Bad News for Chordal Partitions

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Abstract. Reed and Seymour [1998] asked whether every graph has a partition into induced connected subgraphs such that each part is bipartite and the quotient graph is chordal. If true, this would have significant ramifications for Hadwiger's Conjecture. We prove that the answer is 'no'. In fact, we show that the answer is still 'no' for several relaxations of the question.

1 Introduction

Hadwiger's Conjecture says that every graph with no K_{t+1}-minor is t-colourable. This conjecture is easy for $t \leqslant 3$, is equivalent to the 4-colour theorem for $t \in \{4, 5\}$, and is open for $t \geqslant 6$. The best known upper bound is $O(t\sqrt{\log t})$, independently due to Kostochka [12, 13] and Thomason [18, 19]. This conjecture is widely considered to be one of the most important open problems in graph theory; see [17] for a survey. We assume the reader is familiar with basic knowledge about graph minors and treewidth; see [5].

Motivated by Hadwiger's Conjecture, Reed and Seymour [15] introduced the following definitions. A *partition* of a graph G is a set of induced connected subgraphs of G that partition $V(G)$. The *quotient* of a partition of G is the graph obtained by contracting each part into a single vertex. A partition is *chordal* if the quotient is chordal (that is, contains no induced cycle of length at least four). Chordal partitions are a useful tool when studying graphs with no K_{t+1} minor, in which case the quotient contains no K_{t+1} subgraph, and is therefore t-colourable (since chordal graphs are perfect).

Reed and Seymour [15] asked whether every graph has a chordal partition such that each part is bipartite. (This question is repeated in [11, 17].) If true, this would imply that every graph with no K_{t+1}-minor is $2t$-colourable, by taking the product of the t-colouring of the quotient with the 2-colouring of each part. This would be a major breakthrough

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for Hadwiger's Conjecture. The purpose of this note is to answer Reed and Seymour's question in the negative. In fact, we show the following stronger result.

Theorem 1. *For every integer k there is a graph G, such that every chordal partition of G contains K_k in some part. Moreover, for every integer t there is a graph G with treewidth at most $t-1$, and thus K_{t+1}-minor-free, such that every chordal partition of G contains $K_{\Omega(t^{1/3})}$ in some part.*

Given the above motivation, it is natural to consider *perfect partitions* (where the quotient is perfect). Here the quotient of a partition of a K_{t+1}-minor-free graph is still t-colourable. We extend Theorem 1 as follows.

Theorem 2. *For every integer k there is a graph G, such that every perfect partition of G contains K_k in some part. Moreover, for every integer t there is a graph G with treewidth at most $t-1$, and thus K_{t+1}-minor-free, such that every perfect partition of G contains $K_{\Omega(t^{1/3})}$ in some part.*

Theorems 1 and 2 say that it is hopeless to improve on the $O(t\sqrt{\log t})$ bound for the chromatic number of K_t-minor-free graphs using chordal or perfect partitions directly. Indeed, the best possible upper bound on the chromatic number using the above approach would be $O(t^{4/3})$.

Chordal graphs contain no induced 4-cycle, and perfect graphs contain no induced 5-cycle. These are the only properties of chordal and perfect graphs used in the proofs of Theorems 1 and 2. Thus the following result is a qualitative generalisation of both Theorems 1 and 2.

Theorem 3. *For every integer k and graph H, there is a graph G, such that for every partition of G, some part contains K_k or the quotient contains H as an induced subgraph.*

Before presenting the proofs, we mention some applications of chordal decompositions and related topics. Chordal partitions have proven to be a useful tool in the study of the following topics for K_{t+1}-minor-free graphs: cops and robbers pursuit game [1, 2], fractional colouring [11, 15], generalised colouring numbers [20], defective and clustered colouring [21]. These papers show that every graph with no K_{t+1} minor has a chordal partition in which each part has desirable properties.

Several papers [8, 14, 22] have shown that graphs with treewidth k have chordal partitions in which the quotient is a tree, and each part induces a subgraph with treewidth $k-1$, amongst other properties. Such decompositions have been used for queue and track layouts [8] and non-repetitive graph colouring [14]. Tree partitions in which each part is not necessarily connected have also been widely studied [3, 4, 6, 7, 9, 10, 16, 23].

Here the goal is to have few vertices in each part of the partition. The referee of [6] proved that every graph with treewidth k and maximum degree Δ has a tree partition with $O(k\Delta)$ vertices in each part.

2 Chordal Partitions: Proof of Theorem 1

All our proofs depend on the following lemma.

Lemma 4. *Let X be a subgraph of a graph G, such that the neighbourhood of each component of $G - X$ is a clique (in X). Then every partition $\mathcal{P}$ of G, when restricted to X, is a partition of X and the quotient of $\mathcal{P}$ restricted to X equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect X.*

Proof. Since the neighbourhood of each component of $G - X$ is a clique, for each connected subgraph G' of G, the subgraph $G'[V(X)]$ is connected. In particular, $\mathcal{P}$ restricted to X is a partition of X (with connected parts). Moreover, if adjacent parts P and Q of $\mathcal{P}$ both intersect X, then P and Q contain adjacent vertices in X (again since the neighbourhood of each component of $G - X$ is a clique). Thus the quotient of $\mathcal{P}$ restricted to X equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect X. $\square$

The next lemma with $r = 1$ implies Theorem 1.

Lemma 5. *For all integers $k \geqslant 1$ and $r \geqslant 1$, there is a graph $G(k, r)$ with treewidth at most $t(k, r) - 1$, and thus $K_{t(k,r)+1}$-minor-free, where*

$$t(k, r) := \tfrac{1}{6}k(k + 1)(2k + 1) + (r - 1)k + 1,$$

such that for every chordal partition $\mathcal{P}$ of G, either:
(1) G contains a K_{kr} subgraph intersecting each of r distinct parts of $\mathcal{P}$ in k vertices, or
(2) some part of $\mathcal{P}$ contains K_{k+1}.

Proof. Note that $t(k, r)$ is the upper bound on the size of the bags in the tree decomposition of $G(k, r)$ that we construct. We proceed by induction on (k, r). For the base case, the graph with one vertex satisfies (1) for $k = r = 1$ and has a tree decomposition with one bag of size $1 \leqslant t(1, 1)$.

First we prove that the $(k, 1)$ and (k, r) cases imply the $(k, r + 1)$ case. Let $A := G(k, 1)$ and $B := G(k, r)$. Let G be obtained as follows. Start with a copy of A. Then for each k-clique C in A, add a disjoint copy B_C of B to G, where C is complete to B_C. We claim that G satisfies the claimed properties of $G(k, r + 1)$.

By assumption, A has a tree decomposition with bags of size at most $t(k,1)$, and for each k-clique C in A, there is a tree decomposition of B_C with bags of size at most $t(k,r)$. Add an edge (in the tree) between a bag containing C in the tree decomposition of A and any bag of B_C, and add C to every bag of the tree decomposition of B_C. We obtain a tree decomposition of G with bags of size at most $\max\{t(k,1), t(k,r)+k\} = t(k,r)+k = t(k,r+1)$, as desired.

Consider a partition $\mathcal{P}$ of G. By Lemma 4, $\mathcal{P}$ restricted to A is a partition of A, and the quotient of $\mathcal{P}$ restricted to A equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect A. Since the quotient of $\mathcal{P}$ is chordal, the quotient of $\mathcal{P}$ restricted to A is chordal. By induction, since $A = G(k,1)$, the quotient of $\mathcal{P}$ restricted to A satisfies (1) with $r = 1$ or (2). If outcome (2) holds, then some part of $\mathcal{P}$ contains K_{k+1}, and outcome (2) holds for G. Now assume that $\mathcal{P}$ restricted to A satisfies outcome (1) with $r = 1$; that is, some k-clique C of A is contained in some part P of $\mathcal{P}$.

If some vertex of B_C is in P, then P contains K_{k+1}, and outcome (2) holds for G. Now assume that no vertex of B_C is in P. Since each part of $\mathcal{P}$ is connected, the parts of $\mathcal{P}$ that intersect B_C do not intersect $G - V(B_C)$. Thus, $\mathcal{P}$ restricted to B_C is a partition of B_C, and the quotient of $\mathcal{P}$ restricted to B_C equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect B_C, and is therefore chordal. Since $B = G(k,r)$, the quotient of $\mathcal{P}$ restricted to B_C satisfies (1) or (2). If outcome (2) holds, then the same outcome holds for G. Now assume that outcome (1) holds for B_C. Thus B_C contains a K_{kr} subgraph intersecting each of r distinct parts of $\mathcal{P}$ in k vertices. None of these parts are P. Since C is complete to B_C, G contains a $K_{k(r+1)}$ subgraph intersecting each of $r+1$ distinct parts of $\mathcal{P}$ in k vertices, and outcome (1) holds for G. Hence G satisfies the claimed properties of $G(k,r+1)$.

It remains to prove the $(k,1)$ case for $k \geqslant 2$. By induction, we may assume the $(k-1,r)$ case for all r. As illustrated in Figure 1, let G be obtained as follows. Start with a copy of $A := G(k-1,k+1)$. Then for each set $\mathcal{C} = \{C_1,\dots,C_{k+1}\}$ of pairwise-disjoint $(k-1)$-cliques in A, whose union induces $K_{(k-1)(k+1)}$, add a disjoint K_{k+1} subgraph $B_{\mathcal{C}}$, whose i-th vertex is adjacent to every vertex in C_i. We claim that G satisfies the claimed properties of $G(k,1)$.

By assumption, A has a tree decomposition with bags of size at most $t(k-1,k+1)$. For each set $\mathcal{C} = \{C_1,\dots,C_{k+1}\}$ of pairwise-disjoint $(k-1)$-cliques in A, whose union induces $K_{(k-1)(k+1)}$, add a bag containing $V(B_{\mathcal{C}}) \cup C_1 \cup \dots \cup C_{k+1}$ adjacent (in the tree) to a bag of the tree decomposition of A containing $C_1 \cup \dots \cup C_{k+1}$. We obtain a tree decomposition of G with bags of size at most $\max\{t(k-1,k+1), (k+1)k\} = t(k-1,k+1) = t(k,1)$, as desired.

Consider a partition $\mathcal{P}$ of G. By Lemma 4, $\mathcal{P}$ restricted to A is a partition of A and

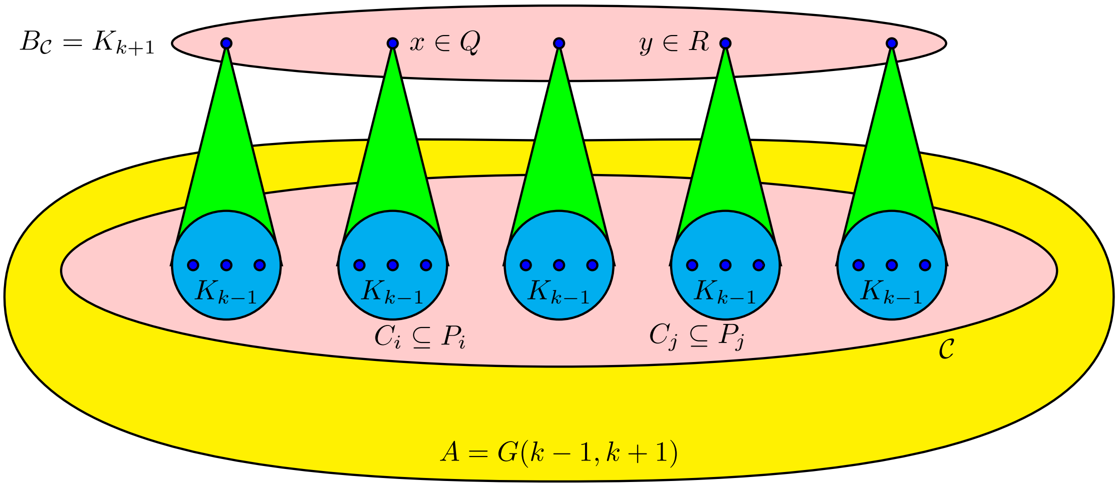

Figure 1: Construction of $G(k, 1)$ in Lemma 5.

the quotient of $\mathcal{P}$ restricted to A equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect A, and is therefore chordal. Recall that $A = G(k-1, k+1)$. If outcome (2) holds for $\mathcal{P}$ restricted to A, then some part of $\mathcal{P}$ contains K_k, and outcome (1) holds for G (with $r = 1$). Now assume that outcome (1) holds for $\mathcal{P}$ restricted to A. Thus A contains a $K_{(k-1)(k+1)}$ subgraph intersecting each of $k+1$ distinct parts $P_1, \dots, P_{k+1}$ of $\mathcal{P}$ in $k-1$ vertices. Let C_i be the corresponding $(k-1)$-clique in P_i. Let $\mathcal{C} := \{C_1, \dots, C_{k+1}\}$.

If for some $i \in [k+1]$, the neighbour of C_i in $B_{\mathcal{C}}$ is in P_i, then P_i contains K_{k+1} and outcome (1) holds for G. Now assume that for each $i \in [k+1]$, the neighbour of C_i in $B_{\mathcal{C}}$ is not in P_i. If some vertex x in $B_{\mathcal{C}}$ is in some P_i, then since P_i is connected, G contains a path between C_i and x avoiding the neighbourhood of C_i in $B_{\mathcal{C}}$. Every such path intersects $C_1 \cup \dots \cup C_{i-1} \cup C_{i+1} \cup \dots \cup C_{k+1}$, but none of these vertices are in P_i. Thus, no vertex in $B_{\mathcal{C}}$ is in $P_1 \cup \dots \cup P_{k+1}$. If $B_{\mathcal{C}}$ is contained in one part, then G contains K_{k+1} in one part, and outcome (2) holds. Now assume that $B_{\mathcal{C}}$ contains vertices x and y in distinct parts Q and R of $\mathcal{P}$. Then x is adjacent to C_i and y is adjacent to C_j, for some distinct $i, j \in [1, k+1]$. Observe that (Q, R, C_j, C_i) is a 4-cycle in the quotient of $\mathcal{P}$. Moreover, there is no QC_j edge in the quotient of $\mathcal{P}$ because $C_1 \cup \dots \cup C_{j-1} \cup C_{j+1} \cup \dots \cup C_{k+1} \cup \{y\}$ separates $x \in Q$ from $C_j \subseteq P_j$, and none of these vertices are in $Q \cup P_j$. Similarly, there is no RC_i edge in the quotient of $\mathcal{P}$. Hence (Q, R, C_j, C_i) is an induced 4-cycle in the quotient of $\mathcal{P}$, which contradicts the assumption that $\mathcal{P}$ is a chordal partition. Therefore G satisfies the claimed properties of $G(k, 1)$. $\square$

3 Perfect Partitions: Proof of Theorem 2

The following lemma with $r = 1$ implies Theorem 2. The proof is very similar to Lemma 5 except that we force C_5 in the quotient instead of C_4.

Lemma 6. *For all integers $k \geqslant 1$ and $r \geqslant 1$, there is a graph $G(k,r)$ with treewidth at most $t(k,r) - 1$, and thus $K_{t(k,r)+1}$-minor-free, where*

$$t(k,r) := \tfrac{1}{3}k(k+1)(2k+1) + (r-1)k,$$

such that for every perfect partition $\mathcal{P}$ of G, either:
(1) G contains a K_{kr} subgraph intersecting each of r distinct parts of $\mathcal{P}$ in k vertices, or
(2) some part of $\mathcal{P}$ contains K_{k+1}.

Proof. Note that $t(k,r)$ is the upper bound on the size of the bags in the tree decomposition of $G(k,r)$ that we construct. We proceed by induction on (k,r). For the base case, the graph with one vertex satisfies (1) for $k = r = 1$ and has a tree decomposition with one bag of size $1 \leqslant t(1,1)$.

First we prove that the $(k,1)$ and (k,r) cases imply the $(k,r+1)$ case. Let $A := G(k,1)$ and $B := G(k,r)$. Let G be obtained as follows. Start with a copy of A. Then for each k-clique C in A, add a disjoint copy B_C of B to G, where C is complete to B_C. We claim that G satisfies the claimed properties of $G(k,r+1)$.

By assumption, A has a tree decomposition with bags of size at most $t(k,1)$, and for each k-clique C in A, there is a tree decomposition of B_C with bags of size at most $t(k,r)$. Add an edge (in the tree) between a bag containing C in the tree decomposition of A and any bag of B_C, and add C to every bag of the tree decomposition of B_C. We obtain a tree decomposition of G with bags of size at most $\max\{t(k,1), t(k,r)+k\} = t(k,r)+k = t(k,r+1)$, as desired.

Consider a partition $\mathcal{P}$ of G. By Lemma 4, $\mathcal{P}$ restricted to A is a partition of A, and the quotient of $\mathcal{P}$ restricted to A equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect A. Since the quotient of $\mathcal{P}$ is perfect, the quotient of $\mathcal{P}$ restricted to A is perfect. By induction, since $A = G(k,1)$, the quotient of $\mathcal{P}$ restricted to A satisfies (1) with $r = 1$ or (2). If outcome (2) holds, then some part of $\mathcal{P}$ contains K_{k+1}, and outcome (2) holds for G. Now assume that $\mathcal{P}$ restricted to A satisfies outcome (1) with $r = 1$; that is, some k-clique C of A is contained in some part P of $\mathcal{P}$.

If some vertex of B_C is in P, then P contains K_{k+1}, and outcome (2) holds for G. Now assume that no vertex of B_C is in P. Since each part of $\mathcal{P}$ is connected, the parts of $\mathcal{P}$ that intersect B_C do not intersect $G - V(B_C)$. Thus, $\mathcal{P}$ restricted to B_C is a partition

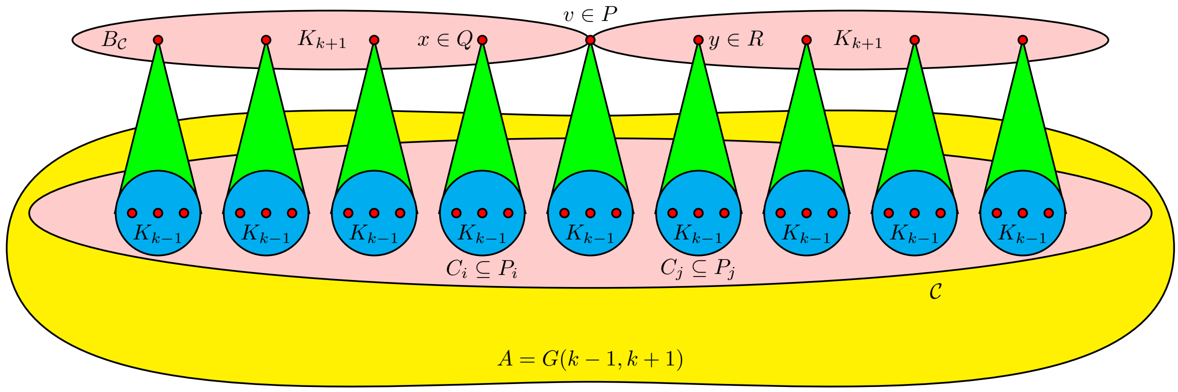

Figure 2: Construction of $G(k, 1)$ in Lemma 6.

of B_C, and the quotient of $\mathcal{P}$ restricted to B_C equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect B_C, and is therefore perfect. Since $B = G(k, r)$, the quotient of $\mathcal{P}$ restricted to B_C satisfies (1) or (2). If outcome (2) holds, then the same outcome holds for G. Now assume that outcome (1) holds for B_C. Thus B_C contains a K_{kr} subgraph intersecting each of r distinct parts of $\mathcal{P}$ in k vertices. None of these parts are P. Since C is complete to B_C, G contains a $K_{k(r+1)}$ subgraph intersecting each of $r+1$ distinct parts of $\mathcal{P}$ in k vertices, and outcome (1) holds for G. Hence G satisfies the claimed properties of $G(k, r+1)$.

It remains to prove the $(k, 1)$ case for $k \geqslant 2$. By induction, we may assume the $(k-1, r)$ case for all r. As illustrated in Figure 2, let G be obtained as follows. Start with a copy of $A := G(k-1, 2k+1)$. Let B be the graph consisting of two copies of K_{k+1} with one vertex in common. Note that $|V(B)| = 2k+1$. For each set $\mathcal{C} = \{C_1, \dots, C_{2k+1}\}$ of pairwise-disjoint $(k-1)$-cliques in A, whose union induces $K_{(k-1)(2k+1)}$, add a disjoint subgraph $B_{\mathcal{C}}$ isomorphic to B, whose i-th vertex is adjacent to every vertex in C_i. We claim that G satisfies the claimed properties of $G(k, 1)$.

By assumption, A has a tree decomposition with bags of size at most $t(k-1, 2k+1)$. For each set $\mathcal{C} = \{C_1, \dots, C_{2k+1}\}$ of pairwise-disjoint $(k-1)$-cliques in A, whose union induces $K_{(k-1)(2k+1)}$, add a bag containing $V(B_{\mathcal{C}}) \cup C_1 \cup \cdots \cup C_{2k+1}$ adjacent (in the tree) to a bag of the tree decomposition of A containing $C_1 \cup \cdots \cup C_{2k+1}$. We obtain a tree decomposition of G with bags of size at most $\max\{t(k-1, 2k+1), (2k+1)k\} = t(k-1, 2k+1) = t(k, 1)$, as desired.

Consider a partition $\mathcal{P}$ of G. By Lemma 4, $\mathcal{P}$ restricted to A is a partition of A and the quotient of $\mathcal{P}$ restricted to A equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect A, and is therefore chordal. Recall that $A = G(k-1, 2k+1)$. If outcome (2) holds for $\mathcal{P}$ restricted to A, then some part of $\mathcal{P}$ contains K_k, and outcome (1) holds for G (with $r = 1$). Now assume that outcome (1) holds for $\mathcal{P}$ restricted to

A. Thus A contains a $K_{(k-1)(2k+1)}$ subgraph intersecting each of $2k+1$ distinct parts $P_1, \dots, P_{2k+1}$ of $\mathcal{P}$ in $k-1$ vertices. Let C_i be the corresponding $(k-1)$-clique in P_i. Let $\mathcal{C} := \{C_1, \dots, C_{2k+1}\}$.

If for some $i \in [2k+1]$, the neighbour of C_i in $B_{\mathcal{C}}$ is in P_i, then P_i contains K_{2k+1} and outcome (1) holds for G. Now assume that for each $i \in [2k+1]$, the neighbour of C_i in $B_{\mathcal{C}}$ is not in P_i. If some vertex x in $B_{\mathcal{C}}$ is in some P_i, then since P_i is connected, G contains a path between C_i and x avoiding the neighbourhood of C_i in $B_{\mathcal{C}}$. Every such path intersects $C_1 \cup \dots \cup C_{i-1} \cup C_{i+1} \cup \dots \cup C_{2k+1}$, but none of these vertices are in P_i. Thus, no vertex in $B_{\mathcal{C}}$ is in $P_1 \cup \dots \cup P_{2k+1}$.

By construction, $B_{\mathcal{C}}$ consists of two $(k+1)$-cliques B^1 and B^2, intersecting in one vertex v. Say v is in part P of $\mathcal{P}$. If B^1 is contained in P, then G contains K_{k+1} in one part, and outcome (2) holds. Now assume that B^1 contains a vertex x in some part Q distinct from P. Similarly, assume that B^2 contains a vertex y in some part R distinct from P. Now, $Q \neq R$, since $C_1 \cup \dots \cup C_{2k+1} \cup \{v\}$ separates x and y, and none of these vertices are in $Q \cup R$. By construction, x is adjacent to C_i and y is adjacent to C_j, for some distinct $i, j \in [1, 2k+1]$. Observe that (Q, P, R, P_j, P_i) is a 5-cycle in the quotient of $\mathcal{P}$. Moreover, there is no QP_j edge in the quotient of $\mathcal{P}$ because $C_1 \cup \dots \cup C_{j-1} \cup C_{j+1} \cup \dots \cup C_{k+1} \cup \{y\}$ separates $x \in Q$ from $C_j \subseteq P_j$, and none of these vertices are in $Q \cup P_j$. Similarly, there is no RP_i edge in the quotient of $\mathcal{P}$. There is no PP_j edge in the quotient of $\mathcal{P}$ because $C_1 \cup \dots \cup C_{j-1} \cup C_{j+1} \cup \dots \cup C_{k+1} \cup \{y\}$ separates $v \in P$ from $C_j \subseteq P_j$, and none of these vertices are in $P \cup P_j$. Similarly, there is no PP_i edge in the quotient of $\mathcal{P}$. Hence (Q, P, R, P_j, P_i) is an induced 4-cycle in the quotient of $\mathcal{P}$, which contradicts the assumption that $\mathcal{P}$ is a perfect partition. Therefore G satisfies the claimed properties of $G(k, 1)$. □

4 General Partitions: Proof of Theorem 3

To prove Theorem 3 we show the following stronger result, in which G only depends on the number of vertices of H and we can force many copies of K_k.

Lemma 7. *For all integers $k, t, r \geqslant 1$, there is a graph $G(k, t, r)$, such that for every partition $\mathcal{P}$ of $G(k, t, r)$ either:*
(1) G contains a K_{kr} subgraph intersecting each of r distinct parts of $\mathcal{P}$ in k vertices,
(2) the quotient of $\mathcal{P}$ contains every t-vertex graph as an induced subgraph, or
(3) some part of $\mathcal{P}$ contains K_{k+1}.

Proof. We proceed by induction on $(k+t, r)$. The $t = 1$ case is trivial.

First we prove that the $(k,t,1)$ and (k,t,r) cases imply the $(k,t,r+1)$ case. Let $A := G(k,t,1)$ and $B := G(k,t,r)$. Let G be obtained as follows. Start with a copy of A. Then for each k-clique C in A, add a disjoint copy B_C of B to G, where C is complete to B_C. We claim that G satisfies the claimed properties of $G(k,t,r+1)$.

Consider a partition $\mathcal{P}$ of G. By Lemma 4, $\mathcal{P}$ restricted to A is a partition of A, and the quotient of $\mathcal{P}$ restricted to A equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect A. Since $A = G(k,t,1)$, the quotient of $\mathcal{P}$ restricted to A satisfies (1) with $r = 1$, (2) or (3). If outcome (3) holds, then some part of $\mathcal{P}$ contains K_{k+1}, and outcome (3) holds for G. If $\mathcal{P}$ restricted to A satisfies outcome (2), then outcome (2) is satisfied for G. We may now assume that $\mathcal{P}$ restricted to A satisfies outcome (1) with $r = 1$; that is, some k-clique C of A is contained in some part P of $\mathcal{P}$.

If some vertex of B_C is in P, then P contains K_{k+1}, and outcome (3) holds for G. Now assume that no vertex of B_C is in P. Since each part of $\mathcal{P}$ is connected, the parts of $\mathcal{P}$ that intersect B_C do not intersect $G - V(B_C)$. Thus, $\mathcal{P}$ restricted to B_C is a partition of B_C, and the quotient of $\mathcal{P}$ restricted to B_C equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect B_C. Since $B = G(k,t,r)$, the quotient of $\mathcal{P}$ restricted to B_C satisfies (1), (2) or (3). If outcome (2) or (3) holds, then the same outcome holds for G. Now assume that outcome (1) holds for B_C. Thus B_C contains a K_{kr} subgraph intersecting each of r distinct parts of $\mathcal{P}$ in k vertices. None of these parts are P. Since C is complete to B_C, G contains a $K_{k(r+1)}$ subgraph intersecting each of $r+1$ distinct parts of $\mathcal{P}$ in k vertices, and outcome (1) holds for G. Hence G satisfies the claimed properties of $G(k,t,r+1)$.

It remains to prove the $(k,t,1)$ case. By induction, we may assume the $(k,t-1,1)$ case and the $(k-1,t,r)$ case for all r. Let $B := G(k,t-1,1)$ and $n := |V(B)|$. Let $B^1,\dots,B^{2^n}$ be the distinct subsets of $V(B)$. Let $A := G(k-1,t,2^n)$. Let G be obtained as follows. Start with a copy of A. Then for each set $\mathcal{C} = \{C_1,\dots,C_{2^n}\}$ of pairwise-disjoint $(k-1)$-cliques in A, whose union induces $K_{(k-1)2^n}$, add a disjoint copy $B_{\mathcal{C}}$ of B to G, where C_i is complete to $B^i_{\mathcal{C}}$ for all $i \in [2^n]$. We claim that G satisfies the claimed properties of $G(k,t,1)$.

Consider a partition $\mathcal{P}$ of G. By Lemma 4, $\mathcal{P}$ restricted to A is a partition of A, and the quotient of $\mathcal{P}$ restricted to A equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect A. Recall that $A = G(k-1,t,2^n)$. If $\mathcal{P}$ restricted to A satisfies outcome (2), then the quotient of $\mathcal{P}$ restricted to A contains every t-vertex graph as an induced subgraph, and outcome (2) is satisfied for G. If outcome (3) holds for $\mathcal{P}$ restricted to A, then some part of $\mathcal{P}$ contains K_k, and outcome (1) holds for G (with $r = 1$). Now assume that outcome (1) holds for $\mathcal{P}$ restricted to A. Thus A contains a $K_{(k-1)2^n}$ subgraph intersecting each of 2^n distinct parts $P_1,\dots,P_{2^n}$ of $\mathcal{P}$ in $k-1$ vertices.

Let C_i be the corresponding $(k-1)$-clique in P_i. Let $\mathcal{C} := \{C_1, \dots, C_{2^n}\}$.

If for some $i \in [2^n]$, some neighbour of C_i in $B_{\mathcal{C}}$ is in P_i, then P_i contains K_k and outcome (1) holds for G. Now assume that for each $i \in [2^n]$, no neighbour of C_i in $B_{\mathcal{C}}$ is in P_i. If some vertex x in $B_{\mathcal{C}}$ is in some P_i, then since P_i is connected, G contains a path between C_i and x avoiding the neighbourhood of C_i in $B_{\mathcal{C}}$. Every such path intersects $C_1 \cup \dots \cup C_{i-1} \cup C_{i+1} \cup \dots \cup C_{2^n}$, but none of these vertices are in P_i. Thus, no vertex in $B_{\mathcal{C}}$ is in $P_1 \cup \dots \cup P_{2^n}$. Hence, no part of $\mathcal{P}$ contains vertices in both $B_{\mathcal{C}}$ and in the remainder of G. Therefore, $\mathcal{P}$ restricted to $B_{\mathcal{C}}$ is a partition of $B_{\mathcal{C}}$, and the quotient of $\mathcal{P}$ restricted to $B_{\mathcal{C}}$ equals the subgraph of the quotient of $\mathcal{P}$ induced by those parts that intersect $B_{\mathcal{C}}$. Since $B = G(k, t-1, 1)$, the quotient of $\mathcal{P}$ restricted to $B_{\mathcal{C}}$ satisfies (1), (2) or (3). If outcome (1) or (3) holds for $\mathcal{P}$ restricted to $B_{\mathcal{C}}$, then the same outcome holds for G. Now assume that outcome (2) holds for $\mathcal{P}$ restricted to $B_{\mathcal{C}}$,

We now show that outcome (2) holds for G. Let H be a t-vertex graph. Let v be a vertex of H. Say $N_H(v) = \{w_1, \dots, w_d\}$. Since outcome (2) holds for $\mathcal{P}$ restricted to $B_{\mathcal{C}}$, the quotient of $\mathcal{P}$ restricted to $B_{\mathcal{C}}$ contains $H - v$ as an induced subgraph. Let $Q_1, \dots, Q_d$ be the parts corresponding to $w_1, \dots, w_d$. Then $B^i_{\mathcal{C}} = V(Q_1 \cup \dots \cup Q_d)$ for some $i \in [2^n]$. Observe that in the quotient of $\mathcal{P}$, the vertex corresponding to P_i is adjacent to $Q_1, \dots, Q_d$ and to no other vertices corresponding to parts contained in $B_{\mathcal{C}}$. Thus, including P_i, the quotient of $\mathcal{P}$ contains H as an induced subgraph, and outcome (2) holds for $\mathcal{P}$. Hence G satisfies the claimed properties of $G(k, t, 1)$. □

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