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Progress on perfect graphs

Abstract. A graph is *perfect* if for every induced subgraph, the chromatic number is equal to the maximum size of a complete subgraph. The class of perfect graphs is important for several reasons. For instance, many problems of interest in practice but intractable in general can be solved efficiently when restricted to the class of perfect graphs. Also, the question of when a certain class of linear programs always have an integer solution can be answered in terms of perfection of an associated graph.

In the first part of the paper we survey the main aspects of perfect graphs and their relevance. In the second part we outline our recent proof of the Strong Perfect Graph Conjecture of Berge from 1961, the following: a graph is perfect if and only if it has no induced subgraph isomorphic to an odd cycle of length at least five, or the complement of such an odd cycle.

Key words. Berge graph – perfect graph – skew partition

1. Introduction

In this paper graphs are finite and simple; that is, they have no loops or multiple edges. Let G be a graph. A *hole* in G is an induced cycle of length at least four. An *antihole* in G is an induced subgraph isomorphic to the complement of a hole. (The *complement* of a graph G is the graph with vertex-set $V(G)$ and edge-set consisting precisely of all distinct pairs of vertices that are not adjacent in G.) A *clique* in G is a set $X \subseteq V(G)$ of pairwise adjacent vertices, and a *stable set* is a set of pairwise non-adjacent vertices. The size of a maximum clique of a graph G is denoted by $\omega(G)$ and the size of a maximum stable set is denoted by $\alpha(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the least number of colors needed to color the vertices of G in such a way that adjacent vertices receive different colors. In other words, $\chi(G)$ is the minimum number k such that the vertex-set $V(G)$ of G can be partitioned into k stable sets. Clearly $\chi(G) \geq \omega(G)$ for every graph G, but equality need not hold. For instance, if G is an odd hole (i.e., a hole with an odd number of vertices), then $\chi(G) = 3 > 2 = \omega(G)$. Similarly, if G is an odd antihole, say on $2k + 1$ vertices, then $\chi(G) = k + 1 > k = \omega(G)$.

A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G; that is, the chromatic number of H is equal to the maximum size of a clique of H. Thus odd holes and odd antiholes are not perfect, and neither is any other graph that has an odd

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hole or an odd hole antihole. Are there any other imperfect graphs? Berge [1] conjectured in 1961 that there are not, and a proof of that is our main result:

1.1. A graph is perfect if and only if it has no odd hole and no odd antihole

In the second half of the paper we will outline our proof. The first part is devoted to discussing the relevance of perfect graphs. This survey is not intended to be exhaustive – we concentrate on the aspects of perfect graphs that we find the most interesting. The subject is much broader than can be covered in our short survey. We refer the reader to the excellent book by Ramírez Alfonsín and Reed [34] for further reading.

2. Examples of perfect graphs

A graph B is *bipartite* if its vertex-set $V(B)$ can be partitioned into two disjoint sets X, Y in such a way that every edge of B has one end in X and the other end in Y. In other words, B is bipartite if and only if $\chi(B) \leq 2$. It follows immediately that every bipartite graph is perfect.

Now let G be the complement of a bipartite graph B, in symbols $G = \overline{B}$. We claim that G is perfect. Since every induced subgraph of G is the complement of an appropriate subgraph of B, it suffices to show that $\chi(G) = \omega(G)$. But $\omega(G) = \alpha(B)$, the maximum size of a stable set of B, and $\chi(G)$ is equal to the minimum number of cliques of B covering the vertices of B. Thus the statement $\chi(G) = \omega(G)$ is equivalent to the assertion that the vertices of the bipartite graph B can be covered by $\alpha(B)$ edges and vertices. The latter is a classical result of König [23] and can be found in almost every graph theory textbook.

Another two classes of perfect graphs can be obtained from bipartite graphs by means of line graphs. Let G be a graph. The *line graph* of G is the graph $L(G)$ with vertex-set $E(G)$ in which $e, f \in E(G)$ are adjacent if they share an end in G. Now if B is a bipartite graph, then $G = L(B)$ is perfect. To see this, it again suffices to show that $\chi(G) = \omega(G)$. Now $\omega(G)$ is equal to the maximum degree of B, and $\chi(G)$ is equal to the *edge-chromatic number* of B, the minimum number of colors needed to color the *edges* of B in such a way that adjacent edges receive different colors. The latter two numbers are equal in any bipartite graph B by a well-known theorem of König [23].

Similarly, if B is a bipartite graph, then $G = \overline{L(B)}$, the complement of $L(B)$, is perfect. To see this it again suffices to show that $\chi(G) = \omega(G)$. But $\chi(G)$ is the minimum number of cliques of $L(B)$ that cover the vertices of $L(B)$, which in turn is equal to the minimum number of vertices of B that meet all the edges of B. On the other hand, $\omega(G)$ is the size of a maximum stable set in $L(B)$, which is equal to the maximum size of a matching of B. But the maximum size of a matching in a bipartite graph is equal to the minimum number of vertices meeting all the edges of B by another classical theorem of Egerváry [13] and König [24].

Thus we have seen four classes of perfect graphs. The perfection of the first class is easy, and for the other three classes their perfection is equivalent to an old and well-known result in graph theory. By now many more classes have been discovered; by the last count there are 96 known classes of perfect graphs.

3. The perfect graph theorem

Notice that in the above examples of perfect or imperfect graphs it was always the case that a graph is perfect if and only if so is its complement. This is not a coincidence, due to the following theorem of Lovász [26], originally conjectured by Berge [1].

3.1 (Perfect Graph Theorem) *A graph is perfect if and only if its complement is perfect.*

For the proof we will need two lemmas. The first is easy.

3.2 *A graph G is perfect if and only if every induced subgraph H of G has a stable set that intersects every maximum clique of H.*

The second lemma we need is due to Lovász [26].

3.3 (Replication Lemma) *Let G be a perfect graph, and let $v \in V(G)$. Define a graph G' by adding a new vertex v' and joining it to v and all the neighbors of v. Then G' is perfect.*

Proof. By (3.2) it suffices to prove that G' itself has a stable set that intersects every maximum clique of G'. Since G is perfect, it has a coloring using $\omega(G)$ colors; let S be the color class containing v. Then S is as desired.   □

In the proof of (3.1) we will give another characterization of perfect graphs in terms of certain polytopes associated with graphs. Let G be a graph. The *stable set polytope* of G, also known as the *vertex packing polytope* of G, denoted by STAB(G), is the convex hull in $\mathbf{R}^{V(G)}$ of all incidence vectors of stable sets of G. A related polytope is the *fractional stable set polytope* or *fractional vertex packing polytope* QSTAB$(G) \subseteq \mathbf{R}^{V(G)}$ defined by the constraints

$$x_v \geq 0 \quad \text{for every } v \in V(G), \tag{1}$$

$$\sum_{v \in V(K)} x_v \leq 1 \quad \text{for every clique } K \text{ in } G. \tag{2}$$

Since the incidence vector of every stable set satisfies (1) and (2), we have STAB$(G) \subseteq$ QSTAB(G). The following theorem, which also incorporates results of Chvátal [2] and Fulkerson [14, 15], implies the Perfect Graph Theorem. Our treatment follows [28].

3.4 *For any graph G, the following conditions are equivalent.*

(i) *G is perfect,*
(ii) *STAB(G) =QSTAB(G),*
(iii) *$\overline{G}$ is perfect,*
(iv) *STAB$(\overline{G})$ =QSTAB$(\overline{G})$.*

Proof. (i) $\Rightarrow$ (ii). Let x satisfy (1) and (2). By rational approximation we may assume that x is rational. Take an integer N such that $y = Nx$ is an integer vector. By (1), $y_v \geq 0$ for all $v \in V(G)$. Let Y_v ($v \in V(G)$) be disjoint sets with $|Y_v| = y_v$, and let H be the graph with vertex–set $\bigcup_{v \in V(G)} Y_v$ in which a vertex of Y_v is adjacent to a vertex of Y_u if and only if either $v = u$ or v is adjacent to u in G. By a repeated application of the Replication Lemma (3.3) the graph H is perfect. By (2) we have $\omega(H) \leq N$, and hence $V(H)$ can be covered by N disjoint stable sets of H. Let $q_1, q_2, \ldots, q_N$ be the incidence vectors of the corresponding stable sets of G. Then $x = \frac{1}{N}(q_1 + q_2 + \cdots + q_N)$, as desired.

(ii) $\Rightarrow$ (iii) Since condition (ii) is inherited by induced subgraphs of G, by (3.2) it suffices to prove that G has a clique that intersects every maximum stable set of G. We prove this assertion by induction on $|V(G)|$. We may assume that every vertex belongs to a maximum stable set of G, for otherwise we may delete such vertex and proceed by induction. Let X be the subset of STAB(G) consisting of all vectors x with $\sum_{v \in V(G)} x_v = \alpha(G)$. Then X is a face of QSTAB(G), and hence one of the inequalities (1), (2) is satisfied with equality for every vector in X. Since every vertex of G belongs to a maximum stable set of G, it follows that none of the inequalities (1) is satisfied with equality for all $x \in X$, and hence one of the inequalities (2) is. Thus there exists a clique of G which meets every maximum stable set of G, as desired.

The remaining two implications follow by applying the previous arguments to the complement of G. □

4. Integrality of polyhedra

Let A be a 0, 1-matrix and consider the following linear program:

$$\max c \cdot x \text{ subject to } x \geq 0 \text{ and } Ax \leq 1. \qquad (3)$$

For which matrices A is it true that for every objective function c, the linear program has integral optimum solution? This is an important question, because solving integer programs is an NP-hard problem, whereas efficient algorithms exist to solve linear programs. It turns out that the answer to our question leads directly to perfect graphs.

We say that the ith row of a matrix $A = (a_{ij})$ is *undominated* if there is no row index $j \neq i$ such that $a_{il} \leq a_{jl}$ for all l. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let $K_1, K_2, \ldots, K_m$ be its (inclusion-wise) maximal cliques. We define the *maximal clique versus vertex incidence matrix* of G to be the $m \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if $v_j \in K_i$, and $a_{ij} = 0$ otherwise. The following is a result of Chvátal [2].

4.1 *The linear program* (3) *has an integral optimum solution for every objective function* c *if and only if the undominated rows of A form the maximal clique versus vertex incidence matrix of a perfect graph.*

Proof. To prove "only if" let $A = (a_{ij})$ be an $m \times n$ matrix, and assume that (3) has an integral optimum for every nonnegative objective function c. We may assume that A has no dominated rows (by deleting them). Let G be the graph with vertex-set $\{1, 2, \ldots, n\}$ in which i is adjacent to j if and only if $a_{li} = a_{lj} = 1$ for some row index l. We first

show that A is a maximal clique versus vertex incidence matrix of G. Indeed, if that were not the case, then G has a clique K such that no row of A has ones in the positions corresponding to the vertices of K and possibly elsewhere. Choose such K with $|K|$ minimal. Then clearly $|K| \geq 3$ by construction. By the minimality of K, for every $v \in K$ there exists a row with ones in columns corresponding to $K - \{v\}$. Let c be the incidence vector of K. Then $c \cdot x \leq 1$ for every integral vector $x \in \mathbf{R}^{V(G)}$ satisfying $x \geq 0$ and $Ax \leq 1$, for such x must be the incidence vector of a stable set. Now let $y \in \mathbf{R}^{V(G)}$ be the vector whose coordinates are $\frac{1}{|K|-1}$ in positions corresponding to elements of K, and all other coordinates are 0. Then $Ay \leq 1$, and yet $c \cdot y = \frac{|K|}{|K|-1} > 1$, a contradiction. This proves that A is a maximal clique versus vertex incidence matrix of G, and hence the polyhedron $\{x \in \mathbf{R}^{V(G)} : x \geq 0 \text{ and } Ax \leq 1\}$ is precisely QSTAB(G). Now the integrality of optima implies that every vertex of QSTAB(G) is integral, and hence is a characteristic vector of a stable set. Thus STAB$(G) = $ QSTAB(G), and hence G is perfect by (3.4).

To prove the "if" part, let G be perfect. We may assume that A itself is the maximal clique versus vertex incidence matrix of G. By (3.4) the linear program (3) is equivalent to maximizing $c \cdot x$ subject to $x \in$ STAB(G), which clearly attains its optimum at a characteristic vector of a stable set, and hence has an integral optimum solution, as desired.   □

5. Berge's motivation: Shannon capacity

Let us go back in history and review what motivated Berge to introduce perfect graphs. Let us consider the transmission of symbols from some finite *input alphabet* Σ through a discrete memoryless channel. Certain symbols may be confused during transmission, and we are interested in how many n-symbol error-free messages there are.

To be more precise, with every $a \in \Sigma$ we associate symbols $a_1, a_2, \ldots$ of some *output alphabet*. The meaning of this is that when a is sent through the channel, one of $a_1, a_2, \ldots$ is received, say according to some probability distribution. However, in this model we are interested in transmissions that are 100% error-free, and so we define confoundability as follows. Let $a \in \Sigma$ be as above, and let $b \in \Sigma$ have $b_1, b_2, \ldots$ as the corresponding members of the output alphabet. We say that a and b are *confoundable* if $a_i = b_j$ for some i, j, and otherwise we say that they are *unconfoundable*. Finally, we say that two words $x, y \in \Sigma^t$ of length t are *unconfoundable* if for at least one coordinate i the corresponding entries x_i and y_i are unconfoundable.

As an example, let $\Sigma = \{a, b, c, d, e\}$, where the confoundable pairs are precisely ab, bc, cd, de, ea. In that case the symbols a, c may be sent without danger of confusion, and hence there are at least 2^n n-symbol error-free messages obtained by taking all n-element sequences of the symbols a and c. But we can do better. The 2-symbol messages ab, bd, ca, dc, ee are pairwise unconfoundable (for instance a may be confused with b, but b cannot be confused with d, and hence the pairs ab and bd are unconfoundable), and by forming arbitrary words composed of those two-letter words we see that there are at least $5^{\lfloor n/2 \rfloor}$ n-symbol error-free messages, an improvement over the earlier bound. An asymptotic study of the bit-per-symbol error-free transmission rate leads to the notion of Shannon capacity of a graph [39], as follows.

Let G be the graph with vertex-set Σ and edge-set all pairs of unconfoundable elements of Σ. Thus we are interested in maximum cliques of G^t, where G^t is the graph with vertex-set all t-tuples of vertices of G in which two such t-tuples are adjacent if and only if for some coordinate the corresponding entries are adjacent in G. The Shannon capacity of G is defined as

$$\lim_{n \to \infty} \frac{1}{n} \log \omega(G^n).$$

It is easy to see that the limit exists. Moreover, we have the following inequalities: $\omega^n(G) \leq \omega(G^n) \leq \chi(G^n) \leq \chi^n(G)$. Thus it follows that if $\chi(G) = \omega(G)$, then the Shannon capacity is equal to the logarithm of this value. This raises the question of what are the minimal graphs that do not satisfy $\chi(G) = \omega(G)$, and that lead Berge to formulate his influential conjectures.

The Shannon capacity is a notoriously difficult parameter to compute. For instance, a celebrated result of Lovász [27] says that the Shannon capacity of the cycle of length five is $\frac{1}{2} \log 5$. The lower bound follows from our discussion two paragraphs above, but the upper bound is a deep result. Lovász' proof uses a new parameter, based on geometric representation of graphs, the so-called theta function of graphs. The theta function played a crucial role in the theory of Grötschel, Lovász and Schrijver that we discuss in the next section.

Incidentally, the Shannon capacity is not known for many small graphs. For instance, it is not known for odd cycles of length at least seven.

6. Miscellaneous connections

Grötschel, Lovász and Schrijver [19] developed a general theory of geometric optimization based on the ellipsoid method [22]. One consequence of this theory is that an optimal coloring and a maximum clique of a perfect graph can be found in polynomial time. More generaly, the theory provided the first theoretically efficient algorithm for semi-definite programming. Semi-definite programs are linear programs over the cones of semi-definite matrices. Practitioners have been formulating and solving semi-definite programs for decades, although they knew of no theoretically efficient algorithm to solve them. As pointed out in [35], algorithms to solve semi-definite programs grew out of the theory of perfect graphs.

An application of perfect graphs to municipal services is described in [41] and [42]. A more recent application area of perfect graphs has been investigated by Gerke, McDiarmid and Reed [16–18, 31, 32]. Motivated by the radio channel assignment problem they introduced and studied a new parameter, called the imperfection ratio. There are several equivalent definitions, but the easiest for us to state is the following: the *imperfection ratio of a graph*, denoted by $\mathrm{imp}(G)$, is the minimum number t such that $\mathrm{QSTAB}(G) \subseteq t\mathrm{STAB}(G)$. Thus by (3.4) $\mathrm{imp}(G) = 1$ if and only if G is perfect. Furthermore, $\mathrm{imp}(G) = \mathrm{imp}(\overline{G})$, generalizing (3.1). We refer to [31] for more details, including background on the channel assignment problem.

7. Graph entropy

Körner [25] studied the following problem. Let G be a graph, and let $P = (p_v : v \in V(G))$ be a probability distribution on $V(G)$. We will think of $V(G)$ as a finite alphabet whose elements are being emitted by a discrete memoryless and stationary information source according to the probability distribution P. Adjacency in G is interpreted as distinguishability. Two t-tuples of vertices of G are *distinguishable* if they are distinguishable in at least one coordinate, and are *indistinguishable* otherwise.

   We wish to examine the performance of a best possible encoding. Here for $\epsilon > 0$ an ϵ-encoding is a mapping from $V(G)^t \to M$, where M is some set, such that there is a set $\Omega \subset V(G)^t$ of probability (in the product space) at least $1 - \epsilon$ such that every two distinguishable t-tuples from Ω are mapped onto different elements of M. The *rate* of this encoding is defined as $\frac{\log|M|}{t}$. Let $R(G, P, t, \epsilon)$ denote the smallest possible rate over all possible ϵ-encodings. Of interest is the value $\liminf_{\epsilon \to 0} \liminf_{t \to \infty} R(G, P, t, \epsilon)$. This is the *graph entropy* $H(G, P)$, introduced by Körner [25]. Körner gave several descriptions of $H(G, P)$, including the following.

7.1 *Let G be a graph, and let P be a probability distribution on $V(G)$. Then $H(G, P) = \min_{x \in \text{STAB}(G)} \sum_{v \in V(G)} p_v \log \frac{1}{p_v}$.*

   If G is the complete graph, then $H(G, P)$ specializes to the classical notion of entropy of a probability distribution, namely $H(P) = \sum_{i=1}^{n} p_i \log \frac{1}{p_i}$. From an information theory point of view it is interesting to study graphs for which $H(G, P) + H(\overline{G}, P) = H(P)$ for every probability distribution P. By a theorem of Cziszár, Körner, Lovász, Marton and Simonyi [12] this holds if and only if G is perfect. Simonyi [40] recently generalized this by showing that $\max_P \left(H(G, P) + H(\overline{G}, P) - H(P) \right) = \log \text{imp}(G)$, where $\text{imp}(G)$ is the imperfection ratio mentioned earlier.

   Finally, let us mention that graph entropy and perfection played important roles in the sorting algorithm of Kahn and Kim [21].

8. Outline of the proof

We now come to the second part of the paper. In the remainder of the text we will outline our proof of (1.1). It is customary to call a graph *Berge* if it has no odd hole and no odd antihole. Thus in order to prove (1.1) we must show that every Berge graph is perfect. We use a strategy originated by Conforti, Cornuéjols and Vušković. They conjectured that every Berge graph either belongs to one of the four classes introduced in Section 2, or has a separation of one of two kinds. This is a reasonable plan, because similar approach was successful for many other graph theory problems, for instance [29, 30, 36, 37, 43], and many others. We were able to prove this conjecture, but to deduce (1.1) we needed to prove a minor modification of it. Let us first introduce the two kinds of separation. A *2-join* in G is a partition (X_1, X_2) of $V(G)$ so that there exist disjoint nonempty sets $A_i, B_i \subseteq X_i$ ($i = 1, 2$) satisfying:

- every vertex of A_1 is adjacent to every vertex of A_2, every vertex of B_1 is adjacent to every vertex of B_2, and there are no other edges between X_1 and X_2,

- for $i = 1, 2$, every component of $G|X_i$ meets both A_i and B_i, and
- for $i = 1, 2$, if $|A_i| = |B_i| = 1$ and $G|X_i$ is a path joining the members of A_i and B_i, then it has odd length ≥ 3.

Cornuéjols and Cunningham [11] (see also [10]) proved that no minimal counterexample to (1.1) has a 2-join. More precisely, they proved:

8.1 *No minimally imperfect graph has a 2-join.*

Moreover, 2-joins afford a decomposition into two smaller graphs such that the original graph is Berge if and only if both the new graphs are Berge, and the same holds for perfection. By (3.1) it follows that the complement of a minimal counterexample to (1.1) also has no 2-join.

Let us discuss skew partitions now. A set $X \subseteq V(G)$ is *connected* if $G|X$ is connected or if X is empty; and *anticonnected* if $\overline{G}|X$ is connected. A *skew partition* in G is a partition (A, B) of $V(G)$ so that A is not connected and B is not anticonnected. Chvátal [3] conjectured that no minimal imperfect graph has a skew partition. This follows from (1.1): by (1.1) the only minimal imperfect graphs are odd holes and odd antiholes, and it is easy to check that those graphs have no skew partition. But in order to be able to use Chvátal's conjecture we would need a proof from first principles, and we were unable to find one. However, it turned out that for our purposes a restricted class of skew partitions sufficed, what we call even skew partitions.

An *antipath* is a subgraph whose complement is a path. The *length* of a path is the number of edges in it and the length of an antipath is the number of edges in its complement. A skew partition (A, B) is *even* if every induced path of length at least two with ends in B and with interior in A is even, and every induced antipath of length at least two with ends in A and with interior in B is even. The significance of even skew partitions is that we can prove a variant of Chvátal's skew partition conjecture for them. Let us say a *minimum imperfect graph* is a Berge graph G such that G is not perfect, but every Berge graph H with $|V(H)| < |V(G)|$ is perfect.

8.2 *No minimum imperfect graph has an even skew partition.*

We give a proof of (8.2) at the end of this section. Our main result is a version of the conjecture of Conforti, Cornuéjols and Vušković, with skew partitions replaced by even skew partitions. It turns out that if we do that, then we need another basic class, the following.

Let H be a bipartite graph, with bipartition (A, B). For each vertex $v \in V(H)$ take two new vertices s_v, t_v, and make a graph G with $V(G) = \{s_v, t_v : v \in V(H)\}$. The edges of G are as follows:

- for $v \in V(H)$, s_v is adjacent to t_v if $v \in A$, and s_v is nonadjacent to t_v if $v \in B$.
- for distinct $u, v \in A$, there are no edges between s_u, t_u and s_v, t_v
- for distinct $u, v \in B$, both s_u, t_u are adjacent to both s_v, t_v.
- for $u \in A$ and $v \in B$, there are exactly two edges joining one of s_u, t_u to one of s_v, t_v; if $uv \in E(H)$ then $s_u s_v$ and $t_u t_v$ are edges of G, and otherwise $s_u t_v$ and $t_u s_v$ are edges of G.

We call such a graph G a *double split graph*. Let us say a graph G is *basic* if either G or $\overline{G}$ is bipartite or is the line graph of a bipartite graph, or is a double split graph. (Note that if G is a double split graph then so is $\overline{G}$.) It is easy to see that all double split graphs are perfect, and thus all basic graphs are perfect by the results discussed in Section 2.

If $X, Y \subseteq V(G)$ are disjoint, we say X is *complete* to Y (or the pair (X, Y) is *complete*) if every vertex in X is adjacent to every vertex in Y; and we say X is *anticomplete* to Y if there are no edges between X and Y. An *M-join* in G is a partition of $V(G)$ into six nonempty sets, (A, B, C, D, E, F), so that:

- every vertex in A has a neighbor in B and a nonneighbor in B, and vice versa
- the pairs (C, A), (A, F), (F, B), (B, D) are complete, and
- the pairs (D, A), (A, E), (E, B), (B, C) are anticomplete.

M-joins are closely related to "homogenous sets" of Chvátal and Sbihi [4], and it follows from their work that no minimally imperfect graph has an M-join. Our main result is as follows.

8.3 *For every Berge graph G, either G is basic, or one of G, $\overline{G}$ admits a 2-join, or G admits an M-join, or G admits an even skew partition.*

The introduction of double split graphs is necessary, because, in general, even though double split graphs have skew partitions, they do not have even ones. The first author has shown in her PhD thesis [5] that M-joins are not needed in the sense that the above theorem remains true if we omit the outcome that G has an M-join. However, to prove this strengthening required an amount of effort comparable to the proof of (8.3) itself. By (8.1), (8.2) and the preceding discussion, theorem (8.3) implies (1.1).

In the remainder of this section we prove (8.2). In fact, we prove something seemingly stronger. Let us say a skew partition (A, B) in a graph G is *rigid* if there exists a maximal anticonnected set $B_1 \subseteq B$ such that the graph obtained from G by adding a new vertex joined precisely to all the vertices in B_1 is Berge. It follows that (A, B) is rigid if and only if every induced path of length at least two with both ends in B_1 and interior in A is even, and every induced antipath of length at least two with both ends in A and interior in B_1 is even. Thus every even skew partition is rigid, and hence (8.2) is implied by the following. Our proof incorporates results of Chvátal [3] and Hoàng [20].

8.4 *No minimum imperfect graph has a rigid skew partition.*

Proof. Suppose for a contradiction that G is a minimum imperfect graph with a rigid skew partition (A, B), and let B_1 be as in the definition of rigid. If one component of A has only one vertex, then by considering the vertex-set of that component we find that (B, A) is a rigid skew partition of $\overline{G}$. Thus we may assume that if some component of A has only one vertex, then $|B_1| = 1$.

Let A_1 be the vertex-set of a component of $G|A$, and let $A_2 = A - A_1$. For $i = 1, 2$ let G_i be the subgraph of G induced by $A_i \cup B$. Let $k = \omega(G)$ and $s = \omega(B_1)$. (For $Z \subseteq V(G)$ we define $\omega(Z)$ to be the size of a largest clique contained in Z.) We need the following claim.

Claim. For $i = 1, 2$ there exists a set $X_i \subseteq V(G_i)$ such that $X_i \cap B = B_1$, $\omega(X_i) = s$ and $\omega(G_i \setminus X_i) \leq k - s$.

It suffices to prove the claim for $i = 1$. Assume first that $|B_1| = 1$. Since G_1 is perfect by the minimality of G, it has a coloring using k colors. Let X_1 be the color class containing the unique element of B_1. Then X_1 is as desired, because B_1 is complete to $B - B_1$. Thus we may assume that $|B_1| > 1$, and hence by the choice of (A, B) above we have $|A_1| \geq 2$ and $|A_2| \geq 2$.

Let G_1' be the graph obtained from G_1 by adding a vertex z joined to precisely the vertices of B_1. Then G_1' is Berge by the rigidity of (A, B), and hence is perfect by the minimality of G and the fact that $|A_2| \geq 2$. Let G_1'' be obtained from G_1' by replacing z by a clique K of size $k - s$, all of whose vertices are adjacent to B_1 and to no other vertex of G_1. Then G_1'' is perfect by (3.3). We have $\omega(G_1'') \leq k$ by construction, and hence G_1'' has a coloring using k colors. Exactly $k - s$ of those colors appear on K, and hence exactly s colors appear on vertices of B_1. Let X_1 be the set of all vertices of G_1 colored using one of those s colors. Since B_1 is complete to $B - B_1$, it follows that X_1 is as desired. This proves the claim.

Now let X_1 and X_2 be as in the claim. It follows that $\omega(X_1 \cup X_2) = s$ and $\omega(G \backslash (X_1 \cup X_2)) \leq k - s$. Thus $G|(X_1 \cup X_2)$ can be colored using s colors, and $G \backslash (X_1 \cup X_2)$ can be colored using $k - s$ colors, because both those graphs are perfect by the minimality of G. It follows that G is k-colorable, contrary to the fact that G is minimum imperfect. □

9. Line graphs

In the rest of the paper we outline the proof of (8.3). The first step is to show that if G contains a line graph of a bipartite subdivision of a "large" 3-connected graph, then either G is itself a line graph of a bipartite graph, or it has one of the separations mentioned in (8.3).

9.1 *Let G be a Berge graph, let J be a 3-connected graph, let H be a bipartite subdivision of J, and let $L(H)$ be an induced subgraph of G. Assume further that $H \neq K_{3,3}$, and that if $J = K_4$, then some edge of every 4-cycle of J is subdivided in H. Then G is a line graph of a bipartite graph, or has a 2-join or has an even skew partition.*

Let us sketch the main steps in the proof of (9.1). First, it is convenient to enlarge $L(H)$ into a certain auxiliary structure, which we call a J-strip system. Let J be 3-connected, and let G be Berge. A *J-strip system* in G is a pair (S, N), where $S_{uv} = S_{vu} \subseteq V(G)$ for each edge uv of J, and $N_v \subseteq V(G)$ for each vertex v of J such that (for $uv \in E(J)$, a *uv-rung* means an induced path R of G with ends s, t say, where $V(R) \subseteq S_{uv}$, and s is the unique vertex of R in N_u, and t is the unique vertex of R in N_v)

  (i) the sets S_{uv} ($uv \in E(J)$) are pairwise disjoint,
 (ii) for each $u \in V(J)$, $N_u \subseteq \bigcup (S_{uv} : v \in V(J)$ adjacent to $u)$,
(iii) for each $uv \in E(J)$, every vertex of S_{uv} is in a uv-rung,
 (iv) if $uv, wx \in E(J)$ with u, v, w, x all distinct, then there are no edges between S_{uv} and S_{wx},
  (v) if $uv, uw \in E(J)$ with $v \neq w$, then $N_u \cap S_{uv}$ is complete to $N_u \cap S_{uw}$, and there are no other edges between S_{uv} and S_{uw}.

Now $L(H)$ gives rise to a J-strip system (S, N) as follows. First notice that $V(J) \subseteq V(H)$, and that each edge $uv \in E(J)$ corresponds to a path in H. Let $f_1, f_2, \ldots, f_k$ be the edges of that path, in order, such that f_1 is incident with u. Then $f_1, f_2, \ldots, f_k$ is the vertex-set of an induced path of $L(H)$. We put f_1 into N_u, f_k into N_v, and define $S_{uv} := \{f_1, f_2, \ldots, f_k\}$. By repeating this for every edge of J we arrive at a J-strip system, which we call the *J-strip system derived from $L(H)$*.

If (S, N) is a J-strip system in G, we define $V(S, N)$ to be the union of S_{uv} over all edges of J. Thus $N_v \subseteq V(S, N)$ for every $v \in V(J)$.

We say that a set X *saturates* the J-strip system (S, N) if for every $u \in V(J)$ there is at most one neighbor v of u in J such that $N_u \cap S_{uv} \nsubseteq X$. A vertex $u \in V(G) - V(S, N)$ is *major* if its neighbors in $V(S, N)$ saturate (S, N), and *minor* otherwise. A set $X \subseteq V(S, N)$ is *local* if either $X \subseteq N_v$ for some $v \in V(J)$, or $X \subseteq S_{uv}$ for some edge $uv \in E(J)$. For the proof of (9.1) we first choose J with $|V(J)|$ maximum, start with the J-strip system derived from $L(H)$, and enlarge it to a J-strip system with $V(S, N)$ maximum. The advantage of that is that now we have the following lemma:

9.2 *Let $F \subseteq V(G) - V(S, N)$ be connected and contain no major vertices. Then the set of neighbors in $V(S, N)$ of vertices in F is a local set.*

We will not give a proof of this lemma, but let us point to Figure 1 for illustration. Let (S, N) be the J-strip system derived from $L(H)$, where J and H are as in Figure 1. There is a unique component F of $V(G) - V(S, N)$, and the neighbors of F do not form a local set. But the strip system is not maximal: the set F can be added to the J-strip system (S, N).

Now for the proof of (9.1). Suppose first that there is a major vertex, and let Y be a maximal anticonnected set of major vertices. Let X be the set of common neighbors of vertices in Y. It can be shown that X saturates the strip system, and from there it is not hard to produce first a skew partition, and then an even skew partition.

Thus we may assume that there are no major vertices. If there is a component F of $G \backslash V(S, N)$ whose set of neighbors is a subset of N_v for some $v \in V(J)$, then $(V(G) - N_v, N_v)$ is a skew partition, and it is not hard to convert it into an even skew partition. On the other hand we may assume that $G \neq L(H)$, and so by (9.2) there is an edge $uv \in E(J)$ such that either S_{uv} is not a path, or some component of $G \backslash V(S, N)$ has all its neighbors in S_{uv}. Let F be the union of all such components. Then the partition $(F \cup S_{uv}, V(G) - F - S_{uv})$ is a 2-join in G. This completes the sketch of the proof of (9.1).

If $H = K_{3,3}$, then the same argument works, except for the step where we deduce that the set of common neighbors of a maximal anticonnected component of major
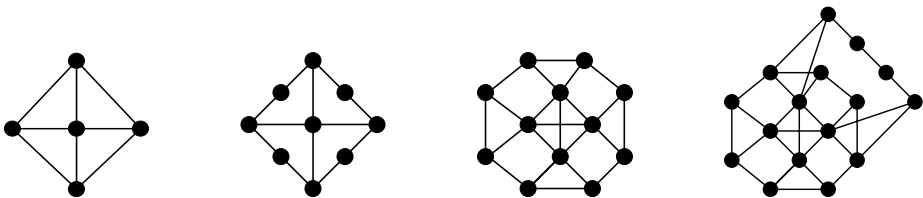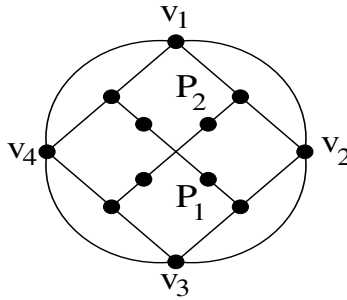

Fig. 1. J, H, $L(H)$ and G

Fig. 2. Line graph of a subdivision of K_4

vertices saturates the strip system. However, if that outcome does not hold, then we find that there is a bipartite subdivision H' of $K_{3,3}$, itself not equal to $K_{3,3}$, such that $L(H')$ is an induced subgraph of the complement of G. In that case we apply (9.1) to H' and $\overline{G}$. This proves

9.3 *Let G be a Berge graph, let J be a 3-connected graph, let H be a bipartite subdivision of J, and let $L(H)$ be an induced subgraph of G. Assume further that if $J = K_4$, then some edge of every 4-cycle of J is subdivided in H. Then G or its complement is a line graph of a bipartite graph, or has a 2-join or has an even skew partition.*

Let us now assume that $J = K_4$, and that no edge of some 4-cycle of J is subdivided in H. Then $L(H)$ is pictured in Figure 2; let P_1, P_2 and v_1, v_2, v_3, v_4 be as in that figure. Let Q_1 denote the antipath with vertex-set $\{v_1, v_3\}$, and let Q_2 denote the antipath with vertex-set $\{v_2, v_4\}$. Notice that if one of P_1, P_2 has length at least two (and hence at least three, because they are odd), then $L(H)$ has a 2-join, whereas if they both have length one, then $L(H)$ is a double split graph. Thus our strategy is as follows. First, we collect all such paths $P_1, P_2, \ldots$ and antipaths $Q_1, Q_2, \ldots$. More generally, we enlarge them into strips and "antistrips", and maximize the resulting structure S. We divide vertices of $V(G) - V(S)$ into *major* and *minor* similarly as before. Here the situation is nicer, because the structure is invariant under taking complements, and taking complements turns minor vertices into major ones, and vice versa. As before, if there is a major vertex, then a maximal anticonnected component of such vertices gives rise to a skew partition, and similarly for connected components of minor vertices. Finally, if all of G falls into the structure, then either the structure proves that G is a double split graph, or G has a 2-join. Thus we have

9.4 *Let G be a Berge graph, let H be a bipartite subdivision of a 3-connected graph, and let $L(H)$ be an induced subgraph of G. Then G is a double split graph, or has an even skew partition, or it or its complement is a line graph of a bipartite graph, or has a 2-join.*

10. Prisms

A *triangle* in a graph G is a cycle of length three. Let $a_1a_2a_3$ and $b_1b_2b_3$ be disjoint triangles, and let P_1, P_2, P_3 be three vertex-disjoint induced paths in G, where P_i has

ends a_i and b_i, and at least one of P_1, P_2, P_3 has length at least two. Assume further that for $1 \leq i < j \leq 3$ there are no edges between P_i and P_j, except for $a_i a_j$ and $b_i b_j$. In those circumstances we say that $P_1 \cup P_2 \cup P_3$ is a *prism* in G. Thus a prism is the line graph of the graph consisting of three internally disjoint paths joining the same pair of vertices. A version of the following result was suggested to the third author by Kristina Vušković.

10.1 *Let G be a Berge graph containing a prism such that neither G nor $\overline{G}$ has an induced subgraph isomorphic to the line graph of a bipartite subdivision of a 3-connected graph. Then either one of G, $\overline{G}$ admits a 2-join, or G admits an even skew partition, or G admits an M-join.*

If G is Berge, and P_1, P_2, P_3 are the paths of a prism in G, then it is easy to see that P_1, P_2, P_3 have the same parity. If they are all even, then (10.1) can be proven by a method analogous to the proof of (9.1). However, the proof is much harder when P_1, P_2, P_3 are odd. We refer to [6] for the details.

There is another graph that can be excluded by similar methods. A *double diamond* is the graph with vertex-set $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$ and the following adjacencies: every two a_i's are adjacent except for $a_3 a_4$, every two b_i's are adjacent except for $b_3 b_4$, and a_i is adjacent to b_i for all $i = 1, 2, 3, 4$.

10.2 *Let G be a Berge graph such that no induced subgraph of G or $\overline{G}$ is isomorphic to a prism or the line graph of a bipartite subdivision of a 3-connected graph. If the double diamond is an induced subgraph of G, then either G or $\overline{G}$ admits a 2-join, or G admits an even skew partition.*

11. Skew partitions revisited

Let us say a Berge graph G is *bipartisan* if no induced subgraph of G or $\overline{G}$ is isomorphic to the double diamond, a prism, or the line graph of a bipartite subdivision of a 3-connected graph. In view of (9.4), (10.1) and (10.2) it suffices to prove (8.3) for bipartisan graphs. From (8.1), (8.2) and the perfection of basic graphs we deduce that every minimum imperfect graph is bipartisan.

It turns out that stronger theorems hold for bipartisan graphs, and in this section we discuss one such result. It can be shown that if a bipartisan graph has a skew partition, then it has an even skew partition. Thus in the remainder of the proof of (8.3) we do not need to concern ourselves with checking whether a particular skew partition is even. Here we confine ourselves to proving the essential part of Chvátal's skew partition conjecture:

11.1 *No minimum imperfect graph admits a skew partition.*

In the proof we will need the following beautiful and powerful theorem of Roussel and Rubio [38]. (We proved it independently, in joint work with Carsten Thomassen, but Roussel and Rubio found it earlier.) We use it many times in our proof of (1.1). So far we have not shown enough detail to demonstrate its usefulness, but its time has come.

11.2 *Let G be a Berge graph, let $X \subseteq V(G)$ be anticonnected, and let P be an induced path in $G \backslash X$ of odd length, such that both ends of P are complete to X. Then either*

  (i) *some internal vertex of P is complete to X, or*
 (ii) *the path P has length at least five and there exists an induced path Q with both ends in X such that P and Q has the same interior, or*
(iii) *the path P has length three and there is an induced odd antipath joining the internal vertices of P with interior in X.*

Please note that if (ii) above holds, then G has a prism. Thus for bipartisan graphs we have the following simpler version.

11.3 *Let G be a bipartisan Berge graph, let $X \subseteq V(G)$ be anticonnected, and let P be an induced path in $G \backslash X$ of odd length, such that both ends and no internal vertex of P are complete to X. Then the path P has length three and there is an induced odd antipath joining the internal vertices of P with interior in X.*

Since, as noted above, every minimum imperfect graph is bipartisan, (11.1) follows from (8.4) and the following result.

11.4 *If a bipartisan Berge graph has a skew partition, then it has a rigid skew partition.*

Proof. Let G be a bipartisan Berge graph, and let (A, B) be a skew partition in G chosen so that B is minimal with respect to inclusion. Assume first that some $v \in B$ has no neighbor in some component of $G|A$. Since $(A \cup \{v\}, B - \{v\})$ is not a skew partition of G by the minimality of B, we deduce that $B - \{v\}$ is anticonnected. Thus v is adjacent to every vertex of $B - \{v\}$, and hence by considering the anticonnected component $\{v\}$ of $G|B$ we deduce that (A, B) is rigid, as desired. Thus we may assume that every $v \in B$ has a neighbor in every component of $G|A$.

We now prove the following claim.

Claim. Let B_1 be a maximal anticonnected subset of B. If some vertex $u \in A$ is adjacent to every vertex of B_1, then (A, B) is rigid.

To prove the claim let first P be an induced path of length at least two with both ends in B_1 and interior in A. We must prove that P is even. The interior of P belongs to a component of $G|A$ with vertex-set A_1, say. If $u \notin A_1$, then P can be completed to a hole by adding u. Since this hole is even, it follows that P is even. Thus we may assume that $u \in A_1$. Let A_2 be the vertex-set of another component of $G|A$. Since every vertex of B has a neighbor in A_2, there exists an induced path P' with the same ends as P and with interior contained in A_2. By completing this path to a hole through the vertex u we see that P' is even. Thus P is even, for otherwise $P \cup P'$ is an odd hole.

Now let Q be an induced antipath of length at least two with both ends in A and interior in B_1, and suppose for a contradiction that Q is odd. The ends of Q belong to the same component of $G|A$. Let the vertex-set of that component be A_1', and let A_2' be the vertex-set of another component of $G|A$. By (11.3) applied in the complement of G to the path $\overline{Q}$ and anticonnected set A_2' we deduce that $\overline{Q}$ has length three. Thus Q is also a path of length three, with both ends in B_1 and interior in A, contrary to the previous paragraph. This proves the claim.

To prove the theorem we may assume that (A, B) is not rigid, and hence there exists a maximal anticonnected set $B_1 \subseteq B$ such that the graph obtained from G by adding a vertex adjacent to precisely the vertices of B_1 is not Berge. This means that there is either an induced odd path of length at least two with both ends in B_1 and interior in A, or an induced odd antipath of length at least two with both ends in A and interior in B_1. Let B_2 be a maximal anticonnected subset of B other than B_1.

Assume first that there exists an induced path P of length at least five with both ends in B_1 and interior in A. By applying (11.3) to P and B_2 we deduce that some internal vertex of P is adjacent to every vertex of B_2, and hence (A, B) is rigid by the claim. Thus we may assume that no such path P exists.

If P is as in the previous paragraph, but of length three, then it is also an antipath with both ends in A and interior in B_1. Thus it follows that G has an induced odd antipath Q_1 with both ends in A and interior in B_1. Let $x_1, y_1 \in A$ be the ends of Q_1. Since they are adjacent, they belong to the same component of $G|A$. Let the vertex-set of that component be A_1, and let A_2 be the vertex-set of another component of $G|A$. By (11.3) applied in the complement of G to the path $\overline{Q_1}$ and anticonnected set A_2 we deduce that Q_1 has length three (let its internal vertices be b_1 and b_1') and that there exists an induced odd path P_2 with ends b_1 and b_1' and all internal vertices in A_2. By the conclusion of the previous paragraph the path P_2 has length three; let its vertices be b_1, x_2, y_2, b_1', in order. By the claim and (11.3) applied to the path P_2 and anticonnected set B_2 we may assume that there exists an induced odd antipath Q_2 with ends x_2, y_2 and interior in B_2. The proof that Q_1 has length three applies to Q_2 as well, and so Q_2 has length three. Thus Q_2 is also a three-edge path; let its vertex set be b_2, x_2, y_2, b_2', in order. Since Q_1 cannot be completed to an odd antihole by adding b_2 or b_2', we deduce that each of them has a neighbor in $\{x_1, y_1\}$. But they do not have a common neighbor in $\{x_1, y_1\}$, for otherwise Q_2 can be completed to an odd antihole using that common neighbor. Thus either b_2 is adjacent to x_1 and not to y_1 and b_2' is adjacent to y_1 and not to x_1, or b_2' is adjacent to x_1 and not to y_1 and b_2 is adjacent to y_1 and not to x_1. In the former case the subgraph of G induced by $\{x_1, y_1, x_2, y_2, b_1, b_1', b_2, b_2'\}$ is a double diamond, and in the latter case it is isomorphic to the line graph of the graph $K_{3,3}$ with one edge deleted. In either case this is a contradiction to the fact that G is bipartisan. $\square$

We remark that (11.1) comes short of proving Chvátal's skew partition conjecture in its full generality, because Chvátal conjectured the same conclusion for *minimal* imperfect graphs. However, (11.1) is sufficiently strong for the proof of (1.1), which in turn implies Chvátal's conjecture.

12. Wheels

As we pointed out earlier, it suffices to prove (8.3) for bipartisan graphs. For bipartisan graphs a stronger result holds.

12.1 *Let G be a bipartisan graph such that G and $\overline{G}$ are not bipartite. Then G has an even skew partition.*

The proof of (12.1) is still fairly long, and perhaps somewhat less natural than the earlier steps. Our strategy, originally initiated by Conforti and Cornuéjols [7], is to use

wheels. If $Y \subseteq V(G)$, then we say that an edge of G is Y-*complete* if both its ends are complete to Y. A *wheel* in a graph G is a pair (C, Y), where C is a hole in G of length at least six and $Y \subseteq V(G) - V(C)$ is a non-empty anticonnected set such that there are two disjoint Y-complete edges of C. We call C the *rim* and Y the *hub* of the wheel.

12.2 *If (C, Y) is a wheel in a Berge graph, then C has an even number of Y-complete edges.*

Proof. If not, then, since C is even, it has an odd subpath P of length at least three with both ends but no internal vertex complete to Y. By (11.2) applied to P and Y there exists either an induced odd path R as in (11.2)(ii) or an induced odd antipath Q as in (11.2)(iii). Since C has two Y-complete edges, it has a vertex $z \notin V(P)$ complete to Y. By adding z to R or Q as appropriate we obtain an odd hole or an odd antihole in G, a contradiction. □

Let (C, Y) be a wheel in a Berge graph G. We say that two vertices $u, v \in V(C)$ have the *same wheel parity* if a subpath of C joining them contains an even number of Y-complete edges (in which case so does the other subpath of C joining them, by (12.2)), and we say they have *opposite wheel parity* otherwise. Now let X be the set of all common neighbors of vertices of Y. Let us say that the wheel (C, Y) is *separable* if some set $Z \subseteq X \cup Y$ intersects both X and Y and separates a vertex of one wheel parity from a vertex of the opposite wheel parity. This is a desirable property, because in this case $(V(G) - Z, Z)$ is a skew partition of G. Unfortunately, not all wheels are separable. First, let us point out the significance of prisms. In the example of Figure 3 the wheel $(C, \{y_1, y_2\})$ (C is shown thick) is not separable. If the path P joining the vertices u and v has length at least two (in fact, it must be odd), then G has a prism, while if P has length one, then the complement of G has a prism. This example illustrates that the absence of prisms eliminates a large class of wheels that are not separable.

But even in bipartisan graphs not all wheels are separable. For instance, consider the graph in Figure 4. It is tempting to replace C by the hole $vv_1v_2v_3v_4v_5$, but then $\{y\}$ is no longer a hub. However, this argument does work when (C, Y) is an "odd wheel". A Y-*segment* is a maximal subpath of C whose vertices are all complete to Y. We say that a wheel (C, Y) is an *odd wheel*, if it has an odd Y-segment. For odd wheels the above argument indeed works: if the difficulty indicated in Figure 4 arises, then we can reroute the rim to obtain another odd wheel, which is better in a certain way. Thus we
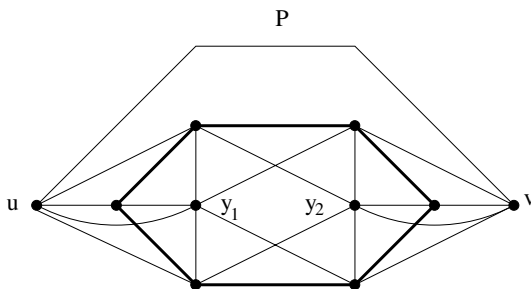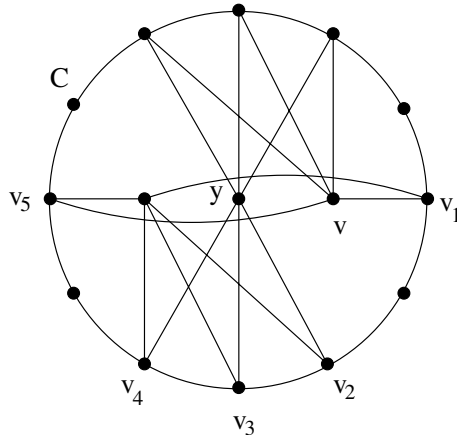

Fig. 3. A non-separable odd wheel

Fig. 4. A non-separable wheel in a bipartisan graph

can show that some carefully selected wheel is separable, and so we have the following result, obtained independently by Conforti, Cornuéjols, Vušković and Zambelli [8].

12.3 *If a bipartisan graph has an odd wheel, then it has an even skew partition.*

How about wheels that are not odd? Those require a different approach. Let (C, Y) be such a wheel in a bipartisan graph. Then there are three consecutive vertices u_1, u_0, u_2 on C, all complete to Y. We now find a maximal sequence $u_1, u_2, \ldots, u_t$ of vertices complete to $Y \cup \{u_0\}$ such that for all $i = 3, 4, \ldots, t$

- there exists a connected subset of $V(G)$ including $V(C) - \{u_0, u_1, u_2\}$, containing a neighbor of u_i, containing no neighbor of u_0 and containing no vertex complete to $\{u_1, u_2, \ldots, u_{i-1}\}$, and
- the vertex u_i is not complete to $\{u_1, u_2, \ldots, u_{i-1}\}$.

Let $U = \{u_1, u_2, \ldots, u_t\}$, and let V be the set of all U-complete vertices other than u_0. Now it is almost true that the set $U \cup V$ separates $\{u_0\}$ from $V(C) - \{u_0, u_1, u_2\}$, in which case $(V(G) - U - V, U \cup V)$ is a skew partition. Thus we show (see [6] for details) that either G has a skew partition, or G has a wheel (C', Y'), where Y is a proper subset of Y'. By choosing the original wheel with Y maximal we therefore obtain an extension of (12.3) to all wheels.

Finally, we must prove that for every bipartisan graph, either it satisfies the conclusion of (12.1), or either it or its complement contains a wheel. We refer to [6] for full details. A weaker result (using a less restricted definition of a wheel) was obtained independently by Conforti, Cornuéjols and Zambelli [9].

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