Rao's degree sequence conjecture

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Abstract

Let us say two (simple) graphs G, G' are *degree-equivalent* if they have the same vertex set, and for every vertex, its degrees in G and in G' are equal. In the early 1980's, S. B. Rao made the conjecture that in any infinite set of graphs, there exist two of them, say G and H, such that H is isomorphic to an induced subgraph of some graph that is degree-equivalent to G. We prove this conjecture.

1 Introduction

Neil Robertson and the second author proved in [7] that the class of all graphs forms a "well-quasi-order" under minor containment, that is, that in every infinite set of graphs, one of its members is a minor of another. The same is not true for induced subgraph containment, but a conjecture of S. B. Rao proposed a way to tweak the latter containment relation to make it a well-quasi-order; and in this paper we prove Rao's conjecture.

Let us be more precise. All graphs and digraphs in this paper are finite and without loops or parallel edges, and digraphs do not have directed cycles of length two. If G is a graph and $X \subseteq V(G)$, we denote by $G|X$ the subgraph of G induced on X (that is, the subgraph with vertex set X and edge set all edges of G with both ends in X); and we say that $G|X$ is an *induced subgraph* of G. Let us say two graphs G, G' are *degree-equivalent* if they have the same vertex set, and for every vertex, its degrees in G and in G' are equal; and H is *Rao-contained* in G if H is isomorphic to an induced subgraph of some graph that is degree-equivalent to G. In the early 1980's, S. B. Rao made the conjecture, the main theorem of this paper, that:

1.1 *In any infinite set of graphs, there exist two of them, say G and H, such that H is Rao-contained in G.*

A *quasi-order* Q consists of a class $E(Q)$ and a transitive reflexive relation which we denote by $\leq$ or $\leq_Q$; and it is a *well-quasi-order* or *wqo* if for every infinite sequence q_i $(i = 1, 2 \ldots)$ of elements of $E(Q)$ there exist $j > i \geq 1$ such that $q_i \leq_Q q_j$. Rao-containment is transitive (this is an easy exercise), and so the following is a reformulation of 1.1:

1.2 *The class of all graphs, ordered by Rao-containment, is a wqo.*

The proof falls into three main parts, and let us sketch them here. A "split graph" is a graph such that there is a partition of its vertex set into a stable set and a clique. For Rao-containment of split graphs, we will require the vertex set injection to preserve this partition. A "k-rooted graph" means (roughly) a graph with k of its vertices designated as roots. For Rao-containment of k-rooted graphs, we require the vertex set injection to respect the roots. (This will all be said more precisely later.) We show three things:

- For every graph H, if G is a graph that does not Rao-contain H, then $V(G)$ can be partitioned into two sets (except for a bounded number of vertices), the first inducing a split graph and the second inducing a graph of bounded degree (or the complement of one), such that the edges between these two sets are under control. This allows us to break G into two parts; but both parts acquire a bounded number of roots, because we need to remember how to hook them back together to form G. This is proved in 4.2.
- For all k, the k-rooted graphs of bounded degree (except for the roots) form a wqo under Rao-containment. This is proved in 6.1
- For all k, the k-rooted split graphs also form a wqo under Rao-containment. This is proved in 7.2.

From these three statements, the truth of 1.2 follows in a few lines, and is given immediately after 7.2. Then the proof of 7.2 occupies the remainder of the paper.

2 Rao-containment in fixed position

We need to study the structure of the graphs that do not Rao-contain a fixed graph H. For there to be a Rao-containment of H in G, there must be an injection of $V(H)$ into $V(G)$, and a graph G' degree-equivalent to G, such that the injection is an isomorphism between H and an induced subgraph of G'. Thus, we need to understand the graphs G such that for every injection of $V(H)$ into $V(G)$ there is no suitable choice of G'. But first, a much easier question. Suppose we are given the injection; then when is it true that no suitable G' exists? For this we can give a good characterization, theorem 2.1 below; either G' exists or there is an obvious reason why it does not exist.

Let G be a graph. If $X, Y \subseteq V$ are disjoint, we define $s(X,Y)$ or $s_G(X,Y)$ to be

$$\sum_{x \in X} deg(x) + \sum_{y \in Y} \overline{deg}(y) - |X||Y|,$$

where $deg(x)$ denotes the degree of x in G and $\overline{deg}(y) = |V(G)| - 1 - deg(y)$ is the degree of y in $\overline{G}$, the complement graph of G.

The main result of this section is the following.

2.1 *Let G, H be graphs with $V(H) \subseteq V(G)$. Then the following are equivalent:*

- *there is a graph G' degree-equivalent to G, such that $G'|V(H) = H$*
- *$s_H(X \cap V(H), Y \cap V(H)) \leq s_G(X,Y)$ for every choice of disjoint $X, Y \subseteq V(G)$.*

The proof needs several steps. We begin with the following. If G is a graph and X, Y are disjoint subsets of $V(G)$, we denote by $E(X,Y)$ or $E_G(X,Y)$ the set of edges of G with one end in X and one end in Y.

2.2 *Let G be a bipartite graph, and let (A, B) be a bipartition. For each vertex v let $d(v)$ be an integer. Then the following are equivalent:*

- *there exists $F \subseteq E(G)$ such that every vertex v is incident with exactly $d(v)$ members of F*
- *$\sum_{u \in A} d(u) = \sum_{v \in B} d(v)$, and for every $X \subseteq A$ and $Y \subseteq B$,*

$$\sum_{u \in X} d(u) + \sum_{v \in Y} (deg(v) - d(v)) \geq |E_G(X,Y)|.$$

Proof. Suppose that F satisfies the first statement. Then $\sum_{u \in A} d(u) = |F|$, and also $\sum_{v \in B} d(v) = |F|$, and so $\sum_{u \in A} d(u) = \sum_{v \in B} d(v)$. Let $X \subseteq A$ and $Y \subseteq B$, and let there be p edges in F between X and Y. Since $\sum_{u \in X} d(u)$ is the total number of edges in F with an end in X, it follows that $\sum_{u \in X} d(u) \geq p$. On the other hand, $\sum_{v \in Y}(deg(v) - d(v))$ is the number of edges in $E(G) \setminus F$ with an end in Y, and there are $|E_G(X,Y)| - p$ such edges between X and Y; and so $\sum_{v \in Y}(deg(v) - d(v)) \geq |E_G(X,Y)| - p$. By adding, we deduce the second statement of the theorem.

For the converse, suppose the second statement of the theorem holds. For $v \in A$, setting $X = \{v\}$ and $Y = \emptyset$ implies that $d(v) \geq 0$, and for $v \in B$, setting $X = A$ and $Y = B \setminus \{v\}$ implies that $d(v) \geq 0$.

Thus $d(v) \geq 0$ for all $v \in V(G)$. Direct every edge of G from A to B, and add two new vertices a, b to G, where a is adjacent to every member of A and b is adjacent from every member of B, forming a digraph H say. For each edge e of H let $c(e) = 1$ if $e \in E(G)$, and let $c(e) = d(v)$ if e is incident with a or b and with one vertex v of G. Since $c(e) \geq 0$ for all $e \in E(H)$, the max-flow min-cut theorem implies that one of the following cases holds:

- there exists $Z \subseteq V(H)$ with $a \in Z$ and $b \notin Z$, such that $\sum_{e \in D} c(e) < \sum_{u \in A} d(u)$, where D is the set of edges e of H with tail in Z and head in $V(H) \setminus Z$
- there is an integer-valued flow ϕ in H from a to b of total value $\sum_{u \in A} d(u)$ such that $0 \leq \phi(e) \leq c(e)$ for every edge e of H.

Suppose that Z is as in the first case, and let $X = A \setminus Z$ and $Y = B \setminus Z$. Then

$$\sum_{u \in X} d(u) + \sum_{v \in B \setminus Y} d(v) + |E(A \setminus X, Y)| = \sum_{e \in D} c(e) < \sum_{v \in B} d(v),$$

and so

$$\sum_{u \in X} d(u) + |E(A \setminus X, Y)| - \sum_{v \in Y} d(v) < 0.$$

Since

$$|E(A \setminus X, Y)| = \sum_{v \in Y} deg_G(v) - |E(X, Y)|,$$

substituting for $|E(A \setminus X, Y)|$ yields

$$\sum_{u \in X} d(u) + + \sum_{v \in Y} deg_G(v) - |E(X, Y)| - \sum_{v \in Y} d(v) < 0,$$

that is,

$$\sum_{u \in X} d(u) + \sum_{v \in Y} (deg_G(v) - d(v)) < |E(X, Y)|,$$

a contradiction. Consequently there is no Z as in the first case.

Thus the second case holds; let ϕ be as in the second case. It follows that $\phi(e) = c(e)$ for every edge e incident with a or b, and setting F to be the set of edges e of G with $\phi(e) = 1$ therefore satisfies the first statement of the theorem. This proves 2.2. ∎

If G is an (undirected) graph, an *arc* of G means an ordered pair (u, v) such that $u, v \in V(G)$ are adjacent, and we call u its *tail* and v its *head*. Let $A(G)$ denote the set of arcs of G; thus, $|A(G)| = 2|E(G)|$.

2.3 *Let G be a graph and for every vertex v let $d(v)$ be an integer. Then the following are equivalent:*

- *there exists $F \subseteq A(G)$, such that for every vertex $v \in V(G)$, there are exactly $d(v)$ members of F with tail v and $d(v)$ members with head v*
- *$\sum_{x \in X} d(x) + \sum_{y \in Y} (deg_G(y) - d(y)) \geq |E_G(X, Y)|$ for every pair of disjoint subsets $X, Y \subseteq V(G)$.*

Proof. Take two new vertices a_x, b_x for each vertex x of G, and let $A = \{a_x : x \in V(G)\}$ and $B = \{b_x : x \in V(G)\}$. Let H be the graph with vertex set $A \cup B$ and edge set $A(G)$, where for each arc $e = (x, y) \in A(G)$, e is incident in H with a_x and b_y. Consequently (A, B) is a bipartition of H. For each $x \in V(G)$, define $d'(a_x) = d'(b_x) = d(x)$. Thus $\sum_{u \in A} d'(u) = \sum_{v \in B} d'(v)$, since $d'(a_v) = d'(b_v)$ for each $v \in V(G)$. Now the first statement of the theorem holds if and only if there exists $F \subseteq E(H)$ such that every vertex $v \in V(H)$ is incident with exactly $d'(v)$ members of F. By 2.2, this is true if and only if

$$\sum_{u \in X'} d'(u) + \sum_{v \in Y'} (deg_H(v) - d'(v)) \geq |E_H(X', Y')|$$

for all $X' \subseteq A$ and $Y' \subseteq B$. By setting $X' = \{a_x : x \in X \cup Z\}$ and $Y' = \{b_y : y \in Y \cup Z\}$, we see that the latter statement is true if and only if

$$\sum_{x \in X \cup Z} d(x) + \sum_{y \in Y \cup Z} (deg_G(y) - d(y)) \geq |E_G(X, Y)| + |E_G(X \cup Y, Z)| + 2|D(Z)|,$$

for all choices of pairwise disjoint subsets $X, Y, Z \subseteq V(G)$, where $D(Z)$ denotes the set of edges of G with both ends in Z. This can be rewritten as

$$\sum_{x \in X} d(x) + \sum_{y \in Y} (deg_G(y) - d(y)) - |E_G(X, Y)| + \sum_{z \in Z} deg_G(z) - |E_G(X \cup Y, Z)| - 2|D(Z)| \geq 0.$$

The last three terms sum to $|E_G(W, Z)|$, where $W = V(G) \setminus (X \cup Y \cup Z)$; and this is minimized when $Z = \emptyset$. Consequently the condition holds for all choices of disjoint X, Y, Z, if and only if it holds for all disjoint X, Y with $Z = \emptyset$; and so the condition is equivalent to the second statement of the theorem. This proves 2.3. ∎

Let us say a graph G is *constricted* if for every two cycles C_1, C_2 of G, both of odd length, the subgraph $G|(V(C_1 \cup C_2))$ is connected. (Thus, either C_1, C_2 share a vertex, or some vertex of C_1 is adjacent to some vertex of C_2.) For constricted graphs we can modify 2.3 as follows:

2.4 *Let G be a constricted graph, and for every vertex v let $d(v)$ be an integer, such that $\sum_{v \in V(G)} d(v)$ is even. Then the following are equivalent:*

- *there exists $F \subseteq E(G)$ such that every vertex $v \in V(G)$ is incident with exactly $d(v)$ members of F*
- *there exists $F' \subseteq A(G)$, such that for every vertex $v \in V(G)$, there are exactly $d(v)$ members of F' with tail v and $d(v)$ members with head v*
- $\sum_{x \in X} d(x) + \sum_{y \in Y} (deg_G(y) - d(y)) \geq |E_G(X, Y)|$ *for every pair of disjoint subsets $X, Y \subseteq V(G)$.*

Proof. The equivalence of the second and third statement follows from 2.3. Moreover if F satisfies the first statement, then the set F' of arcs of G corresponding to the edges in F (thus $|F'| = 2|F|$) satisfies the second statement. Thus it suffices to show that the second statement implies the first. Let F' satisfy the second statement. Let F_2 be the set of all arcs (u, v) in F' such that (v, u) also belongs to F', and let $F_1 = F' \setminus F_2$. Choose F' with F_1 minimal. Let H be the digraph with vertex set $V(G)$ and edge set F_1, with the natural incidence. Since every vertex $v \in V(G)$ is the head of $d(v)$ arcs in F' and the tail of $d(v)$ arcs in F', and also every vertex v is the head of the same number of arcs in F_2 as it is the tail, it follows by subtracting that H is an eulerian digraph, and therefore its edge set can be partitioned into the edge sets of directed cycles $C_1, \dots, C_k$ say.

Suppose first that one of $C_1, \dots, C_k$ has even length, say C_1, and let its vertices be $v_0, v_1, v_2, \dots, v_{2n} = v_0$ in order. Let F'' be obtained from F' by

- removing the arcs (v_{2i-1}, v_{2i}) for $1 \le i \le n$, and
- adding the arcs (v_{2j+1}, v_{2j}) for $0 \le j \le n-1$.

Then F'' also satisfies the second statement of the theorem, contrary to the maximality of F_2. This proves that $C_1, \dots, C_k$ all have odd length.

Next suppose that some two of $C_1, \dots, C_k$ are not vertex-disjoint, say C_1 and C_2, and so we can number the vertices of these two cycles such that C_1 has vertices $u_0, u_1, \dots, u_{2m+1} = u_0$ in order, and C_2 has vertices $v_0, v_1, \dots, v_{2n+1} = v_0$ in order, where $u_0 = v_0$. Note that since C_1, C_2 have no common edges and all their edges belong to F_1, it follows that no edge of C_1 has the same set of ends as an edge of C_2. Let F'' be obtained from F' by

- removing the arcs (u_{2i-1}, u_{2i}) for $1 \le i \le m$, and removing (v_{2j}, v_{2j+1}) for $0 \le j \le n$, and
- adding the arcs (u_{2i+1}, u_{2i}) for $0 \le i \le m$, and adding (v_{2j}, v_{2j-1}) for $1 \le j \le n$.

Again, this contradicts the minimality of F_1. Consequently $C_1, \dots, C_k$ are pairwise vertex-disjoint.

Suppose that $k \ge 2$, and let C_1 have vertices $u_0, u_1, \dots, u_{2m+1} = u_0$ in order, and let C_2 have vertices $v_0, v_1, \dots, v_{2n+1} = v_0$ in order. Since G is constricted, some u_i is adjacent in G to some v_j, and so we may assume that u_0, v_0 are adjacent. Since $C_1, \dots, C_k$ are pairwise vertex-disjoint, it follows that the arcs $(u_0, v_0), (v_0, u_0)$ do not belong to any of $C_1, \dots, C_k$ and hence are not in F_1. There are two cases depending whether they belong to F_2 or not.

First suppose that $(u_0, v_0), (v_0, u_0) \notin F_2$. Let F'' be obtained from F' by

- removing the arcs (u_{2i}, u_{2i+1}) for $0 \le i \le m$, and removing (v_{2j}, v_{2j+1}) for $0 \le j \le n$
- adding the arcs (u_{2i}, u_{2i-1}) for $1 \le i \le m$, and adding (v_{2j}, v_{2j-1}) for $1 \le j \le n$, and adding $(u_0, v_0), (v_0, u_0)$.

This contradicts the minimality of F_1.

Thus F_2 contains one and hence both of $(u_0, v_0), (v_0, u_0)$. Let F'' be obtained from F' by

- removing the arcs (u_{2i-1}, u_{2i}) for $1 \le i \le m$, and removing (v_{2j-1}, v_{2j}) for $1 \le j \le n$, and removing $(u_0, v_0), (v_0, u_0)$
- adding the arcs (u_{2i+1}, u_{2i}) for $0 \le i \le m$, and adding (v_{2j+1}, v_{2j}) for $0 \le j \le n$.

Again, this contradicts the minimality of F_1.

We deduce that $k \le 1$. Since $\sum_{v \in V(G)} d(v)$ is even, and every vertex v is the tail of exactly $d(v)$ members of F', it follows that $|F'|$ is even. But $|F_2|$ is even, and so $|F_1|$ is even, and since $C_1, \dots, C_k$ have odd length, it follows that k is even. Since $k \le 1$ we deduce that $k = 0$, and so $F' = F_2$. But then the first statement of the theorem holds. This proves 2.4. ∎

We are almost ready to prove 2.1; first, one more lemma. If G is a graph and $X \subseteq V(G)$, we denote by $E(X)$ or $E_G(X)$ the set of edges of G with both ends in X; and we remind the reader that if X, Y are disjoint subsets of $V(G)$, we denote by $E(X, Y)$ or $E_G(X, Y)$ the set of edges of G with one end in X and one end in Y. It is convenient to write $F_G(X)$ or $F(X)$ for $E_{\overline{G}}(X)$, and $F_G(X, Y)$ or $F(X, Y)$ for $E_{\overline{G}}(X, Y)$.

2.5 *Let G be a graph and let X, Y, Z be a partition of $V(G)$; then*

$$s_G(X, Y) = 2|E(X)| + |E(X, Z)| + 2|F(Y)| + |F(Y, Z)|.$$

Proof. For $|E(X, Y)| + |F(X, Y)| = |X||Y|$; but

$$\sum_{x \in X} deg(x) = 2|E(X)| + |E(X, Z)| + |E(X, Y)|$$

and

$$\sum_{y \in Y} \overline{deg}(y) = 2|F(Y)| + |F(Y, Z)| + |F(X, Y)|,$$

and adding these three equations yields the statement of the theorem. This proves 2.5. ∎

The main step in the proof of 2.1 is the following.

2.6 *Let V be a finite set, and for every vertex $v \in V$ let $d(v)$ be an integer. Let H be a graph with vertex set a subset of V. Then the following are equivalent:*

- *There is a graph J with vertex set V, such that every $v \in V$ has degree $d(v)$ in J, and $J|V(H) = H$*
- *$\sum_{v \in V} d(v)$ is even, and*

$$\sum_{x \in X} d(x) + \sum_{y \in Y} (|V| - 1 - d(y)) - |X||Y| \geq s_H(X \cap V(H), Y \cap V(H))$$

  for every pair of disjoint subsets X, Y of V.

Proof. Suppose first that J satisfies the first statement. Then $\sum_{v \in V} d(v) = 2|E(J)|$ and therefore is even. Moreover, let $X, Y \subseteq V(G)$ be disjoint. To verify the second statement we must check that $s_J(X, Y) \geq s_H(X \cap V(H), Y \cap V(H))$. But since H is an induced subgraph of J, this is immediate from two applications of 2.5, to $s_J(X, Y)$ and to $s_H(X \cap V(H), Y \cap V(H))$.

Now suppose that the second statement holds. Let $V' = V(H)$. Let G be the graph with vertex set V, in which every two distinct vertices are nonadjacent if and only if they both belong to V'. For each $v \in V$, let $d'(v) = d(v)$ if $v \notin V'$, and $d'(v) = d(v) - deg_H(v)$ if $v \in V'$. If there is a subgraph of G such that every vertex v has degree $d'(v)$ then the first statement of the theorem holds (taking J to be the union of this subgraph with H); so we assume not. Now G is constricted, since every odd cycle of G has at least one vertex not in V'; so by 2.4, there exist disjoint $X, Y \subseteq V$ such that

$$\sum_{x \in X} d'(x) + \sum_{y \in Y} (deg_G(y) - d'(y)) < |E_G(X, Y)|.$$

Let $X' = X \cap V'$ and $Y' = Y \cap V'$. Now

$$\sum_{x \in X} d'(x) = \sum_{x \in X} d(x) - \sum_{x \in X'} deg_H(x),$$

and since every vertex in V' has degree $|V| - |V'|$ in G, and every vertex in $V \setminus V'$ has degree $|V| - 1$ in G, it follows that

$$\sum_{y \in Y} (deg_G(y) - d'(y)) = \sum_{y \in Y} (|V| - 1 - d(y)) - \sum_{y \in Y'} (|V'| - 1 - deg_H(y)).$$

Moreover $|E_G(X, Y)| = |X||Y| - |X'||Y'|$. On substitution we obtain

$$\sum_{x \in X} d(x) - \sum_{x \in X'} deg_H(x) + \sum_{y \in Y} (|V| - 1 - d(y)) - \sum_{y \in Y'} (|V'| - 1 - deg_H(y)) < |X||Y| - |X'||Y'|,$$

that is,

$$\sum_{x \in X} d(x) + \sum_{y \in Y} (|V| - 1 - d(y)) - |X||Y| < \sum_{x \in X'} deg_H(x) + \sum_{y \in Y'} (|V'| - 1 - deg_H(y)) - |X'||Y'| = s_H(X', Y'),$$

a contradiction. This proves that the first statement of the theorem holds, and so proves 2.6. ∎

2.6 extends a result of Koren [4], who proved the same statement with H the null graph.

Proof of 2.1. Let G, H be graphs with $V(H) \subseteq V(G)$. For each vertex $v \in V(G)$, let $d(v) = deg_G(v)$. There is a graph J with vertex set $V(G)$, such that every $v \in V$ has degree $d(v)$ in J, and $J|V' = H$, if and only if the first statement of the theorem holds. But $\sum_{v \in V} d(v)$ is even, since it equals $2|E(G)|$; and so by 2.6, such a graph J exists if and only if

$$\sum_{x \in X} d(x) + \sum_{y \in Y} (|V| - 1 - d(y)) - |X||Y| \geq s_H(X \cap V', Y \cap V')$$

for every pair of disjoint subsets X, Y of V. The left side of this inequality equals $s_G(X, Y)$, and so this proves 2.1. ∎

3 Pairs of bounded surplus

Let H be a fixed graph. We will show in the next section that there are numbers m, θ depending only on H, such that for every graph G that does not Rao-contain H, and for every vertex v of G except at most m, there is a pair (X, Y) of subsets of $V(G)$ with $s_G(X, Y) \leq \theta$ and $v \in X \cup Y$. This in turn will lead to a decomposition theorem for the graphs G that do not Rao-contain H; we will prove they are all "almost" split graphs. In this section we develop some lemmas for that purpose. If $\theta \geq 0$ is an integer, a θ-*shelf* in a graph G is a pair (X, Y) of disjoint subsets of $V(G)$ such that $s_G(X, Y) \leq \theta$. We begin with:

3.1 *Let G be a graph and $\theta \geq 0$ an integer, and for $i = 1, 2$, let (X_i, Y_i) be a θ-shelf. If one of $X_1 \cap Y_2, X_2 \cap Y_1$ is nonempty, then one of $V(G) \setminus (X_1 \cup Y_1), V(G) \setminus (X_2 \cup Y_2)$ has cardinality at most 2θ.*

Proof. Let $Z_i = V(G) \setminus (X_i \cup Y_i)$ for $i = 1, 2$. By 2.5, we have

(1) $2|E(X_i)| + |E(X_i, Z_i)| + 2|F(Y_i)| + |F(Y_i, Z_i)| \leq \theta$ *for* $i = 1, 2$.

We may assume from the symmetry that there exists $w \in Y_1 \cap X_2$. Suppose first that there exists $x \in X_1 \cap Z_2$. For each $v \in Z_1 \cap X_2$, if v, w are adjacent then this edge belongs to $E(X_2)$, and if they are nonadjacent then the edge of $\overline{G}$ joining them belongs to $F(Y_1, Z_1)$; and so in either case the pair $\{v, w\}$ belongs to $E(X_2) \cup F(Y_1, Z_1)$. Similarly if $v \in Z_1 \cap Y_2$ then $\{v, x\} \in E(X_1, Z_1) \cup F(Y_2, Z_2)$, and if $v \in Z_1 \cap Z_2$ then $\{v, w\} \in E(X_2, Z_2) \cup F(Y_1, Z_1)$. Summing, we deduce that

$$|Z_1| = |Z_1 \cap X_2| + |Z_1 \cap Y_2| + |Z_1 \cap Z_2| \leq |E(X_2)| + |F(Y_1, Z_1)| + |E(X_1, Z_1)| + |F(Y_2, Z_2)| + |E(X_2, Z_2)|$$

and so $|Z_1| \leq 2\theta$ by (1), as required.

Thus we may assume that $X_1 \cap Z_2 = \emptyset$. But for each $v \in Y_1 \cap Z_2$, $\{v, w\} \in E(X_2, Z_2) \cup F(Y_1)$; and as we already saw, if $v \in Z_1 \cap Z_2$ then $\{v, w\} \in E(X_2, Z_2) \cup F(Y_1, Z_1)$. Summing, we deduce that

$$|Z_2| = |Y_1 \cap Z_2| + |Z_1 \cap Z_2| \leq |E(X_2, Z_2)| + |F(Y_1)| + |F(Y_1, Z_1)| \leq 2\theta$$

by (1), as required. This proves 3.1. ∎

We need the following.

3.2 *Let G be a graph and let $X_1, X_2, Y_1, Y_2 \subseteq V(G)$ such that $X_1 \cup X_2$ is disjoint from $Y_1 \cup Y_2$. Then*

$$s_G(X_1 \cup X_2, Y_1 \cup Y_2) \leq s_G(X_1, Y_1) + s_G(X_2, Y_2).$$

Proof. Define $X_0 = X_1 \cup X_2$ and $Y_0 = Y_1 \cup Y_2$; and for $i = 0, 1, 2$ let $Z_i = V(G) \setminus (X_i \cup Y_i)$. By 2.5, for $i = 0, 1, 2$ we have

$$s_G(X_i, Y_i) = 2|E(X_i)| + |E(X_i, Z_i)| + 2|F(Y_i)| + |F(Y_i, Z_i)|,$$

and therefore to show that $s_G(X_0, Y_0) \leq s_G(X_1, Y_1) + s_G(X_2, Y_2)$, it suffices to show that

$$2|E(X_1)| + |E(X_1, Z_1)| + 2|E(X_2)| + |E(X_2, Z_2)| \geq 2|E(X_0)| + |E(X_0, Z_0)|$$

and

$$2|F(Y_1)| + |F(Y_1, Z_1)| + 2|F(Y_2)| + |F(Y_2, Z_2)| \geq 2|F(Y_0)| + |F(Y_0, Z_0)|.$$

From the symmetry under replacing G by its complement, it suffices to show the first. For every edge $e = uv$, let us count the contribution of e to the right and left sides. Thus, for $i = 0, 1, 2$ let $p_i = 1$ if $e \in E(X_i)$, and $p_i = 0$ otherwise; and let $q_i = 1$ if $e \in E(X_i, Z_i)$, and $q_i = 0$ otherwise. We will show that

$$2p_1 + q_1 + 2p_2 + q_2 \geq 2p_0 + q_0.$$

Since $q_1 + q_2 \geq q_0$, we may assume that $p_0 = 1$ and hence $q_0 = 0$. If p_1, p_2 are not both zero then the claim holds, so we assume that $p_1 = p_2 = 0$. Since $p_0 = 1$, it follows that one of u, v is in $X_1 \setminus X_2$ and the other is in $X_2 \setminus X_1$; but then $q_1 + q_2 = 2 = 2p_0$ and again the claim holds. This proves 3.2. ∎

3.3 *Let G be a graph and $\theta \geq 0$ an integer, and let L, R be disjoint subsets of $V(G)$. Suppose that for each $v \in L \cup R$, there is a θ-shelf (X, Y) with $X \subseteq L$ and $Y \subseteq R$ and $v \in X \cup Y$. Then either*

- *there exists $V \subseteq L$ such that $s_G(V, R) \leq 4\theta(\theta+1)$, and for every vertex $v \in V \cup R$ there is a θ-shelf (X, Y) with $v \in X \cup Y$ and $X \subseteq V$ and $Y \subseteq R$, or*
- *there exists $V \subseteq R$ such that $s_G(L, V) \leq 4\theta(\theta+1)$, and for every vertex $v \in L \cup V$ there is a θ-shelf (X, Y) with $v \in X \cup Y$ and $X \subseteq L$ and $Y \subseteq V$.*

Proof. By hypothesis, there are θ-shelves (X_i, Y_i) $(i \in I)$ such that $\bigcup_{i \in I} X_i = L$, and $\bigcup_{i \in I} Y_i = R$. If $J \subseteq I$, we say that $A \subseteq L$ is a *left J-transversal* if $A \subseteq \bigcup_{j \in J} X_j$, and $|A \cap X_j| \leq 1$ for each $j \in J$. Similarly, $B \subseteq R$ is a *right J-transversal* if $B \subseteq \bigcup_{j \in J} Y_j$, and $|B \cap Y_j| \leq 1$ for each $j \in J$.

(1) *If $J \subseteq I$ and A, B are left and right J-transversals respectively, then* $\min(|A|, |B|) \leq 2\theta + 2$.

For let $\min(|A|, |B|) = k$ say. Every subset of a left J-transversal is also a left J-transversal, and the same for right J-transversals, and so, by replacing the larger of A, B with a subset of itself with cardinality k, we may assume that $|A| = |B| = k$. Let $a \in A$, and choose $j \in J$ with $a \in X_j$. Since $s_G(X_j, Y_j) \leq \theta$, 2.5 implies that there are at most θ vertices in $B \setminus Y_j$ adjacent to a; and there is at most one vertex in $B \cap Y_j$ adjacent to a, since $|B \cap Y_j| \leq 1$. Consequently a is adjacent to at most $\theta + 1$ members of B, and so (summing over all $a \in A$) we deduce that $|E_G(A, B)| \leq k(\theta + 1)$. Similarly $|F_G(A, B)| \leq k(\theta + 1)$, and since $|E_G(A, B)| + |F_G(A, B)| = |A||B| = k^2$, adding these two inequalities yields that $k^2 \leq 2k(\theta + 1)$. This proves (1).

(2) *There exists $J \subseteq I$ with $|J| \leq 4\theta + 4$, such that either $\bigcup_{j \in J} X_i = L$ or $\bigcup_{j \in J} Y_i = R$.*

For we may assume that I is minimal such that $\bigcup_{i \in I} X_i = L$ and $\bigcup_{i \in I} Y_i = R$. It follows that for each $i \in I$ there exists $v_i \in X_i \cup Y_i$ such that $v_i \notin X_j \cup Y_j$ for all $j \in I$ with $j \neq i$. Let P be the set of all $i \in I$ with $v_i \in L$, and let Q be the set of all $i \in I$ with $v_i \in R$. Thus $\{v_i : i \in P\}$ is a left I-transversal of cardinality $|P|$, and $\{v_i : i \in Q\}$ is a right I-transversal of cardinality Q, and so by (1), $\min(|P|, |Q|) \leq 2\theta + 2$, say $|Q| \leq 2\theta + 2$. (This is without loss of generality, since replacing G by its complement and exchanging L and R will provide a symmetry exchanging P and Q.) Choose $T \subseteq P$ minimal such that $\bigcup_{i \in T} Y_i = \bigcup_{i \in P} Y_i$. Hence for each $i \in T$ there exists $w_i \in Y_i$ such that $w_i \notin Y_j$ for $j \in T \setminus \{i\}$. It follows that $\{w_i : i \in T\}$ is a right T-transversal of cardinality $|T|$. Moreover, $\{v_i : i \in T\}$ is a left T-transversal of cardinality T, and so $|T| \leq 2\theta + 2$ by (1). But

$$\bigcup_{i \in Q \cup T} Y_i = \bigcup_{i \in Q} Y_i \cup \bigcup_{i \in T} Y_i = \bigcup_{i \in Q} Y_i \cup \bigcup_{i \in P} Y_i = \bigcup_{i \in I} Y_i = R.$$

Since $|Q \cup T| \leq 4\theta + 4$, setting $J = Q \cup T$ proves (2).

Let J be as in (2); and from the symmetry we may assume that $\bigcup_{j \in J} Y_i = R$. Let $V = \bigcup_{j \in J} X_j$. By repeated application of 3.2 it follows that $s_G(V, R) \leq 4\theta(\theta+1)$, since $|J| \leq 4(\theta+1)$ and $s_G(X_j, Y_j) \leq \theta$ for each $j \in J$. This proves 3.3. ∎

4 A structure theorem for Rao-containment

In this section we finish the proof that for every graph H, the graphs that do not Rao-contain H are "almost" split graphs. It is convenient to break the proof into two steps. We first prove the following:

4.1 *Let H be a graph, and let $\theta = |V(H)|^2$. If G is a graph that does not Rao-contain H, then there is a partition of $V(G)$ into four sets P, Q, S, T, possibly empty, such that*

- *every vertex in P has at most θ neighbours in $V(G) \setminus Q$, and every vertex in Q has at most θ non-neighbours in $V(G) \setminus P$*
- $s_G(P, Q) \leq 4\theta(\theta + 1)$
- $|S| \leq 2\theta$
- *either every vertex in T has at most θ neighbours in $V(G) \setminus Q$, or every vertex in T has at most θ non-neighbours in $V(G) \setminus P$.*

Proof. Let L, R be the union of the sets X, and the sets Y respectively, over all θ-shelves (X, Y).

(1) *For every θ-shelf (X, Y), every vertex in X has at most θ neighbours in $V(G) \setminus Y$, and every vertex in Y has at most θ non-neighbours in $V(G) \setminus X$. Consequently, every vertex in L has at most θ neighbours in $V(G) \setminus R$, and every vertex in R has at most θ non-neighbours in $V(G) \setminus L$.*

For if (X, Y) is a θ-shelf and $v \in X$, then since $s_G(X, Y) \leq \theta$, 2.5 implies that v has at most θ neighbours in $V(G) \setminus Y$, and similarly, every vertex in Y has at most θ non-neighbours in $V(G) \setminus X$. This proves the first assertion. Now let $v \in L$. Then there is a θ-shelf (X, Y) such that $v \in X$; and since v has at most θ neighbours in $V(G) \setminus Y$, it follows that v has at most θ neighbours in $V(G) \setminus R$. Similarly every vertex in R has at most θ non-neighbours in $V(G) \setminus L$. This proves (1).

(2) *If $L \cap R \neq \emptyset$ then the theorem holds.*

For since $L \cap R \neq \emptyset$, there exist θ-shelves $(X_1, Y_1), (X_2, Y_2)$ such that $X_1 \cap Y_2 \neq \emptyset$. By 3.1, there is a θ-shelf (X, Y) such that $|Z| \leq 2\theta$, where $Z = V(G) \setminus (X \cup Y)$. But then we may take $P = X$, $Q = Y$, $S = Z$ and $T = \emptyset$, and by (1) the theorem is satisfied. This proves (2).

Henceforth we assume that $L \cap R = \emptyset$.

(3) $|V(G) \setminus (L \cup R)| < |V(H)|$.

For suppose not. By replacing H by an isomorphic graph we may assume (to simplify notation) that $V(H) \subseteq V(G) \setminus (L \cup R)$. Since G does not Rao-contain H there is no graph G' degree-equivalent to G, such that $G'|V(H) = H$. By 2.1 it follows that there exist disjoint $X, Y \subseteq V(G)$ such that $s_H(X \cap V(H), Y \cap V(H)) > s_G(X, Y)$. Since $s_H(X \cap V(H), Y \cap V(H)) \leq \theta$ it follows that (X, Y) is a θ-shelf, and so $X \cup Y \subseteq L \cup R$. But $V(H) \cap (L \cup R) = \emptyset$, and so $V(H) \cap (X \cup Y) = \emptyset$. Consequently $s_H(X \cap V(H), Y \cap V(H)) = 0$, which is impossible since $s_H(X \cap V(H), Y \cap V(H)) > s_G(X, Y)$ and $s_G(X, Y) \geq 0$ by 2.5. This proves (3).

By 3.3, either

- there exists $V \subseteq L$ such that $s_G(V, R) \leq 4\theta(\theta+1)$, and for every vertex $v \in V \cup R$ there is a θ-shelf (X, Y) with $v \in X \cup Y$ and $X \subseteq V$ and $Y \subseteq R$, or
- there exists $V \subseteq R$ such that $s_G(L, V) \leq 4\theta(\theta+1)$, and for every vertex $v \in L \cup V$ there is a θ-shelf (X, Y) with $v \in X \cup Y$ and $X \subseteq L$ and $Y \subseteq V$.

In the first case, we set $P = V$, $Q = R$, $S = V(G) \setminus (L \cup R)$, and $T = L \setminus V$. In the second case we set $P = L$, $Q = V$, $S = V(G) \setminus (L \cup R)$, and $T = R \setminus V$. This proves 4.1. ∎

Here is a slightly cleaner version of the same result:

4.2 *Let H be a graph and let $\theta = |V(H)|^2$. If G is a graph that does not Rao-contain H, then there is a partition of $V(G)$ into six sets (possibly empty) A, B, C, A', B', C' such that*

- *A is stable and there are no edges between A and $A' \cup C \cup C'$*
- *B is a clique and every vertex in B is adjacent to $B' \cup C \cup C'$*
- *A', B', C' all have cardinality at most $4\theta(\theta+1)$*
- *every vertex in A' has at most θ neighbours in $V(G) \setminus (B \cup B')$*
- *every vertex in B' has at most θ non-neighbours in $V(G) \setminus (A \cup A')$*
- *either every vertex in C has at most θ neighbours in $V(G) \setminus (B \cup B')$, or every vertex in C has at most θ non-neighbours in $V(G) \setminus (A \cup A')$.*

Proof. Let P, Q, S, T be as in 4.1. Let A' be the set of vertices in P that have a neighbour in $P \cup R \cup S$, and let B' be the set of vertices in Q that have a non-neighbour in $Q \cup R \cup S$. Since $s_G(P, Q) \leq 4\theta(\theta+1)$, it follows that $|A'| + |B'| \leq 4\theta(\theta+1)$. Set $A = P \setminus A'$, and $B = Q \setminus B'$, and $C' = S$, and $C = T$; then the theorem holds. This proves 4.2. ∎

5 Some lemmas about wqos

Now we begin on the second of the three parts sketched in the first section, and it is convenient to assemble here some standard results about wqos that we shall need frequently. For instance:

5.1 *Let Q be a wqo, and let q_i $(i = 1, 2, \dots)$ be an infinite sequence of elements of $E(Q)$. Then there is an infinite sequence $i(1) < i(2) < \dots$ of positive integers such that $q_{i(j)} \leq_Q q_{i(j+1)}$ for all $j \geq 1$.*

If Q_1, Q_2 are quasiorders, then $Q_1 \times Q_2$ is the quasiorder with element set $E(Q_1) \times E(Q_2)$, ordered by the relation $(q_1, q_2) \leq (q_1', q_2')$ if $q_1 \leq_{Q_i} q_i'$ for $i = 1, 2$.

5.2 *If Q_1, Q_2 are wqo's then so is $Q_1 \times Q_2$.*

We need a theorem of Higman [2], which we now describe. Let Q be a quasiorder, and define a quasiorder R as follows. $E(R)$ is the class of all finite sequences of members of Q; and if $a = (u_1, \ldots, u_m)$ and $b = (v_1, \ldots, v_n)$ are members of R, we say $a \leq_R b$ if $m \leq n$ and there exist $j(1), \ldots, j(m)$ with $1 \leq j(1) < j(2) < \cdots < j(m) \leq n$ such that $u_i \leq_Q v_{j(i)}$ for $1 \leq i \leq m$. We denote this quasiorder R by $Q^{<\omega}$. Higman showed

5.3 *If Q is a wqo then so is $Q^{<\omega}$.*

We also need an extension of this. Let Q be a quasiorder, and let $k \geq 0$ be an integer. Define a quasiorder R as follows. $E(R)$ is the class of all finite sequences of odd length, $x_1, \ldots, x_{2n+1}$ say, such that $x_{2i} \in Q$ for $1 \leq i \leq n$, and for $0 \leq i \leq n$, x_{2i+1} is an integer with $0 \leq x_{2i+1} \leq k$. If $a = (u_1, \ldots, u_{2m+1})$ and $b = (v_1, \ldots, v_{2n+1})$ are members of R, we say $a \leq_R b$ if $m \leq n$ and there exist $j(1), \ldots, j(m)$ with $1 \leq j(1) < j(2) < \cdots < j(m) \leq n$ such that

- for $1 \leq i \leq m$, $u_{2i} \leq_Q v_{2j(i)}$
- for $1 \leq i \leq m$, $u_{2i-1} = v_{2j(i)-1}$ and $u_{2i+1} = v_{2j(i)+1}$
- for $0 \leq i \leq m$ and $0 \leq j \leq n$, if either $i = 0$ or $j \geq j(i)$, and either $i = m$ or $j + 1 \leq j(i + 1)$, then $v_{2j+1} \geq u_{2i+1}$.

We denote this quasiorder R by $Q^{<\omega}(k)$. Then we have (see for instance [5]):

5.4 *If Q is a wqo then so is $Q^{<\omega}(k)$, for all $k \geq 0$.*

6 Graphs of bounded degree

Our object in this section is to show that graphs of bounded maximum degree form a wqo under Rao-containment, and some strengthenings of this fact.

A *march* in a set V is a finite sequence of distinct elements of V; and if π is the march $v_1, \ldots, v_k$, we denote the set $\{v_1, \ldots, v_k\}$ by $\bar{\pi}$, and call k the *length* of the march. If η is an injection from V to W say, and π is a march $v_1, \ldots, v_k$ in V, we define $\eta(\pi)$ to be the march $\eta(v_1), \ldots, \eta(v_k)$ in W, and we say that η *takes* π *to* $\eta(\pi)$. Similarly if η is an injection from V to W, and $X \subseteq V$, we define $\eta(X)$ to be the set $\{\eta(v) : v \in X\}$. A *rooted graph* is a pair (G, π) where G is a graph and π is a march in $V(G)$. We call π the *root sequence*, and its terms are the *roots*. A rooted graph is *k-rooted*, or *$(\leq k)$-rooted*, if it has exactly k roots, or at most k roots, respectively. If (G, π) is a rooted graph and $X \subseteq V(G)$ with $\bar{\pi} \subseteq X$, then $(G|X, \pi)$ is a rooted graph and we say it is a *rooted induced subgraph* of (G, π). Two rooted graphs (G, π) and (G', π') are *degree-equivalent* if G, G' are degree-equivalent and $\pi = \pi'$.

A rooted graph (H, ρ) is *Rao-contained* in a rooted graph (G, π) if there is a rooted graph (G', π) degree-equivalent to (G, π), and a rooted induced subgraph (H', π) of (G', π), and an isomorphism from H to H' taking ρ to π. Let $\mathcal{C}(k, D)$ be the class of all $(\leq k)$-rooted graphs (G, π) such that every vertex of G not in $\bar{\pi}$ has at most D neighbours that are not in$\bar{\pi}$. We shall prove:

6.1 *For every two integers $k, D \geq 0$, $\mathcal{C}(k, D)$ is a wqo under Rao-containment.*

Proof. Let $(G_i, \pi_i) \in \mathcal{C}(k, D)$ for $i = 1, 2, \ldots$. We need to show that there exist $i < j$ such that (G_i, π_i) is Rao-contained in (G_j, π_j). By passing to an infinite subsequence, we may assume that all the marches π_i have the same length, and (by reducing k if necessary) we may assume they all have length k. Thus, we may assume, to simplify notation, that all the marches π_i are equal to some fixed march π. Since there are only finitely many possibilities for the graph $G_i|\bar{\pi}$, we may assume (again, by passing to an infinite subsequence) that all these graphs are the same; and so there is a graph H, a common induced subgraph of all the graphs G_i, with $H = G_i|\bar{\pi}$ for each i.

Let $\mu(G)$ denote the size of the largest matching in a graph G.

(1) *We may assume that* $|\mu(G_i)| \geq 2|V(G_{i-1})|^2 + D + k + 1$ *for each* $i \geq 2$.

For if n is fixed, and infinitely many of the G_i's have no matching of size n, then by passing to an infinite subsequence we may assume that in each G_i there is no matching of size n, and consequently in each G_i there is a set of at most $2n$ vertices that contains at least one end of every edge. But it is an easy exercise to show that such (rooted) graphs are well-quasi-ordered by Rao-containment and indeed by induced subgraph containment. Thus we assume that only finitely many have no matching of size n, for each n; and then there is an infinite subsequence satisfying the property of (1). This proves (1).

Let $F_i = G_i \setminus V(H)$ for $i \geq 1$. For each $i \geq 1$, and every $J \subseteq V(H)$, let $Z_i(J)$ be the set of vertices in $V(G_i) \setminus V(H)$ that are adjacent in G_i to every vertex in J and nonadjacent to every vertex in $V(H) \setminus J$. Fix J for the moment. Choose an ordering of each set $Z_i(J)$, arbitrarily, and list the degrees in F_i of the vertices in $Z_i(J)$, in order. This gives a finite sequence of integers for each $i \geq 1$, all at most D; and by 5.3 and 5.1, we may assume (by passing to an infinite subsequence) that for each $i \geq 1$, there is an injection from $Z_i(J)$ into $Z_{i+1}(J)$ such that each $v \in Z_i(J)$ is mapped to a vertex in $Z_{i+1}(J)$ with degree in F_{i+1} equal to the degree of v in F_i.. By repeating this for all J, we deduce that for all $i \geq 1$, there is an injection η_i from $V(G_i)$ into $V(G_{i+1})$ such that

- for $v \in V(H)$, $\eta_i(v) = v$
- for each $J \subseteq V(H)$ and each $v \in Z_i(J)$, $\eta_i(v) \in Z_{i+1}(J)$ and the degree of v in F_i equals the degree of $\eta_i(v)$ in F_{i+1}.

(2) *For every pair of disjoint subsets* X_2, Y_2 *of* $V(F_2)$, *let* $X_1 = \{v \in V(F_1) \ : \eta_1(v) \in X_2\}$ *and let* $Y_1 = \{v \in V(F_1) \ : \eta_1(v) \in Y_2\}$; *then* $s_{F_1}(X_1, Y_1) \leq s_{F_2}(X_2, Y_2)$.

For $s_{F_1}(X_1, Y_1) \leq |V(F_1)|^2$, and so we may assume that $s_{F_2}(X_2, Y_2) < |V(F_1)|^2 \leq |V(G_1)|^2$. Suppose first that $Y_2 \neq \emptyset$, and choose $y \in Y_2$. By 2.5, y has at most $|V(G_1)|^2$ non-neighbours in $V(F_2) \setminus X_2$; but it has at most D neighbours in this set, since F_2 has maximum degree at most D, and so $|V(F_2) \setminus X_2| \leq |V(G_1)|^2 + D$. On the other hand, by 2.5, there are at most $|V(G_1)|^2$ edges with both ends in X_2. Consequently $\mu(F_2) \leq 2|V(G_1)|^2 + D$, and so $\mu(G_2) \leq 2|V(G_1)|^2 + D + k$, contrary to (1).

Thus $Y_2 = \emptyset$, and so $Y_1 = \emptyset$. But then for $i = 1, 2$, $s_{F_i}(X_i, Y_i)$ is the sum of the degrees in F_i of the vertices in X_i; and this is at least as big for F_2 as it is for F_1, since $\eta_1(v)$ has degree in F_2 equal to the degree of v in F_1, for each $v \in V(F_1)$, and so $s_{F_1}(X_1, Y_1) \leq s_{F_2}(X_2, Y_2)$ as required. This proves (2).

From (2) and 2.1, there is a graph F_2' degree-equivalent to F_2, such that the restriction of η_1 to $V(F_1)$ is an isomorphism from F_1 to an induced subgraph of F_2'. Hence there is a graph G_2' degree-equivalent to G_2, such that η_1 is an isomorphism from G_1 to an induced subgraph of G_2'; and so (G_2, π_2) Rao-contains (G_1, π_1), as required. This proves 6.1. ∎

7 Split graphs

A graph G is a *split graph* if there is a partition (A, B) into a stable set A and a clique B. In this section we begin work on the the third of the steps outlined in the first section. Let us mention a convenient lemma:

7.1 *Let G be a split graph and let (A, B) be a partition of its vertex set into a stable set A and a clique B. Let G' be degree-equivalent to G. Then G' is a split graph and A is a stable set and B a clique of G'.*

Proof. Since (A, B) is a partition of $V(G)$ and A is a stable set and B is a clique of G, 2.5 implies that $s_G(A, B) = 0$. But $s_G(A, B) = s_{G'}(A, B)$ since G, G' are degree-equivalent, and so $s_{G'}(A, B) = 0$. By 2.5, we deduce that A is a stable set and B is a clique of G'. This proves 7.1. ∎

We promised in the introduction to prove that $(\leq k)$-rooted split graphs form a wqo under Rao-containment, but in fact we need something a little stronger. The vertex set of a split graph is the union of a stable set and a clique, and we need the Rao-containment to preserve this partition. We shall prove the following:

7.2 *Let $k \geq 0$ be an integer, and for all $i \geq 1$ let (G_i, π_i) be a $(\leq k)$-rooted split graph, and let (A_i, B_i) be a partition of $V(G_i)$ such that A_i is a stable set and B_i is a clique. Then there exist $j > i \geq 1$ and a graph G' degree-equivalent to G_j (and therefore A_j and B_j are respectively a stable set and a clique of G', by 7.1) and an injection $\eta : V(G_i) \to V(G_j)$, with the following properties:*

- *for all distinct $u, v \in V(G_i)$, u, v are adjacent in G_i if and only if $\eta(u), \eta(v)$ are adjacent in G'*
- *$\pi_j = \eta(\pi_i)$*
- *$\eta(A_i) \subseteq A_j$ and $\eta(B_i) \subseteq B_j$.*

The proof of 7.2 will occupy the remainder of the paper, but first let us see that it implies our main result 1.2.

Proof of 1.2, assuming 7.2.
Suppose that 1.2 is false. Then there is a sequence G_i $(i = 0, 1, 2, \ldots)$ of graphs, such that for all $j > i \geq 0$, G_i is not Rao-contained in G_j. In particular, none of $G_1, G_2, \ldots,$ Rao-contains G_0. Let $\theta = |V(G_0)|^2$; then by 4.2, for each $i \geq 1$ there is a partition of $V(G_i)$ into six sets (possibly empty) $A_i, B_i, C_i, A_i', B_i', C_i'$ such that

- A_i is stable and there are no edges between A_i and $A_i' \cup C_i \cup C_i'$

- B_i is a clique and every vertex in B_i is adjacent to every vertex in $B'_i \cup C_i \cup C'_i$
- A'_i, B'_i, C'_i all have cardinality at most $4\theta(\theta+1)$
- every vertex in A'_i has at most θ neighbours in $V(G_i) \setminus (B_i \cup B'_i)$
- every vertex in B'_i has at most θ non-neighbours in $V(G_i) \setminus (A_i \cup A'_i)$
- either every vertex in C_i has at most θ neighbours in $V(G_i) \setminus (B_i \cup B'_i)$, or every vertex in C_i has at most θ non-neighbours in $V(G_i) \setminus (A_i \cup A'_i)$.

Now either there are infinitely many values i such that every vertex in C_i has at most θ neighbours in $V(G_i) \setminus (B_i \cup B'_i)$, or there are infinitely many such that every vertex in C_i has at most θ non-neighbours in $V(G_i) \setminus (A_i \cup A'_i)$. Thus, by replacing the sequence by an infinite subsequence, we may assume that either that the first happens for all i, or the second happens for all i. Now G_i is Rao-contained in G_j if and only if the complement of G_i is Rao-contained in the complement of G_j, and so we may replace each G_i by its complement, and exchange A_i with B_i, and exchange A'_i with B'_i, and thereby obtain another sequence satisfying the same conditions. Thus we may assume that

(1) *For all $i \geq 1$, every vertex in C_i has at most θ neighbours in $V(G_i) \setminus (B_i \cup B'_i)$.*

Since all the sets A'_i, B'_i, C'_i have bounded size, there is an infinite subsequence of the sequence such that all the sets A'_i have the same size, and to simplify notation we may assume that all the sets A'_i are equal. The same applies for the sets B'_i and C'_i; and since there are only finitely many graphs of bounded size, we may assume that for all $i \geq 1$ the subgraph of G_i induced on $A'_i \cup B'_i \cup C'_i$ is the same. In summary, we may assume that

(2) *There are sets A', B', C', and a graph N with vertex set $A' \cup B' \cup C'$, such that $A'_i = A'$, $B'_i = B'$, and $C'_i = C'$, and $G_i|(A' \cup B' \cup C') = N$, for all $i \geq 1$.*

Let us fix a march π with support $A' \cup B' \cup C'$, and a march π' with support $A' \cup B'$. For each $i \geq 1$, let P_i be the graph obtained from $G_i|(A_i \cup A' \cup B_i \cup B')$ by removing all edges with both ends in A' and making B' a clique. Thus $A_i \cup A'$ is a stable set of P_i and $B_i \cup B'$ is a clique of P_i, and so P_i is a split graph. For each $i \geq 1$, let Q_i be $G_i|(A' \cup B' \cup C' \cup C_i)$. Then (G_i, π) is a rooted graph and belongs to $\mathcal{C}(12\theta(\theta+1), \theta)$.

By 7.2, the set of all rooted graphs (P_i, π') is a wqo under the relation described in 7.2, taking $A_i \cup A'_i$ and $B_i \cup B'_i$ to be the corresponding stable set and clique. By 6.1, $\mathcal{C}(12\theta(\theta+1), \theta)$ is a wqo under Rao-containment. By 5.2, there exist $j > i \geq 1$ such that (P_i, π') is contained in (P_j, π') (under the relation of 7.2) and (Q_i, π) is Rao-contained in (Q_j, π). By combining the corresponding two injections (which agree on the intersection of their domains) we deduce that there is an injection η from $V(G_i)$ into $V(G_j)$, such that

- $\eta(v) = v$ for each $v \in A' \cup B' \cup C'$
- $\eta(v) \in A_j$ for each $v \in A_i$, and $\eta(v) \in B_j$ for each $v \in B_i$, and $\eta(v) \in C_j$ for each $v \in C_i$
- there is a graph P'_j degree-equivalent to P_j such that the restriction of η to $A_i \cup A' \cup B_i \cup B'$ is an isomorphism between P_i and an induced subgraph of P'_j

- there is a graph Q'_j degree-equivalent to Q_j such that the restriction of η to $A' \cup B' \cup C' \cup C_i$ is an isomorphism between Q_i and an induced subgraph of Q'_j.

Let X be the set of edges of $N|A'$, and let Y be the set of nonedges of $N|B'$ (that is, the set of unordered pairs of distinct vertices in B' that are nonadjacent in N). Let $R_i = G_i|(A_i \cup A' \cup B_i \cup B')$, and $R_j = G_j|(A_j \cup A' \cup B_j \cup B')$. Now P_i was obtained from R_i by removing the edges in X and adding as edges all the pairs in Y, and so $E(R_i) = (E(P_i) \setminus Y) \cup X$, and $E(R_j) = (E(P_j) \setminus Y) \cup X$. Let R'_j be the graph with vertex set $V(P'_j)$ and with edge set $(E(P'_j) \setminus Y) \cup X$. It follows that R'_j is degree-equivalent to R_j.

Since η fixes every vertex in $V(N)$, it follows that N is an induced subgraph of Q'_j, and $N|(A' \cup B')$ is an induced subgraph of R'_j. Consequently there is a graph G'_j with vertex set $V(G_j)$, such that R'_j and Q'_j are both induced subgraphs of G'_j. But then G'_j is degree-equivalent to G_j, and so G_j Rao-contains G_i, a contradiction. Thus there is no such sequence G_i $(i = 0, 1, 2, \ldots)$. This proves 1.2. ∎

8 Switching-containment

If G is a digraph, the *underlying graph* of G is the graph obtained from G by removing the directions of its edges, and is denoted by G^-. We say digraphs G, G' are *degree-equivalent* if $G^- = G'^-$ (and therefore $V(G) = V(G')$), and every vertex in $V(G)$ has the same outdegree in G and in G' (and consequently has the same indegree in G and in G'). We say a digraph G *switching-contains* a digraph H if there is a digraph G' degree-equivalent to G, such that H is isomorphic to an induced subdigraph of G'.

Before we go on, we remark that switching-containment is not a wqo of the class of all digraphs. For instance if $\mathcal{C}$ is the class of digraphs G such that G^- is a cycle, then $\mathcal{C}$ contains infinitely many non-isomorphic digraphs and none of them switching-contains another. For tournaments, however, switching-containment yields a wqo (this follows from the main theorem of [1], because if a tournament H can be immersed in a tournament G then G switching-contains H). In this paper we show that switching-containment also yields a wqo for the digraphs whose underlying graph is complete bipartite. We prove the following, which implies 7.2:

8.1 *Let $k \geq 0$ be an integer. For all $i \geq 1$ let G_i be a digraph, let (A_i, B_i) be a bipartition of G_i^- such that every vertex in A_i is adjacent in G_i^- to every vertex in B_i, and let π_i be a march in $V(G_i)$ with length at most k. Then there exist $j > i \geq 1$ and a digraph G' degree-equivalent to G_j and an injection $\eta:\ V(G_i) \to V(G_j)$, with the following properties:*

- *for all distinct $u, v \in V(G_i)$, u is adjacent to v in G_i if and only if $\eta(u)$ is adjacent to $\eta(v)$ in G'*
- *$\pi_j = \eta(\pi_i)$*
- *$\eta(A_i) \subseteq A_j$ and $\eta(B_i) \subseteq B_j$.*

Proof of 7.2, assuming 8.1. For each $i \geq 1$ let (G_i, π_i) be a $(\leq k)$-rooted split graph, and let (A_i, B_i) be a partition of $V(G_i)$ as in 7.2. Let H_i be the digraph with vertex set $V(G_i)$, in which $a \in A_i$ is adjacent to $b \in B_i$ if a, b are adjacent in G_i, and a is adjacent from b in H if a, b are nonadjacent in G_i. Thus H_i^- is complete bipartite, and (A_i, B_i) is a bipartition. By 8.1 we deduce that there exist $j > i \geq 1$ and a digraph H' degree-equivalent to H_j and an injection $\eta : V(H_i) \to V(H_j)$, satisfying the three bullets of 8.1 (with G_i, G_j, G' replaced by H_i, H_j, H'). Let G' be the split graph with vertex set $V(G_j)$ in which A_j is stable, B_j is a clique, and $a \in A_j$ and $b \in B_j$ are adjacent in G' if and only if a is adjacent to b in H'. Then G', G_j are degree-equivalent, and it follows that η provides a Rao-containment of (G_i, π_i) in (G_j, π_j). This proves 7.2. ∎

The remainder of the paper is devoted to proving 8.1.

9 Switching-containment in fixed position

Next we need an analogue of 2.1 for switching-containment of directed complete bipartite graphs. In principle this is already solved, because the translation from split graphs will transform 2.1 into a necessary and sufficient condition for switching-containment of directed complete bipartite graphs; but we will derive a much simpler condition (still necessary and sufficient), that holds for general digraphs, not just directed complete bipartite graphs.

We also need an extension of it to what we call "weighted" digraphs. A *weighted digraph* is a triple (G, m, n) such that G is a digraph and m, n are maps from $V(G)$ to the set of nonnegative integers. If v is a vertex of a digraph G, $d^+(v)$ or $d^+_G(v)$ denote the outdegree of v in G, and $d^-(v)$ or $d^-_G(v)$ denote its indegree. Two weighted digraphs $(G, m, n), (G', m', n')$ are *degree-equivalent* if

- $G^- = G'^-$,
- $\sum_{v \in V(G)} m(v) = \sum_{v \in V(G')} m'(v)$, and $\sum_{v \in V(G)} n(v) = \sum_{v \in V(G')} n'(v)$, and
- for every vertex $v \in V(G)$,

$$d^+_G(v) + n(v) - m(v) = d^+_{G'}(v) + n'(v) - m'(v).$$

Let G be a digraph. If $X \subseteq V(G)$, we denote by $D^+_G(X)$ and $D^-_G(X)$ respectively the sets of all edges uv of G with $X \cap \{u, v\} = \{u\}$ and $X \cap \{u, v\} = \{v\}$. The following is an easy consequence of the max-flow min-cut theorem (or of Hoffman's circulation theorem [3]), and we omit its proof.

9.1 *Let G be a digraph and for every vertex v let $t(v)$ be an integer, such that $\sum_{v \in V(G)} t(v) = 0$. Let $F, F' \subseteq E(G)$, with $F \cap F' = \emptyset$. Then the following are equivalent:*

- *there is a map ϕ from $E(G)$ to $\{0, 1\}$ such that $\phi(e) = 0$ for $e \in F$, and $\phi(e) = 1$ for $e \in F'$, and $\sum_{e \in A(v)} \phi(e) - \sum_{e \in B(v)} \phi(e) = t(v)$ for every vertex v, where $A(v)$ and $B(v)$ denote the sets of edges with tail v and head v respectively*
- *for every subset $X \subseteq V(G)$, $|D^-_G(X) \cap F'| + |D^+_G(X) \cap F| + \sum_{v \in X} t(v) \leq |D^+_G(X)|$.*

We deduce

9.2 *Let (G, m, n) and (H, p, q) be weighted digraphs, such that H^- is an induced subgraph of G^-, and $\sum_{v \in V(G)} m(v) = \sum_{v \in V(H)} p(v)$, and $\sum_{v \in V(G)} n(v) = \sum_{v \in V(H)} q(v)$. Then the following are equivalent:*

- *there is a weighted digraph (G', m', n') degree-equivalent to (G, m, n), such that*
  - *(a)* $G'|V(H) = H$
  - *(b)* $m'(v) = p(v)$ *and* $n'(v) = q(v)$ *for every vertex* $v \in V(H)$*, and*
  - *(c)* $m'(v) = n'(v) = 0$ *for every vertex* $v \in V(G) \setminus V(H)$
- *for every subset* $X \subseteq V(G)$*,*

$$|D_G^+(X)| + \sum_{v \in X}(n(v) - m(v)) \geq |D_H^+(X \cap V(H))| + \sum_{v \in X \cap V(H)} (q(v) - p(v)).$$

Proof. For each vertex $v \in V(G)$, let $t(v) = m(v) - n(v) + q(v) - p(v)$ if $v \in V(H)$ and $t(v) = m(v) - n(v)$ otherwise. Thus $\sum_{v \in V(G)} t(v) = 0$. Let F, F' be the sets of edges uv of G such that $uv, vu \in E(H)$ respectively.

(1) *There exists (G', m', n') as in the first statement of the theorem, if and only if there exists ϕ as in 9.1.*

For suppose that ϕ is as in 9.1. For every vertex $v \in V(G)$, let $m'(v) = p(v)$ and $n'(v) = q(v)$ if $v \in V(H)$, and $m'(v) = n'(v) = 0$ if $v \notin V(H)$. Let G' be obtained from G by reversing the direction of all edges $e \in E(G)$ with $\phi(e) = 1$ (and so $G'|V(H) = H$). Thus (G', m', n') is a weighted digraph, and we claim that (G', m', n') and (G, m, n) are degree-equivalent. We must check the three conditions in the definition of "degree-equivalent". The first we have already seen. For the second,

$$\sum_{v \in V(G')} m'(v) = \sum_{v \in V(H)} p(v) = \sum_{v \in V(G)} m(v),$$

and similarly $\sum_{v \in V(G)} n(v) = \sum_{v \in V(G')} n'(v)$. For the third, let $v \in V(G)$. Then $d_{G'}^+(v) = d_G^+(v) + b - a$, where b is the number of edges e of G with head v and with $\phi(e) = 1$, and a is the number of edges e of G with tail v and $\phi(e) = 1$. Hence $a = \sum_{e \in A(v)} \phi(e)$ and $b = \sum_{e \in B(v)} \phi(e)$, with notation as in 9.1. Since ϕ is as in the first statement of 9.1, it follows that

$$\sum_{e \in A(v)} \phi(e) - \sum_{e \in B(v)} \phi(e) = t(v),$$

and so $a - b = t(v)$. Consequently $d_{G'}^+(v) = d_G^+(v) - t(v)$, and so

$$d_G^+(v) + n(v) - m(v) = d_{G'}^+(v) + n'(v) - m'(v).$$

This proves the third condition in the definition of "degree-equivalent", and so proves that (G', m', n') and (G, m, n) are degree-equivalent. Conversely, by reversing this argument it follows that every weighted digraph satisfying the first statement of the theorem arises from some such ϕ in this way. This proves (1).

From (1) and 9.1 we deduce that the first statement of the theorem holds if and only if

$$|D_G^-(X) \cap F'| + |D_G^+(X) \cap F| + \sum_{v \in X} t(v) \leq |D_G^+(X)|$$

for every subset $X \subseteq V(G)$. But $|D^-_G(X) \cap F'| + |D^+_G(X) \cap F| = |D^+_H(X \cap V(H))|$, and

$$\sum_{v \in X} t(v) = \sum_{v \in X} (m(v) - n(v)) + \sum_{v \in X \cap V(H)} (q(v) - p(v)),$$

so the first statement of the theorem holds if and only if

$$|D^+_H(X \cap V(H))| + \sum_{v \in X} (m(v) - n(v)) + \sum_{v \in X \cap V(H)} (q(v) - p(v)) \leq |D^+_G(X)|,$$

that is,

$$|D^+_G(X)| + \sum_{v \in X} (n(v) - m(v)) \geq |D^+_H(X \cap V(H))| + \sum_{v \in X \cap V(H)} (q(v) - p(v)).$$

This proves 9.2. ∎

10 Contests

A *contest* is a seven-tuple (G, A, B, l, m, n, π), where

- (G, m, n) is a weighted digraph
- (A, B) is a bipartition of G^- and every vertex in A is adjacent in G^- to every vertex in B
- π is a march in $V(G)$, and
- $l \geq 0$ is an integer.

The *type* of a contest (G, l, A, B, m, n, π) is the quadruple

$$(|\bar{\pi}|, l, \sum_{v \in V(G)} m(v), \sum_{v \in V(G)} n(v)).$$

Let $\mathcal{C}_1 = (G_1, A_1, B_1, l_1, m_1, n_1, \pi_1)$ and $\mathcal{C}_2 = (G_2, A_2, B_2, l_2, m_2, n_2, \pi_2)$ be contests. We say that $\mathcal{C}_2$ *switching-contains* $\mathcal{C}_1$ if $l_1 = l_2$ and there is a weighted digraph (G', m', n') degree-equivalent to (G_2, m_2, n_2) (and therefore $(G', A_2, B_2, l_2, m', n', \pi_2)$ is a contest) and an injection $\eta : V(G_1) \to V(G_2)$, with the following properties:

- for all distinct $u, v \in V(G_1)$, u is adjacent to v in G_1 if and only if $\eta(u)$ is adjacent to $\eta(v)$ in G'
- $\pi_2 = \eta(\pi_1)$
- $\eta(A_1) \subseteq A_2$ and $\eta(B_1) \subseteq B_2$
- $m_1(v) = m'(\eta(v))$ and $n_1(v) = n'(\eta(v))$ for each $v \in V(G_1)$, and $m'(v) = n'(v) = 0$ for each $v \in V(G_2) \setminus \eta(V(G_1))$.

We will prove the following, which evidently implies 8.1:

10.1 *Let $\mathcal{C}_i$ $(i = 1, 2, \ldots)$ be contests, all of the same type. Then there exist $j > i \geq 1$ such that $\mathcal{C}_j$ switching-contains $\mathcal{C}_i$.*

11 The pieces after a slicing

Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$ be a contest. A *slice* of $\mathcal{C}$ means a partition (X, Y) of $V(G)$, and its *order* is

$$l + |D^+_G(X)| + \sum_{v \in X} n(v) + \sum_{v \in Y} m(v).$$

We need the following:

11.1 *Let (X, Y) and (X', Y') be slices of a contest $\mathcal{C}$, of order h and h' respectively. Then $(X \cap X', Y \cup Y')$ and $(X \cup X', Y \cap Y')$ are slices and the sum of their orders is at most $h + h'$.*

Proof. Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$. The sum of the orders of $(X \cap X', Y \cup Y')$ and $(X \cup X', Y \cap Y')$ is

$$2l + |D^+_G(X \cap X')| + \sum_{v \in X \cap X'} n(v) + \sum_{v \in Y \cup Y'} m(v) + |D^+_G(X \cup X')| + \sum_{v \in X \cup X'} n(v) + \sum_{v \in Y \cap Y'} m(v).$$

But

$$|D^+_G(X \cap X')| + |D^+_G(X \cup X')| \le |D^+_G(X)| + |D^+_G(X')|$$

since every edge contributes at least as much to the right side as it does to the left; and

$$\sum_{v \in X \cap X'} n(v) + \sum_{v \in X \cup X'} n(v) = \sum_{v \in X} n(v) + \sum_{v \in X'} n(v),$$

and

$$\sum_{v \in Y \cup Y'} m(v) + \sum_{v \in Y \cap Y'} m(v) = \sum_{v \in Y} m(v) + \sum_{v \in Y'} m(v).$$

We deduce that the sum of the orders of $(X \cap X', Y \cup Y')$ and $(X \cup X', Y \cap Y')$ is at most

$$2l + |D^+_G(X)| + |D^+_G(X')| + \sum_{v \in X} n(v) + \sum_{v \in X'} n(v) + \sum_{v \in Y} m(v) + \sum_{v \in Y'} m(v) = h + h'$$

as required. This proves 11.1. ∎

Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$ be a contest, and let $(W_1, \ldots, W_t)$ be a sequence of subsets of $V(G)$, pairwise disjoint and with union $V(G)$ (possibly some of the sets W_i are empty). (Thus $t \ge 1$ unless $V(G) = \emptyset$; however, it is useful to permit $t = 0$ when $V(G) = \emptyset$.) We call $(W_1, \ldots, W_t)$ a *slicing* of $\mathcal{C}$. For each i with $0 \le i \le t$, let $X_i = W_1 \cup \cdots \cup W_i$ and let $Y_i = W_{i+1} \cup \cdots \cup W_t$. Then (X_i, Y_i) is a slice for $0 \le i \le t$. For $p \ge 0$, we say the slicing has *order* at most p if each of the slices (X_i, Y_i) $(0 \le i \le t)$ has order at most p.

Let $(W_1, \ldots, W_t)$ be a slicing of $\mathcal{C} = (G, A, B, l, m, n, \pi)$ and define X_i, Y_i for $0 \le i \le t$ as above. For $1 \le i \le t$, let $\mathcal{C}_i = (G_i, A_i, B_i, l_i, m_i, n_i, \pi_i)$ be the contest defined as follows. Let $G_i = G|W_i$, and $A_i = A \cap W_i, B_i = B \cap W_i$. Let

$$l_i = l + \sum_{v \in X_{i-1}} n(v) + \sum_{v \in Y_i} m(v) + |F_i|$$

where F_i denotes the set of edges of G with tail in X_{i-1} and head in Y_i. For each $v \in W_i$, let $m_i(v) = m(v) + x(v)$, where $x(v)$ denotes the number of vertices in X_{i-1} that are adjacent to v in G, and let $n_i(v) = n(v) + y(v)$ where $y(v)$ denotes the number of vertices in Y_i that are adjacent from v in G. Let π_i be the subsequence of π consisting of those terms that belong to W_i. We call $\mathcal{C}_1, \dots, \mathcal{C}_t$ the *pieces* of $\mathcal{C}$ after the slicing $(W_1, \dots, W_t)$.

We observe:

11.2 *Let $(W_1, \dots, W_t)$ be a slicing of $\mathcal{C} = (G, A, B, l, m, n, \pi)$, and let $\mathcal{C}_1, \dots, \mathcal{C}_t$ be the pieces of $\mathcal{C}$ after the slicing. Let $1 \leq i \leq t$, and let (U, V) be a slice of $\mathcal{C}_i$, of order h say. Then*

$$(W_1 \cup \cdots \cup W_{i-1} \cup U, V \cup W_{i+1} \cup \cdots \cup W_t)$$

is a slice of $\mathcal{C}$, and it has the same order h.

Proof. Let $\mathcal{C}_i = (G_i, A_i, B_i, l_i, m_i, n_i, \pi_i)$. Let $U' = W_1 \cup \cdots \cup W_{i-1} \cup U$ and $V' = V \cup W_{i+1} \cup \cdots \cup W_t$, and let the slice (U', V') of $\mathcal{C}$ have order h'. Thus,

$$h' = l + |D_G^+(U')| + \sum_{v \in U'} n(v) + \sum_{v \in V'} m(v),$$

and

$$h = l_i + |D_{G_i}^+(U)| + \sum_{v \in U} n_i(v) + \sum_{v \in V} m_i(v).$$

We need to show that $h' = h$, that is,

$$l - l_i + |D_G^+(U')| - |D_{G_i}^+(U)| + \sum_{v \in U'} n(v) - \sum_{v \in U} n_i(v) + \sum_{v \in V'} m(v) - \sum_{v \in V} m_i(v) = 0.$$

Let $X = W_1 \cup \cdots \cup W_{i-1}$ and $Y = W_{i+1} \cup \cdots \cup W_k$, and for each $v \in W_i$ let $x(v)$ denote the number of vertices in X that are adjacent to v in G, and $y(v)$ denote the number of vertices in Y that are adjacent from v in G; then $m_i(v) = m(v) + x(v)$, and $n_i(v) = n(v) + y(v)$. Let F denote the set of edges of G with tail in X and head in Y. Now

$$l - l_i = -\sum_{v \in X} n(v) - \sum_{v \in Y} m(v) - |F|.$$

But

$$|D_G^+(U')| - |D_{G_i}^+(U)| = \sum_{v \in U} y(v) + \sum_{v \in V} x(v) + |F|$$

and

$$\sum_{v \in U'} n(v) - \sum_{v \in U} n_i(v) = \sum_{v \in U'} n(v) - \sum_{v \in U} (n(v) + y(v)) = \sum_{v \in X} n(v) - \sum_{v \in U} y(v),$$

and similarly

$$\sum_{v \in V'} m(v) - \sum_{v \in V} m_i(v) = \sum_{v \in Y} m(v) - \sum_{v \in V} x(v).$$

The sum of the right sides of these four equations is zero, and so the sum of the left sides is zero. This proves 11.2. ∎

We need the following lemma.

11.3 *Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$ and $\mathcal{D} = (H, A', B', l, p, q, \rho)$ be contests of the same type. Let $(W_1, \ldots, W_{2t+1})$ be a slicing of $\mathcal{C}$, and let $\mathcal{C}_1, \ldots, \mathcal{C}_{2t+1}$ be the pieces of $\mathcal{C}$ after this slicing; define X_i, Y_i $(0 \le i \le 2t+1)$ as before. Let $(U_1, \ldots, U_t)$ be a slicing of $\mathcal{D}$, and let $\mathcal{D}_1, \ldots, \mathcal{D}_t$ be the pieces of $\mathcal{D}$ after this slicing. Suppose that*

- *for $1 \le i \le t$, $\mathcal{D}_i$ is switching-contained in $\mathcal{C}_{2i}$;*
- *for $1 \le i \le t$ and all j, if U_i contains the jth term of ρ, then W_{2i} contains the jth term of π;*
- *for $0 \le i \le t$, the slices (X_{2i}, Y_{2i}) and (X_{2i+1}, Y_{2i+1}) of $\mathcal{C}$ have the same order, say s_i; and every slice (X, Y) of $\mathcal{C}$ with $X_{2i} \subseteq X$ and $Y_{2i+1} \subseteq Y$ has order at least s_i.*

Then $\mathcal{D}$ is switching-contained in $\mathcal{C}$.

Proof. For $1 \le i \le 2t+1$, let $\mathcal{C}_i = (G_i, A_i, B_i, l_i, m_i, n_i, \pi_i)$, and for $1 \le i \le t$ let $\mathcal{D}_i = (H_i, A'_i, B'_i, l'_i, p_i, q_i, \rho_i)$. For $1 \le i \le t$, since $\mathcal{D}_i$ is switching-contained in $\mathcal{C}_{2i}$, there is an injection of $V(H_i)$ into $V(G_{2i})$ with certain properties, and to simplify the notation we may as well assume that this injection is the identity. Thus

(1) *H^- is an induced subgraph of G^-, and $\rho = \pi$, and $A' \subseteq A$ and $B' \subseteq B$, and $U_i \subseteq W_{2i}$ for $1 \le i \le t$. Moreover, for $1 \le i \le t$, $l'_i = l_{2i}$, and there is a weighted digraph $(G'_{2i}, m'_{2i}, n'_{2i})$, degree-equivalent to (G_{2i}, m_{2i}, n_{2i}), such that*

- *$G'_{2i}|U_i = H_i$,*
- *$m'_{2i}(v) = p_i(v)$ and $n'_{2i}(v) = q_i(v)$ for each $v \in U_i$, and*
- *$m'_{2i}(v) = n'_{2i}(v) = 0$ for each $v \in W_{2i} \setminus U_i$.*

For each $v \in V(G)$, with $v \in W_i$ say, let $x(v)$ denote the number of vertices in X_{i-1} that are adjacent to v in G, and $y(v)$ denote the number of vertices in Y_i that are adjacent from v in G.

(2) *For $0 \le i \le t$, there is a digraph G'_{2i+1} with $G'^-_{2i+1} = G^-_{2i+1}$, such that for each $v \in W_{2i+1}$,*

$$d^+_{G'_{2i+1}}(v) = d^+_{G_{2i+1}}(v) - m_{2i+1}(v) + n_{2i+1}(v).$$

We claim first that $\sum_{v \in W_{2i+1}} (m_{2i+1}(v) - n_{2i+1}(v)) = 0$. For the slices (X_{2i}, Y_{2i}) and (X_{2i+1}, Y_{2i+1}) of $\mathcal{C}$ have the same order, and so

$$|D^+_G(X_{2i})| + \sum_{v \in X_{2i}} n(v) + \sum_{v \in Y_{2i}} m(v) = |D^+_G(X_{2i+1})| + \sum_{v \in X_{2i+1}} n(v) + \sum_{v \in Y_{2i+1}} m(v),$$

that is,

$$|D^+_G(X_{2i})| - |D^+_G(X_{2i+1})| + \sum_{v \in W_{2i+1}} (m(v) - n(v)) = 0.$$

But

$$|D^+_G(X_{2i})| - |D^+_G(X_{2i+1})| = \sum_{v \in W_{2i+1}} (x(v) - y(v))$$

and so $\sum_{v \in W_{2i+1}} (x(v) - y(v) + m(v) - n(v)) = 0$, that is, $\sum_{v \in W_{2i+1}} (n_{2i+1}(v) - m_{2i+1}(v)) = 0$. This proves the claim.

Next, we claim that

$$|D^+_{G_{2i+1}}(X)| \geq \sum_{v \in X} (m_{2i+1}(v) - n_{2i+1}(v))$$

for all $X \subseteq W_{2i+1}$. For let $X \subseteq W_{2i+1}$. Since $(X_{2i} \cup X, Y_{2i} \setminus X)$ is a slice of $\mathcal{C}$ and $X_{2i} \subseteq X_{2i} \cup X$ and $Y_{2i+1} \subseteq Y_{2i} \setminus X$, it follows by hypothesis that this slice has order at least that of the slice (X_{2i}, Y_{2i}). Consequently

$$|D^+_G(X_{2i} \cup X)| + \sum_{v \in X_{2i} \cup X} n(v) + \sum_{v \in Y_{2i} \setminus X} m(v) \geq |D^+_G(X_{2i})| + \sum_{v \in X_{2i}} n(v) + \sum_{v \in Y_{2i}} m(v),$$

that is,

$$|D^+_G(X_{2i} \cup X)| - |D^+_G(X_{2i})| + \sum_{v \in X} (n(v) - m(v)) \geq 0.$$

But

$$|D^+_G(X_{2i} \cup X)| - |D^+_G(X_{2i})| = \sum_{v \in X} (y(v) - x(v)) + |D^+_{G_{2i+1}}(X)|,$$

and so

$$|D^+_{G_{2i+1}}(X)| \geq \sum_{v \in X} (x(v) - y(v) + m(v) - n(v)),$$

that is,

$$|D^+_{G_{2i+1}}(X)| \geq \sum_{v \in X} (m_{2i+1}(v) - n_{2i+1}(v)).$$

This proves our second claim.

From these two claims and 9.1 (setting $F_1 = F_2 = \emptyset$ and $t(v) = m_{2i+1}(v) - n_{2i+1}(v)$ for each v), we deduce (by taking L to be the set of edges e with $\phi(e) = 1$) that there is a set $L \subseteq E(G_{2i+1})$ such that for every vertex $v \in W_{2i+1}$, the number of edges in L with tail v minus the number with head v is equal to $m_{2i+1}(v) - n_{2i+1}(v)$. Let G'_{2i+1} be the digraph obtained from G_{2i+1} by reversing the direction of every edge in L; then G'_{2i+1} satisfies (2). This proves (2).

For all odd i, let $m'_i(v) = n'_i(v) = 0$ for all $v \in W_i$. Thus G'_i, m'_i, n'_i are defined for $1 \leq i \leq 2t+1$. Let G' be the digraph with $G'^- = G^-$ defined as follows. Let u, v be adjacent in G^-, and let $u \in W_i$ and $v \in W_j$ say, where $1 \leq i \leq j \leq 2t+1$. If $i = j$ let u be adjacent to v in G' if and only if u is adjacent to v in G'_i. If $i < j$ let u be adjacent to v in G' if and only if

- i, j are even, say $i = 2i'$ and $j = 2j'$ where $1 \leq i', j' \leq t$, and
- $u \in U_{i'}$ and $v \in U_{j'}$, and
- u is adjacent to v in H.

Thus H is a subdigraph of G'. For each $v \in V(H)$, let $m'(v) = p(v)$ and $n'(v) = q(v)$, and for each $v \in V(G) \setminus V(H)$ let $m'(v) = n'(v) = 0$. Thus (G', m', n') is a weighted digraph, and to complete the proof of the theorem it suffices to show that (G', m', n') is degree-equivalent to (G, m, n).

We must check the three conditions in the definition of "degree-equivalent". The first we have already seen. For the second,

$$\sum_{v \in V(G')} m'(v) = \sum_{v \in V(H)} p(v)$$

from the definition of m'; but

$$\sum_{v \in V(H)} p(v) = \sum_{v \in V(G)} m(v)$$

since $\mathcal{C}, \mathcal{D}$ have the same type. We deduce that

$$\sum_{v \in V(G)} m(v) = \sum_{v \in V(G')} m'(v),$$

and similarly

$$\sum_{v \in V(G)} n(v) = \sum_{v \in V(G')} n'(v).$$

This proves the second condition.

For the third condition, we need some preliminaries. For each $v \in V(G)$, if $v \in V(H)$ and $v \in U_i$ say, let $y'(v)$ be the number of vertices in $U_{i+1} \cup \cdots \cup U_t$ that are adjacent from v in H, and let $x'(v)$ be the number of vertices in $U_1 \cup \cdots \cup U_{i-1}$ that are adjacent to v in H. If $v \in V(G) \setminus V(H)$ let $x'(v) = y'(v) = 0$. We claim that for each $v \in V(G)$, if $v \in W_i$ where $1 \leq i \leq 2t+1$, then $n'_i(v) = n'(v) + y'(v)$. To see this there are two cases, depending whether $v \in V(H)$ or not. If $v \notin V(H)$ then $n'_i(v) = 0$, and $n'(v) = 0$, and $y'(v) = 0$ as required. If $v \in V(H)$ (and hence i is even, $i = 2h$ say), then $n'_i(v) = q_h(v)$; but $q_h(v) = q(v) + y'(v)$ and $q(v) = n'(v)$, and so again $n'_i(v) = n'(v) + y'(v)$. This proves the claim. Similarly $m'_i(v) = m'(v) + x'(v)$.

Now to prove the third condition in the definition of "degree-equivalent", let $v \in W_i$ say. We must check that

$$d^+_{G'}(v) + n'(v) - m'(v) = d^+_G(v) + n(v) - m(v)$$

and

$$d^-_{G'}(v) - n'(v) + m'(v) = d^-_G(v) - n(v) + m(v).$$

The first implies the second, since $G'^- = G^-$, so it suffices to prove the first; and from the symmetry we may assume that $v \in A$. Since $v \in A$, v is adjacent in G (to or from) every vertex in $X_{i-1} \cap B$, and to or from none in $X_{i-1} \cap A$, and since v is adjacent from $x(v)$ vertices in X_{i-1} it follows that there are $|X_{i-1} \cap B| - x(v)$ vertices in X_{i-1} that are adjacent from v in G. Consequently

$$d^+_G(v) = d^+_{G_i}(v) + y(v) + |X_{i-1} \cap B| - x(v),$$

and similarly

$$d^+_{G'}(v) = d^+_{G'_i}(v) + y'(v) + |X_{i-1} \cap B| - x'(v).$$

But

$$d^+_{G'_i}(v) + n'_i(v) - m'_i(v) = d^+_{G_i}(v) + n_i(v) - m_i(v)$$

from the choice of G'_i, m'_i, n'_i. Subtracting the second and third of these equations from the first yields that

$$d^+_G(v) - d^+_{G'}(v) - n'_i(v) + m'_i(v) = y(v) - x(v) - y'(v) + x'(v) - n_i(v) + m_i(v).$$

Since $n'_i(v) = n'(v) + y'(v)$ and $m'_i(v) = m'(v) + x'(v)$, and $n_i(v) = n(v) + y(v)$ and $m_i(v) = m(v) + x(v)$, it follows on substitution that

$$d^+_G(v) + n(v) - m(v) = d^+_{G'}(v) + n'(v) - m'(v).$$

This proves the third condition, and hence that (G', m', n') is degree-equivalent to (G, m, n); and so completes the proof of 11.3. ∎

12 Reduction to incoherence

If $p, q \geq 0$, a subset $Z \subseteq V(G)$ is called *(p, q)-coherent* in a contest (G, A, B, l, m, n, π) if $|Z \cap A|, |Z \cap B| \geq q$, and there is no slice (X, Y) with order less than p and with $X \cap Z, Y \cap Z \neq \emptyset$. Let us say that (G, A, B, l, m, n, π) is *(p, q)-incoherent* if the slices $(\emptyset, V(G))$ and $(V(G), \emptyset)$ both have order less than p, and there is no (p, q)-coherent subset of $V(G)$.

The main part of the proof of 10.1 is to prove the following, which is the same statement as 10.1 but with an extra hypothesis:

12.1 *Let $p, q \geq 0$, and let $\mathcal{C}_i$ $(i = 1, 2, \ldots)$ be (p, q)-incoherent contests, all of the same type. Then there exist $j > i \geq 1$ such that $\mathcal{C}_j$ switching-contains $\mathcal{C}_i$.*

For the moment we shall assume the truth of 12.1, and our object in this section is to deduce 10.1 from it. We say two slices $(X_1, Y_1), (X_2, Y_2)$ *cross* if $X_1 \cap Y_2, X_2 \cap Y_1$ are both nonempty. We need the following lemma.

12.2 *Let Z be a (p, q)-coherent set in a contest $\mathcal{C}$; and let $(X_1, Y_1), (X_2, Y_2)$ be slices both of order less than* $\min(p, q/2)$ *that cross. Then Z is a subset of one of $X_1 \cap X_2, Y_1 \cap Y_2$.*

Proof. Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$. Since Z is (p, q)-coherent, not both $Z \cap X_1, Z \cap Y_1$ are nonempty, and so Z is a subset of one of X_1, Y_1, and similarly of one of X_2, Y_2. From the symmetry we may therefore assume (for a contradiction) that $Z \subseteq X_1 \cap Y_2$. Since the two slices cross, there exists $v \in X_2 \cap Y_1$. Since $|Z \cap A|, |Z \cap B| \geq q$ and $\mathcal{C}$ is a contest, it follows that there are at least q edges of G with one end v and the other end in Z. But fewer than $q/2$ of these edges are directed from v to Z, since (X_2, Y_2) has order less than $q/2$; and fewer than $q/2$ are directed from Z to v, since (X_1, Y_1) has order less than $q/2$, a contradiction. This proves 12.2. ∎

We also need:

12.3 *Let $(W_1, \ldots, W_t)$ be a slicing of a contest $\mathcal{C}$, with order at most p; and let $\mathcal{C}_1, \ldots, \mathcal{C}_t$ be the pieces of $\mathcal{C}$ after this slicing. Suppose that Z is $(3p, q)$-coherent in $\mathcal{C}_i$, where $1 \leq i \leq t$. Then Z is (p, q)-coherent in $\mathcal{C}$.*

Proof. Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$. Certainly $|Z \cap A|, |Z \cap B| \geq q$, since Z is $(3p, q)$-coherent in $\mathcal{C}_i$. Suppose that (U, V) is a slice of $\mathcal{C}$, with $U \cap Z, V \cap Z$ both nonempty; we shall prove that (U, V) has order at least p. For let its order be h say. Let $X = W_1 \cup \cdots \cup W_{i-1}$, and $Y = W_{i+1} \cup \cdots \cup W_t$. From 11.1 applied to the slices (U, V) and $X, W_i \cup Y)$, it follows that the slice $(U \cup X, V \cap (W_i \cup Y))$ has order at most $p + h$; and by 11.1 again, applied to this slice and $(X \cup W_i, Y)$, we deduce that the slice $((U \cup X) \cap (X \cup W_i), (V \cap (W_i \cup Y)) \cup Y)$ has order at most $2p + h$, that is, the slice $(X \cup (U \cap W_i), (V \cap W_i) \cup Y)$ has order at most $2p+h$. By 11.2 it follows that the slice $(U \cap W_i, V \cap W_i)$ of $\mathcal{C}_i$ has order at most $2p + h$. But this slice has order at least $3p$ since Z is $(3p, q)$-coherent in $\mathcal{C}_i$, and both $U \cap W_i, V \cap W_i$ have nonempty intersection with Z. We deduce that $h \geq p$ as claimed. This proves 12.3. ∎

Proof of 10.1, assuming 12.1. Let $T = (T_1, T_2, T_3, T_4)$ be a quadruple of non-negative integers. A *bad sequence* for T is an infinite sequence of contests $\mathcal{C}_i$ $(i = 1, 2 \ldots)$, all of type T, such that there do not exist $j > i \geq 1$ such that $\mathcal{C}_j$ switching-contains $\mathcal{C}_i$. We say the quadruple T is *bad* if there exists a bad sequence for T, and *good* otherwise. We need to prove that every quadruple is good.

Suppose not; then we may choose a bad quadruple T as follows:

- first, with T_1 as small as possible
- subject to that, with $T_2 + T_3 + T_4$ as small as possible.

Let $\mathcal{C}_i$ $(i = 1, 2 \ldots)$ be a bad sequence for T, and let $\mathcal{C}_i = (G_i, A_i, B_i, l_i, m_i, n_i, \pi_i)$ for $i \geq 1$.

(1) *We may assume that for all $j > i \geq 1$, the map sending π_i to π_j is an isomorphism from $G_i|\overline{\pi_i}$ to $G_j|\overline{\pi_j}$, and for $1 \leq h \leq T_1$, the* h*th term of π_i belongs to A_i if and only if the* h*th term of π_j belongs to A_j.*

For there are only finitely many possibilities for the (labelled) isomorphism class of

$$(G_i|\overline{\pi_i}, A_i \cap \overline{\pi_i}, B_i \cap \overline{\pi_i}),$$

and so we may assume they are all the same, by passing to an infinite subsequence. This proves (1).

Let $q = |V(G_1)| + 2\max(T_2 + T_3 + T_4)$, and let $p = |E(G_1)| + T_2 + T_3 + T_4$.

(2) *We may assume that for all $i \geq 2$, if (X, Y) is a slice in G_i, and h denotes its order, then:*

- $h \geq \min(T_2 + T_3, T_2 + T_4)$
- *if $h < T_2 + T_3$ then $\overline{\pi_i} \subseteq X$ and some subset of X is (p, q)-coherent; and if $h < T_2 + T_4$ then $\overline{\pi_i} \subseteq Y$ and some subset of Y is (p, q)-coherent.*
- *if $h < p$ then either $\overline{\pi_i} \subseteq X$ and some subset of X is (p, q)-coherent, or $\overline{\pi_i} \subseteq Y$ and some subset of Y is (p, q)-coherent.*

For if there are only finitely many values of i that do not satisfy (2), then we may remove them from the sequence and (2) would follow. Thus we assume there are infinitely many values of $i \geq 2$ with slices that fail to satisfy (2), and so by passing to a subsequence we may assume that for all

$i \geq 2$ there is a slice (X_i, Y_i) in G_i failing to satisfy (2). It follows that (X_i, Y_i) has order at most $\max(T_2 + T_3, T_2 + T_4, p)$, and so by passing to an infinite subsequence we may assume that all the slices (X_i, Y_i) have the same order h say. Let $i \geq 2$, and let $\mathcal{C}'_i, \mathcal{C}''_i$ be the pieces of $\mathcal{C}_i$ under the slicing (X_i, Y_i). Let $\mathcal{C}'_i$ have type $T' = (T'_1, T'_2, T'_3, T'_4)$; then $T'_1 \leq T_1$, and $T'_3 \leq T_3$, and $T'_2 + T'_4 \leq h$. By passing to a subsequence we may assume that for each $i \geq 2$, the type of $\mathcal{C}'_i$ is the same; that is, T' does not depend on i. Similarly, we may assume that for each $i \geq 2$, $\mathcal{C}''_i$ is a contest of type T'', where $T'_1 + T''_1 = T_1$ and $T'_4 \leq T_4$, and $T'_2 + T'_3 \leq h$. Moreover, we may assume that for $1 \leq j \leq T_1$, if there exists $i \geq 2$ such that the jth term of π_i belongs to X_i, then the jth term of π_i belongs to X_i for all $i \geq 2$. (Note that in these arguments where we replace our infinite sequence by an infinite subsequence, it is important that the first term is unchanged, since p, q are defined by means of the first term; and so we cannot assume that the statement of (2) holds for all $i \geq 1$.)

Suppose that switching-containment defines a wqo on the set of all contests $\mathcal{C}'_i$ $(i \geq 2)$, and also on the set of all contests $\mathcal{C}''_i$ $(i \geq 2)$. From 5.3 it follows that there exist $j > i \geq 2$ such that $\mathcal{C}'_i$ is switching-contained in $\mathcal{C}'_j$, and $\mathcal{C}''_i$ is switching-contained in $\mathcal{C}''_j$. But then $\mathcal{C}_i$ is switching-contained in $\mathcal{C}_j$ by 11.3, a contradiction.

From the symmetry, we may therefore assume that switching-containment does not define a wqo on the set of all contests $\mathcal{C}'_i$ $(i \geq 2)$, and consequently T' is a bad quadruple. Since $T'_1 \leq T_1$, and $T'_2 \leq T_2$, and $T'_3 + T'_4 \leq h$, it follows from the choice of T that $T'_1 = T_1$, and so $\overline{\pi_i} \subseteq X_i$ for each $i > 1$. Since $T'_3 \leq T_3$, and $T'_2 + T'_4 \leq h$, the choice of T implies that $T_2 + T_4 \leq h$. For $i \geq 2$, since (X_i, Y_i) does not satisfy (2), it follows that $h < p$ and no subset of X_i is (p, q)-coherent in $\mathcal{C}_i$; and hence, by 12.3, no subset of X_i is $(3p, q)$-coherent in $\mathcal{C}'_i$. But then by 12.1, switching-containment defines a wqo on the set of all $\mathcal{C}'_i$ $(i \geq 2)$, a contradiction. This proves (2).

From the symmetry we may assume that $T_3 \leq T_4$.

(3) *Let $i \geq 2$ and let Z be (p, q)-coherent in G_i. Then either*

- *$T_1 > 0$ and there is a slice (X, Y) of $\mathcal{C}_i$ of order less than p, such that Z is a subset of one of X, Y and $\overline{\pi_i}$ is a subset of the other, or*
- *$T_1 = 0$ and there is a slice (X, Y) of $\mathcal{C}_i$ of order less than $T_2 + T_4$, with $Z \subseteq X$.*

For let us construct an injection $\eta : V(G_1) \to V(G_i)$ as follows. First, let $\eta(\pi_1) = \pi_i$. Let η map $A_1 \setminus \overline{\pi}_1$ injectively into $(Z \cap A_i) \setminus \overline{\pi}_i$ (this is possibly since $|Z \cap A_i| \geq q$) and similarly let η map $B_1 \setminus \overline{\pi}_1$ injectively into $(Z \cap B_i) \setminus \overline{\pi}_i$. Since $\mathcal{C}_1$ and $\mathcal{C}_i$ have the same type, it follows that $l_i = l_1$, and $\sum_{v \in V(G_1)} m_1(v) = \sum_{v \in V(G_i)} m_i(v)$, and $\sum_{v \in V(G_1)} n_1(v) = \sum_{v \in V(G_i)} n_i(v)$. Since $\mathcal{C}_i$ does not switching-contain $\mathcal{C}_1$, there is no weighted digraph (G', m', n') degree-equivalent to (G_i, m_i, n_i) with the following properties:

- for all distinct $u, v \in V(G_1)$, u is adjacent to v in G_1 if and only if $\eta(u)$ is adjacent to $\eta(v)$ in G'
- $m_1(v) = m'(\eta(v))$ and $n_1(v) = n'(\eta(v))$ for each $v \in V(G_1)$, and $m'(v) = n'(v) = 0$ for each $v \in V(G_i) \setminus \eta(V(G_1))$.

From 9.2 we deduce that there exists $X \subseteq V(G_i)$ such that, if we denote $\{v \in V(G_1) : \eta(v) \in X\}$ by X_1, then

$$|D^+_{G_i}(X)| + \sum_{v \in X}(n_i(v) - m_i(v)) < |D^+_{G_1}(X_1))| + \sum_{v \in X_1}(n_1(v) - m_1(v)).$$

Let $Y = V(G_i) \setminus X$ and $Y_1 = V(G_1) \setminus X_1$; then since $\sum_{v \in V(G_i)} m_i(v) = \sum_{v \in V(G_1)} m_1(v)$, we deduce by adding that

$$|D^+_{G_i}(X)| + \sum_{v \in X} n_i(v) + \sum_{v \in Y} m_i(v) < |D^+_{G_1}(X_1))| + \sum_{v \in X_1} n_1(v) + \sum_{v \in Y_1} m_1(v);$$

that is, $h < h_1$, where h is the order of the slice (X, Y) of $\mathcal{C}_i$, and h_1 is the order of the slice (X_1, Y_1) of $\mathcal{C}_1$. Since $h_1 \le p$ from the definition of p, we deduce that $h < p$. Since Z is (p, q)-coherent, it follows that one of X, Y includes Z. By the third assertion of (2), either $\overline{\pi_i} \subseteq X$ or $\overline{\pi_i} \subseteq Y$.

By the first assertion of (2), $h \ge T_2 + T_3$, since $T_3 \le T_4$. Consequently $h_1 > T_2 + T_3$, and so $X_1 \ne \emptyset$. Hence there exists $v \in V(G_1)$ such that $\eta(v) \in X$; and since $\eta(v) \in Z \cup \overline{\pi_i}$ from the construction of η, we deduce that $Z \cup \overline{\pi_i}$ is not a subset of Y, and so at least one of $Z, \overline{\pi_i}$ is a nonempty subset of X. If the other is a nonempty subset of Y, then $T_1 > 0$ and (3) holds, so we may assume that $Z \cup \overline{\pi_i} \subseteq X$. Hence $\eta(v) \in X$ for all $v \in V(G_1)$, and so $X_1 = V(G_1)$ and $Y_1 = \emptyset$. Consequently $h_1 = T_2 + T_4$. Since $h < h_1$, it follows that $h < T_2 + T_4$. By the second assertion of (2) it follows that $\overline{\pi_i} \subseteq Y$, and since we have already seen that $\overline{\pi_i} \subseteq X$, it follows that $T_1 = 0$, and again the claim holds. This proves (3).

(4) $T_1 > 0$.

For suppose that $T_1 = 0$. By 12.1 for some $i \ge 2$ there exists a set Z that is (p, q)-coherent in $\mathcal{C}_i$; and by (3) there there is a slice (X, Y) of G_i of order less than $T_2 + T_4$, with $Z \subseteq X$. Choose such a slice (X, Y) with Y minimal. By the second assertion of (2), there is a (p, q)-coherent set $Z' \subseteq Y$. By (3) there is a slice (X', Y') of $\mathcal{C}_i$ of order less than $T_2 + T_4$, with $Z' \subseteq X'$. Since $Z' \subseteq Y \setminus Y'$, 12.2 implies that $Y' \subseteq Y$, contrary to the minimality of Y. This proves (4).

(5) *For every $i \ge 2$, there is a slicing (L_i, M_i, R_i) of $\mathcal{C}_i$, of order less than p, such that $\overline{\pi_i} \subseteq M_i$ and every (p, q)-coherent set Z is a subset of one of L_i, R_i.*

For let $i \ge 2$. Since $(\emptyset, V(G_i))$ is a slice of order $T_2 + T_3 < p$, it follows that there is a slice (U, V) of $\mathcal{C}_i$ with $\overline{\pi_i} \subseteq V$ of order less than p; choose such a slice (U, V) with U maximal. Similarly choose a slice (U', V') of order less than p with $\overline{\pi_i} \subseteq U'$, with V' maximal. Now $U' \cap V \ne \emptyset$, since it includes $\overline{\pi_i}$. Suppose first that $(U, V), (U', V')$ cross. The sum of the orders of (U, V) and (U', V') is at most $2p - 2$, and so by 11.1, one of the slices $(U \cap U', V \cup V'), (U \cup U', V \cap V')$ has order less than p, and from the symmetry we may assume the first. But then $(U \cap U', U \cap V', V)$ is a slicing of order less than p. Moreover, by 12.2, every (p, q)-coherent set is a subset of one of $U \cap U', V \cap V'$, and in particular is a subset of one of $U \cap U', V$; and so we may set $(L_i, M_i, R_i) = (U \cap U', U \cap V', V)$.

Thus we may assume that $(U, V), (U', V')$ do not cross, and so $U \cap V' = \emptyset$. Hence $(U, U' \cap V, V')$ is a slicing of order less than p. Suppose that there is a (p, q)-coherent set Z that is not a subset of one of U, V'. Since (U, V) and (U', V') both have order less than p and Z is (p, q)-coherent, it follows that $Z \subseteq U' \cap V$. By (3) there is a slice (X, Y) of $\mathcal{C}_i$ of order less than p, such that Z is a subset of one of X, Y and $\overline{\pi_i}$ is a subset of the other; and from the symmetry we may assume that $Z \subseteq X$ and $\overline{\pi_i} \subseteq Y$. Since $Z \subseteq X \cap U'$, 12.2 applied to (U, V) and (X, Y) implies that these two slices do not cross, and so $U \cap Y = \emptyset$; but then $U \subseteq X$, contrary to the maximality of U. This proves that every (p, q)-coherent set is a subset of one of U, V', and hence we may take $(L_i, M_i, R_i) = (U, U' \cap V, V')$. This proves (5).

Now since $T_1 > 0$, our choice of the bad quadruple T implies that every quadruple of non-negative integers with first term zero is good. There are three pieces of $\mathcal{C}_i$ after the slicing described in (5), say $\mathcal{L}_i, \mathcal{M}_i, \mathcal{R}_i$. Since there are only a finite number of possibilities for the type of $\mathcal{L}_i$, we may assume (by passing to an infinite subsequence) that all the contests $\mathcal{L}_i$ $(i \geq 2)$ have the same type; and the same holds for $\mathcal{M}_i$ $(i \geq 2)$ and $\mathcal{R}_i$ $(i \geq 2)$.

Now each $\mathcal{L}_i$ with $i \geq 2$ has type with first term zero, since $\overline{\pi_i} \cap W_i = \emptyset$; and so this type is good, as we already saw. Thus switching-containment defines a wqo on the set of all contests $\mathcal{L}_i$ $(i \geq 2)$, and the same for $\mathcal{R}_i$ $(i \geq 2)$.

Since for $i \geq 2$, no subset of M_i is (p, q)-coherent in $\mathcal{C}_i$, it follows from 12.3 that no subset of M_i is $(3p, q)$-coherent in $\mathcal{M}_i$; and so by 12.1, switching-containment defines a wqo on the set of all contests $\mathcal{M}_i$ $(i \geq 2)$. By 5.3, there exist $j \geq i \geq 2$ such that $\mathcal{L}_i, \mathcal{M}_i, \mathcal{R}_i$ are switching-contained in $\mathcal{L}_j, \mathcal{M}_j, \mathcal{R}_j$ respectively, and then by 11.3, it follows that $\mathcal{C}_i$ is switching-contained in $\mathcal{C}_j$, a contradiction. This proves 10.1. ∎

13 Linked slicings

Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$ be a contest, and let $(W_1, \ldots, W_t)$ be a slicing of $\mathcal{C}$. For $0 \leq i \leq t$, let $X_i = W_1 \cup \cdots \cup W_i$ and $Y_i = W_{i+1} \cup \cdots \cup W_t$. We say this slicing is *linked* if for all h, j with $0 \leq h \leq j \leq t$, if the slices (X_h, Y_h) and (X_j, Y_j) have the same order, say c, and each of the slices (X_i, Y_i) $(h \leq i \leq j)$ has order at least c, then every slice (X, Y) with $X_h \subseteq X$ and $Y_j \subseteq Y$ has order at least c.

If $\mathcal{S}$ is a class of contests, and $(W_1, \ldots, W_t)$ is a slicing of a contest $\mathcal{C}$ such that all the pieces of $\mathcal{C}$ after this slicing belong to $\mathcal{S}$, we say that $\mathcal{C}$ *admits a slicing over* $\mathcal{S}$, and if $(W_1, \ldots, W_t)$ is linked, we say that $\mathcal{C}$ *admits a linked slicing over* $\mathcal{S}$. We need:

13.1 *Let $\mathcal{S}$ be a class of contests that is a wqo under switching-containment, and let T be a quadruple of non-negative integers, and let $p \geq 0$. Then the class of all contests of type T that admit a linked slicing over $\mathcal{S}$ of order at most p is also a wqo under switching-containment.*

Proof. Let $T = (T_1, \ldots, T_4)$, and let $\mathcal{R}$ be the class of all pairs $(\mathcal{C}, J)$, where $\mathcal{C} \in \mathcal{S}$ and $J \subseteq \{1, \ldots, T_1\}$. We say $(\mathcal{C}, J) \leq (\mathcal{C}', J')$ if $\mathcal{C}'$ switching-contains $\mathcal{C}$ and $J' = J$. Since there are only finitely many possibilities for J, this order relation is a wqo on $\mathcal{R}$.

Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$ be a contest of type T that admits a linked slicing $(W_1, \ldots, W_t)$ over $\mathcal{S}$, of order at most p. For $1 \leq i \leq t$, let $(x_1, \ldots, x_{2t+1})$ be a sequence defined as follows. Let $\mathcal{C}_1, \ldots, \mathcal{C}_t$ be the pieces of $\mathcal{C}$ after the slicing $(W_1, \ldots, W_t)$, and for $1 \leq i \leq t$, let $x_{2i} = (\mathcal{C}_i, J_i)$, where J_i is the set of all $j \in \{1, \ldots, T_1\}$ such that the jth term of π belongs to W_i. Thus $x_{2i} \in \mathcal{R}$. For

$0 \leq i \leq t$, let x_{2i+1} be the order of the slicing

$$(W_1 \cup \cdots \cup W_i, W_{i+1} \cup \cdots \cup W_t)$$

of $\mathcal{C}$. (Thus $x_1 = T_2 + T_3$ and $x_{2t+1} = T_2 + T_4$, and $x_1, x_3, \ldots, x_{2t+1} \leq p$, and in particular $T_2 + T_3, T_2 + T_4 \leq p$.) Let us call $(x_1, \ldots, x_{2t+1})$ the *dissection* of $\mathcal{C}$ after $(W_1, \ldots, W_t)$. We see that the dissection $(x_1, \ldots, x_{2t+1})$ belongs to $\mathcal{R}^{<\omega}(p)$ (as defined before 5.4), where $\mathcal{R}$ is ordered as described above.

Now suppose that we have an infinite sequence of contests of type T that admit linked slicings over $\mathcal{S}$, of order at most p. Then we have a corresponding infinite sequence of dissections, that all belong to $\mathcal{R}^{<\omega}(p)$. By 5.4, one of these dissections is at most some later one (where the "less than" relation is the order relation of $\mathcal{R}^{<\omega}(p)$). But then by 11.3, it follows that the first contest is switching-contained in the second. This proves 13.1. ∎

14 Dissecting incoherence

It remains to prove 12.1. Our objective in this section is to show that 12.1 is implied by a special case of itself, 14.2 below. But first we need some definitions.

Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$ be a contest. A *$\mathcal{C}$-slice sequence* is a sequence of slices (X_i, Y_i) $(0 \leq i \leq t)$ of $\mathcal{C}$, satisfying $X_i \subseteq X_j$ and $Y_j \subseteq Y_i$ for $0 \leq i < j \leq t$, and $X_0 = Y_t = \emptyset$. If (X_i, Y_i) $(0 \leq i \leq t)$ is a $\mathcal{C}$-slice sequence, let $W_i = Y_{i-1} \cap X_i$ for $1 \leq i \leq t$; then $(W_1, \ldots, W_t)$ is a slicing, that we call the *corresponding slicing*. Conversely, if $(W_1, \ldots, W_t)$ is a slicing of $\mathcal{C}$, let $X_i = W_1 \cup \cdots \cup W_i$ and $Y_i = W_{i+1} \cup \cdots \cup W_t$ for $0 \leq i \leq t$; then (X_i, Y_i) $(0 \leq i \leq t)$ is a $\mathcal{C}$-slice sequence, that we call the *corresponding $\mathcal{C}$-slice sequence*. Thus a $\mathcal{C}$-slice sequence gives another way to describe a slicing of $\mathcal{C}$, sometimes more convenient.

Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$ be a contest. We call $(\emptyset, V(G))$ and $(V(G), \emptyset)$ its *end-slices*. A subset $Z \subseteq V(G)$ is *$(0,p)$-small* if and only if $Z = \emptyset$. A subset $Z \subseteq V(G)$ is *$(1,p)$-small* if $\min(|A \cap Z|, |B \cap Z|) \leq 2p - 2$. Inductively, for $k \geq 2$, a subset $Z \subseteq V(G)$ is *(k,p)-small* if there is a partition (Z_1, Z_2) of Z such that there are fewer than p edges from Z_1 to Z_2, and Z_1, Z_2 are $(k-1,p)$-small. We observe that every subset of a (k,p)-small set is also (k,p)-small.

14.1 *Let $(X, Y), (X', Y')$ be slices of a contest $\mathcal{C}$ that cross, both of order less than p. Then $X \cap Y', X' \cap Y$ are both $(1,p)$-small.*

Proof. Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$. Since $(X, Y), (X', Y')$ cross, there exists $v \in X \cap Y'$. Since (X, Y) has order less than p, v is adjacent to at most $p-1$ members of $X' \cap Y$, and since (X', Y') has order at most $p-1$, v is adjacent from at most $p-1$ members of $X' \cap Y$. From the symmetry between A and B, we may assume that $v \in A$, and so v is adjacent to or from every vertex in $X' \cap Y \cap B$; and consequently $|X' \cap Y \cap B| \leq 2p - 2$. it follows that $X' \cap Y$ is $(1,p)$-small, and similarly so is $X \cap Y'$. This proves 14.1. ∎

Now, let (X, Y) and (X', Y') be slices of $\mathcal{C}$, that do not cross. We say that the two slices are *(k,p)-close* if they both have order less than p, and $(X_1 \cup X_2) \cap (Y_1 \cup Y_2)$ is (k,p)-small. We say $\mathcal{C}$ is *(k,p)-convex* if both end-slices have order less than p and every slice (X, Y) of $\mathcal{C}$ of order less than p is (k,p)-close to one of the end-slices. The following is 12.1 with an extra hypothesis.

14.2 *Let $p, q \geq 0$, and let $\mathcal{C}_i$ $(i = 1, 2, \ldots)$ be (p,p)-convex (p,q)-incoherent contests, all of the same type. Then there exist $j > i \geq 1$ such that $\mathcal{C}_j$ switching-contains $\mathcal{C}_i$.*

As we said, the objective of this section is to show that 14.2 implies 12.1. The main part of the proof is the following.

14.3 *Let $p, q \geq 0$, and let $\mathcal{S}$ be the class of all contests that are (p,p)-convex and (p,q)-incoherent. Let $\mathcal{C}$ be a (p,q)-incoherent contest. Then $\mathcal{C}$ admits a linked slicing over $\mathcal{S}$ of order less than p.*

Proof. Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$. A slicing $(W_1, \ldots, W_t)$ of $\mathcal{C}$ is *generous* if it satisfies the following, where (X_i, Y_i) $(0 \leq i \leq t)$ is the corresponding $\mathcal{C}$-slice sequence, and h_i is the order of the slice (X_i, Y_i) for $0 \leq i \leq t$:

- the slicing has order less than p, and
- for $1 \leq i < j < t$, the slices (X_i, Y_i) and (X_j, Y_j) are not $(|h_i - h_j|, p)$-close.

(1) *If $(W_1, \ldots, W_t)$ is a generous slicing, then $t \leq |V(G)| + 2$.*

For let $2 \leq i \leq t-1$; then since the slicing is generous, the slices $(W_1 \cup \cdots \cup W_{i-1}, W_i \cup \cdots \cup W_t)$ and $(W_1 \cup \cdots \cup W_i, W_{i+1} \cup \cdots \cup W_t)$ are not $(0,p)$-close, and so $W_i \neq \emptyset$. Since $W_2, \ldots, W_{t-1}$ are all non-empty, this proves (1).

Let $(W_1, \ldots, W_t)$ be a generous slicing of $\mathcal{C}$, and let (X_i, Y_i) $(0 \leq i \leq t)$ be the corresponding $\mathcal{C}$-slice sequence. For $0 \leq i \leq t$, let h_i be the order of (X_i, Y_i). The *spectrum* of this slicing is the sequence $(s_j : j \geq 0)$, where s_j denotes the number of values of $i \in \{1, \ldots, t-1\}$ such that $h_i = j$. (Consequently $s_j = 0$ for all sufficiently large j.) If $(V_1, \ldots, V_s)$ is another generous slicing of $\mathcal{C}$, with spectrum $(r_j : j \geq 0)$, we say that $(V_1, \ldots, V_s)$ is *better* than $(W_1, \ldots, W_t)$ if there exists $j \geq 0$ such that $r_j > s_j$ and $r_i = s_i$ for $0 \leq i < j$. We say that a generous slicing $(W_1, \ldots, W_t)$ is *optimal* if it has order less than p, and no generous slicing of order less than p is better.

Since both end-slices of $\mathcal{C}$ have order less than p (from the definition of (p,q)-incoherent), it follows that $(V(G))$ is a generous slicing. Consequently, (1) implies that there is an optimal generous slicing.

Let $(W_1, \ldots, W_t)$ be an optimal generous slicing. We shall prove that $(W_1, \ldots, W_t)$ satisfies the theorem. We need therefore to show that $(W_1, \ldots, W_t)$ is linked and each piece after the slicing belongs to $\mathcal{S}$. Let (X_i, Y_i) $(0 \leq i \leq t)$ be the corresponding $\mathcal{C}$-slice sequence, and for $0 \leq i \leq t$, let h_i be the order of (X_i, Y_i).

(2) *If (X, Y) is a slice of order $h < p$, then either*

- *there exists i with $1 \leq i \leq t-1$ such that $h_i \leq h$ and (X, Y) crosses (X_i, Y_i), or*
- *there exists i with $1 \leq i \leq t-1$ such that $h_i \leq h$ and (X, Y) is $(h - h_i, p)$-close to (X_i, Y_i).*

For let I be the set of all $i \in \{1, \ldots, t-1\}$ such that (X, Y) does not cross (X_i, Y_i) and (X, Y) is not $(|h - h_i|, p)$-close to (X_i, Y_i). Then the set of slices

$$\{(X_i, Y_i) \ (i \in I \cup \{0, t\})\} \cup (X, Y)$$

can be ordered to be the $\mathcal{C}$-slice sequence of a generous slicing; and from the optimality of $(W_1, \ldots, W_t)$, this generous slicing is not better than $(W_1, \ldots, W_t)$. Since (X, Y) is a slice of this new $\mathcal{C}$-slice sequence, it follows that there exists i with $1 \le i \le t-1$ such that $h_i \le h$ and $i \notin I$. This proves (2).

(3) *Let (X, Y) be a slice of $\mathcal{C}$, of order $h < p$, and let $1 \le i \le t-1$, such that $h < h_i$, and (X, Y) and (X_i, Y_i) do not cross. Then they are not $(h_i - h, p)$-close.*

For suppose the statement is false, and choose i with $1 \le i \le t-1$ and a slice (X, Y), such that

- (X, Y) and (X_i, Y_i) do not cross
- $h < h_i$, where h is the order of (X, Y)
- (X, Y) and (X_i, Y_i) are $(h_i - h, p)$-close
- subject to the previous three conditions, h_i is minimum
- subject to the previous four conditions, $(X \cup X_i) \cap (Y \cup Y_i)$ is minimal.

Suppose first that (X, Y) crosses (X_j, Y_j), for some $j \in \{1, \ldots, t-1\}$ with $h_j \le h$. From 14.1, $X \cap Y_j$ is $(1, p)$-small. From the symmetry we may assume that $i \le j$; and so $i < j$ since (X, Y) and (X_i, Y_i) do not cross. Thus $X_i \subseteq X_j$. Now since (X, Y) crosses (X_j, Y_j), and hence $\emptyset \ne X \cap Y_j \subseteq Y_i$, it follows that $X \not\subseteq X_i$, and so $X_i \subseteq X$. Let the slice $(X \cup X_j, Y \cap Y_j)$ have order h'. If $h' < h_j$, then (X_j, Y_j) and $(X \cup X_j, Y \cap Y_j)$ are $(1, p)$-close and hence $(h_j - h', p)$-close, contrary to the fourth bullet above in the choice of i and (X, Y). Thus $h' \ge h_j$. By 11.1, the slice $(X \cap X_j, Y \cup Y_j)$ has order at most h. Now $X \cap Y_i$ is $(h_i - h, p)$-small, since (X_i, Y_i) and (X, Y) are $(h_i - h, p)$-close. It follows that $X \cap X_j \cap Y_i$ is also $(h_i - h, p)$-small, since it is a subset of an $(h_i - h, p)$-small set; and so $(X \cap X_j, Y \cup Y_j)$ is $(h_i - h, p)$-close to (X_i, Y_i), contrary to the fifth bullet above. Thus, there is no $j \in \{1, \ldots, t-1\}$ such that (X, Y) crosses (X_j, Y_j) and $h_j \le h$.

From (2) it follows that there exists j with $1 \le j \le t-1$ such that $h_j \le h$ and (X, Y) is $(h - h_j, p)$-close to (X_j, Y_j). If $h = h_j$ then (X, Y) is $(0, p)$-close to (X_j, Y_j) and so $(X, Y) = (X_j, Y_j)$, which is impossible since (X, Y) and (X_i, Y_i) are $(h_i - h, p)$-close, and (X_j, Y_j) and (X_i, Y_i) are not $(h_i - h_j, p)$-close. Thus $h_j < h$. From the symmetry we may assume that $i \le j$; and so $i < j$, since $h_j < h < h_i$. Consequently $Y_i \cap X_j$ is not $(h_i - h_j, p)$-small. Since (X, Y) crosses neither of $(X_i, Y_i), (X_j, Y_j)$, there are three cases: $X \subseteq X_i \subseteq X_j$, or $X_i \subseteq X \subseteq X_j$, or $X_i \subseteq X_j \subseteq X$. In the first case, since $Y \cap X_j$ is $(h - h_j, p)$-small, it follows that $Y_i \cap X_j$ is $(h - h_j, p)$-small and hence $(h_i - h_j, p)$-small, a contradiction. In the second case, since $Y_i \cap X$ is $(h_i - h, p)$-small, and $Y \cap X_j$ is $(h - h_j, p)$-small, it follows that $Y_i \cap X$ and $Y \cap X_j$ are both $(h_i - h_j - 1, p)$-small; and since there are fewer than p edges from $Y_i \cap X$ to $Y \cap X_j$ (because (X, Y) has order less than p), we deduce that $Y_i \cap X_j$ is $(h_i - h_j, p)$-small, a contradiction. In the third case, since $Y_i \cap X_j \subseteq Y_i \cap X$, and $Y_i \cap X$ is $(h_i - h, p)$-small, it follows that $Y_i \cap X_j$ is $(h_i - h, p)$-small and hence $(h_i - h_j, p)$-small, a contradiction. This proves (3).

(4) $(W_1, \ldots, W_t)$ *is linked.*

Let $0 \le i \le k \le t$, and suppose the slices (X_i, Y_i) and (X_k, Y_k) have the same order, say c, and each of the slices (X_j, Y_j) $(i \le j \le k)$ has order at least c. We must show that every slice (X, Y)

with $X_i \subseteq X$ and $Y_k \subseteq Y$ has order at least c. Thus, suppose (X,Y) is a slice with $X_i \subseteq X$ and $Y_k \subseteq Y$, of order h say, where $h < c$. Now (X,Y) does not cross (X_j,Y_j) if $j \le i$ or if $j \ge k$, and for $i \le j \le k$, (X_j,Y_j) has order at least $c > h$. Thus, by (2), there exists j with $1 \le j \le t-1$ such that $h_j \le h$ and (X,Y) is $(h-h_j,p)$-close to (X_j,Y_j). Since $h_j \le h < c$ and therefore $j \notin \{i, i+1, \ldots, k\}$, we may assume from the symmetry that $j < i$. Now $Y_j \cap X$ is $(h-h_j,p)$-small, and so $Y_i \cap X$ is $(h-h_j,p)$-small (since $Y_i \cap X \subseteq Y_j \cap X$); and hence $(h_i - h_j, p)$-small (since $h_i > h$). It follows that $(X_i,Y_i),(X_j,Y_j)$ are (h_i-h_j,p)-close, a contradiction. This proves (4).

Let $\mathcal{C}_1, \ldots, \mathcal{C}_t$ be the pieces after the slicing $(W_1,\ldots,W_t)$.

(5) *Let $1 \le i \le t$; then $\mathcal{C}_i$ is (p,p)-convex.*

For let (X,Y) be a slice of $\mathcal{C}_i$, of order $h < p$ say; and let $(X',Y') = (X \cup X_{i-1}, Y \cup Y_i)$. Then (X',Y') is a slice of $\mathcal{C}$, and by 11.2 it also has order h. Now (X',Y') crosses none of the slices (X_j,Y_j) $(0 \le j \le t)$, so by (2), there exists j with $1 \le j \le t-1$ such that $h_j \le h$ and (X',Y') is $(h-h_j,p)$-close to (X_j,Y_j). From the symmetry we may assume that $i \le j$, and so $X_i \subseteq X_j$. Since $Y' \cap X_j$ is $(h-h_j,p)$-small and $Y \subseteq Y' \cap X_j$, it follows that Y is $(h-h_j,p)$-small, and hence (p,p)-small, in $\mathcal{C}$. Thus Y is (p,p)-small in $\mathcal{C}_\rangle$, and so (X,Y) is (p,p)-close to the end-slice $(W_i,\emptyset)$ of $\mathcal{C}_\rangle$. This proves (5).

(6) *Let $1 \le i \le t$; then $\mathcal{C}_i$ is (p,q)-incoherent.*

For let $Z \subseteq W_i$ with $|Z \cap A|, |Z \cap B| \ge q$. Since $\mathcal{C}$ is (p,q)-incoherent, there is a slice (X,Y) of order less than p with $X \cap Z, Y \cap Z \neq \emptyset$. Now (X,Y) may cross either or both of the slices $(X_{i-1},Y_{i-1}),(X_i,Y_i)$; choose (X,Y) so that it crosses as few of these two slices as possible. Suppose that it crosses (X_i,Y_i). Then $0 < i < t$, and $Y_i \cap X$ is $(1,p)$-small by 14.1. If the slice $(X \cup X_i, Y \cap Y_i)$ has order less than h_i, then it is $(1,p)$-close to (X_i,Y_i), contrary to (3). Thus $(X \cup X_i, Y \cap Y_i)$ has order at least h_i, and so by 11.1, $(X \cap X_i, Y \cup Y_i)$ $(= (X',Y')$ say) has order at most h. But then (X',Y') is a slice of order at most h, and $X' \cap Z, Y' \cap Z \neq \emptyset$, and yet (X',Y') crosses fewer of $(X_{i-1},Y_{i-1}),(X_i,Y_i)$ than (X,Y). This proves that (X,Y) does not cross (X_i,Y_i), and similarly it does not cross (X_{i-1},Y_{i-1}). But then $X_{i-1} \subseteq X \subseteq X_i$, and by 11.2, $(X \cap W_i, Y \cap W_i)$ is a slice of $\mathcal{C}_i$ of order h. This proves that Z is not (p,q)-coherent in $\mathcal{C}_i$, and hence $\mathcal{C}_i$ is (p,q)-incoherent. This proves (6).

From (4)–(6), this proves 14.3. ∎

Proof of 12.1, assuming 14.2. Let $p,q \ge 0$, and let $\mathcal{S}$ be the class of all contests that are (p,p)-convex and (p,q)-incoherent. Let $\mathcal{C}_i$ $(i = 1,2,\ldots)$ be (p,q)-incoherent contests, all of the same type. By 14.3, each $\mathcal{C}_i$ admits a linked slicing over $\mathcal{S}$; and the result follows from 13.1 and 14.2. This proves 12.1. ∎

15 Contests with degree constraints

Let $c \ge 0$ be an integer. A contest $\mathcal{C} = (G,A,B,l,m,n,\pi)$ is *c-limited* if there are at most c vertices in B that both have indegree at least c and outdegree at least c. We shall prove:

15.1 *Let $c \geq 0$, and let $\mathcal{C}_i$ $(i = 1, 2, \ldots)$ be c-limited contests, all of the same type. Then there exist $j > i \geq 1$ such that $\mathcal{C}_j$ switching-contains $\mathcal{C}_i$.*

In this section we prove that 15.1 implies 14.2. We begin with the following lemma.

15.2 *Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$ be a contest, let $p \geq 0$ be an integer, and let $\mathcal{F}$ be a set of slices all of order less than p. Let $Z = \bigcup_{(X,Y)\in\mathcal{F}} X$. Then there is a subset $\mathcal{F}' \subseteq \mathcal{F}$ of cardinality at most $6p$, such that one of $(Z \setminus Z') \cap A, (Z \setminus Z') \cap B = \emptyset$, where $Z' = \bigcup_{(X,Y)\in\mathcal{F}'} X$.*

Proof. For $M \subseteq \mathcal{F}$, let $X(M) = \bigcup_{(X,Y)\in M} X$. Choose $M \subseteq \mathcal{F}$ of cardinality at most $4p$, with $|X(M)|$ maximal. Suppose that $|(Z \setminus X(M)) \cap A|, |(Z \setminus X(M)) \cap B| \geq 2p$. Since every member of Z belongs to X for some $(X, Y) \in \mathcal{F}$, there exists $N \subseteq \mathcal{F}$ with cardinality at most $4p$, such that $X(N)$ contains at least $2p$ members of $(Z \setminus X(M)) \cap A$ and at least $2p$ members of $(Z \setminus X(M)) \cap B$. From the choice of M, it follows that $|X(N)| \leq |X(M)|$, and since $|X(N) \setminus X(M)| \geq 4p$, it follows that $|X(M) \setminus X(N)| \geq 4p$. From the symmetry we may assume that there are at least $2p$ members of $A \cap X(M)$ that are not in $X(N)$. Let $P = A \cap (X(M) \setminus X(N))$, and $Q = B \cap (X(N) \setminus X(M))$; then $|P|, |Q| \geq 2p$.

If $u \in P$, then there exists $(X, Y) \in M$ with $u \in X$, and $X \cap Q = \emptyset$ since $Q \cap X(M) = \emptyset$. Since (X, Y) has order less than p, it follows that u is adjacent to at most $p - 1$ members of Q. Hence there are at most $(p - 1)|P| < |P||Q|/2$ edges from P to Q. But similarly, if $v \in Q$ then there exists $(X, Y) \in N$ with $u \in X$ and $X \cap P = \emptyset$; and so v is adjacent to at most $p - 1$ members of P; and so there are at most $(p - 1)|Q| < |P||Q|/2$ edges from Q to P. But there are $|P||Q|$ edges of G^- between P and Q, a contradiction.

This proves that not both $|(Z \setminus X(M)) \cap A|, |(Z \setminus X(M)) \cap B| \geq 2p$, and from the symmetry we may assume that $|(Z \setminus X(M)) \cap A| < 2p$. For each $v \in (Z \setminus X(M)) \cap A$, choose $(X, Y) \in \mathcal{F}$ with $v \in X$, and let N be the set of these (at most $2p$) slices. Then setting $\mathcal{F}' = M \cup N$ satisfies the theorem. This proves 15.2. ∎

15.3 *Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$ be a contest, that is (p, p)-convex and (p, q)-incoherent, for some p, q. Then $\mathcal{C}$ admits a slicing (W_1, W_2, W_3) such that*

- *the slices $(W_1, W_2 \cup W_3)$ and $(W_1 \cup W_2, W_3)$ both have order at most $6p^2$*
- *W_1 and W_3 are both $(6p^2, p)$-small*
- *one of $A \cap W_2$, $B \cap W_2$ contains fewer than $\max(q, 2p)$ vertices with at least p out-neighbours in W_2 and at least p in-neighbours in W_2.*

Proof. Let $\mathcal{F}$ be the set of all slices (X, Y) of order less than p such that X is (p, p)-small, and let $\mathcal{F}'$ be the set of all slices (X, Y) of order less than p such that Y is (p, p)-small. Since $\mathcal{C}$ is (p, p)-convex, every slice of order less than p belongs to one of $\mathcal{F}, \mathcal{F}'$.

(1) *If there exist $(X, Y) \in \mathcal{F}$ and $(X', Y') \in \mathcal{F}'$ such that $X \cap Y' \neq \emptyset$ then the result holds.*

For then let $W_1 = X, W_2 = Y \cap X'$, and $W_3 = Y' \setminus X$. The slice $(W_1, W_2 \cup W_3)$ has order less than $p \leq 6p^2$, and the slice $(W_1 \cup W_2, W_3)$ has order at most $2p \leq 6p^2$ by 11.1. Moreover, W_1 and

W_3 are both (p,p)-small and hence $(4p^2,p)$-small; and $Y \cap X'$ is $(1,p)$-small (since $X \cap Y' \neq \emptyset$), and so one of $A \cap W_2$, $B \cap W_2$ contains fewer than $2p$ vertices. This proves (1).

Let $Z = \bigcup_{(X,Y)\in\mathcal{F}} X$, and $Z' = \bigcup_{(X,Y)\in\mathcal{F}'} Y$. By (1) we may assume that $Z \cap Z' = \emptyset$. By 15.2, there exists $M \subseteq \mathcal{F}$ such that $|M| \leq 6p$ and one of $(Z \setminus W_1) \cap A, (Z \setminus W_1) \cap B = \emptyset$, where $W_1 = \bigcup_{(X,Y)\in M} X$. Similarly there exists $M' \subseteq \mathcal{F}'$ such that $|M'| \leq 6p$ and one of $(Z' \setminus W_3) \cap A, (Z' \setminus W_3) \cap B = \emptyset$, where $W_3 = \bigcup_{(X,Y)\in M} Y$. Let $W_2 = V(G) \setminus (W_1 \cup W_3)$. We claim that (W_1, W_2, W_3) satisfies the theorem. For since $|M| \leq 6p$, it follows that the slice $(W_1, W_2 \cup W_3)$ has order at most $6p^2$, by at most $6p$ applications of 11.1; and similarly $(W_1 \cup W_2, W_3)$ has order at most $6p^2$. For each $(X,Y) \in M$, X is (p,p)-small. Let $M = \{(X_1,Y_1),\ldots,(X_k,Y_k)\}$ say, and for $1 \leq i \leq k$ let Z_i be the set of members of X_i that are not in $X_{i+1} \cup \cdots \cup X_k$. Then each Z_i is (p,p)-small, and $Z_1,\ldots,Z_k$ are pairwise disjoint, and for $1 \leq i \leq k$, there are at most p edges from Z_i to $Z_1 \cup \cdots \cup Z_{i-1}$, since (X_i,Y_i) has order less than p. Consequently W_1 is $(6p^2,p)$-small, and similarly so is W_3.

Finally, let R be the set of vertices in W_2 with at least p out-neighbours in W_2 and at least p in-neighbours in W_2, and suppose that $|A \cap R|, |B \cap R| \geq q$. Since $\mathcal{C}$ is (p,q)-incoherent, there is a slice (X,Y) of order less than p, with $X \cap R, Y \cap R \neq \emptyset$. Since every slice of order less than p belongs to one of $\mathcal{F}, \mathcal{F}'$, we may assume from the symmetry that $(X,Y) \in \mathcal{F}$. Consequently $X \subseteq Z$, and so there exists $v \in X \cap R \cap Z$. From the symmetry we may assume that $v \in A$, and so $(Z \setminus W_1) \cap A \neq \emptyset$. It follows from the choice of M that $(Z \setminus W_1) \cap B = \emptyset$, and so every out-neighbour of v in W_2 belongs to Y; and hence v has at most $p-1$ out-neighbours in W_2 (since (X,Y) has order less than p), contradicting that $v \in R$. This proves that one of $|A \cap R|, |B \cap R| < q$, and hence proves 15.3. ∎

For $k,p \geq 0$, let us say a contest $\mathcal{C} = (G,A,B,l,m,n,\pi)$ is (k,p)-*small* if $V(G)$ is (k,p)-small.

15.4 *Let $k,p \geq 0$, and for each $i \geq 1$ let $\mathcal{C}_i$ be a (k,p)-small contest, all of the same type. Then there exist $j > i \geq 1$ such that $\mathcal{C}_j$ switching-contains $\mathcal{C}_i$.*

Proof. The result is clear if $k = 0$, for if $\mathcal{C} = (G,A,B,l,m,n,\pi)$ is $(0,p)$-small then $V(G) = \emptyset$.

Next we assume that $k = 1$. Let T be a quadruple of non-negative integers, and let $\mathcal{C} = (G,A,B,l,m,n,\pi)$ be a $(1,p)$-small contest of type T, with $|B| \leq 2p$ say. It follows that there are at most T_3 vertices $v \in A$ with $m(v) > 0$, and at most T_4 with $n(v) > 0$; let A' be the set of all vertices $v \in A$ such that either $m(v) > 0$, or $n(v) > 0$, or $v \in \bar{\pi}$. Thus $|A'| \leq T_1 + T_3 + T_4$. We call A' the *core* of $\mathcal{C}$.

For each $i \geq 1$ let $\mathcal{C}_i = (G_i,A_i,B_i,l_i,m_i,n_i,\pi_i)$ be a $(1,p)$-small contest, all of the same type T. For each i, either $|A_i| \leq 2p$ or $|B_i| \leq 2p$, and passing to an infinite subsequence, we may assume that $|B_i| \leq 2p$ for each $i \geq 1$. For each $i \geq 1$, let A_i' be the core of $\mathcal{C}_i$. Since there are only finitely many possibilities for the digraph $G_i|(A_i' \cup B_i)$, we may assume (again passing to a subsequence) that they are all the same, for all $i \geq 1$. Thus there is a digraph H, which is an induced subdigraph of each G_i, and $V(H)$ is the core of each $\mathcal{C}_i$. Let $A_i' = A'$ and $B_i = B$ for each $i \geq 1$. Since there are only finitely many possibilities for the restriction of m_i to $V(H)$, again we can assume they are all equal, and the same holds for n_i; and we may also assume that all the marches π_i are the same. For each $i \geq 1$, and every subset $J \subseteq B$, let $x_i(J)$ be the number of vertices in $A_i \setminus A'$ that are adjacent to every vertex in J and adjacent from every vertex in $B \setminus J$. By passing to a subsequence, we may assume that for every J, the numbers $x_i(J)$ $(i = 1,2,\ldots)$ are non-decreasing. But then $\mathcal{C}_1$ is switching-contained in $\mathcal{C}_2$.

Thus the result holds when $k = 1$ (for all T and p), and now we proceed by induction on k. Let $k \geq 2$, and let T be a quadruple of non-negative integers. Let $\mathcal{F}$ be the class of all $(k-1, p)$-small contests $\mathcal{C}' = (G', A', B', l', m', n', \pi')$ with a type T' that satisfies $T'_1 \leq T_1$, $T'_2 \leq T_2 + T_3 + T_4$, $T'_3 \leq T_3 + p$ and $T'_4 \leq T_4 + p$. From the inductive hypothesis, switching-containment defines a wqo on $\mathcal{F}$.

Now if $\mathcal{C} = (G, A, B, l, m, n, \pi)$ is (k, p)-small, of type T, there is a partition (X, Y) of $V(G)$ such that there are fewer than p edges from X to Y, and X, Y are both $(k-1, p)$-small. The pieces after this slicing are $(k-1, p)$-small, and their types satisfy the four constraints above, and so the pieces both belong to $\mathcal{F}$.

For each $i \geq 1$ let $\mathcal{C}_i = (G_i, A_i, B_i, l_i, m_i, n_i, \pi_i)$ be a (k, p)-small contest of type T. For each $i \geq 1$, let (X_i, Y_i) be a slice as described above. By passing to an infinite subsequence, we may assume that for $1 \leq j \leq T_1$, if the jth term of π_i belongs to X_i for some choice of $i \geq 1$, then the same holds for all choices of i. But then from 11.3, we deduce that there exist $j > i \geq 1$ such that $\mathcal{C}_j$ switching-contains $\mathcal{C}_i$. This proves 15.4. ∎

Proof of 14.2, assuming 15.1. Let $p, q \geq 0$, and let T be a quadruple of nonnegative integers. Let $\mathcal{T}$ be the set of all quadruples of nonnegative integers T' such that $T'_1 \leq T_1$, $T'_2 \leq T_2 + T_3 + T_4$, $T'_3 \leq T_3 + 6p^2$, and $T'_4 \leq T_4 + 6p^2$. Let $\mathcal{F}_1$ be the class of all $(6p^2, p)$-small contests with a type in $\mathcal{T}$. Let $\mathcal{F}_2$ be the class of all $\max(q, 2p)$-limited contests with a type in $\mathcal{T}$. By 15.4, switching-containment defines a wqo on $\mathcal{F}_1$, and from 15.1, the same holds for $\mathcal{F}_2$.

Let $\mathcal{C} = (G, A, B, l, m, n, \pi)$ be a (p, p)-convex (p, q)-incoherent contest of type T. By 15.3, $\mathcal{C}$ admits a slicing (W_1, W_2, W_3) such that

- the slices $(W_1, W_2 \cup W_3)$ and $(W_1 \cup W_2, W_3)$ both have order at most $6p^2$
- W_1 and W_3 are both $(6p^2, p)$-small
- one of $A \cap W_2$, $B \cap W_2$ contains fewer than $\max(q, 2p)$ vertices with at least p out-neighbours in W_2 and at least p in-neighbours in W_2.

There are three pieces after this slicing. All three pieces have a type in $\mathcal{T}$; the first and third are $(6p^2, p)$-small, and so belong to $\mathcal{F}_1$, and the second is $\max(q, 2p)$-limited, and so belongs to $\mathcal{F}_2$.

Now for each $i \geq 1$ let $\mathcal{C}_i = (G_i, A_i, B_i, l_i, m_i, n_i, \pi_i)$ be a (p, p)-convex (p, q)-incoherent contest of type T. For each $i \geq 1$, take a slicing (W_{i1}, W_{i2}, W_{i3}) as just described. By passing to an infinite subsequence, we may assume that for $1 \leq j \leq T_1$ and for $k = 1, 2, 3$, if the jth term of π_i belongs to W_{ik} for some $i \geq 1$, then the same holds for all i. But then the result follows from 11.3. This proves 14.2. ∎

16 The end

So, it remains to prove 15.1. We need a few more easy reductions: first, let us say a contest $\mathcal{C} = (G, A, B, l, m, n, \pi)$ is *clean* if $l = 0$ and m, n are identically zero. (We use 0 loosely to denote the function which is identically zero, so a clean contest may be written $(G, A, B, 0, 0, 0, \pi)$.) We shall prove:

16.1 *Let $c \geq 0$, and let $\mathcal{C}_i$ $(i = 1, 2, \ldots)$ be clean c-limited contests, all of the same type. Then there exist $j > i \geq 1$ such that $\mathcal{C}_j$ switching-contains $\mathcal{C}_i$.*

Proof of 15.1, assuming 16.1. Let T be a quadruple of nonnegative integers, and let $\mathcal{C} = (G, A, B, l, m, n, \pi)$ be a contest of type T. Let π' be a march such that its first T_1 terms are π, and

$$\bar{\pi'} = \bar{\pi} \cup \{v \in V(G) : m(v) + n(v) > 0\}.$$

Then $\mathcal{C}' = (G, A, B, 0, 0, 0, \pi')$ is a clean contest of type T', where $T'_1 \leq T_1 + T_3 + T_4$ and $T'_2, T'_3, T'_4 = 0$, and if $\mathcal{C}$ is c-limited then so is $\mathcal{C}'$. Let us call this an *associated clean contest.*

Now for each $i \geq 1$ let $\mathcal{C}_i = (G_i, A_i, B_i, l_i, m_i, n_i, \pi_i)$ be a c-limited contest, and let $\mathcal{C}'_i$ be an associated clean contest. By moving to an infinite subsequence we may assume that all the contests $\mathcal{C}'_i$ have the same type T' say; and by the same argument, we may assume that there are two T'_1-tuples m, n say, such that for all $i \geq 1$ and for $1 \leq j \leq T'_1$, if v is the jth term of π'_i then $m_i(v) = m(j)$ and $n_i(v) = n(j)$. Moreover, by 16.1, there exist $j > i \geq 1$ such that $\mathcal{C}'_j$ switching-contains $\mathcal{C}'_i$. But then $\mathcal{C}_j$ switching-contains $\mathcal{C}_i$. This proves 15.1. ∎

Let $c, k \geq 0$. A (c, k)-*battle* $\mathcal{B}$ is a five-tuple (G, A, B, C, π) such that

- G is a digraph such that G^- is complete bipartite, and $(A, V(G) \setminus A)$ is a bipartition
- B, C are disjoint subsets of $V(G)$, and $B \cup C = V(G) \setminus (A \cup \bar{\pi})$ (thus, $A, B, C\bar{\pi}$ have union $V(G)$, and they are pairwise disjoint except that $A \cup \bar{\pi}$ may be nonempty)
- every vertex in B has at most c outneighbours in A, and every vertex in C has at most c in-neighbours in A
- π has length at most k.

Let $\mathcal{B}_1 = (G_1, A_1, B_1, C_1, \pi_1)$ and $\mathcal{B}_2 = (G_2, A_2, B_2, C_2, \pi_2)$ be (c, k)-battles. We say $\mathcal{B}_2$ *switching-contains* $\mathcal{B}_1$ if there is a digraph G' degree-equivalent to G_2 and an injection $\eta: V(G_1) \to V(G_2)$, with the following properties:

- for all distinct $u, v \in V(G_2)$, if at least one of u, v belongs to $\bar{\pi_2}$, then uv is an edge of G_2 if and only if uv is an edge of G'
- for all distinct $u, v \in V(G_1)$, u is adjacent to v in G_1 if and only if $\eta(u)$ is adjacent to $\eta(v)$ in G'
- $\eta(\pi_1) = \pi_2$
- $\eta(A_1) \subseteq A_1$ and $\eta(B_1) \subseteq B_2$ and $\eta(C_1) \subseteq C_2$
- for each v in $V(G_1) \setminus \bar{\pi_1}$, and for $1 \leq j \leq |\bar{\pi_1}|$, v is adjacent to the jth term of π_1 if and only if $\eta(v)$ is adjacent to the jth term of π_2
- for each v in $V(G_1) \setminus \bar{\pi_1}$, the degree of v in G_1 is the same as the degree of $\eta(v)$ in G_2.

We shall prove:

16.2 *Let $c, k \geq 0$, and let $\mathcal{B}_i$ $(i = 1, 2, \ldots)$ be (c, k)-battles. Then there exist $j > i \geq 1$ such that $\mathcal{B}_j$ switching-contains $\mathcal{B}_i$.*

be **Proof of 16.1, assuming 16.2.** Let $c \geq 0$, and for each $i \geq 1$ let $\mathcal{C}_i$ be a clean c-limited contest, all of the same type T. For each $i \geq 1$ let $\mathcal{C}_i = (G_i, A_i, B_i, 0, 0, 0, \pi_i)$. For each i there is a partition of B_i into three sets say $D_{i1}1, D_{i2}, D_{i3}$, where every vertex in D_{i1} has outdegree at most c, every vertex in D_{i2} has indegree at most c, and $|D_{i3}| \leq c$. Let π'_i be a march of which π_i is an initial subsequence, and $\bar{\pi}'_i = \bar{\pi}_i \cup D_{i3}$; then $\mathcal{B}_i = (G_i, A_i, D_{i1}, D_{i2}, \pi')$ is a $(c, T_1 + c)$-battle. By 16.2 there exists $j > i \geq 1$ such that $\mathcal{B}_j$ switching-contains $\mathcal{B}_i$; and then $\mathcal{C}_j$ switching-contains $\mathcal{C}_i$. This proves 16.1. ∎

Let G be a digraph and $A, B \subseteq V(G)$. A *matching* in G *from A to B* is a set of directed edges $\{x_1y_1, \ldots, x_ny_n\}$ of G such that $x_1, y_1, \ldots, x_n, y_n$ are all distinct, and $x_1, \ldots, x_m \in A$, and $y_1, \ldots, y_m \in B$. We denote by $\mu(A, B)$ or $\mu_G(A, B)$ the cardinality of the largest matching in G from A to B.

Let $\mathcal{B} = (G, A, B, C, \pi)$ be a (c, k)-battle. For each $v \in A$, we define its *B-spread* to be the maximum n such that there is a matching $\{x_1y_1, \ldots, x_ny_n\}$ of G from B to A such that $x_1, \ldots, x_n$ are all adjacent from v. We define the *C-spread* of $v \in A$ to be the maximum n such that there is a matching $\{x_1y_1, \ldots, x_ny_n\}$ of G from A to C such that $y_1, \ldots, y_n$ are all adjacent to v. We define the *c-pivot* of the battle to be the subset of A consisting of the c members of A with smallest A-spread together with the c members of A with smallest B-spread (if $|A| < c$ we define the c-pivot to be A, and we break ties arbitrarily). A (c, k)-battle (G, A, B, C, π) is *c-pivotal* if its c-pivot is a subset of $\bar{\pi}$. We shall prove

16.3 *Let $c, k \geq 0$, and let $\mathcal{B}_i$ $(i = 1, 2, \ldots)$ c-pivotal (c, k)-battles. Then there exist $j > i \geq 1$ such that $\mathcal{B}_j$ switching-contains $\mathcal{B}_i$.*

Proof of 16.2, assuming 16.3. Let $c, k \geq 0$, and for each $i \geq 1$ let $\mathcal{B}_i$ be a (c, k)-battle. For each $i \geq 1$ let $\mathcal{B}_i = (G_i, A_i, B_i, C_i, \pi_i)$. Let π'_i be a march of length at most $k + 2c$ of which π_i is an initial subsequence, such that every vertex of the c-pivot of $\mathcal{B}_i$ belongs to $\bar{\pi}'$ (since the c-pivot has at most $2c$ vertices, this exists), and let $\mathcal{B}'_i = (G_i, A_i, B_i, C_i, \pi'_i)$. Thus $\mathcal{B}'_i$ is a c-pivotal $(c, k + 2c)$-battle. By 16.3 there exists $j > i \geq 1$ such that $\mathcal{B}'_j$ switching-contains $\mathcal{B}'_i$; and then $\mathcal{B}_j$ switching-contains $\mathcal{B}_i$. This proves 16.2. ∎

Proof of 16.3. The *imbalance* of a battle (G, A, B, C, π) is

- 2 if $A \subseteq \bar{\pi}$
- 1 if $A \not\subseteq \bar{\pi}$ and at least one of $B, C = \emptyset$
- 0 otherwise.

Let $c, k \geq 0$. A (c, k)-*bad sequence* means an infinite sequence $\mathcal{B}_i$ $(i = 1, 2, \ldots)$ of c-pivotal (c, k)-battles, such that there do not exist $j > i \geq 1$ such that $\mathcal{B}_j$ switching-contains $\mathcal{B}_i$. If there is a (c, k)-bad sequence, we say the pair (c, k) is *bad*. If (c, k) is a bad pair, its *imbalance* is the maximum n such that there is a (c, k)-bad sequence each term of which has imbalance n.

We need to show there is no bad pair; thus, suppose that there is a bad pair, and choose a bad pair (c, k) with maximum imbalance. Let $\mathcal{B}_i$ $(i = 1, 2, \ldots)$ be the corresponding (c, k)-bad sequence, and for each $i \geq 1$ let $\mathcal{B}_i = (G_i, A_i, B_i, C_i, \pi_i)$; thus, each $\mathcal{B}_i$ is a c-pivotal (c,k)-battle. By moving to a infinite subsequence, we may assume that either $B_i = \emptyset$ for all i, or $B_i \neq \emptyset$ for all i; and the same for the C_i.

Since there are only finitely many possibilities for $G_i|\bar{\pi}_i$, we may assume they are all the same; and so there is a digraph H, a common subdigraph of $G_1, G_2, \ldots$, and a march π with $\bar{\pi} = V(H)$, such that $\pi_i = \pi$ for each $i \geq 1$. For each $i \geq 1$, let $F_i = G_i \setminus V(H)$.

For every subset $J \subseteq V(H)$, let $N_i(J)$ be the set of all vertices in $V(F_i)$ that are adjacent in G_i to every vertex in J and have no other out-neighbours in $V(H)$. (Thus, $N_i(J) = \emptyset$ unless $J \subseteq A_i$ or $J \cap A_i = \emptyset$.) Let $A_i(J) = A_i \cap N_i(J)$, and define $B_i(J), C_i(J)$ similarly.

For the moment, let us fix $J \subseteq V(H)$. Take an enumeration of the members of $A_i(J) \setminus V(H)$, and for each $v \in A_i(J) \setminus V(H)$ list the pair $(p_i(v), q_i(v))$, where $p_i(v)$ is the number of vertices in B_i adjacent to v, and $q_i(v)$ is the number of vertices in C_i adjacent from v. This gives a finite sequence of pairs of non-negative integers. Pairs of non-negative integers, ordered by component-wise domination, form a wqo; so by 5.3 and 5.1 and passing to an infinite subsequence, we may assume that for each $i \geq 1$ there is an injection η from $A_i(J) \setminus V(H)$ to $A_{i+1}(J) \setminus V(H)$ such that for each $v \in A_i(J) \setminus V(H)$, $\eta(v) \in A_{i+1}(J) \setminus V(H)$, and $p_i(v) \leq p_{i+1}(\eta(v))$, and $q_i(v) \leq q_{i+1}(\eta(v))$. Similarly, take an enumeration of the members of $B_i(J)$, and for each $v \in B_i(J)$ list its outdegree in F_i. This gives a finite sequence of integers, all at most c; and so by 5.3 and 5.1, we may assume that for all $i \geq 1$ there is an injection η from $B_i(J)$ into $B_{i+1}(J)$, such that for each $v \in B_i(J)$, the outdegree of $\eta(v)$ in F_{i+1} equals the outdegree of v in F_i. Similarly we may assume there is an injection η from $C_i(J)$ into $C_{i+1}(J)$, such that for each $v \in C_i(J)$, the indegree of $\eta(v)$ in F_{i+1} equals the indegree of v in F_i. By repeating this for all subsets J of $V(H)$, we may assume

(1) *For each $i \geq 1$, there is an injection η_i from $V(G_i)$ into $V(G_{i+1})$ such that*

- *$\eta_i(v) = v$ for each $v \in V(H)$*
- *for each $v \in V(F_i)$ and $u \in V(H)$, v is adjacent to u in G_i if and oly if $\eta_i(v)$ is adjacent to u in G_{i+1}, and v is adjacent from u in G_i if and oly if $\eta_i(v)$ is adjacent from u in G_{i+1}*
- *for each $v \in V(F_i)$, if $v \in A_i$ then $\eta_i(v) \in A_{i+1}$, and the number of edges of F_{i+1} from B_{i+1} to $\eta(v)$ is at least the number of edges of F_i from B_i to v, and the number of edges of F_{i+1} from $\eta(v)$ to C_{i+1} is at least the number of edges of F_i from v to C_i*
- *for each $v \in B_i$, $\eta_i(v) \in B_{i+1}$, and the outdegree of v in F_i equals the outdegree of $\eta_i(v)$ in F_{i+1}*
- *for each $v \in C_i$, $\eta_i(v) \in C_{i+1}$, and the indegree of v in F_i equals the indegree of $\eta_i(v)$ in F_{i+1}.*

(2) *No pair (c', k') has imbalance 2.*

For suppose there were such a pair; then (c,k) has imbalance 2, from our choice of (c,k). But then by (1), η_1 provides an isomorphism from G_1 to an induced subdigraph of G_2, and so $\mathcal{B}_2$ switching-contains $\mathcal{B}_1$, a contradiction. This proves (2).

(3) *We may assume that $|A_{i+1}| > 2|V(G_i)|^2 + 2c$ for all $i \geq 1$.*

For suppose that for some n there are infinitely many values of i with $|A_i| \leq n$. Then we may assume that this is true for all i, by passing to an infinite subsequence. For each $i \geq 1$, let π'_i be a march of length at most $n+k$ of which π_i is an initial subsequence, and with $A_i \subseteq \bar{\pi}'_i$. Then $\mathcal{B}'_i = (G_i, A_i, B_i, C_i, \pi'_i)$ is a c-pivotal $(c, k+n)$-battle with imbalance 2, for each $i \geq 1$, and so by (2) there exist $j > i \geq 1$ such that $\mathcal{B}'_j$ switching-contains $\mathcal{B}'_i$. But then $\mathcal{B}_j$ switching-contains $\mathcal{B}_i$, a contradiction. So for each n there are only finitely many i with $|A_{i+1}| \leq n$. But then there is an infinite subsequence satisfying (3). This proves (3).

(4) *If $B_1 \neq \emptyset$ then we may assume that for each $i \geq 1$, $\mu_{G_{i+1}}(B_{i+1}, A_{i+1}) > (2|V(G_i)|^2 + c + k)(c+1)$ for each $i \geq 1$. Also, if $C_1 \neq \emptyset$, we may assume that $\mu_{G_{i+1}}(A_{i+1}, C_{i+1}) > (2|V(G_i)|^2 + c + k)(c+1)$ for each $i \geq 1$.*

For suppose that $B_1 \neq \emptyset$, and for some $n \geq 0$, there are infinitely many values of i such that $\mu_{G_i}(B_i, A_i) \leq n$. Then by passing to a subsequence, we may assume that this is the case for all i. By Hall's theorem, for each $i \geq 1$ there is a subset $Z_i \subseteq A_i \cup B_i$ such that every edge from B_i to A_i has at least one end in Z_i. For each $i \geq 1$, let π'_i be a march of length at most $k+n$ such that π_i is an initial subsequence of π'_i, and $Z_i \cup \bar{\pi}_i = \bar{\pi}'_i$. Then $\mathcal{B}'_i = (G_i, A_i, \emptyset, C_i, \pi'_i)$ is a c-pivotal $(c, k+n)$-battle with imbalance greater than that of $\mathcal{B}_i$; and so from our choice of (c,k), there exists $i < j$ such that $\mathcal{B}'_i$ is switching-contained in $\mathcal{B}'_j$, and from (1), it follows that $\mathcal{B}_i$ is switching-contained in $\mathcal{B}_j$ a contradiction. Thus for each n there are only finitely many such i. Similarly, if $C_1 \neq \emptyset$, for each n there are only finitely many i such that $\mu_{G_i}(A_i, C_i) \leq n$; and then there is an infinite subsequence satisfying (4). This proves (4).

(5) *Let $X_2 \subseteq V(F_2)$, and let $X_1 = \{v \in V(F_1) : \eta(v) \in X_2\}$. Then $|D^+_{F_2}(X_2)| \geq |D^+_{F_1}(X_1)|$.*

For suppose that $|D^+_{F_2}(X_2)| < |D^+_{F_1}(X_1)|$, for a contradiction. Since $|D^+_{F_1}(X_1)| \leq |V(G_1)|^2$, it follows that $|D^+_{F_2}(X_2)| < |V(G_1)|^2$. Suppose first that $X_2 \cap C_2 \neq \emptyset$ and $B_2 \not\subseteq X_2$. Let $v \in X_2 \cap C_2$; then v has indegree at most c, and therefore it is outadjacent to all members of $A_2 \setminus (X_2 \cup V(H))$ except at most c. Since it has at most $|V(G_1)|^2$ outneighbours in $A_2 \setminus (X_2 \cup V(H))$ (because $|D^+_{F_2}(X_2)| < |V(G_1)|^2$), it follows that $|A_2 \setminus (X_2 \cup V(H))| \leq |V(G_1)|^2 + c$. Similarly, since $B_2 \not\subseteq X_2$, it follows that $|A_2 \cap X_2| \leq |V(G_1)|^2 + c$, and so $|A_2 \setminus V(H)| \leq 2(|V(G_1)|^2 + c)$, contrary to (1).

Thus not both $X_2 \cap C_2 \neq \emptyset$ and $B_2 \not\subseteq X_2$. From the symmetry we may assume that $X_2 \cap C_2 = \emptyset$. Suppose that $B_2 \subseteq X_2$. Then $|D^+_{F_2}(X_2)|$ is the number of edges of F_2 from B_2 to $A_2 \setminus (X_2 \cup V(H))$, plus the number of edges of F_2 from X_2 to C_2. Now the number of edges of F_2 from X_2 to C_2 is at least the number of edges of F_1 from X_1 to C_1, since for each $v \in X_1$, $\eta(v) \in X_2$, and the number of edges in F_2 from $\eta(v)$ to C_2 is at least the number of edges of F_1 from v to C_1, from the choice of η. Similarly, the number of edges of F_2 from B_2 to $A_2 \setminus (X_2 \cup V(H))$ is at least the number of edges of F_1 from B_1 to $A_1 \setminus (X_1 \cup V(H))$. It follows that $|D^+_{F_2}(X_2)| \geq |D^+_{F_1}(X_1)|$ as required.

Thus, we may assume that $B_2 \not\subseteq X_2$. As before, it follows that $|A_2 \cap X_2| \leq |V(G_1)|^2 + c$. Suppose that $A_2 \cap X_2 = \emptyset$, and so $X_2 \subseteq B_2$. For every vertex $v \in X_1 \cap B_1$, $\eta_1(v)$ belongs to $X_2 \cap B_2$, and all edges of F_2 with tail $\eta_2(v)$ therefore belong to $D^+_{F_2}(X_2)$; and since the outdegree of $\eta(v)$ in F_2 equals the outdegree of v in F_1, it follows that $|D^+_{F_2}(X_2)| \geq |D^+_{F_1}(X_1)|$, a contradiction.

Thus $A_2 \cap X_2 \neq \emptyset$; choose $v \in A_2 \cap X_2$. Let n be the B-spread of v in G_2. Thus there is a matching $\{x_1y_1, \ldots, x_ny_n\}$ of G_2 from B_2 to A_2, such that $x_1, \ldots, x_n$ are all adjacent from v in G_2. Now there are at most $|V(G_1)|^2 + c$ values of $j \in \{1, \ldots, n\}$ such that $y_j \in (A_2 \cap X_2) \setminus V(H)$, since

$|(A_2 \cap X_2) \setminus V(H)| \leq |V(G_1)|^2 + c$; and there are at most k values of j with $y_j \in V(H)$, since $|V(H)| \leq k$. Thus $y_j \in A_2 \setminus (V(H) \cap X_2)$ for at least $n - |V(G_1)|^2 - c - k$ values of $j \in \{1, \ldots, n\}$. But for each such value of j, either $vx_j \in D^+_{F_2}(X_2)$ (if $x_j \notin X_2$) or $x_j y_j \in D^+_{F_2}(X_2)$ (if $x_j \in X_2$), and so there are at most $|V(G_1)|^2$ such values. Consequently $n - |V(G_1)|^2 - c - k \leq |V(G_1)|^2$, and so $n \leq 2|V(G_1)|^2 + c + k$. Since $v \notin V(H)$, it follows that v is not in the c-pivot of $\mathcal{B}_i$, and so there are at least $c+1$ vertices in A_2 with c-spread at most n (counting v as one of them), say $v_1, \ldots, v_{c+1}$. Now $B_2 \neq \emptyset$, and so $B_1 \neq \emptyset$. By (4), there is a matching $\{x_1y_1, \ldots, x_my_m\}$ of G_2 from B_2 to A_2 with $m > n(c+1) \leq (2|V(G_1)|^2 + c + k)(c+1)$. Since $v_1, \ldots, v_{c+1}$ each have B-spread at most n, it follows that each of $v_1, \ldots, v_{c+1}$ is adjacent to at most n of $x_1, \ldots, x_m$. Consequently, there are at least $(m-n)(c+1)$ edges of G_2 from $\{x_1, \ldots, x_m\}$ to $\{v_1, \ldots, v_{c+1}\}$. Since each x_i has outdegree at most c (since it belongs to B_2), it follows that $mc \geq (m-n)(c+1)$, and so $m \leq n(c+1) \leq (2|V(G_1)|^2 + c + k)(c+1)$, a contradiction. This proves that $A_2 \cap X_2 = \emptyset$, and so $X_2 \subseteq B_2 \setminus V(H)$. But for every vertex $v \in X_1 \cap B_1$, $\eta_1(v)$ belongs to $X_2 \cap B_2$, and all edges of F_2 with tail $\eta_2(v)$ therefore belong to $D^+_{F_2}(X_2)$; and since the outdegree of $\eta(v)$ in F_2 equals the outdegree of v in F_1, it follows that $|D^+_{F_2}(X_2)| \geq |D^+_{F_1}(X_1)|$, a contradiction. This proves (5).

Let G_1' be the image of G_1 under η. From (5) and 9.2, applied to the weighted digraphs $(G_1', 0, 0)$ and $(G_2, 0, 0)$ (where 0 denote the function which is identically zero), we deduce that there is a digraph G_2', degree-equivalent to G_2, such that G_1' is an induced subdigraph of G_2'. Consequently $\mathcal{B}_1$ is switching-contained in $\mathcal{B}_2$. This proves 16.3. ∎

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