Erdős-Hajnal for cap-free graphs

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Abstract

A "cap" in a graph G is an induced subgraph of G that consists of a cycle of length at least four, together with one further vertex that has exactly two neighbours in the cycle, adjacent to each other, and the "house" is the smallest, on five vertices. It is not known whether there exists $\varepsilon > 0$ such that every graph G containing no house has a clique or stable set of cardinality at least $|G|^{\varepsilon}$; this is the smallest open case of the Erdős-Hajnal conjecture and has been the subject of much study.

We prove that there exists $\varepsilon > 0$ such that every graph G with no cap has a clique or stable set of cardinality at least $|G|^{\varepsilon}$.

1 Introduction

Graphs in this paper are finite and simple, and $|G|$ denotes the number of vertices of a graph G. A graph is *H-free* if it has no induced subgraph isomorphic to H. The Erdős-Hajnal conjecture [11, 12] asserts:

1.1 Conjecture: *For every graph H, there exists $\varepsilon > 0$ such that every H-free graph G has a clique or stable set of cardinality at least $|G|^{\varepsilon}$.*

This has not yet been proved when H is the five-vertex path P_5, and that problem motivates the work of this paper. The complement of P_5 is the *house*, the graph consisting of a cycle of length four, together with one extra vertex with two neighbours in the cycle, adjacent. By taking complements, we see that proving 1.1 when H is the house is the same problem as proving it when $H = P_5$. The house is the smallest example of a "cap".

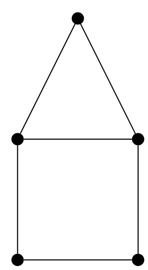

Figure 1: A house.

A *hole* in a graph G is an induced cycle of length at least four. If C is a hole in G, a vertex $v \in V(G) \setminus V(C)$ is said to be a *hat* for C if v has exactly two neighbours $x, y \in V(C)$, and x, y are adjacent. The subgraph induced on $V(C) \cup \{v\}$ is then said to be a *cap* in G; and we say G is *cap-free* if there is no cap in G.

The main result of this paper is:

1.2 *There exists $\varepsilon > 0$ such that for every cap-free graph G, there is a clique or stable set in G of cardinality at least $|G|^{\varepsilon}$.*

Here are some earlier theorems with a similar nature:

- If G contains no hole, there is a clique or stable set in G of cardinality at least $|G|^{1/2}$. (This is immediate because such graphs are perfect.)
- If G contains no house and no hole of odd length, then again there is a clique or stable set in G of cardinality at least $|G|^{1/2}$. (Again, because such graphs are perfect, a consequence of the "strong perfect graph theorem" [6].)
- For each $\ell > 0$, there exists $\varepsilon > 0$ such that if G contains no house and no hole of length at least ℓ, then G has a clique or stable set of cardinality at least $|G|^{\varepsilon}$. (This is a combination of a theorem of Bousquet, Lagoutte, and Thomassé [3] and a theorem of Bonamy, Bousquet and Thomassé [2].)

Since this paper was submitted for publication, we have proved a strengthening of 1.2. In joint work with Alex Scott and Sophie Spirkl [8], we proved:

1.3 *There exists $\varepsilon > 0$ such that for every graph G that contains no five- or six-vertex cap, there is a clique or stable set in G of cardinality at least $|G|^{\varepsilon}$.*

The proof method is quite different from what is used in this paper.

Working in structural graph theory, one always hopes to find a collection of non-crossing decompositions that together break the graph into simpler pieces. That is because the existence of such a collection leads into the well-understood area of tree-decompositions. However, such collections do not often appear in the context of forbidden induced subgraphs. Here we give a weakening of this notion, that is still almost as useful, and does work more often for induced subgraphs.

Cap-free graphs typically admit a certain kind of separation, that we call a "fracture". (This is related to the "amalgam" decomposition of cap-free graphs due to Burlet and Meyniel [4], developed by Conforti, Cornuéjols, Kapoor and Vušković [10].) A fracture is a certain kind of partition of the vertex set of our graph G into three parts (actually four parts, but we merge two of them for this sketch) A, B, C, where A, B are "anticomplete", that is, there are no edges between them. We call A and B the "small" and "big" sides of the fracture. (There is no symmetry between A and B in the full definition of a fracture.) Let S be the union of all small sides of fractures. The graph R obtained from G by deleting S does not admit a fracture with nonempty small side (because such a fracture would extend to one in the whole graph G, and we would have deleted all its small side, including the part in R); so R has a very restricted type. We can base a proof on this, provided we can show that R still contains a substantial part of G: in other words, that S is not too big. And we could show this, if we could prove that:

Every component of S is anticomplete to the big side of some fracture of G.

This is where "non-crossing" would be useful. If it were true that fractures form a set of non-crossing separations, then every component of S would be a component of one small side, and therefore anticomplete to the corresponding big side. This is not true, but we have a substitute: we can show that for any two fractures, if some component of the union of their small sides is not a component of either small side, then the two big sides are equal. It follows from this that every component of S is anticomplete to the big side of some fracture, which is what we needed. A similar idea works in several other situations, and we hope to find further uses for it in the future.

A graph P is *perfect* if chromatic number equals clique number for every induced subgraph of P. We denote the set of nonnegative real numbers by $\mathbb{R}^+$. Let G be a graph and $f : V(G) \to \mathbb{R}^+$; if $X \subseteq V(G)$, we define $f(X) = \sum_{v \in X} f(v)$, and if P is an induced subgraph of G we define $f(P) = f(V(P))$. We say that f is *good on G* if $f(P) \le 1$ for every perfect induced subgraph P of G. Now let $\alpha \ge 1$. We denote by f^{α} the function g on G defined by $g(v) = (f(v))^{\alpha}$ for each $v \in V(G)$. Let us say that G is *α-narrow* if $f^{\alpha}(G) \le 1$ for every good function f on G. Here is a result of Chudnovsky and Safra [7], with a short proof by Chudnovsky and Zwols [9]:

1.4 *If G is α-narrow then G has a clique or stable set of cardinality at least $|G|^{\varepsilon}$, where $\varepsilon = 1/(2\alpha)$.*

Proof. Let P be a perfect induced subgraph of G with as many vertices as possible, and let $p = |P|$. Let $f(v) = 1/p$ for all $v \in V(G)$. Then f is good on G, and so $f^{\alpha}(G) \le 1$, that is, $p^{-\alpha}|G| \le 1$, and so $p \ge |G|^{1/\alpha} = |G|^{2\varepsilon}$. But P is perfect, and so P, and hence G, has a clique or stable set of cardinality at least $p^{1/2} \ge |G|^{\varepsilon}$. This proves 1.4. ∎

In view of 1.4, the following implies 1.2:

1.5 *There exists $\alpha \geq 1$ such that every cap-free graph is α-narrow.*

Let us say a pair (G, f) is *α-critical* if G is a graph and $f : V(G) \to \mathbb{R}^+$ is a good function on G, such that

- every proper induced subgraph of G is α-narrow; and
- $f^\alpha(G) > 1$.

To prove 1.5, choose an appropriately large value of α, and suppose (for a contradiction) that 1.5 is not satisfied, and look at a counterexample G with as few vertices as possible. Hence there is a good function f on G with $f^\alpha(G) > 1$, and so (G, f) is α-critical. Consequently, 1.5 can be reformulated as:

1.6 *There exists $\alpha \geq 1$ such that for every α-critical pair (G, f), there is a cap in G.*

We will prove this at the end of the final section, but let us sketch the proof now. We are proving α-narrowness instead of proving the statement of 1.2 directly, in order to handle homogeneous sets; so vertices will have non-negative weights, but for this sketch the reader could assume that all vertex-weights are one. If there are two disjoint anticomplete sets of vertices, that both have linear total weight, then we win by induction; so we assume there are no two such sets. By a theorem of Rödl, there is a subgraph X containing a linear fraction of the total weight of G, such that either X is sparse (in a weighted sense) or its complement is: and the second is impossible, by a theorem of Bonamy, Bousquet and Thomassé [2]. So the first holds. We look at fractures of X. If A, B, C is such a fracture, then C has very small total weight, and so at least one of A, B has big weight. But by the assumption above, not both A, B have linear total weight, since they are anticomplete; and with some sleight of hand we can arrange that it is always the small side A that has small weight, and therefore that most of the weight of X resides in B.

Let S be the union of all the small sides of fractures of X. By the remarkable fact that we described earlier, every component of S is anticomplete to some big side, and therefore has small weight; and so S itself has small weight (because otherwise we could group its components into two sets both with big weight). That means that deleting S from X gives a graph R that still has big weight. But every fracture in R extends to a fracture in X (this is another useful feature of fractures, and the reason for using "forcers", which we do not explain here), and therefore R has no fracture with nonempty small side. Hence R has a very restricted type, and in particular it is α'-narrow where α' is much less than α; and it follows that G itself is α-narrow, which is what we wanted to show. This completes the sketch.

2 Complete pairs of sets

Cap-free graphs have some convenient structural properties, that we will prove next. Let $A, B \subseteq V(G)$ be disjoint; we say they are *complete* to each other if every vertex in A is adjacent to every vertex in B, and *anticomplete* if there are no edges between A, B. We are concerned in this section with how the remainder of a cap-free graph can attach to a pair of sets of vertices that are complete to each other. A graph is *anticonnected* if its complement is connected; and its *anticomponents* are the complements of the components of its complement. If $X \subseteq V(G)$, we say that X is *connected* if $G[X]$ is connected, and *anticonnected* if $G[X]$ is anticonnected. If $C \subseteq V(G)$, a vertex $v \in V(G) \setminus C$ is *mixed* on C if v is neither complete not anticomplete to C. We begin with:

2.1 *Let G be a cap-free graph. Let C, D be disjoint anticonnected subsets of $V(G)$, complete to each other. Then no vertex of $V(G) \setminus (C \cup D)$ is both mixed on C and mixed on D.*

Proof. Suppose that $v \in V(G) \setminus (C \cup D)$ is both mixed on C and mixed on D. Since C is anticonnected, there exist nonadjacent $c_1, c_2 \in C$ such that v is adjacent to c_1 and not to c_2; and choose $d_1, d_2 \in D$ similarly. Then the subgraph induced on $\{c_1, c_2, d_1, d_2, v\}$ is a house, contradicting that G is cap-free. This proves 2.1. ∎

2.2 *Let G be a cap-free graph. Let C, D be disjoint subsets of $V(G)$, complete to each other, such that C is connected. Let P be a connected subgraph of $G \setminus (C \cup D)$, such that some vertex of P has a neighbour in C, and no vertex of P is complete to C. For every $v \in V(P)$, there exists $u \in V(P)$, mixed on C, such that every vertex in D adjacent to v is also adjacent to u.*

Proof. Suppose the claim is false, and choose a counterexample with P minimal. Choose $v \in V(P)$ such that no vertex of P has a neighbour in C and is adjacent to all neighbours of v in D. Choose $u \in V(P)$ mixed on C. By the minimality of P, P is an induced path between u, v, and u is the only vertex of P with a neighbour in C. Choose $d \in D$ adjacent to v and not to u. By the minimality of P, v is the only vertex of P adjacent to d. Since C is connected, there exist adjacent $c_1, c_2 \in C$ such that u is adjacent to c_1 and not to c_2. But then the subgraph induced on $V(P) \cup \{c_1, c_2, d\}$ is a cap, a contradiction. This proves 2.2. ∎

2.3 *Let G be a cap-free graph. Let C, D be disjoint subsets of $V(G)$, complete to each other, such that C is connected and D is anticonnected. Let P be a connected subgraph of $G \setminus (C \cup D)$, such that some vertex of P has a neighbour in C, and no vertex of P is complete to C. If some vertex of P has a neighbour in D, then some vertex of P is mixed on C and complete to D.*

Proof. If some vertex of P has a neighbour in D, then by 2.2, some vertex $v \in C$ is mixed on C and has a neighbour in D, and therefore by 2.1, v is complete to D. This proves 2.3. ∎

2.4 *Let G be a cap-free graph. Let C, D be disjoint nonempty subsets of $V(G)$, complete to each other, such that C is connected and anticonnected, and D is anticonnected. Let P be a connected subgraph of G with $V(P) \cap (C \cup D) = \emptyset$, such that some vertex of P has a neighbour in C, and no vertex of P is complete to C. Then no vertex of P is mixed on D.*

Proof. Suppose not, and choose C, D, P not satisfying the theorem, with P minimal. Choose $u \in V(P)$ with a neighbour in C, and therefore mixed on C; and choose $v \in V(P)$ mixed on D. From the minimality of P, it follows that P is an induced path with ends u, v, and u is the only vertex of P with a neighbour in C, and no vertex of P different from v is mixed on D. By 2.3 u is complete to D, and in particular $u \neq v$. Let u' be the neighbour of u in P. It follows that $C \cup \{u\}$ is connected and anticonnected, and u' is mixed on it; and this contradicts the minimality of P. This proves 2.4. ∎

2.5 *Let G be a cap-free graph. Let C, D be disjoint nonempty subsets of $V(G)$, complete to each other, such that C is connected and anticonnected, and D is connected and anticonnected. Then there do not exist connected subgraphs P, Q of $G \setminus (C \cup D)$, with $V(P \cap Q) \neq \emptyset$, such that*

- *some vertex of P has a neighbour in C, and no vertex of P is complete to C;*
- *some vertex of Q has a neighbour in D, and no vertex of Q is complete to D.*

Proof. Suppose that such P, Q exist, and choose C, D, P, Q with $|V(P)| + |V(Q)|$ minimal. Choose $w \in V(P \cap Q)$, and choose $p \in V(P)$ with a neighbour in C, and $q \in V(Q)$ with a neighbour in D. From the minimality of $|V(P)| + |V(Q)|$, it follows that P is an induced path with ends p, w, and no vertex of P different from p has a neighbour in C; and similarly for Q; and $V(P \cap Q) = \{w\}$. Suppose that p is complete to D. Hence $p \notin V(Q)$, and so $p \neq w$. Then $C \cup \{p\}$ is connected and anticonnected, and complete to D, and the two paths $P \setminus p$ and Q contradict the minimality of $|V(P)| + |V(Q)|$. Thus p is not complete to D. Since no other vertex of P has a neighbour in C, it follows from 2.2 that no vertex of P has a neighbour in D. Similarly no vertex of Q has a neighbour in C. In particular, no vertex of $P \cup Q$ is complete to C, contrary to 2.4. This proves 2.5. ∎

3 Decomposing cap-free graphs

If $v \in V(G)$, we denote by $N(v) = N_G(v)$ the set of all neighbours of v in G. If $N \subseteq V(G)$, we denote by $G[N]$ the induced subgraph with vertex set N. A *weighted graph* is a pair (G, w), where G is a graph and $w : V(G) \to \mathbb{R}^+$ is a function, such that $w(G) = 1$. Let $\varepsilon > 0$. We say a weighted graph (G, w) is *ε-coherent* if

- for every $v \in V(G)$, $w(v) < \varepsilon$;
- for every $v \in V(G)$, $w(N_G(v)) < \varepsilon$; and
- if $A, B \subseteq V(G)$ are disjoint and anticomplete then $\min(w(A), w(B)) < \varepsilon$.

First we need:

3.1 *Let (G, w) be an ε-coherent weighted graph. If $X \subseteq V(G)$ with $w(X) \geq 3\varepsilon$, there is a component Y of $G[X]$ with $w(Y) > w(X) - \varepsilon$.*

Proof. Let Z be a union of components of $G[X]$, minimal such that $w(Z) \geq \varepsilon$. Since $X \setminus Z$ is anticomplete to Z, it follows that $w(X \setminus Z) < \varepsilon$, and so $w(Z) > w(X) - \varepsilon$. Choose a component Y of $G \setminus X$ with $Y \subseteq Z$. From the minimality of Z, $w(Z \setminus Y) < \varepsilon$, and so $w(Y) \geq (w(X) - \varepsilon) - \varepsilon \geq \varepsilon$, and therefore $Z = Y$ from the minimality of Z. But then $w(Y) = w(Z) > w(X) - \varepsilon$. This proves 3.1. ∎

The component Y of 3.1 satisfies $w(Y) > w(X) - \varepsilon \geq 2\varepsilon$, and since the remainder of $G[X]$ is anticomplete to Y and therefore has weight less than ε, it follows that Y is unique. We call Y the *big component* of $G[X]$.

If $A \subseteq V(G)$, each vertex in $V(G) \setminus A$ with a neighbour in A is called an *attachment* of A. Let $\varepsilon > 0$, with $5\varepsilon \leq 1$, and let (G, w) be an ε-coherent weighted graph. Let C, D be disjoint subsets of $V(G)$, such that:

- $|C| \geq 2$, and $G[C]$ is connected and anticonnected;
- $D \neq \emptyset$, and D is the set of all vertices in $V(G) \setminus C$ that are complete to C; and
- C contains no attachment of the big component of $G \setminus (C \cup D)$.

(Note that since there is a vertex in D complete to C, it follows that $w(C) \leq \varepsilon$, and similarly $w(D) \leq \varepsilon$. Since $5\varepsilon \leq 1$, there is a big component of $G \setminus (C \cup D)$.) In these circumstances we call (C, D) a *split* of (G, w). As we shall see, splits are a useful kind of decomposition in cap-free graphs.

Let us say a *forcer* is a graph F with eight vertices $v_1, \ldots, v_8$, where v_1-v_2-v_3-v_4 and v_5-v_6-v_7-v_8 are induced paths of F, and $\{v_1, \ldots, v_4\}$ is complete to $\{v_5, \ldots, v_8\}$. We call these two paths the *constituent paths* of the forcer. A *forcer in G* means an induced subgraph of G that is a forcer, and G is *forcer-free* if there is no forcer in G . Now we prove the main result of this section. It is a strengthening of the "amalgam" decomposition of cap-free graphs due to Conforti, Cornuéjols, Kapoor and Vušković [10].

3.2 *Let $\varepsilon > 0$, with $5\varepsilon \leq 1$, and let (G, w) be a ε-coherent weighted graph, where G is cap-free. Let F be a forcer in G. Then there is a split (C, D) of G such that $G[C], G[D]$ both contain a constituent path of F.*

Proof. Let F be a forcer, and let P_1, P_2 be the constituent paths of F. Consequently there are disjoint subsets X_1, X_2 of $V(G)$, such that

- $V(P_1) \subseteq X_1$, and X_1 is connected and anticonnected;
- $V(P_2) \subseteq X_2$, and X_2 is connected and anticonnected; and
- X_1, X_2 are complete to one another.

Choose such (X_1, X_2) maximal in the sense that there is no choice of (X_1', X_2') satisfying the same conditions, with $X_i \subseteq X_i'$ for $i = 1, 2$ and $|X_1' \cup X_2'| > |X_1 \cup X_2|$. We call this property the *maximality* of (X_1, X_2). Let X_3 be the set of all vertices in $V(G) \setminus (X_1 \cup X_2)$ that are complete to $X_1 \cup X_2$, and let $R = V(G) \setminus (X_1 \cup X_2 \cup X_3)$. For $i = 1, 2$, let R_i be the set of vertices in R that are complete to X_i.

(1) *R_1 is anticomplete to X_2, and R_2 is anticomplete to X_1, and so $R_1 \cap R_2 = \emptyset$.*

Suppose that $v \in R_1$ has a neighbour in X_2, say. Since $v \notin X_3$, v is mixed on X_2. But then $X_2' = X_2 \cup \{v\}$ is connected and anticonnected, and the pair (X_1, X_2') violates the maximality of (X_1, X_2). This proves (1).

For $i = 1, 2$, let S_i be the union of all components of $G[R \setminus (R_1 \cup R_2)]$ that have an attachment in X_i. Let $S_3 = R \setminus (R_1 \cup R_2 \cup S_1 \cup S_3)$.

(2) $S_1 \cap S_2 = \emptyset$. *Moreover, S_1 is anticomplete to $X_2 \cup R_2 \cup S_2$, and S_2 is anticomplete to $X_1 \cup R_1 \cup S_1$.*

By 2.4, $S_1 \cap S_2 = \emptyset$. By 2.3, S_1 is anticomplete to R_2, and S_2 is anticomplete to R_1. This proves (2).

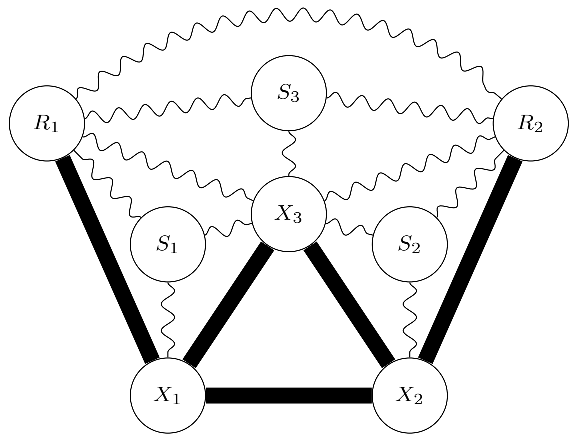

Figure 2: Thick lines indicate complete pairs, and wiggly lines indicate possible edges.

Thus, in summary, the sets $X_1, X_2, X_3, R_1, R_2, S_1, S_2, S_3$ are pairwise disjoint and have union $V(G)$. The pairs

$$(X_1, X_2), (X_1, X_3), (X_2, X_3), (R_1, X_1), (R_2, X_2)$$

are complete to each other; the pairs

$$(R_1, X_2), (R_2, X_1), (S_1, X_2), (S_2, X_1), (S_3, X_1), (S_3, X_2), (S_1, R_2), (S_2, R_1), (S_1, S_2), (S_1, S_3), (S_2, S_3)$$

are anticomplete; and there may be edges between the pairs not listed. Every component of $G[S_i]$ has an attachment in X_i for $i = 1, 2$.

Define $T = X_1 \cup X_2 \cup X_3 \cup R_1 \cup R_2$. Choose $x_1 \in X_1$ and $x_2 \in X_2$. Then every vertex in T is adjacent to one of x_1, x_2, and so $w(T) \leq 2\varepsilon$. Hence $G \setminus T$ has a big component Y, and since the sets S_1, S_2, S_3 are pairwise anticomplete, we may assume that Y is disjoint from S_1, by exchanging X_1, X_2 if necessary. But then $(X_1, X_2 \cup X_3 \cup R_1)$ is a split of G satisfying the theorem. This proves 3.2. ∎

Let us say a split (C, D) of G is *optimal* if there is no split (C', D') with $C \subseteq C'$ and $C' \neq C$. Let (C, D) be an optimal split. Let A be the union of all components of $G \setminus (C \cup D)$ that have an attachment in C; and let B be the union of all other components of $G \setminus (C \cup D)$ (including the big component). Let us call (A, C, D, B) a *fracture* of G. (Note that there are no edges between B and $A \cup C$ but there may well be edges between A and D. Also, $B \neq \emptyset$, since it contains the big component of $G \setminus (C \cup D)$, but A might be empty.) From 3.2 we have immediately:

3.3 *Let $\varepsilon > 0$, with $5\varepsilon \leq 1$, and let (G, w) be an ε-coherent weighted graph, where G is cap-free. Let F be a forcer in G. Then there is a fracture (A, C, D, B) of G such that $G[C]$ contains a constituent path of F.*

We need some observations about fractures.

3.4 *Let $\varepsilon > 0$, with $5\varepsilon \leq 1$, and let (G, w) be an ε-coherent weighted graph, where G is cap-free. Let (A, C, D, B) be a fracture of G.*

- *For each $a \in A$, there is an attachment of the big component of $G \setminus (C \cup D)$ that is nonadjacent to a.*

- *For each $a \in A$, and every anticomponent X of $G[D]$, a is not mixed on X.*

Proof. Let Y be the big component of $G \setminus (C \cup D)$; thus $Y \subseteq B$, and all its attachments belong to D. Suppose that $a \in A$, and a is adjacent to every vertex of D that has a neighbour in Y. Let P be the component of $G[A]$ that contains a; then some attachment of P belongs to C. By 2.2, we may choose $v \in P$ mixed on C, such that every vertex in D adjacent to a is also adjacent to v. Let D' be the set of all neighbours of v in D. Then $(C \cup \{v\}, D')$ is a split (because D' contains all attachments of Y), contradicting that (C, D) is optimal. This proves the first assertion.

Now suppose that $a \in A$ is mixed on an anticomponent X of $G[D]$. Let P be the component of $G[A]$ that contains a. Choose $v \in P$ mixed on C; then P contradicts 2.4 applied to the connected anticonnected set C and the anticonnected set X. This proves 3.4. ∎

4 Multiple fractures

A fracture (A, C, D, B) of G is a kind of separation of G, because deleting $C \cup D$ disconnects A from B. (But A might be empty.) Also the order of this separation is small, since $w(C \cup D) \leq 2\varepsilon$ in the usual notation. It would be nice if these separations did not "cross", so that they give us a tree-decomposition of G, but that is not true. Nevertheless, something like that is true, as we see in this section.

4.1 *Let $\varepsilon > 0$, with $6\varepsilon \leq 1$, and let (G, w) be an ε-coherent weighted graph, where G is cap-free. Let (A, C, D, B) and (A', C', D', B') be fractures in G. Then either*

- *every connected subgraph of $G[A \cup A']$ is contained in one of A, A'; or*

- *the big component of $G \setminus (C \cup D)$ equals the big component of $G \setminus (C' \cup D')$.*

Proof. We suggest that, to follow this argument, the reader imagine a 4×4 matrix with rows labelled A, C, D, B and columns A', C', D', B'. We remind the reader that C is complete to D, and $A \cup C$ is anticomplete to B, and the same for (A', C', D', B').

	A'	C'	D'	B'
A				
C				
D				
B				

Figure 3: Two fractures.

Let Y be the big component of $G \setminus (C \cup D)$, and define Y' similarly. Since $w(Y), w(Y') > 1 - 3\varepsilon \geq 1/2$, it follows that $Y \cap Y' \neq \emptyset$, and since $Y \subseteq B$ and $Y' \subseteq B'$, we deduce that $Y \cap Y' \cap B \cap B' \neq \emptyset$.

(1) *If $C \cap B' \neq \emptyset$ then $D \cap (A' \cup C') = \emptyset$, and if $B \cap C' \neq \emptyset$ then $(A \cup C) \cap D' = \emptyset$.*

Let $u \in C \cap B'$. If $v \in D \cap (A' \cup C')$, then v is adjacent to u (because C is complete to D), and yet v is nonadjacent to u (because $A' \cup C', B'$ are anticomplete), a contradiction. This proves the first statement, and the second follows by symmetry.

(2) *We may assume that $A \cap (C' \cup D') \neq \emptyset$, and $(C \cup D) \cap A' \neq \emptyset$, and at least one of $B \cap D', D \cap B'$ is nonempty.*

If $A \cap (C' \cup D') = \emptyset$, then the first outcome of the theorem holds, and similarly if $(C \cup D) \cap A' = \emptyset$. If $B \cap D', D \cap B'$ are both empty, then $Y, Y' \subseteq B \cap B'$, and so $Y = Y'$ and the second outcome of the theorem holds. This proves (2).

From the third assertion of (2) and the symmetry between the two fractures, we may assume that $B \cap D' \neq \emptyset$. By (1), $(A \cup C) \cap C' = \emptyset$. From (2), $A \cap D' \neq \emptyset$; so by (1) $B \cap C' = \emptyset$. Hence $D \cap C' \neq \emptyset$, because $C' \neq \emptyset$. Every vertex in $C \cap A'$ is complete to $D \cap C'$ and hence to C'; but no vertex in A' is complete to C' from the definition of a fracture, and so $C \cap A' = \emptyset$. By (2), $D \cap A' \neq \emptyset$. By (1), $C \cap B' = \emptyset$, and so $C \cap D' \neq \emptyset$.

	A'	C'	D'	B'
A	?	$\emptyset$	$\bullet$	?
C	$\emptyset$	$\emptyset$	$\bullet$	$\emptyset$
D	$\bullet$	$\bullet$	?	?
B	?	$\emptyset$	$\bullet$	$\bullet$

Figure 4: A solid dot means a nonempty set, and ? means we don't know.

Since $C = C \cap D'$ is anticonnected, and every vertex in A has a nonneighbour in C, and every vertex in $B \cap D'$ has a nonneighbour in $A \cap D'$, it follows that $(A \cup B \cup C) \cap D'$ is anticonnected. But each vertex in $D \cap A'$ has a neighbour in $(A \cup B \cup C) \cap D'$ (namely, in $C \cap D'$), and by 3.4, it follows that $D \cap A'$ is complete to $(A \cup B \cup C) \cap D'$. Similarly, since $D \cap (A' \cup B' \cup C')$ is anticonnected, it follows that $A \cap D'$ is complete to $D \cap (A' \cup B' \cup C')$. (Thus we almost have symmetry between (A, C, D, B) and (A', C', D', B'); but not quite, because we do not know that $D \cap B' \neq \emptyset$.)

Let Q be the set of vertices in $D \cap D'$ that are not complete to $D \cap A'$, and let Q' be the set of vertices in $D \cap D'$ that are not complete to $A \cap D'$. Let $R = (D \cap D') \setminus (Q \cup Q')$.

(3) *$Q \cap Q' = \emptyset$, and Q, Q', R are pairwise complete.*

Since there is a vertex in $A \cap D'$ and it is complete to $D \cap A'$, 3.4 implies that $A \cap D'$ is complete to Q; and so $Q \cap Q' = \emptyset$. If $u \in Q$ and $v \in Q' \cup R$, there is a vertex in $D \cap A'$ adjacent to v and not to u; so 3.4 implies that u, v are adjacent. Thus Q is complete to $Q' \cup R$, and similarly Q' is complete to R. This proves (3).

(4) *$A' \cap B$ is anticomplete to $Q' \cup (B \cap D')$, and $B' \cap A$ is anticomplete to $Q \cup (D \cap B')$.*

Each vertex in $A' \cap B$ is anticomplete to $A \cap D'$, and so by 3.4, also anticomplete to $Q' \cup (B \cap D')$. Similarly $B' \cap A$ is anticomplete to $Q \cup (D \cap B')$. This proves (4).

Choose $v \in D \cap A'$; then by 3.4, there is an attachment q of Y' nonadjacent to v. Since v is adjacent to all vertices of $D' \setminus Q$, it follows that $q \in Q$. Similarly there is an attachment q' of Y with $q' \in Q'$. Let $X = ((A \cup C) \cap D') \cup Q'$, and $X' = (D \cap (A' \cup C')) \cup Q$. Then X, X' are disjoint, and complete to each other, and each of them is both connected and anticonnected. Now some vertex of Y is adjacent to q', and so has a neighbour in X; and no vertex of Y is complete to X (because $Y \subseteq B \cap (B' \cup D')$, since there are no edges between $B \cap A'$ and $B \cap D'$). Similarly some vertex of Y' has a neighbour in X', and no vertex of Y' is complete to X'. Since $Y \cap Y'$ is non-null, this contradicts 2.5. ∎

4.2 *Let $\varepsilon > 0$, with $6\varepsilon \leq 1$, and let (G, w) be an ε-coherent weighted graph, where G is cap-free. Let $\mathcal{F}$ be the set of all fractures of G, and let $\mathcal{A}$ be the union of all the sets A for $(A, C, D, B) \in \mathcal{F}$. Then $w(\mathcal{A}) < 3\varepsilon$.*

Proof. Let Z be the vertex set of a component of $G[\mathcal{A}]$. For each $(A, C, D, B) \in \mathcal{F}$, we call each component of $G[A]$ a *piece*; let H be the set of all maximal pieces (taken over all $(A, C, D, B) \in \mathcal{F}$). More exactly, a piece P is maximal if there is no piece P' (possibly arising from a different choice of (A, C, D, B)) such that P is an induced subgraph of P' and $P \neq P'$.

Thus Z can be expressed as the union of vertex sets of maximal pieces. For each maximal piece X, let $(A, C, D, B) \in \mathcal{F}$ such that X is a component of $G[A]$, and let Y be the big component of $G \setminus (C \cup D)$; we call Y the *fulcrum* of X. (There may be more than one choice of $(A, C, D, B) \in \mathcal{F}$ for a given set X, and correspondingly more than one choice of fulcrum: choose one, arbitrarily).

We observe:

(1) *If X, X' are maximal pieces such that either $V(X \cap X') \neq \emptyset$, or X is not anticomplete to X', then X, X' have the same fulcrum.*

Suppose not. Let X be a component of $G[A]$ where $(A, C, D, B) \in \mathcal{F}$, and define (A', C', D', B') similarly. By 4.1, it follows that every connected subgraph of $G[A \cup A']$ is a subgraph of one of $G[A], G[A']$, and in particular the connected subgraph induced on $V(X) \cup V(X')$ is a subgraph of one of $G[A], G[A']$, say $G[A]$. But X is a component of $G[A]$, so $V(X) = V(X \cup X')$, contradicting that X' is a maximal piece. This proves (1).

Choose a connected subgraph H of $G[Z]$, maximal such that $V(H)$ is the union of maximal pieces all with the same fulcrum Y. Suppose that $V(H) \neq Z$. Since $G[Z]$ is connected, there is a vertex $v_1 \in Z \setminus V(H)$ with a neighbour $v_2 \in V(H)$. Choose a maximal piece X_1 containing v_1, and a maximal piece X_2 containing v_2 with fulcrum Y. By (1), X_1 has fulcrum Y, contrary to the maximality of H. Thus $V(H) = Z$, and so Y is anticomplete to Z. Since $w(Y) \geq \varepsilon$, it follows that $w(Z) < \varepsilon$. Since this holds for each component of $G[\mathcal{A}]$, 3.1 implies that $w(\mathcal{A}) < 3\varepsilon$. This proves 4.2. ∎

Let us say $X \subseteq V(G)$ is a *homogeneous set* of G if for every vertex $v \in V(G) \setminus X$, either v is complete or anticomplete to X. Let G be a graph; we say that G is *guarded* if for every forcer F in G, there is a homogeneous set X of G with $X \neq V(G)$ such that $G[X]$ contains a constituent path of F.

4.3 *Let $\varepsilon > 0$, with $6\varepsilon \leq 1$, and let (G, w) be an ε-coherent weighted graph, where G is cap-free. Then there exists $Z \subseteq V(G)$ with $|Z| > 1$, such that $G[Z]$ is connected and guarded, and $w(Z) > 1 - 4\varepsilon$.*

Proof. Define $\mathcal{F}, \mathcal{A}$ as in 4.2, and let $W = V(G) \setminus \mathcal{A}$. By 4.2, $w(W) > 1 - 3\varepsilon \geq 3\varepsilon$. By 3.1, $G[W]$ has a big component, with vertex set Z say, where $w(Z) \geq 1 - 4\varepsilon$. Hence $|Z| > 1$, since $w(v) \leq \varepsilon < 1 - 4\varepsilon$ for each vertex v. Let F be a forcer in $G[Z]$. Then by 3.3, there is a fracture (A, C, D, B) of G such that $|V(F) \cap C| \geq 4$. Let $X = C \cap Z$. Since C is a homogeneous set of $G \setminus A$, it follows that X is a homogeneous set of $G[Z]$, and it contains a constituent path of F. This proves 4.3. ∎

5 α-critical pairs

In this section we explore the properties of α-critical pairs, and combine these results with 4.3 to prove 1.6.

5.1 *Let $\alpha \geq 2$, and let (G, f) be α-critical. Then $f(w) < 1 - 4^{-1/\alpha}$ for each $w \in V(G)$.*

Proof. Let $w \in V(G)$, and let $c = f(w)$. Let $N = N_G(w)$, and let $M = V(G) \setminus (N \cup \{w\})$. Since (G, f) is α-critical, it follows that $G[N]$ is α-narrow, and so is $G[M]$. Let p be the maximum of $f(P)$ over all perfect induced subgraphs of $G[N]$, and let q be the maximum of $f(Q)$ over all perfect induced subgraphs Q of $G[M]$. We claim that $f^\alpha(N) \leq p^\alpha$. If $f(v) = 0$ for every $v \in N$ then the statement is true, so we may assume that $f(v) > 0$ for some $v \in N$, and hence $p > 0$. So the function $f(v)/p$ $(v \in N)$ is a good function on $G[N]$, and since $G[N]$ is α-narrow, we deduce that $f^\alpha(N) \leq p^\alpha$. Similarly $f^\alpha(M) \leq q^\alpha$.

But if P is a perfect induced subgraph of $G[N]$ then $G[V(P) \cup \{w\}]$ is perfect, and therefore $f(V(P) \cup \{w\}) \leq 1$; and so $p \leq 1 - c$, and similarly $q \leq 1 - c$. Thus

$$1 < f^\alpha(G) = f^\alpha(N) + f^\alpha(M) + f^\alpha(w) \leq p^\alpha + q^\alpha + f^\alpha(w) \leq 2(1-c)^\alpha + c^\alpha.$$

Now for $0 \leq x \leq 1$, the function $g(x) = 2(1-x)^\alpha + x^\alpha$ has the value 1 when $x = 1$, and its value increases with x for $2/3 \leq x \leq 1$, since $\alpha \geq 2$ (as can be seen by taking the derivative). Thus $g(x) \leq 1$ for $2/3 \leq x \leq 1$. Since $g(c) > 1$, it follows that $c < 2/3$, and so $c^\alpha \leq 1/2$; and consequently $2(1-c)^\alpha > 1/2$, that is, $c < 1 - 4^{-1/\alpha}$. This proves 5.1. ∎

5.2 *Let $\alpha \geq 1$, and let (G, f) be α-critical. Let $A, B \subseteq V(G)$ be disjoint and either complete or anticomplete. Then not both $f^\alpha(A), f^\alpha(B) > 2^{-\alpha}$.*

Proof. Let P be a perfect induced subgraph of $G[A]$ with $f(P)$ maximum, and choose Q in $G[B]$ similarly. Since $P \cup Q$ is a perfect induced subgraph of G, it follows that $f(P) + f(Q) \leq 1$, and from the symmetry we may assume that $f(P) \leq 1/2$. We may also assume that $f(P) > 0$, $f(P) = p$ say, and so $f(v)/p$ $(v \in A)$ is a good function on $G[A]$; and we may assume that $Y \neq \emptyset$, and so $G[A]$ is α-narrow, and consequently $f^\alpha(A) \leq p^\alpha \leq 2^{-\alpha}$. This proves 5.2. ∎

Next we need the following consequence of a theorem of Rödl [14]:

5.3 *For all $\varepsilon > 0$ and every graph H, there exists $\delta > 0$ such that for every H-free graph G, there is a subset $X \subseteq V(G)$ with $|X| \geq \delta|V(G)|$ such that one of $G[X], \overline{G}[X]$ has maximum degree less than $\varepsilon|X|$.*

We also need the following theorem of Bousquet, Lagoutte and Thomassé [3]:

5.4 *For every path H, there exists $\varepsilon > 0$ such that for every H-free graph G with $|G| > 1$, either some vertex of G has degree at least $\varepsilon|G|$, or there are disjoint anticomplete subsets $A, B \subseteq V(G)$ with $|A|, |B| \geq \varepsilon|G|$.*

We recall that the *house* is the complement of P_5. Let us say G is *house-free* if G contains no house.

5.5 *For all $\varepsilon > 0$ there exists $\delta > 0$ such that, if G is house-free and $|G| > 1$, then either*

- *there are disjoint sets $A, B \subseteq V(G)$, complete to each other, with $|A|, |B| \geq \varepsilon\delta|G|$, or*
- *there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that $G[X]$ has maximum degree less than $\varepsilon|X|$.*

Proof. Choose $\varepsilon' > 0$ such that 5.4 holds with H, ε replaced by P_5, ε' respectively. Now let $\varepsilon > 0$; we must show that there exists δ as in the theorem. Thus we may assume that $\varepsilon \leq \varepsilon'$, by reducing ε if necessary. Choose δ as in 5.3. Now let G be a house-free graph with $|G| > 1$. The complement $\overline{G}$ of G is P_5-free, and so by 5.3, there is a subset $X \subseteq V(G)$ with $|X| \geq \delta|V(G)|$ such that one of $G[X], \overline{G}[X]$ has maximum degree less than $\varepsilon|X|$.

If $G[X]$ has maximum degree less than $\varepsilon|X|$ then the theorem holds, so we assume that $G[X]$ has maximum degree at least $\varepsilon|X|$ (and so $|X| > 1$), and therefore $\overline{G}[X]$ has maximum degree less than $\varepsilon|X|$. By 5.4 applied to $\overline{G}[X]$, there are disjoint anticomplete (in $\overline{G}$) subsets $A, B \subseteq X$ with $|A|, |B| \geq \varepsilon|X|$. But then A is complete to B in G, and $|A|, |B| \geq \varepsilon|X| \geq \delta\varepsilon|G|$. This proves 5.5. ∎

From 5.5 we deduce:

5.6 *Let $\varepsilon > 0$, and choose $\delta > 0$ satisfying 5.5. Let $\alpha \geq 1$, such that $\varepsilon\delta 2^\alpha > 1$. Let (G, f) be α-critical, where G is house-free. Then there is a subset $X \subseteq V(G)$ and a good function g on $G[X]$ with $g(v) \leq f(v)$ for each $v \in X$, such that $g^\alpha(X) \geq \delta f^\alpha(G)$ and $g^\alpha(N_G(v) \cap X) < \varepsilon g^\alpha(X)$ for every vertex $v \in X$.*

Proof. By rational approximation, we may assume that f^α is rational. Choose an integer $T > 0$ such that $Tf^\alpha(v)$ is an integer for all $v \in V(G)$. Let G' be obtained from G by replacing each vertex v by a clique W_v of cardinality $Tf^\alpha(v)$, where

- the sets W_v ($v \in V(G)$) are pairwise disjoint;
- for all distinct $u, v \in V(G)$ adjacent in G, W_u is complete to W_v in G'; and
- for all distinct $u, v \in V(G)$ nonadjacent in G, W_u is anticomplete to W_v in G'.

It follows that G' is also house-free, and $|G'| = Tf^\alpha(G) > T \geq 1$. From 5.5 applied to G', we deduce that either

- there are disjoint sets $A', B' \subseteq V(G')$, complete to each other, with $|A'|, |B'| \geq \varepsilon\delta|G'|$, or
- there exists $X' \subseteq V(G')$ with $|X'| \geq \delta|G'|$ such that $G'[X']$ has maximum degree less than $\varepsilon|X'|$.

Suppose that the first bullet holds. Let A be the set of vertices $v \in V(G)$ such that $W_v \cap A' \neq \emptyset$, and define B similarly. Then A is complete to B in G. Moreover

$$f^\alpha(A) \geq |A'|/T \geq \varepsilon\delta|G'|/T \geq \varepsilon\delta f^\alpha(G) > \varepsilon\delta,$$

and similarly $f^\alpha(B) \geq \varepsilon\delta f^\alpha(G)$. By 5.2, $\varepsilon\delta \leq 2^{-\alpha}$, contrary to the hypothesis.

Thus the second bullet holds. Let X be the set of all $v \in V(G)$ such that $W_v \cap X' \neq \emptyset$; and for each $v \in V(G)$ let $g(v)$ satisfy $T(g(v))^\alpha = |W_v \cap X'|$. Thus $g^\alpha(X) = |X'| \geq \delta|G'| = \delta f^\alpha(G)$, and $g(v) \leq f(v)$ for each $v \in V(G)$. Let $v \in X$. The union of the sets $W_u \cap X'$ over all $u \in N(v) \cap X$ has cardinality less than $\varepsilon|X'|$ (indeed, less than $\varepsilon|X'| - |W_v \cap X'| + 1$); and so $Tg^\alpha(N(v) \cap X) < \varepsilon|X'| = \varepsilon T g^\alpha(X)$, that is, $g^\alpha(N(v)) < \varepsilon g^\alpha(X)$. This proves 5.6. ∎

Since P_4-free graphs are perfect, a theorem of Erdős and Hajnal [12] (see also Alon, Pach and Solymosi [1]) implies:

5.7 *There exists $\varepsilon > 0$ such that if G is forcer-free then G has a clique or stable set of cardinality at least $|G|^\varepsilon$.*

We also need a theorem of Jacob Fox (he did not publish his proof, but we gave a proof in [5]):

5.8 *Let H be a graph for which there exists a constant $\delta > 0$ such every H-free graph G has a clique or stable set of cardinality at least $|G|^\delta$. Then every H-free graph is $\frac{3}{\delta}$-narrow.*

By combining 5.7 and 5.8 we obtain:

5.9 *There exists $\alpha \geq 1$ such that every forcer-free graph is α-narrow.*

We deduce:

5.10 *Let $\alpha' \geq 1$ such that every forcer-free graph is α'-narrow. Let $\alpha \geq \alpha'$, and let G be a graph such that every proper induced subgraph is α-narrow. Let g be a good function on G. Let $Z \subseteq V(G)$ with $|Z| > 1$, such that $G[Z]$ is connected and guarded. Let d be the maximum of $g^\alpha(N_G(v) \cap Z)$ over all $v \in Z$. Then $g^\alpha(Z) \leq \max(2d, d^{1-\alpha'/\alpha})$.*

Proof. If $G[Z]$ is not anticonnected, there are two vertices $u, v \in Z$ such that $N(u) \cup N(v) = Z$, and so $g^\alpha(Z) \le 2d$ as required. So we may assume that $G[Z]$ is anticonnected. Let us list all subsets X of Z with the properties that X is a homogeneous set of $G[Z]$, and $X \ne Z$, and X is maximal with these two properties; let these subsets be $W_1, \dots, W_k$ say. Thus $W_1 \cup \cdots \cup W_k = Z$, because $|Z| \ge 2$ and so each singleton subset of Z is a subset of one of $W_1, \dots, W_k$.

We claim that $W_1, \dots, W_k$ are pairwise disjoint. Suppose that $W_1 \cap W_2 \ne \emptyset$ say. Choose $w_1 \in W_1 \setminus W_2$ and $w_2 \in W_2 \setminus W_1$. If w_1, w_2 are nonadjacent, then since W_2 is a homogeneous set, w_1 has no neighbours in W_2, and so, since W_1 is homogeneous, each vertex of W_2 has no neighbour in W_1; and so $G[W_1 \cup W_2]$ is not connected. If w_1, w_2 are adjacent, then similarly $G[W_1 \cup W_2]$ is not anticonnected. Since $G[Z]$ is both connected and anticonnected, it follows that $W_1 \cup W_2 \ne Z$. But $W_1 \cup W_2$ is a homogeneous set of $G[Z]$, contrary to the maximality of W_1. This proves that $W_1, \dots, W_k$ form a partition of Z, and so $k > 1$.

Choose $w_i \in W_i$ for $1 \le i \le k$, and let G' be the graph induced on $\{w_1, \dots, w_k\}$. From the hypothesis, G' is forcer-free, and so α'-narrow. Let $t = \alpha/\alpha'$ and $r = 2^{-1/\alpha'}$; thus $r^{\alpha'} = 1/2$, and $r^\alpha = 2^{-t}$. For $1 \le i \le k$, let P_i be a perfect subgraph of $G[W_i]$ with $g(P_i)$ maximum, and let $p(w_i) = g(P_i)$. We claim that p is a good function on G'. Let J' be a perfect induced subgraph of G', and let J be the subgraph of G induced on the union of the sets P_i $(i \in V(J'))$. By Lovász' substitution lemma [13], it follows that J is perfect, and so $g(J) \le 1$; but $g(J) = p(J')$. This proves that p is a good function on G', and so $p^{\alpha'}(G') \le 1$. For $1 \le i \le k$, $G[W_i]$ is α-narrow (because $k > 1$). Since $g(v)/p(w_i)$ $(v \in W_i)$ is good on $G[W_i]$, it follows that $g^\alpha(W_i) \le p(w_i)^\alpha$. But also, since $G[Z]$ is connected, and $W_i \ne Z$, there is a vertex $v \in Z \setminus W_i$ complete to W_i; and so $g^\alpha(W_i) \le d$ by hypothesis. Thus

$$g^\alpha(W_i) \le \min(p(w_i)^\alpha, d) \le p(w_i)^{\alpha'} d^{1-1/t}.$$

Hence

$$g^\alpha(Z) = \sum_{1 \le i \le k} g^\alpha(W_i) \le d^{1-1/t} \sum_{1 \le i \le k} p(w_i)^{\alpha'} = d^{1-1/t} p^{\alpha'}(G') \le d^{1-1/t}.$$

This proves 5.10. ∎

We deduce 1.6, which we restate:

5.11 *There exists $\alpha \ge 1$ such that for every α-critical pair (G, f), there is a cap in G.*

Proof. Let $\varepsilon = 1/6$, and choose $\delta > 0$ satisfying 5.5. From 5.7, there exists $\alpha' \ge 1$ such that every forcer-free graph is α'-narrow. Let $\alpha \ge 2$, such that $\varepsilon \delta 2^{\alpha/\alpha'} > 1$. We claim that α satisfies the theorem. Suppose not; then there is an α-critical pair (G, f), such that G is cap-free. By 5.6, there is a subset $X \subseteq V(G)$ and a good function g on $G[X]$ with $g(v) \le f(v)$ for each $v \in X$, such that $g^\alpha(X) \ge \delta f^\alpha(G) > \delta$ and $g^\alpha(N_G(v) \cap X) < \varepsilon g^\alpha(X)$ for every vertex $v \in X$. Let $g^\alpha(X) = \lambda$; then $\delta < \lambda \le f^\alpha(G)$. Let $H = G[X]$, and define $w(v) = g^\alpha(v)/\lambda$ for each $v \in X$. Then (H, w) is a weighted graph.

(1) *(H, w) is ε-coherent.*

By 5.1, for each $v \in V(H)$, $g(v) \le f(v) < 1 - 4^{-1/\alpha} \le 1/2$, and so $w(v) < 2^{-\alpha}/\lambda \le \varepsilon$ (since $\lambda \ge \delta$ and $\alpha' \ge 1$). Also, for each vertex $v \in V(H)$,

$$\lambda w(N_H(v)) = g^\alpha(N_G(v) \cap X) < \varepsilon g^\alpha(X) = \varepsilon \lambda,$$

and so $w(N_H(v)) < \varepsilon$. Third, by 5.2, if $A, B \subseteq V(H)$ are disjoint and anticomplete, then not both $w(A), w(B) \geq \varepsilon$, since $\varepsilon > 2^{-\alpha}/\lambda$ (because $\lambda \geq \delta$). Consequently (H, w) is ε-coherent. This proves (1).

By 4.3, and since $\varepsilon = 1/6$, there exists $Z \subseteq V(H)$ with $|Z| > 1$, such that $H[Z]$ is connected and guarded, and $w(Z) > 1 - 4\varepsilon$. Let d be the maximum of $\lambda w(N_G(v) \cap Z)$ over all $v \in Z$. Hence $d \leq \varepsilon\lambda$. By 5.10,

$$\lambda w(Z) \leq \max(2d, d^{1-\alpha'/\alpha}) \leq \max(2\varepsilon\lambda, (\varepsilon\lambda)^{1-\alpha'/\alpha}).$$

But $2\varepsilon\lambda \geq (\varepsilon\lambda)^{1-\alpha'/\alpha}$, since $2(\varepsilon\lambda)^{\alpha'/\alpha} \geq 1$ (since $\lambda \geq \delta$). Thus $\lambda w(Z) \leq 2\varepsilon\lambda$, and so $w(Z) \leq 2\varepsilon$, contradicting that $w(Z) > 1 - 4\varepsilon$ and $\varepsilon = 1/6$. This proves 5.11. ∎

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