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GROWING WITHOUT CLONING*

MARIA CHUDNOVSKY† AND PAUL SEYMOUR‡

Abstract. A graph G is *claw-free* if no induced subgraph of it is isomorphic to the complete bipartite graph $K_{1,3}$, and it is *prime* if $|V(G)| \geq 4$ and there is no $X \subseteq V(G)$ with $1 < |X| < |V(G)|$ such that every vertex of $V(G) \setminus X$ with a neighbor in X is adjacent to every vertex of X. In particular, if G is prime, then both G and G^c are connected. This paper has two main results. The first one is that if G is a prime graph that is not a member of a particular family of exceptions, and H is a prime induced subgraph of G, then (up to isomorphism) G can be grown from H, adding one vertex at a time, in such a way that all the graphs constructed along the way are prime induced subgraphs of G. A *simplicial clique* in G is a nonempty clique K such that for every $k \in K$ the set of neighbors of k in $V(G) \setminus K$ is a clique. Our second result is that a prime claw-free graph G has at most $|V(G)| + 1$ simplicial cliques, and we give an algorithm to find them all with running time $O(|V(G)|^4)$. In particular, this answers a question of Prasad Tetali [*private communication*] who asked if there is an efficient algorithm to test if a claw-free graph has a simplicial clique. Finally, we apply our results to claw-free graphs that are not prime. Such a graph may have exponentially many simplicial cliques, so we cannot list them all in polynomial time, but we can in a sense describe them.

Key words. prime graphs, induced subgraphs, splitter theorem, claw-free graphs, simplicial clique

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1. Introduction. All graphs in this paper are finite and simple. Let G be a graph. For $X \subseteq V(G)$, we denote by $G|X$ the subgraph of G induced on X and by $G \setminus X$ the subgraph $G|(V(G) \setminus X)$. We say that $G|X$ is *proper* if $X \neq V(G)$. If $X = \{x\}$, we write $G \setminus x$ for $G \setminus X$. A *clique* in G is a set of vertices all pairwise adjacent. Let A and B be two disjoint subsets of $V(G)$. We say that A is *complete* to B if every vertex in A is adjacent to every vertex in B and that A is *anticomplete* to B if every vertex in A is nonadjacent to every vertex in B. We say that $a \in V(G) \setminus B$ is complete (anticomplete) to B if $\{a\}$ is complete (anticomplete) to B. For $v \in V(G)$, we denote by $N_G(v)$ (or $N(v)$ when there is no danger of confusion) the set of all neighbors of v in G. Two vertices u and v in G are *twins* if $N(u) \cup \{u\} = N(v) \cup \{v\}$ (in particular, u and v are adjacent). A *homogeneous set* in G is a subset X of $V(G)$ such that every vertex of $V(G) \setminus X$ with a neighbor in X is complete to X. A homogeneous set X is *nontrivial* if $1 < |X| < |V(G)|$. Thus if $u, v \in V(G)$ are twins and $|V(G)| > 2$, then $\{u, v\}$ is a nontrivial homogeneous set in G. We say that G is *prime* if $|V(G)| \geq 4$ and G has no nontrivial homogeneous set.

For a graph G, $X \subseteq V(G)$ is a *claw (in G)* if $G|X$ is the complete bipartite graph $K_{1,3}$. A graph is said to be *claw-free* if no subset of its vertex set is a claw. A *simplicial clique* in G is a nonempty clique K, such that for every $k \in K$, the set $N(k) \setminus K$ is a clique.

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†Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027 (mchudnov@columbia.edu). This author was partially supported by NSF grants DMS-0758364 and DMS-1001091.

‡Department of Mathematics, Princeton University, Princeton, NJ 08544 (pds@math.princeton.edu). This author was partially supported by ONR grant N00014-04-1-0062 and NSF grant DMS03-54465.

There are two main results in this paper. Our first goal is to answer a question of Prasad Tetali [6], who asked if there is an efficient algorithm to test if a claw-free graph has a simplicial clique. The idea is to enumerate them all. However, this is impossible in polynomial time since the complete graph on n vertices is a claw-free graph with $2^n - 1$ simplicial cliques. More generally, starting with a claw-free graph, one can replace a vertex by many copies of itself, all pairwise adjacent (thus introducing a large set of twins), and drive the number of simplicial cliques to be exponential in the size of the graph. But what about prime claw-free? Our first main result is the following theorem.

1.1. *A prime claw-free graph G has at most $|V(G)|+1$ distinct simplicial cliques.*

This result is tight, since a k-edge path has $k+1$ vertices and $k+2$ simplicial cliques (namely, all the edges and the two end vertices). We later use Theorem 1.1 to design a polynomial time algorithm that finds a simplicial clique in a prime claw-free graph if one exists (in fact, the algorithm finds all such cliques), answering Tetali's question.

In order to prove Theorem 1.1, we prove a lemma about general graphs (not just claw-free), which we consider to be the second main result of the paper and which gives the paper its title. The lemma is about "growing" prime graphs, starting from a prime induced subgraph and adding vertices one at a time in such a way that all the intermediate subgraphs are prime. Before we can state the lemma precisely, we need to define the class of *obstinate* graphs. Let O_k be the bipartite graph on $2k$ vertices with bipartition $(\{a_1, \dots, a_k\}, \{b_1, \dots, b_k\})$ in which for $i, j \in \{1, \dots, k\}$, a_i is adjacent to b_j if and only if $j \leq i$. A graph G is said to be *obstinate* if there exists a natural number $k > 1$ such that one of G, G^c is isomorphic to O_k. We observe that all obstinate graphs are prime. We can now state the lemma.

1.2. *Let G be a graph, and let H be a proper induced subgraph of G. Assume that both G and H are prime and that G is not obstinate. Then there exists an induced subgraph H' of G, isomorphic to H, and a vertex $v \in V(G) \setminus V(H')$ such that $G|(V(H') \cup \{v\})$ is prime.*

This is closely related to a result of Boudabbous and Ille [1], but the proof we include here is independent of [1]. We remark that the theorem would not be true if we did not allow moving to an isomorphic copy H' for H. To see this, take H to be a three-edge path with vertices a, b, c, d in order, and let $V(G) \setminus V(H) = \{e, f\}$, where e is adjacent to a and c and f is adjacent to b and d.

Repeatedly applying Theorem 1.2, we obtain the following corollary.

1.3. *Let G be a graph, and let H be a proper induced subgraph of G. Assume that both G and H are prime and that G is not obstinate. Then there exists a sequence of prime induced subgraphs $G_0, \dots, G_{|V(G)|-|V(H)|}$ of G such that*

- *G_0 is isomorphic to H,*
- *$G_{|V(G)|-|V(H)|} = G$, and*
- *for every $i \in \{1, \dots, |V(G)| - |V(H)|\}$, there exists $v_i \in V(G_i)$ such that $G_{i-1} = G_i \setminus v_i$.*

This paper is organized as follows. In section 2 we strengthen a theorem from [2] to obtain a structural result, Thereom 2.3, that we need for the proof of Theorem 1.1. In section 3 we apply Theorem 2.3 to prove Theorem 1.1 assuming Theorem 1.2. The proof of Theorem 1.2 occupies section 4. In section 5, we use Theorem 1.1 to give a polynomial time algorithm that finds all simplicial cliques of a prime claw-free graph. In the final section we apply our results to nonprime claw-free graphs; we find that it is possible to "describe" all their simplicial cliques in polynomial time, although there may be too many to list separately.

2. Linear interval graphs revisited. Theorem 5.2 of [2] deals with the structure of claw-free graphs that admit certain types of clique cutsets. (We will define them precisely later.) However, it turns out that that theorem can be easily strengthened with almost no changes to the proof. This strengthening is likely to be useful in future applications, and we also need it in the rest of this paper. Thus modifying Theorem 5.2 of [2] is our first goal here.

To study the structure of graphs with certain forbidden induced subgraphs, it is often helpful to deal with objects slightly more general than graphs, which we call "trigraphs." This concept is also used in [2], and so we start by explaining it. In a graph, every pair of vertices is either adjacent or nonadjacent, but in a trigraph, some pairs may be "undecided." For our purposes, we may assume that this set of undecided pairs is a matching. Thus, let us say a *trigraph* G consists of a finite set $V(G)$ of vertices and a map $\theta_G : V(G)^2 \to \{1, 0, -1\}$, satisfying

- for all $v \in V(G)$, $\theta_G(v, v) = 0$,
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$,
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v), \theta_G(u, w)$ is 0.

We call θ_G the *adjacency function* of G. For distinct u, v in $V(G)$, we say that u, v are *strongly adjacent* if $\theta_G(u, v) = 1$, *strongly antiadjacent* if $\theta_G(u, v) = -1$, and *semiadjacent* if $\theta_G(u, v) = 0$. We say that u, v are *adjacent* if they are either strongly adjacent or semiadjacent and *antiadjacent* if they are either strongly antiadjacent or semiadjacent. Also, we say that u is a *(strong) neighbor* of v if u, v are (strongly) adjacent and u is an *(strong) antineighbor* of v if u, v are (strongly) antiadjacent. For a vertex $v \in V(G)$ we denote by $N(x)$ the set of neighbors of x in G, and $N^*(v)$ denotes the set of strong neighbors of v. We denote by $F(G)$ the set of all pairs $\{u, v\}$ such that $u, v \in V(G)$ are distinct and semiadjacent. Note that a trigraph G is a graph if $F(G) = \emptyset$. We remark that the last condition of the definition of θ_G means that $F(G)$ is a matching.

Let G be a trigraph, and let A and B be two disjoint subsets of $V(G)$. We say that A is *(strongly) complete* to B if every vertex in A is (strongly) adjacent to every vertex in B and that A is *(strongly) anticomplete* to B if every vertex in A is (strongly) antiadjacent to every vertex in B. As in the graph case, if $A = \{a\}$, we will say that a is (strongly) complete or (strongly) anticomplete to B if $\{a\}$ is (strongly) complete or (strongly) anticomplete to B, respectively. If a is neither strongly complete nor strongly anticomplete to B, then a is *mixed* on B. Since every graph is a trigraph with no semiadjacent vertex pairs, the notion of being mixed makes sense for graphs as well as trigraphs. A *(strong) clique* in G is a set of vertices all pairwise (strongly) adjacent, and a *(strongly) stable set* is a set of vertices all pairwise (strongly) antiadjacent. If $X \subseteq V(G)$, we define the trigraph $G|X$ *induced on* X as follows. Its vertex set is X, and its adjacency function is the restriction of θ_G to X^2. We define $G \setminus X = G|(V(G) \setminus X)$. A *homogeneous set* in G is a set of vertices X such that every vertex of $V(G) \setminus X$ is either strongly complete or strongly anticomplete to X. A trigraph G is *claw-free* if there do not exist four vertices $a, b, c, d \in V(G)$, such that a is complete to $\{b, c, d\}$ and $\{b, c, d\}$ is a stable set.

Next we repeat a few definitions from [2]. We say that G admits a *0-join* (X, Y) if X and Y are two disjoint nonempty sets with union $V(G)$ such that X is strongly anticomplete to Y and that G admits a *1-join* (A, B, C, D) if A, B, C, D are four nonempty pairwise disjoint subsets of $V(G)$ such that $A \cup B \cup C \cup D = V(G)$, $B \cup C$ is a strong clique, A is strongly anticomplete to $C \cup D$, and D is strongly anticomplete to B.

Let $A, B \subseteq V(G)$. We call (A, B) a *proper W-join* if

- A, B are disjoint nonempty strong cliques of G and at least one of A, B has at least two members,

- A is a homogeneous set in $G \setminus B$ and B is a homogeneous set in $G \setminus A$,
- no member of A is strongly complete or strongly anticomplete to B and no member of B is strongly complete or strongly anticomplete to A.

A proper W-join (A, B) is *coherent* if the set of the vertices in $V(G) \setminus (A \cup B)$ that are complete to $A \cup B$ is a clique.

We say that G is a *linear interval trigraph* if the vertices of G can be numbered $v_1, \dots, v_n$ such that for all i, j with $1 \leq i < j \leq n$, if v_i is adjacent to v_j, then $\{v_i, v_{i+1}, \dots, v_{j-1}\}$ and $\{v_{i+1}, v_{i+2}, \dots, v_j\}$ are strong cliques.

A *clique cutset* in G is a strong clique C such that $G \setminus C$ is not connected, that is, $V(G) \setminus C$ can be partitioned into two nonempty sets V_1, V_2 such that V_1 is strongly anticomplete to V_2. Let us say the clique cutset C is *internal* if the sets V_1, V_2 can be chosen so that for $i = 1, 2$ either $|V_i| > 1$, or the unique vertex of V_i is semiadjacent to some vertex of C. Note that this definition is different from the one in [2], for here we allow the sets V_i to have size one.

An *antinet* is a trigraph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$ such that $\{a_1, a_2, a_3\}$ is a stable set, a_i, b_i are antiadjacent for $i = 1, 2, 3$, and all other pairs are adjacent. A *strong antinet* is a trigraph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$ such that $\{a_1, a_2, a_3\}$ is a strongly stable set, a_i, b_i are strongly antiadjacent for $i = 1, 2, 3$, and all other pairs are strongly adjacent.

The following is a definition from [3] that plays an important role in the structure theory of claw-free graphs. We say that a trigraph H is a *thickening* of a trigraph G if for every $v \in V(G)$ there is a nonempty subset $X_v \subseteq V(H)$, all pairwise disjoint and with union $V(H)$, satisfying the following:

- for each $v \in V(G)$, X_v is a strong clique of H;
- if $u, v \in V(G)$ are strongly adjacent in G, then X_u is strongly complete to X_v in H;
- if $u, v \in V(G)$ are strongly antiadjacent in G, then X_u is strongly anticomplete to X_v in H;
- if $u, v \in V(G)$ are semiadjacent in G, then X_u is neither strongly complete nor strongly anticomplete to X_v in H.

Let us say that H is a *proper thickening* of G if in addition

- if $u, v \in V(G)$ are semiadjacent in G, then every vertex in X_u has both a neighbor and an antineighbor in X_v in H.

This is a slight variant of a useful lemma from [2].

2.1. *Let G be a claw-free trigraph and let C be a clique cutset in G. Let V_1, V_2 be a partition of $V(G) \setminus C$ such that $V_1, V_2 \neq \emptyset$ and V_1 is anticomplete to V_2. Then*

- *if a vertex $u \in C$ has both a neighbor in V_1 and a neighbor in V_2, then $N(u) \cap V_1$ and $N(u) \cap V_2$ are strong cliques, and*
- *for all $u, v \in C$, either $N(u) \cap V_1 \subseteq N^*(v) \cap V_1$ or $N(u) \cap V_2 \subseteq N^*(v) \cap V_2$.*

Proof. Suppose that for some vertex $u \in C$ with both a neighbor in V_1 and a neighbor in V_2, there exist two antiadjacent vertices x, y in $N(u) \cap V_1$. Let $z \in N(u) \cap V_2$. But now $\{u, x, y, z\}$ is a claw in G, a contradiction. This proves the first assertion of the theorem.

For the second, assume that there exist $v_1 \in (N(u) \setminus N^*(v)) \cap V_1$ and $v_2 \in (N(u) \setminus N^*(v)) \cap V_2$. Since C is a clique, u is adjacent to v. But then $\{u, v, v_1, v_2\}$ is a claw, a contradiction. This proves the second assertion of the theorem and completes the proof of Theorem 2.1. ∎

Next we prove the main result of this section, which is a strengthening of Theorem 5.2 of [2]. The proof is very similar to that in [2], and we apologize to the reader for repeating it. For an integer $k \geq 4$, a *hole of length k* in a trigraph T is a subtrigraph of T with vertices $v_1, \dots, v_k$, such that for $1 \leq i < j \leq k$ the pair v_iv_j is adjacent if $j - i = 1$, the pair v_1v_k is adjacent, and all other pairs are antiadjacent.

2.2. *Let G be a claw-free trigraph with an internal clique cutset such that G does not admit twins, a 0-join, or a 1-join. Then every hole in G has length four; if there is a 4-hole, then G admits a coherent proper W-join, and otherwise G is a linear interval trigraph or a strong antinet.*

Proof. Let G be a claw-free trigraph admitting an internal clique cutset, such that G does not admit twins, a 0-join, or a 1-join. Suppose first that G has no hole. We may assume that G is not a linear interval trigraph. It follows from Theorem 4.1 in [2] that G is an antinet. We claim that G is a strong antinet. Let $a_1, a_2, a_3, b_1, b_2, b_3$ be as in the definition of an antinet. Since $\{b_3, a_1, a_2, b_2\}$ is not a claw in G, it follows that a_1 is strongly adjacent to b_2, and similarly a_i is strongly adjacent to b_j for all $i \neq j \in \{1, 2, 3\}$. Since a_1-b_2-b_1-b_3-a_1 is not a hole in G, it follows that b_2 is strongly adjacent to b_3, and from the symmetry $\{b_1, b_2, b_3\}$ is a strong clique. Since $\{b_1, a_1, a_2, a_3\}$ is not a claw in G, we deduce that a_1 is strongly antiadjacent to b_1, and similarly a_i is strongly antiadjacent to b_i for $i = 2, 3$. Finally, since a_1-a_2-b_1-b_2-a_1 is not a hole, it follows that a_1 is strongly antiadjacent to a_2, and, from the symmetry, $\{a_1, a_2, a_3\}$ is a strongly stable set. This proves that G is a strong antinet. Hence we may assume that there is a hole H in G. Choose H with length at least five if possible.

Let us say a *clique-separation* in G is a triple (C, V_1, V_2) such that

- C is a strong clique of G, and (V_1, V_2) is a partition of $V(G) \setminus C$,
- V_1 is strongly anticomplete to V_2, and
- $V(H) \cap V_2 = \emptyset$.

(1) *There is a clique-separation (C, V_1, V_2) in G with the following properties:*

- *either $|V_2| > 1$, or $|V_2| = 1$ and the unique vertex of V_2 is semiadjacent to some vertex of C, and*
- *subject to that $|V_2|$ is maximum, and*
- *$C \neq \emptyset$, and every vertex in C has a neighbor in V_1 and a neighbor in V_2.*

For since G admits an internal clique cutset, there is a triple (C, V_1, V_2) satisfying the first and second conditions in the definition of a clique-separation and such that for $i = 1, 2$ either $|V_i| > 1$ or the unique vertex of V_i is semiadjacent to some vertex of C. Since C is a strong clique, it follows that $V(H)$ has an empty intersection with one of V_1, V_2. Hence (possibly after exchanging V_1, V_2), it follows that G contains a clique-separation (C, V_1, V_2), where either $|V_2| > 1$ or the unique vertex of V_2 is semiadjacent to some vertex of C. Choose such a clique-separation (C, V_1, V_2) with $|V_2|$ maximum, and subject to that, with C minimal. Since G does not admit a 0-join, it follows that $C \neq \emptyset$. Let $c \in C$. If c has no neighbor in V_2, then $(C \setminus \{c\}, V_1 \cup \{c\}, V_2)$ is also a clique-separation with $|V_2|$ maximum, contradicting the minimality of C; and if c has no neighbor in V_1, then $c \notin V(H)$ (since every vertex in $V(H) \cap C$ has a neighbor in $V(H) \setminus C \subseteq V_1$, because C is a strong clique), and therefore $(C \setminus \{c\}, V_1, V_2 \cup \{c\})$ is a clique-separation contradicting the maximality of $|V_2|$. This proves (1).

For a vertex $c \in C$ and for $i = 1, 2$, let $N_i(c)$ be the set of neighbors of c in V_i, and let $N_i^*(c)$ be the set of strong neighbors of c in V_i. Let J be the digraph with $V(J) = C$ and edge set all pairs (u, v) with $u, v \in C$ (possibly equal) such that $N_1(v) \not\subseteq N_1^*(u)$. Since C is nonempty, there is a strong component of J that is a "sink component"; that is, there exists $X \subseteq C$ such that

- X is nonempty and $J|X$ is strongly connected,
- there is no edge $(u, v) \in E(J)$ with $u \in X$ and $v \notin X$.

(2) *For all distinct* $u, v \in X$, $N_2(u) = N_2^*(u) = N_2(v) = N_2^*(v)$.

Since X is strongly connected, there is a directed path of J from u to v, say, $u = v_1$-$\dots$-$v_k = v$. For $1 \le i < k$, since $(v_i, v_{i+1}) \in E(J)$, it follows that $N_1(v_{i+1}) \not\subseteq N_1^*(v_i)$, and therefore $N_2(v_{i+1}) \subseteq N_2^*(v_i)$ by the second statement of Theorem 2.1. Consequently $N_2(v) \subseteq N_2^*(u)$. Similarly $N_2(u) \subseteq N_2^*(v)$. This proves (2).

Let $Z = \bigcap_{x \in X} N_1^*(x)$.

(3) $X \neq C$.

For suppose that $X = C$. Choose $c \in C$, and let $Y = N_2(c)$. By (1) and Theorem 2.1, Y is a strong clique. There are two cases, depending on whether $N_2(c') = N_2^*(c') = Y$ for all $c' \in C$. Suppose first that $N_2(c') = N_2^*(c') = Y$ for all $c' \in C$. Then $C \cup Y$ is a strong clique. If $V_2 = Y$, then since G admits no twins, it follows that $|V_2| = |Y| = 1$, and yet the unique vertex of Y is not semiadjacent to any vertex of C, a contradiction. Thus $V_2 \neq Y$. But now $(V_1, C, Y, V_2 \setminus Y)$ is a 1-join, a contradiction. Thus we may assume that there exists $c' \in C$ with one of $N_2(c'), N_2^*(c')$ different from Y. By (2), $|C| = 1$, and so $c' = c$ and $N_2(c) \neq N_2^*(c)$. Hence $N_1(c) = N_1^*(c) = Z$ (since c is semiadjacent to a member of V_2 and $F(G)$ is a matching), and by (1) and Theorem 2.1, Z is a strong clique and therefore so is $Z \cup C$. But $Z \neq V_1$, because $G|(V_1 \cup C)$ contains a hole and therefore $V_1 \cup C$ is not a strong clique, and so $(V_1 \setminus Z, Z, C, V_2)$ is a 1-join, a contradiction. This proves (3).

(4) $X \cup Z$ *is a strong clique, and* $N_1(c) \subseteq Z$ *for every vertex* $c \in C \setminus X$*, and* H *is a* 4*-hole, and* $V(H)$ *consists of two vertices of* $C \setminus X$ *and two vertices of* Z.

For (1) and the first statement of Theorem 2.1 imply that Z is a strong clique, and therefore $X \cup Z$ is a strong clique. Let $c \in C \setminus X$ and $x \in X$. Since $(x, c) \notin E(J)$, it follows that $N_1(c) \subseteq N_1^*(x)$. Since this holds for all $x \in X$, we deduce that $N_1(c) \subseteq Z$. From (3) and the maximality of $|V_2|$, $(X \cup Z, V_1 \setminus Z, V_2 \cup (C \setminus X))$ is not a clique-separation of G, and so $V(H) \cap (C \setminus X) \neq \emptyset$. Let H have vertices h_1-$\dots$-h_n-h_1 in order, where $h_1 \in C \setminus X$. Then $h_2, h_n \in C \cup N_1(h_1) \subseteq C \cup Z$, and since C, Z are both strong cliques and h_2, h_n are antiadjacent, we may assume that $h_2 \in C$ and $h_n \in Z$. Since h_2, h_n are antiadjacent, and $X \cup Z$ is a strong clique, it follows that $h_2 \notin X$, and so $h_2 \in C \setminus X$. Thus by the same argument $h_3 \in Z$. Since $h_3, h_n \in Z$ and Z is a strong clique, it follows that $n = 4$, and so H is a 4-hole. This proves (4).

Let us say a *step* is a 4-hole consisting of two vertices of $C \setminus X$ and two vertices of Z. We have seen that H is a step. We say a pair (A, B) is a *step-connected strip* if $A \subseteq Z$ and $B \subseteq C \setminus X$, and for every partition (P, Q) of A or of B with P, Q nonempty, there is a step S with $V(S) \subseteq A \cup B$, and with $P \cap V(S) \neq \emptyset$ and $Q \cap V(S) \neq \emptyset$. Certainly the pair $(V(H) \cap Z, V(H) \cap (C \setminus X))$ is a step-connected strip, so we may choose a step-connected strip (A, B) with $V(H) \subseteq A \cup B$ and with $A \cup B$ maximal.

(5) *Every vertex in* $V(G) \setminus (A \cup B)$ *is either strongly complete or strongly anticomplete to* A *and either strongly complete or strongly anticomplete to* B*. Moreover, the set of vertices* $V(G) \setminus (A \cup B)$ *that are complete to* $A \cup B$ *is a strong clique.*

The proof of (5) is identical to that in Theorem 5.2 of [2], and we omit it.

From (4), H has length four, and so no hole of G has length more than four; and from (5), G admits a coherent proper W-join. This proves Theorem 2.2. ∎

Now we strengthen Theorem 2.2 further.

2.3. *Let* G *be a claw-free trigraph with an internal clique cutset such that* G *does not admit twins, a* 0*-join, or a* 1*-join. Then either* G *is a thickening of a linear interval trigraph or* G *is a strong antinet.*

Proof. Suppose not, and let G be a counterexample to Theorem 2.3 with $|V(G)|$ minimal. By Theorem 2.2, G admits a coherent proper W-join (A, B). Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Then $V(G) = A \cup B \cup C \cup D \cup E \cup F$. Let G' be obtained from $G \setminus (A \cup B)$ by adding two new vertices a and b such that a is strongly complete to $C \cup E$ and strongly anticomplete to $D \cup F$, b is strongly complete to $D \cup E$ and strongly anticomplete to $C \cup F$, and a is semiadjacent to b. Then $|V(G')| < |V(G)|$.

(1) *G' admits an internal clique cutset.*

Let P be an internal clique cutset in G, and let V_1, V_2 be a partition of $V(G) \setminus P$ such that for $i = 1, 2$ either $|V_i| > 1$ or the unique vertex of V_i is semiadjacent to some vertex of P. If possible, choose P disjoint from one of A, B. We may assume that every vertex of P has both a neighbor in V_1 and a neighbor in V_2. Suppose first that $A \subseteq P$. Since no vertex of B is strongly complete to A, it follows that $P \cap B = \emptyset$, and since B is a clique, we may assume from the symmetry that $B \subseteq V_1$. But then $P' = (P \setminus A) \cup \{a\}$ is a clique cutset in G', (V_1', V_2) (where $V_1' = (V_1 \setminus B) \cup \{b\}$) is a partition of $V(G') \setminus P'$, $b \in V_1'$ is semiadjacent to $a \in P$ and either $|V_2| > 1$ or the unique vertex of V_2 is semiadjacent to some vertex of $P \setminus \{a\}$, and therefore G' admits an internal clique cutset. Thus we may assume that $A \setminus P \neq \emptyset$ and $B \setminus P \neq \emptyset$.

Next suppose that $(A \cup B) \cap P = \emptyset$. Since A is a strong clique, we may assume that $A \subseteq V_1$. Since every vertex of A has a neighbor in B, and since B is a strong clique, it follows that $B \subseteq V_1$. But now, letting $V_1' = (V_1 \setminus (A \cup B)) \cup \{a, b\}$, we observe that V_1', V_2 is a partition of $V(G') \setminus P$ into two nonempty sets, $|V_1'| > 1$, and either $|V_2| > 1$ or the unique vertex of V_2 is semiadjacent to some vertex of P. Consequently G' admits an internal clique cutset. Thus we may assume that $A \cap P \neq \emptyset$.

Since $A \setminus P \neq \emptyset$, and since A is a strong clique, we may assume that $A \cap V_1 \neq \emptyset$ and $A \cap V_2 = \emptyset$. Since $A \cap V_1$ is strongly anticomplete to $V_2 \setminus B$, it follows that $A \cap P$ is strongly anticomplete to $V_2 \setminus B$, and therefore, since every vertex of P has a neighbor in V_2, we deduce that $B \cap V_2 \neq \emptyset$. Then $B \cap V_1 = \emptyset$. Since every vertex of A has a neighbor in B, and $A \cap V_1$ is strongly anticomplete to $B \cap V_2$, it follows that $B \cap P \neq \emptyset$. Now from the symmetry, $B \cap P$ is strongly anticomplete to $V_1 \setminus A$. Also, since A is a homogeneous set in $V(G) \setminus B$, it follows that $P \setminus (A \cup B)$ is strongly complete to A. Similarly, $P \setminus (A \cup B)$ is strongly complete to B. We observe that $X = (A \cup P) \setminus B$ is a strong clique, and $(V_1 \setminus A, V_2 \cup B)$ is a partition of $V(G) \setminus X$. It follows from the choice of P that X is not an internal clique cutset in G. Since $V_2 \cup B$ is strongly anticomplete to $V_1 \setminus A$, it follows that $|V_1 \setminus A| \leq 1$ and no vertex of $V_1 \setminus A$ is semiadjacent to a vertex of X.

We claim that $V_1 \setminus A$ is strongly complete to A. Suppose not. Then $V_1 \setminus A \neq \emptyset$; let u_1 be the unique vertex of $V_1 \setminus A$. Since (A, B) is a proper W-join, u_1 is strongly anticomplete to A, and since G is connected, it follows that u_1 has a neighbor in $(A \cup P) \setminus B$. But this contradicts the first statement of Theorem 2.1, since $p \in P \setminus (A \cup B)$ is strongly complete to A. This proves that $V_1 \setminus A$ is strongly complete to A. Similarly, $|V_2 \setminus B| \leq 1$, no vertex of $V_2 \setminus B$ is semiadjacent to a vertex of X, and $V_2 \setminus B$ is strongly complete to B. Let T_1' be the set of vertices of $P \setminus (A \cup B)$ that have a neighbor in $V_1 \setminus A$ and T_2' the set of vertices of $P \setminus (A \cup B)$ that have a neighbor in $V_2 \setminus B$. Define $T_1 = T_1' \setminus T_2'$, $T_2 = T_2' \setminus T_1'$, $Y = T_1' \cap T_2'$, and $Z = P \setminus (A \cup B \cup T_1' \cup T_2')$. By the second assertion of Theorem 2.1, at least one of the sets Y, Z is empty.

Let S be the trigraph with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where the pairs

$$v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4, v_3v_5, v_4v_5, v_4v_6, v_5v_6, v_5v_7, v_6v_7$$

are strongly adjacent, and if $Y \neq \emptyset$, also the pairs

$$v_1 v_4, v_4 v_7$$

are strongly adjacent, the pairs

$$v_2 v_5, v_3 v_6$$

are semiadjacent, and all other pairs are strongly antiadjacent. Let $X_{v_1} = V_1 \setminus A, X_{v_2} = A \cap V_1, X_{v_3} = T_1 \cup (A \cap P), X_{v_4} = Y \cup Z, X_{v_5} = T_2 \cup (B \cap P), X_{v_6} = B \cap V_2, X_{v_7} = V_2 \setminus B$. Then S is a linear interval trigraph. The sets $X_{v_2}, X_{v_3}, X_{v_5}$ and X_{v_6} are nonempty, and either $X_{v_4} = Y$ or $X_{v_4} = Z$. Moreover, if $X_{v_4} = Y$, then X_{v_4} is strongly complete to $X_{v_1} \cup X_{v_7}$, and if $X_{v_4} = Z$, then X_{v_4} is strongly anticomplete to $X_{v_1} \cup X_{v_7}$. Now let $T \subseteq \{v_1, v_7, v_4\}$ be the set of all v_i such that $X_{v_i} = \emptyset$. Then X_{v_i} is a nonempty strong clique for every $v_i \in S \setminus T$. Moreover, for $i, j \in \{1, \ldots, 7\}$, if v_i is strongly complete to v_j, then X_{v_i} is strongly complete to X_{v_j}, and if v_i is strongly anticomplete to v_j, then X_{v_i} is strongly anticomplete to X_{v_j}. Next we claim that the pairs $X_{v_2} X_{v_5}$ and $X_{v_3} X_{v_6}$ are neither strongly complete nor strongly anticomplete. From the symmetry, it is enough to show that X_{v_2} is not strongly complete and not strongly anticomplete to X_{v_5}. Since (A, B) is a proper W-join in G, it follows that no vertex in $B \cap P$ is strongly complete to A, and no vertex of $A \cap V_1$ is strongly anticomplete to B. This implies that $B \cap P$ is not strongly complete to $A \cap V_1$, and $A \cap V_1$ is not strongly anticomplete out $B \cap P$, which proves the claim. Consequently, G is a thickening of $S \setminus T$, contrary to the fact that G is a counterexample to Theorem 2.3. This proves (1).

For $X \subseteq V(G')$ let

$$L(X) = \begin{cases} X & \text{if} \quad a, b \notin X, \\ (X \setminus \{a\}) \cup A & \text{if} \quad a \in X \text{ and } b \notin X, \\ (X \setminus \{b\}) \cup B & \text{if} \quad b \in X \text{ and } a \notin X, \\ (X \setminus \{a, b\}) \cup A \cup B & \text{if} \quad a, b \in X. \end{cases}$$

(2) *G' does not admit a 0-join or a 1-join or twins.*

If G' admits a 0-join (X, Y), then $L(X), L(Y)$ is a partition of $V(G)$ with $L(X)$ strongly anticomplete to $L(Y)$. Consequently, G admits a 0-join, a contradiction. This proves that G' does not admit a 0-join. If G' admits a 1-join (P, Q, R, S), then $L(P), L(Q), L(R), L(S)$ is a 1-join in G, and therefore G' does not admit a 1-join. Finally, if u, v are twins in G', then $\{u, v\} \cap \{a, b\} = \emptyset$, since a is semiadjacent to b; and therefore u, v are twins in G. Thus G' does not admits twins. This proves (2).

By the minimality of $|V(G)|$, one of the outcomes of Theorem 2.3 holds for G'. Since no two vertices of a strong antinet are semiadjacent, we deduce that G' is a thickening of a linear interval trigraph, say, S. Let $\{X_v\}_{v \in V(S)}$ be the subsets of $V(G')$ as in the definition of a thickening. Since a is semiadjacent to b, it follows that there exist two vertices $u, v \in V(S)$ such that $a \in X_u$, $b \in X_v$ and u is semiadjacent to v. But now G is a thickening of S with subsets $\{L(X_v)\}_{v \in V(S)}$. This proves Theorem 2.3. ∎

We finish this section with a lemma that refines Theorem 2.3 further.

2.4. *Let G be a thickening of a linear interval trigraph F. Then there exists a linear interval trigraph F' such that G is a proper thickening of F'.*

Proof. Let F' be a linear interval trigraph such that G is a thickening of F' and subject to that with $|V(F')|$ maximum. Let the vertices of F' be numbered $v_1, \ldots, v_n$

as in the definition of a linear interval trigraph. For $i \in \{1, \dots, n\}$ let $X_i = X_{v_i}$ be a subset of $V(G)$ as in the definition of a thickening. We may assume that for some $1 \leq i < j \leq n$, v_i is semiadjacent to v_j, but some $x \in X_i$ is either strongly complete or strongly anticomplete to X_j. This in particular implies that $|X_i| \geq 2$. Suppose first that x is strongly complete to X_j. Let F'' be the trigraph obtained from F' by adding a new vertex v_i' and making v_i' be strongly adjacent to $N^*(v_i) \cup \{v_i, v_j\}$ and strongly antiadjacent to all the remaining vertices of $V(F')$. Then F'' is a linear interval trigraph, ordering the vertices

$$v_1, \dots, v_i, v_i', v_{i+1}, \dots, v_n,$$

and G is a thickening of F'', where we replace X_i with $X_i \setminus \{x\}$ and define $X_{v_i'} = \{x\}$, contrary to the choice of F'. This proves that x is strongly anticomplete to X_j. Let F'' be the trigraph obtained from F' by adding a new vertex v_i' and making v_i' be strongly adjacent to $N^*(v_i) \cup \{v_i\}$ and strongly antiadjacent to all the remaining vertices of $V(F')$. Then F'' is a linear interval trigraph, ordering the vertices

$$v_1, \dots, v_{i-1}, v_i', v_i, \dots, v_n,$$

and G is a thickening of F'', where we replace X_i with $X_i \setminus \{x\}$ and define $X_{v_i'} = \{x\}$, again contrary to the choice of F'. This proves Theorem 2.4. ◻

3. The proof of Theorem 1.1. The results in this section are about graphs only, though we do use trigraphs in some of the proofs. Our current goal is to prove Theorem 1.1 assuming Theorem 1.2. We start with some definitions. Let G be a graph. A *connected component* of G is a maximal nonnull connected subgraph of G. A vertex $v \in V(G)$ is *simplicial* if $N(v)$ is a clique. Let $s(G)$ denote the number of simplicial cliques in G. We start with an easy lemma.

3.1. *Let G be a claw-free graph, and let X be a nontrivial homogeneous set in G. Let C be the set of vertices of $V(G) \setminus X$ that are complete to X and A the set of vertices of $V(G) \setminus X$ that are anticomplete to X. If X is not a clique, then A is anticomplete to $X \cup C$, and one of G, G^c is not connected.*

Proof. Let $x_1, x_2 \in X$ be nonadjacent. Since $\{c, x_1, x_2, a\}$ is not a claw for any $c \in C$ and $a \in A$, it follows that C is anticomplete to A. This proves the first assertion of the theorem. We may assume that $A = \emptyset$, for otherwise G is not connected. But then $V(G) \setminus X = C \neq \emptyset$, and thus G^c is not connected. This proves Theorem 3.1. ◻

Next we deal with obstinate graphs.

3.2. *Let G be claw-free and obstinate. Then one of the following holds:*

- *G is isomorphic to O_2. Let $V(G) = \{a_1, a_2, b_1, b_2\}$ as in the definition of O_2. Then the simplicial cliques of G are*

$$\{\{a_1\}, \{a_1, b_1\}, \{a_2, b_1\}, \{a_2, b_2\}, \{b_2\}\},$$

and, in particular, $s(G) = |V(G)| + 1$.

- *G^c is isomorphic to O_k with $k \geq 3$. Let $V(G) = \{a_1, \dots, a_k, b_1, \dots, b_k\}$ as in the definition of O_k. Let $A_i = \{a_i, \dots, a_k\}$, $B_i = \{b_1, \dots, b_i\}$. Then the simplicial cliques of G are*

$$A_1, \dots, A_k, B_1, \dots, B_k,$$

and in particular $s(G) = |V(G)|$.

Proof. Since G is obstinate, G or G^c is isomorphic to O_k for some natural number $k \geq 2$. Suppose first that $G = O_k$ for some natural number $k \geq 2$. Let $a_1, \dots, a_k, b_1, \dots, b_k$ be as in the definition of O_k. Since G is claw-free and a_k is complete to $\{b_1, \dots, b_k\}$, it follows that $k = 2$. But now G is a three-edge path a_1-b_1-a_2-b_2, and the first outcome of the theorem holds.

Next assume that $G^c = O_k$ for some $k \geq 2$. If $k = 2$, then G is a 3-edge path, and so G is isomorphic to O_2, and the theorem holds. So we may assume that $k \geq 3$. Let $a_1, \dots, a_k, b_1, \dots, b_k$ be as in the definition of O_k. Let $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$. First we will enumerate all simplicial cliques in G that meet A. Let K be a simplicial clique in G with $K \cap A \neq \emptyset$. Then $b_1 \notin K$.

First we claim that $a_k \in K$. Suppose not. Choose j so that $a_j \in K$. Since a_j is complete to $\{a_k, b_{j+1}, \dots, b_k\}$, and a_k is anticomplete to B, the fact that $N(a_j) \setminus K$ is a clique implies that $\{b_{j+1}, \dots, b_k\} \subseteq K$. Since a_j is anticomplete to $\{b_1, \dots, b_j\}$, it follows that $b_1, \dots, b_j \notin K$. This implies that $K \cap \{a_1, \dots, a_k\} = \{a_j\}$. Let $p \in \{1, 2\} \setminus \{j\}$. Then $a_p, b_1 \notin K$, and b_k is adjacent to both a_p, b_1, contrary to the fact that K is a simplicial clique. This proves that $a_k \in K$.

Since K is a clique, and a_k is anticomplete to B, it follows that $K \subseteq A$. If there exist $1 \leq i < j < k$ such that $a_i \in K$ and $a_j \notin K$, then $a_j, b_j \in N(a_i) \setminus K$, which is a contradiction since a_j is nonadjacent to b_j. Therefore, $K = A_t$ for some $t \in \{1, \dots, k\}$. Moreover, we observe that A_t is a simplicial clique in G for every $t \in \{1, \dots, k\}$. Consequently,

$$A_1, \dots, A_k$$

is the complete list of simplicial cliques in G meeting A (and none of them meet B). From the symmetry,

$$B_1, \dots, B_k$$

is the complete list of simplicial cliques in G meeting B (and none of them meet A). Therefore, $s(G) = |V(G)| = 2k$, and Theorem 3.2 holds. ∎

Proof of Theorem 1.1. The proof is by induction on $|V(G)|$. Let $|V(G)| = n$.

(1) *We may assume that $n \geq 4$ and both G and G^c are connected.*

Since G is prime, it follows that $n \geq 4$. Consequently, again using the fact that G is prime, we deduce that both G and G^c are connected. This proves (1).

From now on, we assume in view of (1) that $n \geq 3$ and both G and G^c are connected.

(2) *If G has a simplicial vertex v such that $G \setminus v$ is prime, then the theorem holds.*

Let v be a simplicial vertex of G. Let $N = N(v)$, $M = V(G) \setminus (N \cup \{v\})$. Let $G' = G \setminus v$. Then N is a clique. Inductively, $s(G') \leq n$. Let $\mathcal{T}$ be the set of all cliques K of G with $v \in K$ such that both K and $K \setminus v$ are simplicial cliques in G. Now, if $K \neq \{v\}$ is a simplicial clique of G, then $K \setminus v$ is a simplicial clique of G', and so

$$s(G) \leq s(G') + 1 + |\mathcal{T}| \leq n + 1 + |\mathcal{T}|.$$

We observe that if some $u \in N$ is anticomplete to M, then $\{u, v\}$ is a nontrivial homogeneous set in G, contrary to the fact that G is prime. This implies that every vertex of N has a neighbor in M, and therefore, since v is anticomplete to M, no subset of N is a simplicial clique of G. Consequently, $\mathcal{T} = \emptyset$. This proves (2).

(3) *If G admits a* 1*-join, then the theorem holds.*

Let (A, B, C, D) be a 1-join. Assume first that some vertex $v \in B \cup C$ is anticomplete to $A \cup D$. Then $N(v) = (B \cup C) \setminus \{v\}$, and therefore v is simplicial. Since G is connected, both $B \setminus \{v\}$ and $C \setminus \{v\}$ are nonempty. We claim that $G \setminus v$ is prime. Suppose not, and let X be a nontrivial homogeneous set in $G \setminus v$. Write $N = B \cup C$, and $M = A \cup D$. Since X is not a nontrivial homogeneous set in G, it follows that both $X \cap M$ and $X \cap N$ are nonempty. Let M_1 be the set of vertices of $M \setminus X$ that are complete to X and M_2 be the set of vertices of $M \setminus X$ that are anticomplete to X. Then $M \setminus X = M_1 \cup M_2$. Since N is a clique, $N \setminus X$ is complete to X. Since $X \cup \{v\}$ is not a homogeneous set in G, it follows that $M_1 \neq \emptyset$. If $M_2 = \emptyset$, then $M = (M \cap X) \cup M_1$, and therefore, since M_1 is complete to X, we deduce that M is connected, contrary to the fact that $M = A \cup D$. This proves that $M_2 \neq \emptyset$. Now, since G is prime, it follows that M_2 is not anticomplete to $V(G) \setminus M_2$, and thus Theorem 3.1 applied in $G \setminus v$ implies that X is a clique. Consequently, $X \cap M$ is complete to N, contrary to the fact that A is anticomplete to C and D to B. This proves that $G \setminus v$ is prime. Now (3) follows from (2). Thus we may assume that every vertex of B has a neighbor in A and every vertex of C has a neighbor in D.

Let $b \in B$ and $c \in C$. Chose $a \in A$ adjacent to b, and $d \in D$ adjacent to c. Define $G_1 = G|(A \cup B \cup \{c, d\})$ and $G_2 = G|(C \cup D \cup \{a, b\})$. Then both G_1 and G_2 are claw-free. We claim that both G_1 and G_2 are prime. Suppose not; then from the symmetry we may assume that there is a nontrivial homogeneous set X in G_1. If $\{c, d\} \cap X = \emptyset$, then X is a nontrivial homogeneous set in G, contrary to the fact that G is prime. So we may assume that at least one of c, d belongs to X. Suppose first that $c \notin X$. Then $d \in X$. Since c is adjacent to d and anticomplete to A, it follows that $X \cap A = \emptyset$, and therefore there is a vertex $b' \in B \cap X$. But b' has a neighbor $a' \in A$, which is nonadjacent to d, contrary to the fact that X is a homogeneous set. Thus $c \in X$, and since c is the unique neighbor of d in G_1, it follows that $d \in X$. Since B is complete to c and anticomplete to d, it follows that $B \subseteq X$. Since d is anticomplete to A, we deduce that $V(G_1) \setminus X$ is anticomplete to X. But $V(G_1) \setminus X = A \setminus X$, and therefore $A \setminus X$ is anticomplete to $B \cup C \cup D \cup (A \cap X)$ in G, contrary to the fact that G is prime. This proves that both G_1 and G_2 are prime.

Let $|V(G_i)| = n_i$. By the inductive hypothesis, $s(G_i) \leq n_i + 1$ for $i = 1, 2$. Since B is a clique cutset in G, and every vertex of B is complete to C and has a neighbor in A, it follows from Theorem 2.1 that $N_{G_1}(v) \cap A$ is a clique for every $v \in B$. Consequently, $B \cup \{c\}$ is a simplicial clique in G_1. We observe that $\{d\}$ and $\{c, d\}$ are also simplicial cliques of G_1. This implies that the number of simplicial cliques of G_1 that are also cliques of $G|(A \cup B)$ is at most $s(G_1) - 3 \leq n_i - 2 = |A| + |B|$. From the symmetry, the number of simplicial cliques of G_2 that are also simplicial cliques of $G|(C \cup D)$ is at most $|C| + |D|$. Let t be the number of simplicial cliques of G that meet both $A \cup B$ and $C \cup D$. We observe that if $K \subseteq A \cup B$ is a simplicial clique of G, then K is a simplicial clique of G_1, and if $K \subseteq C \cup D$ is a simplicial clique of G, then K is a simplicial clique of G_2. Consequently,

$$s(G) \leq |A| + |B| + |C| + |D| + t.$$

It is therefore enough to show that $t \leq 1$. Indeed, let K be a clique of G such that $K \not\subseteq A \cup B$ and $K \not\subseteq C \cup D$. Since K is a clique, it follows that $K \subseteq B \cup C$, and both $K \cap B$ and $K \cap C$ are nonempty. Suppose $B \setminus K \neq \emptyset$, and choose $b' \in B \setminus K$. Let $c' \in K \cap C$, and let $d' \in D$ be a neighbor of c'. But now $b', d' \in N(c') \setminus K$, contrary to the fact that K is a simplicial clique. Thus $B \subseteq K$, and from the symmetry $C \subseteq K$. Consequently $K = B \cup C$ and $t \leq 1$. (In fact, Theorem 2.1 implies that $B \cup C$ is a simplicial clique of G, and so $t = 1$.) This proves (3).

In view of (3) and Theorem 3.2 we may assume that G does not admit a 1-join, and that G is not obstinate. Since both G and G^c are connected, it follows that there exists an induced subgraph of G, isomorphic to the 3-edge path. Let H be such a subgraph. We observe that H is prime. By Theorem 1.3, there exists $v \in V(G)$ such that $G \setminus v$ is prime. By (2), we may assume that v is not simplicial. Let $N = N(v)$, $M = V(G) \setminus (N \cup \{v\})$, and $G' = G \setminus v$. Inductively, $s(G') \leq n$. Let $\mathcal{T}$ be the set of all cliques K of G with $v \in K$, such that both K and $K \setminus v$ are simplicial cliques in G. Since if K is a simplicial clique of G, then $K \setminus v$ is a simplicial clique of G' (since $\{v\}$ is not a simplicial clique in G because v is not a simplicial vertex), it follows that

$$s(G) \leq s(G') + |\mathcal{T}|.$$

It is therefore enough to prove that $|\mathcal{T}| \leq 1$. We may assume that $\mathcal{T} \neq \emptyset$, for otherwise the theorem holds.

(4) *G is an antinet or a thickening of a linear interval trigraph.*

Let $K \in \mathcal{T}$. Then $K \setminus \{v\} \subseteq N$. Since v is complete to $K \setminus \{v\}$ and anticomplete to M, and since $K \setminus \{v\}$ is a simplicial clique of G, it follows that $K \setminus \{v\}$ (and therefore K) is anticomplete to M. Since K is a simplicial clique of G, it follows that $N \setminus K$ is a clique. Since G^c is connected, we deduce that $M \neq \emptyset$, and consequently $N \setminus K$ is a clique cutset in G. If $|M| > 1$, then since G is connected and does not admit a 1-join, it follows from Theorem 2.3 (regarding G as a trigraph with no semiadjacent pairs of vertices) that G is an antinet or a thickening of a linear interval trigraph. So we may assume that $|M| = 1$. Let m be the unique vertex of M. Let $N_1 = N(m) \cap N$, $N_2 = N \setminus (K \cup N_1)$. Let K_1 be the set of vertices of $K \setminus \{v\}$ with a neighbor in N_1, and let $K_2 = K \setminus (K_1 \cup \{v\})$. By the second assertion of Theorem 2.1, K_1 is complete to N_2. Let F be the trigraph with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ such that the pairs

$$v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_3v_5, v_4v_5, v_4v_6, v_5v_6$$

are strongly adjacent, the pair

$$v_2v_5$$

is strongly adjacent if K_1 is complete to N_1 and semiadjacent otherwise, the pair

$$v_3v_6$$

is strongly adjacent if K_2 is complete to N_2 and semiadjacent otherwise, and all other pairs are strongly antiadjacent. Then F is a linear interval trigraph. Now setting

$$X_{v_1} = M, X_{v_2} = N_1, X_{v_3} = N_2, X_{v_4} = \{v\}, X_{v_5} = K_1, X_{v_6} = K_2,$$

we observe that G is a thickening of an induced subgraph of F. This proves (4).

We observe that if G is an antinet, then $|V(G)| = s(G) = 6$, and so in view of (4) and by Theorem 2.4, we may assume from now on that G is a proper thickening of a linear interval trigraph. Let F be a linear interval trigraph of which G is a proper thickening. Since G is connected, it follows that F is connected. Let the vertices of F be $v_1, \ldots, v_n$ numbered as in the definition of a linear interval trigraph, and let $X_i = X_{v_i}$ be subsets of $V(G)$ as in the definition of a proper thickening. Let $i \in \{1, \ldots, n\}$ be such that $v \in X_i$. Let $j \in \{1, \ldots, n\}$ be minimum such that v has a neighbor in X_j, and let $k \in \{1, \ldots, n\}$ be maximum such that v has a neighbor in X_k. Since v is not a simplicial vertex, there exist $x_j \in X_j \cap N(v)$ and $x_k \in X_k \cap N(v)$ nonadjacent, and therefore $j \leq i \leq k$, $j \neq k$, and v_j is not strongly adjacent to v_k in F. Moreover, either $i = j$ or v_i is adjacent to v_j in F and either $i = k$ or v_i is adjacent to v_k in F.

(5) *Either*

- *$j = 1$, v is complete to $X_1 \setminus \{v\}$, and $x_j \in K$ and $K \cap X_k = \emptyset$ for every $K \in \mathcal{T}$, or*
- *$k = n$, v is complete to $X_n \setminus \{v\}$, and $x_k \in K$ and $K \cap X_j = \emptyset$ for every $K \in \mathcal{T}$.*

Let $K \in \mathcal{T}$. Since $v \in K$, and x_j and x_k are both adjacent to v and nonadjacent to each other, it follows that at least one of x_j, x_k is in K. From the symmetry we may assume $x_j \in K$. Since K is a clique, $x_k \notin K$.

Since $K \setminus \{v\}$ is a simplicial clique of G, x_j is adjacent to v, and v is anticomplete to $\bigcup_{p=1}^{j-1} X_p$, it follows that x_j is anticomplete to $\bigcup_{p=1}^{j-1} X_p$. But by the definition of a proper thickening, x_j has a neighbor in every set X_u such that u is adjacent to v_j, and so it follows from the fact that F is connected that $\bigcup_{p=1}^{j-1} X_p = \emptyset$ and so $j = 1$. Since x_j is complete to $X_1 \setminus \{x_j\}$ and $K \setminus \{v\}$ is a simplicial clique, it follows that every vertex of X_1 is either adjacent to v or in K, and therefore v is complete to $X_1 \setminus \{v\}$.

Now suppose there exists $L \in \mathcal{T}$ (possibly $L = K$) such that $L \cap X_k \neq \emptyset$. Let $x'_k \in L \cap X_k$. Since v_1 is not strongly adjacent to v_k in F, it follows that some $x'_1 \in X_1$ is nonadjacent to x'_k. By the argument above applied to L, x'_k, and x'_1, we deduce that $k = n$ and v is complete to X_k. Thus v is complete to $V(G) \setminus \{v\}$, contrary to the fact that G^c is connected. This proves (5).

From the symmetry we may assume that the first outcome of (5) holds. Let $K \in \mathcal{T}$. Then $x_j \in K$, $K \cap X_k = \emptyset$ (and, in particular, $k > i$), $j = 1$, and v is complete to $X_1 \setminus \{v\}$. It follows from the definition of a proper thickening that either $i = 1$ or v_1 is strongly adjacent to v_i in F. Consequently, $\bigcup_{p=1}^{i} X_p$ is a clique. Since v is anticomplete to $\bigcup_{p=k+1}^{n} X_p$, it follows that $K \cap (\bigcup_{p=k}^{n} X_p) = \emptyset$.

(6) *$\bigcup_{p=1}^{i-1} X_p \subseteq K$.*

In this proof we work with both F and G, and whenever we discuss adjacency, we will explicitly mention which graph or trigraph is in question. Suppose there exists $u \in \bigcup_{p=1}^{i-1} X_p \setminus K$. Then $i > 1$. Since K is a simplicial clique in G, it follows that u is adjacent to x_k in G. Let $t \in \{1, \dots, i-1\}$ be such that $u \in X_t$. Then v_t is adjacent to v_k in F. It follows from the definition of a linear interval trigraph that v_i is strongly adjacent to v_k in F, and therefore v is complete to X_k in G. Since v is anticomplete to $\bigcup_{p=k+1}^{n} X_p$ in G, it follows that v_i is strongly anticomplete to $\{v_{k+1}, \dots, v_n\}$ in F, and therefore v_t is strongly anticomplete to $\{v_{k+1}, \dots, v_n\}$ in F. This implies that u is anticomplete to $\bigcup_{p=k+1}^{n} X_p$ in G. The fact that $\{u, v\}$ is not a homogeneous set in G implies that u is not complete to X_k in G. But now, since v is adjacent to u and complete to X_k in G, and K is a simplicial clique of G, we deduce that $K \cap X_k \neq \emptyset$, a contradiction. This proves (6).

(7) *$K \cap (\bigcup_{p=i+1}^{k-1} X_p) = \emptyset$.*

Suppose there exists $u \in K \cap (\bigcup_{p=i+1}^{k-1}) X_p$. Since $K \setminus \{v\}$ is a simplicial clique of G, it follows that u is anticomplete to $\bigcup_{p=k+1}^{n} X_p$. Since K is a clique, u is complete to X_1, and therefore u is complete to $\bigcup_{p=1}^{i} X_i$. Since v has a neighbor in X_k, it follows from the definition of a proper thickening that v_i is adjacent to v_k in F, and therefore $\bigcup_{p=i}^{k-1} X_p$ is a clique. Let $t \in \{1, \dots, n\}$ be such that $u \in X_t$. Since $i < t < k$ we deduce that v_t is strongly adjacent to v_k, and so u is complete to X_k. Now since $\{u, v\}$ is not a homogeneous set in G, it follows that v is not complete to X_k, and so some vertex $w \in X_k$ is adjacent to u and nonadjacent to v. But $K \setminus \{v\}$ is a simplicial clique in G, and $v, w \in N(u) \setminus (K \setminus \{v\})$ are nonadjacent, a contradiction. This proves (7).

Let $X'_k = N(v) \cap X_k$. Let Y be the set of vertices in $X_i \setminus \{v\}$ that are complete to X'_k, and let $Z = X_i \setminus (Y \cup \{v\})$.

(8) *$Y \cap K = \emptyset$.*

Suppose there exists $u \in Y \cap K$. It follows from the maximality of k that v_i is strongly anticomplete to $\{v_{k+1}, \dots, v_t\}$, and therefore u is strongly anticomplete to $\bigcup_{p=k+1}^{n} X_p$. Since $\bigcup_{p=1}^{i} X_p$ and $\bigcup_{p=i}^{k-1} X_p$ are cliques, and since $\{u, v\}$ is not a homogeneous set in G, it follows that some vertex w of X_k is adjacent to u and not to v. But then $v, w \in N(u) \setminus (K \setminus \{v\})$, contrary to the fact that $K \setminus \{v\}$ is a simplicial clique in G. This proves (8).

(9) $Z \subseteq K$.

Suppose there exists $u \in Z \setminus K$. Let $w \in X'_k$ be a nonneighbor of u. Then $u, w \in N(v) \setminus K$, contrary to the fact that K is a simplicial clique in G. This proves (9).

Now it follows from (6)–(9) that $K = (\bigcup_{p=1}^{i-1} X_p) \cup \{v\} \cup Z$ for every $K \in \mathcal{T}$, and therefore $|\mathcal{T}| = 1$, as required. This proves Theorem 1.1. ∎

4. Growing prime graphs. In this section we prove a lemma that we hope is of independent interest. It is similar in spirit to Seymour's splitter theorem for 3-connected graphs [5]; the idea is that a prime graph (that is not obstinate) can be grown by adding one vertex at a time, starting from any of its prime induced subgraphs, and in such a way that all the graphs that are constructed along the way are prime. More precisely, we prove the following (a restatement of Theorem 1.2).

4.1. *Let G be a graph, and let H be a proper induced subgraph of G. Assume that both G and H are prime and that G is not obstinate. Then there exists an induced subgraph H' of G, isomorphic to H, and a vertex $v \in V(G) \setminus V(H')$ such that $G|(V(H') \cup \{v\})$ is prime.*

Here is the outline of the proof. First in Theorem 4.2 we deal with the easy case when H is not "controlling" (the definition is below). Our next step is to show that H can be grown to a prime graph by adding two vertices (this is Theorem 4.3, and we do not need to assume that G is not obstinate). Then we use this result to prove Theorem 4.1 in the case when H is not obstinate or H is a maximal obstinate induced subgraph of G (this is done in Theorem 4.5). Finally, we bridge the remaining gap using Theorem 4.6.

Next we need some definitions. Let H be an induced subgraph of G. Let us say that $v \in V(G) \setminus V(H)$ is *an H-clone* of $x \in V(H)$ if for every $y \in V(H) \setminus \{x\}$, v is adjacent to y if and only if $xy \in E(H)$. For $x \in V(H)$, let V_x^H be the set of H-clones of x. Let A^H be the set of vertices of $V(G) \setminus V(H)$ that are complete to $V(H)$ and B^H the set of vertices of $V(G) \setminus V(H)$ that are anticomplete to $V(H)$. We observe that if H is prime, then the sets A^H, B^H, and V_x^H (where $x \in V(H)$) are all pairwise disjoint. We say that H is *controlling (in G)* if every vertex of $V(G) \setminus V(H)$ either is an H-clone or belongs to $A^H \cup B^H$. We start with the following.

4.2. *Let G be a graph, and let H be a proper induced subgraph of G. Assume that H is prime and not controlling. Then there exists a vertex $v \in V(G) \setminus V(H)$ such that $G|(V(H) \cup \{v\})$ is prime.*

Proof. Let $v \in V(G) \setminus (V(H) \cup A^H \cup B^H)$ be a vertex that is not an H-clone. Suppose that $F = G|(V(H) \cup \{v\})$ is not prime. Then there is a nontrivial homogeneous set X in F. Since X is not a nontrivial homogeneous set in H, and $v \notin A^H \cup B^H$, it follows that $v \in X$. Since $X \setminus \{v\}$ is not a nontrivial homogeneous set in H, it follows that $|X \setminus \{v\}| = 1$. Let x be the unique vertex of $X \setminus \{v\}$. Now v is an H-clone of x, a contradiction. This proves that F is prime and completes the proof of Theorem 4.2. ∎

Let H be an induced subgraph of G, and let $u, v \in V(G) \setminus V(H)$. We call the pair uv *H-conforming* if either

- $u \in V_x^H$ and $v \in V_y^H$ for distinct $x, y \in V(H)$, and $uv \in E(G)$ if and only if $xy \in E(H)$, or

- $u \in A^H$ and $v \in V_x^H$ for some $x \in V(H)$, and u is adjacent to v, or
- $u \in B^H$ and $v \in V_x^H$ for some $x \in V(H)$, and u is nonadjacent to v.

We start with a lemma.

4.3. *Let G be a graph, and let H be a proper induced subgraph of G. Assume that both G and H are prime and that H is controlling. Then there exist $u, v \in V(G) \backslash V(H)$ such that $G|(V(H) \cup \{u, v\})$ is prime and the pair uv is not H-conforming.*

Note that while in Theorem 4.1 we may need to move to an isomorphic copy of H in G (which we denoted by H'), Theorem 4.3 states that we can add two vertices to a fixed subgraph H of G, keeping it prime.

Proof of Theorem 4.3. Let $k = |V(H)|$ and let the vertices of H be $v_1, \dots, v_k$. For $i \in \{1, \dots, k\}$, let $V_i = V_{v_i}^H$, $A = A^H$, and $B = B^H$. Since H is prime, the sets $V_1, \dots, V_k, A^H, B^H$ are all pairwise disjoint, and since H is controlling, $V(G) = V(H) \cup \bigcup_{i=1}^k V_i \cup A^H \cup B^H$. We observe that since H is prime, every vertex of H has both a neighbor and a nonneighbor in H, and therefore every vertex of $\bigcup_{i=1}^k V_i$ is mixed on $V(H)$.

Assume first that $A \cup B \neq \emptyset$. If A is complete to $V(H) \cup \bigcup_{i=1}^k V_i$ and B is anticomplete to $V(H) \cup \bigcup_{i=1}^k V_i$, then $V(H) \cup \bigcup_{i=1}^k V_i$ is a nontrivial homogeneous set in G, contrary to the fact that G is prime. Therefore (by passing to the complement and renumbering the vertices of H if necessary) we may assume that there exist $u \in V_1$ and $v \in B$ such that $uv \in E(G)$. We claim that $G|(V(H) \cup \{u, v\})$ is prime. Suppose not, and let X be a nontrivial homogeneous set in $G|(V(H) \cup \{u, v\})$. If $X \subseteq V(H)$, then, since X is not a nontrivial homogeneous set in H, it follows that $X = V(H)$, contrary to the fact that u is mixed on $V(H)$. This proves that at least one of $u, v \in X$. Since v is adjacent to u and anticomplete to $V(H)$, it follows that if $u \in X$ then $v \in X$. Thus we may assume that $v \in X$. Moreover, since $X \cap V(H)$ is not a nontrivial homogeneous set in H, it follows that either $|X \cap V(H)| \leq 1$ or $V(H) \subseteq X$. If $X \cap V(H) = \emptyset$, then $X = \{u, v\}$, which is a contradiction since every neighbor of v_1 in H is adjacent to u and nonadjacent to v; and if $V(H) \subseteq X$, then $X = V(H) \cup \{v\}$, which is a contradiction since u is mixed on $V(H)$. Thus $|X \cap V(H)| = 1$; let x be the unique vertex of $X \cap V(H)$. But now, since $v \in X$, it follows that every vertex of $V(H) \setminus \{x\}$ is anticomplete to X, and in particular x has no neighbor in H, a contradiction. This proves that $G|(V(H) \cup \{u, v\})$ is prime, and the theorem holds.

Therefore we may assume that $A \cup B = \emptyset$. If for every $i, j \in \{1, \dots, k\}$ and every $u \in V_i$ and $v \in V_j$ the pair $\{u, v\}$ is H-conforming, then each $V_i \cup \{v_i\}$ is a homogeneous set in G; since $V(H) \neq V(G)$, we deduce that at least one of these homogeneous sets is nontrivial, contrary to the fact that G is prime. This implies that there is at least one pair that is not H-conforming. By passing to the complement and renumbering the vertices of H if necessary, we may assume that $v_1v_2 \in E(H)$, and there exist $u_1 \in V_1$ and $u_2 \in V_2$ such that u_1 is nonadjacent to u_2. We claim that $G|(V(H) \cup \{u_1, u_2\})$ is prime. Suppose not, and let X be a nontrivial homogeneous set in $G|(V(H) \cup \{u_1, u_2\})$. Let

$$X' = \begin{cases} X & \text{if} \quad u_1, u_2 \notin X \\ (X \setminus \{u_1\}) \cup \{v_1\} & \text{if} \quad u_1 \in X \text{ and } u_2 \notin X \\ (X \setminus \{u_2\}) \cup \{v_2\} & \text{if} \quad u_2 \in X \text{ and } u_1 \notin X \\ (X \setminus \{u_1, u_2\}) \cup \{v_1, v_2\} & \text{if} \quad u_1, u_2 \in X. \end{cases}$$

Then X' is a homogeneous set in H, and since X' is not a nontrivial homogeneous set in H, it follows that either $|X'| \leq 1$ or $X' = V(H)$. Since u_1u_2 is not a conforming pair, it follows that $\{u_1, v_1\}$ and $\{u_2, v_2\}$ are not homogeneous sets of $G|(V(H) \cup \{u_1, u_2\})$, and therefore $|X'| > 1$. Consequently, $X' = V(H)$. This implies that $V(H) \setminus \{v_1, v_2\} \subseteq X$ and that $|X \cap \{u_1, v_1\}| \geq 1$ and $|X \cap \{u_2, v_2\}| \geq 1$. We observe that $V(H) \not\subseteq X$, since both u_1 and u_2 are mixed on $V(H)$, and so we may assume that $v_1 \notin X$ and $u_1 \in X$. But v_1 is complete to $\{u_2, v_2\}$, and therefore, since X is a homogeneous set, v_1 is complete to $V(H) \setminus \{v_1, v_2\}$. Consequently, u_2 has no nonneighbor in H, a contradiction. This proves that $G|(V(H) \cup \{u_1, u_2\})$ is prime and completes the proof of Theorem 4.3. ∎

Let P, Q be two graphs. Let us call a pair of disjoint sets (A, B) in $V(Q)$ *useful (relative to P, Q)* if

- $|A|, |B| \geq 2$,
- A is a homogeneous set in $Q \setminus B$ and B is a homogeneous set in $Q \setminus A$,
- each of A, B is either a clique or a stable set,
- there exists p such that each of A, B has size p or $p-1$, the vertices of A can be numbered $a_1, \ldots, a_p$ or $a_1, \ldots, a_{p-1}$, the vertices of B can be numbered $b_1, \ldots, b_p$ or $b_2, \ldots, b_p$, and a_i is adjacent to b_j if and only if $j \leq i$, and
- $Q_1 = Q \setminus \{a_1, b_2\}$ is isomorphic to P.

We call p the *order* of the pair (A, B). We observe that p is determined by the pair (A, B). It is not difficult to check that

- if (A, B) is a useful pair of order p relative to P, Q, then (B, A) is a useful pair of order p relative to P, Q;
- if (A, B) is a useful pair of order p relative to P, Q, then (A, B) is a useful pair of order $|A| + |B| + 1 - p$ relative to P^c, Q^c;
- $Q_i = Q \setminus \{a_i, b_{i+1}\}$ is isomorphic to P for all $i \in \{1, \ldots, p-1\}$; and
  - $Q^i = Q \setminus \{a_i, b_i\}$ is isomorphic to P for all $i \in \{2, \ldots, p-1\}$, and
  - if $|A| = p$, then $Q^p = Q \setminus \{a_p, b_p\}$ is isomorphic to P, and
  - if $|B| = p$, then $Q^1 = Q \setminus \{a_1, b_1\}$ is isomorphic to P.

We prove the following easy technical lemma.

4.4. *Let P, Q be graphs, and let (A, B) be a useful pair relative to P, Q. Then, with notation as in the definition of a useful pair,*

- $a_i \in V_{a_{i+1}}^{Q_i}$ *(that is, a_i is a Q_i-clone of a_{i+1}) for $i \in \{1, \ldots, p-2\}$, and*
  - *if $|A| = p$, then $a_{p-1} \in V_{a_p}^{Q_{p-1}}$, and*
  - *$b_{i+1} \in V_{b_i}^{Q_i}$ (that is, b_{i+1} is a Q_i-clone of b_i) for $i \in \{2, \ldots, p-1\}$, and*
  - *if $|B| = p$, then $b_2 \in V_{b_1}^{Q_1}$.*
- $a_i \in V_{a_{i-1}}^{Q^i}$ *and* $b_i \in V_{b_{i+1}}^{Q^i}$ *for all $i \in \{2, \ldots, p-1\}$, and*
  - *if $|A| = p$, then $a_p \in V_{a_{p-1}}^{Q^p}$, and*
  - *if $|B| = p$, then $b_1 \in V_{b_2}^{Q^1}$.*

Proof. Theorem 4.4 follows from the fact that if a_i and a_{i+1} both exist, then b_{i+1} is the only vertex of Q that is mixed on $\{a_i, a_{i+1}\}$, and a similar statement with the roles of A and B exchanged. ∎

Our next step is the following.

4.5. *Let G be a graph, and let H be a proper induced subgraph of G. Assume that both G and H are prime and that no obstinate induced subgraph of G has a proper induced subgraph isomorphic to H. Then there exists an induced subgraph H' of G, isomorphic to H, and a vertex $v \in V(G) \setminus V(H')$ such that $G|(V(H') \cup \{v\})$ is prime.*

Proof. By Theorem 4.2, we may assume that every induced subgraph of G isomorphic to H is controlling. Let $u, v \in V(G) \setminus V(H)$ be as in Theorem 4.3. Let $k = |V(H)|$ and let the vertices of H be $v_1, \ldots, v_k$. For $i \in \{1, \ldots, k\}$, let $V_i = V^H_{v_i}$, $A = A^H$, and $B = B^H$. We observe that both the hypotheses and the conclusion of Theorem 4.5 are invariant under taking complements, and we will make use of the symmetry between G and G^c in the course of the proof.

(1) *We may assume that there exists an induced subgraph H' of G, isomorphic to H, and a pair $u'v'$ such that $u' \in V^{H'}_x$ and $v' \in V^{H'}_y$ for some $x, y \in V(H')$, and $u'v'$ is not H'-conforming.*

By renumbering the vertices of H and passing to the complement if necessary, and since the pair uv is not H-conforming, we may assume that $u \in V_1$, $v \in B$ and u is adjacent to v. Then $H' = G|((V(H) \setminus \{v_1\}) \cup \{u\})$ is isomorphic to H. Let $F = G|(V(H') \cup \{v\})$. Since H' is controlling (because every induced subgraph of G isomorphic to H is) and v is mixed on $V(H')$, we may assume (renumbering $\{v_2, \ldots, v_k\}$ if necessary) that $v \in V^{H'}_{v_2}$ and $v_1 \in V^{H'}_u$. Now since $vv_1 \notin E(G)$ and $uv_2 \in E(H')$, it follows that the pair vv_1 is not H'-conforming, as required. This proves (1).

Let H', u', v' be as in (1). Since in Theorem 4.5 we are allowed to pass to an isomorphic copy of H, we may assume that $H' = H$, $u' = u$, and $v' = v$. By renumbering the vertices of H if necessary, we may assume that $u \in V_1$, $v \in V_2$. Let $F = G|(V(H) \cup \{u, v\})$. We remark that Theorem 4.3 implies that F is prime.

We observe that if $v_1v_2 \in E(H)$, then $(\{u, v_1\}, \{v_2, v\})$ is a useful pair of order two relative to H, F, and if $v_1v_2 \notin E(H)$, then $(\{u, v_1\}, \{v_2, v\})$ is a useful pair of order three relative to H, F. Let (A, B) be a useful pair relative to H, F such that $\{u, v_1\} \subseteq A$, $\{v_2, v\} \subseteq B$ and with $|A \cup B|$ maximum. By passing to the complement and exchanging A and B if necessary, we may assume that (A, B) has order $|A|$, and let $p = |A|$. Let the vertices of A and B be numbered as in the definition of a useful pair, and let F_i and F^i also be as in that definition. Then F_{p-1} is isomorphic to H, and $a_{p-1} \in V^{F_{p-1}}_{a_p}$ and $b_p \in V^{F_{p-1}}_{b_{p-1}}$; and F^p is isomorphic to H and $a_p \in V^{F^p}_{a_{p-1}}$. Let $K = F \setminus a_p$. Let X be the set of vertices of $F \setminus (A \cup B)$ that are complete to A and anticomplete to B, Y the set of vertices of $F \setminus (A \cup B)$ that are complete to B and anticomplete to A, Z the set of vertices of $F \setminus (A \cup B)$ that are complete to $A \cup B$, and W the set of vertices of $F \setminus (A \cup B)$ that are anticomplete to $A \cup B$. Then $V(F) = A \cup B \cup X \cup Y \cup Z \cup W$.

(2) *Either*

- *there exist $x \in V(K) \setminus \{b_p\}$ such that a_p is nonadjacent to x and $N_K(x) \setminus \{b_p\} = N_K(b_p) \setminus \{x\}$, or*
- *$N_F(b_p) = \{a_p\}$.*

Since $K \setminus b_p = F^p$ and is therefore isomorphic to H, it follows that $K \setminus b_p$ is controlling, and therefore either

- b_p is complete to $V(K) \setminus \{b_p\}$, or
- b_p is anticomplete to $V(K) \setminus \{b_p\}$, or
- b_p is a $K \setminus b_p$-clone of some vertex $x \in V(K) \setminus \{b_p\}$.

Since a_1 is nonadjacent to b_p, it follows that b_p is not complete to $V(K) \setminus \{b_p\}$. Since F is prime, and since a_p is adjacent to b_p, we deduce that either

- b_p is anticomplete to $V(K) \setminus \{b_p\}$ (and therefore $N_F(b_p) = \{a_p\}$, and the second outcome of (2) holds), or
- b_p is a $K \setminus b_p$-clone of some vertex $x \in V(K) \setminus \{b_p\}$ and a_p is nonadjacent to x, and the first outcome of (2) holds.

This proves (2).

Suppose first that the first outcome of (2) holds, and let $x \in V(K) \setminus \{b_p\}$ such that a_p is nonadjacent to x, and $N_K(x) \setminus \{b_p\} = N_K(b_p) \setminus \{x\}$. Since x is nonadjacent to a_p, it follows that $x \notin \{b_1, \ldots, b_{p-1}\} \cup X \cup Z$, and therefore $x \in \{a_1, \ldots, a_{p-1}\} \cup Y \cup W$. Assume first that $x \in \{a_1, \ldots, a_{p-1}\}$. Then, since $N_K(x) \setminus \{b_p\} = N_K(b_p) \setminus \{x\}$, it follows that $X \cup Y = \emptyset$, and since $A \cup B$ is not a nontrivial homogeneous set in F, it follows that $Z \cup W = \emptyset$. Now if A is a clique, then a_p is complete to $V(F) \setminus \{a_p\}$, contrary to the fact that F is prime, and therefore A is a stable set. If B is a clique, then either $|B| = p$ and b_1 is complete to $V(F) \setminus \{b_1\}$, or $|B| = p - 1$ and a_1 is anticomplete to $V(G) \setminus \{a_1\}$, in both cases contrary to the fact that F is prime. Consequently, both A and B are stable sets. But now F is obstinate, and F^p is a proper induced subgraph of F, again a contradiction since F^p is isomorphic to H. This proves that $x \notin \{a_1, \ldots, a_{p-1}\}$, and so $x \in Y \cup W$.

Let $B' = B \cup \{x\}$. We claim that (A, B') is a useful pair relative to H, F. The first condition is obvious. Since $N_K(x) \setminus \{b_p\} = N_K(b_p) \setminus \{x\}$, it follows that B' is a homogeneous set in $F \setminus A$. It also follows that $x \in Y$ if and only if B is a clique (and $x \in W$ if and only if B is a stable set), and therefore $B \cup \{x\}$ is either a clique or a stable set, and thus the third condition holds. Let $b_{p+1} = x$; since x is anticomplete to A, the fourth condition is satisfied. The fifth condition satisfied because (A, B) is a useful pair relative to H, F. This proves that the pair (A, B') is useful, contrary to the maximality of $A \cup B$. Thus our assumption that the first outcome of (2) holds, leading to a contradiction.

Therefore the second outcome of (2) holds, namely, $N_F(b_p) = \{a_p\}$. This implies that $Y \cup Z = \emptyset$ and B is a stable set. Suppose that $|B| = p$. It follows from the symmetry that $X = \emptyset$ and A is a stable set, and since $A \cup B$ is not a nontrivial homogeneous set in F, we deduce that $W = \emptyset$. But now F is isomorphic to O_p, which is a contradiction since F^p is a proper induced subgraph of F and F^p is isomorphic to H. This proves that $|B| = p - 1$, and therefore a_1 is anticomplete to B. But now, passing to the complement (with a_1 playing the role of a_p, b_2 playing the role of b_p, and considering the subgraph $K' = F^c \setminus \{a_1\}$ instead of K), we deduce that B is a clique, a contradiction. This proves Theorem 4.5. ∎

Since every prime induced subgraph of O_q is isomorphic to O_p for some $p \leq q$ (this is easy to verify, and we leave the details to the reader), Theorem 4.5 implies that Theorem 4.1 is true in the case when H is not an obstinate graph. The last step in the proof of Theorem 4.1 is to replace the relevant hypothesis of Theorem 4.5 by the assumption that G is not an obstinate graph. To do that, we start with a lemma.

4.6. *Let K be a prime graph, and let $v \in V(K)$ such that $K \setminus v$ is obstinate. Then for every obstinate induced subgraph H of $K \setminus v$, there exists a prime induced subgraph J of K and a vertex $v' \in V(J)$ such that $J \setminus v'$ is isomorphic to H.*

Proof. By Theorem 4.2 we may assume that every induced subgraph of K isomorphic to H is controlling. Let $F = K \setminus v$. If $F = H$, the result is trivial, so we may assume that H is a proper induced subgraph of F. We may also assume, by passing to the complement if necessary, that H is isomorphic to O_p and F is isomorphic to O_q, where p and q are integers and $p < q$. Then $p \geq 2$ and $q \geq 3$. If $|q - p| > 1$, the result follows inductively (by induction on $|q - p|$), so we may assume $q = p + 1$.

Let the vertices of F be numbered $a_1, \ldots, a_{p+1}$ and $b_1, \ldots, b_{p+1}$ as in the definition of O_{p+1}. For $i \in \{1, \ldots, p+1\}$ let $H_i = F \setminus \{a_i, b_i\}$. Then H_i is isomorphic to H. Let $K_i = K|(V(H_i) \cup \{v\})$. Since H_i is controlling, it follows that for every i, either

- v is complete to $V(H_i)$, or
- v is anticomplete to $V(H_i)$, or
- v is an H_i-clone of some vertex of H_i.

We claim that the last bullet holds either for $i = 1$ or for $i = p + 1$. Suppose not. Then v is either complete or anticomplete to each of $V(H_1), V(H_{p+1})$, and since $V(H_1) \cap V(H_{p+1}) \neq \emptyset$ and $V(H_1) \cup V(H_{p+1}) = V(F)$, it follows that v is either complete or anticomplete to $V(F)$, contrary to the fact that K is prime. This proves the claim. From the symmetry we may assume that v is an H_1-clone of a_i for some $i \in \{2, \dots, p + 1\}$. Then v is anticomplete to $\{a_2, \dots, a_{p+1}\} \setminus \{a_i\}$, complete to $\{b_2 \dots, b_i\}$, and anticomplete to $\{b_{i+1}, \dots, b_{p+1}\}$.

Suppose that v is complete or anticomplete to $V(H_{p+1})$. Assume first that $p = i = 2$. Then v is adjacent to b_2, and therefore v is complete to $V(H_{p+1})$. But now the theorem holds setting $J = K|\{a_1, b_1, a_3, b_2, v\}$ and $v' = v$. Thus we may assume that either $p > 2$, or $p = 2$ and $i \neq 2$. Now for some $j \in \{1, \dots, p+1\} \setminus \{1, i, p+1\}$, v is nonadjacent to a_j and has a neighbor in $\{b_1, \dots, b_p\}$; therefore v is not complete and not anticomplete to $V(H_{p+1})$, a contradiction. Thus we may assume that v is not complete and not anticomplete to $V(H_{p+1})$. Consequently, v is an H_{p+1}-clone of some vertex x of H_{p+1}.

Suppose that $x \in \{b_1, \dots, b_p\}$. Since v is complete to $\{b_2, \dots, b_i\}$ and B is a stable set, it follows that $i = 2$ and $x = b_2$. If $p = 2$, then $N_K(v) = \{a_2, b_2\}$, and the theorem holds setting $J = K|\{a_2, a_3, b_1, b_2, v\}$ and $v' = b_2$. Thus we may assume that $p > 2$. Now a_p is adjacent to x and nonadjacent to v, contrary to the fact that v is an H_{p+1}-clone of x. This proves that $x \in \{a_1, \dots, a_p\}$. In particular, v is adjacent to b_1 and nonadjacent to a_1. But now $\{a_i, v\}$ is a homogeneous set in K, a contradiction. This proves Theorem 4.6. ∎

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let G and H be as in Theorem 4.1. If no obstinate induced subgraph of G has a proper induced subgraph isomorphic to H, then the result follows from Theorem 4.5. So we may assume that some obstinate induced subgraph F of G has a proper induced subgraph isomorphic to H. Choose F with $|V(F)|$ maximum. Then $F \neq G$, and no obstinate induced subgraph of G has a proper induced subgraph isomorphic to F. By Theorem 4.5, there exists $v \in V(G) \setminus V(F)$ such that $K = G|(V(F) \cup \{v\})$ is prime. But now, since H is isomorphic to an induced subgraph of F, Theorem 4.6 implies that there is a prime induced subgraph J of K and a vertex $v' \in V(J)$ such that $H' = J \backslash v'$ is isomorphic to H. Now $G|(V(H') \cup \{v'\})$ is prime, as required. This proves Theorem 4.1. ∎

5. Finding simplicial cliques. In this section we use Theorem 1.1 and Theorem 1.3 to give an algorithm that finds all simplicial cliques of a prime claw-free graph. First we show the following.

5.1. *There is an algorithm with the following specifications:*

- Input: *A graph G.*
- Output: *Either*
  1. *a true determination that G is isomorphic to O_k for some $k \geq 2$ and an ordering $a_1, \dots, a_k, b_1, \dots, b_k$ of the vertices of G as in the definition of O_k, or*
  2. *a true determination that G^c is isomorphic to O_k for some $k \geq 3$ and an ordering $a_1, \dots, a_k, b_1, \dots, b_k$ of the vertices of G^c as in the definition of O_k, or*
  3. *a true determination that G is not obstinate.*
- Running time: $O(|V(G)|^2)$.

Proof. If $|V(G)|$ is odd, output "G is not obstinate" and stop. Let $|V(G)| = 2k$, and calculate the degree sequence of G. If the degree sequence of G is not

$$1, 1, 2, 2, \dots, k, k$$

or

$$k-1, k-1, k, k, \dots, 2k-2, 2k-2,$$

output "G is not obstinate" and stop. If the degree sequence of G is

$$1, 1, 2, 2, \dots, k, k,$$

let $H = G$, and if $k \geq 3$ and the degree sequence of G is

$$k-1, k-1, k, k, \dots, 2k-2, 2k-2$$

let $H = G^c$. Let a_k and b_1 be the two vertices of H of degree k. Let $A = N_H(b_1)$ and $B = N_H(a_k)$, and let $D_A = \{deg_H(v)\}_{v \in A}$ and $D_B = \{deg_H(v)\}_{v \in B}$. If one of A, B is not stable or $A \cap B \neq \emptyset$, output "G is not obstinate" and stop. If $D_A \neq \{1, \dots, k\}$ or $D_B \neq \{1, \dots, k\}$, output "G is not obstinate" and stop. For $i \in \{1, \dots, k\}$, let a_i be the vertex of A with degree i, and let b_i be the vertex of B with degree $k+1-i$. Now check whether a_i is adjacent to b_j if and only if $j \leq i$. If not, output "G is not obstinate" and stop. If $G = H$, output "G is isomorphic to O_k"; if $G^c = H$, output "G^c is isomorphic to O_k". In both cases output

$$a_1, \dots, a_k, b_1, \dots, b_k.$$

It is easy to check that the complexity of this algorithm in $O(|V(G)|^2)$ and that the algorithm works correctly. This proves Theorem 5.1. ∎

5.2. *There is an algorithm with the following specifications:*
- Input: *A prime claw-free graph G.*
- Output: *A list $\mathcal{L}$ of all simplicial cliques of G.*
- Running time: $O(|V(G)|^4)$.

Proof. First, run Theorem 5.1 on G. This takes time $O(|V(G)|^2)$. If G is isomorphic to O_2, then output

$$\mathcal{L} = \{\{a_1\}, \{a_1, b_1\}, \{a_2, b_1\}, \{a_2, b_2\}, \{b_2\}\}$$

and stop. This takes constant time. If G^c is isomorphic to O_k for $k \geq 3$, let $A_i = \{a_i, \dots, a_k\}$, $B_i = \{b_1, \dots, b_i\}$, output

$$\mathcal{L} = \{A_1, \dots, A_k, B_1, \dots, B_k\},$$

and stop. This takes time $O(|V(G)|^2)$. Now, for every $v \in V(G)$, check if $G \setminus v$ is prime. This can be done in time $O(|V(G)|^2)$ for each v by [4]. Let v_0 be such that $G' = G \setminus v_0$ is prime. Recursively, run the algorithm on G', and let $\mathcal{L}'$ be the list of all simplicial cliques in G'. Now, for every $K \in \mathcal{L}'$, check if K is simplicial in G, and add K to $\mathcal{L}$ if the answer is yes. This can be done in time $O(|V(G)|^2)$ for each K, and so, since by Theorem 1.1 $|\mathcal{L}'| \leq |V(G)|$, the total running time of this step is $O(|V(G)|^3)$. Next, check if v_0 is a simplicial vertex, and add $\{v_0\}$ to $\mathcal{L}$ if the answer is yes. This takes time $O(|V(G)|^2)$. Finally, for every $K \in \mathcal{L}'$, check if $K \cup \{v_0\}$ is a simplicial clique in G, and add $K \cup \{v_0\}$ to $\mathcal{L}$ if the answer is yes. This can be done in time $O(|V(G)|^2)$ for each K, and again, since $|\mathcal{L}'| \leq |V(G)|$, the total running time of this step is $O(|V(G)|^3)$. This completes the algorithm.

It is clear that the running time of this algorithm is $O(|V(G)|^4)$, so it remains to prove correctness. If G is obstinate, the correctness of the algorithm follows from Theorem 3.2, and so we may assume that G is not obstinate. Since G is prime, it follows that G and G^c are both connected, and therefore there is an induced subgraph H of G isomorphic to the three-edge path. Then, by Theorem 1.3, applied to H and G, there exists $v_0 \in V(G)$ such that $G' = G \setminus v_0$ is prime (taking v_0 to be the vertex $v_{|V(G)|-4}$ in the statement of Theorem 1.3), and so the algorithm will find the graph

G' and the list $\mathcal{L}'$. We observe that if $K \neq \{v\}$ is a simplicial clique in G and $v \in K$, then $K \setminus \{v\}$ is a simplicial clique in G'; and if K is a simplicial clique in G and $v \notin K$, then K is simplicial in G'. From this observation, it follows that $\mathcal{L}$ is indeed the list of all simplicial cliques of G. This proves Theorem 5.2. □

6. Simplicial cliques in general claw-free graphs. In this section we consider the problem of finding the simplicial cliques in a general claw-free graph. We cannot necessarily list them all in polynomial time, because there may be exponentially many; for instance, if G is an n-vertex clique, then it has $2^n - 1$ simplicial cliques. On the other hand, if G is a clique, then it is easy to say what its simplicial cliques are: any subset except the empty set. We might hope for a similar description in general, and indeed it exists, as we shall see. We omit the proofs because they are all easy.

Thus, let G be a claw-free graph with $|V(G)| > 1$. Suppose first that both G and G^c are connected. Then there is a prime graph H such that G is a thickening of H; let $V(H) = \{v_1, \dots, v_k\}$, and let $V_{v_i} = V_i$ be as in the definition of thickening. Then H is prime, and therefore we can enumerate all simplicial cliques of H in polynomial time. For each simplicial clique X of H, and for $1 \leq i \leq k$ with $v_i \in X$, choose $Y_i \subseteq V_i$, where

- if v_i is adjacent to every vertex in $V(H) \setminus X$ with a neighbor in X, then Y_i is an arbitrary nonempty subset of V_i (there is at most one such v_i in X because H is prime and therefore no two members of X are twins);
- otherwise, $Y_i = V_i$.

Let Y be the union of the sets Y_i ($v_i \in X$); then Y is a simplicial clique of G, and every simplicial clique of G arises in this way.

Now suppose that G^c is not connected, and let $G_1, \dots, G_k$ be the connected components of G^c. Certainly if $(X, V(G) \setminus X)$ is a bipartition of G^c (that is, a partition into two stable sets), then X is a simplicial clique of G; we need to describe the other simplicial cliques. We may assume that for some $t \in \{1, \dots, k\}$, each of the components $G_1, \dots, G_t$ has at least two vertices, and each of the components $G_{t+1}, \dots, G_k$ has exactly one vertex. Then every simplicial clique of G meets each of $G_1, \dots, G_t$. Moreover, if K is not contained in $V(G_i)$, where $i \in \{1, \dots, t\}$, then $V(G_i) \setminus K$ is also a clique of G. Thus if $t \geq 2$, then all simplicial cliques of G arise from bipartitions of G^c, and so we may assume that $t = 1$. But now the simplicial cliques of G are those that arise from bipartitions, together with the simplicial cliques of $G|V(G_1)$. ($G|V(G_1)$ might not be connected, but in that case it has exactly two components, both cliques, so in all cases we can describe the simplicial cliques of $G|V(G_1)$.)

Finally, if G is not connected, its simplicial cliques are just those of its components.

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