The edge-density for $K_{2,t}$ minors

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Abstract

Let H be a graph. If G is an n-vertex simple graph that does not contain H as a minor, what is the maximum number of edges that G can have? This is at most linear in n, but the exact expression is known only for very few graphs H. For instance, when H is a complete graph K_t, the "natural" conjecture, $(t-2)n - \frac{1}{2}(t-1)(t-2)$, is true only for $t \leq 7$ and wildly false for large t, and this has rather dampened research in the area. Here we study the maximum number of edges when H is the complete bipartite graph $K_{2,t}$. We show that in this case, the analogous "natural" conjecture, $\frac{1}{2}(t+1)(n-1)$, is (for all $t \geq 2$) the truth for infinitely many n.

1 Introduction

Graphs in this paper are assumed to be finite and without loops or parallel edges. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges.

Mader [5] proved that for every graph H there is a constant C_H such that every graph G not containing H as a minor satisfies $|E(G)| \leq C_H|V(G)|$, but determining the best possible constant C_H for a given graph H is a question that has been answered for very few graphs H.

A particular case that *has* been intensively studied is when H is a complete graph K_t. One natural way to make a large dense graph with no K_t minor is to take a complete graph of size $t-2$, and add $n-t+2$ more vertices each adjacent to all vertices in the complete graph. This produces an n-vertex graph with no K_t minor and with $(t-2)n - \frac{1}{2}(t-1)(t-2)$ edges, and Mader [6] showed that for all $t \leq 7$ and $n \geq t-2$, this is the maximum possible number of edges in an n-vertex graph with no K_t minor. It would be nice if this were true for all t, but Mader also showed that for $t \geq 8$ this is *not* the correct expression, and Kostochka [2, 3] and Thomason [12, 13] showed that for large t and n the maximum number of edges is $O(t(\log t)^{\frac{1}{2}}n)$.

This is disappointing, at least to those with faith in Hadwiger's conjecture. But what about when H is a complete bipartite graph $K_{s,t}$ say? When $s \leq 1$ the problem is very easy, but for $K_{2,t}$ it was open (for $t < 10^{29}$), and is the subject of this paper.

Here is a graph with no $K_{2,t}$ minor (for $t \geq 2$): take a graph each component of which is a t –vertex complete graph, and add one more vertex adjacent to all the previous vertices. This graph has $\frac{1}{2}(t+1)(n-1)$ edges, where n is the number of vertices, and exists whenever t divides $n-1$. We shall show that this is extremal. The following is our main theorem, proved in sections 2–6:

1.1 *Let $t \geq 2$, and let G be a graph with $n > 0$ vertices and with no $K_{2,t}$ minor. Then*

$$|E(G)| \leq \frac{1}{2}(t+1)(n-1).$$

This answers affirmatively a conjecture of Myers [7], who proved 1.1 for all $t \geq 10^{29}$.

As we saw, this is best possible when $n-1$ is a multiple of t, but for other values of n it may not be best possible, and as far as we know, it could be a long way from best possible. For instance, if $n = \frac{3}{2}t$, 1.1 gives an upper bound of about $\frac{1}{2}tn$, but the best lower bound we know is about $\frac{5}{12}tn$.

What if we exclude $K_{1,t}$ instead of $K_{2,t}$? It is easy to see that every n-vertex graph with more than $\frac{1}{2}(t-1)n$ edges contains $K_{1,t}$ as a minor (indeed, as a subgraph), and if t divides n then there is an n-vertex graph with exactly $\frac{1}{2}(t-1)n$ edges with no $K_{1,t}$ minor (the disjoint union of n/t copies of K_t). Thus this question is trivial. Curiously, however, the answer is quite different if we restrict ourselves to connected graphs. The following is shown in [1]:

1.2 *Let $t \geq 3$ and $n \geq t+2$ be integers. If G is an n-vertex connected graph with no $K_{1,t}$ minor, then*

$$|E(G)| \leq n + \frac{1}{2}t(t-3),$$

and for all n, t this is best possible.

We should therefore anticipate some analogous change in the conclusion of 1.1 if we add an appropriate connectivity hypothesis; and versions of 1.1 for higher connectivity are presented in section 8. Assuming G is connected makes no difference (because the extremal example given above is connected anyway); but it turns out that assuming G is 2-connected saves roughly a factor of two, and assuming it is 3-connected makes the bound qualitatively different. To prove the 2-connected result, we need to prove a version of 1.1 when we exclude $K_{2,t}$ as a "rooted" minor, and this is the content of section 7.

More generally, what is the maximum number of edges in graphs with no $K_{s,t}$ minor when $s \geq 1$? If we take a graph each component of which is a clique of size t, and add $s-1$ more vertices each adjacent to all others, then the resulting n-vertex graph has no $K_{s,t}$ minor, and has

$$(t+2s-3)(n-s+1)/2 + (s-1)(s-2)/2$$

edges; is this the maximum? This is true for $s = 1, 2$; and when $s = 3$, Kostochka and Prince have a proof of this for all sufficiently large t (see [9]). It is open for $s = 4, 5$, but for $s \geq 6$ Kostochka and Prince have counterexamples [9]; indeed, Kostochka and Prince [4] proved the following:

1.3 *Let s, t be positive integers with $t \gg s$. Then every graph with average degree at least $t + 3s$ has a $K_{s,t}$ minor, and there are graphs with average degree at least $t + 3s - 5\sqrt{s}$ that do not have a $K_{s,t}$ minor.*

2 The main proof

This and the next four sections are devoted to the proof of 1.1. Let us fix $t \geq 2$ (we can find no advantage in proceeding by induction on t), and suppose the theorem is false for that value of t. Consequently there is a minimal counterexample, that is, a graph G with the following properties:

- G has no $K_{2,t}$ minor
- $|E(G)| > \frac{1}{2}(t+1)(|V(G)|-1)$
- $|E(G')| \leq \frac{1}{2}(t+1)(|V(G')|-1)$ for every graph G' with no $K_{2,t}$ minor and $|V(G')| < |V(G)|$.

We call such a graph G *critical*, and refer to the properties above as the *criticality* of G. Throughout this and the next four sections, let G be a critical graph and let $n = |V(G)|$. Since $|E(G)| > \frac{1}{2}(t+1)(n-1)$, it follows that $n \geq t+2$.

If G is a graph and $X \subseteq V(G)$, $G|X$ denotes the subgraph of G induced on X, and we say X is *connected* if $G|X$ is connected. In this section we prove some preliminary lemmas about critical graphs. In particular, we prove that if G is a critical graph then G is 2-connected, and every edge of G is in at least $\frac{1}{2}t$ triangles, and every two nonadjacent vertices have at least three common neighbours. In order to prove this last statement we first have to show that $t \geq 5$. We begin with:

2.1 *G is 2-connected.*

Proof. For suppose not. Since $n \geq t+2 \geq 3$, there is a partition of $V(G)$ into three nonempty sets $V_1, V_2, \{v\}$ for some vertex v, such that there is no edge between V_1 and V_2. For $i = 1, 2$ let $G_i = G|(V_i \cup \{v\})$; let $|V(G_i)| = n_i$ and $|E(G_i)| = e_i$. From the criticality of G, $e_i \leq \frac{1}{2}(t+1)(n_i - 1)$ for $i = 1, 2$, so, adding, we obtain

$$e_1 + e_2 \leq \frac{1}{2}(t+1)(n_1 + n_2 - 2).$$

But $|E(G)| = e_1 + e_2$ and $n = n_1 + n_2 - 1$, contrary to the criticality of G. This proves 2.1. ∎

If $x, y \in V(G)$ are distinct, an *xy-join* is a vertex z different from x, y and adjacent to both x, y. Let $X(xy)$ denote the set of all xy-joins.

2.2 *For every edge xy of G there are at least $\frac{1}{2}t$ xy-joins, and consequently every vertex has degree at least $\frac{1}{2}t + 1$.*

Proof. Let xy be an edge. Let G' be obtained from G by deleting all edges between x and $X(xy)$, and then contracting the edge xy. (Note that this contraction does not create any parallel edges, and so G' is indeed a "graph" as defined in this paper.) Then $|E(G')| = |E(G)| - |X(xy)| - 1$, and $|V(G')| = n - 1$, and by the criticality of G,

$$|E(G')| \leq \frac{1}{2}(t+1)(|V(G')| - 1).$$

Consequently

$$|E(G)| - |X(xy)| - 1 \leq \frac{1}{2}(t+1)(n-2),$$

and since

$$|E(G)| > \frac{1}{2}(t+1)(n-1)$$

by the criticality of G, it follows that $|X(xy)| \geq \frac{1}{2}t$. This proves the first assertion of 2.2, and the second follows immediately since every vertex is incident with some edge by 2.1. ∎

The *length* of a path or cycle is the number of edges in it. We use $G \setminus x$ to denote the graph obtained from G by deleting x; here x may be a vertex or an edge, or a set of vertices or edges.

2.3 *Let A_1, A_2 be disjoint connected subsets of $V(G)$, such that there is no edge between A_1 and A_2. Let C be the set of all vertices with a neighbour in A_1 and a neighbour in A_2. Then every two nonadjacent vertices in C have a common neighbour in C (and at least two common neighbours in C if t is odd). Consequently if C is nonempty then it is connected.*

Proof. Let $c_1, c_2 \in C$ be nonadjacent; we claim they have a common neighbour in C, and at least two if t is odd. For $i = 1, 2$, there is a path between c_1, c_2 with interior in A_i, since A_i is connected and c_1, c_2 have neighbours in A_i. Choose such a path, P_i say, of minimal length; then it is induced. Let p_i be the neighbour of c_i in P_i, for $i = 1, 2$. No c_1p_1-join belongs to P_1, since P_1 is induced, and none is in P_2 since $p_1 \in A_1$ and all internal vertices of P_2 are in A_2 and there is no edge between A_1 and A_2. Similarly no c_2p_2-join is in P_1 or P_2. Suppose that $|X(c_1p_1) \cup X(c_2p_2)| \geq t$; then by contracting all edges of P_1 except c_1p_1, and all edges of P_2 except c_2p_2, we obtain a $K_{2,t}$ minor, a contradiction. Thus $|X(c_1p_1) \cup X(c_2p_2)| \leq t - 1$. On the other hand, by 2.2, $|X(c_ip_i)| \geq d$, for $i = 1, 2$, where d is the least integer satisfying $d \geq \frac{1}{2}t$. Hence $|X(c_1p_1) \cap X(c_2p_2)| \geq 2d - t + 1$. But every vertex in $X(c_1p_1) \cap X(c_2p_2)$ has neighbours in both A_1 and A_2, and therefore belongs to C, and is a common neighbour of c_1, c_2 in C. This proves 2.3. ∎

A related result is:

2.4 *Let A_1, A_2 be disjoint connected subsets of $V(G)$ with union $V(G)$, and let C be the set of all vertices in A_2 with a neighbour in A_1. Then C is connected.*

Proof. Suppose not; then there is a partition of C into two nonempty subsets X_1, X_2, such that there is no edge between X_1 and X_2. Since A_2 is connected, there is a path of $G|A_2$ with one end in X_1 and the other in X_2. Choose such a path, P_2 say, with minimum length. Let its ends be $c_i \in X_i$ for $i = 1, 2$. Since c_1, c_2 both have neighbours in A_1, there is a minimal path P_1 between c_1, c_2 with interior in A_1. For $i = 1, 2$, let p_i be the neighour of c_i in P_i. By 2.2, $|X(c_ip_i)| \geq t/2$ for $i = 1, 2$, and no c_ip_i-join belongs to P_1 or to P_2, and if $X(c_1p_1) \cap X(c_2p_2) = \emptyset$ then we find a $K_{2,t}$ minor. Thus some vertex $v \in X(c_1p_1) \cap X(c_2p_2)$. Since p_2 does not belong to C, it follows that p_2 has no neighbour in A_1 and so $v \notin A_1$. Consequently $v \in A_2$, since $A_1 \cup A_2 = V(G)$; and v is adjacent to $p_1 \in A_1$, and so $v \in C$; yet v has neighbours in both X_1, X_2, which is impossible. This proves 2.4. ∎

It follows from 2.4 that for every vertex v, the set of neighbours of v is connected (taking $A_1 = \{v\}$ and $A_2 = V(G) \setminus \{v\}$; the latter is connected by 2.1).

2.5 *For every two nonadjacent vertices x, x' there are at least three xx'-joins, and so G is 3-connected.*

Proof. Suppose there are at most two. Since G is 2-connected, there are two induced paths P, Q between x, x', vertex-disjoint except for their ends; and since there are at most two xx'-joins, we may choose P, Q such that every xx'-join is a vertex of one of P, Q. Let p, q be the neighbours of x in P, Q respectively, and define p', q' similarly for x'. Let N be the set of all neighbours of x, and define N' similarly. Let $d = \lceil \frac{1}{2}t \rceil$.

Let us suppose that:

(1) *There do not exist disjoint connected subsets $A, B, C_1, \ldots, C_d$ of $N \cup \{x\}$ with the following properties:*

- *for $1 \le i \le d$ there is an edge of G between C_i and A, and an edge of G between C_i and B*
- *$p \in A$ and $q \in B$.*

We shall derive several consequences of this, and eventually reach a contradiction.

Let H be the subgraph $G|N$. Every vertex of H has degree at least d in H, since for each $v \in V(H)$, there are at least d xv-joins in G, by 2.2. If p has d neighbours in H different from q, we may set $A = \{p\}, B = \{q, x\}$, and let $C_1, \ldots, C_d$ each consist of some neighbour of p different from q, contrary to (1). So p has degree exactly d in H, and p, q are adjacent; let the other neighbours of p be $v_1, \ldots, v_{d-1}$ say. If q is adjacent in H to each of $v_1, \ldots, v_{d-1}$, we may set $A = \{p\}$, $B = \{q\}$, $C_i = \{v_i\}$ for $1 \le i \le d-1$ and $C_d = \{x\}$, contrary to (1). Thus we may assume that $d \ge 2$ and q is not adjacent to v_{d-1}. Let $Y = N \setminus \{p, q, v_1, \ldots, v_{d-1}\}$.

(2) *If r_1-$\cdots$-r_k is a path R of H with $r_1 \in \{v_1, \ldots, v_{d-1}\}$ and $r_2, \ldots, r_k \in Y$, then r_k has at most one neighbour in Y different from $r_2, \ldots, r_{k-1}$.*

For suppose it has two, say y_1, y_2. Let $r_1 = v_j$ say. Then we may set $A = \{p\} \cup V(R), B = \{q, x\}, C_i = \{v_i\}$ for $1 \le i \le d-1$ with $i \neq j$, $C_j = \{y_1\}$, and $C_d = \{y_2\}$, contrary to (1). This proves (2).

Suppose first that $d = 2$; thus every vertex in H has degree at least two. If the edge pq does not belong to a cycle of H, then (by taking a maximal path containing p and not q) it follows that there is a path between p and some vertex of H with degree at least three, not passing through q; but a minimal such path is contrary to (2). Thus there is a cycle of H containing pq, say $p = p$-p_1-$\cdots$-p_k-q-p; but then we may set $A = \{p\}, B = \{p_2, \ldots, p_k, q\}$, $C_1 = \{x\}$, and $C_2 = \{p_1\}$, contrary to (1).

Thus $d \ge 3$. By taking $k = 1$ and $r_1 = v_{d-1}$ we deduce that v_{d-1} has at most one neighbour in H different from all of $p, v_1, \ldots, v_{d-2}$. But v_{d-1} has degree at least d in H, and so v_{d-1} is adjacent to all of $p, v_1, \ldots, v_{d-2}$, and has exactly one more neighbour in H, say v_d.

By taking $k = 2$, $r_1 = v_{d-1}$ and $r_2 = v_d$, we deduce from (2) that v_d has at most one neighbour in Y. Suppose that v_d is not adjacent to q in H. Since v_d has degree at least d in H, v_d is adjacent to all of $v_1, \ldots, v_{d-1}$ and it has exactly one other neighbour in H, say v_{d+1}.

By (2) with $k = 3$ and $r_1 = v_{d-1}, r_2 = v_d$ and $r_3 = v_{d+1}$, we deduce that v_{d+1} has at most one neighbour in Y different from v_d. But each of $v_1, \ldots, v_{d-1}$ has at most one neighbour in Y, and they are adjacent to $v_d \in Y$, as we already saw, so v_{d+1} has at most two neighbours in H different from q. Since v_{d+1} has at least $d \geq 3$ neighbours in H, we deduce that q, v_{d+1} are adjacent. But then we may set $A = \{p\}, B = \{q, v_{d+1}, v_d\}$, $C_i = \{v_i\}$ for $1 \leq i \leq d-1$, and $C_d = \{x\}$, contrary to (1). This proves that v_d is adjacent to q.

If v_d is adjacent to all of $v_1, \ldots, v_{d-1}$, we may set $A = \{p\}$, $B = \{q, v_d\}$, $C_i = \{v_i\}$ for $1 \leq i \leq d-1$ and $C_d = \{x\}$, contrary to (1). So we may assume that v_d is nonadjacent to v_1 say. We already saw that v_d has at most one neighbour in Y; and since it has degree at least d in H, v_d is adjacent to $v_2, \ldots, v_{d-1}, q$ and to one new vertex. If q is adjacent to v_1, we may set $A = \{p\}$, $B = \{q, v_d\}$, $C_i = \{v_i\}$ for $1 \leq i \leq d-1$, and $C_d = \{x\}$, contrary to (1). Thus q is nonadjacent to v_1. By the same argument (with v_1, v_{d-1} exchanged) we deduce that v_1 has a unique neighbour (say v_{d+1}) in Y, and is adjacent to all of $v_2, \ldots, v_{d_1}$, and v_{d+1} is adjacent to all except one of $v_2, \ldots, v_{d-1}$. Now $v_{d+1} \neq v_d$ since v_d is nonadjacent to v_1, and at least $d-3$ of $v_1, \ldots, d_{d-1}$ are adjacent to both v_d, v_{d+1}. Since $v_1, \ldots, v_{d-1}$ each have at most one neighbour in Y, we deduce that $d = 3$. But then we may set $A = \{p\}$, $B = \{q, v_3, v_4\}$, $C_1 = \{v_1\}$, $C_2 = \{v_2\}$ and $C_3 = \{x\}$. This proves that our assumption of (1) was false.

Consequently there exist disjoint connected subsets $A, B, C_1, \ldots, C_d$ of $N \cup \{x\}$ with the following properties:

- for $1 \leq i \leq d$ there is an edge of G between C_i and A, and an edge of G between C_i and B
- $p \in A$ and $q \in B$.

Similarly, if N' denotes the set of neighbours of x', and p', q' are the neighbours of x' in P, Q respectively, there exist disjoint connected subsets $A', B', C'_1, \ldots, C'_d$ of $N' \cup \{x'\}$ with the following properties:

- for $1 \leq i \leq d$ there is an edge of G between C'_i and A', and an edge of G between C'_i and B'
- $p' \in A'$ and $q' \in B'$.

But then contracting all edges with both ends in one of

$$A \cup A' \cup (V(P) \setminus \{x, x'\}), B \cup B' \cup (V(Q) \setminus \{x, x'\}), C_1, \ldots, C_d, C'_1, \ldots, C'_d$$

gives a $K_{2,t}$ minor, a contradiction. This proves 2.5. ∎

3 Vertices of large degree

In this section we prove some results about vertices of degree at least $t+1$, and particularly about vertices with degree close to n. We denote the complement graph of G by $\overline{G}$. A *cut* of G is a partition (A_1, A_2, C) of $V(G)$ such that A_1, A_2 are nonempty, and there is no edge between A_1 and A_2; and if $|C| = k$ we call it a *k-cut*. If $X \subseteq V(G)$, by a *component* of X we mean the vertex set of a component of $G|X$. First we need:

3.1 $n \geq t+4$.

Proof. We are given that $t \geq 2$, and since $|E(G)| > \frac{1}{2}(t+1)(n-1)$ it follows that $t+1 < n$. Suppose that $n = t+2$. Then the complement $\overline{G}$ has fewer than

$$\frac{1}{2}n(n-1) - \frac{1}{2}(n-1)^2 = \frac{1}{2}(n-1)$$

edges, and so some two vertices have degree 0 in $\overline{G}$; so in G these two vertices are both adjacent to all others, and G has a $K_{2,t}$ subgraph, a contradiction.

Now suppose that $n = t+3$. Then $\overline{G}$ has fewer than

$$\frac{1}{2}n(n-1) - \frac{1}{2}(n-2)(n-1) = n-1$$

edges, and so at most $n-2$. Thus there are two vertices of $\overline{G}$ both with degree at most one. If some vertex has degree zero in $\overline{G}$, choose another with degree at most one; then in G they have at least t common neighbours and so G has a $K_{2,t}$ subgraph, a contradiction. So every vertex has degree at least one in $\overline{G}$. Let $v_1, \dots, v_k$ be those with degree one, and $u_1, \dots, u_k$ their respective neighbours. Thus $k \geq 2$. If $u_1 = u_2$ or $u_1 = v_2$, then in G, v_1, v_2 have t common neighbours, a contradiction. Consequently $u_1, \dots, u_k, v_1, \dots, v_k$ are all distinct. If u_1 has only two neighbours in $\overline{G}$, say v_1, w_1, then u_1, v_1 have t common neighbours in G; so each u_i has degree at least three in $\overline{G}$. Hence the sum of the degrees of all vertices in $\overline{G}$ is at least $2n$, a contradiction. This proves 3.1. ∎

3.2 *If x_1, x_2 are nonadjacent vertices then* $\deg(x_1) + \deg(x_2) \leq n+t-4$*, while if x_1, x_2 are adjacent then* $\deg(x_1) + \deg(x_2) \leq n+t-2$.

Proof. Let G_0 be the graph obtained from G by deleting the edge x_1x_2 if it exists (and $G_0 = G$ if not). For $i = 1, 2$ let d_i be the degree of x_i in G_0. We need to show that $d_1 + d_2 \leq n+t-4$. There do not exist t paths in G_0 between x_1, x_2, disjoint except for their ends, because then G would contain a $K_{2,t}$ minor. Thus by Menger's theorem there is a partition of $V(G)$ into three sets A_1, A_2, C with $x_1 \in A_1, x_2 \in A_2$, such that $|C| \leq t-1$ and there are no edges between A_1 and A_2. Now for $i = 1, 2$, $d_i \leq |A_i| + |C| - 1$, and so

$$d_1 + d_2 \leq |A_1| + |A_2| + 2|C| - 2 = n + |C| - 2 \leq n+t-3.$$

We may therefore assume that equality holds, and so $|C| = t-1$ and for $i = 1,2$ x_i is adjacent to every other vertex in $A_i \cup C$. By 2.5 $|C| \geq 3$ and so $t \geq 4$.

By 3.1, $|A_1| + |A_2| \geq 5$ since $|C| \leq t-1$, and so we may assume that $|A_1| \geq 3$. If some $c \in C$ is adjacent to two members a, a' of $A_1 \setminus \{x_1\}$, then contracting the edge x_2c gives a $K_{2,t}$ minor, a contradiction. Thus each vertex in C has at most one neighbour in $A_1 \setminus \{x_1\}$.

Suppose that $A_1 \setminus \{x_1\}$ is stable. Choose distinct $a, a' \in A_1 \setminus \{x_1\}$; then $\deg(a)+\deg(a') \leq |C|+2 = t+1$, contrary to 2.2. Thus there is an edge aa' with $a, a' \in A_1 \setminus \{a_1\}$. By 2.5 there is an ax_2-join, and so there exists $c \in C$ adjacent to a. By 2.2 there are at least $\frac{1}{2}t$ aa'-joins, and so at least two, since $t \geq 3$; let b be an aa'-join different from x_1. Then $b \notin C$, and so $b \in A_1 \setminus \{x_1\}$. Since both a', b are adjacent to both x_1, a, it follows that contracting the edges x_2c and ac gives a $K_{2,t}$ minor, a contradiction. This proves 3.2. ∎

For each vertex $v \in V(G)$, let us define $\text{surplus}(v) = \deg(v)-t$, and for a subset $X \subseteq V(G)$, $\text{surplus}(X)$ denotes the sum of $\text{surplus}(v)$ over all $v \in X$.

3.3 $\text{surplus}(V(G)) \geq n-t$, *and at least three vertices have positive surplus.*

Proof. By the criticality of G, $2|E(G)| \geq (t+1)(n-1)+1$, and so $2|E(G)| - nt \geq n-t$. Consequently

$$\text{surplus}(V(G)) = \sum_{v \in V(G)} (\deg(v)-t) = 2|E(G)| - nt \geq n-t.$$

This proves the first assertion. For the second, note that 3.2 implies that for every two vertices x_1, x_2, $\text{surplus}(x_1)+\text{surplus}(x_2) \leq n-t-2$, and so at least three vertices have positive surplus. This proves 3.3. ∎

3.4 *For every vertex v of G there are at least two vertices nonadjacent to v.*

Proof. Suppose there is at most one such vertex, and so $|A| \geq n-2$, where A is the set of neighbours of v. By 3.3 there are at least three vertices with degree at least $t+1$, so at least one of them is in A, say u. Thus u has at least $t-1$ neighbours in A. Now u, v have at most $t-1$ common neighbours, since G has no $K_{2,t}$ subgraph; and so $|N| = t-1$, where N is the set of neighbours of u in A. By 3.1, $n \geq t+4$, and so $|A| \geq t+2$. Let $M = A \setminus (N \cup \{u\})$. Now $|M| \geq 2$; choose $m_1, m_2 \in M$, distinct. By 2.5 and by 2.2, there are at least three m_1m_2-joins, and u is not any of them, so at least one is in $A \setminus \{u\}$. If $w \in N$ is an m_1m_2-join, then contracting the edge uw gives a $K_{2,t}$ minor. Thus some $m_3 \in M$ is an m_1m_2-join. By 2.5, there exists $x \in N$ adjacent to m_3. But then contracting the edges ux, xm_3 gives a $K_{2,t}$ minor. This proves 3.4. ∎

3.5 *G is 5-connected, and so $t \geq 6$.*

Proof. Let (A_1, A_2, C) be a cut of G, chosen with $|C|$ minimum. Suppose that $|C| \leq 4$. For each $a_1 \in A_1$ and $a_2 \in A_2$, since a_1, a_2 have three common neighbours by 2.5, it follows that they both have at least three neighbours in C. Thus every vertex in $V(G) \setminus C$ has at least three neighbours in C. Choose $c, c' \in C$; then since $|V(G) \setminus C| \geq n - 4 \geq t$ by 3.1, some vertex in $V(G) \setminus C$ is not adjacent to one of c, c'. Consequently $|C| = 4$.

Suppose that $C = \{c_1, c_2, c_3, c_4\}$ where c_1c_2 and c_3c_4 are edges. Every vertex in $V(G) \setminus C$ is adjacent to one of c_1, c_2 and to one of c_3, c_4, and it follows that contracting the edges c_1c_2 and c_3c_4 gives a $K_{2,t}$ minor. Hence no two edges of $G|C$ are disjoint. But C is connected, by 2.3, and so we may assume that some vertex $c \in C$ is adjacent to every vertex in $C \setminus \{c\}$, and the other vertices in C are pairwise nonadjacent. By 3.4 there is a vertex nonadjacent to c, say $a_1 \in A_1$. Choose $a_2 \in A_2$; then $C \setminus \{c\}$ is the set of all a_1a_2-joins, and yet $C \setminus \{c\}$ is not connected, contrary to 2.3. Thus $|C| \geq 5$. This proves that G is 5-connected. By 3.4 there are two nonadjacent vertices, and therefore there are five paths joining them, with disjoint interiors. Since G has no $K_{2,t}$ minor it follows that $t \geq 6$. This proves 3.5. ∎

4 Neighbour sets of little subsets

If $W \subseteq V(G)$, we denote by $N(W)$ the set of all vertices of G not in W but with a neighbour in W, and $M(W)$ the set of vertices not in W with no neighbour in W. For a vertex v, we write $N(v), M(v)$ for $N(\{v\}), M(\{v\})$. In this section we give the central argument of the proof of 1.1; we show that either $t \leq 10$ or there is no edge w_1w_2 with $|N(\{w_1, w_2\})| \geq t + 4$. Then the remainder of the proof of 1.1 consists of handling the cases left open by this result.

Several of the steps to come depend on finding a small (at most four vertices) connected subset W, such that $|N(W)|$ is large (at least $t + 3$ and preferably larger), and trying to find a connected subset W' disjoint from W such that $N(W')$ has at least t vertices in common with $N(W)$ (for this would yield a $K_{2,t}$ minor). We begin with some lemmas. We denote by $\lambda(W)$ the minimum k such that for every nonempty subset $X \subseteq W$, some vertex in X has at most k neighbours in X. (This is sometimes called the *degeneracy* of $G|W$.)

4.1 *Let $W \subseteq V(G)$.*

- *If W is connected and $|W| \leq 4$ then $N(W)$ is connected.*
- *Every vertex in $N(W)$ has at least $\frac{1}{2}t - \lambda(W)$ neighbours in $N(W)$.*

Proof. To prove the first statement, suppose that W is connected and $|W| \leq 4$. By 3.5, $V(G) \setminus W$ is connected. But also W is connected, so $N(W)$ is connected by 2.4. For the second statement, let $v \in N(W)$. Let X be the set of neighbours of v in W. Since X is nonempty, some vertex $x \in X$ has at most $\lambda(W)$ neighbours in X. But there are at least $\frac{1}{2}t$ vx-joins by 2.2, and at most $\lambda(W)$ of them are in W, since x has at most $\lambda(W)$ neighbours in X. Thus all the others are in $N(W)$. This proves 4.1. ∎

If $X \subseteq V(G)$ we say an edge is *within* X if it has both ends in X. Let us say a *grasp* is a pair (X, Y) of disjoint subsets of $V(G)$, such that X is nonempty and connected and every vertex in Y has a neighbour in X.

4.2 *Let $W \subseteq V(G)$ be connected with $|W| \leq 4$. Let (X, Y) be a grasp where $X \cap W = \emptyset$ and $Y \subseteq N(W)$. Let $Z = N(W) \setminus (X \cup Y)$.*

- *If $|W| \leq 2$ then $|Z| < 2(t - |Y|)$.*
- *If $3 \leq |W| \leq 4$ and $G|W$ is not isomorphic to K_4, and $t \geq 11$, then $|Z| \leq 2(t - |Y|)$.*

Proof. With G, W fixed, we prove both claims simultaneously by induction on $|V(G)| - |X \cup Y|$. If some $z \in Z$ has a neighbour in X, then the result follows from the inductive hypothesis applied to the grasp $(X, Y \cup \{z\})$; while if some $v \in M(W) \setminus X$ has a neighbour in X, the result follows from the inductive hypothesis applied to the grasp $(X \cup \{v\}, Y)$. Thus we may assume that

(1) $N(X) \subseteq Y \cup W$.

We may also assume that

(2) *If $z_1, z_2 \in Z$ are distinct then every z_1z_2-join belongs to $Z \cup W$.*

For suppose that u is a z_1z_2-join that is not in $Z \cup W$. Thus either $u \in X \cup Y$, or $u \in M(W) \setminus X$. Certainly $u \notin X$ since $z_1 \notin N(X)$ by (1). If $u \in Y$, the result follows from the inductive hypothesis applied to the grasp

$$(X \cup \{u\}, (Y \setminus \{u\}) \cup \{z_1, z_2\}).$$

Thus $u \in M(W) \setminus X$, and so $u \notin N(X)$ by (1). Choose $x \in X$, and let y be a ux-join. Since $u \notin W \cup N(W)$, it follows that $y \notin W$, and so $y \in Y$ by (1). But then the result follows from the inductive hypothesis applied to the grasp

$$(X \cup \{y, u\}, (Y \setminus \{y\} \cup \{z_1, z_2\}).$$

This proves (2).

We may assume that

(3) *Every vertex in Z with a neighbour in Y has at most two neighbours in Z, and has no neighbours in Z if $t \geq 11$.*

For suppose some $z \in Z$ has neighbours $z_1, \ldots, z_d \in Z$, where $d \geq 1$, and a neighbour $y \in Y$. If $d \geq 3$ then the result follows from the inductive hypothesis applied to the grasp

$$(X \cup \{y, z\}, (Y \setminus \{y\}) \cup \{z_1, z_2, z_3\}),$$

so we may assume that $d \leq 2$; and hence we may also assume that $t \geq 2|W|+3$. There are at least $\frac{1}{2}t$ zz_1-joins in G; they all belong to $Z \cup W$, by (2); but at most $d-1$ are in Z, and so $d-1+|W| \geq t/2$. Since $d \leq 2$, this proves (3). This proves (3).

(4) *Every vertex in Z has a neighbour in Y.*

For suppose first that $|W| \leq 2$, and let $x \in X$. For each $z \in Z$, there are at least three xz-joins by 2.5, and at least one, y say, is not in W. By (1) $y \in Y$, and so z has a neighbour in Y as claimed. Thus we may assume that $|W| \geq 3$, and so $t \geq 11$ by hypothesis. Suppose that some vertex in Z has no neighbour in Y. Since $Y \neq \emptyset$ and $N(W)$ is connected by 4.1, there are distinct vertices $z, z' \in Z$ and $y \in Y$ such that z' has no neighbours in Y and z is adjacent to both y, z'; but this contradicts the final assertion of (3). This proves (4).

Now let us complete the proof of the first assertion of the theorem. Let $|W| \leq 2$, and suppose for a contradiction that $|Z| \geq 2(t-|Y|)$. Since $|Y| < t$ (because otherwise contracting all edges within X and within W produces a $K_{2,t}$ minor), it follows that $|Z| \geq 2$. If $z_1, z_2 \in Z$ are distinct, 2.2 and 2.5 imply that there is a z_1z_2-join $u \notin W$, and therefore in Z by (2). It follows that every two vertices in Z have a common neighbour in Z. In particular, we may choose z_1, z_2 adjacent, and so there are three vertices in Z, pairwise adjacent, say z_1, z_2, z_3. By (3) and (4), no other vertex in Z has a common neighbour with z_1, and so $Z = \{z_1, z_2, z_3\}$. Since $|Z| \geq 2(t-|Y|)$, it follows that $|Y| = t-1$. Choose $y \in Y$ adjacent to z_3. Then contracting all edges within $X \cup \{y, z_3\}$ and W yields a $K_{2,t}$ minor, a contradiction. This completes the proof of the first assertion.

Now we prove the second assertion. Thus, $t \geq 11$; $G|W$ is not isomorphic to K_4 (and so $\lambda(w) \leq 2$); Z is stable by (3) and (4); and we suppose for a contradiction that $|Z| \geq 2(t-|Y|)+1$. Since every vertex in Z has at least $t/2 - \lambda(W) \geq t/2-2$ neighbours in $N(W)$ from 4.1, and all these neighbours belong to Y by (4), it follows that there are at least $|Z|(t/2-2)$ edges between Y and Z. But there are at most $|Y|$ such edges, by (2), and so $|Z|(t/2-2) \leq |Y|$. Now $|Z| \geq 2(t-|Y|)+1$, and so $(2(t-|Y|)+1)(t/2-2) \leq |Y|$, that is $(2t+1)(t/2-2) \leq |Y|(t-3) \leq (t-1)(t-3)$, a contradiction since $t \geq 11$. This proves 4.2. ∎

The proof of the next theorem is the central argument of the paper, disposing of "most" possibilities for a critical graph G.

4.3 *Let $W \subseteq V(G)$ be connected with $|W| \leq 2$. If $t \geq 11$ then $|N(W)| \leq t+3$.*

Proof. Suppose that $t \geq 11$ and $|N(W)| \geq t+4$. By 3.4 we may assume that $|W| = 2$, $W = \{w_1, w_2\}$ say. Let $A = N(W)$ and $B = M(W)$. For each vertex $v \in A \cup B$, let $d(v)$ denote the number of neighbours of v in $A \cup B$.

(1) *Let $v_1, v_2 \in A \cup B$ be distinct. Then $d(v_1)+d(v_2) \leq 2t-2$; and if $d(v_1)+d(v_2) \geq 2t-3$ then v_1, v_2 are adjacent and there is no v_1v_2-join in B.*

For we may assume that $d(v_1) + d(v_2) \geq 2t - 3$. For $i = 1, 2$, let A_i denote the set of vertices in A different from v_1, v_2 that are adjacent to v_i, and let B_i be the set of vertices in B different from v_1, v_2 that are adjacent to v_i. For $i = 1, 2$ let $u_i = v_i$ if $v_i \in A$ and let $u_i \in A \setminus \{v_1, v_2\}$ be adjacent to v_i if $v_i \in B$. (Such vertices u_i exist by 2.5.)

By the second assertion of 4.2, applied taking $W' = W \cup \{u_1, v_1\}$ to be the set called W in that theorem, $X = \{v_2\}$, Y the set of neighbours of v_2 in $N(W')$, and $Z = N(W') \setminus (X \cup Y)$, we deduce that $|Z| \leq 2(t - |Y|)$, since $t \geq 11$. For $i = 1, 2$, let $a_i = 1$ if $v_i \in A$ and $a_i = 0$ otherwise; and let $b_1 = 1$ if $u_1 \in A_2$ (and therefore $u_i \neq v_i$ and $v_i \in B$), and $b_1 = 0$ otherwise, and define b_2 similarly. Now

$$|Z| \geq |A \setminus (\{u_1, v_2\} \cup A_2)| + |B_1 \setminus B_2| \geq t + 3 - |A_2| + b_1 - a_2 + |B_1 \setminus B_2|,$$

since $|A| \geq t + 4$; and $|Y| \geq |A_2| - b_1 + |B_1 \cap B_2|$. Consequently

$$t + 3 - |A_2| + b_1 - a_2 + |B_1 \setminus B_2| \leq 2(t - |A_2| + b_1 - |B_1 \cap B_2|),$$

that is,

$$|A_2| + |B_1| + |B_1 \cap B_2| \leq t + b_1 + a_2 - 3.$$

By exchanging v_1, v_2 and adding, we obtain

$$|A_1| + |A_2| + |B_1| + |B_2| + 2|B_1 \cap B_2| \leq 2t - 6 + a_1 + a_2 + b_1 + b_2.$$

Now for $i = 1, 2$, $d(v_i) = |A_i| + |B_i| + x$, where $x = 1$ if v_1, v_2 are adjacent and otherwise $x = 0$. Let $d(v_1) + d(v_2) = 2t - 3 + y$, where $y \geq 0$; we deduce that

$$|A_1| + |A_2| + |B_1| + |B_2| + 2x = 2t - 3 + y.$$

Combining this with the previous inequality, we deduce that

$$2t - 3 + y - 2x + 2|B_1 \cap B_2| \leq 2t - 6 + a_1 + a_2 + b_1 + b_2,$$

that is, $3 + y + 2|B_1 \cap B_2| \leq 2x + a_1 + a_2 + b_1 + b_2$. Now if $v_1 \in A$ then $v_1 \notin A_2$ from the definition of A_2, and so $a_1 + b_1 \leq 1$, and similarly $a_2 + b_2 \leq 1$; and so $a_1 + a_2 + b_1 + b_2 \leq 2$, and therefore $y + 1 + 2|B_1 \cap B_2| \leq 2x$. Consequently $x = 1$ and $|B_1 \cap B_2| = 0$, and $y \leq 1$. This proves (1).

(2) *$d(v) \leq t - 1$ for each $v \in A \cup B$.*

For suppose that $d(v_1) \geq t$ for some $v_1 \in A \cup B$; say $d(v_1) = t + x$ where $x \geq 0$. By (1), $d(v_2) \leq t - x - 2$ for every $v_2 \in A \cup B$ different from v_1, and if v_1, v_2 are nonadjacent then $d(v_2) \leq t - x - 4$. Thus one vertex of $G|(A \cup B)$ has degree $t + x$; $t + x$ more have degree at most $t - x - 2$; and the remaining $n - t - x - 3$ vertices have degree at most $t - x - 4$. Consequently the sum over all $v \in A \cup B$ of $d(v)$ is at most

$$t + x + (t + x)(t - x - 2) + (n - t - x - 3)(t - x - 4) = tn - x(n - 6) - 4(n - 3) \leq tn - 4n + 12.$$

By 3.2, $\deg(w_1) + \deg(w_2) \leq n + t - 2$, and so

$$2|E(G)| \leq tn - 4n + 12 + 2(n + t - 2) - 2 = tn - 2n + 6 + 2t.$$

But from the criticality of G, $2|E(G)| > (t + 1)(n - 1)$, and so $3n < 7 + 3t$, contrary to 3.1. This proves (2).

By (2), every vertex in A has degree at most $t + 1$, and every vertex in B has degree at most $t - 1$. Let X be the set of all vertices $v \in A$ with $\deg(v) = t + 1$. By the first assertion of 4.2, every vertex in A has at most $t - 2$ neighbours in A (in fact, at most $t - 4$, though we do not need this); and consequently every vertex in X has a neighbour in B. But if $v \in X$ then $d(v) \geq t - 1$, and so no two members of $X \cap A$ are adjacent to the same member of B. It follows that $|X| \leq |B|$. But surplus$(A) \leq |X|$, and surplus$(B) \leq -|B|$, and so surplus$(A \cup B) \leq 0$. Since surplus$(V(G)) \geq n - t$ by 3.3, it follows that surplus(w_1) + surplus$(w_2) \geq n - t$, contrary to 3.2. This proves 4.3. ∎

5 Small t cases

In this section we focus on strengthening 4.3 when t is small. We make a start on this with the following corollary of 4.2:

5.1 *$t \geq 7$.*

Proof. By 3.3 there is a vertex w of degree at least $t + 1$. Let C be a component of $M(w)$ (this exists, by 3.4); then $N(C) \subseteq N(w)$. By 3.5, $|N(C)| \geq 5$. By the first assertion of 4.2 applied to the grasp $(C, N(C))$, we deduce that $|N(W) \setminus N(C)| < 2(t - |N(C)|)$, and so $2t > |N(W)| + |N(C)| \geq (t + 1) + 5$. This proves 5.1. ∎

We need an elaboration of this. Given integers $h \geq 3$ and $z \geq 0$, we define $\beta_0 = 0$, and for $1 \leq i \leq h - 2$, we define inductively

$$\beta_i = \beta_{i-1} + \lceil 3(z - \beta_{i-1})/(h - i + 1) \rceil.$$

We write $\beta_i(h, z)$ for β_i to show the dependence on h, z. Note that $\beta_i(h, z) \leq z$ and $\beta_i(h, z)$ is monotone nondecreasing in z. (To see the latter, prove inductively that if z is increased by 1 then either $\beta_i(h, z)$ remains the same or increases by 1.)

5.2 *Let $W \subseteq V(G)$ be connected with $|W| \leq 2$. Then there exists h with $5 \leq h \leq t - 2$ such that*

$$\beta_i(h, z) - 2i < 2t - h - |N(W)|$$

for all i with $0 \leq i \leq h - 2$, where $z = |N(W)| - h$.

Proof. If $|N(W)| \leq t$, then every choice of h with $5 \leq h \leq t-2$ satisfies the theorem (and there is such a choice by 5.1), since $\beta_i(h,z) \leq z = |N(W)| - h$ for $i > 0$. Thus we may assume that $|N(W)| > t$.

Suppose first that $M(W) = \emptyset$. By 3.3, some vertex $v \in N(W)$ has degree at least $t+1$, and hence has at least $t-1$ neighbours in $N(W)$. By 4.2 applied to the grasp $(\{v\}, N(v) \cap N(W))$, we deduce that

$$|N(W)| - (1 + |N(v) \cap N(W)|) < 2(t - |N(v) \cap N(W)|),$$

and so

$$|N(W)| \leq 2t - |N(v) \cap N(W)| \leq t + 1.$$

Thus $n \leq t+3$, contrary to 3.1. Therefore $M(W)$ is nonempty; let C be a component of $M(W)$. Let $Z = N(W) \setminus N(C)$, let $h = |N(C)|$, and let $z = |Z| = |N(W)| - h$; we will show that h, z satisfy the theorem. Certainly $h \geq 5$ since G is 5-connected by 3.5. By 4.2 applied to the grasp $(C, N(C))$, it follows that

$$|N(W)| - |N(C)| < 2(t - |N(C)|),$$

and since $|N(W)| > t$, we deduce that $h = |N(C)| \leq t-2$.

(1) *For $0 \leq i \leq h-2$, there exists $X_i \subseteq N(C)$ with $|X_i| = i$ such that at least $\beta_i(h,z)$ vertices in $N(W) \setminus N(C)$ have neighbours in X_i.*

This is trivial for $i = 0$, since $\beta_0(h,z) = 0$. We proceed by induction on i. Thus, assume that $1 \leq i \leq h-2$ and there exists $X_{i-1} \subseteq N(C)$ with $|X_i| = i-1$ such that $|Y| \geq \beta_{i-1}(h,z)$, where Y is the set of vertices in $N(W) \setminus N(C)$ with a neighbour in X_{i-1}. Choose $c \in C$; then every vertex in $Z \setminus Y$ has at least three common neighbours with c by 2.5, and therefore has at least three neighbours in $N(C)$, and therefore in $N(C) \setminus X_{i-1}$, since it has no neighbour in X_{i-1}. Consequently there exists $x \in N(C) \setminus X_{i-1}$ with at least $\lceil 3|Z \setminus Y|/(h-i+1) \rceil$ neighbours in $Z \setminus Y$. Let $X_i = X_{i-1} \cup \{x\}$; then there are at least $|Y| + \lceil 3(z - |Y|)/(h-i+1) \rceil$ vertices in Z with a neighbour in X_i. Since this expression is increasing with $|Y|$ (because $h - i + 1 \geq 3$), and $|Y| \geq \beta_{i-1}(h,z)$, it follows that there are at least

$$\beta_{i-1}(h,z) + \lceil 3(z - \beta_{i-1}(h,z))/(h-i+1) \rceil = \beta_i(h,z)$$

such vertices. This proves (1).

Now let i satisfy $0 \leq i \leq h-2$, and let X_i be as in (1). Let Y_i be the set of vertices in Z with a neighbour in X_i. Thus $|Y_i| \geq \beta_i(h,z)$. From the first assertion of 4.2, applied to the grasp $(C \cup X_i, (N(C) \setminus X_i) \cup Y_i)$, we deduce that

$$|N(W)| - |N(C)| - |Y_i| < 2(t - (h - |X_i|) - |Y_i|),$$

that is, $z - |Y_i| < 2t - 2h + 2i - 2|Y_i|$. Since $|Y_i| \geq \beta_i(h,z)$ and $z = |N(W)| - h$, it follows that $|N(W)| + \beta_i(h,z) < 2t - h + 2i$. This proves 5.2. ∎

From 5.2 we deduce the following strengthening of 4.3 (note that the case of small t is still exceptional, but now it is a good exception rather than a bad one):

5.3 *Let $W \subseteq V(G)$ be connected with $|W| \leq 2$. Then $|N(W)| \leq t+3$, and if $t \leq 13$ then $|N(W)| \leq t+2$.*

Proof. We may assume that $|N(W)| \geq t+3$. We show first that $t \geq 14$. Choose h, z as in 5.2; then $5 \leq h \leq t-2$, and

$$\beta_i(h,z) - 2i < 2t - h - |N(W)|$$

for all i with $0 \leq i \leq h-2$. Consequently

$$\beta_i(h, t+3-h) - 2i \leq t-h-4,$$

for all i with $0 \leq i \leq h-2$, since $\beta_i(h,z)$ is a nondecreasing function in z. Setting $i=0$, we deduce that $h \leq t-4$. In particular $t \geq 9$, since $h \geq 5$. Also we may assume $h \leq 9$, for otherwise it follows that $t \geq 14$ as required. Setting $i=1$ gives

$$\beta_1(h, t+3-h) \leq t-h-2,$$

and so $3(t+3-h)/h \leq t-h-2$, that is, $3(t+3)/h \leq t-h+1$. If $h=5$ this implies $29 \leq 2t$, and so $t \geq 15$ as required. If $h=9$ this implies $27 \leq 2t$ as required. We may therefore assume that $6 \leq h \leq 8$. Setting $i=2$ gives $\beta_2(h, t+3-h) \leq t-h$, and so

$$\lceil 3(t+3-h)/h \rceil + \lceil 3(t+3-h-\lceil 3(t+3-h)/h \rceil)/(h-1) \rceil \leq t-h,$$

that is,

$$3(t+3)/h + \lceil 9/(h-4) \rceil \leq t-(h-3).$$

If $h=6$ this gives $19 \leq t$ as required. If $h=7$ this gives $29 \leq 2t$ as required. If $h=8$ this gives $73 \leq 5t$ as required. This proves that $t \geq 14$. From 4.3 it follows that $|N(W)| = t+3$. This proves 5.3. ∎

6 Finding an edge with a large neighbourhood

Now we can complete the main proof.

Proof of 1.1.

An edge uv is *dominating* if every vertex of G is adjacent or equal to one of u, v. Take a vertex w of maximum degree $t+s$ say, chosen if possible such that there is a dominating edge not incident with w. Let $A = N(w)$, and $B = M(w)$.

(1) *Every vertex in A has at most $4-s$ neighbours in B, and at most $3-s$ if $t \leq 13$.*

For let $a \in A$, with say d neighbours in B. Then $|N(\{w,a\})| = t+s-1+d$, and so by 5.3, $t+s-1+d \leq t+3$, and $t+s-1+d \leq t+2$ if $t \leq 13$. This proves (1).

(2) *Every vertex in B has at least $\max(3, \frac{1}{2}t+s-2)$ neighbours in A, and at least $\max(3, \frac{1}{2}t+s-1)$ if $t \leq 13$.*

For let $b \in B$. Since w, b have at least three common neighbours by 2.5, it remains (for the first assertion) to show that b has at least $\frac{1}{2}t+s-2$ neighbours in A. Choose $a \in A$ adjacent to b. There are at least $\frac{1}{2}t$ ab-joins by 2.2, and at most $3-s$ of them belong to B, since a has at most $4-s$ neighbours in B ; so at least $\frac{1}{2}t+s-3$ of them belong to A and are different from a. Thus b has at least $\frac{1}{2}t+s-2$ neighbours in A. This proves the first assertion of (2), and the second follows similarly.

(3) *Every vertex in A has at most $t-s$ neighbours in A.*

For let $v \in A$, let Y be the set of its neighbours in A, and $Z = A \setminus (Y \cup \{v\})$. By the first assertion of 4.2, $|Z| < 2(t-|Y|)$, and since $|Z| = s+t-1-|Y|$, this proves (3).

(4) $s \leq 2$.

For (1) implies that $s \leq 4$. If $s = 4$, then since G is connected, (1) implies that B is empty, contrary to 3.4. Suppose that $s = 3$. By (2), every vertex in B has at least $\frac{1}{2}t+1$ neighbours in A, and so (1) implies that $|B| \leq 2$, and so $|B| = 2$ by 3.4. The two members of B have no common neighbour, contrary to 2.2 and 2.5. This proves (4).

Let e_1 denote the number of edges between A and B, and e_2 the number of edges with both ends in B.

(5) *If $s = 2$, then $t \geq 14$ and $e_2 \leq 1$ and $|B| \leq 3$.*

For suppose that $s = 2$. Suppose first that $t \leq 13$. By (1) and (2), $|A| \geq e_1 \geq (\frac{1}{2}t+1)|B|$, and since $|A| = t+2$ and $t \geq 7$ by 5.1, it follows that $|B| \leq 2$, and so $|B| = 2$ by 3.4; let $B = \{b_1, b_2\}$. By (1), no vertex in A is adjacent to both b_1, b_2, contrary to 2.2 and 2.5. This proves that $t \geq 14$.

By (1) and (2), $2|A| \geq e_1 \geq \lceil \frac{1}{2}t \rceil |B|$, and since $|A| = t+2$ and $t \geq 9$ it follows that $|B| \leq 4$.

Suppose that there are three vertices $b_1, b_2, b_3 \in B$, pairwise adjacent. Now by 2.2 there are at least $\frac{1}{2}t$ b_1b_2-joins, and so there are at least $\frac{1}{2}t-2$ b_1b_2-joins in A. The same holds for b_1b_3- and b_2b_3-joins, and all these vertices are different by (1). Thus at least $3(\frac{1}{2}t-2)$ vertices in A have neighbours in $\{b_1, b_2, b_3\}$, and since $3(\frac{1}{2}t-2) > t-1$ (since $t \geq 11$), it follows that G has a $K_{2,t}$ minor, a contradiction. Thus no three members of B are pairwise adjacent.

Next suppose that there exist $b_1, b_2, b_3 \in B$ such that b_1b_2 and b_2b_3 are edges. There are at least $\frac{1}{2}t$ b_1b_2-joins, all in A, and the same for b_2b_3-joins, and they are all different by (1), so there are at least t vertices in A with neighbours in $\{b_1, b_2, b_3\}$, and contracting the edges within B gives a $K_{2,t}$ minor, a contradiction. Thus every vertex in B has at most one neighbour in B.

Suppose that $e_2 \geq 2$. Then it follows that $e_2 = 2$ and $|B| = 4$, and we may assume that b_1b_2 and b_3b_4 are edges, where $B = \{b_1, b_2, b_3, b_4\}$. There are at least $\frac{1}{2}t$ b_1b_2-joins, all in A, and the same for b_3b_4-joins; and at least three b_1b_3-joins, by 2.5. All these vertices are different, by (1), so $|A| \geq t + 3$, a contradiction. This proves that $e_2 \leq 1$.

Suppose that $|B| = 4$, and so $n = t + 7$. Now the sum of the degrees of the four vertices in B is $e_1 + 2e_2$; and we have seen that $e_1 \leq 2(t + 2)$ and $e_2 \leq 1$. Thus

$$\text{surplus}(B) \leq (2t + 6) - 4t = 6 - 2t.$$

By (1) and (3), every vertex in A has degree at most $t+1$, and so $\text{surplus}(A\cup\{w\}) \leq t+4$. Thus $\text{surplus}(V(G)) \leq (6 - 2t) + (t + 4) = 10 - t$. But by 3.3, $\text{surplus}(V(G)) \geq n - t = 7 > 10 - t$, a contradiction. Consequently $|B| \leq 3$. This proves (5).

(6) *If* $s = 2$ *then* $|B| = 2$.

For suppose that $s = 2$; then $2 \leq |B| \leq 3$ from 3.4 and (5). Suppose that $|B| = 3$, $B = \{b_1, b_2, b_3\}$ say. Then $n = t + 6$. By (5), $e_2 \leq 1$.

Suppose that $e_2 = 1$, and let b_1b_2 be an edge say. There are at least $\frac{1}{2}t$ b_1b_2-joins in A by 2.2, and at least $\frac{1}{2}t + 1$ neighbours of b_3, also by 2.2, and all these vertices are different by (1). So there are at least $t + 1$ vertices in A with a neighbour in B. By 2.5, some vertex $a \in A$ is adjacent to both b_1, b_3; so contracting the edges b_1b_2, b_1a, b_3a gives a $K_{2,t}$ minor, a contradiction. This proves that $e_2 = 0$.

Suppose that every vertex in A has a neighbour in B. Choose a b_1b_2-join $a_1 \in A$, and a b_2b_3-join $a_2 \in A$. Then by contracting the edges $b_1a_1, a_1b_2, b_2a_2, a_2b_3$ we obtain a $K_{2,t}$ minor, a contradiction. This proves that some vertex in A has no neighbour in B, and so $e_1 \leq 2(t+1)$. Then $\text{surplus}(B) \leq 2 - t$, and so

$$\text{surplus}(A) \geq t - 2 - \text{surplus}(w) + (n - t) = n - 4 = t + 2$$

by 3.3. By (3), every vertex in A has degree at most $t+1$, so all $t+2$ members of A have degree $t + 1$. But some one of them has no neighbour in B as we already saw, and this contradicts (3). This proves (6).

(7) $s = 1$, *and therefore every vertex in* G *has degree at most* $t + 1$, *and* $t \geq |B| - 1$.

For suppose that $s = 2$, and therefore $|B| = 2$, by (6), and so $n = t + 5$. Let $B = \{b_1, b_2\}$ say. Let X be the set of all vertices in $V(G) \setminus \{w\}$ with degree at least $t + 1$. By 3.2, $X \cup \{w\}$ is a clique, and so $X \subseteq A$. By (1) and (3), every vertex in X has degree exactly $t + 1$, and has exactly $t-2$ neighbours in A, and is adjacent to both b_1, b_2. By 3.3, $|X| \geq n-t-2=3$ since surplus$(w)=2$. Let $a_0 \in X$, and let N be its set of neighbours in A. Let a_1, a_2, a_3 be the three vertices in A nonadjacent to a_0. Since each of a_1, a_2, a_3 has at least $\frac{1}{2}t$ neighbours in A by 2.2, there are at least $3t/2-6$ edges between $\{a_1, a_2, a_3\}$ and N. Since $3t/2-6 > t-2 = |N|$ since $t \geq 9$, some vertex $a_4 \in N$ is adjacent to two of a_1, a_2, a_3, say to a_1, a_2. Choose $a_5 \in X$ different from a_0, a_4; then $a_5 \in N$, and contracting the edges wa_5, a_0a_4 gives a $K_{2,t}$ minor, a contradiction. This proves the first statement of (7). The second follows from the choice of w. For the third, we observe from (1) that $e_1 \leq 3|A| = 3(t+1)$, and from (2) that $e_1 \geq 3|B|$, and so $|B| \leq t+1$. This proves (7).

Let $\kappa(B)$ be the number of components of B, and let A_0 be the set of vertices in A with no neighbour in B.

(8) $|A_0| + \kappa(B) \geq 3$, *and for every component C of B, at most $t-2$ vertices in A have neighbours in C. (In particular, if B is connected then $|A_0| \geq 3$.)*

For choose $T \subseteq B$ containing exactly one vertex of each component of B. Since every two members of T have a common neighbour in A by 2.5, it follows that there is a set $S \subseteq A$ with $|S| \leq |T|-1$ such that $B \cup S$ is connected. Since contracting all edges within $B \cup S$ does not produce a $K_{2,t}$ minor, it follows that $|A \setminus (S \cup A_0)| < t$. Thus $t+1-(\kappa(B)-1)-|A_0| \leq t-1$, and this proves the first assertion. For the second, let C be a component of B. Let $Y = N(C) \subseteq A$, and $Z = A \setminus Y$. By the first assertion of 4.2, $|Z| < 2(t-|Y|)$, and since $|Z| = t+1-|Y|$ this proves (8).

Let X be the set of all vertices in A with degree $t+1$. Let $d=2$ if $t \leq 13$ and $d=3$ otherwise. By (1), every vertex in A has at most d neighbours in B.

(9) $|X| + e_1 + 2e_2 \geq (t+1)|B| + 1$, *and* $|X| + |A_0| \leq t+1$, *and so*

$$2e_2 \geq (t+1)(|B|-d-1) + (d+1)|A_0| + 1.$$

For since every vertex in A has degree at most $t+1$, it follows that surplus$(A \cup \{w\}) \leq |X|+1$. But surplus$(B) = e_1 + 2e_2 - t|B|$, and by 3.3, surplus$(V(G)) \geq n-t = |B|+2$, so

$$|X| + 1 + e_1 + 2e_2 - t|B| \geq |B| + 2.$$

This proves the first assertion. For the second, since no vertex in A has t neighbours in A by (3), it follows that $X \cap A_0 = \emptyset$, and so $|X| + |A_0| \leq t+1$. But $e_1 \leq d(t+1-|A_0|)$ by (1), and so $|X| + e_1 \leq (d+1)(t+1-|A_0|)$. Substituting in the first assertion, we deduce that $(d+1)(t+1-|A_0|) + 2e_2 \geq (t+1)|B| + 1$. This proves (9).

(10) $|B| \leq 5$, *and if $t \leq 13$ then $|B| \leq 4$.*

First suppose that $t \leq 13$. By (1) and (2), $2(t+1) \geq e_1 \geq \lceil \frac{1}{2}t \rceil |B|$ and so $|B| \leq 4$ since $t \geq 7$. Thus we may assume that $t \geq 14$. By (1) and (2), $3(t+1) \geq (\frac{1}{2}t - 1)|B|$, and it follows that $|B| \leq 7$. But (9) implies that $2e_2 \geq (t+1)(|B|-4)+1 \geq 15(|B|-4)+1$. If $|B| = 7$, this implies that $2e_2 \geq 46$, a contradiction since $e_2 \leq 21$. If $|B| = 6$, this implies that $2e_2 \geq 31$, again a contradiction since $e_2 \leq 15$. This proves (10).

(11) $|B| \leq 4$.

For suppose that $|B| = 5$. By (10), $t \geq 14$ and so $d = 3$. By (9), $2e_2 \geq t + 4|A_0| + 2 \geq 16$, and so B is connected. Thus $|A_0| \geq 3$ by (8), and $2e_2 \geq t + 14 \geq 28$, which is impossible. This proves (11).

(12) $|B| \leq 3$.

For suppose that $|B| = 4$. By (9), $2e_2 \geq (3-d)(t+1) + (d+1)|A_0| + 1$. If B is connected then $|A_0| \geq 3$ by (8), and so $12 \geq 2e_2 \geq (3-d)(t+1) + 3(d+1) + 1$, which is impossible (since either $d = 3$, or $d = 2$ and $t \geq 7$). Thus B is not connected, and so $e_2 \leq 3$. Consequently $6 \geq (3-d)(t+1) + (d+1)|A_0| + 1$, and so $d = 3$ and therefore $t \geq 14$, and $|A_0| \leq 1$.

Suppose that some vertex in B has more than one neighbour in B. Since B is not connected, it follows that B has two components C_1, C_2, where $|C_1| = 3$ and $|C_2| = 1$. At least three vertices in A have no neighbour in C_1, by (8), and so (1) implies $e_1 \leq 3(t+1) - 6$. Since (9) implies $|X| + e_1 + 2e_2 \geq 4t + 5$, we deduce that $|X| + 2e_2 \geq t + 8$, which is impossible since $|X| \leq t+1$ and $e_2 \leq 3$. Thus $G|B$ has maximum degree at most one, and in particular $e_2 \leq 2$.

Since $2e_2 \geq 4|A_0| + 1$, we deduce that $A_0 = \emptyset$. For every edge uv of $G|B$, at least two (indeed, at least three) vertices of A are nonadjacent to both u, v, by (8), and since no two edges within B share an end, and every vertex in A has a neighbour in B, it follows that there are at least $2e_2$ vertices in A with at most two neighbours in B. Consequently $e_1 \leq 3(t+1) - 2e_2$; but $|X| + e_1 + 2e_2 \geq 4t + 5$ by (9), and so $|X| \geq t + 2$, which is impossible. This proves (12).

(13) *There is a dominating edge.*

For suppose not; then every vertex in A has at most $|B| - 1$ neighbours in B, and so $e_1 \leq (t + 1 - |A_0|)(|B| - 1)$. By (9),

$$t + 1 - |A_0| + e_1 + 2e_2 \geq |X| + e_1 + 2e_2 \geq (t+1)|B| + 1,$$

and so

$$2e_2 \geq 1 + |A_0||B| \geq 1 + |B|(3 - \kappa(B))$$

by (8). In particular, $e_2 > 0$, and so $\kappa(B) \leq 2$; and consequently $2e_2 \geq 1 + |B|$, and therefore $|B| = 3$. We deduce that $2e_2 \geq 1 + 3(3 - \kappa(B))$; so $e_2 \geq 2$, and therefore $\kappa(B) = 1$, and $2e_2 \geq 1 + 3 \times 2$, which is impossible. This proves (13).

(14) *At most two vertices in A have more than one neighbour in B.*

For since there are at least three vertices of degree $t+1$ by 3.3, it is possible to choose one such that some dominating edge is not incident with it; and so from our choice of w, there is a dominating edge v_1v_2 say with $v_1, v_2 \neq w$. If there is a vertex $a \in A$ different from v_1, v_2 with at least two neighbours in B, then contracting the edges v_1v_2 and wa gives a $K_{2,t}$ minor, a contradiction. Thus every vertex in A different from v_1, v_2 has at most one neighbour in B. This proves (14).

By 3.4, we may choose distinct $b_1, b_2 \in B$, adjacent if possible. There are at least three b_1b_2-joins by 2.5 and 2.2, and only two of them are in A by (14), and so the third is in B. Consequently $|B| = 3$, and b_1, b_2 are adjacent (from the choice of b_1, b_2), and $e_2 = 3$. By (8), $|A_0| \geq 3$, and by (14), $e_1 \leq t-1-|A_0|+6 \leq t+2$. By (9), $(t+1-|A_0|)+e_1+2e_2 \geq (t+1)|B|+1$, and so $(t-2)+(t+2)+6 \geq 3(t+1)+1$, a contradiction. This proves 1.1. ∎

7 Rooted minors

Now we come to the second topic of the paper, "rooted $K_{2,t}$ minors". Let us say an *expansion* of H in G is a function ϕ with domain $V(G) \cup E(G)$, satisfying:

- for each vertex v of H, $\phi(v)$ is a nonnull connected subgraph of G, and the subgraphs $\phi(v)$ $(v \in V(H))$ are pairwise vertex-disjoint
- for each edge $e = uv$ of H, $\phi(e)$ is an edge of G with one end in $V(\phi(u))$ and the other in $V(\phi(v))$.

It is easy to see that H is a minor of G if and only if there is an expansion of H in G.

Now let G be a graph, let $r, r' \in V(G)$ be distinct, and let $t \geq 0$. We say that G contains an *rr'-rooted $K_{2,t}$ minor* if there is an expansion ϕ of $K_{2,t}$ in G, such that $\phi(s), \phi(s')$ each contain one of r, r', where s, s' are two nonadjacent vertices of $K_{2,t}$ of degree t.

The result of this section is an analogue of 1.1 for rr'-rooted $K_{2,t}$ minors, but it needs a little care to formulate. In particular, if there is a cut (A_1, A_2, C) with $|C| \leq 1$ and $r, r' \in A_1 \cup C$, then G contains an rr'-rooted $K_{2,t}$ minor if and only if $G|(A_1 \cup C)$ contains such a minor, and therefore the number of edges within $A_2 \cup C$ is irrelevant. Let us say that G is *2-connected to rr'* if there is no cut (A_1, A_2, C) with $|C| \leq 1$ and $r, r' \in A_1 \cup C$. For $t \geq 2$, define $\delta(t) = \frac{1}{2}(t+3-\frac{4}{t+2})$. We shall prove the following.

7.1 *Let $t \geq 2$, let G be a graph with n vertices, let $r, r' \in V(G)$ be distinct, and let G be 2-connected to r, r'. If G contains no rr'-rooted $K_{2,t}$ minor then*

$$|E(G)| \leq \delta(t)(n-1) - 1;$$

and for all $t \geq 2$ there are infinitely many such G that attain equality.

The proof requires several steps. First let us see the last claim, that there are infinitely many such graphs G that attain equality. Let $k \geq 1$ be an integer, and let $p_1\text{-}\cdots\text{-}p_k$ be a path. Add a new vertex p_0 adjacent to each of $p_1, \ldots, p_k$. For $1 \leq i \leq k$, take a set X_i of $t+1$ new vertices, and choose distinct $x_i, x_i' \in X_i$; and make every two vertices in $X_i \cup \{p_{i-1}, p_i\}$ adjacent except for the pairs $p_{i-1}x_i, x_ix_i'$ and $x_i'p_i$. This graph G has n vertices, where $n = k(t+2)+1$, and has

$$(\frac{1}{2}(t+2)(t+3)-2)k - 1 = \delta(t)(n-1) - 1$$

edges. Moreover, it has no p_0p_k-rooted $K_{2,t}$ minor (we leave the reader to check this, but here is a hint: the edge p_0p_k is useless and can be deleted, and then p_{k-1} is a cutvertex.) This proves the last claim of the theorem.

The remainder of this section is devoted to proving the first claim. Suppose it is false; then there is a smallest graph G that is a counterexample (for some t). Moreover, if G is such a graph, and r, r' are nonadjacent in G, then we may add the edge rr' and delete some other edge, and the graph we produce is another counterexample. Thus it suffices to prove that there is no "minimum counterexample", where we say a 5-tuple (G, t, r, r', n) is a *minimum counterexample* if it has the following properties:

- G is a graph with n vertices, and $t \geq 2$
- $r, r' \in V(G)$ are distinct and adjacent, G is 2-connected to rr', and G contains no rr'-rooted $K_{2,t}$ minor
- $|E(G)| > \delta(t)(n-1) - 1$
- For all t' with $2 \leq t'$, and for every graph G', and all distinct $s, s' \in V(G')$, if G' is 2-connected to ss' and G' contains no ss'-rooted $K_{2,t'}$ minor, and $|V(G')| < |V(G)|$, then
$$|E(G')| \leq \delta(t')(|V(G')| - 1) - 1.$$

We proceed to prove several statements about minimum counterexamples, that eventually will lead to a contradiction and thereby complete the proof of 7.1. The first is:

7.2 *If (G, t, r, r', n) is a minimum counterexample then $n \geq t+3$.*

Proof. Suppose that $n \leq t+2$. Since $\delta(t) \geq t/2+1$, we have $|E(G)| > (t/2+1)(n-1)-1$. In particular, $|E(G)| \geq 2$, since $n, t \geq 2$, and therefore $n \geq 3$. Let $|E(G)| = n(n-1)/2 - x$ say, where $x \geq 0$ is an integer. Then

$$n(n-1)/2 - x > (t/2+1)(n-1) - 1,$$

that is,

$$(t+2-n)(n-1)/2 + x < 1;$$

and since $n-1 \geq 2$ and $t+2-n, x \geq 0$, we deduce that $x = 0$ and $n = t+2$. Consequently G is isomorphic to the complete graph K_{t+2}, and therefore has an rr'-rooted $K_{2,t}$ minor, a contradiction. This proves 7.2. ∎

A notational convention: when we produce a minor H of G by contracting some edges, naming the vertices of H is sometimes a little awkward. Some of them may correspond to single vertices of G, in which case it is natural to give them the same name as that vertex of G, but some may be formed by identifying several vertices of G. In our case, when we have two distinguished vertices r, r', we adopt the convention that if a vertex of H is formed by identifying r with other vertices of G, we give this vertex the name r (and the same for r', and we will be careful not to identify r and r' under contraction).

Let H be a graph, and let u, v be distinct vertices of H. Let H' be the graph obtained from H by adding the edge uv if u, v are nonadjacent in H, and otherwise $H' = H$. We say that H' is obtained from H by *adding* uv.

7.3 *If (G, t, r, r', n) is a minimum counterexample then there is no 2-cut (A_1, A_2, C) with $r, r' \in A_1 \cup C$.*

Proof. Suppose that there is, and choose it with A_2 maximal, and let $C = \{c, c'\}$. For $i = 1, 2$, let $n_i = |A_i|$ and let e_i be the number of edges of G with at least one end in A_i.

Suppose first that $C = \{r, r'\}$. Since $A_1 \neq \emptyset$, and the graph $G|(A_1 \cup C)$ therefore has an rr'-rooted $K_{2,1}$ minor, it follows that $G|(A_2 \cup C)$ has no rr'-rooted $K_{2,t-1}$ minor (and so $t \geq 3$). The minimality of (G, t, r, r', n) (applied to $G|(A_2 \cup C)$) implies that $e_2 + 1 \leq \delta(t-1)(n_2+1) - 1$. A similar inequality holds for e_1, n_1, and adding the two gives

$$e_1 + e_2 + 2 \leq \delta(t-1)(n_1 + n_2 + 2) - 2.$$

But $e_1 + e_2 + 1 = |E(G)| > \delta(t)(n-1) - 1$, and $n_1 + n_2 + 2 = n$, and so $\delta(t-1)n - 2 > \delta(t)(n-1)$. Since $\delta(t) \geq \delta(t-1) + \frac{1}{2}$, it follows that $(\delta(t) - \frac{1}{2})n - 2 > \delta(t)(n-1)$, that is, $n + 4 < 2\delta(t)$. Thus

$$\frac{1}{2}n(n-1) \geq |E(G)| > \delta(t)(n-1) - 1 > \frac{1}{2}(n+4)(n-1) - 1,$$

and so $n \leq 1$, a contradiction. This proves that $C \neq \{r, r'\}$.

Let $y = 1$ if c, c' are adjacent, and $y = 0$ otherwise. We claim that $n_2 \geq 3$. For let F be the graph obtained from $G|(A_1 \cup C)$ by adding cc'. Then $|E(F)| = e_1 + 1$; but F is 2-connected to rr', and F has no rr'-rooted $K_{2,t}$ minor, so from the minimality of (G, t, r, r', n), $e_1 + 1 \leq \delta(t)(n_1 + 1) - 1$. But

$$e_1 + e_2 + y = |E(G)| > \delta(t)(n_1 + n_2 + 1) - 1,$$

and subtracting yields $e_2 + y - 1 > \delta(t)n_2$. Since $y \leq 1$, we deduce that $e_2 > \delta(t)n_2$. In particular, since $\delta(t) \geq 2$ and $n_2 \geq 1$, it follows that $e_2 \geq 3$, and so $n_2 \geq 2$. Suppose that $n_2 = 2$. Then $e_2 \leq 5$, and yet $e_2 > 2\delta(t)$, and so $5 > 2\delta(t)$, that is, $t = 2$, and $e_2 = 5$. In particular both members of A_2 are adjacent to both members of C; but then G has an rr'-rooted $K_{2,t}$ minor, by choosing two disjoint paths between $\{r, r'\}$ and C and contracting their edges, a contradiction. This proves that $n_2 \geq 3$.

Let X be the set of vertices in A_1 adjacent to both c, c'. Since G is 2-connected to rr', there are two disjoint paths P_1, P_2 of $G|(A_1 \cup C)$ between $\{r, r'\}$ and $\{c, c'\}$; choose them to contain as few members of X as possible. Let there be x vertices in X that do not belong to $P_1 \cup P_2$. Let H be the graph obtained from $G|(A_2 \cup C)$ by adding cc'. Then H has no cc'-rooted $K_{2,t-x}$ minor (for otherwise we could contract the edges of P_1, P_2 and obtain an rr'-rooted $K_{2,t}$ minor in G). In particular, since $A_2 \neq \emptyset$ and H therefore has a cc'-rooted $K_{2,1}$ minor, it follows that $t - x \geq 2$. Since H is 2-connected to cc', and $|E(H)| = e_2 + 1$, the minimality of (G, t, r, r', n) implies that

$$e_2 \leq \delta(t-x)(n_2+1) - 2.$$

Let $e_2 = \delta(t-x)(n_2+1) - 2 - z$ say, where $z \geq 0$. Let J be the graph obtained from G by deleting all edges between X and c, and then contracting all edges within $A_2 \cup C$ (note that this graph has no parallel edges, since we deleted the edges between X and c). The maximality of A_2 implies that J is 2-connected to r, r'. (We use here that not both r, r' belong to C.) Since $|E(J)| = e_1 - |X|$ and $|V(J)| = n_1 + 1$, the minimality of (G, t, r, r', n) implies that $e_1 - |X| \leq \delta(t)n_1 - 1$. Summing these two inequalities yields

$$e_1 + e_2 - |X| \leq \delta(t)n_1 + \delta(t-x)(n_2+1) - 3 - z.$$

Since $e_1 + e_2 + y = |E(G)| > \delta(t)(n-1) - 1$, it follows that

$$\delta(t)n_1 + \delta(t-x)(n_2+1) - 3 - z > \delta(t)(n-1) - 1 - y - |X|,$$

that is,

$$|X| + y - z > (\delta(t) - \delta(t-x))(n_2+1) + 2.$$

Since $y \leq 1$ and $\delta(t) - \delta(t-x) \geq x/2$, we deduce that $|X| - z > x(n_2+1)/2 + 1$, and in particular $|X| - z > 2x + 1$ since $n_2 \geq 3$. Since $|X| \leq x + 2$, it follows that $x = 0$ and $|X| = 2$ and $z < 1$.

We deduce that P_1, P_2 both contain members of X, and therefore $r, r' \notin C$. Let $X = \{x_1, x_2\}$ where $x_i \in V(P_i)$ for $i = 1, 2$. We may assume that $r \in V(P_1)$ and $r' \in V(P_2)$; for $i = 1, 2$ let Q_i be the maximal subpath of P_i disjoint from $C \cup X$. Suppose first that $\{r, r'\} \neq \{x_1, x_2\}$. From the maximality of A_2, there is a path of $G|(A_1 \cup C)$ between C and $\{r, r'\}$ with no vertex in X. Consequently there is a path of $G|(A_1 \cup C)$ between C and $V(Q_1 \cup Q_2)$ with no vertex in X. Choose a minimal such path Q, say between c and $V(Q_1)$. Then in $Q_1 \cup Q$ there is a path P_1' between c and r, containing no vertex of X and disjoint from $V(P_2) \setminus \{c\}$; and in $G|(V(Q_2) \cup \{x_2, c'\})$ there is a path P_2' between c' and r', disjoint from P_1'. But this contradicts the choice of P_1, P_2.

We deduce that $\{r, r'\} = \{x_1, x_2\}$. Since G has an rr'-rooted $K_{2,2}$ minor (indeed, subgraph), it follows that $t \geq 3$. Suppose that $A_1 = \{r, r'\}$. Then $e_1 = 5$, and we recall that $e_2 \leq \delta(t)(n_2+1) - 2$ (since $x = 0$), and so $|E(G)| \leq \delta(t)(n_2+1) + 4$; and since $|E(G)| > \delta(t)(n-1) - 1$ and $n = n_2 + 4$, we deduce that

$$\delta(t)(n_2+1) + 4 > \delta(t)(n_2+3) - 1,$$

that is, $5 > 2\delta(t)$, which is impossible since $t \geq 3$. Thus $n_1 > 2$. From the maximality of A_2, there is therefore a path Q with nonnull interior between X and C, with interior in $A_1 \setminus X$. Let Q be c-q_1-$\cdots$-q_k-r' say. By contracting the edges $cx_1, c'x_2$, and all the edges of the path q_1-$\cdots$-q_k, we deduce that the graph H (defined earlier) has no cc'-rooted $K_{2,t-1}$ minor; and so $e_2 + 1 \leq \delta(t-1)(n_2+1) - 1$. But $e_2 > \delta(t)(n_2+1) - 3$ since $z < 1$, and so

$$\delta(t-1)(n_2+1) - 2 > \delta(t)(n_2+1) - 3,$$

that is, $1 > (\delta(t) - \delta(t-1))(n_2+1)$, and since $\delta(t) - \delta(t-1) \geq 1/2$, this is impossible. This proves 7.3. ∎

7.4 *If (G, t, r, r', n) is a minimum counterexample and $u, v \in V(G)$ are adjacent and $\{u, v\} \neq \{r, r'\}$ then $|X(uv)| \geq \frac{1}{2}(t+1)$. Moreover, if $u, v, w, x \in V(G)$ are pairwise adjacent, and $\{u, v\}, \{w, x\} \neq \{r, r'\}$, then $|X(uv)| + |X(wx)| \geq t+2$.*

Proof. Let G' be obtained from G by deleting all edges between u and $X(uv)$, and then contracting the edge uv. From 7.3 it follows that G' is 2-connected to rr'; and since G' has no rr'-rooted $K_{2,t}$ minor, the minimality of (G, t, r, r', n) implies that $|E(G')| \leq \delta(t)(n-2) - 1$. But $|E(G)| > \delta(t)(n-1) - 1$, and $|E(G)| - |E(G')| = |X(uv)| + 1$, and so

$$|X(uv)| + 1 > \delta(t) = \frac{1}{2}(t + 3 - 4/(t+2)).$$

Hence $|X(uv)| + 1 \geq \frac{1}{2}(t+3)$, that is, $|X(uv)| \geq \frac{1}{2}(t+1)$. This proves the first assertion.

For the second, let $u, v, w, x \in V(G)$ be pairwise adjacent, and let G'' be obtained from G by deleting all edges between u and $X(uv)$, and between w and $X(wx)$, and then contracting the edges uv and wx. From 7.3, G'' is 2-connected to rr', and so the minimality of (G, t, r, r', n) implies that $|E(G'')| \leq \delta(t)(n-3) - 1$. But $|E(G)| - |E(G'')| = |X(uv)| + |X(wx)| + 1$ (since the edge uw is both between u and $X(uv)$ and between w and $X(wx)$); consequently

$$|X(uv)| + |X(wx)| + 1 > 2\delta(t) \geq t + 2,$$

and so $|X(uv)| + |X(wx)| \geq t + 2$. This proves 7.4. ∎

7.5 *If (G, t, r, r', n) is a minimum counterexample, then there are two paths P_1, P_2 between r, r', both with nonempty interior, and disjoint except for their ends. Consequently $t \geq 3$.*

Proof. Suppose not. Let G' be the graph obtained from G by deleting the edge rr'. By Menger's theorem there is a cut (A_1, A_2, C) of G' with $r \in A_1$ and $r' \in A_2$, and with $|C| \leq 1$. By 7.3, $(A_1, A_2 \setminus \{r'\}, C \cup \{r'\})$ is not a cut of G, since $r, r' \in A_1 \cup C \cup \{r'\}$; and so $A_2 = \{r'\}$. Similarly $A_1 = \{r\}$, and so $|V(G)| \leq 3$, and yet $|E(G)| > \delta(t)(n-1) - 1 \geq 2n - 3$ which is impossible. This proves 7.5. ∎

7.6 *If (G, t, r, r', n) is a minimum counterexample, then $X(rr') \neq \emptyset$.*

Proof. Suppose that $X(rr') = \emptyset$. Let P_1, P_2 be as in 7.5. We cannot choose P_1, P_2 to be induced paths, since r, r' are adjacent; but we can choose them induced except for the edge rr'. More precisely, we may choose P_1, P_2 such that for $i = 1, 2$, every pair of vertices of P_i that are adjacent in G are also adjacent in P_i, except for the pair rr'. If P_1, P_2 are chosen in this way we say the pair P_1, P_2 is 1-*optimal*. We say the pair is 2-*optimal* if $|V(P_1)| + |V(P_2)|$ is minimized over all pairs satisfying 7.5. (Thus every 2-optimal pair is also 1-optimal.)

Below, we prove several statements about a 1-optimal pair P_1, P_2. For $i = 1, 2$, let p_i be the neighbour of r in P_i, and let p_i' be the neighbour of r' in P_i.

(1) *t is odd, and for every 1-optimal pair P_1, P_2, with p_1, p_2, p_1', p_2' defined as above, it follows that p_1, p_2 are adjacent, and p_1', p_2' are adjacent, and the edges $rp_1, rp_2, r'p_1', r'p_2'$ are each in exactly $(t+1)/2$ triangles.*

For by contracting all edges of P_1 except rp_1, and all edges of P_2 except $r'p_2'$, we do not produce an rr'-rooted $K_{2,t}$ minor, and so there are at most $t - 1$ vertices not in $V(P_1 \cup P_2)$ that are either rp_1-joins or $r'p_2'$-joins. Now there are at least $(t+1)/2$ rp_1-joins, and at most one of them is in $V(P_1 \cup P_2)$ (namely p_2, and only if p_1, p_2 are adjacent; here we use that $p_1 \notin X(rr')$), so at least $(t-1)/2$ are not in $V(P_1 \cup P_2)$. Similarly there are at least $(t-1)/2$ $r'p_2'$-joins that are not in $V(P_1 \cup P_2)$. But no rp_1-join is also an $r'p_2'$-join, since $X(rr') = \emptyset$; and so we have equality throughout. In particular, t is odd, and p_1, p_2 are adjacent, and so are p_1', p_2'. This proves (1).

(2) *If P_1, P_2 is a 1-optimal pair, then P_1, P_2 both have at least four edges.*

Since $X(rr') = \emptyset$, it follows that P_1, P_2 both have at least three edges; suppose that P_1 has exactly three, and its vertices are r-p_1-p_1'-r' in order. Let G' be the graph obtained from G by deleting p_1' and deleting all edges between p_1 and $X(rp_1)$, and then contracting rp_1. Since t is odd and $|X(rp_1)| = (t+1)/2$ by (1), it follows that

$$|E(G')| = |E(G)| - (t+3)/2 - \deg(p') > \delta(t)(n-1) - (t+5)/2 - \deg(p_1').$$

We claim that G' is 2-connected to rr'. For suppose not; then there is a component C of $V(G) \setminus V(P_1 \cup P_2)$ such that no vertex of $P_1 \cup P_2$ has a neighbour in C except possibly r, p_1, p_1'. By 7.3, both r and p_1' have neighbours in C. Consequently there is a path Q between r, r', with interior in $(V(P_1 \setminus p_1) \cup V(C)$, induced except for the edge rr'. Then Q, P_2 form a 1-optimal pair, and the neighbours of r in P_2, Q are nonadjacent, contrary to (1). This proves that G' is 2-connected to rr'. Now G' contains no rr'-rooted $K_{2,t-1}$ minor; and so from the minimality of (G, t, r, r', n), we deduce that $|E(G')| \leq \delta(t-1)(n-3) - 1$, and so

$$\delta(t)(n-1) - (t+5)/2 - \deg(p_1') < \delta(t-1)(n-3) - 1,$$

that is,

$$2\deg(p_1') > n + t + 4\frac{n-5-2t}{(t+1)(t+2)}.$$

Since $n \geq t+3$, it follows that

$$4\frac{n-5-2t}{(t+1)(t+2)} \geq -4/(t+1) \geq -1,$$

and so $2\deg(p_1') \geq n+t$. The same holds for $\deg(p_1)$, and so $\deg(p_1) + \deg(p_1') \geq n+t$. Consequently there are at least t p_1p_1'-joins, and they all belong to $V(G) \setminus V(P_1)$, so contracting the edges rp_1 and $r'p_1'$ produces an rr'-rooted $K_{2,t}$ minor, a contradiction. This proves (2).

(3) *If P_1, P_2 is a 1-optimal pair, and C is a connected subgraph of $G \setminus V(P_1 \cup P_2)$, and for $i = 1,2$ some vertex of the interior of P_i has a neighbour in $V(C)$, then one of r, r' has a neighbour in $V(C)$.*

For suppose that r, r' are anticomplete to $V(C)$. Define p_1, p_2, p_1', p_2' as before. At most one member of $X(rp_1)$ belongs to $V(P_1 \cup P_2)$ (namely, p_2), since the pair P_1, P_2 is 1-optimal, and none of them belong to $V(C)$ since r is anticomplete to $V(C)$. Thus by 7.4, at least $(t-1)/2$ members of $X(rp_1)$ do not belong to $V(P_1 \cup P_2 \cup C)$. Similarly at least $(t-1)/2$ members of $X(r'p_2')$ do not belong to $V(P_1 \cup P_2 \cup C)$. Since $X(rr') = \emptyset$, and therefore $X(rp_1) \cap X(r'p_2') = \emptyset$, we deduce that there are at least $t-1$ members of $X(rp_1) \cup X(r'p_2')$ that do not belong to $V(P_1 \cup P_2 \cup C)$. Consequently contracting all edges of $P_1 \cup P_2$ except rp_1 and $r'p_2'$ (and contracting some edges of C) produces an rr'-rooted $K_{2,t}$ minor, a contradiction. This proves (3).

(4) *If P_1, P_2 is a 2-optimal pair, then for every edge uv of P_1, some member of $X(uv)$ belongs to $V(P_2)$.*

For suppose not. By (1) it follows that $u, v \neq r, r'$. We may assume that r, u, v, r' occur in this order in P_1. Since we do not produce an rr'-rooted $K_{2,t}$ minor by contracting all edges of $P_1 \cup P_2$ except uv and rp_2, it follows that there are at most $t-1$ members of $X(rp_2) \cup X(uv)$ that do not belong to $V(P_1 \cup P_2)$. Since $V(P_1 \cup P_2)$ contains only one member of $X(rp_2)$, and no member of $X(uv)$, 7.4 implies that there exists $w \in X(rp_2) \cap X(uv)$. Thus w is adjacent to both r, v, and does not belong to P_2. From the 2-optimality of the pair P_1, P_2, it follows that no path between r, r' with nonempty interior in $V(P_1 \cup \{w\})$ has strictly fewer edges than P_1, and in particular r, u are adjacent. Similarly r', v are adjacent; but then P_1 has only three edges, contrary to (2). This proves (4).

(5) *If P_1, P_2 is a 2-optimal pair, then P_1, P_2 both have exactly four edges.*

For by (2) they both have at least four edges; suppose that P_1 has at least five, and choose an edge uv of P_1 such that u, v are both nonadjacent to both of r, r'. We may assume that r, u, v, r' are in order in P_1. Suppose first that some uv-join w does not belong to $V(P_2)$. By 7.3, there is a path between w and $V(P_1 \cup P_2)$ containing neither of u, v; and so there is a path $w = q_0$-q_1-$\cdots$-q_k say, such that $q_0, \ldots, q_k \notin V(P_1 \cup P_2)$, and q_k is adjacent to some $y \in V(P_1 \cup P_2) \setminus \{u, v\}$. Choose such a path with k minimum. (Possibly $k = 0$.) It follows that for $0 \le i < k$, q_i has no neighbour in $V(P_1 \cup P_2) \setminus \{u, v\}$.

We claim that q_k has a neighbour in $V(P_1) \setminus \{u, v\}$, and we may therefore assume that $y \in V(P_1)$. For suppose not; then y belongs to the interior of P_2, and in particular r, r' are nonadjacent to q_k. Hence r, r' have no neighbours in $\{q_0, \ldots, q_k\}$, contrary to (3). This proves that we may choose $y \in V(P_1)$. From the symmetry we may assume that y belongs to the subpath of P_1 between r and u.

Now there is a path with nonempty interior, between r, r', with interior contained in $(V(P_1) \setminus \{u\}) \cup \{q_0, \ldots, q_k\}$; choose such a path, P_3 say, minimal. Thus the pair P_3, P_2 is 1-optimal. Some vertex of P_3 does not belong to P_1, and so we may choose $i \le k$ minimum such that $q_i \in V(P_3)$. Let C be the subgraph induced on $\{u, q_0, \ldots, q_{i-1}\}$. Thus C is connected, and disjoint from both P_2, P_3, and r, r' both have no neighbours in C (since $q_k \notin V(C)$). Moreover, q_i belongs to the interior of P_3, and has a neighbour in $V(C)$; and by (4), some vertex of the interior of P_2 is adjacent to u and therefore has a neighbour in $V(C)$. But this contradicts (3) applied to C and the 1-optimal pair P_2, P_3.

This proves that there is no such vertex w, and so every uv-join belongs to $V(P_2)$. Since P_1, P_2 is 2-optimal, it follows that every two uv-joins in $V(P_2)$ are adjacent (for otherwise we could choose another pair of paths with smaller union), and in particular there are at most two uv-joins. By 7.4 there are at least $(t+1)/2$ uv-joins, and so $t = 3$, and there are exactly two uv-joins x, y say, and x, y are adjacent members of the interior of P_2. Thus u, v, x, y are pairwise adjacent, and so by the second statement of 7.4, $|X(uv)| + |X(xy)| \ge t + 2 = 5$. Since $|X(uv)| = 2$, it follows that $|X(xy)| \ge 3$, and so there is an xy-join z different from u, v. But then contracting all edges of P_2 except xy gives an rr'-rooted $K_{2,3}$ minor, a contradiction. This proves (5).

Now by 7.5 there is a 2-optimal pair P_1, P_2. By (5), P_1 and P_2 both have four edges; for $i = 1, 2$, let P_i have vertices r-p_i-q_i-p'_i-r' in order.

(6) $\deg(q_1), \deg(q_2) \ge (n + t - 2)/2$.

For let G' be obtained from G by deleting the edges between p_1 and $X(rp_1)$, and between p'_1 and $x(r'p'_1)$, and deleting q_1, and contracting the edges rp_1 and $r'p'_1$. As in the proof of (2), it follows that G' is 2-connected to rr'. Since G' has no rr'-rooted $K_{2,t-1}$ minor, the minimality of (G, t, r, r', n) implies that $|E(G')| \le \delta(t-1)(n-4) - 1$. But

$$|E(G')| = |E(G)| - |X(rp_1)| - |X(r'p'_1)| - 2 - \deg(q_1),$$

and by (1) $|X(rp_1)| = |X(r'p'_1)| = (t+1)/2$. Consequently

$$|E(G)| - (t+1) - 2 - \deg(q_1) \le \delta(t-1)(n-4) - 1,$$

that is, $|E(G)| \le \delta(t-1)(n-4)+t+2+\deg(q_1)$. But $|E(G)| > \delta(t)(n-1)-1$, and therefore

$$\delta(t)(n-1)-1 < \delta(t-1)(n-4)+t+2+\deg(q_1),$$

that is,

$$n+t-1+4\frac{n-3t-7}{(t+2)(t+1)} < 2\deg(q_1).$$

Since $n \ge t+3$, it follows that

$$4\frac{n-3t-7}{(t+2)(t+1)} \ge -8/(t+1) \ge -2,$$

and so $n+t-2 \le 2\deg(q_1)$. This proves (6).

There are at least $(t-1)/2$ $r'p_2'$-joins that are not in $V(P_1 \cup P_2)$, and at least $(t-1)/2$ rp_1-joins with the same property. If all these rp_1-joins are adjacent to q_1, then (since p_1 is adjacent to r, q_1) contracting the edges $q_1p_1', p_1'r', rp_2, p_2q_2, q_2p_2'$ yields an rr'-rooted $K_{2,t}$ minor, a contradiction. We deduce that some rp_1-join s_1 say is not in $V(P_1 \cup P_2)$ and is not adjacent to q_1. Similarly some $r'p_2'$-join s_2 is not in $V(P_1 \cup P_2)$ and is nonadjacent to q_2.

Let $X_1 = X(q_1q_2) \setminus V(P_1 \cup P_2)$, and $X_2 = X(q_1q_2) \cap V(P_1 \cup P_2)$. Let Z be the set of all vertices different from r, r' that are nonadjacent to both q_1, q_2 (with $q_1, q_2 \in Z$ if q_1, q_2 are nonadjacent). Let $A_1 = \{r, p_1, q_1\}$ and $A_2 = \{r', p_2', q_2\}$. Let B be the set of all vertices not in $V(P_1 \cup P_2) \cup X_1$ with a neighbour in A_1 and a neighbour in A_2. Since G does not contain an rr'-rooted $K_{2,t}$ minor obtained by contracting the edges of $G|A_1$ and $G|A_2$, and since every vertex in $B \cup X_1 \cup \{p_1', p_2\}$ has a neighbour in A_1 and one in A_2, it follows that $|B| \le t-3-|X_1|$.

Now if s_1 is nonadjacent to q_2 then $s_1 \in Z$, and if s_1 is adjacent to q_2 then $s_1 \in B$, and similarly s_2 belongs to one of Z, B_1. Since $s_1 \ne s_2$, we deduce that $|B|+|Z| \ge 2$, and therefore $2-|Z| \le t-3-|X_1|$, that is, $|X_1| \le |Z|+t-5$. Since $X_2 \subseteq \{p_1, p_1', p_2, p_2'\}$ and therefore $|X_2| \le 4$, it follows that $|X(q_1q_2)| = |X_1|+|X_2| \le |Z|+t-1$. But

$$|X(q_1q_2)| + (n-|Z|-2) = \deg(q_1)+\deg(q_2),$$

and so $\deg(q_1)+\deg(q_2) \le n+t-3$, contrary to (6). This proves 7.6. ∎

7.7 *If (G, t, r, r', n) is a minimum counterexample, then there is exactly one rr'-join x, and* $\deg(x) > \delta(t) + (\delta(t)-\delta(t-1))(n-2)$.

Proof. By 7.6 there is an rr'-join x. We prove first that $\deg(x) > \delta(t)+(\delta(t)-\delta(t-1))(n-2)$. For let G' be obtained from G by deleting x. By 7.3, G' is 2-connected to rr', and has no rr'-rooted $K_{2,t-1}$ minor (for otherwise this could be extended to an rr'-rooted $K_{2,t}$ minor in G, using x). From the minimality of (G, t, r, r', n), $|E(G')| \le \delta(t-1)(n-2)-1$. But $|E(G)| >$

$\delta(t)(n-1)-1$, and $|E(G)|-|E(G')| = \deg(v)$, and so $\deg(x) > \delta(t)(n-1)-\delta(t-1)(n-2)$. This proves the claim.

Now suppose that y is another rr'-join. If there are t vertices different from x,y,r,r' and adjacent to both x,y, then contracting the edges $rx, r'y$ gives an rr'-rooted $K_{2,t}$ minor, a contradiction. Thus there are at most $t-1$ such vertices, and hence $\deg(x)+\deg(y) \le 6+(n-4)+(t-1) = n+t+1$. But we have seen that $\deg(x), \deg(y) > \delta(t)+(\delta(t)-\delta(t-1))(n-2)$, and so $2\delta(t)+2(\delta(t)-\delta(t-1))(n-2) < n+t+1$, which on substituting the expressions for $\delta(t)$ and $\delta(t-1)$ simplifies down to $n < t+3$, a contradiction. This proves 7.7. ∎

In view of 7.7, it remains to handle the case when $|X(rr')| = 1$. This will take several more lemmas, but first let us set up some notation. In what follows in this section, (G,t,r,r',n) is a minimum counterexample; there is a unique rr'-join x; and N, N' are the sets of vertices in $V(G)\setminus\{x,r,r'\}$ adjacent to r,r' respectively. (Since $X(rr') = \{x\}$, it follows that $N \cap N' = \emptyset$.) Let $W = V(G)\setminus(N \cup N' \cup \{x,r,r'\})$. We fix $p \in N$ and $p' \in N'$ and a path P, such that P is between p,p' and its interior is a subset of W. (This is possible by 7.5.) We partition $N\setminus\{p\}$ into four sets A,B,C,D as follows. A vertex in $N\setminus\{p\}$ belongs to $A\cup C$ if and only if it is adjacent to p, and it belongs to $B\cup C$ if and only if it is adjacent to x. (Thus, A is the set of vertices in $N\setminus\{p\}$ adjacent to p and not to x, and so on.) We define A',B',C',D' similarly with r,r' exchanged. Let $e=1$ if x,p are adjacent, and $e=0$ otherwise; and let $e'=1$ if x,p' are adjacent, and $e'=0$ otherwise.

7.8 *The following inequalities hold:*

$$|A|+|C|+|B'|+|C'| \le t-1;$$

$$|A'|+|C'|+|B|+|C| \le t-1;$$

$$\begin{aligned}
(t+1)/2 - e \le |A|+|C| &\le (t-1)/2 + e';\\
(t+1)/2 - e' \le |A'|+|C'| &\le (t-1)/2 + e;\\
(t-1)/2 - e \le |B|+|C| &\le (t-3)/2 + e';\\
(t-1)/2 - e' \le |B'|+|C'| &\le (t-3)/2 + e.
\end{aligned}$$

Proof. Since contracting $rx, r'p'$ and all edges of P does not produce an rr'-rooted $K_{2,t}$ minor, the first statement holds, and the second follows by exchanging r,r'. The four remaining lower bounds are consequences of 7.4 applied to the edges $rp, r'p', rx, r'x$; and the upper bounds follow from these and the first two statements. This proves 7.8. ∎

7.9 *If $a \in A$ has no neighbour in N', then there is an integer $h \ge (t+1)/2$ and disjoint subsets $X_1, X_2, \ldots, X_h, Y_1, Y_2 \subseteq V(G)\setminus(N'\cup\{r',x\})$, satisfying:*

- *each of $X_1,\ldots,X_h,Y_1,Y_2$ induces a connected subgraph of G*

- $r \in Y_1, p \in Y_2$
- *for* $1 \le i \le h$ *there is an edge of* G *between* X_i *and* Y_1*, and an edge of* G *between* X_i *and* Y_2*, and*
- *every vertex of each of* $X_1, \dots, X_h, Y_1, Y_2$ *either belongs to* $N \cup \{r\}$ *or is adjacent to* a*.*

Proof. If $|A \cup C| \ge (t+1)/2$, we may take $h = |A \cup C|$, and let $X_1, \dots, X_h$ be the singleton subsets of $A \cup C$, and $Y_1 = \{r\}$ and $Y_2 = \{p\}$. Thus we may assume that $|A \cup C| \le t/2$. By 7.8, $|A \cup C| \ge (t+1)/2 - e$, and so $e = 1$ (that is, x, p are adjacent) and $|A \cup C| \ge (t-1)/2$. Let $h = |A \cup C| + 1$, and for $3 \le i \le h$ let X_i be a singleton subset of $C \cup (A \setminus \{a\})$. It remains to select X_1, X_2, Y_1 and Y_2, and we do this as follows. If a has two neighbours $w_1, w_2 \in B \cup D$, we may take $X_1 = \{w_1\}, X_2 = \{w_2\}$, $Y_1 = \{r\}$, and $Y_2 = \{p, a\}$. Thus we may assume that a has at most one neighbour in $B \cup D$. Now $|X(ar)| \ge (t+1)/2$ by 7.4, and since $|A \cup C| \le t/2$, it follows that a has a unique neighbour in $B \cup D$, say u_1. Choose a sequence $u_1, \dots, u_k$ of distinct vertices, maximal with the following properties (where $u_0 = r$):

- $u_2, \dots, u_k \in W$,
- u_1-$\cdots$-u_k is a path, and a is adjacent to all of $u_1, \dots, u_k$
- p is nonadjacent to all of $u_1, \dots, u_k$, and
- for $1 \le i \le k-1$, $X(au_i) \subseteq \{u_{i-1}, u_{i+1}\} \cup A \cup C$.

Now $|X(au_k)| \ge (t+1)/2$ by 7.4. Since $|A \cup C| \le t/2$, it follows that there is a vertex $u_{k+1} \notin A \cup C \cup \{u_{k-1}, u_k\}$ such that a, u_k, u_{k+1} are pairwise adjacent. Since u_k is nonadjacent to p, and a is nonadjacent to x and has no neighbour in $N' \cup \{r'\}$, it follows that $u_{k+1} \notin N' \cup \{r', x\}$. Suppose that $u_{k+1} \in \{u_0, \dots, u_k\}$, and let $u_{k+1} = u_i$ where $0 \le i \le k$. Then $i \le k-2$ (since $u_{k+1} \ne u_{k-1}, u_k$), and so $k \ge 2$ and therefore $u_k \notin N$, and so $i > 0$. Consequently $X(au_i) \subseteq \{u_{i-1}, u_{i+1}\} \cup A \cup C$, which is impossible since $u_k \in X(au_i)$. Thus $u_{k+1} \ne u_0, \dots, u_k$. Since $u_{k+1} \ne u_1$, and u_1 is the unique neighbour of a in $B \cup D$, it follows that $u_{k+1} \notin B \cup D$, and so $u_k \notin N$. From the maximality of the sequence $u_1, \dots, u_k$, we deduce that either p is adjacent to u_{k+1}, or $X(au_k) \not\subseteq \{u_{k-1}, u_{k+1}\} \cup A \cup C$. In the first case, we may take $X_1 = \{a\}, X_2 = \{u_1, \dots, u_k, u_{k+1}\}$, $Y_1 = \{r\}$, and $Y_2 = \{p\}$. In the second case, let $w \in X(au_k)$ with $w \notin \{u_{k-1}, u_{k+1}\} \cup A \cup C$; then we may take $X_1 = \{u_{k+1}\}$, $X_2 = \{w\}$, $Y_1 = \{r, u_1, \dots, u_k\}$ and $Y_2 = \{p, a\}$. This proves 7.9. ∎

7.10 *x is adjacent to both p, p'.*

Proof. For suppose there is some choice of P, p, p' such that x is nonadjacent to one of p, p'; and choose such P, p, p' with P of minimum length. Let x, p' be nonadjacent, say. By 7.8, x is adjacent to p, and $|A| + |C| = (t-1)/2$, $|A'| + |C'| = (t+1)/2$, $|B| + |C| = (t-3)/2$, and $|B'| + |C'| = (t-1)/2$. In particular, since $|A| + |C| > |B| + |C|$, it follows that $A \ne \emptyset$; choose

$a \in A$. It follows that a has no neighbour in P different from p, since otherwise we could choose a new path P' between a and p', and this is impossible by 7.8 since x is nonadjacent to both a, p'.

Suppose that $a \in A$ has no neighbour in N'. Since $|X(xr')| \geq (t+1)/2$ by 7.4, and $X(xr') \subseteq N' \cup \{r\}$, there are at least $(t-1)/2$ xr'-joins in N', and none of them is in P. Moreover, since no vertex of P belongs to N or is adjacent to a except p, 7.9 implies that contracting $rx, p'r'$ and the edges of P (and the edges of the $h+2$ subgraphs given by 7.9) yields an rr'-rooted $K_{2,t}$ minor, a contradiction.

Thus there exists $a' \in N'$ adjacent to a. Since a has no neighbour in P different from p, it follows that a, p' are nonadjacent, and in particular $a' \neq p'$. The path a-a' satisfies our hypotheses for the choice of P, and so from the minimality of the length of P, we deduce that P has only one edge, and so p, p' are adjacent. From 7.8, x is adjacent to a'. Now $|A' \cup C'| = (t+1)/2$ as we already saw, and so there are at least $(t-1)/2$ vertices not in $\{x, r, r', p, p', a, a'\}$ and adjacent to both p', r'; and similarly there are at least $(t-1)/2$ such vertices adjacent to both a, r. But then contracting the edges $rp, pp', aa', a'r'$ gives an rr'-rooted $K_{2,t}$ minor, a contradiction. This proves 7.10. ∎

7.11 *P has length at least two.*

Proof. Suppose not; then p, p' are adjacent. Suppose there is a 3-cut $(L, M, \{r, p, p'\})$, where $x, r' \in M$. Then there is a path between r and p' with interior in L, by 7.3, and x has no neighbour in the interior of this path; and hence there is a choice of P, p, p' that violates 7.10, a contradiction. Thus there is no such 3-cut. Let G' be the graph obtained from G by deleting all edges between p and $X(pr)$, deleting the vertex p', and contracting pr. It follows that G' is 2-connected to rr'.

Now G' has no rr'-rooted $K_{2,t-1}$ minor, and so from the minimality of (G, t, r, r', n), it follows that $|E(G')| \leq \delta(t-1)(n-3) - 1$. But $|E(G)| - |E(G')| = \deg(p') + |A| + |C| + 2$, and $|C| \leq |B| + |C| \leq (t-1)/2$ by 7.8, and so

$$|E(G)| \leq \delta(t-1)(n-3) + \deg(p') + |A| + (t+1)/2.$$

Since $|E(G)| > \delta(t)(n-1) - 1$, we deduce that

$$\delta(t)(n-1) - 1 < \delta(t-1)(n-3) + \deg(p') + |A| + (t+1)/2,$$

and so

$$\deg(p') > 2\delta(t) + (\delta(t) - \delta(t-1))(n-3) - |A| - (t+3)/2.$$

But since contracting the edges $rx, p'r'$ does not produce an rr'-rooted $K_{2,t}$ minor, it follows that x, p' have at most $t-2$ common neighbours that are not in $V(P) \cup \{x, r, r'\}$, and therefore at most t common neighbours in total. Since every vertex in A is nonadjacent to x (by definition) and to p' (by 7.10), it follows that $\deg(p') + \deg(x) \leq n - |A| + t$. But from 7.7, $\deg(x) > \delta(t) + (\delta(t) - \delta(t-1))(n-2)$; and so

$$2\delta(t) + (\delta(t) - \delta(t-1))(n-3) - |A| - (t+3)/2 + \delta(t) + (\delta(t) - \delta(t-1))(n-2)) < n - |A| + t,$$

which simplifies to

$$(t-3)(t+2)+8(n-t-3)<0,$$

a contradiction. This proves 7.11. ∎

7.12 *A, A' are both nonempty.*

Proof. Suppose that $A' = \emptyset$, say. By 7.8, $|A'|+|C'| \geq (t-1)/2$, and $|B'|+|C'| \leq (t-1)/2$; so t is odd, $|C'| = (t-1)/2$, and $B' = \emptyset$. If there exists $a \in A$, then (since a is anticomplete to $N' \cup (V(P) \setminus \{p\})$ by 7.10), 7.9 implies that contracting the edges $rx, p'r'$ and all edges of P (and the edges of the subgraphs provided by 7.9) yields an rr'-rooted $K_{2,t}$ minor, a contradiction. Thus $A = \emptyset$, and so similarly $B = \emptyset$ and $|C| = (t-1)/2$.

If every member of C has a neighbour in $V(P \setminus p)$, then we may obtain an rr'-rooted $K_{2,t}$ minor by contracting $rx, r'p'$ and all edges of $P \setminus p$, a contradiction. Thus there exists $c \in C$ with no neighbour in $V(P \setminus p)$. Now $|X(rp)| = (t+1)/2$, and since r, p, x, c are pairwise adjacent, 7.4 implies that $|X(cx)| \geq (t+3)/2$. Hence there is a vertex $u_1 \notin C \cup \{p, r\}$ and adjacent to c, x. Since $u_1 \notin C$ and $B = \emptyset$, it follows that r, u_1 are nonadjacent, and so $u_1 \notin N$; and since N is anticomplete to N' by 7.11, it follows that $u_1 \in W$. We claim that $X(cx) \subseteq C \cup \{p, r, u_1\}$; for if not, there is a second vertex u_1' that satisfies the defining condition for u_1, and then contracting the edges $rx, r'p', pc$ and all edges of P gives an rr'-rooted $K_{2,t}$ minor, a contradiction. Let $u_0 = x$, and choose a maximal sequence $u_1, \ldots, u_k$ of distinct members of W with the following properties:

- $u_1\text{-}\cdots\text{-}u_k$ is a path, and c is adjacent to all of $u_1, \ldots, u_k$, and
- for $1 \leq i < k$, $X(cu_i) \subseteq C \cup \{u_{i-1}, u_{i+1}\}$.

Now by 7.4, $|X(cu_k)| \geq (t+1)/2$, and so there exists a vertex $u_{k+1} \neq u_{k-1}, u_k$ such that $u_k \notin C$. If $u_{k+1} \in V(P)$, then contracting $rx, r'p'$, all edges of P, and the edges of the path $u_2\text{-}\cdots\text{-}u_{k+1}$ gives an rr'-rooted $K_{2,t}$ minor, a contradiction. If $u_{k+1} \in D$, then contracting $rp, r'x$, all edges of P, and the edges of the path $x\text{-}u_1\text{-}\cdots\text{-}u_k$ gives an rr'-rooted $K_{2,t}$ minor. Moreover, $u_{k+1} \notin N'$, since c is anticomplete to N'; and so $u_{k+1} \in W \cup \{x\}$. Suppose that $u_{k+1} = u_i$ for some $i \in \{0, \ldots, k\}$; then $i \leq k-2$, and so $k \geq 2$, and $u_k \in X(cu_i)$. But $X(cu_0) \subseteq C \cup \{p, r, u_1\}$, so $i \neq 0$; hence $X(cu_i) \subseteq C \cup \{u_{i-1}, u_{i+1}\}$, a contradiction. Thus $u_{k+1} \in W$ and is different from $u_0, \ldots, u_k$. From the maximality of the sequence $u_1, \ldots, u_k$, it follows that $X(cu_k) \not\subseteq C \cup \{u_{k-1}, u_{k+1}\}$, and so there is a vertex w adjacent to c, u_k and not in $C \cup \{u_{k-1}, u_{k+1}\}$. Thus w satisfies the defining conditions for u_{k+1}, and so by the same argument $w \in W$ and is different from $u_0, \ldots, u_k$. But then contracting $rx, r'p', pc$, all edges of P, and all edges of the path $x\text{-}u_1\text{-}\cdots\text{-}u_k$ gives an rr'-rooted $K_{2,t}$ minor, a contradiction. This proves 7.12. ∎

Now we complete the proof of the second main result.

Proof of 7.1 We may assume that P is an induced path. Let q be the neighbour of p in P. By 7.12, both A, A' are nonempty. Choose $a' \in A'$. Since a' is anticomplete to N by 7.10, 7.9 (with r, r' exchanged) yields that there is an integer $h \geq (t+1)/2$ and disjoint subsets $X_1, X_2, \dots, X_h, Y_1, Y_2 \subseteq V(G) \setminus (N \cup \{r, x\})$, satisfying:

- each of $X_1, \dots, X_h, Y_1, Y_2$ induces a connected subgraph of G
- $r' \in Y_1, p' \in Y_2$
- for $1 \leq i \leq h$ there is an edge of G between X_i and Y_1, and an edge of G between X_i and Y_2, and
- every vertex of each of $X_1, \dots, X_h, Y_1, Y_2$ either belongs to $N' \cup \{r'\}$ or is adjacent to a'.

It follows that all these subsets are disjoint from $V(P)$ except that $p' \in Y_2$, by 7.10. Let F be the union of the edge sets of $X_1, X_2, \dots, X_h, Y_1, Y_2$. By contracting rp, all edges of P, and all edges of F, it follows that $(t+3)/2 \leq t-1$, and so $t \geq 5$. By contracting $rp, r'x$, all edges of P, and all edges of F, we deduce that $|B \cup C| \leq (t-3)/2$, and so equality holds, by 7.8. Moreover, the same contraction shows that every vertex in $X(xp)$ belongs to C, except for r and possibly q; and so $|C| = (t-3)/2$ and $B = \emptyset$ and $|X(xp)| = (t+1)/2$. Since $t \geq 4$, there exists $c \in C$. Now c, p, r, x are pairwise adjacent, and so 7.4 implies that $|X(rc)| \geq (t+3)/2$. Since $|B \cup C| = (t-3)/2$, there are at least two members of $X(rc)$ not in $B \cup C \cup \{x, p\}$, say w_1, w_2; thus $w_1, w_2 \in A \cup D$. In particular, $w_1, w_2 \notin V(P)$, and so contracting $rp, r'x, xc$, all edges of P, and all edges of F produces an rr'-rooted $K_{2,t}$ minor, a contradiction. Thus there is no minimum counterexample (G, t, r, r', n). This completes the proof of 7.1. ∎

8 Higher connectivity

If we add to 1.1 the hypothesis that G is k-connected, we should expect a change in the extremal function (depending on k), and in this section we study this. First, a result of G. Ding (private communication):

8.1 *For every $t \geq 0$, there exists $n(t) \geq 0$ such that every 5-connected graph with no $K_{2,t}$ minor has at most $n(t)$ vertices.*

If we replace 5-connected by 4-connected, this is no longer true. For instance, let n be even, $n = 2m$ say, and let G be the graph with n vertices $u_1, \dots, u_m, v_1, \dots, v_m$, in which for $1 \leq i \leq m$, u_i, v_i are adjacent, and $\{u_i, v_i\}$ is complete to $\{u_{i+1}, v_{i+1}\}$ (where u_{m+1}, v_{m+1} mean u_1, v_1) and with no other edges. Then G is 4-connected and has no $K_{2,5}$ minor. Note that in this graph, every vertex has degree 5, and so $|E(G)| = 5n/2$. This shows that the next result is also best possible in a sense. The next result was proved in joint work with Sergey Norin and Robin Thomas, and is more or less an analogue of 1.2.

8.2 *For every $t \geq 0$, there exists $c(t) \geq 0$ such that every 3-connected n-vertex graph with no $K_{2,t}$ minor has at most $5n/2 + c(t)$ edges.*

Proof. The proof is a fairly standard "bounded treewidth" argument, using the methods of [8], and so we just sketch it. Let G be a 3-connected graph with no $K_{2,t}$ minor. We prove by induction on $|V(G)|$ that $|E(G)| \leq 5n/2 + c(t)$, where $n = |V(G)|$ and $c(t)$ is a large constant.

A *tree-decomposition* of G is a pair $(T, (X_s : s \in V(T)))$, where T is a tree and each X_s is a subset of $V(G)$, satisfying:

- $\bigcup_{s \in V(T)} = V(G)$, and for every edge uv of G there exists $s \in V(T)$ with $u, v \in X_s$
- for all $s_1, s_2, s_3 \in V(T)$, if s_2 belongs to the path of T between s_1, s_3, then $X_{s_1} \cap X_{s_3} \subseteq X_{s_2}$.

Let us say that a tree-decomposition $(T, (X_s : s \in V(T)))$ is *proper* if

- for every leaf s of T (that is, a vertex with degree one in T) there is a vertex $v \in X_s$ such that $v \notin X_{s'}$ for all $s' \in V(T) \setminus \{s\}$,
- $X_s \neq X'_s$ for every edge ss' of T, and
- for every edge $f \in E(T)$, if S is the vertex set of a component of $T \setminus f$, then $\cup_{s \in S} X_s$ is connected.

We define the *order* of an edge ss' of T to be $|X_s \cap X_{s'}|$. Let us say $(T, (X_s : s \in V(T)))$ is *linked* if it is proper, and for every two distinct vertices $s_1, s_2 \in V(T)$, and every integer $k \geq 0$, either

- there are k vertex-disjoint paths in G between X_{s_1} and X_{s_2}, or
- there is an edge of the path of T between s_1, s_2 with order less than k.

Finally, we say a tree-decomposition $(T, (X_s : s \in V(T)))$ is a *path-decomposition* if T is a path.

Since $K_{2,t}$ is planar, it follows from the main theorem of [10] that there is a number c_1 (depending on t, but independent of G) such that G admits a tree-decomposition $(T, (X_s : s \in V(T)))$ with $|X_s| \leq c_1$ for all $s \in V(T)$. From a theorem of Thomas [11] we may choose this tree-decomposition so that in addition it is linked. If some vertex s of T has degree more than $(t-1)c_1(c_1-1)/2$, then $G \setminus X_s$ has more than $(t-1)c_1(c_1-1)/2$ components, each with at least two attachments in X_t (indeed, with at least three, since G is 3-connected); so some t of them share the same two attachment vertices, and G has a $K_{2,t}$ minor, a contradiction. Thus the maximum degree in T is bounded.

On the other hand, by choosing the constant $c(t)$ in the theorem large enough, we can ensure that $|V(G)|$ is at least any desired function of t, and so $|V(T)|$ is large; and consequently standard tree-decomposition methods yield a linked path-decomposition of G, $(P, (Y_i : i \in V(P)))$ say, where P has vertices $0, 1, \dots, m$ in order, say, such that m is large (at least some large function of t) and all the sets $Y_i \cap Y_{i+1}$ have the same size k say, where $3 \le k \le c_1$. (The sets Y_i may have unbounded cardinality.) The linkedness of this decomposition provides disjoint paths $P_1, \dots, P_k$ from Y_0 to Y_m, and we may choose them with total length minimum. For $1 \le i \le m$ each P_j has a unique vertex in $Y_{i-1} \cap Y_i$. Let G_i be the subgraph $G|Y_i$.

Let I_1 be the set of all $i \in \{1, \dots, m-1\}$ such that some vertex of Y_i is not in $V(P_1 \cup \dots \cup P_k)$. For each $i \in I_1$, there is a component C of $G_i \setminus (P_1 \cup \dots \cup P_k)$, and at least one of $P_1, \dots, P_k$ contains an attachment of C; and by rerouting the portions of $P_1, \dots, P_k$ within G_i (using the 3-connectivity of G) we can arrange that at least two of $P_1, \dots, P_k$ contain attachments of some such C. By contracting the edges of (the rerouted) $P_1, \dots, P_k$, since G has no $K_{2,t}$ minor, we deduce that $|I_1|$ is at most some function of t.

Since m is at least some (much bigger) function of t, there is a large subpath of P containing no member of I_1; and so we may assume that $I_1 = \emptyset$, by replacing P by this subpath and adjusting the constants accordingly.

Now either P_1 contains an edge of only a bounded number of $G_1, \dots, G_{m-1}$ (at most an appropriate function of t) or it does not. In the first case we can find a large subpath of P such that all the graphs G_i for i in this subpath contain no edge of P_1; and in this case we may replace P by this subpath. In the second case, we may group the terms of the path-decomposition so that P_1 has an edge in every group (indeed, at least two edges in every group), and so obtain a new linked path-decomposition such that P_1 has at least two edges in every term. By repeating this for all P_j, we may assume that for $1 \le j \le k$, if P_j has positive length then P_j has at least two edges in each G_i.

Let I_2 be the set of all $i \in \{1, \dots, m-1\}$ such that for some $j \in \{1, \dots, k\}$, P_j has positive length and there are at least two values of $j' \ne j$ such that there is an edge of G_i between $V(P_j)$ and $V(P_{j'})$. For each $i \in I_2$, there are only k^3 possibilities for the value of j and the two values of j', so there are at least $|I_2|/k^3$ values of $i \in I_2$ giving the same triple, say $j = 1$ and the j' values are $2, 3$. By taking every second one of these, we arrange that the subpaths of P_1 in these various G_i are vertex-disjoint; and then by contracting the edges of P_2, P_3, and using that G has no $K_{2,t}$ minor, we deduce that $|I_2| \le 2k^3(t-1)$. Thus $|I_2|$ is bounded, and so by replacing P by a large subpath, we may assume that $I_2 = \emptyset$.

Now some P_i has positive length, say P_1. Then the intersection of P_1 with each G_i has length at least two, and therefore has an internal vertex v_i say. Since G is 3-connected and so v_i has degree at least three, v_i has a neighbour u_i different from its two neighbours in P_1. Since every neighbour of v_i in G belongs to Y_i, and P_1 is induced, and $I_1 = \emptyset$, there exists $j(i) \in \{2, \dots, k\}$ such that $u_i \in V(P_{j(i)} \cap G_i)$. Since $i \notin I_2$, it follows that $j(i)$ is independent of the choice of v_i; and so every internal vertex of $P_1 \cap G_i$ has a neighbour in $P_{j(i)} \cap G_i$, and has no neighbour in $P_h \cap G_i$ for $1 \le h \le k$ with $h \ne 1, j(i)$. Suppose that there is a large number (at least a large function of t) of $i \in \{1, \dots, m-2\}$ such that $j(i) \ne j(i+1)$. Then we may group some of the terms of our path-decomposition into pairs, and obtain a new linked path-decomposition in which $|I_2|$ is large, and obtain a $K_{2,t}$ minor, a contradiction. Thus there are only a bounded number of $i \in \{1, \dots, m-2\}$ such that $j(i) \ne j(i+1)$; and so we may replace P by a large subpath and assume that $j(i)$ is the same for all i. Since $I_2 = \emptyset$, we may assume that every internal vertex of P_1 has neighbours in P_2, and has no neighbours in any P_h for $3 \leq h \leq k$. We repeat the same for P_2; thus, we may assume that every internal vertex of P_2 has neighbours in P_1, and has no neighbours in any P_h for $3 \leq h \leq k$. (Possible P_2 has zero length, however, in which case this statement is vacuous.)

We recall that for $1 \leq i \leq m-1$, $P_1 \cap G_i$ has at least two edges, and hence at least one internal vertex. We may arrange that $m \geq 5$. Let the vertices of $P_1 \cap G_3$ be $p_1, \ldots, p_s$ in order, where $p_1 \in Y_2 \cap Y_3$ and $p_s \in Y_3 \cap Y_4$. Since $m \geq 5$, it follows that $p_1, \ldots, p_s$ have no neighbours in $Y_0 \cup Y_m$ (except possibly the vertex of P_2 if P_2 has length zero). Let p_0 be the neighbour of p_1 in P_1 different from p_2, and define p_{s+1} similarly. Thus p_0 is an internal vertex of G_2, and p_{s+1} of G_4. Let $h \in \{1, \ldots, s-1\}$, and let $u = p_h$ and $v = p_{h+1}$. Let $X = V(P_2 \cap (G_2 \cup G_3 \cup G_4))$. Every neighbour of p_h is in $\{p_{h-1}\} \cup X$, and every neighbour of v is in $X \cup \{p_{h+2}\}$. Suppose that for some vertex w of G, G admits a 3-cut $(A, B, \{u, v, w\})$. Since G is 3-connected, both u, v have neighbours in both A, B, and so both A, B meet the connected sets $\{p_{h-1}\} \cup X$ and $X \cup \{p_{h+2}\}$. Consequently $w \in X$. It follows that P_2 has positive length, and w belongs to the interior of P_2. Hence $w \notin Y_0 \cup Y_m$; but Y_0, Y_m are both connected (since the path-decomposition is proper), and so $G \setminus \{u, v, w\}$ is connected, a contradiction. Thus there is no such 3-cut, and so the graph obtained by contracting the edge uv is 3-connected (and this is true for every edge of $P_1 \cap G_3$). Consequently there are at least two uv-joins w_1, w_2 say, since otherwise contracting uv would give a smaller counterexample. It follows that $w_1, w_2 \in V(P_2 \cap G_3)$, and so P_2 has nonzero length. From the minimality of the union of $P_1, \ldots, P_k$, we deduce that w_1, w_2 are adjacent in $P_2 \cap G_3$. In particular, there are exactly two uv-joins, and similarly exactly two w_1w_2-joins. But then contracting the edges uv and w_1w_2 gives a smaller counterexample. (Here is where the number 5/2 appears.) This proves 8.2. ∎

We can apply 8.2 to the 2-connected case, and prove the following. (The idea of this proof is due to A. Kostochka, and he kindly gave us permission to include it here.) We recall that $\delta(s) = \frac{1}{2}(s + 3 - 4/(s+2))$.

8.3 *Let $t \geq 0$ be odd, $t = 2s - 1$ say, and let $c(t)$ be as in 8.2. Then every 2-connected n-vertex graph with no $K_{2,t}$ minor has at most $\delta(s)n + c(t)$ edges.*

Proof. We proceed by induction on n. The result is easy for $t \leq 3$, so we may assume that $t \geq 5$, and $s \geq 3$. If G is 3-connected, the claim follows from 8.2, so we may assume that G admits a 2-cut $(A_1, A_2, \{r_1, r_2\})$ say. For $i = 1, 2$, let $|A_i| = n_i$, and let there be e_i edges with an end in A_i. For $i = 1, 2$, let G_i be the graph obtained from $G|(A_i \cup \{r_1, r_2\})$ by adding the edge r_1r_2; and choose s_i minimum such that G_i has no r_1r_2-rooted K_{2,s_i} minor. Thus $2 \leq s_i \leq n_i + 1$. We assume for a contradiction that $e_1 + e_2 + 1 > \delta(s)(n_1 + n_2 + 2) + c(t)$.

(1) *For $i = 1, 2$, $e_i \leq \delta(s_i)(n_i + 1) - 2$, and $e_i > \delta(s)n_i$.*

The first claim follows from 7.1 applied to G_i. From the inductive hypothesis applied to the 2-connected graph G_i, we deduce that $e_i \leq \delta(s)(n_i + 2) + c(t) - 1$ for $i = 1, 2$, and since $e_1 + e_2 + 1 > \delta(s)(n_1 + n_2 + 2) + c(t)$, subtracting yields the second claim. This proves (1).

(2) *One of* $s_1, s_2 > s$, *and* $s_1 + s_2 \leq t + 1$.

If $s_1, s_2 \leq s$, then summing the first inequalities of (1) for $i = 1, 2$ yields

$$|E(G)| \leq e_1 + e_2 + 1 \leq \delta(s)(n_1 + n_2 + 2) - 3,$$

a contradiction; so one of $s_1, s_2 > s$, and this proves the first claim. Since for $i = 1, 2$, G_i has an r_1r_2-rooted K_{2,s_i-1} minor, and yet combining these does not give a $K_{2,t}$ minor of G, it follows that $(s_1 - 1) + (s_2 - 1) \leq t - 1$. This proves the second claim, and so proves (2).

In view of (2) we assume henceforth that $s_1 > s$, and therefore $s_2 < t + 1 - s = s$. Since $e_2 \leq (n_2 + 2)(n_2 + 1)/2 - 1$, and (1) implies that $e_2 > \delta(s)n_2$, it follows that

$$\delta(s)n_2 < (n_2 + 2)(n_2 + 1)/2 - 1,$$

that is, $s-4/(s+2) < n_2$, and so $n_2 \geq s$. The inequalities of (1) yield $\delta(s)n_2 < \delta(s_2)(n_2+1)-2$, that is,

$$\delta(s) > (\delta(s) - \delta(s_2))(n_2 + 1) + 2.$$

But $\delta(s) \leq (s + 3)/2$, and $\delta(s) - \delta(s_2) \geq (s - s_2)/2 \geq 1/2$, and $n_2 \geq s$, and we deduce that $(s + 3)/2 > (s + 1)/2 + 2$, a contradiction. This proves 8.3. ∎

This result is best possible except for the constant $c(t)$, since there is a 2-connected n-vertex graph with no $K_{2,t}$ minor with $\delta(s)n - 3$ edges. (To see this, take two copies of the graph defined after the statement of 7.1, with t replaced by s, and identify the roots of the first with those of the second.) We have confined ourself to the case when t is odd because the even case seems to be more difficult.

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