac{1}{5}(e/d)^d + c2^{2t}(2t)!) = 2^{-\epsilon t}(2t)^d(\frac{1}{4}(e/d)^d) < 1/4,$$

and the result follows from 4.2. This proves 4.3. ∎

5 Odd girth

By analogy with graphs, let us say the *odd girth* of a tournament T is the minimum k such that some k-vertex subtournament is not 2-colourable (and it is undefined if T is 2-colourable).

If $X, Y \subseteq V(T)$, we say that X is *complete to* Y and Y is *complete from* X if $X \cap Y = \emptyset$ and every vertex in Y is adjacent from every vertex in X. For $v \in V(T)$, we say v is complete to X if $\{v\}$ is complete to X, and so on. Let C_3 denote the three-vertex tournament which is a cyclic triangle; and let S^* be the tournament obtained from C_3 by substituting a copy of C_3 for two of its three vertices. (In other words, $V(S^*)$ is the disjoint union of three sets X, Y, Z, where $|Z| = 1$, the subtournaments induced on X and on Y are both cyclic triangles, and X is complete to Y, Y is complete to Z, and Z is complete to X.) It is easy to see that S^* is not 2-colourable. Our main theorem is

5.1 *S^* is a rebel.*

To prove this we must show that the class of all S^*-free tournaments has bounded domination. Certainly all tournaments with odd girth at least 8 are S^*-free; thus, the following assertion appears much weaker than 5.1, but as we show below, the two statements are in fact equivalent. More precisely, 5.2 contains infinitely many assertions, one for each value of $k \geq 8$; they all evidently are implied by 5.1, and we will show that any one of them implies 5.1.

5.2 *For $k \geq 8$, the class of tournaments with odd girth at least k has bounded domination.*

Proof of 5.1, assuming 5.2 for some k. Let $k \geq 0$, and suppose that c is such that every tournament with odd girth at least k has domination number at most c. We will show that every S^*-free tournament has domination number at most $c + k - 1$. Let T be an S^*-free tournament. Let us say a *brick* of T is a subset $X \subseteq V(T)$ with $|X| < k$ such that $T[X]$ is not 2-colourable. If there is no brick in T then its odd girth is at least k, so its domination number is at most c as required; and so we may assume there is a brick in T. Consequently we may choose a sequence $X_1, \ldots, X_n$ of bricks of T, pairwise disjoint, such that X_i is complete to X_{i+1} for $1 \leq i \leq n$, and with $n \geq 1$ maximum.

(1) *There is no vertex in $X_2 \cup \cdots \cup X_n$ that is complete to X_1.*

Let $v \in X_m$ say, where $2 \leq m \leq n$, and suppose that v is complete to X_1. Then v is complete to the vertex set of a cyclic triangle in X_1 (because $T[X_1]$ is not transitive); choose $i < m$ maximum such that v is complete to the vertex set of a cyclic triangle in X_i. Let Y_1 be the vertex set of such a triangle. Now $i \leq m - 2$, since X_m is complete from X_{m-1}. From the maximality of i, the set of out-neighbours of v in X_{i+1} does not include a cyclic triangle and hence is transitive; and since $G[X_{i+1}]$ is not 2-colourable, the set of in-neighbours of v in X_{i+1} is not transitive, and so includes a cyclic triangle Y_2 say. But then $T[Y_1 \cup Y_2 \cup \{v\}]$ is isomorphic to S^*, a contradiction. This proves (1).

Let C be the set of vertices of T that are complete to X_1. By (1), $C \cap (X_1 \cup \cdots \cup X_n) = \emptyset$. From the maximality of n, there is no brick included in C; and so $T[C]$ has odd girth at least k, and hence has domination number at most c. But X_1 dominates $V(T) \setminus C$, and so the domination number of T is at most $c + |X_1| < c + k$. This proves 5.1. ∎

Thus, henceforth we may confine ourselves to tournaments with odd girth at least any constant that we choose. Incidentally, it seems that there is a fundamental difference between tournaments and graphs: graphs of large girth can be transformed into tournaments with large odd girth (via the use of an enumeration with which we interpret edges as backward directed edges), but graphs with large odd girth (such as Borsuk graphs) do not seem suitable for transformation into tournaments with large odd girth. Let us turn this into a problem.

A tournament is a *forest* if one can enumerate its vertices so that its backward edges form a forest. The *girth* of a tournament T is the smallest k for which there exists a subtournament S of T with k vertices which is not a forest. The following problem is not very well-posed, but nonetheless interesting:

5.3 Problem: *Find constructions of tournaments with large odd girth and large chromatic number which are not based on tournaments with large girth and large chromatic number.*

6 Domination in tournaments with large odd girth

We will prove the following, which therefore implies 5.1 and 5.2.

6.1 *The class of tournaments with odd girth at least* 74 *has bounded domination.*

It is helpful first to prove the following somewhat weaker statement.

6.2 *The class of tournaments T with odd girth at least* 74 *and such that $|N^-_T(v)| \geq |V(T)|/6$ for every vertex v has bounded domination.*

Proof. The seven-vertex Paley tournament has vertex set $\{v_0, \dots, v_6\}$, and for $0 \le i, j \le 6$, we say v_i is adjacent to v_j if $j = i+1, i+2$ or $i+4$ modulo 7. Let P^* denote the tournament obtained by reversing any one of its edges, say v_4v_1 (its automorphism group is transitive on edges, so it makes no difference which edge we reverse). Then $\{v_1, v_2, v_3, v_4\}$ is the only 4-vertex transitive set in P^* (we leave checking this to the reader), and in particular there is a unique 2-colouring, say $\{C, D\}$, where $|C| = 4$. Choose d to satisfy 3.3, taking P^* for S. Choose c to satisfy 4.3, taking $\epsilon = 1/6$, and let $t = \lceil 12d \log(36) \rceil$. Now choose δ such that $\frac{1}{6}\delta^2 d^3 2^{2t}(2t)! = 1$. We will prove that every tournament with odd girth at least 74 and such that $|N_T^-(v)| \ge |V(T)|/6$ for every vertex v has domination number at most $t + 2\delta^{-1}$.

Let T be such a tournament. Let us say a *domino* in T is a 7-vertex subtournament of T isomorphic to P^*. Let G be the graph with vertex set $V(T)$, in which two vertices are adjacent if some domino contains them both. For each edge uv of G, there may be more than one domino that contains both u, v; but since the union of any two such dominoes induces a 2-colourable tournament (since T has odd girth at least 74) and both dominoes are uniquely 2-colourable, it follows that either u, v have the same colour in the 2-colouring of every domino containing them both, or they have different colours in the 2-colouring of every domino containing them both. Edges uv of the latter kind we call *odd* edges.

For each $v \in V(T)$ and $i \ge 0$, let $N_G^i(v)$ denote the set of all vertices in $V(T)$ with distance (in G) from v at most i. It turns out that the set of vertices v such that $|N_G^2(v)|$ is large (linear in $|V(T)|$) can be dominated by a bounded subset for one reason, and the set of v with $|N_G^2(v)|$ not so large can be dominated by a bounded subset for another reason. Proving these statements will take several steps.

(1) *For each $v \in V(T)$, $T[N_G^4(v)]$ is 2-colourable.*

For every $u \in N_G^4(v)$, there is a path of G of length at most four between u, v; let A be the set of all such u such that some such path contains an even number of odd edges, and B the set such that some such path has an odd number of odd edges. We claim that $\{A, B\}$ is a 2-colouring of $T[N_G^4(v)]$. To show this we must show that $A \cap B = \emptyset$, and $T[A], T[B]$ are both transitive. For the first, suppose that $u \in A \cap B$, and take two paths of G between u, v, one with an odd number of odd edges and one with an even number. The union of these two paths has at most eight edges; for each of these edges, choose a domino containing both its ends. The union of these dominoes contains at most 48 vertices, and thus induces a 2-colourable subtournament, contradicting that each of the dominoes is uniquely 2-colourable. Thus $A \cap B = \emptyset$.

Suppose that $T[A]$ is not transitive; then there is a cyclic triangle in $T[A]$ with vertices u_1, u_2, u_3 say. For each of u_1, u_2, u_3 take a path of G of length at most four between it and v (necessarily containing an even number of odd edges); then the union of these paths has at most twelve edges. For each of these edges, choose a domino containing both ends of the edge; then the union of these dominoes has at most 73 vertices and so is 2-colourable, again a contradiction. Similarly $T[B]$ is transitive. This proves (1).

Let $n = |V(T)|$, and let S be the set of vertices v of T such that $|N_G^2(v)| \ge \delta n$.

(2) *There is a set dominating S of cardinality at most $2\delta^{-1}$.*

Choose S_0 in S maximal such that every two members of S_0 have distance at least five in G. It follows that the sets $N^2_G(v)$ $(v \in S_0)$ are pairwise disjoint, and since they all have cardinality at least δn, there are at most $1/\delta$ of them, that is, $|S_0| \leq \delta^{-1}$. Now from the maximality of S_0, every vertex in S has distance at most four in G from some member of S_0, and hence there are at most δ^{-1} 2-colourable subtournaments of T with union including S, by (1). But every 2-colourable tournament has domination number at most two, and so there is a set of at most $2\delta^{-1}$ vertices dominating S. This proves (2).

In view of (2), it remains to find a t-set that dominates $V(T) \setminus S$. We may therefore assume that $n \geq d$. Let H be the hypergraph with vertex set $V(T)$ and edge set all edges $N^-(v)$ $(v \in V(T) \setminus S)$. We need to show that $\tau_H \leq t$. To do so we will apply 4.3, and for the latter we need to show that the d-subsets of $V(T)$ shattered by H are sparse. We prove that as follows. A domino has a unique 2-colouring $\{A, B\}$, and one of A, B has cardinality three, say A; we call A the *outside* of the domino, and B its *inside*. A domino is *normal* if its inside is disjoint from S. Let G' be the graph with vertex set $V(T)$ in which u, v are adjacent if there is a normal domino such that u, v both belong to its outside. We see that G' is a subgraph of the graph G defined earlier.

(3) *Every vertex $v \in V(T)$ has degree at most δn in G'.*

For we may assume that v has degree at least one in G', and therefore belongs to the outside of some normal domino; and so v is adjacent in G to a vertex $u \in V(T) \setminus S$. Since $u \notin S$, $|N^2_G(u)| < \delta n$; but every neighbour of v in G' belongs to $N^2_G(u)$, and so v has at most δn neighbours in G' (in fact fewer). This proves (3).

(4) *The number of d-subsets of $V(T)$ shattered by H is at most $\frac{1}{6}\delta^2 d^3 \binom{n}{d}$.*

Let K be the set of all outsides of normal dominoes. (We remark that K is a set, not a multiset; two normal dominoes with the same outside contribute only one member to K.) Every member of K is a triangle of G', and it is an easy exercise to check that an n-vertex graph with maximum degree at most k has at most $nk(k-1)/6$ triangles. Consequently it follows from (3) that $|K| \leq \frac{1}{6}\delta^2 n^3$. Each set in K is a subset of $\binom{n-3}{d-3}$ d-subsets of $V(T)$, and so, since $n \geq d$, there are at most

$$\frac{1}{6}\delta^2 n^3 \binom{n-3}{d-3} = \frac{1}{6} d\delta^2 \frac{(d-1)(d-2)}{(n-1)(n-2)} n^2 \binom{n}{d} \leq \frac{1}{6}\delta^2 d^3 \binom{n}{d}$$

d-subsets of $V(T)$ that include members of K. But from 3.3, every d-set shattered by H includes a member of K. This proves (4).

From 4.3, taking $\epsilon = 1/6$, we deduce that $\tau_H \leq t$. Consequently there is a set of cardinality at most t dominating $V(T) \setminus S$; and from (2), we deduce that T has domination number at most $t + 2\delta^{-1}$. This proves 6.2. ∎

Proof of 6.1. Let c be as in 6.2; we claim every tournament T with odd girth at least 74 has domination number at most c. We may assume that every vertex of T has an in-neighbour; and so, by the theorem of [4], for each vertex v there is a rational number $f(v)$ with $0 \leq f(v) \leq 1/3$, summing to 1, such that for each v, the sum of $f(u)$ over all in-neighbours u of v is at least $1/2 - f(v)/2$, and in particular is at least $1/3$.

Let $n = |V(T)|$, and choose an integer $M \geq n$ such that $Mf(v)$ is an integer for each vertex v. For each v with $f(v) > 0$, substitute a transitive tournament T_v say with $Mf(v)$ vertices for v (vertices v with $f(v) = 0$ are not deleted). Let the tournament just constructed be T'. Then for every vertex v of T with $f(v) = 0$, v has at least $M/2$ in-neighbours in T'; and for each $v \in V(T)$ with $f(v) > 0$, every vertex of T_v has at least $M/3$ in-neighbours in T'. Since $|V(T')| \leq M + n \leq 2M$, it follows that every vertex of T' has at least $|V(T')|/6$ in-neighbours in T'. Moreover, the domination number of T is at most that of T', and the odd girth of T' is at least 74. The result follows from 6.2. This proves 6.1. ∎

7 Growth rates

Finally, let us mention a curiosity, not connected to domination, but lending credence to the false conjecture that all rebels are 2-colourable. There is a dramatic difference between excluding a 2-colourable tournament and excluding one that is not 2-colourable, because of the following.

7.1 *Let S be a tournament, let V be a set of size n say, where n is even, and let $f(n)$ be the number of S-free tournaments with vertex set V.*

- *If S is not 2-colourable then $f(n) \geq 2^{n^2/4}$;*
- *If S is 2-colourable then for all $\epsilon > 0$, $f(n) \leq 2^{\epsilon n^2}$ if n is sufficiently large in terms of ϵ.*

Proof. (Sketch) If S is not 2-colourable, then no 2-colourable tournament contains it, and there are at least $2^{n^2/4}$ 2-colourable tournaments on V. Now we assume S is 2-colourable, and let $0 < \epsilon \leq 1$. We claim that there is a constant c such that every S-free tournament on n vertices (with n sufficiently large) can be obtained from a tournament with some d vertices, where $\epsilon^{-1} \ll d < c$, by first replacing each vertex by a transitive set of size $\lfloor n/d \rfloor$ or $\lceil n/d \rceil$ (and directing the edges within each transitive set according to some fixed ordering of V) and then reversing the direction of at most $\frac{1}{2}\epsilon n^2$ edges. To see this we use the regularity lemma. Let T be an S-free tournament with vertex set V. Choose $\delta > 0$ much smaller than ϵ, and take a δ-regular partition of V into say d sets all of size $\lfloor n/d \rfloor$ or $\lceil n/d \rceil$, where $\epsilon^{-1} \ll d$, and d is at most a constant depending on ϵ. Since the VC-dimension of the in-neighbourhood hypergraph of T is bounded (by 3.2), it follows that for each δ-regular pair (X, Y) of sets in this partition, either there are at most $\delta|X||Y|$ edges from X to Y, or at most $\delta|X||Y|$ from Y to X. Consequently T can be obtained from a d-vertex tournament as described earlier; and so the number of such T is at most $2^{\epsilon n^2}$. This proves 7.1. ∎

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