A survey of χ-boundedness

Alex Scott[1]

Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour[2]

Princeton University, Princeton, NJ 08544

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Abstract

If a graph has bounded clique number and sufficiently large chromatic number, what can we say about its induced subgraphs? András Gyárfás made a number of challenging conjectures about this in the early 1980's, which have remained open until recently; but in the last few years there has been substantial progress. This is a survey of where we are now.

1   Introduction

Let G be a graph. (All graphs in this paper are finite and simple.) We denote the chromatic number of G by $\chi(G)$, and its clique number (the cardinality of its largest clique) by $\omega(G)$. If $X \subseteq V(G)$, we denote the subgraph induced on X by $G[X]$, and write $\chi(X)$ for $\chi(G[X])$ when there is no danger of ambiguity. A *hole* in G is an induced cycle of length at least four, and an *odd* hole is one with odd length; an *antihole* is an induced subgraph whose complement graph is a hole of the complement graph $\overline{G}$ of G.

Certainly $\chi(G) \geq \omega(G)$, and if we are told that $\chi(G) > \omega(G)$, we can deduce something about the induced subgraphs of G:

1.1 *If $\chi(G) > \omega(G)$ then some induced subgraph of G is an odd hole or an odd antihole.*

This is the strong perfect graph theorem [26], and it settled a long-standing open question [13] about perfect graphs. It is in a sense the complete answer, because if we are told nothing else about G except that $\chi(G) > \omega(G)$, we cannot deduce anything more about the induced subgraphs of G, because G might itself be the odd hole or antihole.

But what if we are told that $\chi(G)$ is much bigger than $\omega(G)$? More precisely, fix some bound κ and let us consider graphs G with $\omega(G) \leq \kappa$. If we choose $\chi(G)$ sufficiently large, can we then deduce more about the induced subgraphs of G? A theorem of this type appears in [125], but the question was first investigated systematically by András Gyárfás in [59], and then in a beautiful and influential paper, "Problems from the world surrounding perfect graphs" [60]. Gyárfás raised a number of interesting questions. For instance, one of his questions was: in this situation can we strengthen the conclusion of the strong perfect graph theorem? Perhaps such a graph must contain an odd hole? And indeed this is true: we proved in [107] that:

1.2 *For all $\kappa \geq 0$, if G is a graph with $\omega(G) \leq \kappa$ and $\chi(G) > 2^{2^{\kappa+2}}$ then G has an odd hole.*

It is convenient to express questions of this type in the framework of *ideals* and χ-*boundedness*. An *ideal* (or *hereditary class*) is a class of graphs closed under isomorphism and under taking induced subgraphs. We say that a graph is H-*free* if does not contain an induced subgraph isomorphic to H. Thus the class of H-free graphs is an ideal; and every ideal $\mathcal{I}$ is defined by the set of (minimal) graphs H such that $\mathcal{I}$ is H-free.

An ideal $\mathcal{I}$ is χ-*bounded* if there is a function f such that $\chi(G) \leq f(\omega(G))$ for each graph $G \in \mathcal{I}$: in this case we say that f is a χ-*binding function* for $\mathcal{I}$. The ideal of all graphs is not χ-bounded, because as is well known (and we will see in the next section), there are triangle-free graphs with arbitrarily large chromatic number; but what about subideals? Not all of them are χ-bounded (for instance, the ideal of all triangle-free graphs is not χ-bounded) but some are. For instance, 1.2 says that the ideal of all graphs with no odd hole is χ-bounded, with χ-binding function $f(x) = 2^{2^{x+2}}$. Which other ideals are χ-bounded?

There are a number of interesting conjectures and results, and to survey them we break them into three classes: results on forests, results on holes, and other stuff. We will cover topics roughly in that order, alongside related questions and topics. We begin in the next section with some examples of graphs with small ω and large χ; and at the end of the paper there is a selection of open problems (in addition to those discussed elsewhere in the paper).

2 Examples

Before we begin on any of the three main topics, let us give some useful graphs: different kinds of graphs, that are triangle-free (that is, they have clique number two) and have arbitrarily large chromatic number. Most of them are explicit constructions. (The *girth* of a graph is the minimum length of its cycles.)

Tutte's construction

The first proof that triangle-free graphs of large chromatic number exist, is due to Tutte (writing as Blanche Descartes [42, 43]), as follows. Let G_1 be a 1-vertex graph. Inductively, having defined G_k, let G_k have n vertices say; now take a set Y of $k(n-1)+1$ vertices, and for each n-subset X of Y take a copy of G_k (disjoint from everything else), and join it to X by a matching. This makes G_{k+1}. It follows inductively that G_k is not $(k-1)$-colourable, and triangle-free (indeed, it has girth at least six).

Erdős' random graph

Erdős [46] proved that for all $g, k \geq 1$, and all sufficiently large n, there is a graph G with n vertices and girth more than G, in which every stable set has fewer than n/k vertices. This can be shown as follows. Choose a function $p = p(n)$ such that $np \to \infty$ and $(np)^g = o(n)$. Now take a random graph G on $2n$ vertices, in which every pair of vertices is joined independently at random with probability p; then with probability tending to 1 as $n \to \infty$, G has no stable set of cardinality at least n/k, and has at most n cycles of length at most g. Consequently, there is a set X of n vertices that intersects every cycle of length at most g, and by deleting X we obtain a graph with the desired properties. (Erdős actually made a more efficient argument deleting edges instead of vertices.) In particular, if we take $g = 3$, the graph we obtain is triangle-free and has chromatic number more than k.

Later, explicit constructions for graphs with large girth and chromatic number were given by Lovász [87], Nešetřil and Rödl [93], and by Alon, Kostochka, Reiniger, West, and Zhu [9].

Mycielski's construction

Mycielski [92] gave the following construction. Let G_2 be the two-vertex complete graph, and inductively, having defined G_k, define G_{k+1} as follows. Let G_k have vertex set $\{v_1, \ldots, v_n\}$. Let G_{k+1} have $2n+1$ vertices $a_1, \ldots, a_n, b_1, \ldots, b_n, c$, where

- for $1 \leq i < j \leq n$, if $v_i v_j$ is an edge of G_k, then $a_i a_j, a_i b_j, b_i a_j$ are all edges of G_{k+1}; and

- for $1 \leq i \leq n$, $b_i c$ is an edge of G_{k+1}.

Then G_k is triangle-free and has chromatic number k. It is easy to see by induction that every triangle-free graph is an induced subgraph of G_k for some k, so this is not a good source of graphs with forbidden induced subgraphs.

Kneser graphs

The following grew from a problem of Kneser [78]. Let n, k be integers with $n > 2k > 0$, and let $K(n, k)$ be the graph with vertex set the set of all k-subsets of $\{1, \ldots, n\}$, in which two such sets are adjacent if they are disjoint. This graph has chromatic number $n - 2k + 2$, as was shown by Lovász [85]. The graph $K(n, k)$ has short even cycles, but all its odd cycles have length at least $n/(n - 2k)$, so if we take $n = 2k(1 - 1/g)^{-1}$ (assuming appropriate divisibility) we obtain a triangle-free graph in which all odd cycles have length at least g, and with chromatic number at least $2k/g + 2$.

Shift graphs

Let n, k be integers with $n > 2k > 2$, and let G be the graph with vertex set all k-tuples $(a_1, \ldots, a_k)$ such that $1 \le a_1 < \cdots < a_k \le n$, in which $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are adjacent if $a_{i+1} = b_i$ for $1 \le i < k$, or vice versa. This graph is triangle-free, and for fixed k its chromatic number tends to infinity with n, as was shown by Erdős and Hajnal [51]. Moreover, all its odd cycles have length at least $2k + 1$.

For $k = 3$, this has a remarkable property. Colour each vertex (a_1, a_2, a_3) by its middle element a_2; then this is a proper colouring (although not optimal), and yet for every vertex, only two colours appear on its neighbours.

Zykov's construction

Here is a construction due to Zykov [126]: let G_1 be a one-vertex graph, and inductively, having defined G_k, define G_{k+1} as follows. Take the disjoint union of $G_1, \ldots, G_k$ and for each k-tuple $(v_1, \ldots, v_k)$ where $v_i \in V(G_i)$ for $1 \le i \le k$, add a new vertex with neighbours $v_1, \ldots, v_k$. This makes G_{k+1}. Then each G_k is triangle-free and has chromatic number k.

This has a pretty variant due to Kierstead and Trotter [74]. Take the graph G_k just constructed, and orient its edges so that for each i, the new vertices in G_{i+1} are out-adjacent to their neighbours in the copies of G_i, forming a digraph H say. Now for all u, v, if there is a directed path from u to v of odd length (more than one) in H, add another edge from u to v. The digraph produced is still triangle-free, and it has no four-vertex induced directed path.

Ramsey graphs

The Ramsey number $R(3, t)$ is the smallest n such that every triangle-free graph with at least n vertices has a t-vertex stable set, and in order to bound it, constructions have been found of n-vertex triangle-free graphs without t-vertex stable sets, which therefore have chromatic number at least $n/(t - 1)$. For instance, Kim's proof [76] that $R(3, t) \ge O(t^2 \log t)$ produced triangle-free graphs with $O(t^2 \log t)$ vertices and chromatic number $\Omega(t \log t)$. Kim's proof is non-constructive, but there are explicit constructions of triangle-free graphs with $\Omega(t^{3/2})$ vertices and with no stable set of cardinality t, and so with chromatic number $\Omega(t^{1/2})$, by Alon [7], and by Codenotti, Pudlák and Resta [40].

The Burling graph

Finally, here is a construction of Burling [20]. Let G_1 be the complete graph K_2, and let $T_1 \subseteq V(G_1)$ with $|T_1| = 1$. Inductively, suppose that we have defined G_k and T_k, and T_k is a stable subset of the

vertex set of G_k. We define G_{k+1} and T_{k+1} as follows. Let $T_k = \{a_1, \ldots, a_s\}$ say, and for $1 \le i \le s$ let N_i be the set of neighbours of a_i in G_k. Take a graph consisting of $s+1$ disjoint copies of $G_k \setminus T_k$, say $A_0, \ldots, A_s$. For $0 \le i, j \le s$, let the isomorphism from $G_k \setminus T$ to A_i map N_j to N_{ij}. Now add to this $3s^2$ new vertices, namely x_{ij}, y_{ij}, z_{ij} for all i, j with $1 \le i, j \le s$. Also add edges so that x_{ij}, y_{ij} are both adjacent to every vertex in $N_{0,i}$, and x_{ij}, z_{ij} are both adjacent to every vertex in N_{ij}, and $y_{ij} z_{ij}$ an edge, for $1 \le i, j \le s$. Let G_{k+1} be the resulting graph, and let T_{k+1} be the set

$$\{x_{ij}, y_{ij} : 1 \le i, j \le s\}.$$

It is easy to check that G_k has no triangles, and for every colouring of G_k with any number of colours, some vertex in T_k has neighbours of k different colours, and in particular $\chi(G_k) \ge k+1$.

3 The Gyárfás-Sumner conjecture

An ideal is defined by the minimal graphs that it does not contain. If we fix a graph H, when is the ideal of all H-free graphs χ-bounded? If so, let us say H is χ-*bounding*. For instance, the complete graph K_2 is obviously χ-bounding, but K_3 is not. Indeed, Erdős' random graph of section 2 shows that no graph with a cycle is χ-bounding; because if H has a cycle of length g, then all graphs of girth more than g are H-free, and they are triangle-free and can have arbitrarily large chromatic number. Thus, only forests are χ-bounding. A famous conjecture, proposed independently by Gyárfás [59] and Sumner [119], asserts the converse:

3.1 The Gyárfás-Sumner conjecture: *All forests are χ-bounding.*

One can reduce this to trees, because a forest is χ-bounding if and only if all its components are χ-bounding. To see this, let H be the disjoint union of non-null forests H_1, H_2, which are both χ-bounding; we prove that all H-free graphs with clique number at most κ have bounded chromatic number, by induction on κ. Let G be an H-free graph with $\omega(G) \le \kappa$; we may assume G contains a subgraph G_1 isomorphic to H_1. The set of vertices of G with no neighbours in $V(G_1)$ is H_2-free and so has bounded chromatic number; and for each $v \in V(G_1)$, the set of vertices adjacent to v has bounded chromatic number by induction on κ. The claim follows.

Incidentally, what we just proved is a special case of something much more general. A forest that is not a tree is disconnected, and the complete graph on $\kappa + 1$ vertices is disconnected in the complement, so we could have applied the following curious result (joint with Maria Chudnovsky [28]):

3.2 *Let H, J be graphs such that H is the disjoint union of non-null graphs H_1, H_2, and $\overline{J}$ is the disjoint union of non-null graphs $\overline{J_1}, \overline{J_2}$. Then there exists t such that for every $\{H, J\}$-free graph G, its vertex set can be partitioned into t subsets, such that for each of them, say X, $G[X]$ is either H_1-free, H_2-free, J_1-free or J_2-free.*

Back to the point: for trees the conjecture is a lot tougher, and it remains open. The following weakening is known [106]:

3.3 *For every tree T, the ideal of all graphs containing no subdivision of T as an induced subgraph is χ-bounded.*

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But the Gyárfás-Sumner conjecture itself is only known to be true for some simple types of tree. For paths there is a very pretty proof by Gyárfás [59, 60] that is worth giving here.[1]

3.4 *Every path is χ-bounding.*

Proof. (Sketch.) $N(v)$ denotes the neighbour set of a vertex v. We prove first the following lemma: If G is a connected graph such that

- $\chi(N(v)) \le c$ for every $v \in V(G)$; and

- for some vertex v, there is no ℓ-vertex induced path with first vertex v

then $\chi(G)$ is at most some function of c, ℓ. This we prove by induction on ℓ. Let v be as in the second bullet. Then $\chi(N(v)) \le c$; and for each component H of non-neighbours of v, some $u \in N(v)$ has a neighbour in $V(H)$, and so there is no $(\ell-1)$-vertex induced path in $G[V(H) \cup \{v\}]$ with first vertex u. By the inductive hypothesis, $\chi(V(H) \cup \{u\})$ is bounded, and so $\chi(H)$ is bounded, and therefore G has bounded chromatic number. This proves the lemma.

Then, with ℓ fixed, we prove by induction on κ that for all κ, every graph G with $\omega(G) \le \kappa$ and no ℓ-vertex induced path has bounded chromatic number. By the inductive hypothesis, $\chi(N(v))$ is bounded for each vertex v; and so from the lemma, each component of G, and hence G itself, has bounded chromatic number. ■

Until recently the complete list of cases when the Gyárfás-Sumner conjecture is known was the following:

- Stars (a *star* is a tree in which one vertex is adjacent to all the others); this is just an easy application of Ramsey's theorem.

- Paths and brooms (see Gyárfás [59, 60]; a *broom* is obtained by identifying an endvertex of a path with the central vertex of a star).

- Generalizing these, subdivisions of stars [106] (that is, trees in which at most one vertex has degree more than two).

- Trees of radius two (Gyárfás, Szemerédi and Tuza [67] for the triangle-free case, and Kierstead and Penrice [72] for the general case).

- Trees that can be obtained from a tree of radius two by subdividing once every edge incident with the root (Kierstead and Zhu [75]).

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[1]For paths, Gyárfás gives a proof for the triangle-free case in theorem 8 of [59]; he also says that "J. Gerlits proved first (oral communication)" the triangle-free case, and that Lovász proved the general case. But [60] seems to be the first place a general proof is published, and this doesn't cite anybody.
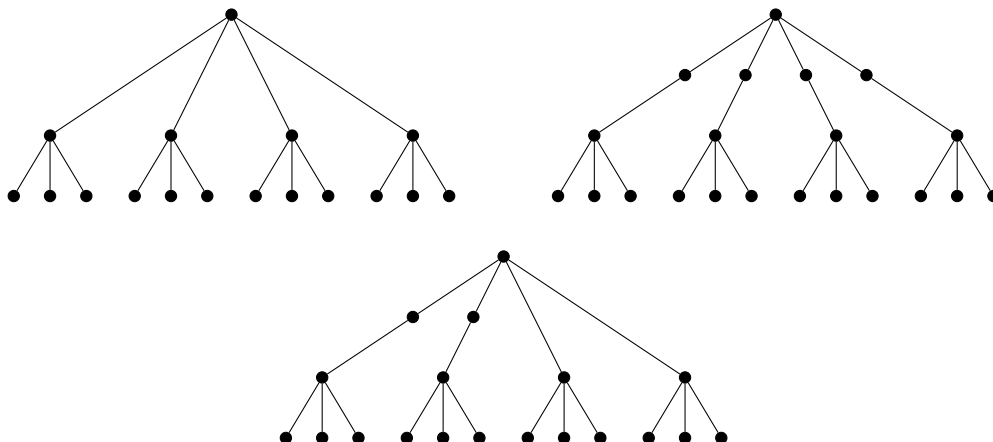
Figure 1: Kierstead-Penrice and Kierstead-Zhu trees, and our generalization

We unified the last two statements above, proving [113] that:

3.5 *Trees that can be obtained from a tree of radius two by subdividing once some edges incident with the root are χ-bounding.*

This uses the same proof method as [67, 72, 75], with more bells and whistles, the method of "templates". Here is the basic idea of that proof. By 12.10 (see section 12) and Ramsey's theorem, one can assume that there is a $K_{n,n}$ induced subgraph, with n a large constant. There might or might not be a $K_{n',n',n'}$ induced subgraph (the complete 3-partite graph), where n' is a large constant (much smaller than n), and so on; look at the largest k such that G has a subgraph $G[X]$ partitioned into k stable parts, all of some specified large size and complete to one another (k is bounded since the clique number is bounded). Then we ask how the other vertices attach to X. Not many vertices have a large number of neighbours in each part of X, from the maximality of X; and every other vertex v with a neighbour in X is a leaf of a large star of $G[X \cup \{v\}]$, and these stars are useful for growing the desired tree. Delete X and some appropriate set $Y(X)$ of vertices with neighbours in X (deciding which vertices to delete is the delicate part of the proof) and do it again, and so on; when the process stops, what remains is a set of vertices with no $K_{n,n}$ induced subgraph and with bounded clique number, so with bounded chromatic number by 12.10, and we can delete it. In effect the vertex set of the graph is partitioned into "templates", the various sets $X \cup Y(X)$. One can assume that each template has bounded chromatic number (because we can assume that for every vertex, its first neighbourhood has bounded chromatic number, by induction on clique number; and each template $X \cup Y(X)$ is a union of boundedly many first neighbourhoods, since each X has bounded size). So if G has huge chromatic number, there must be many edges between the templates; and now the proof gets harder and we omit any more details.

The trees mentioned so far have the property that their vertices of degree more than two are close together, and indeed at this stage the conjecture still remained open for every tree with two vertices of degree more than two, far apart. But now we have several theorems proving that some such trees are χ-bounding. Here are three (the first two are joint with Chudnovsky [32]; the third is joint with Chudnovsky and Sophie Spirkl, and appears in Spirkl's PhD thesis [116]):

3.6 *The following are χ-bounding:*

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- *trees obtained from a star and a star subdivision by adding a path joining their centres;*

- *trees obtained from a star subdivision by adding one vertex.*

- *trees obtained from two disjoint paths by adding an edge between them.*
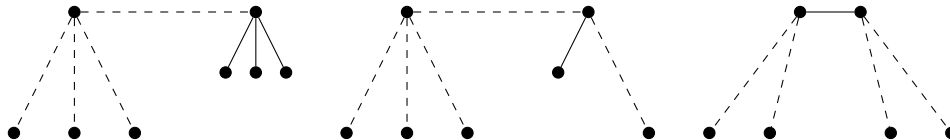

Figure 2: The trees of 3.6 (dashed lines are paths of arbitrary length)

The proofs of the last three results are all by variants of the following idea. First, we work by induction on clique number, so we may assume that the set of neighbours of each vertex has bounded chromatic number. Second, choose a vertex v_0, and classify all vertices by their distance from v_0. (This gives us a collection of disjoint subsets $L_0, L_1, \ldots$ of $V(G)$ such that $|L_0| = 1$ and for all $i > 1$, all the vertices in L_i have a neighbour in L_{i-1} and have no neighbours in L_h for $h < i - 1$; we call this a *levelling.*) In this levelling, one of the levels L_k say has chromatic number at least $\chi(G)/2$, and now we focus on it. Order the vertices in L_{k-1}, say $L_{k-1} = \{u_1, \ldots, u_m\}$; and for each i, let W_i be the set of vertices in L_k that are adjacent to u_i and nonadjacent to $u_1, \ldots, u_{i-1}$. This partitions L_k into the sets $W_1, W_2, \ldots, W_m$, and each of the W_i has bounded chromatic number (since W_i has a common neighbour). But the union of all the W_i has large chromatic number, and so (for instance) there must exist some i and a vertex $v \in W_i$ with many neighbours in $W_{i+1} \cup \cdots \cup W_m$, pairwise nonadjacent. Since u_i is adjacent to v and nonadjacent to all these neighbours, we have a little bit of a tree, that we can combine with other parts grown elsewhere. Incidentally, we call $(W_1, \ldots, W_m)$ a *grading*; it is a surprisingly useful concept, and is used both here and in some of the results discussed in section 6.

Finally, two more results proving that some trees with far-apart vertices of degree more than two are χ-bounding: we proved different strengthenings of the theorems of Kierstead-Penrice and of Kierstead-Zhu (though these are not written down, so perhaps "we believe we proved" is more correct). Let T_1 be a tree of radius two, with root v_1; let T_2 be a star, with centre v_2; take the disjoint union of T_1 and T_2; and join v_1, v_2 by a path, of any length. Then the tree just constructed is χ-bounding. The same holds if T_1 is one of the Kierstead-Zhu trees. We were not able to prove that it holds when T_1 is one of the trees of 3.5.

4 Variants of Gyárfás-Sumner: rainbow subgraphs

Suppose G is a graph with huge chromatic number and bounded clique number, and now we colour it (not necessarily optimally). Say a subgraph is *rainbow* if all its vertices have different colours. Which graphs H have the property that for all such G, and all colourings of G, there is a rainbow induced copy of H in G? Not if H has a cycle (because then H need not appear at all, rainbow or not); and not if H has a vertex of degree more than two, as shown by Kierstead and Trotter [74]. To see the latter, let G be the shift graph, described in section 2, with $k = 3$. We showed earlier that

G can be coloured so that the neighbours of each vertex receive only two colours, and so G contains no rainbow copy of H if H has a vertex of degree more than two.

So all graphs H with the desired property are paths and disjoint unions of paths; what about the converse? Here we have a complete solution [111] (see Gyárfás and Sárközy [66] for earlier results). As every disjoint union of paths is an induced subgraph of a larger path, it is enough to prove the following:

4.1 *For every $\kappa, s \geq 1$ there exists c such that if G is a graph with $\chi(G) \geq c$ and $\omega(G) \leq \kappa$, then for every proper colouring of G, there is an s-vertex rainbow induced path of G.*

The proof is neat, so we sketch it. For a vertex v, let $R(v)$ be the set of vertices that are joined to v by a rainbow path (or equivalently, by a rainbow induced path).

Suppose first that for some v, $R(v)$ has large chromatic number. Enumerate the neighbours of v, say $u_1, \ldots, u_m$, and for each i let U_i be the set of vertices that are reached by a rainbow induced path starting from v with first edge vu_i. If one of these still has large chromatic number (not so large as before, but still large), then enumerate the neighbours of u_i that are not neighbours of v, and repeat. Eventually (since we do not get a long induced rainbow path), we find some induced path P from v to u say, such that the set of vertices reachable by induced rainbow paths starting with P has large chromatic number, but P is maximal (in the sense that we cannot extend P even if we somewhat reduce "large"). Let Z be the set of all vertices that have no neighbour in $V(P)$ except possibly u. Enumerate the neighbours of u in Z, and let U_i be the set of vertices of Z that are reachable by a rainbow path that starts with P and then uses the ith neighbour of u, and its further vertices all belong to Z and are nonadjacent to u. For each i, let $W_i = U_i \setminus (U_1 \cup \cdots \cup U_{i-1})$. This giving a grading $(W_1, \ldots, W_m)$ say. Let $W = W_1 \cup \cdots \cup W_m$, and direct every edge of $G[W]$ towards its end with the larger colour. The Gallai-Roy theorem [55, 103] implies that there is a long directed path in this digraph (not induced), necessarily rainbow since it is directed; and a theorem of Galvin, Rival and Sands [56] implies that the vertex set of this directed path includes the vertex set of a large complete bipartite graph K, which is therefore also rainbow. Let w be the earliest vertex in K (that is, belonging to W_i for i minimum). It follows that w has many neighbours, all of different colours, in later sets W_j, and not in W_i (not quite; we have to arrange that each W_i is stable, but that is easy); and then this gives a contradiction, because the rainbow path to w could be extended to one of these neighbours, contradicting that this neighbour is not in U_i.

The second case, when each set $R(v)$ has small chromatic number, is very similar (again using Gallai-Roy and Galvin-Rival-Sands), and we omit it.

Incidentally, there is a conjecture of Aravind (see [12]):

4.2 Conjecture: *Let G be a triangle-free graph. Then for every colouring (not necessarily optimal) of G, there is a $\chi(G)$-vertex rainbow induced path in G.*

This remains open, but some special cases have been proved. For instance, if we just ask for an induced path (not necessarily rainbow), then it holds by a theorem of Gyárfás [60]. Or if we just ask for a rainbow path (not necessarily induced), then it holds by the Gallai-Roy theorem [55, 103], even without the bound on clique number: if we direct every edge of G towards the end with higher colour, then every directed path of the digraph obtained is rainbow. The conjecture also holds if the girth of G equals its chromatic number, by a result of Babu, Basavaraju, Chandran and Francis [12]: in particular, if (G, ϕ) is a triangle-free coloured graph with $\chi(G) \geq 4$, then some induced four-vertex path of G is rainbow.

5 Variants of Gyárfás-Sumner: orientations

Suppose G is a graph with huge chromatic number and bounded clique number, and now we direct its edges, obtaining a digraph (which we also call G). Which digraphs H must be present in G as an induced subdigraph? Only (directed) forests, as usual; but which ones? This question was raised by Gyárfás [63], and he showed that the orientation $\rightarrow\leftarrow\rightarrow$ of the four-vertex path does not have the property (to see this, take the shift graph with $k = 2$, and take its natural orientation). Kierstead and Trotter [74] proved that the orientation $\rightarrow\rightarrow\rightarrow$ of the same path also does not have the property (to see this, take the variant of Zykov's construction given in section 2).

There are two other orientations of the four-vertex path, $\rightarrow\leftarrow\leftarrow$ and $\leftarrow\rightarrow\rightarrow$; these are equivalent up to reversing edges, so it is enough to consider the first. In the case of *acyclic* digraphs, Chvátal [39] showed that if G has no induced $\rightarrow\leftarrow\leftarrow$ then its underlying graph is perfect; so $\chi(G) = \omega(G)$. In the general case, Kierstead and Rödl [73] raised the question of whether the ideal of graphs that can be oriented with no induced $\rightarrow\leftarrow\leftarrow$ is χ-bounded, and Aboulker et al. [2] conjectured that it is. We (with Chudnovsky) resolved the question in [31]:

5.1 *The ideal of graphs that can be oriented with no induced $\rightarrow\leftarrow\leftarrow$ is χ-bounded.*

The proof is not particularly difficult, but also not very exciting.

A much more challenging question is, what about stars? It is easy to see that in-directed stars and out-directed stars have the property, but what about the star H when some edges are directed in and some out? Again, this question was raised by Gyárfás [63]. Kierstead and Rödl [73] proved that the ideal of digraphs with no induced H and no cyclic triangle is χ-bounded; and Aboulker et al. [2] showed that digraphs with no induced H and no transitive triangle have bounded chromatic number (note that every orientation of K_4 has a transitive triangle, so if G has no transitive triangle then $\omega(G)$ is at most 3).

We (with Chudnovsky) answered the question for stars in [31]:

5.2 *If H is a digraph obtained from directing the edges of a star, then the ideal of graphs that can be oriented so that no induced subgraph is isomorphic to H is χ-bounded.*

This does not yet answer the question of which oriented trees in general have this property. Only those that do not contain a four-vertex path directed as $\rightarrow\leftarrow\rightarrow$ or as $\rightarrow\rightarrow\rightarrow$; but perhaps that is the answer? For futher discussion, see the paper of Aboulker et al. [2].

6 Holes

Along with the Gyárfás-Sumner conjecture, the best known of Gyárfás' conjectures were these three:

- The ideal of graphs with no odd hole is χ-bounded.

- For all $\ell \geq 0$, the ideal of graphs with no hole of length $> \ell$ is χ-bounded.

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(The third implies the other two, of course.) These have all now been proved, in [107, 29, 35] respectively, partly in joint work with Chudnovsky and Spirkl; indeed an even stronger result is now

known [112] (see 6.5 below). The proofs all use levellings, and focus on a level with large chromatic number. The proof of the first was quite pretty, although too long to sketch here, and used some methods that we were not able to use again for later papers. The second and third were both long and complicated, and both used an approach that we used again in several other papers: to look at the maximum chromatic number of balls of bounded radius. Prove the result assuming that every ball of some appropriate radius r has bounded chromatic number; and use that as a lemma. Now we can assume that some ball of radius r has large chromatic number, as large as we like; and whenever we find an induced subgraph of large chromatic number, it also must have an r-ball of large chromatic number.

Let us say this more exactly. An r-*ball* in G means a subset $B \subseteq V(G)$ such that every vertex in X is joined by a path of $G[B]$ of length at most r to some fixed vertex in B. If ϕ is a function, we say that a graph G is (r, ϕ)-*controlled* if for every induced subgraph H of G, $\chi(H) \leq \phi(k)$ where k is the maximum chromatic number of r-balls in H. (It is helpful to assume that ϕ is non-decreasing, without loss of generality.)

We fix some κ and we are trying to show that every graph G with clique number at most κ and very large chromatic number has a hole of some desired length (let us call this a "good" hole). We assume the result holds for all graphs with clique number less than κ. Say we prove that for all graphs G with clique number $\leq \kappa$ and no good hole, and all k, if every r-ball has chromatic number $\leq k$, then G has bounded chromatic number, say at most $\phi(k)$. That tells us that, if G is a general graph with clique number at most κ and with no good hole, then G is (r, ϕ)-controlled. This idea is very helpful and we used it many times in different situations.

For instance, the simplest use is an idea of the first author from several years ago [105], that we used to prove the theorem of [29]:

6.1 *Let $\ell \geq 0$, and let G be a graph with no hole of length $> \ell$. Let every 2-ball in G have chromatic number at most k. Then G has chromatic number at most $4k\ell$.*

Proof. (Sketch.) Suppose not, choose a component with chromatic number $\chi(G)$, take a levelling of it with levels $L_0, L_1 \ldots$, and choose j such that $\chi(L_j) \geq \chi(G)/2 > 2k\ell$. Let A_1 be a component of $G[L_j]$ with maximum chromatic number, and choose $v_0 \in L_{j-1}$ with a neighbour in A_1. Let A_2 be a component with maximum chromatic number of the graph obtained from A_1 by deleting all neighbours of v_0; and choose a neighbour $v_1 \in A_0$ of v_0, with a neighbour in A_2. Now let A_3 be a component with maximum chromatic number of the graph obtained from A_2 by deleting all neighbours of v_1; and choose a neighbour $v_2 \in A_1$ of v_1 with a neighbour in A_3. We continue this for ℓ steps. (This is Gyárfás's argument to prove 3.4.) At each stage, $\chi(A_i)$ is at least $\chi(A_{i-1}) - k$, and so, since $\chi(A_1) > 2k\ell$, we have $\chi(A_\ell) > k\ell$. It follows that there is some vertex in A_ℓ that has distance at least three from each of $v_0, v_1, \ldots, v_{\ell-1}$; and this vertex has a neighbour $u \in L_{j-1}$. Consequently, u has distance at least two from each of $v_1, \ldots, v_{\ell-1}$; and so the induced path v_0-v_1-$\cdots$-$v_{\ell-1}$ can be extended through A_ℓ to become an induced path between v_0, u of length at least ℓ. Now join v_0, u by a path via $L_0 \cup \cdots \cup L_{j-2}$, and we have a hole of length $> \ell$, a contradiction. This proves 6.1. ∎

So far, we have wanted holes of odd length, but there are other avenues to explore. For instance, what about graphs with no even holes? Addario-Berry, Chudnovsky, Havet, Reed and Seymour [5] claimed to prove that such graphs have "bisimplicial" vertices, vertices whose neighbour set is the union of two cliques. Unfortunately there is a mistake in the paper and it has been withdrawn [6], but a paper [38] by Chudnovsky and the second author (currently being refereed) gives a different

proof. If correct, the result implies that the ideal of graphs with no even holes is χ-bounded, and indeed:

6.2 *If a graph has no even hole then its chromatic number is at most twice its clique number.*

The length of holes modulo 3 is also interesting. For instance, if G is a cycle, the number of stable sets in G of even cardinality minus the number of those of odd cardinality is ± 2 if the cycle has length a multiple of three, and 0 or ± 1 otherwise. Kalai and Meshulam [69, 70] conjectured the following, which was recently proved in [34]:

6.3 *If G has no induced cycle of length divisible by three (and so, no triangles) then the number of stable sets of odd cardinality and the number of even cardinality differ by at most one.*

Kalai and Meshulam also conjectured that the ideal of graphs with no induced cycle of length divisible by three does not contain graphs of arbitrarily large chromatic number. This was proved in a breakthrough paper of Bonamy, Charbit and Thomassé [15]:

6.4 *Every graph with sufficiently large chromatic number contains either a triangle or a hole of length 0 modulo 3.*

(It might be true that all these graphs are three-colourable, and that remains open.) Note that 6.4 implies that if $\mathcal{G}$ is the ideal of graphs G for which, in every induced subgraph, the number of stable sets of odd cardinality and the number of even cardinality differ by at most one, then the graphs in $\mathcal{G}$ have bounded chromatic number. Motivated by topological considerations, Kalai and Meshulam [69, 70] made several other beautiful conjectures connecting graph-theoretic properties with homological ones, some of which we discuss later.

More recently we proved a much stronger theorem that contains all three of the Gyárfás conjectures and 6.4. We have the following [112]:

6.5 *For all integers $k \geq 0$ and $\ell \geq 1$, the ideal of all graphs with no hole of length k modulo ℓ is χ-bounded.*

In fact, we could prove something even more general:

6.6 *Let $n \geq 0$ be an integer, and for $1 \leq i \leq n$ let $p_i \geq 0$ and $q_i \geq 1$ be integers. Let $\mathcal{C}$ be the ideal of all graphs that do not contain n pairwise anticomplete holes $H_1, \ldots, H_n$ where H_i has length p_i modulo q_i for $1 \leq i \leq n$. Then $\mathcal{C}$ is χ-bounded.*

As an application, we used this in [112] to prove two further conjectures of Kalai and Meshulam [69, 70]. For a graph G, we write $I(G)$ for the independence complex of G (that is, the collection of stable sets in G); the *Euler characteristic* of $I(G)$ is the number of stable sets of even cardinality minus the number of odd cardinality. The following, one of the Kalai-Meshulam conjectures, is proved in [112]:

6.7 *For every integer $k \geq 0$ there exists c such that the following holds. Let G be a graph such that the Euler characteristic of every induced subgraph of G has modulus at most k. Then G has chromatic number at most c.*

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The other conjecture of Kalai and Meshulam concerns the "total Betti number" of a graph, that is, the sum of the Betti numbers:

6.8 *For every integer $k \geq 0$ there exists c such that the following holds. If the total Betti number of every induced subgraph of G is at most k, then $\chi(G) \leq c$.*

In both 6.7 and 6.8, the key point was to look for sets of pairwise anticomplete holes of length divisible by three.

Despite the generality of 6.5, its proof is reasonable.[2] Again, we look at the chromatic number of r-balls, and here $r = 8$ turns out to be important. If all 8-balls have bounded chromatic number, we obtain a bound on $\chi(G)$ by an argument using gradings, chaining together in a cycle many pairs of paths whose lengths differ by one modulo ℓ. Next we look at 7-balls; if all 7-balls have bounded chromatic number, but some 8-ball has huge chromatic number, again we use gradings to win (with more-or-less the same argument, except now we can do a little better; the pairs of paths have lengths differing by one, not just by one modulo ℓ); and so on, until we have proved a bound on $\chi(G)$ if all 2-balls have bounded chromatic number. Now we are in much the same situation as when we were using 6.1 to prove the theorem of [29], and the proof is completed the same way. The proof of 6.6 uses the same ideas, with a little more work.

6.5 is very strong, but there are even stronger statements that might be true. We cannot hope to specify the lengths of holes exactly, because there are graphs with large chromatic number and arbitrarily large girth, but perhaps a graph with huge chromatic number and bounded clique number must contain many holes with consecutive lengths. More exactly:

6.9 Conjecture: *For every integer $\ell \geq 0$, the ideal of graphs G such that G does not contain holes of ℓ consecutive lengths is χ-bounded.*

This remains open, but it is true in the triangle-free case. We proved [108]:

6.10 *For every integer $\ell \geq 0$, there exists k such that if G is triangle-free and $\chi(G) > k$, then G has ℓ holes of consecutive lengths.*

The proof of this was surprisingly difficult, and so far we see no way to extend it to graphs that have triangles.

7   Subdivisions

One way to formulate the result of [29] is:

7.1 *For every cycle C, the ideal of all graphs containing no subdivision of C as an induced subgraph is χ-bounded.*

We already mentioned the result from [106], that:

7.2 *For every tree T, the ideal of all graphs containing no subdivision of T as an induced subgraph is χ-bounded.*

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[2]One of the referees said: "Reading the proofs was even occasionally an enjoyable activity" – we are very proud of that!

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What about other graphs, not cycles or trees? Let us say H is *weakly pervasive* if it has the property of these two theorems, that the ideal of all graphs containing no subdivision of H as an induced subgraph is χ-bounded. The first author [106] made the conjecture that, in fact, all graphs are weakly pervasive.

Unfortunately this is false; Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter and Walczak [96] showed that the Burling graph (described in section 2) is a counterexample. If H is obtained from a graph G by subdividing every edge exactly ℓ times, we say that H is an ℓ-*subdivision* of G; and similarly if every edge is subdivided at least ℓ times, we call it a $(\geq \ell)$-*subdivision*). Let H be a 1-subdivision of K_5; then the Burling graph contains no subdivision of H as an induced subgraph, and so H is not weakly pervasive, and this disproves the conjecture.

It would be nice to characterize all the weakly pervasive graphs, but this is still open. A more tractable question is, which graphs H have the property that every subdivision of H is weakly pervasive? We call such a graph H *pervasive*. (And extend the definition to multigraphs, for convenience.) This we can come much closer to answering.

Note that 7.1 above is equivalent to saying that the multigraph with two vertices and two parallel edges is pervasive. Let us say a multigraph is a *banana tree* if it can be obtained by adding parallel edges to a tree. In [109], we proved the following, which contains both 7.1 and 7.2:

7.3 *Every banana tree is pervasive.*

In the reverse direction, which graphs do we know not to be pervasive? We have seen that K_5 is not, because its 1-subdivision is not contained as an induced subdivision in the Burling graph. In fact there are smaller graphs with the same property. For instance, let H be obtained from K_4 by subdividing once every edge of a cycle of length four; then Chalopin, Esperet, Li and Ossona de Mendez proved in [21] that the Burling graph contains no induced subdivision of H. Indeed, in that paper they figured out exactly which graphs H had the property that, for every subdivision H' of H, the Burling graph contains an induced subdivision of H'. We need to describe their result.

Say a *chandelier* is a graph obtained from a tree by adding a new vertex (called the *pivot*) adjacent to all the leaves of the tree. A *tree of chandeliers* is a graph that can be obtained recursively, by starting with a chandelier G_1, and at each step identifying the pivot of some new chandelier with a vertex of the graph G_i that we have already constructed, forming G_{i+1}. A *forest of chandeliers* is a graph in which every component is a tree of chandeliers. It is proved in [21] that:

7.4 *A graph H has the property that, for every subdivision H' of H, a (large enough) Burling graph contains an induced subdivision of H', if and only if H is a forest of chandeliers.*

In view of 7.4, every graph that is pervasive in the ideal of all graphs must be a forest of chandeliers. This suggests:

7.5 Conjecture: *A graph is pervasive in the ideal of all graphs if and only if it is a forest of chandeliers.*

Conveniently, every subdivision of a forest of chandeliers is another forest of chandeliers, so to prove that they are all pervasive, it is enough to prove that they are all weakly pervasive. We have some evidence for this, in addition to the banana trees result above.

We recall that ϕ is a function, a graph G is (r, ϕ)-*controlled* if for every induced subgraph H of G, $\chi(H) \leq \phi(k)$ where k is the maximum chromatic number of r-balls in H. Say an ideal $\mathcal{I}$ is

r-controlled if there is a function ϕ such that every graph in the ideal is (r, ϕ)-controlled. We (with Chudnovsky) proved in [30] that:

7.6 *Every forest of chandeliers is pervasive in every r-controlled ideal, for all $r \geq 2$.*

In fact we proved more than this, we showed the next two results:

7.7 *Let $m \geq 0$, let $r \geq 2$, and let $\mathcal{I}$ be an r-controlled ideal. The ideal of all graphs in $\mathcal{I}$ that do not contain the s-subdivision of the complete bipartite graph $K_{m,m}$ as an induced subgraph, for $1 \leq s \leq r + 2$, is 2-controlled. Consequently, for all H and $r \geq 2$, every r-controlled ideal of H-subdivision-free graphs is 2-controlled.*

7.8 *Let $\kappa, m \geq 0$, and let H be a forest of chandeliers. Let $\mathcal{I}$ be a 2-controlled ideal of graphs all with clique number at most κ. Then every graph in $\mathcal{I}$ with sufficiently large chromatic number contains either H or the 1-subdivision of $K_{m,m}$ as an induced subgraph.*

The first of these reduces our problem from r-controlled ideals to 2-controlled ideals (because if G contains an induced s-subdivision of the complete bipartite graph $K_{m,m}$ for m large enough, then it contains an induced subdivision of any fixed graph); and the second says that in the 2-controlled case we can do better than just getting an induced subdivision of H; we can either get H itself as an induced subgraph, or the 1-subdivision of $K_{m,m}$. (In fact we proved 7.8 for graphs H that are much more general than forests of chandeliers, but we omit the details here.) Together, 7.7 and 7.8 imply that forests of chandeliers are weakly pervasive in every r-controlled ideal, and so prove 7.6.

The proof of 7.7 was straightforward Ramsey theory, but 7.8 was intricate. It used a refinement of the methods used in [29]; let us sketch some of the ideas. We can assume there is a 2-ball with huge chromatic number, say with centre z_1. The neighbours of z_1 have bounded chromatic number, by induction on clique number; so the set B_1 of vertices with distance exactly two from z_1 has huge chromatic number. In $G[B_1]$, there is therefore a 2-ball with huge (not quite so huge) chromatic number, say with centre z_2; and again we look at its second neighbours B_2 in $G[B_1]$. This contains a 2-ball with huge chromatic number, and so on. Eventually we have a sequence of vertices $z_1, z_2, \ldots, z_m$, pairwise nonadjacent, and a set of neighbours A_i of z_i for $1 \leq i \leq m$, pairwise disjoint, and a set C with large chromatic number, disjoint from everything else, such that each A_i covers C, and each A_i covers $A_j \cup \{z_j\}$ for $j > i$, and z_i has no neighbours in $A_j \cup \{z_j\}$ for $j > i$. Now z_j has neighbours in A_i. Perhaps the vertices in A_i nonadjacent to z_j still cover a large part of C, or perhaps not; and by Ramsey's theorem, we can assume the same happens for all pairs $i < j$. In one case we can arrange to have the same structure as before except that now z_j is anticomplete to A_i for $i \neq j$; and this case is fairly straightforward, and we obtain a 1-subdivision of $K_{m,m}$. In the other case, we can arrange that every vertex in A_i is adjacent to z_j for $j > i$ (at least in the triangle-free case; when there are triangles, we need to replace the z_i's by cliques and look at the common neighbours of these cliques). After a lot of intricate analysis, we find a surprisingly rich supply of induced subgraphs, and in particular we can find H itself as an induced subgraph.

A *string graph* is the intersection graph of a set of curves in the plane. String graphs are particularly nice in this context, for three reasons:

- Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter and Walczak [96] showed that the Burling graph is a string graph;

- the ideal of all string graphs is 2-controlled (see 8.3); and

- no string graph contains a (≥ 1)-subdivision of $K_{3,3}$ as an induced subgraph.

Consequently, we have

7.9 *For every forest of chandeliers H and all $\kappa \geq 0$, every string graph G with $\omega(G) \leq \kappa$ and $\chi(G)$ sufficiently large contains H as an induced subgraph.*

This is much more than saying that H is pervasive in the ideal of string graphs; we get not only an induced subdivision of H, but H itself.

What happens for ideals that are not r-controlled? Let us say a multigraph H is *widespread* if for every subdivision H' of H, there exists $r \geq 0$ such that for every ideal of H'-subdivision-free graphs is r-controlled. Being widespread is roughly complementary to being pervasive in r-controlled ideals; a graph is both pervasive in all r-controlled ideals for all r, and widespread, if and only if it is pervasive in the ideal of all graphs.
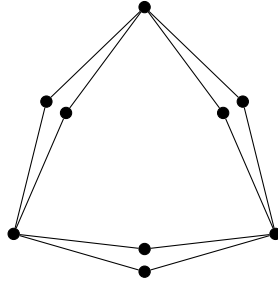

Figure 3: A widespread graph that is not a forest of chandeliers

So we would like to determine which graphs are widespread. In view of the conjecture 7.5, we expected the answer to be, again, forests of chandeliers, but this is false. In [109], we found a multigraph that is not a forest of chandeliers, that is widespread: the multigraph obtained from a triangle by adding many parallel edges between two of the three pairs of vertices, and adding one parallel edge (so, two total) between the last pair. (This can be converted to a graph with the same properties by subdividing edges.) This does not disprove the conjecture 7.5, but it does mean we have no idea which graphs are widespread. Perhaps they all are? As discussed in [109], the case of 2-controlled graphs is critical here; indeed, the conjecture that all graphs are widespread is equivalent to the following.

7.10 Conjecture: *For every graph H, the ideal of all H-subdivision-free graphs is 2-controlled.*

8   Graphs with geometric representations

A particularly interesting class of problems arises when we consider graphs that arise from geometric constructions. Many of these are *intersection graphs*: given a collection $\mathcal{F}$ of sets, the intersection graph $I(\mathcal{F})$ has vertex set $\mathcal{F}$, and distinct $X, Y \in \mathcal{F}$ are adjacent whenever $X \cap Y$ is nonempty. For example, if we begin with a collection of intervals in the real line, then the corresponding intersection

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graph is an *interval graph.* These are well known to be perfect, and therefore χ-bounded. More generally, a *d-dimensional box graph* is an intersection graph of a collection of boxes (i.e. products of intervals) in $\mathbb{R}^d$. So 1-dimensional box graphs are interval graphs. Asplund and Grünbaum [11] showed that the ideal of 2-dimensional box graphs is χ-bounded. Surprisingly, the ideal of 3-dimensional box graphs is not χ-bounded: this follows from the construction of Burling discussed in section 2, which can be realised as a 3-dimensional box graph. See Gyárfás and Lehel [64] and Gyárfás [62] for related results.

Burling's construction has repeatedly proved useful. For example, another generalization of interval graphs is the ideal of intersection graphs of line segments in the plane. In the 1970s, Erdős asked whether this ideal contains triangle-free graphs with arbitrarily large chromatic number. This was resolved only recently: the beautiful paper of Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter and Walczak [96] (see also [95]) already mentioned in section 7 shows that Burling's construction can be realised as an intersection graph of line segments in the plane, thus answering the question of Erdős in the positive, and showing that this ideal is not χ-bounded.

An important, and more general, ideal is the ideal of string graphs (also discussed in the previous section). A *string graph* is the intersection graph of a collection of curves in the plane (see, for instance, [88]). The ideal of string graphs is clearly not χ-bounded, as it contains the ideal of intersection graphs of line segments in the plane, but various interesting subideals are. McGuinness (see [91, 90, 89]) made a significant study of intersection graphs of objects in the plane. Among other things, he showed that the ideal of triangle-free intersection graphs of simple curves that cross a fixed line in exactly one point have bounded chromatic number. This was generalized by Suk [117], who showed that the ideal of intersection graphs of simple families of curves that intersect the y-axis and do not intersect any vertical line in more than one point is χ-bounded. A nice consequence of this is the following:

8.1 *The ideal of intersection graphs of unit segments in the plane is χ-bounded.*

Further progress was made by Lasoń, Micek, Pawlik and Walczak [81], Rok and Walczak [99] and Suk and Walczak [118]. A very general result was proved by Rok and Walczak [100]:

8.2 *For every integer $t \geq 1$, the ideal of intersection graphs of curves each crossing a fixed curve in at least one and at most t points is χ-bounded.*

A particularly useful property of string graphs in our context is that if they have large chromatic number then some small ball has large chromatic number. The following was proved in [30], building on and simplifying ideas of McGuinness [90].

8.3 *The ideal of string graphs is 2-controlled.*

This is an important ingredient in the proof of 7.9.

There are other interesting ways to define graphs from geometric objects. For example, the *visibility graph* of a set S of points in the plane has vertex set S, with vertices x, y adjacent if the line segment from x to y contains no other points from the set. The class of visibility graphs is not an ideal, because visibility graphs can have induced subgraphs that are not themselves visibility graphs (deleting a vertex can make two other vertices visible to each other). Kára, Pór and Wood [71] showed that visibility graphs with clique number at most four have bounded chromatic number, and conjectured that for all κ, all visibility graphs with clique number at most κ have bounded

chromatic number. But this was disproved by Pfender [98], who showed there are visibility graphs with arbitrarily large chromatic number and with clique number six.

9   Connections: the Erdős-Hajnal conjecture

In this section, we look at cliques and stable sets of graphs in ideals defined by forbidden subgraphs. It is well known from Ramsey theory [52] that every graph on n vertices contains a clique or stable set of size at least $\frac{1}{2} \log n$. This is tight up to a constant factor: considering random graphs shows that most graphs on n vertices contain no clique or stable set of size more than $2 \log n$ [47]. The celebrated Erdős-Hajnal conjecture asserts that H-free graphs (that is, graphs that do not contain an induced copy of H) have much larger cliques or stable sets. Let us say that an ideal $\mathcal{I}$ has the *Erdős-Hajnal property* if there is some $\epsilon > 0$ such that every graph $G \in \mathcal{I}$ has a clique or stable set of size at least $|G|^{\epsilon}$. The Erdős-Hajnal conjecture [48, 49] is the following:

9.1 Conjecture: *For every graph H, the ideal of H-free graphs has the Erdős-Hajnal property.*

There is a natural connection here to χ-boundedness: if an ideal $\mathcal{I}$ is χ-bounded with a *polynomial* χ-binding function f then $\mathcal{I}$ satisfies the Erdős-Hajnal property, as every graph $G \in \mathcal{I}$ satisfies

$$\alpha(G) \geq |G|/\chi(G) \geq |G|/f(\omega(G))$$

and so $\alpha(G)f(\omega(G)) \geq |G|$. There is no implication in the other direction. For example, the ideal of triangle-free graphs has the Erdős-Hajnal property, but is not χ-bounded.

There is an interesting property that is stronger than the Erdős-Hajnal property: an ideal $\mathcal{I}$ has the *strong Erdős-Hajnal property* if there exists $\varepsilon > 0$ such that for every graph $G \in \mathcal{I}$ with $|G| > 1$, there exist disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq \varepsilon|G|$ such that A, B are complete or anticomplete. Here, we say that two disjoint sets A, B are *complete* if every vertex in A is adjacent to every vertex in B, and *anticomplete* if there are no edges between A, B. It is not hard to show that if an ideal has the strong Erdős-Hajnal property then it has the Erdős-Hajnal property (see [10, 54, 37]).

Which ideals have the strong Erdős-Hajnal property? Let us start with ideals defined by excluding one graph. The random graph of Erdős described in section 2 has, with high probability, no pair of linear-sized sets that are either complete or anticomplete; so for all g, there is a graph not in $\mathcal{I}$ and with no cycle of length at most g. Consequently, if the ideal of all H-free graphs has the strong Erdős-Hajnal property, then H has no cycles, and nor does its complement; so H has at most four vertices. Thus, for ideals defined by excluding one graph, the strong Erdős-Hajnal property is not very interesting.

What about ideals defined by excluding a finite set of induced subgraphs? Once again, because of Erdős' random graph, one of the excluded subgraphs must be a forest and one must be the complement of a forest. In this case, an interesting result was proved by Bousquet, Lagoutte and Thomassé [18]:

9.2 *For every path H, the ideal of all graphs that contain neither H nor $\overline{H}$ has the strong Erdős-Hajnal property.*

This was extended by Choromanski, Falik, Liebenau, Patel and Pilipczuk [23], who proved the same for trees H formed from a path by adding a leaf adjacent to its third vertex. Which other graphs H

can we take here? It is clearly necessary for one of $H, \overline{H}$ to be a forest. Liebenau and Pilipczuk [82] conjectured that this is sufficient: that for every forest H, the ideal of all $\{H, \overline{H}\}$-free graphs has the strong Erdős-Hajnal property. This has recently been proved, as we discuss below.

A useful tool in these problems is a theorem of Rödl [101], which says:

9.3 *For every graph H and all $\varepsilon > 0$ there exists $\delta > 0$ such that for every H-free graph G, there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that in one of $G[X], \overline{G}[X]$, every vertex in X has degree at most $\varepsilon|X|$.*

The proof is a straightforward regularity lemma argument, but the result is important as it allows us to concentrate on the sparse and dense cases. If we are excluding a pair $H, \overline{H}$ then the problem is invariant under complementing G. So to prove that the ideal of all $\{H, \overline{H}\}$-free graphs has the strong Erdős-Hajnal property, it suffices to show the "one-sided" result that for some $\varepsilon > 0$, if $G \in \mathcal{I}$ has at least two vertices and has maximum degree less than $\varepsilon|G|$, then there exist disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq \varepsilon|G|$ such that A, B are anticomplete ("complete" is not an outcome since the maximum degree is less than $\varepsilon|G|$).

In [83], Liebenau, Pilipczuk, Spirkl and the second author proposed a strengthening of the Liebenau-Pilipczuk conjecture, the statement 9.4 below; and proved it for "subdivided caterpillars", that is, trees such that all their vertices of degree more than two lie in a path. Very recently, we (with Chudnovsky and Spirkl) [36] have proved this conjecture:

9.4 *For every forest H, there exists $\varepsilon > 0$ such that for every H-free graph G with at least two vertices, either some vertex has degree at least $\varepsilon|G|$ or there exist disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq \varepsilon|G|$ such that A, B are anticomplete.*

Once again, this is best possible, because no graph H that is not a forest has this property. Using 9.3, this implies the Liebenau-Pilipczuk conjecture, in full; that is:

9.5 *For all forests H, K, the ideal of all graphs that contain neither H nor $\overline{K}$ has the strong Erdős-Hajnal property.*

We feel this gives a little support to the Gyárfás-Sumner conjecture. It shows that something major happens when a forest is excluded; not what Gyárfás-Sumner predicts, but something. It also suggests that it might be fruitful to look at the following weakening of the Gyárfás-Sumner conjecture, raised by Gyárfás in [60].

9.6 Conjecture: *For all forests H, the ideal of all graphs that contain neither H nor $\overline{H}$ is χ-bounded.*

Another interesting parallel with χ-boundedness comes when we exclude induced subdivisions of a graph. Recall from section 7 that if H is a tree or a cycle (or more generally a banana tree) then the ideal of graphs with no induced subdivision of H is χ-bounded; and that this does not hold unless H is a forest of chandeliers. An intriguing result of Bonamy, Bousquet and Thomassé, extending 9.2, shows:

9.7 *For every cycle C, the ideal of all graphs G such that neither G nor $\overline{G}$ contains an induced subdivision of C has the strong Erdős-Hajnal property.*

We recently (with Chudnovsky and Spirkl) [37] proved the following very substantial extension of this result.

9.8 *For all graphs H, K, the ideal of all graphs G such that G does not contain an induced subdivision of H and $\overline{G}$ does not contain an induced subdivision of K has the strong Erdős-Hajnal property.*

Once again, this follows from a (stronger) "one-sided" result analogous to 9.4:

9.9 *For every graph H, there exists $\varepsilon > 0$ such that every graph G with at least two vertices contains one of the following:*

- *an induced subdivision of H;*

- *a vertex of degree at least $\varepsilon|G|$; or*

- *disjoint anticomplete sets A, B of size at least $\varepsilon|G|$.*

Interestingly, the proof strategy is in part adapted from some of the χ-boundedness arguments discussed in sections 6 and 7. For further discussion on the connections, see [37].

10   Gyárfás' complementation conjecture

Let us mention one other result about χ-boundedness. The idea of χ-boundedness grew as a generalization of perfect graphs; and the complement of a perfect graph is perfect. But if an ideal $\mathcal{I}$ is χ-bounded, it does not follow that the ideal of all complements of members of $\mathcal{I}$ is χ-bounded. For instance, the ideal of all graphs with stability number at most two is χ-bounded $(\chi(G) \leq |G| \leq \omega(G) + \omega(G)^2$, for all graphs G in the ideal); but the ideal of their complements is not; while the ideal of intersection graphs of line segments in the plane is not χ-bounded, but the ideal of complements of these graphs is (this follows from work of Pach and Törőcsik [94]).

Let f be a function from the set of nonnegative integers into itself. We recall that f is a χ-*binding function* for an ideal $\mathcal{I}$ if $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{I}$. Let $\mathcal{I}_f$ be the maximal ideal with χ-binding function f (it is unique). We say f has a *complementary χ-binding function* if the ideal of complements of members of $\mathcal{I}_f$ is χ-bounded. Thus, Lovász's theorem [86], that the complement of a perfect graph is perfect, implies that the function $f(x) = x$ has a complementary χ-binding function, and we might ask which other functions do.

Gyárfás [60] proved that if f has a complementary χ-binding function then $\inf_{x \to \infty} f(x)/x = 1$ (see [65] for a sharpening), and conjectured that for all c the function $f(x) = x + c$ has a complementary χ-binding function. We proved this in [110]:

10.1 *Let $c \geq 0$, and let $\mathcal{I}$ be an ideal such that $\chi(G) \leq \omega(G) + c$ for all $G \in \mathcal{I}$. Let $\mathcal{I}'$ be the ideal of complements of members of $\mathcal{I}$. Then $\mathcal{I}'$ is χ-bounded.*

The proof is quite nice, so here is a sketch. No graph in $\mathcal{I}'$ has $c + 1$ odd holes, pairwise anticomplete; for otherwise there would be a graph in $\mathcal{I}$ consisting of $c + 1$ odd antiholes, pairwise complete, and this graph has chromatic number $c + 1$ more than its clique number, contrary to the hypothesis. This turns out to be all we need; we prove that every ideal of graphs without $c + 1$ odd holes, pairwise anticomplete, is χ-bounded. Thus, we need to show that if G does not have $c + 1$

odd holes, pairwise anticomplete, and has clique number at most κ, then $\chi(G) \leq f(c, \kappa)$, for some appropriate function $f(c, \kappa)$. This follows from 6.6, but at the time we had not proved the latter, and the direct proof we found is worth explaining. We use induction on $c + \kappa$. The result holds if $c = 0$, by 1.2, and trivially if $\kappa = 0$, so we may assume that $c, \kappa > 0$.

Thus we can assume that G has an odd hole; let C be a shortest odd hole. From the inductive hypothesis, the set of vertices with no neighbour in C has chromatic number at most $f(c - 1, \kappa)$; so we just need to show that the set of vertices that have neighbours in C has bounded chromatic number. If C has bounded length, then this is true, since for every vertex in C, its set of neighbours has bounded chromatic number, from the inductive hypothesis. So we can assume $|C|$ is as large as we like. But C is a shortest odd hole, so we can say a lot about the vertices with neighbours in it. Indeed, the algorithm of [24] was mostly concerned with these neighbours, and we can use many of the same ideas here. We omit further details.

11   Operations on χ-bounded ideals

In this section we consider the closure of ideals under various operations. We shall be interested in operations that act on a single graph, or a finite set of graphs, and produce a graph as output. The *closure* of an ideal $\mathcal{G}$ under an operation ϕ is the smallest ideal that is closed under ϕ and contains $\mathcal{G}$ (or equivalently, the smallest class that contains $\mathcal{G}$ and is closed under ϕ and taking induced subgraphs). The main question is, under which operations does the closure of a χ-bounded ideal $\mathcal{G}$ remain χ-bounded? In many cases, this is equivalent to considering an ideal $\mathcal{G}^*$ and some way to decompose elements of $\mathcal{G}^*$ into smaller graphs in $\mathcal{G}^*$. If we write $\mathcal{G}$ for the basic (indecomposable) graphs in $\mathcal{G}^*$, when does the χ-boundedness of $\mathcal{G}$ imply the χ-boundedness of $\mathcal{G}^*$?

For example, it is easy to see that if we take the closure of a χ-bounded ideal $\mathcal{G}$ under the operation of gluing along cliques, then the resulting ideal is χ-bounded (with the same χ-binding function).

A more interesting example is given by substitution. Given graphs G, H and a vertex $v \in V(G)$, the graph obtained by *substituting H for v* is obtained from G by deleting v from G, adding a vertex-disjoint copy of H, and joining every vertex of H to every neighbour of v in G. Lovász [84] showed that the ideal of perfect graphs is closed under substitution. In general, χ-bounded classes are not closed under substitution, but the following was shown in [25]:

11.1 *If $\mathcal{G}$ is a χ-bounded ideal then the closure of $\mathcal{G}$ under substitution is also χ-bounded. Furthermore, if $\mathcal{G}$ has a polynomial χ-binding function then so does its closure; and if $\mathcal{G}$ has an exponential χ-binding function then so does its closure.*

Other operations are known to preserve χ-boundedness. For example, the closure of an ideal under the operation of gluing along at most k vertices (where the two graphs being glued together have the same induced subgraph on the overlap) preserves χ-boundedness (this is proved in [25], or can be deduced from from earlier work of Alon, Kleitman, Saks, Seymour, and Thomassen [8]). The closure of a χ-bounded ideal under 1-joins is also χ-bounded (see Dvořák and Král' [45], Bonamy and Pilipczuk [17], and Kim, Kwon, Oum and Sivaraman [77]). It would be interesting to know what happens with other graph compositions (see [25] for discussion).

A little is known about combining operations. It is known that the closure under both substitution and gluing along cliques preserves χ-boundedness, as does the closure under gluing along sets of

bounded size and cliques [25]. A more general result was proved by Penev [97], who shows that it is possible to add in amalgams as well. However the following question from [25] is open.

11.2 Question: *Is the closure of a χ-bounded ideal under substitution and gluing along a bounded number of vertices χ-bounded?*

In general, given two operations that separately preserve χ-boundedness of ideals when taking the closure, it is not clear that χ-boundedness is preserved when taking the closure under both operations together. However, the following generalizes a problem raised in [25].

11.3 Problem: *Suppose that ϕ_1 and ϕ_2 are operations such that the closure of every χ-bounded ideal under either ϕ_1 or ϕ_2 is χ-bounded. Is the closure of every χ-bounded ideal under ϕ_1 and ϕ_2 χ-bounded?*

Note that it is important that the closure is an ideal: if we instead consider just the smallest class closed under applying the operations and do not demand that it is closed under taking induced subgraphs then the problem has a negative answer (see [25], which uses this weaker form of closure).

12 Open problems

Here are some open questions, different from those discussed earlier. (There are yet more in [114].)

Triangle-free subgraphs

There is a fundamental conjecture, due to Louis Esperet (unpublished), although it might have been asked before. (See [120], where they say "This has been asked several times by researchers, but we could not find a reference.")

12.1 Conjecture: *For all κ, n, every graph with sufficiently large chromatic number and clique number at most κ has a triangle-free induced subgraph with chromatic number at least n.*

If we relax our requirements and do not demand that the subgraph be induced, then the conjecture is true: Rödl [102] proved that, for all n, every graph with sufficiently large chromatic number contains a triangle-free subgraph with chromatic number at least n. (The stronger conjecture of Erdős and Hajnal [50] that every graph with huge chromatic number contains a subgraph with large girth and large chromatic number is still open.)

Conjecture 12.1 would serve to reduce many questions about χ-boundedness to questions about triangle-free graphs, which might be much easier. More exactly, it would imply that an ideal is χ-bounded if and only if the triangle-free graphs in it have bounded chromatic number.[3] For instance, every triangle-free graph with chromatic number at least three has an odd hole, so 1.2 would be immediate; and the conjecture before 6.10 would be reduced to 6.10 itself. This is a great question, and we do not even know whether it holds for $n = 4$ (it is true for $n = 3$ by 1.2).

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[3]Note that it is important here that we are considering an ideal: for example the class of graphs of form $K_3 \cup H$, where H is triangle-free, is not χ-bounded, and it contains no triangle-free graphs; but then it is not closed under taking induced subgraphs. A similar trick, taking a sequence of graphs with increasing clique number and much more rapidly increasing chromatic number, can be used to construct a class of graphs where the optimal χ-binding function grows at arbitrary rate.

Polynomial χ-boundedness

Another question from Esperet [53]: an ideal is *polynomially χ-bounded* if it has a polynomial χ-binding function (defined in section 10). We have no example of an ideal that is known to be χ-bounded and not polynomially χ-bounded (although there are many that are known to be χ-bounded and not known to be polynomially χ-bounded). Could it be true that every χ-bounded ideal is polynomially χ-bounded? This would imply that every χ-bounded ideal satisfies the Erdős-Hajnal conjecture.

Some ideals of graphs are known to be polynomially χ-bounded (see for instance Schiermeyer and Randerath [104], and Bonamy and Pilipczuk [17]). But it is easy to see that there is no d such that every χ-bounded ideal is bounded by a polynomial of degree d. (Consider the ideal of graphs with no stable set of size t.) As noted by Trotignon and Pham [122] (which also contains interesting further discussion), even the following problem is open.

12.2 Problem. *Is it true that, for every t, the ideal $\mathcal{I}_t$ of graphs with no induced t-vertex path is polynomially χ-bounded?*

(This is open even for $t = 5$, and its truth in that case would settle the smallest open case of the Erdős-Hajnal conjecture.) Gyárfás [60] shows that every χ-binding function f for $\mathcal{I}_t$ must satisfy

$$f(\omega) \geq \frac{R(\lceil t/2 \rceil, \omega + 1) - 1}{\lceil t/2 \rceil - 1},$$

as a graph G with no stable set of size $\lceil t/2 \rceil$ does not contain an induced path on t vertices and has chromatic number at least $|G|/\alpha(G) \geq |G|/(\lceil t/2 \rceil - 1)$. By results of Spencer [115] on off-diagonal Ramsey numbers (see Bohman and Keevash [14] for a strengthening), $f(\omega) = \Omega((\omega/\log \omega)^{(t+1)/4})$ for fixed t, so any polynomial χ-binding function for $\mathcal{I}_t$ must have degree at least $(t+1)/4$. From the other side, 3.4 implies that $\mathcal{I}_t$ is χ-bounded, and that proof gives a singly-exponential χ-binding function for $\mathcal{I}_t$. The best result we know on these lines is due to Gravier, Hoàng and Maffray [58]:

12.3 *For all $t \geq 4$, if a graph G has no induced t-vertex path, then $\chi(G) \leq (t-2)^{\omega(G)-1}$.*

Trotignon and Pham [122] raised another intriguing question on the growth of χ-binding functions. Given a χ-bounded ideal $\mathcal{G}$, we can define an "optimal" χ-binding function $f_{\mathcal{G}}$ by

$$f_{\mathcal{G}}(k) = \max\{\chi(G) : G \in \mathcal{G}, \omega(G) \leq k\}.$$

Suppose we know the value of $f_{\mathcal{G}}(2)$. Does this tell us anything about the possible values of $f_{\mathcal{G}}(3)$? If $f_{\mathcal{G}}(2) = 2$ then $\mathcal{G}$ is contained in the ideal of graphs with no odd hole, and so $f_{\mathcal{G}}(3)$ is bounded (in fact, a result of Chudnovsky, Robertson, Seymour and Thomas [27] implies that $f_{\mathcal{G}}(3) \leq 4$). But what if $f_{\mathcal{G}}(2) = 3$? Can $f_{\mathcal{G}}(3)$ be arbitrarily large? This would be answered by 12.1, but might be easier than solving 12.1 in general.

Rainbow subgraphs

When we were discussing rainbow subgraphs before, we coloured the graph with an arbitrary number of colours; but what if we colour it optimally? Thus, which graphs H have the following property? Let G be a graph with huge chromatic number and bounded clique number, coloured with $\chi(G)$ colours; then a rainbow copy of H is present as an induced subgraph.

Again, all such graphs H are forests, but more than just paths; at least all stars and all paths have this property. Could it be that every forest has the property?

Hole lengths

As discussed in section 6, the three Gyárfás conjectures on holes are now resolved; indeed, 6.5 gives a much stronger result. But many further questions remain. Let us say that a set S of integers is *constricting* if the ideal of graphs that do not contain a hole with length in S is χ-bounded. What sets are constricting? Conjecture 6.9 would imply the following:

12.4 Conjecture: *Every infinite set S of natural numbers with bounded gaps is constricting.*

Could we make do with even sparser sets of integers? Perhaps the following wild conjecture is true:

12.5 Conjecture: *A set of integers is constricting if and only if it has strictly positive lower density.*

Note that we cannot replace lower density by upper density here: choose a sequence $G_1, G_2, \ldots$ of triangle-free graphs with $\chi(G_i) > i$ and with girth growing sufficiently rapidly (say $g(G_{i+1}) > 2^{|G_i|}$), and let $\mathcal{G}$ be the set of all induced subgraphs of the G_i. Then $\mathcal{G}$ is clearly not χ-bounded, but the set of hole lengths that do not occur in $\mathcal{G}$ has upper density 1.

We are far from proving 12.5 in either direction, even for triangle-free graphs. At the moment, we do not even know whether the following conjecture (which contradicts 12.5!) is true.

12.6 Conjecture: *There is a set of integers with upper density 0 that is constricting.*

It would be very interesting to resolve this. Note that we can assume that A contains all integers between 3 and some large constant, so that we are considering graphs of large girth. In the case where we do not demand that our cycles are induced, more is known: Verstraete [124] showed that there is a set A of density 0 such that every graph with sufficiently large minimum degree contains (as a subgraph) a cycle whose length belongs to A.

Another intriguing line of inquiry is to look at whether the holes we get are short or long. Let us focus on triangle-free graphs (as the problems are already hard enough there). Of course, a triangle-free graph with large chromatic number may not contain short holes; but if it does not, then perhaps we get holes of many different lengths. One way to say this is the following:

12.7 Problem: *Are there functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(t) \to 0$ and $g(t) \to \infty$ as $t \to \infty$ and the following holds? Let G be a triangle-free graph and let C be the set of lengths of holes in G. Then*

$$\sum_{t \in C} f(t) \geq g(\chi(G)).$$

Similarly, perhaps there is a version of 6.10 that takes account of the length of the holes:

12.8 Problem: *Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(t) \to \infty$ as $t \to \infty$ and the following holds: for every triangle-free graph G of sufficiently large chromatic number, there is some t such that G contains holes of $f(t)$ consecutive lengths, each at most t?*

As a first step towards 12.7 and 12.8, surely the following variant of 6.10 must be true:

12.9 Conjecture: *There is an integer k such that the following holds. For every integer t, every graph with chromatic number at least k and sufficiently large girth contains holes of t consecutive lengths.*

Excluding $K_{n,n}$

A helpful weakening of several of these problems is to exclude $K_{n,n}$ as well as K_n (or equivalently, to just exclude $K_{n,n}$ as a subgraph rather than an induced subgraph). For instance, the following weakening of the Gyárfás-Sumner conjecture was proved by Hajnal and Rödl independently (see [67]).

12.10 *For every tree T and every integer $n \geq 0$, all T-free graphs not containing $K_{n,n}$ as a subgraph (not necessarily as an induced subgraph) have bounded chromatic number.*

There is also a beautiful theorem of Kühn and Osthus [80]:

12.11 *For all integers $n \geq 0$, every graph G with large enough average degree has either a $K_{n,n}$ subgraph or a subdivision of K_n as an induced subgraph.*

This is best possible in some sense, because G itself could be a complete bipartite graph and then it contains nothing else (see also [44]). But if we assume that G has large chromatic number instead of just large average degree, then we might hope to prove something stronger. Here is one possibility. By an "odd subdivision" we mean a subdivision where every edge is replaced by a path of odd length. We hope that

12.12 Conjecture: *For every graph G, if its chromatic number is large enough, then either G has a $K_{n,n}$ subgraph or G contains an odd subdivision of K_n as an induced subgraph.*

Using our standard approach via r-controlled ideals, the problem breaks into subproblems, and we have handled all except one of them, so this might come out.

Colouring graphs with no long holes

Say G is *short-holed* if every hole in G has length four. Here is a nice question, suggested by Vaidy Sivaraman (private communication). We know that the ideal of short-holed graphs is χ-bounded, but how well can we bound chromatic number in terms of clique number? We were able to prove (thanks to Sivaraman for discussions on this):

12.13 *If G is short-holed, then $\chi(G) \leq 10^{20} 2^{\omega(G)^2}$.*

This is no doubt still far from the truth; can we at least get a bound that is singly-exponential? As far as we know, maybe $\chi(G) \leq \omega(G)^2$ for all short-holed graphs (or even for all graphs with no odd hole).

On a related topic, the Hoàng-McDiarmid conjecture [68] asserts the following:

12.14 *If G has at least one edge and has no odd hole, then its vertex set can be partitioned into two sets A, B such that every maximum clique intersects both A and B.*

(Note that odd cycles of length more than three do not admit such a partition.) This is not proved even for short-holed graphs. 12.14 would imply a singly-exponential bound on χ in terms of ω.

Recently (private communication) Hoàng has proposed a stronger conjecture, that if G has no odd hole, its vertex set can be partitioned into $\omega(G)$ sets each of which induces a perfect graph. This would imply that $\chi(G) \leq \omega(G)^2$.

Cycles with chords

We know from the results of section 6 that, for every ℓ, the ideal of graphs with no induced cycle of length at least ℓ is χ-bounded. What about cycles with chords?

Trotignon and Vušković [123] showed that the ideal of graphs that do not contain as an induced subgraph a cycle with a unique chord is χ-bounded (such graphs satisfy $\chi(G) = \omega(G)$ if $\omega(G) > 2$, so they are very close to perfect); and Trotignon and Pham [122] showed that this also holds for the ideal of graphs that do not contain an induced subgraph that is a cycle of length at least five with a unique chord (both papers use structural decompositions and obtain polynomial bounds). Aboulker and Bousquet [3] prove that, for $k = 2, 3$, the ideal of graphs that do not contain an induced subgraph that is a cycle with exactly k chords is χ-bounded. More generally, they conjecture:

12.15 Conjecture: *For every k, the ideal of graphs that do not contain an induced subgraph that is a cycle with exactly k chords is χ-bounded.*

It seems possible that a stronger statement holds.

12.16 Conjecture: *For every k, the ideal of graphs that do not contain an induced cycle such that some vertex has at least k neighbours on the cycle is χ-bounded.*

If we do not require that the cycle is induced, then Aboulker [1] conjectures that the chromatic number is at most k (see Trotignon [121], Aboulker, Radovanović, Trotignon and Vušković [4], Bousquet and Thomassé, [19] and Aboulker [1] for results and discussion related to both conjectures).

Vertex-minors

We say H is a *vertex-minor* of G if H can be obtained from an induced subgraph of G by (repeatedly) choosing a vertex v and replacing the subgraph induced on the neighbour set of v by its complement. A "circle graph" is the intersection graph of chords of a circle. The ideal of all circle graphs is closed under taking vertex-minors, and is χ-bounded [61, 62, 79], and indeed Davies and McCarty [41] proved that it is polynomially χ-bounded, with a quadratic binding function. (Incidentally, a result of Kim, Kwon, Oum and Sivaraman [77] shows that for every $n \geq 3$, the ideal of graphs with no vertex-minor isomorphic to the cycle C_n is polynomially χ-bounded.)

Jim Geelen (see [45]) conjectured the following (an ideal is *proper* if it is not the ideal of all graphs):

12.17 Conjecture: *Every proper ideal that is closed under taking vertex-minors is χ-bounded.*

Choi, Kwon, Oum and Wollan [22] proved that every ideal closed under vertex-minors not containing all wheels is χ-bounded. This was extended by a recent result of Geelen, Kwon, McCarty and Wollan [57], proving the same for every ideal closed under vertex-minors that does not include all circle graphs. These result are both superceded by a very recent result by James Davies, claiming that 12.17 is true in general.

Geelen also conjectures (private communication) that for every proper ideal closed under vertex-minors, there are polynomial-time algorithms to compute maximum clique and maximum stable set for its members.

Algorithms

What about algorithms for all the χ-boundedness theorems? For instance, we proved that graphs with no odd hole have chromatic number bounded by a function of their clique number, but can we find such a colouring in polynomial time? There is now a poly-time algorithm to test if a graph has an odd hole [33], but even without the use of that, we observed in [107] that there is a poly-time algorithm that will either find an odd hole or find a clique and a colouring, with the number of colours at most doubly-exponential in the size of the clique, as in the theorem. It would be good to develop results like this for the other χ-boundedness theorems.

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References

[1] P. Aboulker, "Excluding 4-wheels", *J. Graph Theory* **75** (2014), 311–322.

[2] P. Aboulker, J. Bang-Jensen, N. Bousquet, P. Charbit, F. Havet, F. Maffray and J. Zamora, "χ-bounded families of oriented graphs", *J. Graph Theory* 89 (2018), 304–326, `arXiv:1605.07411`.

[3] P. Aboulker and N. Bousquet, "Excluding cycles with a fixed number of chords", *Discrete Applied Mathematics* **180** (2015), 11–24.

[4] P. Aboulker, M. Radovanović, N. Trotignon and K. Vušković, "Graphs that do not contain a cycle with a node that has at least two neighbors on it", *SIAM J. Discrete Math.* **26** (2012), 1510–1531.

[5] L. Addario-Berry, M. Chudnovsky, F. Havet, B. Reed and P. Seymour, "Bisimplicial vertices in even-hole-free graphs", *J. Combinatorial Theory, Ser. B* **98** (2008), 1119–1164.

[6] L. Addario-Berry, M. Chudnovsky, F. Havet, B. Reed and P. Seymour, "Corrigendum to 'Bisimplicial vertices in even-hole-free graphs' ", *J. Combinatorial Theory, Ser. B*, **142** (2020), 374–375.

[7] N. Alon, "Explicit Ramsey graphs and orthonormal labelings", *Electronic Journal of Combinatorics* **1** (1994), article R12.

[8] N. Alon, D. J. Kleitman, M. Saks, P. Seymour, and C. Thomassen. "Subgraphs of large connectivity and chromatic number in graphs of large chromatic number", *J. Graph Theory* **11** (1987), 367–371.

[9] N. Alon, A. Kostochka, B. Reiniger, D. B. West and X. Zhu, "Coloring, sparseness and girth", *Israel J. Math.* **214** (2016), 315–331.

[10] N. Alon, J. Pach, R. Pinchasi, R. Radoičić and M. Sharir, "Crossing patterns of semi-algebraic sets", *J. Combinatorial Theory, Ser. A* **111** (2005), 31–326.

[11] E. Asplund and B. Grünbaum, "On a Coloring Problem", *Mathematica Scandinavica* **8** (1960), 181–188.

[12] J. Babu, M. Basavaraju, L. Chandran and M. Francis, "On induced colorful paths in triangle-free graphs", *Discrete Applied Mathematics* **255** (2019), 109–116, `arXiv:1604.06070`.

[13] C. Berge, "Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind", Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **10** (1961), 114.

[14] T. Bohman and P. Keevash, "The early evolution of the H-free process", *Invent. Math.* **181** (2010), 291–336.

[15] M. Bonamy, P. Charbit and S. Thomassé, "Graphs with large chromatic number induce $3k$-cycles", submitted for publication, `arXiv 1408.2172`.

[16] M. Bonamy, N. Bousquet and S. Thomassé, "The Erdős-Hajnal conjecture for long holes and antiholes", *SIAM J.Discrete Math.* **30** (2016), 1159–1164.

[17] M. Bonamy and M. Pilipczuk, "Graphs of bounded cliquewidth are polynomially χ-bounded", `arXiv:1910.00697`.

[18] N. Bousquet, A. Lagoutte, and S. Thomassé, "The Erdős-Hajnal conjecture for paths and antipaths", *J. Combinatorial Theory, Ser. B* **113** (2015), 261–264.

[19] N. Bousquet and S. Thomassé, "On the chromatic number of wheel-free graphs with no large bipartite graphs", *manuscript*.

[20] J. Burling, *On Coloring Problems of Families of Polytopes*, PhD thesis, University of Colorado (1965).

[21] J. Chalopin, L. Esperet, Z. Li and P. Ossona de Mendez, "Restricted frame graphs and a conjecture of Scott", *Electronic J. Combinatorics* **23** (2016), #P1.30, `arXiv:1406.0338`.

[22] H. Choi, O. Kwon, S. Oum and P. Wollan, "Chi-boundedness of graph classes excluding wheel vertex-minors", *J. Combinatorial Theory, Ser. B* **135** (2019), 319–348.

[23] K. Choromanski, D. Falik, A. Liebenau, V. Patel, and M. Pilipczuk, "Excluding hooks and their complements", *Electronic J. Combinatorics* **25** (2018), #P3.27, `arXiv:1508.00634`.

[24] M. Chudnovsky, G. Cornuéjols, X. Liu, K. Vušković and P. Seymour, "Recognizing Berge graphs", *Combinatorica* **25** (2005), 143–186.

[25] M. Chudnovsky, I. Penev, A. Scott and N. Trotignon, "Substitution and χ-boundedness", *J. Combinatorial Theory, Ser. B* **103** (2013), 567–586.

[26] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, "The strong perfect graph theorem", *Annals of Math.* **164** (2006), 51–229.

[27] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, "K_4-free graphs with no odd holes", *J. Combinatorial Theory, Ser. B* **100** (2010), 313–331,

[28] M. Chudnovsky, A. Scott and P. Seymour, "Excluding pairs of graphs", *J. Combinatorial Theory, Ser. B* **106** (2014), 15–29.

[29] M. Chudnovsky, A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. III. Long holes", *Combinatorica* **37** (2017), 1057–72.

[30] M. Chudnovsky, A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. V. Chandeliers and strings", submitted for publication, `arXiv:1609.00314`.

[31] M. Chudnovsky, A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. XI. Orientations", *European Journal of Combinatorics* **76** (2019), 53–61, `arXiv:1711.07679`.

[32] M. Chudnovsky, A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. XII. Distant stars", *J. Graph Theory* **92** (2019), 237–254, `arXiv:1711.08612`.

[33] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Detecting an odd hole", *J. Assoc. Comput. Mach.*, **67** (2020), 1–12, `arXiv:1903.00208`.

[34] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Proof of the Kalai-Meshulam conjecture", submitted for publication, `arXiv:1810.00065`.

[35] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Induced subgraphs of graphs with large chromatic number. VIII. Long odd holes", *J. Combinatorial Theory, Ser. B* **140** (2020), 84–97, `arXiv:1701.07217`.

[36] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Pure pairs. I. Trees and linear anticomplete pairs", submitted for publication, `arXiv:1809.00919`.

[37] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Pure pairs. II. Excluding all subdivisions of a graph", `arXiv:1804.01060`.

[38] M. Chudnovsky and P. Seymour, "Even-hole-free graphs still have bisimplicial vertices", `arXiv:1909.10967`.

[39] V. Chvátal, "Perfectly ordered graphs", in *Ann. Discrete Math.* **21** (1989), 63–65.

[40] B. Codenotti, P. Pudlák and G. Resta, "Some structural properties of low-rank matrices related to computational complexity", *Theoretical Computer Science* **235** (2000), 89–107.

[41] J. Davies and R. McCarty, "Circle graphs are quadratically χ-bounded", `arXiv:1905.11578`.

[42] B. Descartes, "A three colour problem", *Eureka* **21** (April 1947). (Solution March 1948.)

[43] B. Descartes, "Solution to Advanced Problem no. 4526", *Amer. Math. Monthly* **61** (1954), 352.

[44] Z. Dvořák, "Induced subdivisions and bounded expansion", *European J. Combinatorics* **69** (2018), 143–148, `arXiv:1706.05766`

[45] Z. Dvořák and D. Král', "Classes of graphs with small rank decompositions are χ-bounded", *Europ. J. Comb.* **33** (2012), 679–683.

[46] P. Erdős, "Graph theory and probability", *Canad. J. Math.* **11** (1959), 34–38.

[47] P. Erdős, "Some remarks on the theory of graphs", *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

[48] P. Erdős and A. Hajnal, "On spanned subgraphs of graphs", *Graphentheorie und Ihre Anwendungen* (Oberhof, 1977), www.renyi.hu/~p_erdos/1977-19.pdf.

[49] P. Erdős and A. Hajnal, "Ramsey-type theorems", *Discrete Applied Math.* **25** (1989), 37–52.

[50] P. Erdős, "Problems and results in graph theory and combinatorial analysis", in *Graph Theory and Related Topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977)*, 153–163, Academic Press, New York-London, 1979.

[51] P. Erdős and A. Hajnal, "On chromatic number of infinite graphs", in *Theory of Graphs* (Proc. Coll. Tihany 1966, P. Erdős and G. Katona, eds.), Akadémiai Kiadó, 1968, 83–89. *Acta Math. Acad. Sci. Hungar.* **17** (1966), 61–99.

[52] P. Erdős and G. Szekeres, "A combinatorial problem in geometry", *Compositio Mathematica* **2** (1935), 463–470.

[53] L. Esperet, *Graph Colorings, Flows and Perfect Matchings*, Habilitation thesis, Université Grenoble Alpes (2017), 24.

[54] J. Fox and J. Pach, "Erdős-Hajnal-type results on intersection patterns of geometric objects", *Horizon of Combinatorics (G.O.H. Katona et al., eds.)*, Bolyai Society Studies in Mathematics, Springer, 79–103, 2008.

[55] T. Gallai, "On directed graphs and circuits", in *Theory of Graphs* (Proc. Colloquium Tihany, 1966), Academic Press, 1968, 115–118.

[56] F. Galvin, I. Rival and B. Sands, "A Ramsey-type theorem for traceable graphs", *J. Combinatorial Theory, Ser. B* **33** (1982), 7–16.

[57] J. Geelen, O-J. Kwon, R. McCarty and P. Wollan, "The grid theorem for vertex-minors", https://arxiv.org/pdf/1909.08113.pdf.

[58] S. Gravier, C. T. Hoàng and F. Maffray, "Coloring the hypergraph of maximal cliques of a graph with no long path", *Discrete Math.* **272** (2003), 285–290.

[59] A. Gyárfás, "On Ramsey covering-numbers", *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pp. 801–816. Colloq. Math. Soc. Janos Bolyai **10**, North-Holland, Amsterdam, 1975.

[60] A. Gyárfás, "Problems from the world surrounding perfect graphs", *Proceedings of the International Conference on Combinatorial Analysis and its Applications*, (Pokrzywna, 1985), *Zastos. Mat.* **19** (1987), 413–441.

[61] A. Gyárfás, "On the chromatic number of multiple interval graphs and overlap graphs", *Discrete Math.* **55** (1985), 161–166.

[62] A. Gyárfás, Corrigendum: "On the chromatic number of multiple interval graphs and overlap graphs" *Discrete Math.* **62** (1986), 333.

[63] A. Gyárfás, "Problem 115", *Discrete Math.* **79** (1990), 109–110.

[64] A. Gyárfás and J. Lehel, "Covering and coloring problems for relatives of intervals", *Discrete Mathematics* **55** (1985), 167–180.

[65] A. Gyárfás, Z. Li, R. Machado, A. Sebo, S. Thomassé, and N. Trotignon, "Complements of nearly perfect graphs", *Journal of Combinatorics* **4** (2013), 299–310.

[66] A. Gyárfás and G. Sárközy, "Induced colorful trees and paths in large chromatic graphs", *Electronic J. Combinatorics* **23** (2016), Paper P4.46.

[67] A. Gyárfás, E. Szemerédi and Zs. Tuza, "Induced subtrees in graphs of large chromatic number", *Discrete Math.* **30** (1980), 235–344.

[68] C. Hoàng and C. McDiarmid, "On the divisibility of graphs", *Discrete Math.* **242** (2002), 145–156.

[69] G. Kalai, https://gilkalai.wordpress.com/2014/12/19/when-a-few-colors-suffice/

[70] G. Kalai, https://gilkalai.files.wordpress.com/2010/10/es.pdf

[71] J. Kára, A. Pór and D. Wood, "On the chromatic number of the visibility graph of a set of points in the plane", *Discrete and Comput. Geom.* **34** (2005), 497–506.

[72] H. A. Kierstead and S. G. Penrice, "Radius two trees specify χ-bounded classes", *J. Graph Theory* **18** (1994), 119–129.

[73] H. A. Kierstead and V. Rödl, "Applications of hypergraph coloring to coloring graphs not inducing certain trees", *Discrete Math.* **150** (1996), 187–193.

[74] H. A. Kierstead and W. T. Trotter, "Colorful induced subgraphs", *Discrete Math.* **101** (1992) 165–169.

[75] H. A. Kierstead and Y. Zhu, "Radius three trees in graphs with large chromatic number", *SIAM J. Disc. Math.* **17** (2004), 571–581.

[76] J. H. Kim, "The Ramsey number $R(3, t)$ has order of magnitude $t^2/\log t$", *Random Structures and Algorithms* **7** (1995), 173–207.

[77] R. Kim, O. Kwon, S. Oum and V. Sivaraman, "Classes of graphs with no long cycle as a vertex-minor are polynomially χ-bounded", *J. Combinatorial Theory, Ser. B* **140** (2020), 372–386, `arXiv:1809.04278`

[78] M. Kneser, "Aufgabe 300", *Jahresbericht der Deutschen Math.-Verein.* **58** (1955).

[79] A. Kostochka and J. Kratochvíl, "Covering and coloring polygon-circle graphs", *Discrete Mathematics* **163** (1997), 299–305,

[80] D. Kühn and D. Osthus, "Induced subdivisions in $K_{s,s}$-free graphs of large average degree", *Combinatorica* **24** (2004), 287–304.

[81] M. Lasoń, P. Micek, A. Pawlik and B. Walczak, "Coloring intersection graphs of arc-connected sets in the plane", *Discrete Comput. Geom.* **52** (2014), 399–415.

[82] A. Liebenau and M. Pilipczuk, "The Erdős-Hajnal conjecture for caterpillars and their complements", `arXiv:1710.08701`.

[83] A. Liebenau, M. Pilipczuk, P. Seymour and S. Spirkl, "Caterpillars in Erdős-Hajnal", *J. Combinatorial Theory, Ser. B, J. Combinatorial Theory, Ser. B,* **136** (2019), 33–43, `arXiv:1810.00811`.

[84] L. Lovász, "Perfect graphs", in: *Selected Topics in Graph Theory, Volume 2* (L. W. Beineke, R. J. Wilson, eds.), Academic Prcess, London-New York, 1983, 55–87.

[85] L. Lovász, "Kneser's conjecture, chromatic number, and homotopy", *J. Combinatorial Theory, Ser. A* **25** (1978), 319–324.

[86] L. Lovász, "Normal hypergraphs and the perfect graph conjecture", *Discrete Math.* **2** (1972), 253–267.

[87] L. Lovász, "On chromatic number of finite set-systems", *Acta Math. Acad. Sci. Hungar.* **19** (1968), 59–67.

[88] J. Matoušek, "String graphs and separators", in *Geometry, structure and randomness in combinatorics*, CRM Series **18**, Ed. Norm., Pisa, (2015), 61–97.

[89] S. McGuinness, "Colouring arcwise connected sets in the plane I", *Graphs Combin.* **16** (2000), 429–439.

[90] S. McGuinness, "Colouring arcwise connected sets in the plane II", *Graphs Combin.* **17** (2001), 135–148.

[91] S. McGuinness, "On bounding the chromatic number of L-graphs", *Discrete Math.* **154** (1996), 179–187.

[92] J. Mycielski, "Sur le coloriage des graphes", *Colloq. Math.* **3** (1955), 161–162.

[93] J. Nešetřil and V. Rödl, "A short proof of the existence of highly chromatic graphs without short cycles", *J. Combinatorial Theory, Ser. B* **27** (1979), 225–227.

[94] J. Pach and J. Törőcsik, "Some geometric applications of Dilworth's theorem", *Discrete Comput. Geom.* **12** (1994), 1–7.

[95] A. Pawlik, J. Kozik, T. Krawczyk, M. Lasoń, P. Micek, W. T. Trotter, B. Walczak, "Triangle-free geometric intersection graphs with large chromatic number", *Discrete Comput. Geom.* **50** (2013), 714–726.

[96] A. Pawlik, J. Kozik, T. Krawczyk, M. Lasoń, P. Micek, W. T. Trotter and B. Walczak, "Triangle-free intersection graphs of line segments with large chromatic number", *J. Combinatorial Theory, Ser. B* **105** (2014), 6–10.

[97] I. Penev, I. (2017), "Amalgams and χ-boundedness", *J. Graph Theory* **84** (2017), 57–92.

[98] F. Pfender, "Visibility graphs of point sets in the plane", *Discrete and Computational Geometry*, 39 (2008), 455–459.

[99] A. Rok and B. Walczak, "Outerstring graphs are χ-bounded", *SIAM J. Discrete Mathematics* **33** (2019), 2181–2199, `arXiv:1312.1559`.

[100] A. Rok and B. Walczak, "Coloring curves that cross a fixed curve", *Discrete and Computational Geometry* **61** (2019), 830–851, `arXiv:1512.06112`.

[101] V. Rödl, "On universality of graphs with uniformly distributed edges", *Discrete Math.* **59** (1986), 125–134.

[102] V. Rödl, "On the chromatic number of subgraphs of a given graph", *Proc. Amer. Math. Soc.* **64** (1977), 370–371.

[103] B. Roy, "Nombre chromatique et plus longs chemins d'un graphe" , *Rev. Française Informat. Recherche Opérationnelle* **1** (1967), 129–132.

[104] I. Schiermeyer and B. Randerath, "Polynomial χ-binding functions and forbidden induced subgraphs: a survey", *Graphs and Combinatorics* **35** (2019), 1–31.

[105] A. Scott, "Induced cycles and chromatic number", *J. Combinatorial Theory, Ser. B* **76** (1999), 150–154.

[106] A. Scott, "Induced trees in graphs of large chromatic number", *J. Graph Theory* **24** (1997), 297–311.

[107] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. I. Odd holes", *J. Combinatorial Theory, Ser. B* **121** (2016), 68–84.

[108] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. IV. Consecutive holes", *J. Combinatorial Theory, Ser. B*, **132** (2018), 180–235, `arXiv:1509.06563`.

[109] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. VI. Banana trees", in press, `arXiv:1701.05597`.

[110] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. VII. Gyárfás' complementation conjecture", *J. Combinatorial Theory, Ser. B* 142 (2020), 43-55, `arXiv:1701.06301`.

[111] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. IX. Rainbow paths", *Electronic J. Combinatorics* **24.2**:#P2.53, 2017.

[112] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. X. Holes with specific residue", *Combinatorica* **39** (2019), 1105–1132, `arXiv:1705.04609`.

[113] A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. XIII. New brooms", *European J. Combinatorics* **84** (2020), 103024, `arXiv:1807.03768`.

[114] V. Sivaraman, "Some problems on induced subgraphs", *Discrete Applied Math.* **236** (2018), 422–427.

[115] J. Spencer, "Asymptotic lower bounds for Ramsey functions", *Discrete Math.* **20** (1997), 69–76.

[116] S. Spirkl, *Cliques, Stable Sets, and Coloring in Graphs with Forbidden Induced Subgraphs*, Ph.D. thesis, Princeton University (2018).

[117] A. Suk, "Coloring intersection graphs of x-monotone curves in the plane", *Combinatorica* **34** (2014), 487–505.

[118] A. Suk and B. Walczak, "New bounds on the maximum number of edges in k-quasi-planar graphs", *Comput. Geom.* **50** (2015), 24–33.

[119] D. P. Sumner, "Subtrees of a graph and chromatic number", in *The Theory and Applications of Graphs*, (G. Chartrand, ed.), John Wiley & Sons, New York (1981), 557–576.

[120] S. Thomassé, N. Trotignon and K. Vušković, "A polynomial Turing-kernel for weighted independent set in bull-free graphs", *Algorithmica* **77** (2017), 619–641.

[121] N. Trotignon, "Perfect graphs: a survey", in: Topics in Chromatic Graph Theory, Cambridge University Press, 2015, pp. 137-160.

[122] N. Trotignon and L. A. Pham, "χ-bounds, operations, and chords", *J. Graph Theory* **88** (2018), 312–336.

[123] N. Trotignon and K. Vušković, "A structure theorem for graphs with no cycle with a unique chord and its consequences", *J. Graph Theory* **63** (2010), 31–67.

[124] J. Verstraete, "Unavoidable Cycle Lengths in Graphs", *Journal of Graph Theory* **49** (2005), 151–167.

[125] S. Wagon, "A bound on the chromatic number of graphs without certain induced subgraphs", *J. Combinatorial Theory, Ser. B* **29** (1980), 345–346.

[126] A. Zykov, "On some properties of linear complexes" (in Russian), *Mat. Sbornik N.S.* **24** (1949), 163–188; English translation in: *Amer. Math. Soc. Transl.* 79 (1952).