# Rooted grid minors

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**Abstract**

Intuitively, a *tangle* of large order in a graph is a highly-connected part of the graph, and it is known that if a graph has a tangle of large order then it has a large grid minor. Here we show that for any $k$, if $G$ has a tangle of large order and $Z$ is a set of vertices of cardinality $k$ that cannot be separated from the tangle by any separation of order less than $k$, then $G$ has a large grid minor containing $Z$, in which the members of $Z$ all belong to the outside of the grid. This is a lemma for use in a later paper.

# 1 Introduction

A *separation* of *order* $k$ in a graph $G$ is a pair $(A, B)$ of subgraphs of $G$ such that $A \cup B = G$, $E(A \cap B) = \emptyset$, and $|V(A \cap B)| = k$. A *tangle* in $G$ of *order* $\theta \geq 1$ is a set $\mathcal{T}$ of separations of $G$, all of order less than $\theta$, such that

- for every separation $(A, B)$ of order less than $\theta$, $\mathcal{T}$ contains one of $(A, B), (B, A)$
- if $(A_i, B_i) \in \mathcal{T}$ for $i = 1, 2, 3$, then $A_1 \cup A_2 \cup A_3 \neq G$
- if $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

Let $G, H$ be graphs. A *pseudomodel* of $H$ in $G$ is a map $\eta$ with domain $V(H) \cup E(H)$, where

- for every $v \in V(H)$, $\eta(v)$ is a non-null subgraph of $G$, all pairwise vertex-disjoint
- for every edge $e$ of $H$, $\eta(e)$ is an edge of $G$, all distinct
- if $e \in E(H)$ and $v \in V(H)$ then $e \notin E(\eta(v))$
- for every edge $e = uv$ of $H$, if $u \neq v$ then $\eta(e)$ has one end in $V(\eta(u))$ and the other in $V(\eta(v))$; and if $u = v$, then $\eta(e)$ is an edge of $G$ with all ends in $V(\eta(v))$.

If in addition we have

- $\eta(v)$ is connected for each $v \in V(H)$

then we call $\eta$ a *model* of $H$ in $G$. Thus, $G$ contains $H$ as a minor if and only if there is a model of $H$ in $G$. If $\eta$ is a pseudomodel of $H$ in $G$, and $F \subseteq V(H)$, we denote

$$\bigcup(V(\eta(v)) \ : v \in F)$$

by $\eta(F)$; and if $F$ is a subgraph of $H$, $\eta(F)$ denotes the subgraph of $G$ formed by the union of all the subgraphs $\eta(v)$ for $v \in V(F)$ and all the edges $\eta(e)$ for $e \in E(F)$.

For $g \geq 1$, the $g \times g$*-grid* has vertex set $\{v_{i,j} \ : 1 \leq i, j \leq g\}$, and vertices $v_{i,j}, v_{i',j'}$ are adjacent if $|i' - i| + |j' - j| = 1$. We denote this graph by $\mathcal{G}_g$. For $1 \leq i \leq g$, we call $\{v_{i,1}, v_{i,2}, \dots, v_{i,g}\}$ a *row* of the grid, and define the *columns* of the grid similarly.

The following was proved in [2, 3]:

**1.1** *For all $g \geq 1$ there exists $K \geq 1$ with the following property. Let $\mathcal{T}$ be a tangle of order at least $K$ in a graph $G$. Then there is a model $\eta$ of $\mathcal{G}_g$ in $G$, such that for each $(A, B) \in \mathcal{T}$, if $\eta(R) \subseteq V(A)$ for some row $R$ of the grid, then $(A, B)$ has order at least $g$.*

Our objective here is an analogous result, for graphs with some vertices distinguished, the following:

**1.2** *For all $k, g$ with $1 \leq k \leq g$ there exists $K \geq 1$ with the following property. Let $\mathcal{T}$ be a tangle of order at least $K$ in a graph $G$, and let $Z \subseteq V(G)$ with $|Z| = k$. Suppose that there is no separation $(A, B) \in \mathcal{T}$ of order less than $k$ with $Z \subseteq V(A)$. Then there is a model $\eta$ of $\mathcal{G}_g$ in $G$, such that*

- *for* $1 \leq i \leq k$, $V(\eta(v_{i,1}))$ *contains a member of* $Z$

- *for each* $(A, B) \in \mathcal{T}$, *if* $\eta(R) \subseteq V(A)$ *for some row* $R$ *of the grid, then* $(A, B)$ *has order at least* $g$.

A form of this result is implicit in a paper of Bruce Reed (statement 5.5 of [1]), but what we need is not explicitly proved there, so it seems necessary to do it again. It has as an immediate corollary the following (the proof of which is clear):

**1.3** *Let* $H$ *be a planar graph, drawn in the plane, and let* $v_1, \ldots, v_k$ *be distinct vertices of* $H$, *each incident with the infinite region. Then there exists* $K$ *with the following property. Let* $\mathcal{T}$ *be a tangle of order at least* $K$ *in a graph* $G$, *and let* $Z \subseteq V(G)$ *with* $|Z| = k$ *such that there is no separation* $(A, B) \in \mathcal{T}$ *of order less than* $k$ *with* $Z \subseteq V(A)$. *Then there is a model* $\eta$ *of* $H$ *in* $G$ *such that for* $1 \leq i \leq k$, $\eta(v_i)$ *contains a vertex of* $Z$.

We remark that this is best possible, in the sense that the hypotheses that $H$ is planar and $v_1, \ldots, v_k$ are all incident with a common region are both necessary for the conclusion to hold. To see this, let $G$ be an arbitrarily large grid, let $\mathcal{T}$ be the corresponding tangle, and let $Z$ be some set of $k$ vertices of $G$ all incident with the infinite region of $G$. If there is a model $\eta$ as in 1.3 then $H$ is planar and $v_1, \ldots, v_k$ all belong to the same region of $H$.

# 2 The main proof

To prove 1.2, it is convenient to prove something a little stronger, which we explain next. Let $H$ be a subgraph of $G$. We define $\beta_G(H)$ to be the set of vertices of $H$ incident with an edge of $G$ that does not belong to $E(H)$, and call $\beta_G(H)$ the *boundary* of $H$ in $G$. If $f \in E(G)$, $G/f$ denotes the graph obtained from $G$ by contracting $f$.

Let $G$ be a graph and $Z \subseteq V(G)$ with $|Z| = k$. Let $\eta$ be a model of $\mathcal{G}_g$ in $G$. We say $\eta$ is *$Z$-augmentable* in $G$ if there is a model $\eta'$ of $\mathcal{G}_g$ in $G$, and we can label the vertices of $\mathcal{G}_g$ as usual, such that

- for $1 \leq i \leq g$ and $2 \leq j \leq g$, $\eta'(v_{i,j}) = \eta(v_{i,j})$

- for $1 \leq i \leq g$, $\eta'(v_{i,1}) = \eta(v_{i,1})$ if $i > k$, and $\eta'(v_{i,1}) \supseteq \eta(v_{i,1})$ if $i \leq k$

- for $1 \leq i \leq k$, $V(\eta'(v_{i,1}))$ contains a member of $Z$

- for each $e \in E(\mathcal{G}_g)$, $\eta'(e) = \eta(e)$.

In this case we call $\eta'$ a *$Z$-augmentation* of $\eta$ in $G$.

**2.1** *Let* $g, k$ *be integers with* $g \geq k \geq 1$, *and let* $n$ *be an integer such that* $n > (k+1)(g+2k)$. *Let* $G$ *be a graph, and let* $Z \subseteq V(G)$ *with* $|Z| = k$. *Let* $J$ *be a subgraph of* $\mathcal{G}_n$, *with boundary* $\beta$, *including at least one row of* $\mathcal{G}_n$. *Let* $\eta$ *be a pseudomodel of* $J$ *in* $G$. *Suppose that*

*(i) for each* $v \in V(J)$, *either* $\eta(v)$ *is connected and* $v \notin \beta$, *or every component of* $\eta(v)$ *contains a vertex of* $Z$

*(ii) there is no separation $(A, B)$ of $G$ of order less than $k$ such that $Z \subseteq V(A)$ and there is a row $R$ of $\mathcal{G}_n$ with $R \subseteq V(J)$ and $\eta(R) \subseteq V(B)$.*

*Then there is a subgraph $H$ of $J$, isomorphic to $\mathcal{G}_g$, such that $Z \cap V(\eta(H))$ is null, and the restriction of $\eta$ to $H$ is $Z$-augmentable.*

**Proof.** To aid the reader's intuition, we point out first that since $\eta(v)$ meets $Z$ for each $v \in \beta$, and $|Z| = k$, it follows that $|\beta| \leq k$; and since $J$ includes a row of $\mathcal{G}_n$ and its boundary has cardinality at most $k$, it follows that $J$ consists of "almost all" of $\mathcal{G}_n$. We will say this more precisely later. We proceed by induction on $|V(G)| + |E(G)|$.

(1) *We may assume that there is no separation $(A, B)$ of $G$ of order $k$ with $B \neq G$ such that $Z \subseteq V(A)$ and there is a row $R$ of $\mathcal{G}_n$ with $R \subseteq V(J)$ and $\eta(R) \subseteq V(B)$.*

For suppose that $(A, B)$ is such a separation. Let $J'$ be the subgraph of $J$ with vertex set those $v \in V(J)$ with $\eta(v) \cap B$ non-null, and with edge set all edges $e$ of $J$ such that $\eta(u) \cap A$ is null for some end $u$ of $e$. (Note that if $e = uv \in E(J)$ and $\eta(u) \cap A$ is null, then both ends of $\eta(e)$ belong to $V(B)$, and so $\eta(u) \cap B, \eta(v) \cap B$ are both nonempty; and hence $J'$ is well-defined.) Let $\beta'$ be the boundary of $J'$ in $\mathcal{G}_n$. By the assumption that $\eta(R) \subseteq V(B)$, it follows that $V(J')$ includes at least one row of $\mathcal{G}_n$. Let $Z' = V(A \cap B)$. Then $|Z'| = k$. For each $v \in V(J')$, let $\eta'(v) = \eta(v) \cap B$ (note that $\eta(v) \cap B$ is non-null from the definition of $V(J')$), and for each $e \in E(J')$, let $\eta'(e) = \eta(e)$ (note that $\eta(e) \in E(B)$ from the definition of $E(J')$.) Thus $\eta'$ is a pseudomodel of $J'$ in $B$.

We claim that there is no separation $(A', B')$ of $A$, of order less than $k$, with $Z \subseteq V(A)$ and $Z' \subseteq V(B)$. For suppose that there is such a separation $(A', B')$. Then $(A', B' \cup B)$ is a separation of $G$ of the same order (and hence of order less than $k$), with $Z \subseteq V(A')$; and moreover, the row $R$ satisfies $R \subseteq V(J)$ and $\eta(R) \subseteq V(B) \subseteq V(B' \cup B)$, contrary to hypothesis (ii) of the theorem. This proves there is no such separation $(A', B')$. Since $|Z| = |Z'| = k$, there are $k$ paths of $A$, pairwise vertex-disjoint, each with one end in $Z$ and the other in $Z'$, and each with no other vertex in $Z'$ (and therefore with no vertex in $B$ except its end in $Z'$). Let us name these paths $P_{z'}$ ($z' \in Z'$), where $z' \in Z'$ is the unique vertex of $P_{z'}$ in $V(B)$.

We claim that there is no separation $(A', B')$ of $B$ of order less than $k$ such that $Z' \subseteq V(A')$ and there is a row $R$ of $\mathcal{G}_n$ with $R \subseteq V(J')$ and $\eta(R) \subseteq V(B')$. For suppose there is such a separation $(A', B')$. Then $(A \cup A', B')$ is a separation of $G$. Moreover, $(A \cup A') \cap B' = A' \cap B'$, since

$$A \cap B' \subseteq A \cap B = Z' \subseteq A'.$$

But this contradicts hypothesis (ii) of the theorem.

Now let $v \in V(J')$. We will show that either $\eta'(v)$ is connected and $v \notin \beta'$, or every component of $\eta'(v)$ contains a vertex of $Z'$. We may assume that some component $C'$ of $\eta'(v)$ is disjoint from $Z'$. Let $C$ be the component of $\eta(v)$ containing $C'$. If $C \neq C'$, then some vertex $u \in V(C')$ is adjacent in $C$ to some vertex $v \in V(C) \setminus V(C')$, and consequently $v \notin V(B)$; but then $u \in V(A \cap B) = Z'$, a contradiction. So $C = C'$. If some vertex of $C$ is in $Z$, then that vertex belongs to $V(A)$ and hence to $Z'$, a contradiction. Thus no vertex of $C$ is in $Z$. It follows from hypothesis (i) of the theorem that $\eta(v)$ is connected and $v \notin \beta$. In particular, since

$$C' \subseteq \eta'(v) \subseteq \eta(v) = C = C'$$

it follows that $\eta'(v)$ is connected. It remains to check that $v \notin \beta'$. Thus, suppose $v \in \beta'$, and so $v$ is incident in $\mathcal{G}_n$ with some edge $f = uv$ of $\mathcal{G}_n$ where $f \notin E(J')$. Since $v \notin \beta$, it follows that $f \in E(J)$ and $u \in V(J)$. Since $f \notin E(J')$, both of $\eta(u), \eta(v)$ have non-null intersection with $A$. But then $V(\eta(v))$ meets $Z'$, a contradiction. This proves that for every $v \in V(J')$, either $\eta'(v)$ is connected and $v \notin \beta'$, or every component of $\eta'(v)$ contains a vertex of $Z'$.

Consequently, we may apply the inductive hypothesis with $G, Z, J, \eta, \beta$ replaced by $B, Z', J', \eta', \beta'$. We deduce that there is a subgraph $H$ of $J'$, isomorphic to $\mathcal{G}_g$, such that $Z' \cap V(\eta'(H))$ is null, and the restriction of $\eta'$ to $H$ is $Z'$-augmentable in $B$. Let $v \in V(H)$. Since $Z' \cap V(\eta'(v)) = \emptyset$, it follows that $\eta'(v) = \eta(v)$. We deduce that the restriction of $\eta$ to $H$ is $Z'$-augmentable in $B$. Consequently there is a model $\eta_1$ of $H$ in $B$, satisfying the four statements in the definition of "$Z$-augmentable" (with $G, Z, \eta, \eta'$ replaced by $B, Z', \eta', \eta_1$). In particular for each vertex $v$ of $H$, if $v$ is the first vertex of one of the first $k$ rows of $H$ then $\eta_(v)$ contains a unique vertex of $Z'$, and for all other $v$, $V(\eta_1(v)) \cap Z' = \emptyset$. For each $v \in V(H)$, define $\eta_2(v)$ as follows: if $V(\eta_1(v)) \cap Z' = \{z'\}$ for some $z' \in Z'$, let $\eta_2(v) = \eta_1(v) \cup P_{z'}$, and if $V(\eta_1(v)) \cap Z' = \emptyset$ let $\eta_2(v) = eta_1(v)$. It follows that $\eta_2$ is a $Z$-augmentation of $\eta_1$ and hence of $\eta$ in $G$, and so the theorem holds. This proves (1).

(2) *We may assume that for every $f \in E(G)$, there exists $e \in E(J)$ such that $f = \eta(e)$. Consequently for each $v \in V(J)$, either $\eta(v)$ has only one vertex, or $V(\eta(v)) \subseteq Z$.*

Let $f \in E(G)$, and suppose there is no such $e$. Suppose first that there is no $u \in V(J)$ with $f \in E(\eta(u))$. It follows that $\eta$ is a pseudomodel of $J$ in $G \setminus f$. By (1), hypothesis (ii) of the theorem holds for $G \setminus f, Z, J, \eta, \beta$; and the other hypothesis holds trivially. Thus from the inductive hypothesis, the theorem holds for $G \setminus f, Z, J, \eta, \beta$ and hence for $G, Z, J, \eta, \beta$. We may therefore assume that there exists $u \in V(J)$ with $f \in E(\eta(u))$. If $f$ is a loop or both ends of $f$ belong to $Z$, define $\eta'(u) = \eta(u) \setminus f$, and $\eta'(v) = \eta(v)$ for every other vertex $v$ of $J$; then $\eta'$ is a pseudomodel of $J$ in $G \setminus f$, and again the result follows from the inductive hypothesis. Thus we may assume that $f$ is not a loop and some end of $f$ does not belong to $Z$. Let $f = ab$ say, and let $c$ be the vertex of $G/f$ formed by identifying $a, b$ under contraction. Define $\eta'(u) = \eta(u)/f$, and $\eta'(v) = \eta(v)$ for every other vertex $v$ of $J$; then $\eta'$ is a pseudomodel of $J$ in $G/f$. Let $Z'$ be the set of vertices $z'$ of $G/f$ such that either $z' \in Z$ and $z' \neq a, b$, or $z' = c$ and one of $a, b \in Z$. Then $|Z'| = |Z| = k$ since not both $a, b \in Z$. Suppose that there is a separation $(A', B')$ of $G/f$ of order less than $k$ such that $Z' \subseteq V(A')$ and there is a row $R$ of $\mathcal{G}_n$ with $R \subseteq V(J)$ and $\eta'(R) \subseteq V(B)$. If $c \notin V(A')$ let $A = A'$, and if $c \in A'$ let $A$ be the subgraph of $G$ with $f \in E(A)$ such that $A/f = A'$; and define $B$ similarly. Then $(A, B)$ is a separation of $G$, of order at most one more than the order of $(A', B')$, and hence at most $k$. Since $Z' \subseteq V(A')$ and $|Z'| = k$, it follows that $Z' \not\subseteq V(B')$, and so $V(B') \neq V(G')$; and therefore $V(B) \neq V(G)$. But $Z \subseteq V(A)$, and by (1) this is impossible. It follows that there is no such $(A', B')$; and so the result follows from the inductive hypothesis. This proves the first assertion of (2), and the second follows.

Now let us label the vertices of $\mathcal{G}_n$ as usual. Let $Z'$ be the set of all vertices $v$ of $\mathcal{G}_n$ such that $Z \cap V(\eta(v)) \neq \emptyset$. Since $|Z| = k$ it follows that $|Z'| \leq k$, and $\beta \subseteq Z'$ from hypothesis (i).

(3) *There is a subgraph $H_0$ of $J$, isomorphic to $\mathcal{G}_{g+2k}$, such that every row of $\mathcal{G}_n$ that intersects $V(H_0)$ is a subset of $V(J) \setminus Z'$.*

From the choice of $n$, there are $k+1$ subgraphs of $\mathcal{G}_n$, each isomorphic to $\mathcal{G}_{g+2k}$, such that no row of $\mathcal{G}_n$ meets more than one of them. Consequently there is a subgraph $H_0$ of $\mathcal{G}_n$, isomorphic to $\mathcal{G}_{g+2k}$, such that no row of $\mathcal{G}_n$ meets both $V(H_0)$ and $Z'$. Let $H'$ be the subgraph of $\mathcal{G}_n$ induced on the union of the rows of $\mathcal{G}_n$ that meet $V(H_0)$. We claim that every vertex of $H'$ belongs to $J$. For suppose not; then none of them belong to $J$, since $H'$ is connected and none of its vertices belong to $\beta \subseteq Z'$. Since there is a row $R$ of $\mathcal{G}_n$ with $R \subseteq V(J)$, it follows that every column of $\mathcal{G}_n$ meets both $V(J)$ and $V(H')$, and therefore meets $\beta$ and hence $Z'$. But $|Z'| \leq k < n$, a contradiction. This proves (3).

Since $V(H_0) \cap Z' = \emptyset$, (2) implies that $|V(\eta(v))| = 1$ for each $v \in V(H_0)$. Choose $i_0, j_0$ such that

$$V(H_0) = \{v_{i,j} : \ i - i_0, j - j_0 \in \{-k, \ldots, k+g-1\}\}$$

Let $H$ be the subgraph of $H_0$ induced on the vertex set

$$\{v_{i,j} : i - i_0, j - j_0 \in \{0, 1, \ldots, g-1\}\}.$$

and let

$$L = \{v_{i,j_0} : \ i - i_0 \in \{0, 1, \ldots, k-1\}\}.$$

Thus $H$ is isomorphic to $\mathcal{G}_g$, and $L$ is the set of first vertices of the first $k$ rows of $H$. Let $G^* = G \setminus \eta(V(H) \setminus L)$. We may assume (for a contradiction) that

(4) *There is a separation $(A, B)$ of $G^*$ of order less than $k$, such that $Z \subseteq V(A)$ and $\eta(L) \subseteq V(B)$.*

For if not, then since $|\eta(L)| = k$, by Menger's theorem there are $k$ vertex-disjoint paths $P_v$ ($v \in L$) of $G^*$ between $\eta(L)$ and $Z$, where for each $v \in L$, the unique vertex of $\eta(v)$ belongs to $P_v$. For each $v \in V(H)$, let $\eta'(v) = \eta(v)$ if $v \notin L$, and $\eta'(v) = \eta(v) \cup P_v$ if $v \in L$. Let $\eta(e) = \eta(e)$ for each edge $e$ of $H$. Then $\eta'$ is a $Z$-augmentation of $H$, and the theorem holds. This proves (4).

Let $X = V(A \cap B)$, and let

$$\begin{aligned} A' = &\ \{v \in V(J) : \quad \eta(v) \cap V(A) \neq \emptyset\} \\ B' = &\ \{v \in V(J) : \quad \eta(v) \cap V(B) \neq \emptyset\} \\ X' = &\ \{v \in V(J) : \quad \eta(v) \cap X \neq \emptyset\}. \end{aligned}$$

(5) *The following hold:*

- *If $v \in A'$, then every component of $\eta(v)$ contains a vertex of $V(A)$.*
- *If $v \in B' \setminus A'$, then $\eta(v)$ is connected.*
- *$A' \cap B' = X'$.*
- *If $a' \in A' \setminus B'$ and $b' \in B' \setminus A'$ then $a', b'$ are not adjacent in $J$.*
- *If $C$ is a connected subgraph of $\mathcal{G}_n$ disjoint from $X' \cup (V(H) \setminus L)$ and with non-empty intersection with $B'$ then $C$ is a subgraph of $J$ and $V(C) \subseteq B' \setminus A'$.*

For the first bullet, let $v \in A'$; the assertion is true if $\eta(v)$ is connected, and otherwise every component of $\eta(v)$ contains a vertex of $Z \subseteq V(A)$ as required. For the second bullet, let $v \in B' \setminus A'$; then $Z \cap \eta(v) = \emptyset$, and so $\eta(v)$ is connected. For the third bullet, clearly $X' \subseteq A' \cap B'$. For the converse, let $v \in A' \cap B'$, and choose $b \in V(B) \cap \eta(v)$. By the first bullet, the component of $\eta(v)$ containing $b$ has a vertex in $V(A)$, and therefore a vertex in $X$, since every path between $V(A), V(B)$ in $G^*$ contains a vertex of $X$; and therefore $v \in X'$.

For the fourth bullet, suppose that $a' \in A' \setminus B'$ and $b' \in B' \setminus A'$ are adjacent in $J$, joined by an edge $f'$. Let $\eta(f') = f$ say; then $f$ has an end in $\eta(a')$ and an end in $\eta(b')$. Let $C$ be the component of $\eta(a')$ containing an end of $f$. By the first two bullets, the subgraph formed by the union of $C$, $\eta(b')$, and $f$ is connected, and since it meets both $V(A)$ and $V(B)$, it also meets $X$, and so one of $a', b' \in X'$, contrary to the third bullet.

Finally, for the fifth bullet, let $C$ be a connected subgraph of $\mathcal{G}_n$ disjoint from $X' \cup (V(H) \setminus L)$ and with non-empty intersection with $B'$. If the claim does not hold, then since $V(C) \cap A' \cap B' = \emptyset$ (by the third bullet), there are adjacent vertices $a', b'$ of $C$ with $b' \in B' \setminus A'$ and

$$a' \in (A' \setminus B') \cup (V(J) \setminus (A' \cup B')) \cup (V(\mathcal{G}_n) \setminus V(J)).$$

But

- $a' \notin A' \setminus B'$ by the fourth bullet;
- $a' \notin V(J) \setminus (A' \cup B')$, since $V(J) \setminus (A' \cup B') = V(H) \setminus L$ and $C$ is disjoint from $V(H) \setminus L$; and
- $a' \notin V(\mathcal{G}_n) \setminus V(J)$, since $b' \notin A' \supseteq Z' \supseteq \beta$.

This is a contradiction, and so completes the proof of (5).

For $i_0 \le i \le i_0 + k - 1$, let $R_i$ be the $i$th row of $\mathcal{G}_n$, that is, the set

$$\{v_{i,j} \ : 1 \le j \le n\},$$

and let $Q_i$ be the subgraph of $\mathcal{G}_n$ induced on

$$\{v_{i,j} \ : 1 \le j \le j_0\}.$$

Thus each $Q_i$ is a connected subgraph of $\mathcal{G}_n$ containing a vertex of $L$ but disjoint from $V(H) \setminus L$. Since $|X'| < k$, there exists $r$ with $i_0 \le r \le i_0 + k - 1$ such that $R_r \cap X' = \emptyset$, and in particular $V(Q_r) \cap X' = \emptyset$. Since $L \subseteq B'$, it follows from the fifth bullet of (5) that $Q_r \subseteq B'$, that is, $v_{r,j} \in B'$ for $1 \le j \le j_0$.

For $1 \le s \le k$, let $S_s$ be the set of all $v_{i,j}$ where $(i,j)$ belongs to

$$\begin{aligned} &\{(i, j_0 - k + s - 1): & i_0 - k + s - 1 \le i \le i_0 + k + g - s\} \\ \cup&\{(i_0 - k + s - 1, j): & j_0 - k + s - 1 \le j \le j_0 + k + g - s\} \\ \cup&\{(i, j_0 + k + g - s): & i_0 - k + s - 1 \le i \le i_0 + k + g - s\} \\ \cup&\{(i_0 + k + g - s, j): & j_0 - k + s - 1 \le j \le j_0 + k + g - s\}. \end{aligned}$$

Thus, for $1 \le s \le k$, $S_s$ is the vertex set of a cycle of $H_0$ "surrounding" $H$; and the sets $S_1, \dots, S_k$ are pairwise disjoint and each is disjoint from $V(H)$. Since $|X'| < k$, there exists $s$ with $1 \le s \le k$

such that $S_s \cap X' = \emptyset$. Since $v_{r,j_0-k+s-1} \in B' \cap S_s$ it follows from the fifth bullet of (5) that $S_s \subseteq B'$.

(6) *There is a path of $G$ between $Z$ and $\eta(v_{r,j_0})$ disjoint from $X$.*

Suppose first that $R_r \cap Z' \neq \emptyset$, and let $P$ be a minimal subpath of $\mathcal{G}_n$ between $Z'$ and $v_{r,j_0}$ with $V(P) \subseteq R_r$. It follows that no vertex of $P$ except possibly one end belongs to $\beta$, since $\beta \subseteq Z'$; and so $P$ is a path of $J$, and $\eta(v)$ is defined for every vertex $v$ of $P$, and therefore the desired path can be chosen in $G|\eta(P)$. We may therefore assume that $R_r \cap Z' = \emptyset$, and so $R_r \subseteq V(J)$. By hypothesis, there is no separation $(C, D)$ of $G$ of order less than $k$ such that $Z \subseteq V(C)$ and $V(D)$ includes $\eta(R_r)$. In particular, since $|X| < k$, there is a path $T$ of $G \setminus X$ between $Z$ and $\eta(R_r)$. But then the union of $T$ and $G|\eta(R_r)$ includes the required path. This proves (6).

Let $Y'$ be the union of $S_{s+1}, \ldots, S_k$ and $V(H)$; that is, the set of vertices of $\mathcal{G}_n$ "surrounded" by $S_s$. By (6), there is a minimal path $Q$ of $G \setminus X$ between $Z$ and $\eta(Y')$; let its ends be $z \in Z$ and $y \in \eta(Y')$. It follows that no vertex of $Q \setminus y$ is in $\eta(V(H) \setminus L)$, and hence $Q \setminus y$ is a path of $G^*$. Choose $y' \in Y'$ with $y \in V(\eta(y'))$. Let $x$ be the neighbour of $y$ in $Q$, and let $x \in \eta(x')$. From (2), the edge $xy$ of $G$ equals $\eta(f')$ for some edge $f'$ of $J$ incident with $x', y'$, and since $x' \notin Y'$, it follows that $x' \in S_s$. Consequently $Q \setminus y$ is a path of $G^*$ between $Z$ and $\eta(S_s)$ disjoint from $X$, which is impossible since $(A, B)$ is a separation of $G^*$, and $\eta(S_s) \subseteq B$. Thus there is no $(A, B)$ as in (4). This proves 2.1. ∎

Finally, let us deduce 1.2, which we restate:

**2.2** *For all $k, g$ with $1 \leq k \leq g$ there exists $K \geq 1$ with the following property. Let $\mathcal{T}$ be a tangle of order at least $K$ in a graph $G$, and let $Z \subseteq V(G)$ with $|Z| = k$. Suppose that there is no separation $(A, B) \in \mathcal{T}$ of order less than $k$ with $Z \subseteq V(A)$. Then there is a model $\eta$ of $\mathcal{G}_g$ in $G$, such that*

- *for $1 \leq i \leq k$, $V(\eta(v_{i,1}))$ contains a member of $Z$*
- *for each $(A, B) \in \mathcal{T}$, if $\eta(R) \subseteq V(A)$ for some row $R$ of the grid, then $(A, B)$ has order at least $g$.*

**Proof.** Let $n$ be as in 2.1. Choose $K$ to satisfy 1.1 (with $g$ replaced by $n$.) We claim that this choice of $K$ satisfies 2.2. For let $\mathcal{T}$ be a tangle of order at least $K$ in a graph $G$, and let $Z \subseteq V(G)$ with $|Z| = k$. Suppose that there is no separation $(A, B) \in \mathcal{T}$ of order less than $k$ with $Z \subseteq V(A)$. By 1.1 there is a model $\eta$ of $\mathcal{G}_n$ in $G$, such that for each $(A, B) \in \mathcal{T}$, if $\eta(R) \subseteq V(A)$ for some row $R$ of $\mathcal{G}_n$, then $(A, B)$ has order at least $n$.

(1) *There is no separation $(A, B)$ of $G$ of order less than $k$ such that $Z \subseteq V(A)$ and there is a row $R$ of $\mathcal{G}_n$ with $R \subseteq V(J)$ and $\eta(R) \subseteq V(B)$.*

For suppose that $(A, B)$ is such a separation. Since $k \leq n \leq K$, it follows that one of $(A, B), (B, A) \in \mathcal{T}$. But there is no separation $(A, B) \in \mathcal{T}$ of order less than $k$ with $Z \subseteq V(A)$, so $(A, B) \notin \mathcal{T}$; and for each $(C, D) \in \mathcal{T}$, if $\eta(R) \subseteq V(C)$ for some row $R$ of $\mathcal{G}_n$, then $(C, D)$ has order at least $n$, so $(B, A) \notin \mathcal{T}$, a contradiction. This proves (1).

From (1) and 2.1, taking $J = \mathcal{G}_n$, we deduce that there is a subgraph $H$ of $\mathcal{G}_n$, isomorphic to $\mathcal{G}_g$, such that the restriction of $\eta$ to $H$ is $Z$-augmentable.

(2) *For each* $(A, B) \in \mathcal{T}$, *if* $\eta(R) \subseteq V(A)$ *for some row* $R$ *of* $\mathcal{G}_g$, *then* $(A, B)$ *has order at least* $g$.

For since $\eta(R) \subseteq V(A)$, and $J$ is a subgraph of $\mathcal{G}_n$, it follows that there are at least $g$ columns $C$ of $\mathcal{G}_n$ such that $C \cap V(A) \neq \emptyset$. If each of them contains a vertex of $A \cap B$ then $|A \cap B| \geq g$ as required, and otherwise some column $C$ of $\mathcal{G}_n$ is included in $V(A)$. But then every row of $\mathcal{G}_n$ contains a vertex in $V(A)$; if they all meet $A \cap B$ then $|A \cap B| \geq n \geq g$ as required, and otherwise some row of $\mathcal{G}_n$ is included in $V(A)$. But then from the choice of $\eta$, $(A, B)$ has order at least $n \geq g$. This proves (2).

This proves 2.2. ∎

## References


[1] B.Reed, "Mangoes and blueberries", *Combinatorica* 19 (1999), 267–296.

[2] N.Robertson and P.D.Seymour, "Graph minors. V. Excluding a planar graph", *J. Combinatorial Theory, Ser. B,* 41 (1986), 92–114.

[3] N.Robertson and P.D.Seymour, "Graph minors. X. Obstructions to tree-decomposition", *J. Combinatorial Theory, Ser. B,* 52 (1991), 153–190.