Claw-free Graphs. V. Global structure

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Abstract

A graph is *claw-free* if no vertex has three pairwise nonadjacent neighbours. In earlier papers of this series we proved that every claw-free graph either belongs to one of several basic classes that we described explicitly, or admits one of a few kinds of decomposition. In this paper we convert this "decomposition" theorem into a theorem describing the global structure of claw-free graphs.

1 Introduction

The main goal of this series of papers is to prove a theorem describing how to build the "most general" claw-free graph. In earlier papers, particularly in [4], we proved that every claw-free graph either belongs to one of a few basic classes that were we able to describe explicitly, or it admits one of a few kinds of decomposition. The decompositions all have inverses that are constructions producing larger claw-free graphs from smaller ones, so one might think we were done; the most general claw-free graph can be built by iterating these constructions, starting from graphs in our basic classes. But with care we can obtain a much more informative result. For instance, one of our constructions does not need to be iterated; it can be performed just in one round. A second only applies to very restricted classes of graphs, and for such graphs it is essentially the only construction needed. A third does need to be iterated, but the graphs that result from this process have a sort of generalized line graph structure and are better viewed in that light. Some of the constructions cannot be applied to graphs in some basic classes at all. The goal of this paper is to sort out all these issues.

2 Trigraphs

All graphs in this paper are finite and simple, that is, have no loops or parallel edges. As in earlier papers, it is helpful to work with "trigraphs" rather than with graphs. A *trigraph* G consists of a finite set $V(G)$ of vertices, and a map $\theta_G : V(G)^2 \to \{1, 0, -1\}$, satisfying:

- for all $v \in V(G)$, $\theta_G(v, v) = 0$
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v), \theta_G(u, w) = 0$.

We call θ_G the *adjacency function* of G. For distinct u, v in $V(G)$, we say that u, v are *strongly adjacent* if $\theta_G(u, v) = 1$, *strongly antiadjacent* if $\theta_G(u, v) = -1$, and *semiadjacent* if $\theta_G(u, v) = 0$. We say that u, v are *adjacent* if they are either strongly adjacent or semiadjacent, and *antiadjacent* if they are either strongly antiadjacent or semiadjacent. Also, we say u is adjacent to v and u is a *neighbour* of v if u, v are adjacent (and a *strong neighbour* if u, v are strongly adjacent); u is antiadjacent to v and u is an *antineighbour* of v if u, v are antiadjacent. We denote by $F(G)$ the set of all pairs $\{u, v\}$ such that $u, v \in V(G)$ are distinct and semiadjacent. Thus a trigraph G is a graph if $F(G) = \emptyset$.

For a vertex a and a set $B \subseteq V(G) \setminus \{a\}$, we say that a is *complete* to B or *B-complete* if a is adjacent to every vertex in B; and that a is *anticomplete* to B or *B-anticomplete* if a has no neighbour in B. For two disjoint subsets A and B of $V(G)$ we say that A is *complete*, respectively *anticomplete*, to B, if every vertex in A is complete, respectively anticomplete, to B. (We sometimes say A is B-complete, or the pair (A, B) is complete, meaning that A is complete to B.) Similarly, we say that a is *strongly complete* to B if a is strongly adjacent to every member of B, and so on.

Let G be a trigraph. A *clique* in G is a subset $X \subseteq V(G)$ such that every two members of X are adjacent, and a *strong clique* is a subset such that every two of its members are strongly adjacent. A clique with cardinality three is a *triangle*. A set $X \subseteq V(G)$ is *stable* if every two of its members are antiadjacent, and *strongly stable* if every two of its members are strongly antiadjacent. A *triad* in a trigraph G means a stable set with cardinality three.

We say a trigraph H is a *thickening* of a trigraph G if for every $v \in V(G)$ there is a nonempty subset $X_v \subseteq V(H)$, all pairwise disjoint and with union $V(H)$, satisfying the following:

- for each $v \in V(G)$, X_v is a strong clique of H
- if $u, v \in V(G)$ are strongly adjacent in G then X_u is strongly complete to X_v in H
- if $u, v \in V(G)$ are strongly antiadjacent in G then X_u is strongly anticomplete to X_v in H
- if $u, v \in V(G)$ are semiadjacent in G then X_u is neither strongly complete nor strongly anticomplete to X_v in H.

If $X \subseteq V(G)$, we define the trigraph $G|X$ *induced on* X as follows. Its vertex set is X, and its adjacency function is the restriction of θ_G to X^2. We define $G \setminus X = G|(V(G) \setminus X)$. Isomorphism for trigraphs is defined in the natural way, and if G, H are trigraphs, we say that G *contains* H and H is an *induced subtrigraph* of G if there exists $X \subseteq V(G)$ such that H is isomorphic to $G|X$.

A *claw* is a trigraph with four vertices a_0, a_1, a_2, a_3, such that $\{a_1, a_2, a_3\}$ is stable and a_0 is complete to $\{a_1, a_2, a_3\}$. If $X \subseteq V(G)$ and $G|X$ is a claw, we often loosely say that X is a claw; and if no induced subtrigraph of G is a claw, we say that G is *claw-free*. It is easy to check that if H is a thickening of a claw-free trigraph G, then H is also claw-free.

Let us say a trigraph G is *connected* if there is no partition (V_1, V_2) of $V(G)$ such that $V_1, V_2 \neq \emptyset$ and V_1 is strongly anticomplete to V_2. To understand all claw-free trigraphs, it evidently suffices to understand those that are connected, because the others can be built from the connected ones by taking disjoint unions in the natural way. We find that connected claw-free trigraphs fall naturally into three classes:

- those that admit a partition of their vertex set into three strong cliques
- those that admit a "nontrivial strip-structure" (defined later), and
- those for which neither of the above holds.

We shall see that for each of the three classes, the trigraphs in that class can be built starting from certain basic trigraphs and piecing them together by means of certain operations; but the basic trigraphs and the operations are different for the three classes, and it is most convenient to treat the three classes separately. Our characterization for the first class is given in 4.1, and for the second and third in 7.2, but let us state rudimentary versions of those results now. What follows is somewhat vague, because we are postponing a number of important definitions, but we hope that it will give the reader a better idea of the goal of this paper. Here is a construction.

- Start with a trigraph G_0 that is a disjoint union of strong cliques. Take a partition of $V(G_0)$ into strongly stable sets $X_1, \ldots, X_k$, such that each X_i satisfies $1 \leq |X_i| \leq 2$.
- For $1 \leq i \leq k$, take a trigraph G_i (where $G_0, G_1, \ldots, G_k$ are pairwise vertex-disjoint) and a subset $Y_i \subseteq V(G_i)$ with $|Y_i| = |X_i|$. (When we apply this, the pairs (G_i, Y_i) will be taken from an explicit list of allowed pairs. In particular, G_i is claw-free, Y_i is strongly stable in G_i, and for each vertex in Y_i, its neighbour set in G_i is a strong clique.) Take the disjoint union of $G_0, \ldots, G_k$.

- For $1 \leq i \leq k$, take a bijection between X_i and Y_i; and for each $x \in X_i$ and its mate $y \in Y_i$, make the neighbour set of x strongly complete to the neighbour set of y, and then delete x, y. (The order of these operations does not affect the final outcome.)

Then we can state the essentials of our main theorems as follows. (We leave the reader to check that this follows from the full versions 7.2 and 4.1.)

2.1 *For every connected claw-free trigraph G, one of the following holds:*

- *G can be constructed as above, where all the pairs (G_i, Y_i) are taken from an explicit list described later in the paper (either G_i is a three-vertex path and Y_i consists of its two ends, or (G_i, Y_i) is a so-called thickening of a member of one of $\mathcal{Z}_1, \ldots, \mathcal{Z}_{15}$)*
- *G belongs to one of three classes (the thickenings of members of $\mathcal{S}_1, \mathcal{S}_3, \mathcal{S}_7$) described explicitly later*
- *the vertex set of G can be partitioned into three strong cliques. If so, then G can be expressed as a sequence of disjoint pieces, where the adjacency between pieces is specified by a rule given later, and each piece belongs to one of five classes (the thickenings of permutations of members of $\mathcal{TC}_1, \ldots, \mathcal{TC}_5$) described explicitly later.*

3 Some trigraphs

Here are some types of trigraphs that will be important for us. These constitute most of the building blocks that we need for the results just mentioned (it is convenient to collect them in one place, although we shall not need them all immediately.)

- **Line trigraphs.** Let H be a graph, and let G be a trigraph with $V(G) = E(H)$. We say that G is a *line trigraph* of H if for all distinct $e, f \in E(H)$:
  - if e, f have a common end in H then they are adjacent in G, and if they have a common end of degree at least three in H, then they are strongly adjacent in G
  - if e, f have no common end in H then they are strongly antiadjacent in G.

  We say that $G \in \mathcal{S}_0$ if G is isomorphic to a line trigraph of some graph. It is easy to check that any line trigraph is claw-free.

- **Trigraphs from the icosahedron.** The *icosahedron* is the unique planar graph with twelve vertices all of degree five. Let it have vertices $v_0, v_1, \ldots, v_{11}$, where for $1 \leq i \leq 10$, v_i is adjacent to v_{i+1}, v_{i+2} (reading subscripts modulo 10), and v_0 is adjacent to v_1, v_3, v_5, v_7, v_9, and v_{11} is adjacent to $v_2, v_4, v_6, v_8, v_{10}$. Let this graph be G_0, regarded as a trigraph (so no pairs are semiadjacent in G_0). Let G_1 be obtained from G_0 by deleting v_{11}. Let G_2 be obtained from G_1 by deleting v_{10}, and possibly making v_1 semiadjacent to v_4 or making v_6 semiadjacent to v_9 (or both). All these trigraphs (G_0, G_1 and the several possibilities for G_2 are claw-free); let $\mathcal{S}_1$ be the class of all such trigraphs.

- **The trigraphs $\mathcal{S}_2$.** Let H be the trigraph with vertex set $\{v_1, \dots, v_{13}\}$, with adjacency as follows. v_1-$\cdots$-v_6-v_1 is a hole in G of length 6. Next, v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5 and possibly to v_7; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; v_{12} is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$; and v_{13} is adjacent to $v_1, v_2, v_4, v_5, v_7, v_8$. No other pairs are adjacent, and all adjacent pairs are strongly adjacent except possibly for v_7, v_8 and v_9, v_{10}. (Thus the pair v_7v_8 may be strongly adjacent, semiadjacent or strongly antiadjacent; the pair v_9v_{10} is either strongly adjacent or semiadjacent.) We say $G \in \mathcal{S}_2$ if G is isomorphic to $H \setminus X$, where $X \subseteq \{v_7, v_{11}, v_{12}, v_{13}\}$.

- **Long circular interval trigraphs.** Let Σ be a circle, and let $F_1, \dots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$, such that no two of $F_1, \dots, F_k$ share an end-point, and no three of them have union Σ. Now let $V \subseteq \Sigma$ be finite, and let G be a trigraph with vertex set V in which, for distinct $u, v \in V$,

  - if $u, v \in F_i$ for some i then u, v are adjacent, and if also at least one of u, v belongs to the interior of F_i then u, v are strongly adjacent
  - if there is no i such that $u, v \in F_i$ then u, v are strongly antiadjacent.

  Such a trigraph G is called a *long circular interval trigraph*. We write $G \in \mathcal{S}_3$ if G is a long circular interval trigraph.

- **Modifications of $L(K_6)$.** Let H be a graph with seven vertices $h_1, \dots, h_7$, in which h_7 is adjacent to h_6 and to no other vertex, h_6 is adjacent to at least three of $h_1, \dots, h_5$, and there is a cycle with vertices h_1-h_2-$\cdots$-h_5-h_1 in order. Let $J(H)$ be the graph obtained from the line graph of H by adding one new vertex, adjacent precisely to those members of $E(H)$ that are not incident with h_6 in H. Then $J(H)$ is a claw-free graph. Let G be either $J(H)$ (regarded as a trigraph), or (in the case when h_4, h_5 both have degree two in H), the trigraph obtained from $J(H)$ by making the vertices $h_3h_4, h_1h_5 \in V(J(H))$ semiadjacent. Let $\mathcal{S}_4$ be the class of all such trigraphs G.

- **The trigraphs $\mathcal{S}_5$.** Let $n \geq 2$. Construct a trigraph G as follows. Its vertex set is the disjoint union of four sets A, B, C and $\{d_1, \dots, d_5\}$, where $|A| = |B| = |C| = n$, say $A = \{a_1, \dots, a_n\}, B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$. Let $X \subseteq A \cup B \cup C$ with $|X \cap A|, |X \cap B|, |X \cap C| \leq 1$. Adjacency is as follows: A, B, C are strong cliques; for $1 \leq i, j \leq n$, a_i, b_j are adjacent if and only if $i = j$, and c_i is strongly adjacent to a_j if and only if $i \neq j$, and c_i is strongly adjacent to b_j if and only if $i \neq j$. Moreover

  - a_i is semiadjacent to c_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $b_i \in X$
  - b_i is semiadjacent to c_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $a_i \in X$
  - a_i is semiadjacent to b_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $c_i \in X$
  - no two of $A \setminus X$, $B \setminus X$, $C \setminus X$ are strongly complete to each other.

  Also, d_1 is strongly $A \cup B \cup C$-complete; d_2 is strongly complete to $A \cup B$, and either semiadjacent or strongly adjacent to d_1; d_3 is strongly complete to $A \cup \{d_2\}$; d_4 is strongly complete to $B \cup \{d_2, d_3\}$; d_5 is strongly adjacent to d_3, d_4; and all other pairs are strongly antiadjacent. Let the trigraph just constructed be G, and let $H = G|(V(G) \setminus X)$. Then H is claw-free; let $\mathcal{S}_5$ be the class of all such trigraphs H.

- **Near-antiprismatic trigraphs.** Let $n \geq 2$. Construct a trigraph H as follows. Its vertex set is the disjoint union of three sets A, B, C, where $|A| = |B| = n + 1$ and $|C| = n$, say $A = \{a_0, a_1, \ldots, a_n\}, B = \{b_0, b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$. Adjacency is as follows. A, B, C are strong cliques. For $0 \leq i, j \leq n$ with $(i, j) \neq (0, 0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. a_0, b_0 may be semiadjacent or strongly antiadjacent. All other pairs not specified so far are strongly antiadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{a_0, b_0\}$ with $|C \setminus X| \geq 2$. Let all adjacent pairs be strongly adjacent except:

  - a_i is semiadjacent to c_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $b_i \in X$
  - b_i is semiadjacent to c_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $a_i \in X$
  - a_i is semiadjacent to b_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $c_i \in X$

  Let the trigraph just constructed be H, and let $G = H \setminus X$. Then G is claw-free; let $\mathcal{S}_6$ be the class of all such trigraphs G. We call such a trigraph G *near-antiprismatic*.

- **Antiprismatic trigraphs.** Let us say a trigraph is *antiprismatic* if for every $X \subseteq V(G)$ with $|X| = 4$, X is not a claw and there are at least two pairs of vertices in X that are strongly adjacent. If G is a trigraph, let H be the graph with vertex set $V(G)$ in which vertices are adjacent in H if and only if they are antiadjacent in G; then G is antiprismatic if and only if H is what we called a "prismatic" graph in [1, 2] and every semiadjacent pair of vertices of G is what we called a "changeable" edge of H. (See those papers for a definition of "prismatic" and "changeable".) In these two papers we listed all prismatic graphs and all ways in which an edge of such a graph can be changeable, and so that provides a description of all antiprismatic trigraphs. Let $\mathcal{S}_7$ be the class of all antiprismatic trigraphs.

4 Three strong cliques

As we said, we plan to tackle the three classes of connected claw-free trigraphs separately; and first we handle the trigraphs G such that $V(G)$ can be partitioned into three strong cliques. A *three-cliqued trigraph* (G, A, B, C) consists of a trigraph G and three strong cliques A, B, C of G, pairwise disjoint and with union $V(G)$. If G is also claw-free we say that (G, A, B, C) is a *three-cliqued claw-free trigraph.*

If (G, A, B, C) is a three-cliqued trigraph, and H is a thickening of G, let X_v $(v \in V(G))$ be the corresponding strong cliques of H; then $\cup_{v \in A} X_v$ is a strong clique A' say of H, and if we define B', C' from B, C similarly, then (H, A', B', C') is a three-cliqued trigraph, that we say is a *thickening* of (G, A, B, C).

Let $n \geq 0$, and for $1 \leq i \leq n$, let (G_i, A_i, B_i, C_i) be a three-cliqued trigraph, where $V(G_1), \ldots, V(G_n)$ are all nonempty and pairwise vertex-disjoint. Let $A = A_1 \cup \cdots \cup A_n$, $B = B_1 \cup \cdots \cup B_n$, and $C = C_1 \cup \cdots \cup C_n$, and let G be the trigraph with vertex set $V(G_1) \cup \cdots \cup V(G_n)$ and with adjacency as follows:

- for $1 \leq i \leq n$, $G|V(G_i) = G_i$;
- for $1 \leq i < j \leq n$, A_i is strongly complete to $V(G_j) \setminus B_j$; B_i is strongly complete to $V(G_j) \setminus C_j$; and C_i is strongly complete to $V(G_j) \setminus A_j$; and

- for $1 \le i < j \le n$, if $u \in A_i$ and $v \in B_j$ are adjacent then u, v are both in no triads; and the same applies if $u \in B_i$ and $v \in C_j$, and if $u \in C_i$ and $v \in A_j$.

In particular, A, B, C are strong cliques, and so (G, A, B, C) is a three-cliqued trigraph; we call the sequence (G_i, A_i, B_i, C_i) $(i = 1, \dots, n)$ a *worn hex-chain* for (G, A, B, C). When $n = 2$ we say that (G, A, B, C) is a *worn hex-join* of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2). Note also that every triad of G is a triad of one of $G_1, \dots, G_n$, and if each G_i is claw-free then so is G. If we replace the third condition above by the strengthening

- for $1 \le i < j \le n$, the pairs $(A_i, B_j), (B_i, C_j)$ and (C_i, A_j) are strongly anticomplete

we call the sequence a *hex-chain* for (G, A, B, C); and if $n = 2$, (G, A, B, C) is a *hex-join* of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2).

Here are some examples of three-cliqued claw-free trigraphs.

- **A type of line trigraph.** Let v_1, v_2, v_3 be distinct nonadjacent vertices of a graph H, such that every edge of H is incident with one of v_1, v_2, v_3. Let v_1, v_2, v_3 all have degree at least three, and let all other vertices of H have degree at least one. Moreover, for all distinct $i, j \in \{1, 2, 3\}$, let there be at most one vertex different from v_1, v_2, v_3 that is adjacent to v_i and not to v_j in H. Let A, B, C be the sets of edges of H incident with v_1, v_2, v_3 respectively, and let G be a line trigraph of H. Then (G, A, B, C) is a three-cliqued claw-free trigraph; let $\mathcal{TC}_1$ be the class of all such three-cliqued trigraphs such that every vertex is in a triad.

- **Long circular interval trigraphs.** Let G be a long circular interval trigraph, and let Σ be a circle with $V(G) \subseteq \Sigma$, and $F_1, \dots, F_k \subseteq \Sigma$, as in the definition of long circular interval trigraph. By a *line* we mean either a subset $X \subseteq V(G)$ with $|X| \le 1$, or a subset of some F_i homeomorphic to the closed unit interval, with both end-points in $V(G)$. Let L_1, L_2, L_3 be pairwise disjoint lines with $V(G) \subseteq L_1 \cup L_2 \cup L_3$; then $(G, V(G) \cap L_1, V(G) \cap L_2, V(G) \cap L_3)$ is a three-cliqued claw-free trigraph. We denote by $\mathcal{TC}_2$ the class of such three-cliqued trigraphs with the additional property that every vertex is in a triad.

- **Near-antiprismatic trigraphs.** Let H be a near-antiprismatic trigraph, and let A, B, C, X be as in the definition of near-antiprismatic trigraph. Let $A' = A \setminus X$ and define B', C' similarly; then (H, A', B', C') is a three-cliqued claw-free trigraph. We denote by $\mathcal{TC}_3$ the class of all three-cliqued trigraphs with the additional property that every vertex is in a triad.

- **Antiprismatic trigraphs.** Let G be an antiprismatic trigraph and let A, B, C be a partition of $V(G)$ into three strong cliques; then (G, A, B, C) is a three-cliqued claw-free trigraph. We denote the class of all such three-cliqued trigraphs by $\mathcal{TC}_4$. (In [1] we described explicitly all three-cliqued antiprismatic graphs, and their "changeable" edges; and this therefore provides a description of the three-cliqued antiprismatic trigraphs.) Note that in this case there may be vertices that are in no triads.

- **Sporadic exceptions.**
  - Let H be the trigraph with vertex set $\{v_1, \dots, v_8\}$ and adjacency as follows: v_i, v_j are strongly adjacent for $1 \le i < j \le 6$ with $j - i \le 2$; the pairs v_1v_5 and v_2v_6 are

strongly antiadjacent; v_1, v_6, v_7 are pairwise strongly adjacent, and v_7 is strongly antiadjacent to v_2, v_3, v_4, v_5; v_7, v_8 are strongly adjacent, and v_8 is strongly antiadjacent to $v_1, \ldots, v_6$; the pairs v_1v_4 and v_3v_6 are semiadjacent, and v_2 is antiadjacent to v_5. Let $A = \{v_1, v_2, v_3\}, B = \{v_4, v_5, v_6\}$ and $C = \{v_7, v_8\}$. Let $X \subseteq \{v_3, v_4\}$; then $(H \setminus X, A \setminus X, B \setminus X, C)$ is a three-cliqued claw-free trigraph, and all its vertices are in triads.

– Let H be the trigraph with vertex set $\{v_1, \ldots, v_9\}$, and adjacency as follows: the sets $A = \{v_1, v_2\}$, $B = \{v_3, v_4, v_5, v_6, v_9\}$ and $C = \{v_7, v_8\}$ are strong cliques; v_9 is strongly adjacent to v_1, v_8 and strongly antiadjacent to v_2, v_7; v_1 is strongly antiadjacent to v_4, v_5, v_6, v_7, semiadjacent to v_3 and strongly adjacent to v_8; v_2 is strongly antiadjacent to v_5, v_6, v_7, v_8 and strongly adjacent to v_3; v_3, v_4 are strongly antiadjacent to v_7, v_8; v_5 is strongly antiadjacent to v_8; v_6 is semiadjacent to v_8 and strongly adjacent to v_7; and the adjacency between the pairs v_2v_4 and v_5v_7 is arbitrary. Let $X \subseteq \{v_3, v_4, v_5, v_6\}$, such that

  * v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$
  * v_7 is not strongly anticomplete to $\{v_5, v_6\} \setminus X$
  * if $v_4, v_5 \notin X$ then v_2 is adjacent to v_4 and v_5 is adjacent to v_7.

  Then $(H \setminus X, A, B \setminus X, C)$ is a three-cliqued claw-free trigraph.

We denote by $\mathcal{TC}_5$ the class of such three-cliqued trigraphs (given by one of these two constructions) with the additional property that every vertex is in a triad.

If (G, A, B, C) is a three-cliqued trigraph, and $\{A', B', C'\} = \{A, B, C\}$, then (G, A', B', C') is also a three-cliqued trigraph, that we say is a *permutation* of (G, A, B, C). Now we can state our theorem about three-cliqued claw-free trigraphs.

4.1 *Every three-cliqued claw-free trigraph admits a worn hex-chain into terms each of which is a thickening of a permutation of a member of one of $\mathcal{TC}_1, \ldots, \mathcal{TC}_5$.*

5 Decompositions

Our next goal is the proof of 4.1, in the section following this. The main part of the proof is an application of the decomposition theorem of [4], and before we can apply that we need to state it precisely, and in particular to describe the decompositions that it uses; and that is the purpose of this section.

Two strongly adjacent vertices of a trigraph G are called *twins* if (apart from each other) they have the same neighbours in G, and the same antineighbours, and if there are two such vertices, we say "G admits twins". If $X \subseteq V(G)$ is a strong clique and every vertex in $V(G) \setminus X$ is either strongly complete or strongly anticomplete to X, we call X a *homogeneous set*. Thus, G admits twins if and only if some homogeneous set has more than one member.

Let A, B be disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair* in G if A, B are strong cliques, and for every vertex $v \in V(G) \setminus (A \cup B)$, v is either strongly A-complete or strongly A-anticomplete and either strongly B-complete or strongly B-anticomplete. Let (A, B) be a homogeneous pair, such that A is neither strongly complete nor strongly anticomplete to B, and at least one of A, B has at least two members. In these circumstances we call (A, B) a *W-join*.

Next, suppose that V_1, V_2 is a partition of $V(G)$ such that V_1, V_2 are nonempty and V_1 is strongly anticomplete to V_2. We call the pair (V_1, V_2) a *0-join* in G. Thus, G admits a 0-join if and only if it is not connected.

Next, suppose that V_1, V_2 is a partition of $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- $A_i, V_i \setminus A_i \neq \emptyset$ for $i = 1, 2$
- $A_1 \cup A_2$ is a strong clique, and
- $V_1 \setminus A_1$ is strongly anticomplete to V_2, and V_1 is strongly anticomplete to $V_2 \setminus A_2$.

In these circumstances, we say that (V_1, V_2) is a *1-join*.

Next, suppose that V_0, V_1, V_2 are disjoint subsets with union $V(G)$, and for $i = 1, 2$ there are subsets A_i, B_i of V_i satisfying the following:

- $V_0 \cup A_1 \cup A_2$ and $V_0 \cup B_1 \cup B_2$ are strong cliques, and V_0 is strongly anticomplete to $V_i \setminus (A_i \cup B_i)$ for $i = 1, 2$;
- for $i = 1, 2$, $A_i \cap B_i = \emptyset$ and A_i, B_i and $V_i \setminus (A_i \cup B_i)$ are all nonempty; and
- for all $v_1 \in V_1$ and $v_2 \in V_2$, either v_1 is strongly antiadjacent to v_2, or $v_1 \in A_1$ and $v_2 \in A_2$, or $v_1 \in B_1$ and $v_2 \in B_2$.

We call the triple (V_0, V_1, V_2) a *generalized 2-join*, and if $V_0 = \emptyset$ we call the pair (V_1, V_2) a *2-join*.

Finally, we say that G *admits a hex-join* if there are three strong cliques A, B, C such that (G, A, B, C) is a three-cliqued trigraph that is expressible as a hex-join. Let us say that a trigraph G is *indecomposable* if it does not admit twins, a W-join, a 0-join, a 1-join, a generalized 2-join, or a hex-join. (In [4] we were careful only to use "nondominating" W-joins, but now it no longer matters.) The main theorem of [4] is the following.

5.1 *Every indecomposable claw-free trigraph belongs to* $\mathcal{S}_0 \cup \cdots \cup \mathcal{S}_7$.

We shall also need the following theorem of [4].

5.2 *Let G be claw-free, and let B_1, B_2, B_3 be strong cliques in G. Let $B = B_1 \cup B_2 \cup B_3$. Suppose that:*

- $B \neq V(G)$,
- *there are two triads $T_1, T_2 \subseteq B$ with $|T_1 \cap T_2| = 2$, and*
- *there is no triad T in G with $|T \cap B| = 2$.*

Then either

- *there exists $V \subseteq B$ with $T_1, T_2 \subseteq V$ such that V is a union of triads, and G is a hex-join of $G|V$ and $G|(V(G) \setminus V)$, where $(V \cap B_1, V \cap B_2, V \cap B_3)$ is the corresponding partition of V into strong cliques, or*
- *there is a homogeneous set with at least two members, included in one of B_1, B_2, B_3, such that all its members are in triads, or*
- *there is a homogeneous pair (V_1, V_2) with $\max(|V_1|, |V_2|) \geq 2$, such that V_1 is a subset of one of B_1, B_2, B_3 and V_2 is a subset of another.*

6 Proof of the three-cliques result

We need several lemmas. First, we observe:

6.1 *Let G be a trigraph.*

- *Let $u, v \in V(G)$ be twins. If $G \setminus \{u\}$ is a thickening of a trigraph H, then G is also a thickening of H. If (G, A, B, C) is a three-cliqued graph and u, v belong to the same member of A, B, C, say A, and $(G \setminus \{u\}, A \setminus \{u\}, B, C)$ is a thickening of a three-cliqued trigraph, then (G, A, B, C) is a thickening of the same three-cliqued trigraph.*

- *Let (P, Q) be a W-join, let $p \in P$ and $q \in Q$, and let G' be obtained from G by deleting $(P \setminus \{p\}) \cup (Q \setminus \{q\})$ and making p, q semiadjacent. If G' is a thickening of some trigraph H, then G is also a thickening of H. If (G, A, B, C) is a three-cliqued trigraph and $P \subseteq A$ and $Q \subseteq B$, and $(G', A \setminus (P \setminus \{p\}), B \setminus (Q \setminus \{q\}), C)$ is a thickening of a three-cliqued trigraph, then (G, A, B, C) is a thickening of the same three-cliqued trigraph.*

- *If (G, A, B, C) is a three-cliqued trigraph expressible as a worn hex-join of (G_i, A_i, B_i, C_i) for $i = 1, 2$, and for $i = 1, 2$, (G_i, A_i, B_i, C_i) admits a worn hex-chain with all terms in some class $\mathcal{C}$, then (G, A, B, C) also admits a worn hex-chain with all terms in $\mathcal{C}$.*

Proof. For the first statement, note that since u, v are twins, it follows that no vertex is semiadjacent to v. Now suppose that $G \setminus \{u\}$ is a thickening of H, and let X_w $(w \in V(H))$ be the corresponding partition of $V(G \setminus \{u\})$ into strong cliques. Let $v \in X_w$ say. Then $X_w \cup \{u\}$ is a strong clique of G, and if we replace X_w by $X_w \cup \{u\}$, the partition of $V(G)$ we obtain shows that G is a thickening of H. The statement for three-cliqued trigraphs follows similarly.

For the second statement, G' is a thickening of H; let X_w $(w \in V(H))$ be the corresponding partition of $V(G')$. Since p, q are semiadjacent in G', they do not belong to the same clique; say $p \in X_u$ and $q \in X_v$. Since X_u, X_v are neither strongly complete nor strongly anticomplete, it follows that u, v are semiadjacent in H. If we replace X_u, X_v by $(X_u \setminus \{p\}) \cup P$ and $(X_v \setminus \{q\}) \cup Q$ respectively, the partition of $V(G)$ we obtain shows that G is a thickening of H. The statement for three-cliqued trigraphs follows similarly.

For the third statement, by hypothesis for $i = 1, 2$ (G_i, A_i, B_i, C_i) admits a worn hex-chain with all terms in $\mathcal{C}$; and the concatenation of the two corresponding sequences is a worn hex-chain of (G, A, B, C) with all terms in $\mathcal{C}$. This proves 6.1. ■

6.2 *Let (G, A, B, C) be a three-cliqued claw-free trigraph, and suppose that G admits a* 0-, 1- *or generalized* 2*-join, and does not admit a hex-join, and every vertex of G belongs to a triad. Then (G, A, B, C) is a permutation of a thickening of a member of $\mathcal{TC}_1 \cup \mathcal{TC}_2 \cup \mathcal{TC}_3 \cup \mathcal{TC}_5$.*

Proof. Suppose G admits a 0-join (V_1, V_2) say. Thus $V_1, V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V(G)$, and V_1 is strongly anticomplete to V_2. Since each of A, B, C is a strong clique, each is a subset of one of V_1, V_2, so we may assume that $V_1 = A \cup B$ and $V_2 = C$. If $A, B \neq \emptyset$, G is a thickening of the trigraph H with three vertices a, b, c, in which a, b are antiadjacent and c is strongly antiadjacent to both a, b. Since $(H, \{a\}, \{b\}, \{c\}) \in \mathcal{TC}_2$, in this case the claim holds. If say $B = \emptyset$, then $A = V_1$ is a nonempty strong clique, and now G is a thickening of a two-vertex trigraph, and again the theorem holds.

Next, suppose that G admits a 1-join and no 0-join, and let $V(G)$ be the union of the four disjoint nonempty sets Y_1, Y_2, Z_1, Z_2, where $Z_1 \cup Z_2$ is a strong clique, and Y_1 is strongly anticomplete to $Y_2 \cup Z_2$, and Y_2 is strongly anticomplete to $Y_1 \cup Z_1$. Since $Y_1, Y_2 \neq \emptyset$, we may assume that $A \cap Y_1, C \cap Y_2 \neq \emptyset$, and so $A \subseteq Y_1 \cup Z_1$ and $C \subseteq Y_2 \cup Z_2$. Suppose that $B \cap Y_1 \neq \emptyset$. Then also $B \subseteq Y_1 \cup Z_1$, and so $Y_2 \cup Z_2 = C$. Let A' be the set of vertices in $A \cap Y_1$ with a neighbour in $B \cap Z_1$, and let B' be the set of vertices in $B \cap Y_1$ with a neighbour in $A \cap Z_1$. Since $(B \cap Z_1) \cup Z_2 \cup A' \cup (B \cap Y_1)$ includes no claw, it follows that A' is strongly complete to $B \cap Y_1$, and similarly B' is strongly complete to $A \cap Y_1$. If $A \cap Z_1 = \emptyset$, then $B' = \emptyset$ and (G, A, B, C) is a thickening of a member of $\mathcal{TC}_2$ with at most six vertices, so we may assume that $A \cap Z_1$ and similarly $B \cap Z_1$ are nonempty. Since the members of Z_2 belong to triads, it follows that $A \cap Y_1$ is not strongly complete to $B \cap Y_1$. But then (G, A, B, C) is a thickening of a member of $\mathcal{TC}_5$. This completes the argument when $B \cap Y_1 \neq \emptyset$, and so we may assume that $B \subseteq Z_1 \cup Z_2$. Hence $Y_1 \subseteq A$ and $Y_2 \subseteq C$; and (G, A, B, C) is a thickening of a member of $\mathcal{TC}_2$ with at most six vertices.

Now suppose that G admits a generalized 2-join and no 0- or 1-join. Let V_0, V_1, V_2 be disjoint subsets with union $V(G)$ as in the definition of 2-join, choosing them with V_0 nonempty if possible. For $i = 1, 2$, let V_i be the union of the three disjoint nonempty sets X_i, Y_i, Z_i, where X_1 is strongly complete to X_2, and Y_1 to Y_2, and otherwise V_1 is strongly anticomplete to V_2; and V_0 is strongly complete to X_1, X_2, Y_1, Y_2 and strongly anticomplete to Z_1, Z_2.

(1) *If one of A, B, C has nonempty intersection with both V_1, V_2 then the theorem holds.*

For suppose that B intersects both V_1, V_2 say. It follows that B is a subset of one of $X_1 \cup X_2 \cup V_0, Y_1 \cup Y_2 \cup V_0$, and we may assume the first from the symmetry. Since Z_1, Z_2 are both nonempty, and are strongly anticomplete, we may assume that $Z_1 \subseteq A$ and $Z_2 \subseteq C$. Hence every vertex in $(X_1 \cup X_2 \cup V_0) \setminus B$ is strongly complete to B, and since every vertex is in a triad it follows that $B = X_1 \cup X_2 \cup V_0$, and therefore $A = Y_1 \cup Z_1$ and $C = Y_2 \cup Z_2$. For $i = 1, 2$, every vertex in X_i with a neighbour in Y_i is strongly complete to Z_i, since otherwise $Y_i \cup X_i \cup Z_i \cup Y_{3-i}$ includes a claw. Moreover, Z_1 is not strongly anticomplete to X_1 since G does not admit a 1-join; and similarly Z_2 is not strongly anticomplete to X_2. For $i = 1, 2$, no vertex in X_i is strongly complete to Y_i since every vertex in X_i is in a triad. Consequently if $V_0 = \emptyset$ then (G, A, B, C) is a thickening of a member of $\mathcal{TC}_2$, and if $V_0 \neq \emptyset$ then (G, A, B, C) is a thickening of a member of $\mathcal{TC}_5$. This proves (1).

In view of (1) we may assume that $A, B \subseteq V_0 \cup V_1$, and so $V_2 \subseteq C$. Since $Z_2 \neq \emptyset$ and every vertex of $V_0 \cup V_1$ is strongly anticomplete to Z_2, it follows that $C = V_2$, and therefore $A \cup B = V_0 \cup V_1$.

(2) *If $A \cap Z_1$ is strongly anticomplete to $B \setminus Z_1$, and $B \cap Z_1$ is strongly anticomplete to $A \setminus Z_1$, then the theorem holds.*

For since $Z_1 \neq \emptyset$, we may assume that $A \cap Z_1 \neq \emptyset$. Since

$$(Y_1 \cap A) \cup (Z_1 \cap A) \cup (X_1 \cap B) \cup Y_2$$

includes no claw, it follows that $Y_1 \cap A$ is strongly anticomplete to $X_1 \cap B$, and similarly $X_1 \cap A$ is strongly anticomplete to $Y_1 \cap B$. Since $A \cap Z_1 \neq \emptyset$ it follows that $A \cap V_0 = \emptyset$, and so $V_0 \subseteq B$. If $V_0 \neq \emptyset$, then $B \cap Z_1 = \emptyset$, and $Z_1 \subseteq A$, and so (G, A, C, B) is a thickening of a member of $\mathcal{TC}_3$. On the other hand, if $V_0 = \emptyset$, then (G, A, B, C) is a thickening of a member of $\mathcal{TC}_1$. This proves (2).

(3) *Let* $b \in B \cap Z_1$. *Then*

- $V_0 \subseteq A$;
- *if* b *has a neighbour in* $A \cap (X_1 \cup Y_1)$, *then* b *is strongly complete to* $A \cap Z_1$;
- *if* b *has a neighbour in* $A \cap (X_1 \cup Y_1)$ *and* $A \cap X_1, A \cap Y_1$ *are both nonempty then* b *is strongly complete to* $A \cap V_1$.

For since $b \in B$ is strongly anticomplete to V_0 it follows that $B \cap V_0 = \emptyset$, and so $V_0 \subseteq A$. This proves the first assertion, and for the other two we may assume that b is adjacent to some $a \in A \cap X_1$ say. Since $\{a, b\} \cup X_2 \cup (A \cap (Y_1 \cup Z_1))$ includes no claw, it follows that b is strongly complete to $A \cap (Y_1 \cup Z_1)$, and in particular, this proves the second assertion. For the third, let $a' \in A \cap Y_1$; then b is adjacent to a', and so by the same argument with X_1, Y_1 exchanged, it follows that b is strongly complete to $A \cap (V_1 \setminus Y_1)$ and therefore to $A \cap V_1$. This proves (3).

(4) *If* $B \cap Z_1$ *is not strongly anticomplete to* $A \cap (X_1 \cup Y_1)$, *and both* $A \cap X_1, A \cap Y_1$ *are nonempty, then the theorem holds.*

For let P be the set of vertices in $B \cap Z_1$ that are strongly complete to $A \cap V_1$, and let $Q = (B \cap Z_1) \setminus P$. From (3), every vertex in Q is strongly anticomplete to $A \cap (X_1 \cup Y_1)$, and so $P \neq \emptyset$ by hypothesis. Consequently some vertex of P is in a triad, and therefore $\emptyset \neq V_0 \subseteq A$, and so $A \cap Z_1 = \emptyset$. Since the vertices of X_2 are in triads, it follows that $Q \neq \emptyset$. Since $(B \cap X_1) \cup Q \cup (A \cap Y_1) \cup X_2$ includes no claw, it follows that $B \cap X_1$ is strongly anticomplete to $A \cap Y_1$, and similarly $A \cap X_1$ is strongly anticomplete to $B \cap Y_1$. But then (G, B, C, A) is a thickening of a member of $\mathcal{TC}_3$. This proves (4).

(5) *We may assume that* $V_0 = \emptyset$ *and* $A \cap Z_1, B \cap Z_1 \neq \emptyset$.

For in view of (2) and (4), we may assume that $A \cap X_1$ is not strongly anticomplete to $B \cap Z_1$, and $A \cap Y_1 = \emptyset$. Since vertices in X_2 are in triads, it follows that X_2 is not strongly complete to A, and so $A \cap Z_1 \neq \emptyset$. By (3) it follows that $V_0 = \emptyset$. This proves (5).

(6) *If* $A \cap X_1$ *is not strongly anticomplete to* $B \cap Y_1$ *then the theorem holds.*

For let A_1 be the set of vertices in $A \cap X_1$ with a neighbour in $B \cap Y_1$, and let B_1 be the set of vertices in $B \cap Y_1$ with a neighbour in $A \cap X_1$. Since $A_1 \cup B_1 \cup (A \cap Z_1) \cup X_2$ includes no claw, it follows that B_1 is strongly complete to $A \cap Z_1$, and similarly A_1 is strongly complete to $B \cap Y_1$. Since $A \cap Z_1 \neq \emptyset$, we may assume by (4) that $B \cap X_1 = \emptyset$, and similarly $A \cap Y_1 = \emptyset$. Then the sets

$$X_1 \setminus A_1, A_1, A \cap Z, B \cap Z, B_1, Y_1 \setminus B_1, Y_2, Z_2, X_2$$

(in circular order) show that (G, A, B, C) is a thickening of a member of $\mathcal{TC}_2$. This proves (6).

In view of (2) and (4), we may assume that $A \cap X_1$ is not strongly anticomplete to $B \cap Z_1$, and $A \cap Y_1 = \emptyset$. Since $Y_1 \neq \emptyset$, it follows that $B \cap Y_1 \neq \emptyset$, and so by (4) with A, B exchanged, we may assume that $A \cap Z_1$ is strongly anticomplete to $B \cap X_1$.

(7) *If $A \cap Z_1$ is not strongly anticomplete to $B \cap Y_1$, then the theorem holds.*

For then by (4) we may assume that $B \cap X_1 = \emptyset$, and by (6) we may assume that $A \cap X_1$ is strongly anticomplete to $B \cap Y_1$. Every member of $B \cap Z_1$ with a neighbour in $A \cap X_1$ is strongly complete to $A \cap Z_1$; and every member of $A \cap Z_1$ with a neighbour in $B \cap Y_1$ is strongly complete to $B \cap Z_1$. But then (G, A, B, C) is a thickening of a member of $\mathcal{TC}_2$. This proves (7).

Thus by (7) we may assume that $A \cap Z_1$ is strongly anticomplete to $B \cap Y_1$, and by (6) that $A \cap X_1$ is strongly anticomplete to $B \cap Y_1$. Hence $(B \cap X_1, A \cup (B \cap Z_1), C \cup (B \cap Y_1))$ is a generalized 2-join, and so $B \cap X_1 = \emptyset$, from the choice of V_0, V_1, V_2. Since by (3) every vertex in $B \cap Z_1$ is either strongly anticomplete to $A \cap X_1$ or strongly complete to $A \cap Z_1$, it follows that (G, A, B, C) is a thickening of a member of $\mathcal{TC}_2$. This completes the proof of 6.2. ■

Next we need the following.

6.3 *Let (G, A, B, C) be a three-cliqued claw-free graph, such that G belongs to one of the classes $\mathcal{S}_0, \dots, \mathcal{S}_7$, and G does not admit a hex-join, and every vertex of G is in a triad. Then (G, A, B, C) is a permutation of a thickening of a member of $\mathcal{TC}_1 \cup \dots \cup \mathcal{TC}_5$.*

Proof. First suppose that $G \in \mathcal{S}_0$; let G be a line trigraph of some graph H. Since A is a strong clique, either there is a vertex $a \in V(H)$ such that every edge in A is incident with a, or there is a set of three vertices of H such that every edge in A joins two of these vertices. The same holds for B, C. Suppose that there exist $a, b, c \in V(H)$ such that a is incident with every edge in A, and b with B, and c with C. Let X be the set of edges of H with both ends in $\{a, b, c\}$. If $X = \emptyset$, then a, b, c are pairwise nonadjacent in H, and (G, A, B, C) is a thickening of a permutation of a member of $\mathcal{TC}_1 \cup \dots \cup \mathcal{TC}_4$. (To see this, let us temporarily omit the condition that a, b, c have degree at least three, in the definition of $\mathcal{TC}_1$, and let $\mathcal{TC}_1'$ be the class of trigraphs that we thereby define. It is easy to see that (G, A, B, C) is a thickening of a a member of $\mathcal{TC}_1'$; but any member of $\mathcal{TC}_1'$ not in $\mathcal{TC}_1$ is a permutation of a member of $\mathcal{TC}_2 \cup \mathcal{TC}_3 \cup \mathcal{TC}_4$, as can easily be verified using 6.2.) Thus we may assume that $X \neq \emptyset$. If no edge in X is semiadjacent in G to any member of $E(H) \setminus X$, then G admits a hex-join (take the cliques $A \setminus X, B \setminus X, C \setminus X$ together with the singleton subsets of X), a contradiction. Thus we may assume that $e = ab \in X$ say is semiadjacent in G to $f = ad \in E(H) \setminus X$. Since H is a line trigraph, it follows that a has degree two in H, and $d \neq a, b, c$. If $e \notin B$, then since e is strongly complete to B, it follows that e is in no triad, a contradiction; so $e \in B$. Hence $A = \{f\}$, and so every vertex of G is in a triad with a. Consequently d has degree one in H. But then (G, A, B, C) is a permutation of a thickening of a member of $\mathcal{TC}_2$.

Next suppose that $a \in V(H)$ is incident with all edges in A, and $b \in V(H)$ is incident with all edges in B, and $|C| = 3$ and the three edges in C form a triangle with vertex set Z say. For each $z \in Z$, the edges of H incident with z are pairwise strongly adjacent in G; for if z has degree at least three, this is true since G is a line trigraph, and if z has degree two then it is true since C is a strong clique. Since every edge of C is in a triad, it follows that $a, b \notin Z$. Suppose that $e \in A$ is incident with a, b in H. Since e is in a triad, it is antiadjacent to some $f \in B$, and since e, f are both incident with b in H it follows that b has degree two in H and e, f are semiadjacent in G. Consequently $B = \{f\}$, and so every edge of H different from f is antiadjacent to f in G. Let f be incident with

b, c say. Since the edges of C are antiadjacent to f, we deduce that $c \notin Z$, and so c has degree one in H. But then (G, A, B, C) is a permutation of a thickening of a member of $\mathcal{TC}_2$. We may therefore assume that a, b are not adjacent in H. Consequently A is the set of all edges of H incident with a, and similarly for B, b. Since the edges of C belong to triads, there is an edge of H incident with no vertex in Z and incident with exactly one of a, b, say with a. If also there is such an edge incident with b, then (G, A, B, C) is a thickening of a member of $\mathcal{TC}_3 \cup \mathcal{TC}_4$; so we may assume that every edge of H incident with b is incident with a vertex in Z. Then no edge of A incident with a vertex in Z belongs to a triad of G, and so H is disconnected, and (G, A, B, C) is a thickening of a member of $\mathcal{TC}_2$.

Next suppose that $a \in V(H)$ is incident with all edges in A, and the edges in B and in C both form triangles of H. Let $Y \subseteq V(H)$ with $|Y| = 3$ so that the three members of B each join two vertices of Y, and define Z similarly for C. Since every member of B is in a triad of G, it follows that $a \notin Y$, and $|Y \cap Z| \leq 1$, and similarly $a \notin Z$. If $Y \cap Z = \emptyset$ then (G, A, B, C) is a thickening of a member of $\mathcal{TC}_2$, so we may assume that $Y \cap Z = \{y\}$ say. But then (V_0, V_1, V_2) is a generalized 2-join, where V_0 is the set of (at most one) edge of H between a, y, and V_1 is the set of edges of H with an end in $Y \setminus \{y\}$, and $V_2 = V(G) \setminus (V_0 \cup V_1)$. From 6.2 the claim follows.

Finally, suppose that all three of A, B, C consist of three edges forming a triangle of H; and so $|E(H)| = 9$. Let the vertex sets of these three triangles of H be X, Y, Z respectively. Since every edge of H is in a triad of G, it follows that $|X \cap Y| \leq 1$, and similarly $|X \cap Z|, |Y \cap Z| \leq 1$. If at most one vertex of X belongs to $Y \cup Z$, then G admits a 0- or 1-join and the result follows from 6.2; so we may assume that $X \cap Y = \{z\}$ and $Y \cap Z = \{x\}$ and $X \cap Z = \{y\}$ say, where x, y, z are all distinct. But then the edge of C incident with x, y in H belongs to no triad of G, a contradiction. This completes the proof when $G \in \mathcal{S}_0$.

No trigraph in $\mathcal{S}_1 \cup \mathcal{S}_2$ has a vertex set that can be partitioned into three cliques, so next we assume that $G \in \mathcal{S}_3$. Let Σ be a circle, and let $V, F_1, \ldots, F_k \subseteq \Sigma$ be as in the definition of long circular interval trigraph, where $V(G) = V$. Define "line" as before; we claim that there is a line L with $L \cap V(G) = A$. For this is trivial if $|A| \leq 1$, so we assume $|A| \geq 2$, and since every two members of A are adjacent and therefore both belong to some one of $F_1, \ldots, F_k$, there is a line L with both endpoints in A. Choose such a line with $L \cap V(G)$ maximal. We claim that $L \cap V(G) = A$. For first, suppose that there exists $a \in A$ with $a \notin L$. Let L have endpoints a_1, a_2 say, and let $L \subseteq F_1$ say. From the maximality of L, $a \notin F_1$, and since a, a_1 are adjacent, there exists i with $a, a_1 \in F_i$; $i = 2$ say. From the maximality of L and since $F_1 \cup F_2 \neq \Sigma$, it follows that $a_2 \notin F_2$. Similarly, since a, a_2 are adjacent, we may assume that $a, a_2 \in F_3$ and $a_1 \notin F_1$. But then $F_1 \cup F_2 \cup F_3 = \Sigma$, a contradiction. This proves that $A \subseteq L$. Now suppose that there exists $b \in L \cap V(G)$ with $b \notin A$, say $b \in B$. Then b is strongly adjacent to all members of A, since $A \subseteq F_1$ and b is in the interior of F_1; and so b is in no triad, a contradiction. Thus $L \cap V(G) = A$. Similarly there are lines for B, C; they are pairwise disjoint and have union $V(G)$, and so $(G, A, B, C) \in \mathcal{TC}_2$.

No trigraph in $\mathcal{S}_4$ has a vertex set that is the union of three cliques, so next we assume that $G \in \mathcal{S}_5$. Let $d_1, \ldots, d_5$ be as in the definition of $\mathcal{S}_5$; let P, Q, R be the sets of vertices different from $\{d_1, \ldots, d_5\}$ adjacent to d_3 and not d_4, d_4 and not d_3, and neither of d_3, d_4, respectively (that is, the sets called A, B, C in the definition of $\mathcal{S}_4$). Since d_1 is in a triad, it follows that d_1, d_2 are semiadjacent. We may assume that $d_5 \in A$, and so $A \subseteq \{d_3, d_4, d_5\}$; and so $d_2 \notin A$; we may assume that $d_2 \in B$. Hence $R \subseteq C$, since d_2, d_5 are both anticomplete to R. Let P_1 be the set of vertices in P with an antineighbour in R, and define $Q_1 \subseteq Q$ similarly. Then $P_1 \cup Q_1 \subseteq B$, and so is a strong clique. On the other hand, since d_3 is in a triad, it follows that $Q_1 \neq \emptyset$, and similarly $P_1 \neq \emptyset$; and since every vertex in P has at most one neighbour in Q and vice versa, we deduce that $|P_1| = |Q_1| = 1$. Let $P_1 = \{p_1\}$ and $Q_1 = \{q_1\}$. Hence $B = \{p_1, q_1, d_2\}$; so $A = \{d_3, d_4, d_5\}$ and $C = \{d_1\} \cup (P \setminus P_1) \cup (Q \setminus Q_1) \cup R$. Let R_1 be the set of vertices in R with an antineighbour in $P \cup Q$, necessarily in $\{p_1, q_1\}$. From the definition of $\mathcal{S}_4$ it follows that R_1 is strongly anticomplete to both P_1, Q_1, and $|R_1| \leq 1$; and $R_1 \neq \emptyset$ since $P_1, Q_1 \neq \emptyset$. Let $R_1 = \{r_1\}$ say. Then $A \setminus \{d_5\}$ is the set of neighbours of d_5, and $C \setminus \{r_1\}$ is the set of neighbours of r_1; and since $(\{d_2\}, \{d_1\} \cup (R \setminus R_1))$ is a homogeneous pair, it follows that (G, A, B, C) is a permutation of a thickening of a member of $\mathcal{TC}_3$ (taking d_5, r_1 as the vertices called a_0, b_0 in the definition of $\mathcal{TC}_3$).

Next, we assume that $G \in \mathcal{S}_6$. Let a_0, b_0 be as in the definition of $\mathcal{S}_6$. Since they are antiadjacent, we may assume that $a_0 \in A$ and $b_0 \in B$. (Note that A, B, C here refer to the three cliques of our three-cliqued graph, and may be different from the sets called A, B, C in the definition of $\mathcal{S}_6$.) Hence every vertex adjacent to a_0 except b_0 is strongly complete to A, and since every vertex is in a triad, it follows that every such vertex belongs to A. So in fact the sets called A, B, C here are the same as the sets called A, B, C in the definition of $\mathcal{S}_6$, and therefore $(G, A, B, C) \in \mathcal{TC}_3$.

Finally, we assume that $G \in \mathcal{S}_7$. Then $(G, A, B, C) \in \mathcal{TC}_4$ from the definition of $\mathcal{TC}_4$. This proves 6.3. ■

Proof of 4.1.

Let (G, A, B, C) be a three-cliqued claw-free graph; we prove that the result holds for (G, A, B, C) by induction on $|V(G)|$. By 6.1, we may assume that

(1) *No two members of the same set A, B, C are twins; and there is no W-join (X, Y) with X a subset of one of A, B, C and Y a subset of another; and (G, A, B, C) is not expressible as a worn hex-join.*

(2) *There is no choice of cliques P, Q, R with the following properties: (G, P, Q, R) is a three-cliqued trigraph, expressible as a hex-join of two three-cliqued trigraphs (G_1, P_1, Q_1, R_1) and (G_2, P_2, Q_2, R_2), and $P_1 \subseteq A, Q_1 \subseteq B$, $R_1 \subseteq C$, and $V(G_1)$ is a union of triads.*

For suppose that such P, Q, R exist. Now $P_2 \cap C$ is strongly complete to R_1 since $R_1 \subseteq C$ and C is a strong clique; and yet P_2 is strongly anticomplete to R_1 from the definition of a hex-join. Since R_1 is nonempty (because $V(G_1)$ is a nonempty union of triads) it follows that $P_2 \cap C = \emptyset$, and similarly $Q_2 \cap A$ and $R_2 \cap B$ are empty. Let $Y = V(G_1)$ and let

$$X = (A \cap R_2) \cup (B \cap P_2) \cup (C \cap Q_2).$$

Let $Z = V(G_2) \setminus X$. Suppose that $X \neq \emptyset$; we claim that (G, A, B, C) is a worn hex-join of $(G|X, X \cap A, X \cap B, X \cap C)$ and $(G \setminus X, A \setminus X, B \setminus X, C \setminus X)$. To see this, we observe that both of these are three-cliqued trigraphs; and certainly $X \cap A$ is strongly complete to $A \setminus X$ since A is a strong clique. We must therefore check that $X \cap A$ is complete to $C \setminus X$, and that if $u \in X \cap A$ and $v \in B \setminus X$ are adjacent then u, v are both not in triads. (Also we must check similar statements with A replaced by B, C, which follow from the symmetry.) Since $P_2 \cap C = \emptyset$, it follows that $(X \cap A) \cup (Z \cap C) \subseteq R_2$, and since R_2 is a strong clique we deduce that $X \cap A$ is strongly complete to $Z \cap C$; and from definition of a hex-join, $X \cap A$ is strongly complete to $Y \cap C$. Thus $X \cap A$ is strongly complete to $C \setminus X$. Now suppose that $u \in X \cap A$ and $v \in B \setminus X$ are adjacent. Then $v \notin Y \cap B$

from the definition of a hex-join, and so $v \in B \cap Z$. Suppose that $u \in T$ for some triad T. Since $T \cap C \neq \emptyset$, and u is strongly complete to $C \setminus X$, it follows that there exists $c \in T \cap X \cap C$; since c is strongly complete to $B \setminus X$ (by the argument above, with A, B, C replaced by B, C, A), it follows that $T \subseteq X$; and then $\{v\} \cup T$ is a claw, a contradiction. Thus u is in no triad, and similarly v is in no triad. This proves our claim; but that is contradictory to (1). Hence $X = \emptyset$, and similarly $Z = \emptyset$, a contradiction since $V(G_2) \neq \emptyset$. Thus there do not exist such P, Q, R. This proves (2).

(3) *We may assume that G admits no hex-join, and every vertex of G is in a triad.*

For if G is antiprismatic then the theorem holds. We assume then that there are two triads T_1, T_2 with $|T_1 \cap T_2| = 2$. Suppose that some vertex v is in no triad, with $v \in A$ say. Then (1) and (2) contradict 5.2 applied to the three cliques $A \setminus \{v\}$, B, C. Thus every vertex is in a triad. Suppose that G admits a hex-join (V_1, V_2); then every triad is a subset of one of V_1, V_2, and we may assume that $T_1, T_2 \subseteq V_1$, and again (1) and (2) contradict 5.2 applied to the three cliques $A \cap V_1, B \cap V_1, C \cap V_1$. This proves (3).

If G admits a 0, 1- or generalized 2-join, the result holds by 6.2; so we may assume that G is indecomposable. By 5.1, $G \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_7$. But then the result follows from 6.3. This proves 4.1. ■

7 Statement of the main theorem

We have completed the description of the claw-free trigraphs G such that $V(G)$ is the union of three strong cliques, and now we begin the study of the others. Our goal in this section is to state the counterpart of 4.1.

A vertex v of a trigraph is *simplicial* if $N \cup \{v\}$ is a strong clique, where N is the set of all neighbours of v. Let us say (G, Z) is a *stripe* if G is a claw-free trigraph, and $Z \subseteq V(G)$ is a set of simplicial vertices, such that Z is strongly stable and no vertex has two neighbours in Z.

Let (G', Z') be a stripe, and let G be a thickening of G', with sets X_v ($v \in V(G')$), such that $|X_z| = 1$ for each $z \in Z'$. Let $Z = \cup_{z \in Z'} X_z$. Then (G, Z) is also a stripe, and we say it is a *thickening* of (G', Z').

Let (G, A, B, C) be a three-cliqued claw-free trigraph, and let $z \in A$ such that z is strongly anticomplete to $B \cup C$. Let V_1, V_2, V_3 be three disjoint sets of new vertices, and let G' be the trigraph obtained by adding V_1, V_2, V_3 to G with the following adjacencies:

- V_1 and $V_2 \cup V_3$ are strong cliques
- V_1 is strongly complete to $B \cup C$ and strongly anticomplete to A
- V_2 is strongly complete to $C \cup A$ and strongly anticomplete to B
- V_3 is strongly complete to $A \cup B$ and strongly anticomplete to C.

(The adjacency between V_1 and $V_2 \cup V_3$ is unspecified.) It follows that G' is a claw-free trigraph, and z is a simplicial vertex of it. In this case we say that (G', z) is a *hex-expansion* of (G, A, B, C).

Here are some types of stripes. We call the corresponding sets of pairs (G, Z) $\mathcal{Z}_1$–$\mathcal{Z}_{15}$, and we define $\mathcal{Z}_0$ to be the set of all members of $\mathcal{Z}_1$–$\mathcal{Z}_{15}$ that are not thickenings of members of $\mathcal{Z}_1$–$\mathcal{Z}_{15}$ with fewer vertices.

$\mathcal{Z}_1$: Let G be a trigraph with vertex set $\{v_1, \dots, v_n\}$, such that for $1 \le i < j < k \le n$, if v_i, v_k are adjacent then v_j is strongly adjacent to both v_i, v_k. We call G a *linear interval trigraph.* (Every linear interval trigraph is also a long circular interval trigraph.) Also, let $n \ge 2$, let v_1, v_n be strongly antiadjacent, and let there be no vertex adjacent to both v_1, v_n, and no vertex semiadjacent to either v_1 or v_n. Let $Z = \{v_1, v_n\}$.

$\mathcal{Z}_2$: Let $G \in \mathcal{S}_6$, let a_0, b_0 etc. be as in the definition of $\mathcal{S}_6$, with a_0, b_0 strongly antiadjacent, and let $Z = \{a_0, b_0\}$.

$\mathcal{Z}_3$: Let H be a graph, and let h_1-h_2-h_3-h_4-h_5 be the vertices of a path of H in order, such that h_1, h_5 both have degree one in H, and every edge of H is incident with one of h_2, h_3, h_4. Let G be obtained from a line trigraph of H by making the edges h_2h_3 and h_3h_4 of H (vertices of G) either semiadjacent or strongly antiadjacent to each other in G. Let $Z = \{h_1h_2, h_4h_5\}$.

$\mathcal{Z}_4$: Let G be the trigraph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2\}$ and adjacency as follows: $\{a_0, a_1, a_2\}, \{b_0, b_1, b_2, b_3\}$, $\{a_2, c_1, c_2\}$ and $\{a_1, b_1, c_2\}$ are strong cliques; b_2, c_1 are strongly adjacent; b_2, c_2 are semiadjacent; b_3, c_1 are semiadjacent; and all other pairs are strongly antiadjacent. Let $Z = \{a_0, b_0\}$.

$\mathcal{Z}_5$: Let $G \in \mathcal{S}_2$, and let $v_1, \dots, v_{13}, X, H$ be as in the definition of $\mathcal{S}_2$, where $G = H \setminus X$; let v_7, v_8 be strongly antiadjacent in H, and let $Z = \{v_7, v_8\} \setminus X$.

$\mathcal{Z}_6$: Let G be a long circular interval trigraph, and let $\Sigma, F_1, \dots, F_k$ be as in the corresponding definition. Let $z \in V(G)$ belong to at most one of $F_1, \dots, F_k$, and not be an endpoint of any of $F_1, \dots, F_k$. Then z is a simplicial vertex of G; let $Z = \{z\}$.

$\mathcal{Z}_7$: Let $G \in \mathcal{S}_4$, let $H, h_1, \dots, h_7$ be as in the definition of $\mathcal{S}_4$, let e be the edge h_6h_7 of H, and let $Z = \{e\}$.

$\mathcal{Z}_8$: Let $G \in \mathcal{S}_5$, let $d_1, \dots, d_5, A, B, C$ be as in the definition of $\mathcal{S}_5$, and let $Z = \{d_5\}$.

$\mathcal{Z}_9$: Let G have vertex set partitioned into five sets $\{z\}, A, B, C, D$, with $|A| = |B| > 0$, say $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ where $n \ge 1$, such that

- $\{z\} \cup D$ is a strong clique and z is strongly antiadjacent to $A \cup B \cup C$,
- $A \cup C$ and $B \cup C$ are strong cliques,
- for $1 \le i \le n$, a_i, b_i are antiadjacent, and every vertex in D is strongly adjacent to exactly one of a_i, b_i and strongly antiadjacent to the other, and
- for $1 \le i < j \le n$, $\{a_i, b_i\}$ is strongly complete to $\{a_j, b_j\}$.

(The adjacency between C and D is arbitrary.) Every such trigraph G is antiprismatic. Let $Z = \{z\}$.

$\mathcal{Z}_{10}$: Let G' be the trigraph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2, d\}$ and adjacency as follows: $A = \{a_0, a_1, a_2, d\}$, $B = \{b_0, b_1, b_2, b_3\}$, $C = \{c_1, c_2\}$ and $\{a_1, b_1, c_2\}$ are strong cliques; a_2 is strongly adjacent to b_0 and semiadjacent to b_1; b_2, c_2 are semiadjacent; b_2, c_1 are strongly adjacent; b_3, c_1 are either semiadjacent or strongly adjacent; b_0, d are either semiadjacent or strongly adjacent; and all other pairs are strongly antiadjacent. Then (G', A, B, C) is a three-cliqued trigraph (not clawfree) and a_0 is a simplicial vertex of G'. Let $X \subseteq \{a_2, b_2, b_3, d\}$ such that either $a_2 \in X$ or $\{b_2, b_3\} \subseteq X$, let $Z = \{a_0\}$, and let (G, Z) be a hex-expansion of $(G' \setminus X, A \setminus X, B \setminus X, C)$.

$\mathcal{Z}_{11}$: Let $G_1 \in \mathcal{S}_6$, and let $a_0, b_0, A, B, C, A', B', C'$ be as in the definition of $\mathcal{S}_6$, with a_0, b_0 semi-adjacent. Let G_2 be obtained from G_1 by adding a new vertex z strongly complete to A and strongly anticomplete to $B \cup C$, and possibly making a_0, b_0 strongly adjacent, and let $Y \subseteq \{a_0\}$; then $(G_2 \setminus Y, (A \setminus Y) \cup \{z\}, B, C)$ is a three-cliqued trigraph and z is a simplicial vertex of $G_2 \setminus Y$. Let $Z = \{z\}$, and let (G, Z) be a hex-expansion of $(G_2 \setminus Y, (A \setminus Y) \cup Z, B, C)$.

$\mathcal{Z}_{12}$: Let $H, v_1, \ldots, v_9, X$ be as in the second construction of $\mathcal{TC}_5$, where v_2, v_4 are adjacent. Add a new vertex z to H, strongly adjacent to v_3, v_4, v_5, v_6, v_9, forming H_2 say. Then

$$(H_2 \setminus X, \{v_3, v_4, v_5, v_6, v_9, z\} \setminus X, \{v_7, v_8\}, \{v_1, v_2\})$$

is a three-cliqued trigraph and z is a simplicial vertex of $H_2 \setminus X$. Let $Z = \{z\}$, and let (G, Z) be a hex-expansion of this three-cliqued trigraph.

$\mathcal{Z}_{13}$: Let $(H, V(H) \cap L_1, V(H) \cap L_2, V(H) \cap L_3) \in \mathcal{TC}_2$, where $\Sigma, F_1, \ldots, F_k, L_1, L_2, L_3$ are as in the corresponding definition. Let $z \in L_1$ belong to exactly one of $F_1, \ldots, F_k$, say F_1; thus $L_1 \subseteq F_1$. Let z belong to the interior of F_1. Thus z is a simplicial vertex of H. Let $Z = \{z\}$, and let (G, Z) be a hex-expansion of $(H, V(H) \cap L_1, V(H) \cap L_2, V(H) \cap L_3)$.

$\mathcal{Z}_{14}$: Let v_0, v_1, v_2, v_3 be distinct vertices of a graph H, such that: v_1 is the only neighbour of v_0 in H; every vertex of H different from v_0, v_1, v_2, v_3 is adjacent to both v_2, v_3, and at most one of them is nonadjacent to v_1; and v_1, v_2, v_3 are pairwise nonadjacent, and each has degree at least three. For $i = 1, 2, 3$, let A_i be the set of edges of H incident with v_i, and let z be the edge $v_0 v_1$. Let G_1 be a line trigraph of H; thus (G_1, A_1, A_2, A_3) is a three-cliqued claw-free trigraph, and z is a simplicial vertex of G_1. Let $Z = \{z\}$, and let (G, Z) be a hex-expansion of (G_1, A_1, A_2, A_3).

$\mathcal{Z}_{15}$: Let $v_1, \ldots, v_8, H, A, B, C, X$ be as in the first construction of $\mathcal{TC}_5$. Then (H, A, B, C) is a three-cliqued trigraph and v_8 is a simplicial vertex of H; let $Z = \{v_8\}$, and let (G, Z) be a hex-expansion of (H, A, B, C).

These, and thickenings of them, will be the building blocks of our theorem describing how to construct all the connected claw-free trigraphs that admit a strip-structure. Now we explain how these building blocks fit together.

A *hypergraph* H consists of a finite set $V(H)$, a finite set $E(H)$, and an incidence relation between $V(H)$ and $E(H)$ (that is, a subset of $V(H) \times E(H)$). For the statement of the main theorem, we only need hypergraphs such that every member of $E(H)$ is incident with either one or two members of $V(H)$ (thus, these hypergraphs are graphs if we permit "graphs" to have loops and parallel edges), but it is helpful for the proof, later, to permit general hypergraphs. If $F \in E(H)$ we write $\overline{F}$ to denote the set of all $h \in V(H)$ incident with F.

Let G be a trigraph. A *strip-structure* (H, η) of G consists of a hypergraph H with $E(H) \neq \emptyset$, and a function η mapping each $F \in E(H)$ to a subset $\eta(F)$ of $V(G)$, and mapping each pair (F, h) with $F \in E(H)$ and $h \in \overline{F}$ to a subset $\eta(F, h)$ of $\eta(F)$, satisfying the following conditions.

(SD1) The sets $\eta(F)$ ($F \in E(H)$) are nonempty and pairwise disjoint and have union $V(G)$.

(SD2) For each $h \in V(H)$, the union of the sets $\eta(F, h)$ for all $F \in E(H)$ with $h \in \overline{F}$ is a strong clique of G.

(SD3) For all distinct $F_1, F_2 \in E(H)$, if $v_1 \in \eta(F_1)$ and $v_2 \in \eta(F_2)$ are adjacent in G, then there exists $h \in \overline{F_1} \cap \overline{F_2}$ such that $v_1 \in \eta(F_1, h)$ and $v_2 \in \eta(F_2, h)$.

(SD4) For each $F \in E(H)$, the family $\eta(F, h)$ ($h \in \overline{F}$) is a circus in $\eta(F)$.

(We postpone the definition of a "circus" to the next section, and the reader should ignore condition **(SD4)** through the remainder of this section.)

Let (H, η) be a strip-structure of a trigraph G, and let $F \in E(H)$, where $\overline{F} = \{h_1, \ldots, h_k\}$. Let $v_1, \ldots, v_k$ be new vertices, and let J be the trigraph obtained from $G|\eta(F)$ by adding $v_1, \ldots, v_k$, where v_i is strongly complete to $\eta(F, h_i)$ and strongly anticomplete to all other vertices of J. We call $(J, \{v_1, \ldots, v_k\})$ the *strip of* (H, η) *at* F. (Thus, it is uniquely defined except for the choice of the new vertices $v_1, \ldots, v_k$, and we speak of "the" strip at F without serious ambiguity.) These strips are not necessarily stripes; but soon we will only need to consider strip-structures in which every strip is either a stripe or is very simple.

This provides a way to piece together claw-free trigraphs to make larger claw-free trigraphs, because of the following.

7.1 *Let (H, η) be a strip-structure of a trigraph G. If J is claw-free for every strip (J, Z) of this strip-structure, then G is claw-free.*

Proof. Suppose that $\{a, b, c, d\}$ is a claw in G, where a is adjacent to b, c, d and $\{b, c, d\}$ is stable. By **(SD1)**, we may choose $F \in E(H)$ with $a \in \eta(F)$. Let $\overline{F} = \{h_1, \ldots, h_k\}$, and let J be the strip at F, with new vertices $v_1, \ldots, v_k$. We shall show that J contains a claw. If $b, c, d \in \eta(F)$ then $\{a, b, c, d\}$ is a claw of J, so we may assume that $b \notin \eta(F)$. Choose $F_1 \in E(H)$ with $b \in \eta(F_1)$; thus $F_1 \neq F$. Since a, b are adjacent, we may assume by **(SD3)** that $h_1 \in \overline{F} \cap \overline{F_1}$ and $a \in \eta(F, h_1)$ and $b \in \eta(F_1, h_1)$. Neither of c, d belongs to $\eta(F, h_1)$ by **(SD2)** since they are antiadjacent to b; and if they are both in $\eta(F) \setminus \eta(F, h_1)$ then $\{a, v_1, c, d\}$ is a claw of J, so we may assume that $c \notin \eta(F)$. Choose $F_2 \in E(H)$ with $c \in \eta(F_2)$. Then $F_2 \neq F$, and there exists $h \in \overline{F} \cap \overline{F_2}$ with $a \in \eta(F, h)$ and $c \in \eta(F_2, h)$. Since b, c are antiadjacent, it follows that $c \notin \eta(F_1, h_1)$, and so $h \neq h_1$ and we may assume that $h = h_2$. If $d \in \eta(F) \setminus (\eta(F, h_1) \cup \eta(F, h_2))$, then $\{a, v_1, v_2, d\}$ is a claw in J, so we assume not; and certainly $d \notin \eta(F, h_1) \cup \eta(F, h_2)$ since it is antiadjacent to b, c. Hence $d \notin \eta(F)$, and we deduce that there exists $F_3 \in E(H) \setminus \{F\}$ and $h_3 \in \overline{F} \cap \overline{F_3}$ with $a \in \eta(F, h_3)$ and $d \in \eta(F_3, h_3)$. Also by the same argument, $h_3 \neq h_1, h_2$; but then $\{a, v_1, v_2, v_3\}$ is a claw in J. This proves 7.1. ■

We say a strip-structure (H, η) is *nontrivial* if $|E(H)| \geq 2$. Now we can at least state the counterpart of 4.1, which together with 4.1 has been the goal of this series of papers, although we are not ready to prove it yet.

7.2 *Let G be a connected claw-free trigraph, such that $V(G)$ is not the union of three strong cliques. Then either*

- *G is a thickening of a member of $\mathcal{S}_1 \cup \mathcal{S}_3 \cup \mathcal{S}_7$, or*
- *G admits a nontrivial strip-structure such that for each strip (J, Z), $1 \leq |Z| \leq 2$, and either*

– $|V(J)| = 3$ *and* $|Z| = 2$, *or*
– (J, Z) *is a thickening of a member of* $\mathcal{Z}_0$.

8 Optimal strip-structures

We have not yet explained condition **(SD4)**; let us do so now. Let G be a trigraph, let $Y \subseteq V(G)$, and let X_i $(1 \leq i \leq k)$ be a family of subsets of Y. We say that this family is a *circus* in Y if

(CS1) For $1 \leq i \leq k$ and all $x \in X_i$, the set of all neighbours of x in $Y \setminus X_i$ is a strong clique.

(CS2) For $1 \leq i < j \leq k$, $X_i \cap X_j$ is strongly anticomplete to $Y \setminus (X_i \cup X_j)$.

(CS3) For $1 \leq h < i < j \leq k$, $X_h \cap X_i \cap X_j = \emptyset$.

It is easy to see that condition **(SD4)** is equivalent (assuming that G is claw-free) to assuming that all the strips are claw-free; and since in the statement of 7.2, all the strips are claw-free anyway, deleting **(SD4)** from the definition of a strip-structure would have no effect on the meaning of 7.2. The reason for retaining **(SD4)** is to facilitate the proof of 7.2; because our proof method is to choose a nontrivial strip-structure (H, η) (with all strips claw-free), that cannot be "refined" any further, and prove that it has the properties we require. And the reason for using "circus" instead of just saying that the strips are claw-free is because we thereby avoid having to refer to the extra vertices $v_1, \dots, v_k$.

If (H, η) is a strip-structure, its *nullity* is the number of pairs (F, h) with $F \in E(H)$ and $h \in \overline{F}$ such that $\eta(F, h) = \emptyset$. For a fixed trigraph G, a strip-structure (H, η) of G is said to be *optimal* if there is no strip-structure (H', η') of G with $|E(H')| \geq |E(H)|$, such that either $|E(H')| > |E(H)|$, or the nullity of (H', η') is strictly smaller than that of (H, η). Every strip-structure (H, η) of G satisfies $|E(H)| \leq |V(G)|$ (since the sets $\eta(F)$ $(F \in E(H))$ are nonempty and pairwise disjoint), and every trigraph admits a strip-structure (H, η) where H is a hypergraph with $|V(H)| = 0$ and $|E(H)| = 1$, and so every trigraph admits an optimal strip-structure. As we shall see, optimal strip-structures have a number of attractive properties.

If a strip-structure (H, η) also satisfies the following condition, we call the strip-structure *purified*:

- for each $F \in E(H)$, either all the sets $\eta(F, h)$ $(h \in \overline{F})$ are pairwise disjoint, or $|\overline{F}| = 2$ and $|\eta(F)| = 1$ and $\eta(F, h_1) = \eta(F, h_2) = \eta(F)$ where $\overline{F} = \{h_1, h_2\}$.

In other words, all the corresponding strips are either stripes or spots, where a *spot* is a pair (J, Z) such that J has three vertices say v, z_1, z_2, and v is strongly adjacent to z_1, z_2, and z_1, z_2 are strongly antiadjacent, and $Z = \{z_1, z_2\}$.

8.1 *Let G be a claw-free trigraph. Then every optimal strip-structure of G has nullity zero and is purified.*

Proof. Let (H, η) be an optimal strip-structure of G. Let $F \in E(H)$, and let $\overline{F} = \{h_1, \dots, h_k\}$. Suppose first that $\eta(F, h_1) = \emptyset$. Let H' be the hypergraph obtained from H by making F not incident with h_1, and leaving the incidence of all other pairs unchanged. For each $F_0 \in E(H)$ let $\eta'(F_0) = \eta(F_0)$, and for each h incident with F_0 in H', let $\eta'(F_0, h) = \eta(F_0, h)$. Then (H', η') is a strip-structure of G, and $E(H') = E(H)$, and its nullity is smaller than that of (H, η), contrary to the optimality of (H, η). This proves that (H, η) has nullity zero.

Now again let $F \in E(H)$ with $\overline{F} = \{h_1, \ldots, h_k\}$, and suppose that $k \geq 2$ and $\eta(F, h_1) \cap \eta(F, h_2) \neq \emptyset$. Write $W = \eta(F, h_1) \cap \eta(F, h_2)$. Define (H', η') as follows. Let $F' \notin V(H) \cup E(H)$ be a new element. Then

- $V(H') = V(H)$, and $E(H') = E(H) \cup \{F'\}$
- for each $F_0 \in E(H)$ and $h \in V(H)$, F_0 is incident with h in H' if and only if they are incident in H;
- F' is incident in H' with h_1, h_2 and with no other member of $V(H)$;
- for each $F_0 \in E(H) \setminus \{F\}$, $\eta'(F_0) = \eta(F_0)$ and $\eta'(F_0, h) = \eta(F_0, h)$ for all $h \in \overline{F_0}$;
- $\eta'(F) = \eta(F) \setminus W$;
- for $i = 1, 2$, $\eta'(F, h_i) = \eta(F, h_i) \setminus W$, and for $3 \leq i \leq k$, $\eta'(F, h_i) = \eta(F, h_i)$;
- $\eta'(F') = \eta'(F', h_1) = \eta'(F', h_2) = W$.

We see that for $3 \leq i \leq k$, $\eta'(F, h_i) \subseteq \eta'(F)$, since no vertex belongs both to W and to some $\eta(F, h_i)$ with $i \geq 3$ from **(CS3)**. Also, from **(CS2)**, W is strongly anticomplete to $\eta(F) \setminus (\eta(F, h_1) \cup \eta(F, h_2))$, and so (H', η') satisfies **(SD3)**. But from the optimality of (H, η), (H', η') cannot be a strip-structure of G, and therefore does not satisfy **(SD1)**; and so $\eta'(F) = \emptyset$, that is, $\eta(F, h_1) = \eta(F, h_2) = \eta(F)$, and in particular, $\eta(F)$ is a strong clique. Since (H, η) has zero nullity, it follows that $k = 2$.

Suppose that $|\eta(F)| > 1$, and choose a partition (X_1, X_2) of $\eta(F)$ with $X_1, X_2 \neq \emptyset$. Define (H', η') as follows. Let $F' \notin V(H) \cup E(H)$ be a new element. Then

- $V(H') = V(H)$, and $E(H') = E(H) \cup \{F'\}$
- for each $F_0 \in E(H)$ and $h \in V(H)$, F_0 is incident with h in H' if and only if they are incident in H;
- F' is incident in H' with h_1, h_2 and with no other member of $V(H)$;
- for each $F_0 \in E(H) \setminus \{F\}$, $\eta'(F_0) = \eta(F_0)$ and $\eta'(F_0, h) = \eta(F_0, h)$ for all $h \in \overline{F_0}$;
- $\eta'(F) = \eta'(F, h_1) = \eta'(F, h_2) = X_1$ and $\eta'(F') = \eta'(F', h_1) = \eta'(F', h_2) = X_2$.

Then (H', η') is a strip-structure of G, contrary to the optimality of (H, η). Hence $|\eta(F)| = 1$, and therefore (H, η) is purified. This proves 8.1. ■

9 Unbreakable stripes

Let us say a trigraph G admits a *pseudo*-1-*join* if there is a partition V_1, V_2 of $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- for $i = 1, 2$, V_1, V_2 are not strongly stable

- $A_1 \cup A_2$ is a strong clique, and
- $V_1 \setminus A_1$ is strongly anticomplete to V_2, and V_1 is strongly anticomplete to $V_2 \setminus A_2$.

It is easy to check that if G admits a 1-join and no 0-join, then it admits a pseudo-1-join, and the converse is false.

Let us say that G admits a *pseudo-2-join* if there is a partition V_0, V_1, V_2 of $V(G)$ (where V_0 may be empty), and for $i = 1,2$ there are disjoint subsets A_i, B_i of V_i satisfying the following:

- $V_0 \cup A_1 \cup A_2$ and $V_0 \cup B_1 \cup B_2$ are strong cliques, and V_0 is strongly anticomplete to $V_i \setminus (A_i \cup B_i)$ for $i = 1,2$;
- for $i = 1,2$, V_i is not strongly stable; and
- for all $v_1 \in V_1$ and $v_2 \in V_2$, either v_1 is strongly antiadjacent to v_2, or $v_1 \in A_1$ and $v_2 \in A_2$, or $v_1 \in B_1$ and $v_2 \in B_2$.

Again, if G admits a 2-join or generalized 2-join then it admits a pseudo-2-join, and the converse is not true.

Finally, let us say G admits a *biclique* if there is a partition V_1, V_2, V_3, V_4 of $V(G)$, and

- $V_1 \neq \emptyset$, and $V_1 \cup V_2, V_1 \cup V_3$ are strong cliques
- V_1 is strongly anticomplete to V_4
- either $|V_1| \geq 2$, or $V_2 \cup V_3$ is not a strong clique
- $V_2 \cup V_3 \cup V_4$ is not strongly stable
- if $v_2 \in V_2$ and $v_3 \in V_3$ are adjacent then they have the same neighbours in V_4 and neither of them is semiadjacent to any member of V_4.

It is convenient to say that a stripe (J, Z) is a *clique* if $V(J)$ is a strong clique. A stripe (J, Z) is said to be *unbreakable* if

- J does not admit a 0-join, a pseudo-1-join, a pseudo-2-join or a biclique,
- there are no twins $u, v \in V(J) \setminus Z$,
- there is no W-join (A, B) in J such that $Z \cap A, Z \cap B = \emptyset$, and
- Z is the set of all vertices that are simplicial in J.

The reason for interest in this concept is the following.

9.1 *Every claw-free trigraph admits a strip-structure such that all its strips are either spots or cliques or thickenings of unbreakable stripes.*

Proof. Let G be a claw-free trigraph, and let (H, η) be an optimal strip-structure. Let $F \in E(H)$, and let (J, Z) be the corresponding strip. We claim that either (J, Z) is a spot or a clique or a thickening of an unbreakable stripe. We may assume that (J, Z) is not a spot; so by 8.1 (J, Z) is a stripe; and we may assume it is not a clique. Let $\overline{F} = \{h_1, \ldots, h_k\}$, and let $Z = \{v_1, \ldots, v_k\}$ as usual.

(1) *J does not admit a 0-join.*

For suppose that J admits a 0-join (V_1, V_2). For $j = 1, 2$, let $Q_j = Z \cap V_j$, and

$$P_j = \{h_i : 1 \leq i \leq k \text{ and } v_i \in V_j\}.$$

Since (V_1, V_2) is a 0-join, it follows that $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = \{h_1, \ldots, h_k\}$. If $V_1 = Q_1$, then since $V_1 \neq \emptyset$, we may assume that $v_1 \in Q_1$; and since (H, η) has nullity zero, v_1 has a neighbour in J, which is necessarily in V_1 and not in Q_1, a contradiction. Thus $V_1 \setminus Q_1 \neq \emptyset$, and similarly $V_2 \setminus Q_2 \neq \emptyset$.

Let (H', η') be defined as follows. Let F'_1, F'_2 be new elements not in $V(H) \cup E(H)$. Then

- $V(H') = V(H)$ and $E(H') = (E(H) \setminus \{F\}) \cup \{F'_1, F'_2\}$;
- for each $F_0 \in E(H) \setminus \{F\}$ and $h \in V(H)$, F_0 is incident with h in H' if and only if they are incident in H;
- for $j = 1, 2$ and $h \in V(H)$, F'_j is incident with h in H' if and only if $h \in P_j$;
- for all $F_0 \in E(H) \setminus \{F\}$, $\eta'(F_0) = \eta(F_0)$ and $\eta'(F_0, h) = \eta(F_0, h)$ for all $h \in \overline{F_0}$;
- for $j = 1, 2$, $\eta'(F'_j) = V_j \setminus Q_j$; and
- for $j = 1, 2$ and $h \in P_j$, $\eta'(F'_j, h) = \eta(F, h)$.

Then (H', η') is a strip-structure of G, contradicting the optimality of (H, η). This proves (1).

(2) *J does not admit a pseudo-1-join.*

For suppose that it does; then there is a partition (V_1, V_2) of $V(J)$, and for $i = 1, 2$, a subset $A_i \subseteq V_i$, such that:

- for $i = 1, 2$, V_i is not strongly stable
- $A_1 \cup A_2$ is a strong clique, and
- $V_1 \setminus A_1$ is strongly anticomplete to V_2, and V_1 is strongly anticomplete to $V_2 \setminus A_2$.

Since J does not admit a 0-join, it follows that $A_i \neq \emptyset$ for $i = 1, 2$. For $j = 1, 2$, since V_j is not strongly stable, there is a vertex of V_j that is not in Z. Suppose first that none of $v_1, \ldots, v_k$ belong to $A_1 \cup A_2$. For $j = 1, 2$, let $Q_j = V_j \cap Z$, and let

$$P_j = \{h_i : 1 \leq i \leq k \text{ and } v_i \in V_j\}.$$

It follows that $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = \{h_1, \ldots, h_k\}$, and $V_j \setminus Q_j \neq \emptyset$ for $j = 1, 2$, as we already saw. Let (H', η') be defined as follows. Let F'_1, F'_2, h' be new elements not in $V(H) \cup E(H)$. Then

- $V(H') = V(H) \cup \{h'\}$ and $E(H') = (E(H) \setminus \{F\}) \cup \{F'_1, F'_2\}$;
- for each $F_0 \in E(H) \setminus \{F\}$ and $h \in V(H)$, F_0 is incident with h in H' if and only if they are incident in H;
- no member of $E(H)$ is incident with h' in H';
- for $j = 1, 2$ and $h \in V(H)$, F'_j is incident with h in H' if and only if $h \in P_j$;
- for $j = 1, 2$, F'_j is incident with h';
- for all $F_0 \in E(H) \setminus \{F\}$, $\eta'(F_0) = \eta(F_0)$ and $\eta'(F_0, h) = \eta(F_0, h)$ for all $h \in \overline{F_0}$;
- for $j = 1, 2$, $\eta'(F'_j) = V_j \setminus Q_j$;
- for $j = 1, 2$ and $h \in P_j$, $\eta'(F'_j, h) = \eta(F, h)$; and
- for $j = 1, 2$, $\eta'(F'_j, h') = A_j$.

We claim that (H', η') is a strip-structure of G. Certainly for each $h_i \in P_j$, $\eta'(F'_j, h_i) = \eta(F, h_i) \subseteq V_j \setminus Q_j = \eta'(F'_j)$ since $v_i \notin A_j$. Also, **(SD1)**,**(SD2)**,**(SD3)** are clear (note that since (V_1, V_2) is a pseudo-1-join, **(SD2)** and **(SD3)** are satisfied for pairs of vertices both in $\eta(F)$); let us check **(SD4)**. It suffices to check that the family $\eta'(F'_1, h)$ $(h \in P_1 \cup \{h'\})$ is a circus in $\eta'(F'_1)$. Choose $a_2 \in A_2$; then a_2 is strongly anticomplete to $\eta'(F'_1) \setminus \eta'(F'_1, h')$. Let $v \in \eta'(F'_1, h')$; since $\eta(F, h)$ $(h \in Z)$ is a circus in $\eta(F)$, it suffices to check that

- the set of neighbours of v in $\eta'(F'_1) \setminus \eta'(F'_1, h')$ is a strong clique
- if $v \in \eta'(F'_1, h)$ for some $h \in P_1$, then every neighbour of v in $\eta'(F'_1)$ belongs to $\eta'(F'_1, h) \cup \eta'(F'_1, h')$
- v belongs to $\eta'(F'_1, h)$ for at most one $h \in P_1$.

The first assertion holds since v is adjacent to a_2, and G is claw-free. The second holds since v is adjacent to a_2 and the family $\eta(F, h) :$ $(h \in \overline{F})$ satisfies **(CS1)**; and the third holds since (H, η) is purified. Consequently, (H', η') is a strip-structure of G, contradicting the optimality of (H, η).

Next, suppose that one of $v_1, \ldots, v_k$ belong to $A_1 \cup A_2$, $v_1 \in A_1$ say. Since $A_1 \cup A_2$ is a strong clique, and $v_1, \ldots, v_k$ are pairwise antiadjacent and no two of them have a common neighbour (since (H, η) is purified), it follows that $v_2, \ldots, v_k \notin A_1 \cup A_2$ and are strongly anticomplete to $A_1 \cup A_2$. Also, v_1 is strongly anticomplete to $(V_1 \cup V_2) \setminus (A_1 \cup A_2)$, since every vertex in this set has an antineighbour in $A_1 \cup A_2$ and v_1 is simplicial. For $j = 1, 2$, let $Q_j = Z \cap V_j$ and let

$$P_j = \{h_i :\ 2 \leq i \leq k \text{ and } v_i \in V_j\}.$$

Then $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = \{h_2, \ldots, h_k\}$. As we already saw, $V_1 \setminus (Q_1 \cup \{v_1\}), V_2 \setminus Q_2 \neq \emptyset$. Define (H', η') as follows. Let F'_1, F'_2 be two new elements not in $V(H) \cup E(H)$. Then

- $V(H') = V(H)$ and $E(H') = (E(H) \setminus \{F\}) \cup \{F'_1, F'_2\}$;
- for each $F_0 \in E(H) \setminus \{F\}$ and $h \in V(H)$, F_0 is incident with h in H' if and only if they are incident in H;

- for $j = 1,2$ and $h \in V(H)$, F'_j is incident with h in H' if and only if $h \in P_j \cup \{h_1\}$;
- for all $F_0 \in E(H) \setminus \{F\}$, $\eta'(F_0) = \eta(F_0)$ and $\eta'(F_0, h) = \eta(F_0, h)$ for all $h \in \overline{F_0}$;
- $\eta'(F'_1) = V_1 \setminus (Q_1 \cup \{v_1\})$ and $\eta'(F'_2) = V_2 \setminus Q_2$;
- for $j = 1,2$ and $h \in P_j$, $\eta'(F'_j, h) = \eta(F, h)$; and
- $\eta'(F'_1, h_1) = A_1 \setminus \{v_1\}$ and $\eta'(F'_2, h_1) = A_2$.

Note that $A_1 \setminus \{v_1\}, A_2 \neq \emptyset$ since J does not admit a 0-join; so we can prove as before that (H', η') is a strip-structure of G, contradicting the optimality of (H, η). This proves (2).

(3) *J does not admit a pseudo-2-join.*

For suppose that it does. The previous case divided into two subcases depending whether one of $v_1, \dots, v_k$ belongs to $A_1 \cup A_2$ or not, and the same happens for pseudo-2-joins, except there are more subcases. Let us therefore handle them all simultaneously. We have three sets V_0, V_1, V_2 as in the definition of a pseudo-2-join, where V_0 may be empty. Also, for $i = 1,2$ we will have subsets A_i, B_i of V_i as in the definition of pseudo-2-join. We may assume that J does not admit a pseudo-1-join, and so A_1, A_2, B_1, B_2 are all nonempty. Possibly one vertex in $A_1 \cup A_2$ belongs to Z (at most one since $\{v_1, \dots, v_k\}$ is stable), and if say $v_1 \in Z \cap (A_1 \cup A_2)$ then v_1 has no neighbours in $(V_1 \cup V_2) \setminus (A_1 \cup A_2)$ since A_1, A_2 are nonempty and v_1 is simplicial. The same applies to B_1, B_2. Also none of $v_1, \dots, v_k$ belong to V_0 since V_0 is complete to A_1 and to B_2, and A_1 is not complete to B_2. For $i = 1,2$, since V_i is not strongly stable, at least one vertex of V_i does not belong to $\{v_1, \dots, v_k\}$.

For $i = 1,2$, let $Q_i = Z \cap (V_i \setminus (A_i \cup B_i))$; and let $Q_3 = Z \cap (A_1 \cup A_2)$ and $Q_4 = Z \cap (B_1 \cup B_2)$. For $i = 1,2$, let $C_i = A_i \setminus Q_3$, let $D_i = B_i \setminus Q_4$, and let $W_i = V_i \setminus Z$. In summary, then, we have a partition V_0, W_1, W_2, Q_3, Q_4 of $V(J)$, and for $i = 1,2$ there are subsets C_i, D_i, Q_i of W_i satisfying the following:

- $Q_1 \cup Q_2 \cup Q_3 \cup Q_4 = Z$, and $|Q_3|, |Q_4| \leq 1$
- $V_0 \cup C_1 \cup C_2 \cup Q_3$ and $V_0 \cup D_1 \cup D_2 \cup Q_4$ are strong cliques, and V_0 is strongly anticomplete to $W_i \setminus (C_i \cup D_i)$ for $i = 1,2$;
- for $i = 1,2$, the sets C_i, D_i, Q_i are pairwise disjoint, and $W_i \setminus Q_i \neq \emptyset$;
- for all $w_1 \in W_1$ and $w_2 \in W_2$, either w_1 is strongly antiadjacent to w_2, or $w_1 \in C_1$ and $w_2 \in C_2$, or $w_1 \in D_1$ and $w_2 \in D_2$;
- Q_3 is strongly anticomplete to $W_1 \cup W_2 \setminus (C_1 \cup C_2)$, and Q_4 is strongly anticomplete to $W_1 \cup W_2 \setminus (D_1 \cup D_2)$.

Since J does not admit a pseudo-1-join, it follows that C_1, C_2, D_1, D_2 are all nonempty. Define (H', η') as follows. Let F'_1, F'_2 be two new elements. If $Q_3 = \emptyset$ let h'_1 be another new element, and otherwise let $h'_1 = h_i$ where $Q_3 = \{v_i\}$. If $Q_4 = \emptyset$ let h'_2 be another new element, and otherwise let $h'_2 = h_i$ where $Q_4 = \{v_i\}$. If $V_0 \neq \emptyset$ let F'_3 be another new element. Then

- $V(H') = V(H) \cup \{h'_1, h'_2\}$, and $E(H') = (E(H) \setminus \{F\}) \cup \{F'_1, F'_2, F'_3\}$ if F'_3 exists, and otherwise $E(H') = (E(H) \setminus \{F\}) \cup \{F'_1, F'_2\}$;
- for $F_0 \in E(H) \setminus \{F\}$ and each $h \in V(H)$, F_0 is incident with h in H' if and only if they are incident in H;
- for $F_0 \in E(H) \setminus \{F\}$, F_0 is not incident with any of h'_1, h'_2 that are not in $V(H)$;
- for $j = 1, 2$, F'_j is incident with $h \in V(H')$ if and only if $h \in Q_j \cup \{h'_1, h'_2\}$;
- if F'_3 exists, F'_3 is incident with h'_1, h'_2 and with no other member of $V(H')$;
- for all $F_0 \in E(H) \setminus \{F\}$, $\eta'(F_0) = \eta(F_0)$ and $\eta'(F_0, h) = \eta(F_0, h)$ for all $h \in \overline{F_0}$;
- for $j = 1, 2$, $\eta'(F'_j) = W_j \setminus Q_j$;
- for $j = 1, 2$ and $h \in P_j$, $\eta'(F'_j, h) = \eta(F, h)$;
- for $j = 1, 2$, $\eta'(F'_j, h'_1) = C_j$, and $\eta'(F'_j, h'_2) = D_j$; and
- if F'_3 exists, $\eta'(F'_3) = \eta'(F'_3, h'_1) = \eta'(F'_3, h'_2) = V_0$.

Since C_1, C_2, D_1, D_2 are all nonempty, we can prove as before that (H', η') is a strip-structure of G, contradicting the maximality of (H, η). This proves (3).

(4) *J does not admit a biclique.*

For suppose it does; then there is a partition V_1, V_2, V_3, V_4 of $V(J)$, such that

- $V_1 \neq \emptyset$, and $V_1 \cup V_2, V_1 \cup V_3$ are strong cliques
- V_1 is strongly anticomplete to V_4
- either $|V_1| \geq 2$, or $V_2 \cup V_3$ is not a strong clique
- $V_2 \cup V_3 \cup V_4$ is not strongly stable
- if $v_2 \in V_2$ and $v_3 \in V_3$ are adjacent then they have the same neighbours in V_4 and the same antineighbours in V_4.

Choose such $V_1, \ldots, V_4$ with V_1 maximal. Suppose first that $V_2 \cup V_3$ is a strong clique. Then $|V_1| \geq 2$ by hypothesis, and so J admits a pseudo-1-join, since V_1 and $V_2 \cup V_3 \cup V_4$ are both not strongly stable, a contradiction. Thus $V_2 \cup V_3$ is not a strong clique, and in particular, V_2, V_3 are both nonempty.

For $i = 1, \ldots, 4$, let $Q_i = V_i \cap Z$. Hence $Q_1 = \emptyset$. Morover, not both Q_2, Q_3 are nonempty, since (H, η) is purified; and so we may assume that $Q_3 = \emptyset$. Also $|Q_2| \leq 1$ since Z is stable. Define (H', η') as follows. Let F', h'_1 be new elements. If $Q_2 = \emptyset$ let h'_2 be another new element, and otherwise let $h'_2 = h_i$ where $Q_2 = \{v_i\}$. Then

- $E(H') = E(H) \cup \{F'\}$ and $V(H') = V(H) \cup \{h'_1, h'_2\}$;
- for $F_0 \in E(H)$ and each $h \in V(H)$, F_0 is incident with h in H' if and only if they are incident in H;

- for $F_0 \in E(H) \setminus \{F\}$, F_0 is not incident with h'_1, and not with h'_2 if $h'_2 \notin V(H)$;
- F is incident with h'_1, h'_2; and F' is incident with h'_1, h'_2 and with no other member of $V(H')$;
- for all $F_0 \in E(H) \setminus \{F\}$, $\eta'(F_0) = \eta(F_0)$ and $\eta'(F_0, h) = \eta(F_0, h)$ for all $h \in \overline{F_0}$;
- $\eta'(F) = (V_2 \cup V_3 \cup V_4) \setminus (Q_2 \cup Q_4)$, and $\eta(F') = V_1$;
- $\eta'(F, h'_1) = V_3$, $\eta'(F, h'_2) = V_2 \setminus Q_2$, and for all other $h \in V(H')$ incident with F, $\eta'(F, h) = \eta(F, h)$;
- $\eta'(F', h'_1) = \eta'(F', h'_2) = V_1$.

We claim that (H', η') is a strip-structure of G, and since $\eta'(F') \neq \emptyset$ and $\eta'(F) \neq \emptyset$ (since $V_2 \cup V_3 \cup V_4$ is not strongly stable), it suffices to show that the family

$$(\eta'(F, h) : \ h \text{ is incident with } F \text{ in } H')$$

is a circus in $G|\eta'(F)$. Since $(\eta(F, h) : \ h \in \overline{F})$ is a circus in $\eta(F)$ (where $\overline{F}$ denotes the set of vertices of H incident with F in H), and $\eta'(F, h'_1) \cap \eta'(F, h'_2) = \emptyset$, it suffices to check that:

- for $j = 1, 2$ and $x \in \eta'(F, h'_j)$, the neighbours of x in $\eta'(F) \setminus \eta'(F, h'_j)$ are a strong clique
- for $j = 1, 2$ and $h \in \overline{F} \setminus \{h'_1, h'_2\}$, if $x \in \eta'(F, h) \cap \eta'(F, h'_j)$ then x is strongly anticomplete to $\eta'(F) \setminus (\eta'(F, h) \cup \eta'(F, h'_j))$
- for $j = 1, 2$, if $x \in \eta'(F, h'_j)$ then x belongs to $\eta'(F, h)$ for at most one $h \in \overline{F} \setminus \{h'_1, h'_2\}$.

The third assertion is clear since (H, η) is purified. For the first assertion, let $x \in \eta'(F, h'_j)$ where $j \in \{1, 2\}$. Then x has a neighbour in V_1, and this neighbour is anticomplete to V_4, and so the set of neighbours of x in V_4 is a strong clique. Let $\{j, k\} = \{1, 2\}$. If $y \in V_k$ is adjacent to x, then x, y have the same neighbours in V_4, from the definition of a biclique, and so the set of neighbours of x in $V_k \cup V_4$ is a strong clique. This proves the first assertion.

Now let us check the second assertion. First let $j = 1$; then $x \in V_3 \cap \eta(F, h)$ where $h \in \overline{F} \setminus \{h'_1, h'_2\}$; $h = h_1$ say. Then $v_1 \in V_4$, and x is adjacent in J to v_1, and v_1 is anticomplete to V_1. Since x has a neighbour in V_1, and its neighbours in $\eta(F) \setminus \eta(F, h)$ are a strong clique, it follows that every neighbour of x in V_4 is in $\eta(F, h)$. From the final condition in the definition of a biclique, all neighbours of x in V_2 are adjacent in J to v_1, and therefore belong to $\eta(F, h)$. Thus the second assertion holds when $j = 1$. Now let $j = 2$. If $h'_2 \notin V(H)$, then the same proof applies, exchanging h'_1, h'_2; so we assume that $h'_2 \in V(H)$. But then x is strongly anticomplete to $\eta(F) \setminus (\eta(F, h) \cup \eta(F, h'_2))$, since $(\eta(F, h) : \ h \in \overline{F})$ is a circus in $\eta(F)$, and again the second assertion holds. Hence (H', η') is a strip-structure of G, contradicting the maximality of (H, η). This proves (4).

(5) *Z is the set of all simplicial vertices of J.*

For suppose that some $v \in V(J) \setminus Z$ is simplicial. Suppose first that v is adjacent in J to one of $v_1, \ldots, v_k$, say to v_1. Since v_1, v are both simplicial in J, they are twins; let N be the set of vertices of J different from v, v_1 and adjacent to v. Then $\{v, v_1\}$ and N are strong cliques, and $\{v, v_1\}$ is strongly complete to N and strongly anticomplete to $V(J)\setminus(N\cup\{v,v_1\})$. Since $(\{v,v_1\},V(J)\setminus\{v,v_1\})$ is not a pseudo-1-join by (2), it follows that $V(J)\setminus\{v,v_1\}$ is strongly stable; in particular, $\{v,v_1\}\cup N = V(J)$, since any vertex not in $\{v,v_1\}\cup N$ would have no neighbours and J would admit a 0-join; and so $V(J)$ is a strong clique, a contradiction.

This proves that v is antiadjacent to $v_1,\dots,v_k$. Let N be the set of neighbours of v in J. Define (H',η') as follows. Let F',h' be new elements; then

- $V(H') = V(H)\cup\{h'\}$ and $E(H') = E(H)\cup\{F'\}$;
- for $F_0\in E(H)$ and each $h\in V(H)$, F_0 is incident with h in H' if and only if they are incident in H;
- F' is incident with h' and with no other vertex of H';
- for $F_0\in E(H)$, F_0 is incident with h' if and only if $F_0 = F$;
- for all $F_0\in E(H)\setminus\{F\}$, $\eta'(F_0)=\eta(F_0)$ and $\eta'(F_0,h)=\eta(F_0,h)$ for all $h\in\overline{F_0}$;
- $\eta'(F)=\eta(F)\setminus\{v\}$, and $\eta(F')=\{v\}$;
- $\eta'(F,h')=N$, and for all other $h\in V(H')$ incident with F, $\eta'(F,h)=\eta(F,h)$;
- $\eta'(F',h')=\{v\}$.

Note that $\eta(F)\neq\{v\}$ since $V(J)$ is not a strong clique, and so $\eta'(F)\neq\emptyset$. It follows that (H',η') is a strip-structure of G, contrary to the optimality of (H,η). This proves (5).

Choose a stripe (J',Z') with $|V(J')|$ minimum such that (J,Z) is a thickening of (J',Z'), and let X_v $(v\in V(J'))$ be the corresponding subsets. Hence no two vertices in $V(J')\setminus Z'$ are twins in J', and there is no W-join (A,B) in J' with $Z'\cap A, Z'\cap B=\emptyset$. Moreover, by (1) – (4) it follows that J' does not admit a 0-join, a pseudo-1-join, a pseudo-2-join, or a biclique, since J does not. If v is a simplicial vertex of J' with $v\notin Z'$, then every vertex in X_v is simplicial in J, contrary to (5). Thus Z' is the set of all simplicial vertices of J'. This proves that (J,Z) is a thickening of an unbreakable stripe, and so completes the proof of 9.1. ■

Thus, in order to prove 7.2 it would suffice to identify all the unbreakable stripes and check that they are consistent with 7.2, and that is the goal of the remainder of the paper.

10 Preliminaries on unbreakable stripes

We begin with some useful lemmas about unbreakable stripes. First, we have

10.1 *If (G,Z) is an unbreakable stripe, then G does not admit twins.*

Proof. Suppose that u,v are twins of G; then one of $u,v\in Z$ since (G,Z) is unbreakable. Let $u\in Z$ say; thus u is simplicial. Since u,v are twins it follows that v is simplicial, and therefore $v\in Z$ since Z is the set of all simplicial vertices of G. But then Z is not stable, a contradiction. This proves 10.1. ■

Second, we observe:

10.2 *Let (G, Z) be an unbreakable stripe with $|V(G)| > 2$, such that $V(G)$ is the union of two strong cliques. Then $|V(G)| \leq 4$ and $(G, Z) \in \mathcal{Z}_1 \cup \mathcal{Z}_6$.*

Proof. By 10.1, G does not admit twins. Let A, B be strong cliques in G with $A \cup B = V(G)$, chosen with $A \cap B$ maximal. Then $A, B \neq V(G)$ since G does not admit twins. Since $(A, B \setminus A)$ is a homogeneous pair, and one of $|A|, |B \setminus A| > 1$, and (G, Z) is unbreakable, it follows that $Z \neq \emptyset$. Any two members of $A \cap B$ are twins, so $|A \cap B| \leq 1$. From the maximality of $A \cap B$, every member of $A \setminus B$ has an antineighbour in $B \setminus A$, and vice versa. Therefore no vertex in $A \cap B$ is simplicial, and so $Z \subseteq V(G) \setminus (A \cap B)$. Suppose that $A \cap B \neq \emptyset$. Then since the four sets $A \cap B, A \setminus B, B \setminus A, \emptyset$ do not form a biclique, it follows that $(A \setminus B) \cup (B \setminus A)$ is strongly stable. Since A is a strong clique, it follows that $|A \setminus B| = 1$, and similarly $|B \setminus A| = 1$. Then $|V(G)| = 3$, and $|Z| = 1$ (since no two members of Z have a common neighbour), and $(G, Z) \in \mathcal{Z}_6$. Thus we may assume that $A \cap B = \emptyset$. Let $a \in Z$, say $a \in A$. If a has a neighbour $b \in B$, then since a is simplicial, it follows that b is strongly complete to A, contrary to the maximality of $A \cap B$. So a is strongly anticomplete to B, and so $(A \setminus \{a\}, B)$ is a homogeneous pair. If $Z = \{a\}$ then since (G, Z) is unbreakable, it follows that $|A| \leq 2$ and $|B| = 1$ and again $(G, Z) \in \mathcal{Z}_6$. We may therefore assume that there exists $b \in Z$ with $b \neq a$. Since Z is strongly stable, it follows that $b \in B$, and as before b is strongly anticomplete to A, and $|Z| = 2$. Since $(A \setminus \{a\}, B \setminus \{b\})$ is a homogeneous pair, it follows that $|A|, |B| \leq 2$, and $(G, Z) \in \mathcal{Z}_1$. This proves 10.2. ■

And third, we have:

10.3 *Let (G, Z) be an unbreakable stripe with $|V(G)| > 2$. If G is a thickening of a line trigraph then $V(G)$ is the union of two strong cliques and therefore $(G, Z) \in \mathcal{Z}_1 \cup \mathcal{Z}_6$.*

Proof. Let G' be a line trigraph of some graph H say, and let G be a thickening of G'; let X_v ($v \in V(G')$ be the corresponding subsets of $V(G)$. Since G admits no 0-join, it follows that H is connected. Suppose first that some vertex h of H has degree two; let $e = hx$ and $f = hy$ be the two edges of H incident with h. In particular, if some two vertices of G' are semiadjacent, choose h such that e, f are semiadjacent in G'. Let P be the set of all edges of H incident with x and with neither of h, y, and similarly let Q be the set of all edges of H incident with y and with neither of h, x. Write $A_1 = X_e$, $B_1 = X_f$, $A_2 = \bigcup_{p \in P} X_p$, and $B_2 = \bigcup_{q \in Q} X_q$. Let $V_1 = A_1 \cup B_1$, and $V_0 = X_t$ if there is an edge t of H incident with x, y, and otherwise $V_0 = \emptyset$. Let $V_2 = V(G) \setminus (V_1 \cup V_0)$. We claim that $A_1 \cup A_2 \cup V_0$ is a strong clique. Suppose not; then x has degree two in H (from the definitions of a line trigraph and thickening), and e is semiadjacent in G' to some edge of H incident with x. Consequently e is not semiadjacent to f in G', since $F(G')$ is a matching, contrary to our choice of h. This proves that $A_1 \cup A_2 \cup V_0$ and similarly $B_1 \cup B_2 \cup V_0$ are strong cliques. Since e, f are both incident with x in H, it follows that e, f are adjacent in G', and therefore $A \cup B$ is not strongly stable in G. Since G does not admit a pseudo-2-join, it follows that V_2 is strongly stable; but then $V_2 = A_2 \cup B_2$, since G does not admit a 0-join, and therefore $V(G)$ is the union of two strong cliques and the result follows from 10.2.

We may assume therefore that no vertex of H has degree two; and so no pair of vertices of G' are semiadjacent. Consequently no pair of vertices of G are semiadjacent. If every edge of H is incident with a vertex of degree one, then since H is connected, it follows that some vertex of H is incident with every edge of H, and so G' and hence G is a strong clique and the result follows from 10.2.

Thus we may assume that there is an edge $e = uv$ of H such that u, v both have degree at least three in H. Let P, Q be the sets of edges of H incident with u and not v, and with v and not u, respectively. Let $V_1 = X_e$, $V_2 = \bigcup_{p \in P} X_p$, $V_3 = \bigcup_{q \in Q} X_q$, and $V_4 = V(G) \setminus (V_1 \cup V_2 \cup V_3)$. Then $V_1 \neq \emptyset$, and $V_1 \cup V_2, V_1 \cup V_3$ are strong cliques; V_1 is strongly anticomplete to V_4; since $|P| \geq 2$, V_2 is not strongly stable; if $v_2 \in V_2$ and $v_3 \in V_3$ are adjacent then they belong to sets X_p, X_q where $p \in P$ and $q \in Q$ share an end in H, and therefore v_2, v_3 have the same neighbours in V_4; and since there exist $p \in P$ and $q \in Q$ with no common end in H, it follows that $V_2 \cup V_3$ is not a strong clique. Consequently G admits a biclique, a contradiction. This proves 10.3. ■

We also need the following observation:

10.4 *For every claw-free trigraph G, if G does not admit a 0-join, 1-join or generalized 2-join, then either G is a thickening of an indecomposable member of $\mathcal{S}_0 \cup \cdots \cup \mathcal{S}_7$, or G admits a hex-join.*

Proof. We proceed by induction on $|V(G)|$. Suppose first that G contains twins u, v, and let $G' = G \setminus \{v\}$. Then G' is claw-free, and does not admit a 0-join, 1-join or generalized 2-join (for adding a twin to a trigraph that admits a 0-,1- or generalized 2-join produces a trigraph that still admits the same decomposition). From the inductive hypothesis, either G' is a thickening of a member of $\in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_7$ (and therefore so is G, by 6.1), or G' admits a hex-join (and therefore so does G). Therefore we may assume that G does not admit twins.

Suppose that G admits a W-join (A, B). Choose $a \in A$ and $b \in B$, and let G' be obtained from G by deleting $(A \setminus \{a\}) \cup (B \setminus \{b\})$ and making a, b semiadjacent. Then G' is claw-free and does not admit a 0-join, 1-join or generalized 2-join, (for if say G' admits a generalized 2-join (V_0, V_1, V_2), then since a, b are semiadjacent, they do not belong to V_0, and belong to the same one of V_1, V_2, and restoring A, B in place of a, b yields a generalized 2-join in G, a contradiction). From the inductive hypothesis, either G' is a thickening of a member of $\mathcal{S}_0 \cup \cdots \cup \mathcal{S}_7$ (and therefore so is G, by 6.1), or G' admits a hex-join (and therefore so does G). Consequently we may assume that G does not admit a W-join, and hence is indecomposable. But then the result holds by 5.1. This proves 10.4. ■

Our approach to 7.2 is via the following.

10.5 *Let (G, Z) be an unbreakable stripe. Then either*

- *$|V(G)| \leq 4$ and $V(G)$ is the union of two strong cliques, or*
- *G is a thickening of an indecomposable member of $\mathcal{S}_i$ for some $i \in \{1, \ldots, 7\}$, or*
- *G admits a hex-join.*

Proof. Choose a trigraph G' with $|V(G')|$ minimum such that G is a thickening of G'. Suppose first that G' is indecomposable. By 5.1, $G' \in \mathcal{S}_i$ for some $i \in \{0, \ldots, 7\}$. If $i \geq 1$ then the theorem holds, so we may assume that $i = 0$. By 10.2 and 10.3, $|V(G)| \leq 4$ and $V(G)$ is the union of two strong cliques, and the theorem holds.

Thus we may assume that G' is not indecomposable. Since (G, Z) is unbreakable, G does not admit a 0-join, 1-join or generalized 2-join, and hence neither does G'. From the minimality of $|V(G')|$ and 6.1, G' does not admit twins or a W-join. Thus G' admits a hex-join, and therefore so does G. This proves 10.5. ■

11 Simplicial vertices in indecomposable trigraphs

We need to study the simplicial vertices of indecomposable trigraphs in $\mathcal{S}_i$ where $1 \le i \le 7$; and it is convenient to study which vertices are "near-simplicial", at the same time. Let us say a vertex of a trigraph G is *near-simplicial* if v is semiadjacent to some vertex, and N^* is a strong clique, where N^* is the set of strong neighbours of v. The following answers the question above, except for $\mathcal{S}_3$, which will be handled in a different way later.

11.1 *Let $G \in \mathcal{S}_i$ for some $i \in \{1, 2, 4, 5, 6, 7\}$, and suppose that G is indecomposable, and $V(G)$ is not expressible as the union of two strong cliques. Let z be a simplicial or near-simplicial vertex of G.*

- *If z is simplicial, let Z be the set of all simplicial vertices of G; then $|Z| \le 2$ and $(G, Z) \in \mathcal{Z}_j$ for some $j \in \{2, 5, 7, 8, 9\}$.*
- *If $z \in V(G)$ is near-simplicial, semiadjacent to z' say, let $Z = \{z, z'\}$; then $(G', Z) \in \mathcal{Z}_2 \cup \mathcal{Z}_5$, where G' is the trigraph obtained from G by making z, z' strongly antiadjacent.*

Proof. First, let $v_0, \ldots, v_{11}$ and G_0, G_1, G_2 be as in the definition of $\mathcal{S}_1$. Then no vertex of G_0, G_1 is simplicial or near-simplicial. Moreover, each of $v_0, \ldots, v_9$ has two strong neighbours in G (and therefore in G_2) different from v_{10}, v_{11}, that are antiadjacent; and consequently no vertex of G_2 is simplicial or near-simplicial. Hence $G \notin \mathcal{S}_1$.

Next, suppose that $G \in \mathcal{S}_2$, and let $H, v_1, \ldots, v_{13}$ and $X \subseteq \{v_7, v_{11}, v_{12}, v_{13}\}$ be as in the definition of $\mathcal{S}_2$ (where $G = H \setminus X$). From the hole $v_1\text{-}\cdots\text{-}v_6\text{-}v_1$, it follows that no vertex is simplicial or near-simplicial except possibly v_7 or v_8. Thus $Z \subseteq \{v_7, v_8\}$. If z is simplicial then $(G, Z) \in \mathcal{Z}_5$, and if say $z = v_8 \in Z$ is near-simplicial then $z' = v_7 \notin X$ and v_7, v_8 are semiadjacent, and $(G', \{v_7, v_8\}) \in \mathcal{Z}_5$, where G' is obtained by making v_7, v_8 strongly antiadjacent.

Next suppose that $G \in \mathcal{S}_4$. Let $H, J(H)$ and $h_1, \ldots, h_7$ be as in the definition of $\mathcal{S}_4$, and let w be the vertex of G that is not an edge of H. The edges of the cycle $h_1\text{-}\cdots\text{-}h_5\text{-}h_1$ of H form a hole in G, and so w is not simplicial or near-simplicial in G, and nor is any edge of H with both ends in $\{h_1, \ldots, h_5\}$. The edge $h_i h_6$ (where $1 \le i \le 5$) is strongly adjacent in G to the edges $h_6 h_7$ and $h_i h_{i+1}$, and so is not simplicial or near-simplicial in G. Thus z is the edge $h_6 h_7$, and z is simplicial and $Z = \{z\}$, and $(G, Z) \in \mathcal{Z}_7$.

Next suppose that $G \in \mathcal{S}_5$. Let n, A, B, C, D, X etc. be as in the definition of $\mathcal{S}_5$. We may assume (by decreasing n) that X contains at most two of a_i, b_i, c_i for $1 \le i \le n$. Since d_1, d_3 are antiadjacent, it follows that no vertex in A is simplicial or near-simplicial, and similarly none in B. Since $A \setminus X$ is not strongly complete to $B \setminus X$, it follows that d_1, d_2 are not simplicial or near-simplicial; and since d_2, d_5 are antiadjacent it follows that d_3, d_4 are not simplicial or near-simplicial. Suppose that c_1 is simplicial or near-simplicial in G, and in particular $c_1 \notin X$. Since c_1 is strongly complete to $(A \setminus \{a_1\}) \cap X$ and to $(B \setminus \{b_1\}) \cap X$, and $A \setminus X$ is not complete to $B \setminus X$, we may assume that $a_1, b_2 \notin X$. Since c_2, b_2 are antiadjacent, $c_2 \in X$. If $n \ge 3$ then $a_3 \in X$ (since a_3, b_2 are antiadjacent), so $a_2 \notin X$ (since $|A \cap X| \le 1$), and hence $b_3 \in X$ (since b_3, a_2 are antiadjacent), and so $c_3 \notin X$ (since X contains at most two of a_3, b_3, c_3), and therefore $(\{d_2\}, \{d_1, c_3\})$ is a homogeneous pair, a contradiction. Thus $n = 2$ and $C \setminus X = \{c_1\}$. Since $C \setminus X$ is not strongly complete to $B \setminus X$, it follows that $b_1 \notin X$, and so c_1 is strongly antiadjacent to a_1, b_1. Hence c_1 is simplicial, and no vertex of G is near-simplicial, and therefore $Z = \{c_1, d_5\}$; but then $(G,Z) \in \mathcal{Z}_2$. Thus we may assume that no member of C is simplicial or near-simplicial, and so $Z = \{d_5\}$ and $(G,Z) \in \mathcal{Z}_8$.

Next suppose that $G \in \mathcal{S}_6$, and let H, a_0, b_0, A, B, C, X etc. be as in the definition of $\mathcal{S}_6$. Thus $G = H \setminus X$. Since $|C \setminus X| \geq 2$ and every member of $A \setminus \{a_0\}$ has at most one antineighbour in C, it follows that no member of $A \setminus \{a_0\}$ is simplicial or near-simplicial in G, and similarly for $B \setminus \{b_0\}$. Suppose that $c_1 \in C \setminus X$ is simplicial or near-simplicial in G. Since $|C \setminus X| \geq 2$, we may assume that $c_2 \in C \setminus X$. Since c_1 is strongly adjacent in H to c_2, a_2, b_2, and the strong neighbours of c_1 in G are a strong clique, it follows that $a_2, b_2 \in X$. Hence any two members of $C \setminus (X \cup \{c_1\})$ are twins in G, and so $|C \setminus X| = \{c_1, c_2\}$, since G is indecomposable. Let $A' = A \setminus (X \cup \{a_0, a_1\})$, and $B' = B \setminus (X \cup \{b_0, b_1\})$. Since G does not admits a 1-join, it follows that $A' \cup B' \neq \emptyset$. But

$$\{a_0, a_1\} \setminus X, \{b_0, b_1\} \setminus X, \{c_1, c_2\}, A', B', \emptyset$$

are pairwise disjoint strong cliques, with union $V(G)$, and since A' is strongly complete to $\{a_0, a_1\} \setminus X$ and to $\{c_1, c_2\}$, and strongly anticomplete to $\{b_0, b_1\} \setminus X$, and similarly for B', it follows that G is the hex-join of $G|(A' \cup B')$ and $G \setminus (A' \cup B')$, a contradiction. Thus no member of $C \setminus X$ is simplicial or near-simplicial in G, and so $z \in \{a_0, b_0\}$. If z is simplicial then a_0, b_0 are strongly antiadjacent, and $Z = \{a_0, b_0\}$, and $(G,Z) \in \mathcal{Z}_2$; and if z is near-simplicial, then a_0, b_0 are semiadjacent, and $(G', \{a_0, b_0\}) \in \mathcal{Z}_2$, where G' is obtained from G by making a_0, b_0 strongly antiadjacent.

Finally, suppose that $G \in \mathcal{S}_7$. Let $z \in Z$, and let D, D^* be the sets of neighbours and strong neighbours of z respectively; then D^* is a strong clique. Since G is antiprismatic, every vertex in $V(G) \setminus (D \cup \{z\})$ has at most one antineighbour in the same set; let $V(G) \setminus (D \cup \{z\}) = A \cup B \cup C$, where A, B, C are disjoint, $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, and for $1 \leq i \leq n$ a_i, b_i are antiadjacent, and otherwise every two members of $A \cup B \cup C$ are strongly adjacent. Suppose first that z is near-simplicial, semiadjacent to z' say; then $Z = \{z, z'\}$ and $D \setminus D^* = \{z'\}$. If $n > 0$ then $\{z, z', a_1, b_1\}$ is a claw; so $n = 0$. Let N be the set of all neighbours of z' different from z. Then the following sets are six strong cliques, pairwise disjoint and with union $V(G)$:

$$\{z\}, \{z'\}, C \setminus N, C \cap N, D \setminus N, D \cap N.$$

Moreover z is strongly complete to $D \setminus N, D \cap N$ and strongly anticomplete to $C \cap N$; z' is strongly complete to $C \cap N, D \cap N$ and strongly anticomplete to $D \setminus N$; and $C \setminus N$ is strongly complete to $C \cap N, D \setminus N$ (the latter since G is antiprismatic) and strongly anticomplete to $D \cap N$ (since $\{z, z'\} \cup (C \setminus N) \cup (D \cap N)$ includes no claw). Since G does not admit a hex-join, we deduce that $C \cap N, D \setminus N, D \cap N$ are all empty. Since G does not admit a 0-join, it follows that $C \setminus N = \emptyset$, and so $V(G) = \{z, z'\}$ and $V(G)$ is the union of two strong cliques, a contradiction. This proves that z is simplicial, and so $D = D^*$. Since $V(G)$ is not the union of the two strong cliques $C, D \cup \{v\}$, it follows that $n > 0$. For $1 \leq i \leq n$ and each $d \in D$, since $\{d, z, a_i, b_i\}$ is not a claw it follows that d is strongly antiadjacent to one of a_i, b_i, and strongly adjacent to the other and to z since G is antiprismatic. Thus $(G, \{z\}) \in \mathcal{Z}_9$. Every vertex in D is strongly adjacent to z and to one of a_1, b_1, and so is not simplicial or near-simplicial. No vertex in C is simplicial or near-simplicial since every such vertex is strongly adjacent to a_1, b_1 and they are antiadjacent. Suppose that say a_1 is simplicial or near-simplicial. Then $n = 1$, since if $n > 1$ then a_1 would be strongly adjacent to both a_2, b_2. Let P be the set of vertices in D adjacent to a_1, and $Q = D \setminus P$; then $D \neq \emptyset$ (since G does not admit a 0-join), and the six cliques

$$P, Q, C, \{b_1\}, \{a_1\}, \{z\}$$

show that G is expressible as a hex-join of $G|\{z, a_1, b_1\}$ and $G|(C \cup D)$, a contradiction. Thus no vertex except z is simplicial or near-simplicial, so $Z = \{z\}$ and $(G, Z) \in \mathcal{Z}_9$. This proves 11.1. ■

12 Unbreakable thickenings of basic trigraphs

Our current objective is to catalogue all the unbreakable stripes (G, Z) where G is a thickening of a member of one of $\mathcal{S}_1, \ldots, \mathcal{S}_7$. We begin with:

12.1 *Let (G, Z) be an unbreakable stripe, such that $Z \neq \emptyset$. If G is a thickening of a long circular interval trigraph then $(G, Z) \in \mathcal{Z}_1 \cup \mathcal{Z}_6$.*

Proof. Let G be a thickening of a long circular interval trigraph G', and let X_v ($v \in V(G')$) be the corresponding subsets of $V(G)$. By 10.1, G does not admit twins. By theorem 2.1 of [3], we may choose G' such that:

(1) *For every semiadjacent pair u, v of vertices of G', every vertex in X_u has a neighbour and an antineighbour in X_v and vice versa.*

Let $Z = \{z_1, \ldots, z_t\}$, and for $1 \leq i \leq t$ let $z'_i \in V(G')$ such that $z_i \in X_{z'_i}$. The vertices $z'_1, \ldots, z'_t$ are all distinct since Z is stable in G; let $Z' = \{z'_1, \ldots, z'_t\}$.

(2) $|X_v| = 1$ *for all $v \in V(G')$.*

For suppose that $|X_v| > 1$ for some $v \in V(G')$. Since G does not admit twins, it follows that some vertex in $V(G) \setminus X_v$ is neither strongly complete nor strongly anticomplete to X_v; and hence v is semiadjacent in G' to some $u \in V(G')$. If there exists $z \in Z \cap X_u$, then by (1) z has a neighbour $y \in X_v$, and by (1) again y has an antineighbour $x \in X_u$; but then x, y are both adjacent to z, and antiadjacent to each other, contradicting that z is simplicial. Thus $Z \cap X_u = \emptyset$, and similarly $Z \cap X_v = \emptyset$. But (X_u, X_v) is a W-join in G, contradicting that (G, Z) is unbreakable. This proves (2).

From (2), it follows that G is isomorphic to G', and in particular, G is a long circular interval trigraph; let Σ and $F_1, \ldots, F_k$ be as in the definition of $\mathcal{S}_3$. Let the vertices of G be $v_1, \ldots, v_n$ in cyclic order in Σ. We may assume that $|Z| \geq 2$, for otherwise $(G, Z) \in \mathcal{Z}_6$ and the theorem holds. Thus we may assume that $v_1, v_i \in Z$, where $2 \leq i \leq n$. Suppose first that v_1, v_n are antiadjacent, and therefore strongly antiadjacent (since $v_1 \in Z$). Then G is a linear interval trigraph with vertices $v_1, \ldots, v_n$ in order. If $|Z| = 2$ and $i = n$ then $(G, Z) \in \mathcal{Z}_1$, so we may assume that $i < n$. For $2 \leq j \leq n$, since G does not admit a 0-join it follows that $\{v_1, \ldots, v_{j-1}\}$ is not strongly anticomplete to $\{v_j, \ldots, v_n\}$, and so v_{j-1}, v_j are adjacent for $2 \leq j \leq n$. In particular v_1, v_2 are adjacent, and so $i \geq 3$. Choose h, j with $1 \leq h < i < j \leq n$ minimum and maximum such that v_i is adjacent to v_h, v_j. Thus $\{v_1, \ldots, v_{h-1}\}$ is strongly anticomplete to $\{v_i, \ldots, v_n\}$ (since v_{h-1} is strongly antiadjacent to v_i), and similarly $\{v_h, \ldots, v_{i-1}\}$ is strongly anticomplete to $\{v_{j+1}, \ldots, v_n\}$, and $\{v_h, \ldots, v_j\}$ is a strong clique (since v_i is simplicial). Hence G admits a pseudo-1-join, a contradiction. Thus if v_1, v_n are antiadjacent then the theorem holds. We may assume therefore that v_1, v_n are strongly adjacent, and similarly v_1, v_2 are strongly adjacent (and so $3 \leq i \leq n-1$), and v_i is strongly adjacent to v_{i-1}, v_{i+1}. Choose g with $1 < g < i$ maximum such that v_1, v_g are adjacent, and m with $i < m \leq n$ minimum such that v_1, v_m are adjacent. Choose h, j with $1 < h < i < j \leq n$ minimum and maximum such that v_i is adjacent to v_h, v_j. Since no vertex of G' is adjacent to both v_1, v_i, it follows that $g < h$ and similarly $j < m$. Let $V_1 = \{v_1, \ldots, v_{i-1}\}$ and $V_2 = \{v_i, \ldots, v_n\}$. Since v_1, v_2 are adjacent, V_1 is not strongly stable, and similarly V_2 is not strongly stable. For $x_1 \in V_1$ and $x_2 \in V_2$, x_1, x_2 are strongly adjacent if either $x_1 \in \{v_1, \ldots, v_g\}$ and $x_2 \in \{v_m, \ldots, v_n\}$, or $x_1 \in \{v_h, \ldots, v_{i-1}\}$ and $x_2 \in \{v_i, \ldots, v_j\}$, and otherwise x_1, x_2 are strongly antiadjacent. Hence G admits a pseudo-2-join, a contradiction. This proves 12.1. ■

Now we can prove the main result of this section, the following.

12.2 *Let (G, Z) be an unbreakable stripe with $Z \neq \emptyset$. Suppose that G is a thickening of an indecomposable member of $\mathcal{S}_i$, where $i \in \{1, \ldots, 7\}$. Then $(G, Z) \in \mathcal{Z}_0$.*

Proof. Let G be a thickening of G', where $G' \in \mathcal{S}_i$ is indecomposable, and let the corresponding subsets of $V(G)$ be X_v ($v \in V(G')$). If $V(G')$ is the union of two strong cliques, then the same is true for G and the theorem holds by 10.2. We assume that $V(G')$ is not so expressible. By 12.1, we may assume that $i \neq 3$.

(1) $|X_v| = 1$ *for all* $v \in V(G')$.

For suppose not; then since G does not admit twins, we may choose $u, v \in V(G')$, semiadjacent in G', such that (X_u, X_v) is a W-join in G'. Since (G, Z) is unbreakable we may assume that there exists $z \in Z \cap X_u$, and so u is near-simplicial in G'. Since G' is indecomposable, 11.1 implies that u, v are both near-simplicial in G', and no vertex of G' is adjacent to both u, v. Since setting $V_1 = X_u \cup X_v, V_0 = \emptyset$, and $V_2 = V(G) \setminus V_1$ does not define a pseudo-2-join in G, and V_1 is not strongly stable in G (from the definition of "thickening", since u, v are semiadjacent in G'), it follows that V_2 is strongly stable in G. Since G does not admit a 0-join, it follows that every vertex in V_2 has a neighbour in V_1. But then $V(G)$ is the union of two strong cliques, a contradiction. This proves (1).

From (1), G' is isomorphic to G, and so $G \in \mathcal{S}_i$. By 11.1 applied to (G, Z), it follows that (G, Z) belongs to $\mathcal{Z}_j$ for some $j \in \{2, 5, 7, 8, 9\}$. This proves 12.2. ■

13 Unbreakable stripes with hex-joins

It remains to catalogue the unbreakable stripes (G, Z) such that G admits a hex-join. We begin with:

13.1 *Let (G, Z) be an unbreakable stripe such that G admits a hex-join. Then $|Z| \leq 2$.*

Proof. Suppose that $|Z| \geq 3$, and let $z_1, z_2, z_3 \in Z$ be distinct. Let G be a hex-join of $G|V_1$ and $G|V_2$. From the symmetry we may assume that at least two of z_1, z_2, z_3 belong to V_1. Since $\{z_1, z_2, z_3\}$ is a triad and every triad is a subset of one of V_1, V_2, it follows that $z_3 \in V_1$. Let $v \in V_2$. Then v is antiadjacent to at least two of z_1, z_2, z_3, since (G, Z) is a stripe; and so v therefore belongs to a triad that contains two of z_1, z_2, z_3, and consequently contains a vertex of V_1. Hence this triad is not a subset of either of V_1, V_2, a contradiction. This proves 13.1. ■

13.2 *Let (G, Z) be an unbreakable stripe with $|Z| = 2$, such that $V(G)$ is the union of three strong cliques. Then $(G, Z) \in \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$.*

Proof. We may assume that $V(G)$ is not the union of two strong cliques, for otherwise the result follows from 10.2. ($(G, Z) \notin \mathcal{Z}_6$ since $|Z| = 2$.) Let $Z = \{z_1, z_2\}$.

(1) *The set of vertices in G antiadjacent to both z_1, z_2 is a strong clique of G.*

For there are three strong cliques with union $V(G)$, and since z_1, z_2 are antiadjacent, only one of these three cliques does not contain either of z_1, z_2; and this clique contains all vertices antiadjacent to them both. This proves (1).

(2) *There is no W-join (X, Y) in G with $z_1 \in X$ and $z_2 \in Y$.*

For suppose that (A_1, B_1) is a W-join with $z_1 \in A_1$ and $z_2 \in B_1$. Let $V_1 = A_1 \cup B_1$ and $V_2 = V(G) \setminus V_1$. Let A_2, B_2 be the sets of vertices in V_2 strongly complete to A_1 and strongly complete to B_1 respectively. Since no vertex is adjacent to both z_1, z_2, it follows that $A_2 \cap B_2 = \emptyset$. Every vertex in V_2 with a neighbour in A_1 belongs to A_2, and similarly for B_1; and $A_1 \cup A_2, B_1 \cup B_2$ are strong cliques since z_1, z_2 are simplicial. Since (A_1, B_1) is a W-join and G does not admit a pseudo-2-join, it follows that V_2 is strongly stable, and so $V_2 = A_2 \cup B_2$ since G admits no 0-join. But then $V(G)$ is the union of two strong cliques, a contradiction. This proves (2).

Let G' be the trigraph obtained from G by making z_1, z_2 semiadjacent. Since z_1, z_2 are simplicial in G, it follows that G' is claw-free. If (X, Y) is a W-join of G', then since z_1, z_2 are semiadjacent, it follows that $z_1 \in X$ if and only if $z_2 \in Y$, and so by (2), and since (G, Z) is unbreakable, G' does not admit a W-join. By theorem 11.1 of [4] applied to G', z_1, z_2, we deduce that one of the six outcomes of that theorem hold. The first is contrary to what we just showed; the second implies that $(G, Z) \in \mathcal{Z}_1$; the third contradicts 10.3; the fourth implies that $(G, Z) \in \mathcal{Z}_3$; since G' does not admit a generalized 2-join, the fifth implies that $(G, Z) \in \mathcal{Z}_4$; and the sixth implies that $(G, Z) \in \mathcal{Z}_2$. This proves 13.2. ■

13.3 *Let (G, Z) be an unbreakable stripe with $|Z| = 1$, such that $V(G)$ is the union of three strong cliques. Then $(G, Z) \in \mathcal{Z}_0$.*

Proof. We proceed by induction on $|V(G)|$. Let $Z = \{z\}$, and let N be the set of neighbours of Z. Thus, $N \cup \{z\}$ is a strong clique. Let $R = V(G) \setminus (N \cup \{z\})$. We may assume that R is not a strong clique, for otherwise the result follows from 10.2. Hence $|V(G)| \geq 4$.

(1) *Every vertex in N has a neighbour in R.*

For by 10.1, no vertex in N is a twin of z. This proves (1).

An *anticomponent* of R means a maximal subset of R that cannot be partitioned into two nonempty subsets that are strongly complete to each other. Let $D_1, \ldots, D_k$ be the anticomponents of R; then $D_1, \ldots, D_k$ are pairwise disjoint and have union R, and are all strongly complete to each other. Since R is not a strong clique, at least one of $D_1, \ldots, D_k$ has more than one vertex.

Now $V(G)$ is the union of three strong cliques. Since one of these cliques contains z and therefore is a subset of $N \cup \{z\}$, it follows that R can be partitioned into two strong cliques, say L, M. Thus $V(G)$ is partitioned into three strong cliques $L, M, N \cup \{z\}$.

Let T be the union of all triads of G, and $S = V(G) \setminus T$. Thus $z \in T$, and $T \cap R$ is the union of all the sets $D_1, \ldots, D_k$ that have more than one member.

(2) *If $n \in N \cap S$, then for all i with $1 \leq i \leq k$ such that $|D_i| > 1$, n is strongly complete to one of $L \cap D_i, M \cap D_i$ and strongly anticomplete to the other. Moreover, $R \setminus T$ is strongly anticomplete to $N \cap T$.*

For let $x, y \in D_i$ be antiadjacent. Now n is not adjacent to both x, y (since $\{n, z, x, y\}$ is not a claw) and n is not antiadjacent to both x, y (since $\{n, x, y\}$ is not a triad, because $n \in S$). Hence n is strongly adjacent to one of x, y and strongly antiadjacent to the other. In particular, if $x' \in D_i$ and x', y are antiadjacent, then n is either strongly complete or strongly anticomplete to $\{x, x'\}$. This proves the first assertion of (2). For the second, let $n \in N \cap T$ and $r \in R \setminus T$. Then n is in a triad with two members of $T \cap R$, and r is adjacent to these two members of $T \cap R$ and therefore strongly antiadjacent to n (since G is claw-free). This proves (2).

(3) *We may assume that $N \cap T \neq \emptyset$.*

For suppose not. Thus $N \subseteq S$, and by (2), for all i with $1 \leq i \leq k$ such that $|D_i| > 1$, every vertex in N is either strongly complete to $D_i \cap L$ and strongly anticomplete to $D_i \cap M$, or vice versa. Thus if $|D_i| > 1$ then $(L \cap D_i, M \cap D_i)$ is a homogeneous pair, and so $|L \cap D_i|, |M \cap D_i| = 1$ since (G, Z) is unbreakable. It follows that $(G, Z) \in \mathcal{Z}_9$, and the theorem holds. This proves (3).

(4) *There is exactly one value of i with $1 \leq i \leq k$ such that $|D_i| > 1$.*

For there is at least one such value, since R is not a strong clique; suppose there are two. By (3), there exists $n \in N$ with two antineighbours $x_1, y_1 \in R$, that are antiadjacent to each other. We may assume that $x_1, y_1 \in D_1$. Let $x_2 \in R \setminus D_1$; then x_2 is adjacent to x_1, y_1, and therefore strongly antiadjacent to n since $\{x_2, n, x_1, y_1\}$ is not a claw. So n is strongly anticomplete to $R \setminus D_1$. We may assume that $|D_2| > 1$, and so n has two antineighbours in D_2 that are antiadjacent; and so by exchanging D_1 and D_2, it follows that n is strongly anticomplete to $R \setminus D_2$, and therefore to R. But this is contrary to (1), and so proves (4).

In view of (4), we may assume that $|D_1| > 1$, and $|D_i| = 1$ for $2 \leq i \leq k$. Thus $T \cap R = D_1$.

(5) *$G|T$ does not admit twins. If (P, Q) is a W-join in $G|T$, then $z \in P \cup Q$, say $z \in P$. Moreover, $|P| = 2$, say $P = \{z, p\}$, and $|Q| = 1$, say $Q = \{q\}$; and $q \in R \cap T$ belongs to one of $L \cap T, M \cap T$ and is strongly anticomplete to the other; and q is the unique neighbour of p in R.*

For suppose that u, v are twins in $G|T$. If $u, v \in N \cup \{z\}$, then by (2) they are also twins in G, contrary to (1). Thus we may assume that $v \in R$. Since u, v are adjacent, $u \neq z$; and since z is therefore strongly complete or strongly anticomplete to both u, v, we deduce that $u \in R$. We may assume that $u \in L$. Since D_1 is anticonnected, u has an antineighbour $m \in M \cap T$; and therefore $m \neq v$, and m is strongly antiadjacent to both u, v. Consequently $v \notin M$, and so $u, v \in L$. But then u, v are twins in G, by (2), contrary to 10.1. This proves the first assertion.

Now suppose that (P, Q) is a W-join in $G|T$, and suppose first that $P, Q \subseteq R$. Since (P, Q) is not a W-join in G, and since $R \setminus T$ is strongly complete to $P \cup Q$, we may assume that some vertex of $N \setminus T$ is neither strongly complete nor strongly anticomplete to Q. By (2) it follows that $Q \cap L, Q \cap M$ are both nonempty. Since every vertex in $(L \cap T) \setminus (P \cup Q)$ has a neighbour in $Q \cap L$, it follows that $(L \cap T) \setminus (P \cup Q)$ is strongly complete to Q, and similarly $(R \cap T) \setminus (P \cup Q)$ is strongly complete to Q. Now every vertex in $Q \cap L$ belongs to a triad, and therefore has an antineighbour in $M \cap T$, which consequently belongs to P. So no vertex in $Q \cap L$ is strongly complete to $P \cap M$, and similarly no vertex in $Q \cap M$ is strongly complete to $P \cap L$; and in particular, $P \cap L, P \cap M$ are both nonempty. By exchanging P, Q we deduce that $(R \cap T) \setminus (P \cup Q)$ is strongly complete to P; and so $(R \cap T) \setminus (P \cup Q)$ is strongly complete to $P \cup Q$. Moreover, $(P \cap L) \cup (Q \cap M)$ is strongly complete to $(P \cap M) \cup (Q \cap L)$, since P, Q, L, M are all strong cliques. But this is impossible since D_1 is an anticomponent.

Consequently not both P, Q are subsets of R. Since P is not strongly complete to Q, not both P, Q are subsets of $N \cup \{z\}$; and since P, Q are both nonempty, we may assume that $P \cap (N \cup \{z\})$ and $Q \cap R$ are both nonempty. In particular, $z \notin Q$, since Q is a strong clique. We claim that $P \subseteq N \cup \{z\}$; for if $z \in P$ then $P \subseteq N \cup \{z\}$ since P is a strong clique, and if $z \notin P$ then since z has a neighbour in P, it follows that z is strongly complete to P, and again $P \subseteq N \cup \{z\}$.

We may assume that there exists $q \in Q \cap M$. Since D_1 is anticonnected, q has an antineighbour $q' \in L$; and hence $q' \notin Q$, since Q is a strong clique. Consequently q' is strongly anticomplete to Q, and in particular to $Q \cap L$; and since L is a strong clique and $q' \in L$, we deduce that $Q \cap L = \emptyset$.

Suppose that $z \notin P$. Since z has an antineighbour in Q, it follows that z is strongly anticomplete to Q, and so $Q \subseteq R$, and therefore $Q \subseteq M$. But then (P, Q) is a W-join in G, contrary to (1). This proves that $z \in P$.

Every vertex in $(M \cap T) \setminus Q$ has a neighbour in $Q \cap M$ and is therefore strongly complete to Q; and so $M \cap T$ is strongly complete to $Q \cap N$. But every vertex in $Q \cap N$ belongs to a triad, and therefore has an antineighbour in $M \cap T$. Consequently $Q \cap N = \emptyset$, and so $Q \subseteq M$. If $P = \{z\}$, then $|Q| > 1$ and Q is a homogeneous set in $G|T$ and hence in G, a contradiction. Thus $|P| > 1$.

By (2), it follows that $(P \setminus \{z\}, Q)$ is a homogeneous pair in G, and so $|P| = 2$ and $|Q| = 1$; let $P = \{z, p\}$ and $Q = \{q\}$ say. Since (P, Q) is a W-join in $G|T$, every vertex in $R \setminus Q$ is strongly anticomplete to P (since every such vertex is antiadjacent to $z \in P$), and so q is the only neighbour of p in R. We claim that $L \cap T$ is strongly anticomplete to q; for suppose that some $y \in L \cap T$ is adjacent to q. Since y belongs to a triad, it has an antineighbour $y' \in M \cap T$; but then $\{q, p, y, y'\}$ is a claw, a contradiction. This proves that $L \cap T$ is strongly anticomplete to q, and therefore proves (5).

(6) *There exist j with $1 \leq j \leq 5$ and $(G', A', B', C') \in \mathcal{TC}_j$ such that one of the following holds:*

- *$G' = G|T$ and $\{A', B', C'\} = \{(N \cap T) \cup \{z\}, L \cap D_1, M \cap D_1\}$;*
- *some $p \in N \cap T$ has a unique neighbour $q \in R$; q belongs to one of $L \cap T, M \cap T$ and is strongly anticomplete to the other; G' is the trigraph obtained from $G|T$ by deleting z and making p, q semiadjacent; and $\{A', B', C'\} = \{N \cap T, L \cap D_1, M \cap D_1\}$.*

For by (2), there is a partition (S_1, S_2) of $N \cap S$ such that S_1 is strongly complete to $L \cap D_1$ and strongly anticomplete to $M \cap D_1$, and S_2 is strongly complete to $M \cap D_1$ and strongly anticomplete to $L \cap D_1$. Now $(G|T, (N \cap T) \cup \{z\}, L \cap D_1, M \cap D_1)$ is a three-cliqued claw-free trigraph, and it does not admit a hex-join since every vertex in D_1 is in a triad containing z, and every vertex in $N \cap T$ is in a triad meeting D_1. Moreover, each of its vertices is in a triad, and so it does not admit a worn hex-join. Hence, by 4.1, $(G|T, (N \cap T) \cup \{z\}, L \cap D_1, M \cap D_1)$ is a permutation of a thickening of a member (G', A', B', C') of $\mathcal{TC}_j$ for some $j \in \{1, \ldots, 5\}$. Let X_v $(v \in V(G'))$ be the corresponding subsets, where $z \in X_{z'}$ say; then z' is either simplicial or near-simplicial in G'. If z' is simplicial in G', then $|X_v| = 1$ for every $v \in V(G')$ by (5), and so $(G|T, (N \cap T) \cup \{z\}, L \cap D_1, M \cap D_1)$ is a permutation of (G', A', B', C') and the claim holds. If z' is near-simplicial in G', then by (5), $|X_v| = 1$ for each $v \in V(G') \setminus \{z'\}$; $X_{z'} = \{z, p\}$ where $p \in N \cap T$ has a unique neighbour $q \in R \cap T$ as in (5), and again the claim holds. This proves (6).

Let j and (G', A', B', C') be as in (6). If the first case of (6) holds, let $p = z$, and otherwise let p be as in the second case of (6).

(7) *If $j = 1$ then the theorem holds.*

For suppose that $j = 1$. Let $H, v_1, v_2, v_3, A, B, C$ be as in the definition of $\mathcal{TC}_1$, where G' is a line trigraph of H, and $A = (N \cap T)$ or $(N \cap T) \cup \{z\}$, and $B = L \cap D_1$, and $C = M \cap D_1$. Then $p \in A$ and so p is incident with v_1 in H. Let the ends of p in H be v_1, v_0 say; then $v_0 \neq v_2, v_3$. Moreover, since p is simplicial or near-simplicial in G', it follows that every edge of H incident with v_0 different from p shares an end with every edge incident with v_1. Since there are at least three edges incident with v_1, it follows that p is the only edge of H incident with v_0, and therefore p is simplicial in G', and so the first case of (6) holds. But then $(G, Z) \in \mathcal{Z}_{14}$ and the theorem holds. This proves (7).

(8) *If $j = 2$ then the theorem holds.*

For let $j = 2$. It is easy to see that for any trigraph J and $y \in V(J)$, if y is a simplicial vertex of J and $(J, X, Y, Z) \in \mathcal{TC}_2$, then $(J, \{y\}) \in \mathcal{Z}_{13}$. In particular, if the first case of (6) holds, then $(G, Z) \in \mathcal{Z}_{13}$; so we may assume that the second case holds, and so p is near-simplicial in G'. Let q be as in (6). Then $(G', N \cap T, L \cap D_1, M \cap D_1) \in \mathcal{TC}_2$; let Σ be a circle with $V(G') \subseteq \Sigma$, and let $F_1, \ldots, F_k \subseteq \Sigma$, as in the definition of long circular interval trigraph. Let L_1, L_2, L_3 be pairwise disjoint lines with $V(G') \subseteq L_1 \cup L_2 \cup L_3$, and with

$$V(G') \cap L_1 = N \cap T, V(G') \cap L_2 = L \cap D_1, V(G') \cap L_3 = M \cap D_1.$$

Since p, q are semiadjacent in G', we may assume they are both ends of F_1 say, and no other F_i contains both p, q. But then $(G|T, (N \cap T) \cup \{z\}, L \cap D_1, M \cap D_1) \in \mathcal{TC}_2$, as we see by inserting z into Σ consecutive with p and not in F_1; and so again $(G, Z) \in \mathcal{Z}_{13}$. This proves (8).

(9) *If $j = 3$ then the theorem holds.*

For let $j = 3$. Then G' is a near-antiprismatic trigraph; let H, A, B, C, X, a_0, b_0 be as in the definition of near-antiprismatic trigraph, such that $G' = H \setminus X$, and $A' = A \setminus X$ and similarly for B', C'. As in the proof of 11.1, it follows that a_0, b_0 are the only simplicial or near-simplicial vertices of G', so we may assume that $p = a_0$. If the first case of (6) holds, then p is simplicial in G', and a_0, b_0 are strongly antiadjacent in G', and $G' = G|T$; so $A = N$, and $\{B, C\} = \{L \cap D_1, M \cap D_1\}$. But then $(G, \{z\}) \in \mathcal{Z}_{11}$ and the theorem holds. We may therefore assume that the second case of (6) holds, and so p is near-simplicial in G'; so a_0, b_0 are semiadjacent in G', and $q = b_0$, where q is as in (6). But then $(G, \{z\}) \in \mathcal{Z}_{11}$ and the theorem holds. This proves (9).

(10) *If $j = 4$ then the theorem holds.*

For suppose that G' is antiprismatic. Suppose first that $G' = G|T$. By (3), there exists $n \in N \cap T$, and it belongs to a triad $\{n, x, y\}$, where $x, y \in R$. But then only one pair of vertices in $\{n, x, y, z\}$ is strongly adjacent, contradicting that G' is antiprismatic. Thus the second case of (6) holds, and $V(G') = T \setminus \{z\}$. Let p, q be as in (6). Then p, q are semiadjacent in G', and p is strongly adjacent in G' to only one vertex of each triad of G' that does not contain p, and since G' is antiprismatic, there is no such triad, that is, $N \cap T = \{p\}$. We may assume that $q \in M \cap T$ and q is strongly anticomplete to $L \cap T$. If there exists $x \in M \cap T$ different from q, then x has an antineighbour $y \in L$, and only one pair of vertices in $\{p, q, x, y\}$ are strongly adjacent in G', contradicting that G' is antiprismatic. Hence $M \cap T = \{q\}$. If $x, y \in L \cap T$ are distinct, then only one pair of vertices in $\{p, q, x, y\}$ are strongly adjacent, again a contradiction; so $|L \cap T| = 1$. But then $(G, \{z\}) \in \mathcal{Z}_{13}$. This proves (10).

(11) *If $j = 5$ then the theorem holds.*

For let $H, X, v_1, \ldots, v_8, A, B, C$ be as in the first case of the definition of $\mathcal{TC}_5$, where

$$(H \setminus X, A, B \setminus X, C) = (G', A', B', C').$$

Thus $X \subseteq \{v_3, v_4\}$. Hence v_1, v_3, v_4, v_6, v_7 are not simplicial or near-simplicial in G', so p is one of v_2, v_5, v_8. Suppose that $p = v_2$. Since v_1, v_4 are semiadjacent in H, it follows that $v_4 \in X$. If v_2 is simplicial in G', then v_2, v_5 are strongly antiadjacent, and $G' = G|T$, and so $(G, \{z\}) \in \mathcal{Z}_{11}$ and the theorem holds. If v_2 is not simplicial in G', then v_2, v_5 are semiadjacent in G', and $G|T$ is obtained from G' by adding z strongly adjacent to $\{v_1, v_2, v_3\} \setminus X$ and strongly antiadjacent to all other vertices of G', and possibly making v_2, v_5 strongly adjacent. But then $(G, \{z\}) \in \mathcal{Z}_{10}$ and the theorem holds. Hence we may assume that $p \neq v_2$, and similarly $p \neq v_5$, and therefore $p = v_8$. Hence p is not semiadjacent to any vertex in G', and so $G' = G|T$. Consequently $(G, \{z\}) \in \mathcal{Z}_{15}$, and the theorem holds.

Now let $H, X, v_1, \ldots, v_9, A, B, C$ be as in the second case of the definition of $\mathcal{TC}_5$, where

$$(H \setminus X, A, B \setminus X, C) = (G', A', B', C').$$

Thus $X \subseteq \{v_3, v_4, v_5, v_6\}$ and contains at most one of v_3, v_4 and at most one of v_5, v_6. Moreover, v_2 is adjacent in G' to one of v_3, v_4, and v_7 is adjacent to one of v_5, v_6. Hence none of v_1, v_3, v_6, v_8, v_9 are simplicial or near-simplicial in G', so from the symmetry we may assume that p is one of v_2, v_4. Suppose first that $p = v_2$. One of $A, B \setminus X, C$ consists of $\{p\}$ together with all strong neighbours of p in G', and this must be A since $p \in A$; so $v_3 \in X$ (because otherwise v_3 is a strong neighbour of p in G' that is not in A), and v_2 is semiadjacent to v_4, and so $q = v_4$ and the second case of (6) holds. Then $G|T$ is obtained from G' by adding a vertex z strongly adjacent to v_1, v_2 and strongly antiadjacent to all other vertices of G', and possibly making v_2, v_4 strongly adjacent. But then $(G, Z) \in \mathcal{Z}_{10}$ and the theorem holds.

We may therefore assume that $p = v_4$, and in particular, $v_4 \notin X$. Since v_4 is simplicial or near-simplicial in G', it follows that v_2, v_4 are antiadjacent. If they are strongly antiadjacent, then $G|T = G'$ and $(G, Z) \in \mathcal{Z}_{12}$. If they are semiadjacent, then $G|T$ is obtained from G' by adding a vertex z strongly adjacent to $\{v_3, v_4, v_5, v_6, v_9\} \setminus X$ and strongly antiadjacent to all other vertices of G', and possibly making v_2, v_4 strongly adjacent. But then again $(G, Z) \in \mathcal{Z}_{12}$. This proves (11).

From (6)–(11), this proves 13.3. ■

Proof of 7.2. Let G be a connected claw-free trigraph such that $V(G)$ is not the union of three strong cliques. By 9.1, there is a strip-structure (H, η) of G such that all its strips are either spots or cliques or thickenings of unbreakable stripes. Suppose first that (H, η) is trivial. Then the unique strip of the strip-structure is $(G, \emptyset)$, and so this is either a spot or a clique or a thickening of an unbreakable stripe. It is not a spot since $|Z| = 2$ for every spot (J, Z). If it is a clique then G is a strong clique and hence a thickening of a one-vertex trigraph (which belongs to $\mathcal{S}_3$ for instance), and the theorem holds. We assume then that $(G, \emptyset)$ is an unbreakable stripe. But G does not admit a hex-join, since $V(G)$ is not the union of three cliques; and G does not admit twins, a W-join, a 0-join, a 1-join or a generalized 2-join since $(G, \emptyset)$ is an unbreakable stripe. Thus G is indecomposable. By 5.1, $G \in \mathcal{S}_i$ for some $i \in \{0, \dots, 7\}$. If $|V(G)| \leq 2$ then $G \in \mathcal{S}_3$ and the theorem holds, so we assume that $|V(G)| \geq 3$; and so by 10.3, $i > 0$. Moreover, G has no simplicial vertex, since $(G, \emptyset)$ is unbreakable, and so $i \neq 4, 5$. Suppose that $i \in \{2, 6\}$. Then there are two vertices a_0, b_0 of G, either strongly antiadjacent to each other and both simplicial, or semiadjacent to each other and both near-simplicial. The first is impossible since no vertices of G are simplicial. In the second case, let $V_1 = \{a_0, b_0\}$ and V_0 the set of all vertices in $V(G) \setminus \{a_0, b_0\}$ that are strongly adjacent to both a_0, b_0, and $V_2 = V(G) \setminus (V_0 \cup V_1)$; since V_1, V_2 are not strongly stable, these define a pseudo-2-join, a contradiction. Thus $i \in \{1, 3, 7\}$ and the theorem holds.

Thus we may assume that (H, η) is nontrivial. Let (J, Z) be a strip of the strip-structure. We must show that either (J, Z) is a spot, or (J, Z) is a thickening of a member of $\mathcal{Z}_0$. Suppose then that (J, Z) is not a spot. Certainly $Z \neq \emptyset$ since G is connected and the strip-structure is nontrivial. If (J, Z) is a clique, then $|Z| = 1$ and (J, Z) is a thickening of a member of $\mathcal{Z}_6$. We assume therefore that (J, Z) is not a clique, and so $|V(J)| > 2$. It follows that (J, Z) is a thickening of an unbreakable stripe (J', Z'). If $V(J')$ is the union of two strong cliques, then by 10.2 $(J', Z') \in \mathcal{Z}_1 \cup \mathcal{Z}_6$ as required, so we may assume that $V(J')$ is not the union of two strong cliques. By 10.5 either J' is a thickening of an indecomposable member of $\mathcal{S}_i$ for some $i \in \{1, \dots, 7\}$, or J' admits a hex-join. In the first case, by 12.2 we deduce that $(J', Z') \in \mathcal{Z}_0$ as required. In the second case, 13.1 implies that $|Z'| = 1$ or 2. If $|Z'| = 2$ then 13.2 implies that $(J', Z') \in \mathcal{Z}_0$ as required. If $|Z'| = 1$ then 13.3 implies that $(J', Z') \in \mathcal{Z}_0$ again as required. This proves 7.2. ■

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