Induced subgraphs of graphs with large chromatic number. XI. Orientations

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Abstract

Fix an oriented graph H, and let G be a graph with bounded clique number and very large chromatic number. If we somehow orient its edges, must there be an induced subdigraph isomorphic to H? Kierstead and Rödl [12] raised this question for two specific kinds of digraph H: the three-edge path, with the first and last edges both directed towards the interior; and stars (with many edges directed out and many directed in). Aboulker et al. [1] subsequently conjectured that the answer is affirmative in both cases. We give affirmative answers to both questions.

1 Introduction

All graphs in this paper are finite and simple. If G is a graph, $\chi(G)$ denotes its chromatic number, and $\omega(G)$ denotes its clique number, that is, the cardinality of the largest clique of G. This paper is concerned with the digraphs that can be obtained by orienting the edges of a graph, and in particular, digraphs in this paper have no "antiparallel" pairs of edges, that is, no directed cycles of length two, as well as no loops or parallel edges. If G is a digraph, G^* means the underlying graph. For a digraph G, we say u is G-adjacent *to* v or *from* v to indicate the direction of the edge between u and v, and u is G^*-adjacent *with* v to mean adjacency in G^*. The *chromatic number* $\chi(G)$ and *clique number* $\omega(G)$ of a digraph G mean the corresponding quantities for G^*.

Let H be a graph. We say that H is χ-*bounding* if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every graph G not containing H as an induced subdigraph. If H is χ-bounding then it cannot contain a cycle, since (as shown by Erdős [5]) there are graphs with large girth and large chromatic number. Thus the only possible χ-bounding graphs are forests. The Gyárfás-Sumner conjecture [6, 17] asserts that every forest is χ-bounding. Despite considerable work, the conjecture is only known for comparatively few families (see [7, 9, 10, 11, 15, 16, 3]).

Now let H be an oriented graph. When is there a function f such that $\chi(G) \leq f(\omega(G))$ for every oriented graph G not containing H as an induced subdigraph? As in the graph case, we call a digraph H with this property χ-*bounding*. Then every χ-bounding digraph H is an oriented forest, because we can take G to be any orientation of a graph with large girth and large chromatic number. However, for digraphs it is *not* the case that H is χ-bounding whenever H^* is a forest. Indeed, this is false even for digraphs H such that H^* is a three-edge path.

There are four ways to orient the edges of a three-edge path, up to reversing the path, and we denote the corresponding digraphs by

$$\to\to\to, \to\leftarrow\to, \to\leftarrow\leftarrow, \leftarrow\to\to$$

with the natural meaning. Then

- Kierstead and Trotter [13] showed that $\to\to\to$ is not χ-bounding, by constructing triangle-free graphs with arbitrarily large chromatic number together with a suitable orientation;
- Gyárfás [8] noted that $\to\leftarrow\to$ is not χ-bounding: let D be the natural orientation of the shift graph on pairs, so D has vertex set $[n]^{(2)}$, with edges from $\{i, j\}$ to $\{j, k\}$ whenever $i < j < k$. Then D is triangle-free, has large chromatic number, and does not contain an induced copy of $\to\leftarrow\to$.

The two remaining orientations are equivalent under reversing all edges, so both or neither are χ-bounding. Thus it is enough to consider $\to\leftarrow\leftarrow$.

Oriented graphs with no induced $\to\leftarrow\leftarrow$ have been considered by several authors. In the special case of *acyclic* orientations, Chvátál [4] showed that if G is an acyclic oriented graph with no induced copy of $\to\leftarrow\leftarrow$ then G is perfect, and so $\chi(G) = \omega(G)$. Kierstead and Rödl [12] asked whether $\to\leftarrow\leftarrow$ is χ-bounding, and showed that the class of oriented graphs with no induced copy of $\to\leftarrow\leftarrow$ and no cyclic triangle is χ-bounded. Aboulker et al. [1] conjectured that $\to\leftarrow\leftarrow$ is in fact χ-bounding, and proved some further special cases. Our first main result resolves the question.

1.1 *The digraphs $\to\leftarrow\leftarrow$ and $\leftarrow\to\to$ are χ-bounding.*

This is proved in the next section.

Which forests have the property that every orientation is χ-bounding? The results of Gyárfás and of Kierstead and Trotter mentioned above show that such a forest cannot contain a three-edge path, and so every component must be a star.

A digraph H is an *oriented star* if H^* is a star, that is, isomorphic to the complete bipartite graph $K_{1,t}$ for some $t \geq 0$. As with paths, there have been several previous results on the chromatic number of graphs with a forbidden oriented star. Gyárfás [8] asked whether, for every oriented star H, the class of acyclic oriented graphs with no induced H is χ-bounded. Kierstead and Rödl [12] proved the stronger result that the class of oriented graphs with no induced H and no cyclic triangle is χ-bounded; they further asked whether every oriented star is χ-bounding. Aboulker et al. [1] conjectured that oriented stars are indeed χ-bounding, and showed that for every oriented star H the class of oriented graphs with no induced H and no transitive triangle has bounded chromatic number (note that every orientation of K_4 has a transitive triangle, so if G has no transitive triangle then $\omega(G)$ is at most 3). Our second main result answers this question.

1.2 *Every oriented star is χ-bounding.*

It is easy to prove this for stars in which every edge is directed away from the centre, or every edge is directed towards the centre, but the case when there are edges of both types is more difficult. It follows from Theorem 1.2 that if F is a forest such that every component is a star then every orientation of F is χ-bounding.

2 An oriented three-edge path

If $X \subseteq V(G)$, $G[X]$ denotes the subgraph or subdigraph induced on X, and we write $\chi(X)$ for $\chi(G[X])$ when there is no danger of ambiguity. If G is a digraph and $v \in V(G)$, we denote the set of vertices with distance at most r (in G^*) from v by $N^r[v]$ or $N^r_G[v]$, and the set with distance exactly r by $N^r(v)$. We denote by $\chi^r(G)$ the maximum of $\chi(N^r[v])$ over all $v \in V(G)$ (or zero for the null digraph.)

In this section we prove our first main result, that $\rightarrow\leftarrow\leftarrow$ is χ-bounding. In fact, with a very little extra work we can prove a stronger statement, which we now explain. A *hole* in a graph is an induced cycle of length at least four, and when G is a digraph, by a "hole" of G we mean an induced subgraph C such that C^* is a hole of G^*. By a *long* hole we mean (just in this paper) a hole of length at least five. A hole of a digraph C is

- *directed* if each of its vertices has outdegree one in C;
- *alternating* if each of its vertices has outdegree two or zero in C (and therefore C has even length); and
- *disoriented* if it is neither directed nor alternating.

It is easy to see that if some long hole of G is disoriented, then G contains $\rightarrow\leftarrow\leftarrow$ as an induced subdigraph. (Some two consecutive edges of G make a two-edge directed path, but C is not a directed cycle; grow the path to a maximal directed path of C and look at its ends.) Thus the following theorem implies that $\rightarrow\leftarrow\leftarrow$ is χ-bounding. (A useful feature of this strengthening is that now we are proving something invariant under reversing all edges of G, which reduces the case analysis.)

2.1 *For all κ there exists c such that if G is a digraph with $\omega(G) \leq \kappa$ and $\chi(G) > c$ then some long hole of G is disoriented.*

Proof. We proceed by induction on κ; thus we may assume that $\chi(J) \leq \tau$ for every digraph J with with $\omega(J) < \kappa$ and no disoriented long hole. Let $c = 2(3\tau)^5$; we claim that c satisfies the theorem. Let G be a digraph with $\omega(G) \leq \kappa$ and with no disoriented long hole.

(1) *For each vertex z and integer $r \geq 0$, $\chi(N^r(z)) \leq 3\tau\chi(N^{r-1}(z))$. Consequently $\chi(N^r(z)) \leq \tau(3\tau)^{r-1}$ and $\chi(N^s(z)) \leq (3\tau)^{s-r}\chi(N^r(z))$ for all $s \geq r$.*

Let us write L_i for $N^i(z)$ $(i \geq 0)$. Since $\omega(G[L_1]) < \kappa$ the result holds if $r = 1$, so we assume $r \geq 2$. Let $I \subseteq L_{r-1}$ be stable. Let I_1 be the set of vertices in I with no in-neighbours in L_{r-2}; I_2 the set with no out-neighbours in L_{r-2}; and $I_3 = I \setminus (I_1 \cup I_2)$. Let J_i be the set of vertices in L_r with a neighbour in I_i for $i = 1, 2, 3$. Let $i \in \{1, 2, 3\}$, and suppose that $\omega(G[J_i]) = \kappa$. Choose a clique K of $G[J_i]$ with cardinality κ, and take a minimal subset I_0 of I_i such that every vertex in K has a neighbour in I_0. Since $\omega(G) = |K|$, it follows that $|I_0| \geq 2$; choose distinct $v_1, v_2 \in I_0$. From the minimality of I_0, there exists $u_1 \in K$ G^*-adjacent with v_1 and not with v_2, and u_2 G^*-adjacent with v_2 and not with v_1. Thus v_1-u_1-u_2-v_2 is an induced path of G^*. By reversing all edges of G if necessary (this is legitimate since what we are proving is invariant under this reversal) we may assume that $i \in \{1, 3\}$.

Since $G^*[L_0 \cup \cdots \cup L_{r-2}]$ is connected, there is an induced path of G^* joining v_1, v_2 with interior in this set, and its union with v_1-u_1-u_2-v_2 is a hole. We may assume this hole is either directed or alternating in G, and in either case exactly one of the edges u_1v_1, u_2v_2 of G^* is oriented in G from L_r to L_{r-1}. Consequently we may assume that v_1u_1 and u_2v_2 are edges of G. Since $i \in \{1, 3\}$, both v_1, v_2 have out-neighbours in L_{r-2}, say w_1, w_2 respectively. If w_1 is G^*-adjacent with v_2, then adding w_1 to u_1, v_1, v_2, u_2 gives a hole of length five that is not directed, a contradiction; so w_1, v_2 are G^*-nonadjacent. In particular $w_1 \neq w_2$, so $r \geq 3$. If v_1, w_2 are G^*-nonadjacent, there is an induced path of G^* between w_1 and w_2 with interior in $L_0 \cup \cdots \cup L_{r-3}$, and its union with w_1-v_1-u_1-u_2-v_2-w_2 yields a disoriented hole of G, a contradiction; so v_1, w_2 are G^*-adjacent. This provides a hole of length five, which is therefore directed; so u_1u_2 and w_2v_1 are edges of G. In particular v_1 has both an in-neighbour and an out-neighbour in L_{r-2}, and so $i = 3$, and therefore v_2 has an in-neighbour x_2 say in L_{r-2}. Since the path $u_2v_2x_2$ is not directed, it follows that x_2, v_1 are G^*-nonadjacent, and in particular $x_2 \neq w_1$. Join w_1, x_2 by an induced path with interior in $L_0 \cup \cdots \cup L_{r-3}$; then the union of this with the path w_1-v_1-u_1-u_2-v_2-x_2 yields a disoriented hole, a contradiction. This proves that $\omega(G[J_i]) < \kappa$, and so $\chi(J_i) \leq \tau$. Consequently $\chi(J_1 \cup J_2 \cup J_3) \leq 3\tau$. Applying this to each colour class of a $\chi(L_{r-1})$-colouring of $G[L_{r-1}]$, we deduce the first assertion of (1). The second follows from the first by induction on r, since $\omega(G[L_1]) < \kappa$ and so $\chi(L_1) \leq \tau$; and the third follows from the first by induction on $s - r$. This proves (1).

Suppose that $\chi(G) > c = 2(3\tau)^5$. We may assume that G^* is connected; choose a vertex z, and let $L_i = N^i(z)$ for all $i \geq 0$. Choose s such that $\chi(L_s) \geq \chi(G)/2$. Since $\chi(G) > 2(3\tau)^5$, it follows that $\chi(L_s) > (3\tau)^5$ and so $s \geq 6$ by (1). Let S be the vertex set of a component of $G[L_s]$ with maximum chromatic number. Let $r = s - 4$, and choose $R \subseteq L_r$ minimal such that every vertex in S is joined to a vertex in R by a path in G^* of length 4. Let $G' = G \setminus (L_r \setminus R)$. Thus $N^r_{G'}(z) = R$, and $S \subseteq N^s_{G'}$. By the third assertion of (1) applied to G', $\chi(N^s_{G'}) \leq (3\tau)^4(\chi(N^r_{G'}))$, and since $\chi(N^s_{G'}) > 3(3\tau)^4$ it follows that $\chi(R) > 2$ (indeed, $\chi(R) > 3$).

If $a \in R$ and $v \in S$, we say that a is an *ancestor* of v if there is a path of G^* between v and a of length 4. From the minimality of R, for each $a \in R$ there is a vertex v in S such that a is its unique ancestor; let P_a be a path between a and some such v of length 4.

Let R_1 be the set of vertices $a \in R$ such that the edge of P_a incident with a has head a, and R_2 the set for which this edge has tail a. Since $\chi(R) > 2$, not both R_1, R_2 are stable, and by reversing all edges if necessary, we may assume that R_1 is not stable. Let $a_r, b_r \in R_1$ be G^*-adjacent. Let the vertices of P_{a_r} be a_r-a_{r+1}-$\cdots$-a_s in order, and let those of P_{b_r} be b_r-b_{r+1}-$\cdots$-b_s in order. Since b_r is the unique ancestor of b_s, b_i is G^*-nonadjacent with a_{i-1} for $r+1 \le i \le s$, and similarly a_i is G^*-nonadjacent with b_{i-1} for $r+1 \le i \le s$. In particular, $a_i \neq b_i$ for $r \le i \le s$. (However, a_i, b_i may be adjacent in G^*.)

(2) *There is an induced path of G^* between a_{r+2} and b_r containing no neighbours of $a_{r+1}, b_{r+1}, b_{r+2}$, and an induced path between b_{r+2} and a_r containing no neighbours of $a_{r+1}, b_{r+1}, a_{r+2}$.*

Since $\chi(S) > 3\chi^5(G)$, there is a vertex $v \in S$ with distance at least 6 from each of $a_{r+1}, b_{r+1}, b_{r+2}$. Let $u \in R$ be an ancestor of v, and let P be a path of length 4 between u and v. Thus none of $a_{r+1}, b_{r+1}, b_{r+2}$ have neighbours in $V(P)$. Also there is an induced path between u and b_r with interior in $L_0 \cup \cdots \cup L_{r-1}$, an induced path between a_s and v with interior in S, and the path a_{r+2}-$\cdots$-a_s. The union of these paths gives a path of G^* (not necessarily induced) between a_{r+2} and b_r, and so there is an induced path of G^* using a subset of the same vertices between a_{r+2} and b_r. None of $a_{r+1}, b_{r+1}, b_{r+2}$ has a neighbour in any of these paths, and this proves the first statement. The second follows by symmetry. This proves (2).

Since $G^*[S]$ is connected, there is an induced path of G^* between a_{r+1} and b_{r+1} with interior in $L_{r+2} \cup L_{r+3} \cup S$, and since the union of this path with the path a_{r+1}-a_r-b_r-b_{r+1} does not give a disoriented hole, it follows that a_{r+1}, b_{r+1} are G^*-adjacent. By exchanging a_r, b_r if necessary, we may assume that the edge $a_{r+1}b_{r+1}$ has head b_{r+1}. If the edge $a_{r+2}a_{r+1}$ has head a_{r+2}, the path b_r-b_{r+1}-a_{r+1}-a_{r+2} together with the first path of (2) gives a disoriented hole, a contradiction. So $a_{r+2}a_{r+1}$ has head a_{r+1}. If the edge $b_{r+2}b_{r+1}$ has head b_{r+2}, then the path a_r-a_{r+1}-b_{r+1}-b_{r+2} together with the second path of (2) gives a disoriented hole; so $b_{r+2}b_{r+1}$ has head b_{r+1}. There is an induced path joining a_{r+2}, b_{r+2} with interior in $L_{r+3} \cup S$, and its union with a_{r+2}-a_{r+1}-b_{r+1}-b_{r+2} does not give a disoriented hole; so a_{r+2}, b_{r+2} are G^*-adjacent. If the edge $a_{r+2}b_{r+2}$ has head a_{r+2}, the union of the path a_{r+2}-b_{r+2}-b_{r+1}-b_r with the first path of (2) gives a disoriented hole; while if $a_{r+2}b_{r+2}$ has head b_{r+2}, the union of b_{r+2}-a_{r+2}-a_{r+1}-a_r with the second path of (2) gives a disoriented hole. This proves 2.1. ∎

3 Oriented stars

Now we turn to the proof of 1.2. If v is a vertex of a digraph G, $N^+(v)$ denotes the set of out-neighbours of V, and $N^-(v)$ denotes the set of in-neighbours. Let us say a digraph G is *λ-spread* if for every vertex v of G, and for all $A \subseteq N^+(v)$ and $B \subseteq N^-(v)$ with $|A| = |B| = \lambda$, some vertex of A is G^*-adjacent with some vertex of B. If S is an oriented star and $\kappa \ge 0$, choose λ such that every graph with at least λ vertices has either a clique of cardinality more than κ or a stable set of cardinality $|V(S)|$. It follows then that every digraph G with $\omega(G) \le \kappa$ not containing S as an induced subdigraph is λ-spread. (For otherwise, with v, A, B as above, there are stable subsets of A, B each of cardinality $|V(S)|$, and an appropriate subset of $A \cup B$ together with $\{v\}$ induces S, a contradiction.) It is more convenient to replace the hypothesis that G does not contain S as an induced subdigraph with the hypothesis that G is λ-spread. Thus, now we need to prove: for all $\kappa, \lambda \ge 0$, if G is a λ-spread digraph with $\omega(G) \le \kappa$ then the chromatic number of G is bounded by some function of κ, λ. We will prove this by induction on κ, for fixed λ; so we will assume (throughout this section) that κ, λ and τ are fixed integers satisfying

- $\kappa \ge 2$, $\lambda \ge 0$, and $\chi(J) \le \tau$ for every λ-spread digraph J with $\omega(J) < \kappa$.

If G is a graph and $A, B \subseteq V(G)$, we say A is *G-complete with B* if $A \cap B = \emptyset$ and every vertex in A is G-adjacent with every vertex in B. If G is a digraph, we say A is *G-complete to B* and B *is complete from A* if $A \cap B = \emptyset$ and every vertex in A is G-adjacent to every vertex of B. We need the following, which is an easy application of Ramsey's theorem [14] and its bipartite version [2], and we omit its proof:

3.1 *For all k, m there exists $n \ge 0$ with the following property. Let $A_1, \dots, A_n, B_1, \dots, B_n$ be pairwise disjoint subsets of the vertex set of a graph G, each of cardinality m. Then either*

- *there exist $A \subseteq A_1 \cup \dots \cup A_n$ and $B \subseteq B_1 \cup \dots \cup B_n$ with $|A| = |B| = \lambda$ such that no vertex in A has a neighbour in B, or*
- *there exist $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J| = k$ such that $\bigcup_{i \in I} A_i$ is G-complete with $\bigcup_{j \in J} B_j$.*

A *k-clique* means a clique of cardinality k. If X is a clique of a digraph G, a vertex in X is a *source* of X if it is G-adjacent to every other vertex in X, and a *sink* if it is G-adjacent from every other vertex in X. If $k, m \ge 1$ are integers, a vertex v of a digraph G is *(k, m)-rich* if there exist k pairwise disjoint m-cliques $A_1, \dots, A_k \subseteq N^+(v)$, and k pairwise disjoint m-cliques $B_1, \dots, B_k \subseteq N^-(v)$, such that $A_1 \cup \dots \cup A_k$ is G^*-complete with $B_1 \cup \dots \cup B_k$.

3.2 *For all integers $k, m \ge 1$ there exists t with the following property. Let G be a λ-spread digraph such that no vertex of G is (k, m)-rich. Then $V(G)$ can be partitioned into t sets $X_1, \dots, X_t$ such that for $1 \le i \le t$, either no $(m+1)$-clique of $G[X_i]$ has a source or no $(m+1)$-clique of $G[X_i]$ has a sink.*

Proof. Choose n such that 3.1 holds, and let $t = 4nm$. We claim that t satisfies the theorem. For let G be as in the theorem. Let P be the set of vertices of G such that there do not exist n pairwise disjoint m-cliques in $N^+(v)$, and let Q be the set such that there do not exist n pairwise disjoint m-cliques in $N^-(v)$. Suppose first that some vertex v belongs to neither of P, Q. Then there exist n pairwise disjoint m-cliques $A_1, \dots, A_n \subseteq N^+(v)$, and there exist n pairwise disjoint m-cliques $B_1, \dots, B_n \subseteq N^-(v)$. Since G is λ-spread, 3.1 implies that there exist $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J| = k$ such that $\bigcup_{i \in I} A_i$ is G^*-complete with $\bigcup_{j \in J} B_j$, that is, v is (k, m)-rich, a contradiction. This proves that $P \cup Q = V(G)$.

For each vertex $v \in P$, choose a maximal set of pairwise disjoint m-cliques included in $N^+(v)$, and let the union of the members of this set be P_v. Then each $|P_v| < nm$, and has nonempty intersection with every m-clique included in $N^+(v)$. Let H be the digraph with vertex set P and edge set the edges with tail v and head in P_v, for each $v \in P$. Then every vertex of H has outdegree less than nm, and so $\chi(H) \le 2nm$. Let X be a stable set of H^*. It follows that for each $v \in X$, there is no m-clique included in $N^+(v) \cap X$ (since P_v has nonempty intersection with every such clique, and $P_v \cap X = \emptyset$ because X is stable in H^*). Consequently there is no $(m+1)$-clique with a source included in X. But P can be partitioned into $\chi(H) \le 2nm = t/2$ such sets X, and similarly we can partition Q. This proves 3.2. ∎

Choose a function ϕ such that for all $k, m \ge 0$, setting $t = \phi(k,m)$ satisfies 3.2. Let ϕ be fixed for the remainder of this section.

Next we prove 1.2 for acyclic digraphs (a digraph is *acyclic* if it has no directed cycle).

3.3 *There exists c_0 such that $\chi(G) \le c_0$ for every acyclic λ-spread digraph G with $\omega(G) \le \kappa$.*

Proof. Let $t = \phi(1, \kappa - 1)$, and let $c_0 = t\tau$. We claim that c_0 satisfies the theorem.

Let G be a λ-spread acyclic digraph with $\omega(G) \le \kappa$. Now no vertex of G is $(1, \kappa-1)$-rich, because then G would have a clique of cardinality $2\kappa - 1 > \kappa$. By 3.2, $V(G)$ can be partitioned into t sets $X_1, \dots, X_t$, such that for $1 \le i \le t$, either no κ-clique of $G[X_i]$ has a source or no κ-clique of $G[X_i]$ has a sink. But every κ-clique has both a source and a sink, since G is acyclic, and so $\omega(G[X_i]) < \kappa$, and consequently $\chi(X_i) \le \tau$. Hence $\chi(G) \le t\tau = c_0$. This proves 3.3. ∎

A digraph G is (h,k)-*out-orderable* if there is a partition $X_1, \dots, X_n$ of its vertex set, such that for $1 \le i \le n$, $\chi(X_i) \le h$, and each vertex of X_i has at most $k-1$ out-neighbours in $X_{i+1} \cup \dots \cup X_n$. We define (h,k)-in-orderable similarly. If G is a digraph, we say $X \subseteq V(G)$ is *acyclic* if $G[X]$ is acyclic.

3.4 *If the digraph G is (h,k)-out-orderable, then there is a partition of $V(G)$ into hk acyclic sets.*

Proof. Let $X_1, \dots, X_n$ be as in the definition of (h,k)-out-orderable. Let J be the graph with vertex set $V(G)$ in which u, v are J-adjacent if u is G-adjacent to v and $i \le j$ where $u \in X_i$ and $v \in X_j$. Since $J[X_i]$ is h-colourable for each i, there is a partition $Y_1, \dots, Y_h$ of $V(J)$ such that $X_i \cap Y_j$ is stable for $1 \le i \le n$ and $1 \le j \le h$. But every nonempty induced subgraph of $J[Y_j]$ has a vertex with degree in J less than k (choose a vertex of Y_j in X_i for the smallest i with $X_i \cap Y_j$ nonempty); and so $J[Y_j]$ is $(k-1)$-degenerate and hence k-colourable. Consequently $\chi(J) \le hk$. But for each stable set Y of J, $G[I]$ is acyclic. This proves 3.4. ∎

A digraph G is (h,k)-*robust* if for every nonempty subset $Z \subseteq V(G)$ with $\chi(Z) \le h$, some vertex of Z has at least k out-neighbours in $V(G) \setminus Z$ and at least k in-neighbours in $V(G) \setminus Z$.

3.5 *Let $h, k \ge 0$; then for every digraph G there is a partition of $V(G)$ into three sets P, Q, R such that $G[P]$ is (h,k)-out-orderable, $G[Q]$ is (h,k)-in-orderable and $G[R]$ is (h,k)-robust.*

Proof. We proceed by induction on $|V(G)|$. If G is (h,k)-robust we are done, so we may assume that there is a nonempty subset $Z \subseteq V(G)$ with $\chi(Z) \le h$, such that for each $v \in Z$, either $|N^+(v) \setminus Z| < k$ or $|N^-(v) \setminus Z| < k$. Let X_1 be the set of vertices $v \in Z$ such that $|N^+(v) \setminus X| < k$, and $Y_1 = Z \setminus X_1$. From the inductive hypothesis there is a partition P, Q, R of $V(G) \setminus Z$ such that $G[P]$ is (h,k)-out-orderable, $G[Q]$ is (h,k)-in-orderable and $G[R]$ is (h,k)-robust. Let $X_2, \dots, X_n$ be a partition of

P such that for $2 \le i \le n$, $\chi(X_i) \le h$, and each vertex of X_i has at most $k-1$ out-neighbours in $X_{i+1} \cup \cdots \cup X_n$. Then the sequence $X_1, \ldots, X_n$ shows that $G[P \cup X_1]$ is (h,k)-out-orderable. Similarly $G[Q \cup Y_1]$ is (h,k)-in-orderable, and so the partition $P \cup X_1, Q \cup Y_1, R$ satisfies the theorem. This proves 3.5. ∎

We recall that κ, λ and τ are fixed integers satisfying $\kappa \ge 2$, $\lambda \ge 0$, and $\chi(J) \le \tau$ for every λ-spread digraph J with $\omega(J) < \kappa$. Let us define $\Lambda = 2\lambda^2 + \lambda$ (throughout the remainder of this section).

3.6 *Let G be a λ-spread digraph and let $X \subseteq V(G)$ be nonempty. If every vertex in X has at least Λ out-neighbours in X and at least Λ in-neighbours in X, then G is not $(|X|\tau, |X| + \Lambda)$-robust.*

Proof. We claim first:

(1) *For each vertex v of G, if $A \subseteq N^+(v)$ and $B \subseteq N^-(v)$ with $|A| = \lambda$, then some vertex of A is G^*-adjacent with at least $|B|/\lambda - 1$ members of B.*

For there are fewer than λ members of B that have no G^*-neighbour in A, since G is λ-spread. So all the others have at least one G^*-neighbour in A; and so some vertex in A is G^*-adjacent with at least $(|B| - \lambda)/\lambda$ of them. This proves (1).

Now let $X \subseteq V(G)$ be nonempty, such that every vertex in X has at least Λ out-neighbours in X and at least Λ in-neighbours in X. Let P be the set of vertices not in X with at least 2λ G^*-neighbours in X.

(2) *For each $u \in X$, u is G^*-adjacent with fewer than 2λ vertices in $V(G) \setminus (P \cup X)$.*

For suppose not; then from the symmetry we may assume that there is a set A of in-neighbours of u in $V(G) \setminus (P \cup X)$, with $|A| = \lambda$. But u has at least Λ out-neighbours in X; and so by (1), some vertex in $V(G) \setminus (P \cup X)$ has at least 2λ neighbours in X, and therefore belongs to P, a contradiction. This proves (2).

Suppose that there exists $v \in P$ with at least $|X| + \Lambda$ out-neighbours in $V(G) \setminus P$ and at least $|X| + \Lambda$ in-neighbours in $V(G) \setminus P$. Since v has at least 2λ G^*-neighbours in X, from the symmetry we may assume that v has at least λ out-neighbours in X. Let Y be the set of vertices in $V(G) \setminus (P \cup X)$ that are in-neighbours of v. Then $|Y| \ge \Lambda$. Since v has at least λ out-neighbours in X, (1) implies that one of these out-neighbours, say u, is G^*-adjacent with at least $|Y|/\lambda - 1 \ge 2\lambda$ vertices in Y, contrary to (2). Thus there is no such v. But $\chi(P) \le |X|\tau$ since every vertex in P has a neighbour in X; and so G is not $(|X|\tau, |X| + \Lambda)$-robust. This proves 3.6. ∎

If G is a digraph and u, v, w are vertices, pairwise G^*-adjacent, such that one of them is G-adjacent from the other two, we call $\{u, v, w\}$ a *transitive triangle*. Next we need:

3.7 *There exists k_0 with the following property. Every non-null $(3\Lambda\tau, k_0)$-robust λ-spread digraph has a transitive triangle.*

Proof. By the bipartite version of Ramsey's theorem [2], for all $n \geq 0$ there exists $f(n) \geq n$ such that for every partition of the edges of the complete bipartite graph $K_{f(n),f(n)}$ into two sets, either the first set includes the edge set of a $K_{n,n}$ subgraph, or the second set includes the edges of a $K_{\lambda,\lambda}$-subgraph. Let $k_0 = f(f(f(\Lambda)))$. Suppose that G is a $(3\Lambda\tau, k_0)$-robust λ-spread digraph with no transitive triangle. Since G is $(3\Lambda\tau, k_0)$-robust, every vertex has at least k_0 out-neighbours and k_0 in-neighbours. Let $v \in V(G)$; then since G is λ-spread, there exist $A_1 \subseteq N^+(v)$ and $B_1 \subseteq N^-(v)$ with $|A_1| = |B_1| = f(f(\Lambda))$ such that A_1 is G^*-complete with B_1; and since there is no transitive triangle it follows that every vertex in A_1 is G-adjacent to every vertex in B_1. Choose $a \in A_1$. Since a has at least k_0 in-neighbours, and none of them belong to $A_1 \cup B_1$ (because A_1 is stable since there is no transitive triangle) there is a set C of vertices in $V(G) \setminus (A_1 \cup B_1)$ all G-adjacent to a, with $|C| = k_0$. Since a is G-adjacent to every vertex in B_1, and $|B_1|, |C| \geq f(f(\Lambda))$, and there is no transitive triangle, there exist $B_2 \subseteq B_1$ and $C_1 \subseteq C$ with $|B_2| = \Lambda$ and $|C_1| = f(\Lambda)$ such that every vertex in B_2 is G-adjacent to every vertex in C_1. Choose $b \in B_2$. Since b is G-adjacent to every vertex in C_1 and from every vertex in A_1, and $|A_1|, |C_1| \geq f(\Lambda)$, there exist $A_2 \subseteq A_1$ and $C_2 \subseteq C_1$ with $|A_2|, |C_2| = \Lambda$ such that A_2 is G-complete from C_2. Since $|A_2|, |B_2|, |C_2| = \Lambda$, every vertex in $A_2 \cup B_2 \cup C_2$ has at least Λ out-neighbours and Λ in-neighbours in $A_2 \cup B_2 \cup C_2$, contrary to 3.6. This proves 3.7. ∎

A tournament H is *regular* if all its vertices have the same outdegree, and they all have the same indegree; and it follows that $|V(H)|$ is odd, $|V(H)| = 2m+1$ say, and all vertices have indegree and outdegree m. A tournament H is *cyclic* if it has an odd number of vertices, say $2m+1$, and its vertex set can be ordered as $\{v_1, \ldots, v_{2m+1}\}$ such that for $1 \leq i < j \leq 2m+1$, v_i is H-adjacent to v_j if and only if $j - i \leq m$.

3.8 *Let H be a regular tournament with $2m+1$ vertices, and let $v \in V(H)$; and suppose there is no directed cycle with vertices p-q-r-s-p in order such that p, r are out-neighbours of v and q, s are in-neighbours of v. Then H is cyclic.*

Proof. Let J be the subdigraph of H with vertex set $V(H)$ and edge set all edges between $N^+(v)$ and $N^-(v)$. If J has a directed cycle, take the shortest such directed cycle C; then C is induced and so has length four, a contradiction. Thus J has no directed cycle, and so $V(H) \setminus \{v\}$ can be ordered as $\{v_1, \ldots, v_{2m}\}$ such that for every edge of J, its head is earlier than its tail. We claim that

- for i odd, $v_i \in N^+(v)$ and v_i is H-adjacent from all vertices of $N^+(v)$ except $v_1, v_3, \ldots, v_{i-2}$; and
- for i even, $v_i \in N^-(v)$ and v_i is H-adjacent from all vertices of $N^-(v)$ except $v_2, v_4, \ldots, v_{i-2}$.

We prove this claim by induction on i. Suppose then that it holds for all smaller values of i, and first suppose that $v_i \in N^+(v)$. Then v_i is H-adjacent to v_h for all odd $h < i$ (from the inductive hypothesis applied to v_h), and there are $\lfloor i/2 \rfloor$ such values of h. Moreover, v_i is H-adjacent to all vertices $v_j \in N^-(v)$ with $j > i$, from the property of the ordering. But $|N^-(v)| = m$, and there are exactly $\lceil i/2 \rceil - 1$ values of j with $j < i$ such that $v_j \in N^-(v)$, from the inductive hypothesis, and so v_i has at least $m + 1 - \lceil i/2 \rceil$ outneighbours in $N^-(v)$. Since v_i has outdegree exactly m in H, it must be the case that $\lfloor i/2 \rfloor + m + 1 - \lceil i/2 \rceil \leq m$. So i is odd, and moreover v_i has no further outneighbours; and so v_i is H-adjacent from all vertices of $N^+(v)$ except $v_1, v_3, \ldots, v_{i-2}$ as claimed.

Now suppose that $v_i \in N^-(v)$. Thus v_i is H-adjacent to v_h for all even $h < i$ by the inductive hypothesis, and to every vertex of $N^+(v)$ except for the vertices v_j with j odd and $j < i$, because of the ordering. There are $\lceil i/2\rceil - 1$ outneighbours of the first kind, and $m - \lfloor i/2\rfloor$ of the second kind; and in addition v_i is H-adjacent to v. Consequently $\lceil i/2\rceil - 1 + m - \lfloor i/2\rfloor + 1 \le m$, and so i is even, and v_i is H-adjacent from every vertex of $N^-(v)$ except $v_2, v_4, \ldots, v_{i-2}$. This proves the inductive statement. But then the result follows. This proves 3.8. ∎

3.9 *There exist $k_1, c_1 \ge 0$ with the following property. Let G be a $(4\Lambda\tau, k_1)$-robust λ-spread digraph with $\omega(G) \le \kappa$, such that no $(\lceil\kappa/2\rceil + 1)$-clique of G has a source. Then $\chi(G) \le c_1$.*

Proof. Let k_0 satisfy 3.7, and let $k_1 = \max(k_0, 5\Lambda)$. Choose $n \ge 0$ such that for every partition of the edges of the complete bipartite graph $K_{n,n}$ into two sets, either the first set includes the edge set of a $K_{\Lambda,\Lambda}$ subgraph, or the second set includes the edges of a $K_{\lambda,\lambda}$-subgraph.

Let $m = \lfloor\kappa/2\rfloor$. Choose k such that for every partition of the edges of $K_{k,k}$ into $m^4 + 1$ sets, one of the sets includes all the edges of some $K_{n,n}$ subgraph. Let $c_1 = \phi(k, m)\tau$.

Now let G be as in the theorem. We claim that $\chi(G) \le c_1$. If κ is even then since no clique of G has a vertex of outdegree $\kappa/2$ (because no $(\lceil\kappa/2\rceil + 1)$-clique of G has a source), it follows that $\omega(G) < \kappa$ and so $\chi(G) \le \tau \le c_1$. We may therefore assume that κ is odd, and so $\kappa = 2m + 1$. If $\kappa = 3$, then since no $(\lceil\kappa/2\rceil + 1)$-clique of G has a source, it follows that G has no transitive triangle, contrary to 3.7. Thus $\kappa > 3$ and so $m \ge 2$.

(1) *No vertex of G is (k, m)-rich.*

Because suppose that v say is (k, m)-rich. Let $A_1, \ldots, A_k, B_1, \ldots, B_k$ be cliques as in the definition of (k, m)-rich. For $1 \le i \le k$ let $A_i = \{a^i_1, \ldots, a^i_m\}$ and $B_i = \{b^i_1, \ldots, b^i_m\}$, choosing the numbering such that if A_i is a transitive tournament, then a^i_p is G-adjacent to a^i_q for all $p < q$, and if B_i is transitve then b^i_p is G-adjacent to b^i_q for all $p < q$. For $1 \le i, j \le m$, if there is a directed cycle of length four, with vertices a^i_p-b^j_q-a^i_r-b^j_s-a^i_p in order, we say the pair (i, j) has *type* (p, q, r, s) (choosing some such quadruple arbitrarily if there is more than one), and type 0 otherwise. By the choice of k, we may assume that all the pairs (i, j) for $1 \le i, j \le n$ have the same type. If this type is nonzero, say (p, q, r, s), let X be the set

$$\bigcup_{1\le i\le\Lambda} \{a^i_p, a^i_r\} \cup \bigcup_{1\le j\le\Lambda} \{b^j_q, b^j_s\}.$$

Every vertex in X has at least Λ out-neighbours and Λ in-neighbours in X, and $|X| = 4\Lambda$, contrary to 3.6.

Thus for $1 \le i, j \le n$, (i, j) has type 0. From 3.8, $G[A_i \cup B_j \cup \{v\}]$ is cyclic, and therefore both A_i, B_j are transitive tournaments, and from the choice of numbering, a^i_p is G-adjacent to a^i_q for all $p < q$, and similarly b^j_p is G-adjacent to b^j_q for all $p < q$. Since $G[A_i \cup B_j \cup \{v\}]$ is cyclic, it follows that for $1 \le p, q \le m$, a^i_p is G-adjacent to b^j_q if and only if $q \le p$.

Suppose that for some $p, q \in \{1, \ldots, n\}$, a^p_1 is G-adjacent from a^q_m. Then the subdigraph induced on $\{a^p_1, a^q_m\} \cup B_1$ is an $(m+2)$-clique with a source (namely a^q_m), a contradiction. Now b^1_1 is G-adjacent to each of $a^1_1, \ldots, a^n_1$ and G-adjacent from each of $a^1_m, \ldots, a^n_m$. Consequently since G is λ-spread, from the definition of t there exist $A \subseteq \{a_1^1, \dots, a_1^n\}$ and $C \subseteq \{a_m^1, \dots, a_m^n\}$ with $|A| = |C| = \Lambda$, such that A is G-complete to C. Let $B = \{b_1^1, \dots, b_\Lambda^1\}$; then B is G-complete to A, and C is G-complete to B. Consequently, every vertex in $A \cup B \cup C$ has at least Λ out-neighbours and Λ in-neighbours in this set. But this contradicts 3.6. This proves (1).

By 3.2, $V(G)$ can be partitioned into $\phi(k, m)$ subsets such that for each such subset Y say, either no $(m+1)$-clique of $G[Y]$ has a source, or none has a sink. In either case it follows that $\omega(G[Y]) < \kappa$ and so $\chi(Y) \le \tau$ and hence $\chi(G) \le \phi(k, m)\tau = c_1$. This proves 3.9. ∎

Proof of 1.2. As discussed at the beginning of this section, it suffices to show that for some $c \ge 0$, $\chi(G) \le c$ for every λ-spread digraph G with $\omega(G) \le \kappa$. Let c_0 satisfy 3.3, let c_1, k_1 satisfy 3.9, and let $c = 4\Lambda\tau k_1 c_0 + \phi(1, \lceil\kappa/2\rceil)c_1$.

Now let G be a λ-spread digraph G with $\omega(G) \le \kappa$. By 3.5 and 3.4 there is a subset $R \subseteq V(G)$ such that $G[R]$ is $(4\Lambda\tau, k_1)$-robust and $V(G) \setminus R$ can be partitioned into $4\Lambda\tau k_1$ acyclic sets; and each of the latter induces a c_0-colourable digraph by 3.3. Thus $\chi(G) \le 4\Lambda\tau k_1 c_0 + \chi(R)$, so it remains to bound $\chi(R)$.

No vertex is $(1, \lceil\kappa/2\rceil)$-rich, since that would imply that G contains a $(\kappa + 1)$-clique. By 3.2, $V(G)$ can be partitioned into $\phi(1, \lceil\kappa/2\rceil)$ subsets such that for each such subset Y say, either no $(\lceil\kappa/2\rceil + 1)$-clique of $G[Y]$ has a source or none has a sink. From 3.9 it follows that $\chi(Y \cap R) \le c_1$ for each Y, and so $\chi(R) \le \phi(1, \lceil\kappa/2\rceil)c_1$. Consequently $\chi(G) \le 4\Lambda\tau k_1 c_0 + \phi(1, \lceil\kappa/2\rceil)c_1 = c$. This proves 1.2. ∎

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