Solution of three problems of Cornuéjols

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Abstract

A graph is *balanced* if it is bipartite and every induced cycle has length divisible by four. In his book [6], Gérard Cornuéjols proposed a number of open questions, offering \$5000 for the solution of any of them. Here we solve three of them, about balanced graphs.

1 Introduction

A graph is said to be *balanced* if it is bipartite, and every induced cycle has length divisible by four. In his excellent book [6], Gérard Cornuéjols proposed eighteen conjectures, and offered $5000 for a proof or counterexample for any of them. Two, concerned with perfect graphs, were settled by the solution of the strong perfect graph conjecture [2]. Now we are happy to report the solution of three more, concerned with balanced graphs; conjectures 9.23, 9.28 and 9.29 of [6]. We give a counterexample to the first two, and a proof of the third.

2 A counterexample to conjectures 9.23 and 9.28 of [6]

Conjecture 9.23 on page 98 of [6] asserts:

2.1 Conjecture *(Conforti, Cornuéjols and Rao [4]) If G is a balanced graph that is not totally unimodular, then G is either a W_{pq} or has a biclique cutset or a 2-join.*

We need to explain these terms. A graph is *totally unimodular* if it admits a bipartition (A, B) such that every square submatrix of the matrix $(m_{ab} : \; a \in A, b \in B)$ has determinant ± 1 or 0, where $m_{ab} = 1$ if a, b are adjacent and 0 otherwise. The graphs W_{pq} are a particular class of balanced graphs that we do not need to define here (they are essentially a special case of what we call crossmatchings below). A *biclique cutset* is a pair of disjoint nonempty sets $A, B \subseteq V(G)$, such that every vertex in A is adjacent to every vertex in B, and $G \setminus (A \cup B)$ is disconnected. A graph G has a 2-*join* if its vertex set can be partitioned into V_1, V_2 in such a way that, for each $i = 1, 2$, there exist disjoint nonempty subsets $A_i, B_i \subseteq V_i$, such that

- every vertex of A_1 is adjacent to every vertex of A_2,
- every vertex of B_1 is adjacent to every vertex of B_2,
- there are no other adjacencies between V_1 and V_2,
- for $i = 1, 2$ V_i contains at least one path from A_i to B_i, and
- for $i = 1, 2$, if $|A_i| = |B_i| = 1$ then the graph induced by V_i is not a chordless path between A_i and B_i.

(Remark: the definition of a 2-join in [6] contains a minor error, and the fifth condition above has been amended to fix this error.)

Two disjoint subsets A, B of the vertex set $V(G)$ are said to be *matched* in G if A, B are stable sets in G and each member of A has a unique neighbour in B and vice versa. Here is a class of balanced graphs. Let $p, q \geq 1$ be integers, and let C be a cycle with vertices

$$a_1, \ldots, a_{4p-3}, b_1, \ldots, b_{4q-3}, c_1, \ldots, c_{4p-3}, d_1, \ldots, d_{4q-3}, a_1$$

in order. Take $p + q$ new vertices $x_1, \ldots, x_p, y_1, \ldots, y_q$, and add edges as follows:

- x_i, y_j are adjacent for all i, j with $1 \leq i \leq p$ and $1 \leq j \leq q$

- for $1 \le i \le p$, x_i and c_{4i-3} are adjacent
- for $1 \le j \le q$, y_j and d_{4j-3} are adjacent
- $\{a_1, a_5, a_9, \dots, a_{4p-3}\}$ and $\{x_1, \dots, x_p\}$ are matched
- $\{b_1, b_5, b_9, \dots, b_{4q-3}\}$ and $\{y_1, \dots, y_q\}$ are matched

and there are no other edges. Let us call such a graph a *crossmatching.* It is easy to check that every crossmatching is balanced.

In particular, let $p = 3, q = 2$, and take a crossmatching such that the pairs

$$a_1x_1, a_9x_2, a_5x_3, b_1y_1, b_5y_2$$

are edges. This is balanced, and does not satisfy 2.1 (we leave it to the reader to check this). The same graph is also a counterexample to conjecture 9.28 of [6] (we do not state this in full, because it is just a strengthening of 2.1, and needs several further definitions).

3 Conjecture 9.29 of [6]

The goal of the remainder of this paper is to prove conjecture 9.29 on page 100 of [6], which asserts the following:

3.1 Conjecture *(Conforti, Cornuéjols, Kapoor and Vušković [3]) Every balanceable bipartite graph that is not regular has a double star cutset.*

We need first to define these terms. A graph is *eulerian* if every vertex has even degree (we do not require it to be connected). If G is a graph and $w : E(G) \to \{-1, 1\}$ is a map, and H is a subgraph of G, we denote $\sum_{e \in E(H)} w(e)$ by $w(H)$. A bipartite graph G is *balanceable* if there is a map $w : E(G) \to \{-1, 1\}$ such that $w(C)$ is a multiple of four for every induced cycle C of G. A bipartite graph G is *regular* if there is a map $w : E(G) \to \{-1, 1\}$ such that $w(H)$ is a multiple of four for every induced eulerian subgraph H of G. (This definition of "regular" is more convenient for us than the definition used in [6]; they are equivalent, because of Camion's theorem [1].) Any such map w is called a *t.u. signing* of G.

A *cutset* in G is a subset $X \subseteq V(G)$ such that $G \setminus X$ has at least two components. (This is not quite the definition from [6], but the difference is not significant.) A *star cutset* in G is a cutset X such that some $u \in X$ is adjacent to all other members of X. Then u is called a *centre* of the star cutset. A *double star cutset* in G is a cutset X such for some edge uv with $u, v \in X$, every member of X is adjacent to one of u, v; and then uv is called a *centre* of the double star cutset.

A remark: the definition of "double star cutset" above is the standard definition used in many of Cornuéjols' papers, such as [3]. However, in [6] the definition is different; he requires in addition that the subgraph induced on the cutset is a tree. This is presumably a mistake in [6], because with this definition it is easy to give counterexamples to 3.1; for instance, take the graph with ten vertices

$$a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, d_1, d_2,$$

and adjacency as follows: a_ib_i is an edge for $i = 1, 2, 3$; $\{b_1, b_2, b_3\}$ is complete to $\{c_1, c_2\}$; $\{c_1, c_2\}$ is complete to $\{d_1, d_2\}$; and $\{d_1, d_2\}$ is complete to $\{a_1, a_2, a_3\}$. Then this graph is a counterexample to 3.1 using the definition of “double star cutset” from [6]. Henceforth then, we use the standard definition.

If $v \in V(G)$ we denote the union of $\{v\}$ and the set of neighbours of v by $N[v]$; and if uv is an edge of G then $N[uv]$ denotes $N[u] \cup N[v]$. If $v, w \in V(G)$ are distinct, we say that v *dominates* w if every vertex adjacent to w is also adjacent to v (and hence v, w are nonadjacent). We observe:

3.2 *Let G be a bipartite graph with $|V(G)| \geq 5$ and $E(G) \neq \emptyset$ and with no double star cutset. Then*

- *G is connected*
- *G has no star cutset*
- *no vertex of G dominates another*
- *for every edge uv, the subgraph induced on $V(G) \setminus N[uv]$ is nonnull and connected, and every vertex in $N[uv] \setminus \{u, v\}$ has a neighbour in $V(G) \setminus N[uv]$.*

Proof. Suppose that G is not connected, and let uv be an edge, chosen from the component C of G that has most vertices. Then either G has at least three components, or $|V(C)| \geq 3$, and in either case $\{u, v\}$ is a double star cutset, a contradiction. Thus G is connected.

Suppose that X is a star cutset with centre u. If there exists $v \in X \setminus \{u\}$, then X is also a double star cutset with centre uv, a contradiction; so $X = \{u\}$. Let A_1 be a component of $G \setminus X$, and let $A_2 = V(G) \setminus (A_1 \cup X)$; thus $A_2 \neq \emptyset$. Since G is connected, u has neighbours $v_i \in A_i$ for $i = 1, 2$. Since $X \cup \{v_i\}$ is not a double star cutset, it follows that $A_i = \{v_i\}$ for $i = 1, 2$, and so $|V(G)| = 3$, a contradiction. Thus G has no star cutset.

Now suppose that v dominates w. Let X be the union of $\{v\}$ and the set of all neighbours of w. Since X is not a star cutset with centre v, it follows that $X \cup \{w\} = V(G)$. Let u be adjacent to w. Since $\{u, v, w\}$ is not a star cutset with centre u, we deduce that $|V(G)| \leq 4$, a contradiction. Thus no vertex dominates another.

Finally, let uv be an edge. Suppose first that $N[uv] = V(G)$. Then u dominates every neighbour of v different from u, so by what we just proved, v has degree one, and similarly u has degree one, a contradiction. Thus $N[uv] \neq V(G)$. Since $N[uv]$ is not a double star cutset, it follows that the subgraph induced on $V(G) \setminus N[uv]$ is connected. Now let $w \in N[uv]$ with $w \neq u, v$; say w is adjacent to u. Since v does not dominate w, it follows that w has a neighbour in $V(G) \setminus N[uv]$. This completes the proof of 3.2. ∎

4 Operations preserving regularity

In this section we discuss some lemmas stating that if we piece two regular graphs together in prescribed ways, then the graph we produce is also regular.

If $X \subseteq V(G)$, we denote the subgraph induced on X by $G|X$. Let G be a connected bipartite graph that admits a 2-join, and let V_i, A_i, B_i $(i = 1, 2)$ be as in the definition of a 2-join. Let G_1 be the graph obtained from $G|V_1$ by adding a path p_1-p_2-$\cdots$-p_k of new vertices, where p_1 is adjacent to every vertex in A_1, p_k is adjacent to every vertex in B_1, and there are no other edges between V_1 and $\{p_1, \ldots, p_k\}$, and $k \geq 3$, and k is chosen so that G_1 is bipartite. Define G_2 similarly, adding a path to $G|V_2$. We call G_1, G_2 a pair of *blocks* of the 2-join. We need first:

4.1 *Let G be a connected bipartite graph that admits a 2-join, and let V_i, A_i, B_i $(i = 1, 2)$ be as before, and let G_1, G_2 be a pair of blocks of the 2-join. If G_1, G_2 are both regular then so is G.*

This result is well-known, and related to the fact that 3-sums in matroid theory preserve matroid regularity. Another closely related result is that if G_1, G_2 are totally unimodular then so is G, and that is proved in lemma 2.3 of [4]. The proof given there can easily be adapted to prove 4.1, and we omit the details.

Let G be a bipartite graph. A partition (V_1, V_2) of $V(G)$ is a *6-join* if $|V_1|, |V_2| \geq 4$ and there exist disjoint nonempty subsets $A_1, A_3, A_5 \subseteq V_1$ and $A_2, A_4, A_6 \subseteq V_2$, satisfying:

- for $i = 1, \dots, 6$ A_i is complete to A_{i+1}, where A_7 means A_1
- there are no other edges between V_1 and V_2.

In this case, let G_1 be obtained from $G|V_1$ by adding three new vertices b_2, b_4, b_6, where for $i = 2, 4, 6$, b_i is adjacent to every vertex of $A_{i-1} \cup A_{i+1}$ (reading subscripts modulo 6), and there are no other new edges. Similarly, define G_2 by adding four vertices b_1, b_3, b_5 to $G|V_2$, where for $i = 1, 3, 5$, b_i is adjacent to every vertex of $A_{i-1} \cup A_{i+1}$, and there are no other new edges. We call G_1, G_2 a pair of *blocks* of the 6-join. We need a result analogous to 4.1 for 6-joins, but first a lemma:

4.2 *Let $a_1, \dots, a_6$ be integers such that a_1, a_3, a_5 are all even or all odd, and a_2, a_4, a_6 are all even or all odd. Then*

$$a_1a_2 - a_2a_3 + a_3a_4 - a_4a_5 + a_5a_6 - a_6a_1$$

is a multiple of four.

Proof. Changing the value of a_1 by two does not change the value of the expression modulo four, since a_1 multiplies $a_2 - a_6$, which is even. Thus we may assume that $a_1 \in \{0, 1\}$, and similarly $a_2, \dots, a_6 \in \{0, 1\}$. Since a_1, a_3, a_5 are all even or all odd, they are all equal, and so are a_2, a_4, a_6; and hence the expression is zero. ∎

The analogue of 4.1 is:

4.3 *Let (V_1, V_2) be a 6-join in a connected bipartite graph G, and let G_1, G_2 be a pair of blocks of this 6-join. If G_1, G_2 are both regular then so is G.*

Proof. Let $A_1, \dots, A_6$ be as in the definition of a 6-join, and let $b_1, \dots, b_6$ be the new vertices of G_1, G_2 as above. (Throughout this proof we read subscripts modulo 6.) Let $a_i \in A_i$ for $i = 1, 3, 5$. Let w_1 be a t.u. signing of G_1. If $Y \subseteq V(G_1)$ and we replace $w_1(e)$ by $-w_1(e)$ for every edge e of G_1 with exactly one end in Y, we obtain another t.u. signing of G_1; and we may therefore choose w_1 such that:

- for $j = 2, 4, 6$, $w_1(e) = 1$ for every edge e incident with b_j and a vertex of A_{j-1}, and
- for $j = 2, 4$, $w_1(e) = -1$ for the edge $e = b_ja_{j+1}$.

Since the subgraph of G_1 induced on $\{a_1, b_2, a_3, b_4, a_5, b_6\}$ is eulerian and w_1 is a t.u. signing, it follows that also $w_1(e) = -1$ for the edge $e = b_6a_1$. Also, for each choice of $a'_1 \in A_1$, since the subgraph induced on $\{a'_1, b_2, a_3, b_4, a_5, b_6\}$ is eulerian, it follows that $w_1(e) = -1$ for the edge $e = b_6a'_1$. Similarly for $j = 2, 4$ and for each $a'_{j+1} \in A_{j+1}$, it follows that $w_1(e) = -1$ for the edge $e = b_ja'_{j+1}$. Thus in summary we have:

- for $j = 2,4,6$, $w_1(e) = 1$ for every edge e incident with b_j and a vertex of A_{j-1}, and
- for $j = 2,4,6$, $w_1(e) = -1$ for every edge e incident with b_j and a vertex of A_{j+1}.

Similarly we may choose a t.u. signing w_2 of G_2 such that:

- for $j = 1,3,5$, $w_2(e) = 1$ for every edge e incident with b_j and a vertex of A_{j+1}, and
- for $j = 1,3,5$, $w_2(e) = -1$ for every edge e incident with b_j and a vertex of A_{j-1}.

For each edge e of G, either $e \in E(G|V_i)$ for some $i \in \{1,2\}$, or $e = uv$ where $u \in A_i$ and $v \in A_{i+1}$ for some $i \in \{1,\dots,6\}$. In the first case let $w(e) = w_i(e)$, and in the second case let $w(e) = 1$ if i is odd and -1 if i is even. We claim that w is a t.u. signing of G. For let $X \subseteq V(G)$ such that $G|X$ is eulerian.

Let $x_i = |X \cap A_i|$ for $1 \le i \le 6$. We say that for $1 \le i \le 6$, x_i is *exceptional* if $x_i + x_{i+2}, x_i + x_{i-2}$ are both odd (and therefore $x_{i+2} + x_{i-2}$ is even). Thus at most one of x_1, x_3, x_5 is exceptional, and at most one of x_2, x_4, x_6; and if there is one of each, say x_i and x_j, we claim that $j \ne i+1, i-1$. To see the last assertion, suppose that x_1, x_2 are exceptional, say. Thus $x_1 + x_3, x_2 + x_4$ are odd, and $x_4 + x_6$ is even; and so

$$(x_1 + x_3)(x_2 + x_4) + (x_1 + x_5)(x_4 + x_6)$$

is odd. But this equals

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_1$$

modulo 2, and so the total number of edges between $V_1 \cap X$ and $V_2 \cap X$ is odd, contradicting that $G|X$ is eulerian. This proves our assertion. Consequently, from the symmetry we may assume that x_1, x_5, x_2, x_4 are not exceptional, that is, $x_1 + x_5$ and $x_2 + x_4$ are both even.

Let $X_1 \subseteq V(G_1)$ be defined as follows. Let $X_1 \cap V_1 = X \cap V_1$, let $b_2, b_4 \notin X_1$, and let $b_6 \in X_1$ if and only if x_6 is exceptional. Similarly, let $X_2 \subseteq V(G_2)$ where $X_2 \cap V_2 = X \cap V_2$, $b_1, b_5 \notin X_2$, and $b_3 \in X_2$ if and only if x_3 is exceptional.

(1) $G_1|X_1$ *is eulerian.*

For let $v \in X_1$; we must check that its degree d_1 say in $G_1|X_1$ is even. If $v = b_6$ then its degree is $x_1 + x_5$, which is even, so we may assume that $v \in X$; let its degree in $G|X$ be d. Thus d is even. If $v \in V_1 \setminus (A_1 \cup A_3 \cup A_5)$ then $d_1 = d$ and therefore is even; if $v \in A_3$ then $d_1 = d - (x_2 + x_4)$, and therefore is even; if $v \in A_1$ then $d_1 = d - (x_2 + x_6)$ if $b_6 \notin X_1$ (that is, if $x_2 + x_6$ is even), and $d_1 = d - (x_2 + x_6) + 1$ if $b_6 \in X_1$ (that is, if $x_2 + x_6$ is odd), and in either case d_1 is even; and similarly d_1 is even if $v \in A_5$. This proves (1).

Define $X_2 \subseteq V(G_2)$ similarly. Then $w_1(G_1|X_1)$ and $w_2(G_2|X_2)$ are multiples of four, so let us examine $w(G|X) - w_1(G_1|X_1) - w_2(G_2|X_2)$. First,

$$w(G|X) = w(G|(X \cap V_1)) + w(G|(X \cap V_2)) + \sum_{i=1,3,5} x_i(x_{i+1} - x_{i-1}).$$

Let $y_6 = 1$ if $b_6 \in X_1$, and $y_6 = 0$ otherwise, and define y_3 similarly; then

$$w_1(G_1|X_1) = w(G|(X \cap V_1)) + y_6(x_5 - x_1)$$

and

$$w_2(G_2|X_2) = w(G|(X \cap V_2)) + y_3(x_4 - x_2).$$

Thus $w(G|X) - w_1(G_1|X_1) - w_2(G_2|X_2) = R$, where by definition

$$R = x_1x_2 - x_2(x_3 - y_3) + (x_3 - y_3)x_4 - x_4x_5 + x_5(x_6 - y_6) - (x_6 - y_6)x_1.$$

But since x_1, x_5 are not exceptional, the definition of y_3 ensures that $x_1, x_3 - y_3, x_5$ are all odd or all even; and similarly $x_2, x_4, x_6 - y_6$ are all odd or all even. By 4.2, it follows that R is a multiple of four. Consequently $w(G|X)$ is a multiple of four, and so w is a t.u. signing of G. This proves 4.3. ▮

Third, we need the following. Let us say distinct vertices of G are *twins* if they have the same neighbour sets (and consequently are nonadjacent to each other).

4.4 *Let u, v be twins in G, and suppose that $G \setminus \{v\}$ is regular. Then G is regular.*

Proof. Let w be a t.u. signing of $G \setminus \{v\}$, and extend the domain of w to $E(G)$ by defining $w(vx) = w(ux)$ for each edge vx of G. We claim that w is a t.u. signing of G. For let $X \subseteq V(G)$ such that $G|X$ is eulerian. If $v \notin X$ then $w(G|X)$ is a multiple of four since w is a t.u. signing of $G \setminus \{v\}$, and if $u \notin X$ the same conclusion follows from the symmetry between u, v. Thus we may assume that $u, v \in X$. Let $X' = X \setminus \{u, v\}$; then $G|X'$ is eulerian, and so $w(G|X')$ is a multiple of four. But $w(G|X) = w(G|X') + 2z$, where z is the sum of $w(ux)$ over all edges ux with $x \in X'$; and since $G|X$ is eulerian, it follows that z is even. Hence $w(G|X)$ is a multiple of four, and so w is a t.u. signing of G, and therefore G is regular. This proves 4.4. ▮

5 Some 6-join lemmas

A 6-join (V_1, V_2) in a bipartite graph G is said to be *skeletal* if $|V_2| = 7$, and V_2 can be numbered as $\{a_2, a_4, a_6, c_2, c_4, c_6, c_8\}$ such that

- c_8 has degree three in G, with neighbours c_2, c_4, c_6
- for $i = 2, 4, 6$, c_i has degree two in G, with neighbours a_i, c_8
- there are disjoint nonempty subsets $A_1, A_3, A_5 \subseteq V_1$ such that for $i = 2, 4, 6$, a_i is complete to $A_{i-1} \cup A_{i+1}$ (where A_7 means A_1) and there are no other edges between V_1 and V_2.

An induced subgraph of G that is a cycle is called a *hole* in G, and a hole of length k is a *k-hole*. If G is a balanceable bipartite graph, an induced subgraph H is said to be an *irregularity* in G if H is not regular, and every induced subgraph of G with fewer vertices than H is regular. We need:

5.1 *Let G be balanceable, and let (V_1, V_2) be a skeletal 6-join. Let $V_2 = \{a_2, a_4, a_6, c_2, c_4, c_6, c_8\}$ as in the definition of "skeletal". Let H be an irregularity in G; then $c_2, c_4, c_6, c_8 \notin V(H)$.*

Proof. Let $A_1, \ldots, A_6$ be as in the definition of 6-join, where $A_i = \{a_i\}$ for $i = 2, 4, 6$. Let $w : E(G) \to \{-1, 1\}$ such that $w(C)$ is a multiple of four for every induced cycle C of G. As usual we may assume that $w(a_ia_{i+1}) = 1$ for $i = 1, 3, 5$ and all $a_i \in A_i$, and $w(a_ia_{i-1}) = -1$ for $i = 1, 3, 5$ and all $a_i \in A_i$, where A_0 means A_6. Now w induces a t.u. signing of J for every regular induced subgraph J of G. Since H is an irregularity in G, it follows that H is eulerian; and since w induces a t.u. signing of every proper induced subgraph of H and not of H itself, we deduce that $w(H)$ is not a multiple of four. Suppose that one of $c_2, c_4, c_6, c_8 \in V(H)$. Hence we may assume that $a_2, c_2, c_8, c_4, a_4 \in V(H)$ and $c_6 \notin V(H)$. For $i = 1, 3, 5$, let $x_i = |V(H) \cap A_i|$.

Let $Y = \{a_2, c_2, c_8, c_4, a_4\}$; then $w(G|Y)$ is a multiple of four, since $w(C)$ is a multiple of four where C is the hole c_8-c_2-a_2-a_3-a_4-c_4-c_8 for some $a_3 \in A_3$. Suppose first that $a_6 \in V(H)$, and let $X = V(H) \cap V_1$. Then $G|X$ is eulerian, and therefore regular from the minimality of $|V(H)|$, and so $w(G|X)$ is a multiple of four. But $w(H) = w(G|X) + w(G|Y)$, and so $w(H)$ is a multiple of four, a contradiction. Thus $a_6 \notin V(H)$. Let $X = (V(H) \cap V_1) \cup \{a_6\}$. Then again $G|X$ is eulerian, and has fewer vertices than H, and so $w(G|X)$ is a multiple of four. But

$$w(H) = w(G|X) + w(G|Y) - 2x_5 + 2x_1,$$

and $x_5 - x_1$ is even since a_6 has even degree in $G|X$. It follows again that $w(H)$ is a multiple of four, a contradiction. This proves 5.1. ∎

A 6-join (V_1, V_2) in a bipartite graph G is said to be *internal* if $|V_1|, |V_2| \geq 8$. We need several results saying that balanceable graphs containing certain induced subgraphs admit either double star cutsets or internal 6-joins.

If $X, Y \subseteq V(G)$, we say that X is *anticomplete* to Y if $X \cap Y = \emptyset$ and there is no edge xy with $x \in X$ and $y \in Y$. The proof of theorem 6.3 of [3] also proves the following:

5.2 *Let G be balanceable, and let a_1-b_2-a_3-b_1-a_2-b_3-a_1 be a 6-hole C in G. Suppose that there are subsets $A, B \subseteq V(G)$ with the following properties:*

- *$A, B, V(C)$ are pairwise disjoint, and $G|A, G|B$ are connected;*
- *a_1, a_2, a_3 have neighbours in A, and b_1, b_2, b_3 do not;*
- *b_1, b_2, b_3 have neighbours in B, and a_1, a_2, a_3 do not; and*
- *A is anticomplete to B.*

Then either G admits a double star cutset, or G admits a 6-join (V_1, V_2) with $A \cup \{a_1, a_2, a_3\} \subseteq V_1$ and $B \cup \{b_1, b_2, b_3\} \subseteq V_2$.

6 Big dominoes

A triple (ab, C_1, C_2) is a *domino* in G if C_1, C_2 are holes in G, and ab is an edge, and $V(C_1) \cap V(C_2) = \{a, b\}$, and $V(C_1) \setminus \{a, b\}$ is anticomplete to $V(C_1) \setminus \{a, b\}$. An *odd theta* is a graph consisting of two nonadjacent vertices u, v and three odd length paths between u, v, such that the interiors of these three paths are pairwise disjoint and pairwise anticomplete. An *odd wheel* is a graph consisting of a cycle C and another vertex $v \notin V(C)$, such that v has an odd number, at least three, of neighbours in $V(C)$. We need the following easy and well-known lemma (we omit the proof).

6.1 *If G is a balanceable bipartite graph, then no induced subgraph of G is an odd theta or an odd wheel.*

Let us say two vertices u, v in the same component of a bipartite graph G have the *same biparity* if every path between them has even length, and otherwise they have *opposite biparity* (and therefore every path between them has odd length). We begin with a lemma.

6.2 *Let (a_0b_0, C_1, C_2) be a domino in a balanceable graph G, such that C_1, C_2 both have length at least six. For $i = 1, 2$, let $P_i = C_i \setminus \{a_0, b_0\}$; then P_i is a chordless path of length at least three with ends a_i, b_i say, where a_i is adjacent to b_0 and b_i to a_0. Suppose that G does not admit a double star cutset, and does not admit a 6-join (V_1, V_2) such that $V(C_i) \setminus \{a_0, b_0\} \subseteq V_i$ for $i = 1, 2$, and V_1, V_2 each contain exactly one of a_0, b_0. Let q_1-$\cdots$-q_n be a chordless path such that*

- *for $1 \leq i \leq n$, q_i has a neighbour in the interior of P_1 if and only if $i = 1$, and q_i has a neighbour in the interior of P_2 if and only if $i = n$, and*
- *$q_1, \ldots, q_n$ are all nonadjacent to both a_0, b_0.*

Then either

(a) a_1 is adjacent to one of $q_2, \ldots, q_n$, and a_2 is adjacent to one of $q_1, \ldots, q_{n-1}$, and b_1 is nonadjacent to $q_2, \ldots, q_n$, and b_2 is nonadjacent to $q_1, \ldots, q_{n-1}$, or

(b) b_1 is adjacent to one of $q_2, \ldots, q_n$, and b_2 is adjacent to one of $q_1, \ldots, q_{n-1}$, and a_1 is nonadjacent to $q_2, \ldots, q_n$, and a_2 is nonadjacent to $q_1, \ldots, q_{n-1}$.

Moreover, if (a) holds then either

- *q_1 is adjacent to both a_1, a_2, and a_2 is nonadjacent to $q_2, \ldots, q_{n-1}$, or*
- *q_n is adjacent to both a_1, a_2, and a_1 is nonadjacent to $q_2, \ldots, q_{n-1}$.*

An analogous statement holds if (b) is true.

Proof. Let us say a_1 or b_1 is *active* if it is adjacent to one of $q_2, \ldots, q_n$, and a_2 or b_2 is active if it is adjacent to one of $q_1, \ldots, q_{n-1}$.

(1) *At least one of a_1, b_1, a_2, b_2 is active, and so $n \geq 2$.*

For suppose not. We may assume that q_1, a_0 have opposite biparity. If q_1 has more than one neighbour in P_1, there are three paths between q_1, a_0 forming an odd theta, namely two with interior in $V(C_1)$ and the third with interior in $\{q_2, \ldots, q_n\} \cup (V(P_2) \setminus \{a_2\})$, a contradiction. Thus q_1 has a unique neighbour, p_1 say, in P_1. Since q_1 has a neighbour in the interior of P_1 it follows that $p_1 \neq a_1$; and there are three paths between p_1, b_0 forming an odd theta, namely two with interior in $V(C_1)$ and the third with interior in $\{q_2, \ldots, q_n\} \cup (V(P_2) \setminus \{b_2\})$, a contradiction. This proves (1).

(2) *Not both a_1, b_1 are active.*

For if they are, there are three paths between a_1, b_1 forming an odd theta, namely P_1, a_1-b_0-a_0-b_1 and a path with interior in $\{q_2, \ldots, q_n\}$, a contradiction.

(3) *Not both* b_1, b_2 *have neighbours in* $\{q_2, \dots, q_{n-1}\}$.

For if they do, let R be a chordless path between b_1, b_2 with interior in $\{q_2, \dots, q_{n-1}\}$. Then a_1, a_2 are both anticomplete to $V(R)$, by (2), and so

$$b_1\text{-}P_1\text{-}a_1\text{-}b_0\text{-}a_2\text{-}P_2\text{-}b_2\text{-}R\text{-}b_1$$

is a hole and a_0 has three neighbours in it, a contradiction. This proves (3).

In view of (1) and (2) we may assume that b_1 is active and a_1 is not. Let $j \in \{2, \dots, n\}$ be maximum such that q_j, b_1 are adjacent.

(4) *One of* a_2, b_2 *is adjacent to one of* $q_1, \dots, q_{j-1}$.

For suppose not. Let R be a chordless path between b_0 and q_j with interior in $(V(P_1) \setminus \{b_1\}) \cup \{q_1, \dots, q_{j-1}\}$. Let S, T be chordless paths between q_j, b_0 with interior in $\{q_{j+1}, \dots, q_n\} \cup (V(P_2) \setminus \{b_2\})$ and in $\{q_{j+1}, \dots, q_n, a_0\} \cup (V(P_2) \setminus \{a_2\})$ respectively. Then b_0-R-q_j-S-b_0 and b_0-R-q_j-T-b_0 are holes, and b_1 has one more neighbour in the second hole than in the first; and so by 6.1, b_1 has exactly one neighbour in R, namely q_j. But then there are three paths between q_j and b_0 that form an odd theta, namely q_j-R-b_0, q_j-S-b_0 and q_j-b_1-a_0-b_0, a contradiction. This proves (4).

(5) a_2 *is not active.*

For suppose that a_2 is active. Then b_2 is not, by (2); and so by (4), a_2 is adjacent to one of $q_1, \dots, q_{j-1}$. Let $i \in \{1, \dots, j-1\}$ be minimum such that a_2, q_i are adjacent. If $j > i+1$, there are three paths between b_1, a_2 forming an odd theta, namely one with interior in $V(P_1) \cup \{q_1, \dots, q_i\}$, one with interior in $\{q_j, \dots, q_n\} \cup V(P_2)$, and b_1-a_0-b_0-a_2. Thus $j = i+1$; but then the 6-hole a_0-b_1-q_j-q_i-a_2-b_0-a_0, and the two subsets $(V(P_2) \setminus \{a_2\}) \cup \{q_{j+1}, \dots, q_n\}$ and $(V(P_1) \setminus \{b_1\}) \cup \{q_1, \dots, q_{i-1}\}$ satisfy the hypotheses of 5.2, and consequently there is either a double star cutset or a 6-join that violates the hypothesis of the theorem. This proves (5).

From (4) and (5), it follows that b_2 is adjacent to one of $q_1, \dots, q_{j-1}$, and so there is symmetry between b_1 and b_2. By (3) one of b_1, b_2 is nonadjacent to $q_2, \dots, q_{n-1}$, so by exchanging C_1, C_2 if necessary, we may assume that b_2 is nonadjacent to $q_2, \dots, q_{n-1}$. Consequently b_2 is adjacent to q_1. (Note that possibly b_2 is adjacent to q_n, and possibly $j = n$.)

(6) b_1 *is adjacent to* q_1.

For suppose not. If q_1 has at least two neighbours in P_1, there are three paths between q_1 and b_1 forming an odd theta, namely two with interior in $V(C_1)$ and one with interior in $\{q_2, \dots, q_j\}$, a contradiction. If q_1 has a unique neighbour p_1 in P_1, then p_1, a_0 are nonadjacent and there are three paths between p_1, a_0 forming an odd theta, namely two paths of C_1 and a path with interior in $\{q_1, \dots, q_j\}$, again a contradiction. This proves (6), and completes the proof of 6.2. ∎

The lemma is used for the following.

6.3 *Let (a_0b_0, C_1, C_2) be a domino in a balanceable graph G, such that C_1, C_2 both have length at least six. Then either G admits a double star cutset, or G admits a 6-join (V_1, V_2) such that $V(C_i) \setminus \{a_0, b_0\} \subseteq V_i$ for $i = 1, 2$, and V_1, V_2 each contain exactly one of a_0, b_0.*

Proof.

For $i = 1, 2$, let $P_i = C_i \setminus \{a_0, b_0\}$; then P_i is a chordless path of length at least three with ends a_i, b_i say, where a_i is adjacent to b_0 and b_i to a_0. We assume that G does not admit a double star cutset and does not admit a 6-join satisfying the theorem. Hence there is a chordless path $q_1\text{-}\cdots\text{-}q_n$ as in 6.2, and again from 6.2 we may assume that q_1 is adjacent to b_1, b_2, and b_1 is adjacent to one of $q_2, \ldots, q_n$, and b_2 is nonadjacent to $q_2, \ldots, q_{n-1}$, and a_1 is nonadjacent to $q_2, \ldots, q_n$, and a_2 is nonadjacent to $q_1, \ldots, q_{n-1}$.

Since b_1 is adjacent to q_1 and to one of $q_2, \ldots, q_n$, it follows that $n \geq 3$. Let p_2 be the neighbour of b_2 in P_2. Since G does not admit a double star cutset, there is a chordless path R between q_2 and some vertex r such that r has a neighbour in $V(C_1) \cup V(C_2) \setminus \{a_0, b_1, b_2, p_2\}$, and a_0, b_2 are both nonadjacent to every vertex of R. By choosing R minimal, it follows that r is the only vertex of R with a neighbour in $V(C_1) \cup V(C_2) \setminus \{a_0, b_1, b_2, p_2\}$. (However, b_1, p_2 may have neighbours in $V(R) \setminus \{r\}$.)

(1) *r is adjacent to b_0.*

For suppose not. Let $\mathbf{S_1}$ be the statement that r has a neighbour in $V(P_1) \setminus \{b_1\}$, and $\mathbf{S_2}$ the statement that some vertex of R has a neighbour in $V(P_2) \setminus \{b_2\}$ (in other words, either r has a neighbour in $V(P_2) \setminus \{b_2\}$ or some vertex of R is adjacent to p_2). Thus at least one of $\mathbf{S_1}, \mathbf{S_2}$ holds. We claim that if $\mathbf{S_1}$ holds then r has a neighbour in $V(P_1) \setminus \{a_1, b_1\}$. For suppose that a_1 is the unique neighbour of r in $V(P_1) \setminus \{b_1\}$. Then there are three paths between b_1, a_1 that form an odd theta, namely $b_1\text{-}a_0\text{-}b_0\text{-}a_1$, a path with interior in $\{q_2, \ldots, q_n\} \cup V(R)$, and P_1, a contradiction.

We claim also that if $\mathbf{S_2}$ holds then some vertex of R has a neighbour in $V(P_2) \setminus \{a_2, b_2\}$. For suppose that a_2 is the unique neighbour of r in $V(P_2) \setminus \{b_2\}$, and there are no other edges between $V(R)$ and $V(P_2) \setminus \{b_2\}$. Then there are three paths between b_2, a_2 that form an odd theta, namely $b_2\text{-}a_0\text{-}b_0\text{-}a_2$, a path with interior in $\{q_1\} \cup V(R)$, and P_2, a contradiction.

Now suppose that both $\mathbf{S_1}$ and $\mathbf{S_2}$ hold. Then there is a subpath T of R that satisfies the initial hypotheses for the path $q_1\text{-}\cdots\text{-}q_n$. Moreover, no vertex of T is adjacent to both b_1, b_2, and no vertex of $V(T) \setminus \{r\}$ is adjacent to a_1 or to a_2, contrary to 6.2. This proves that not both $\mathbf{S_1}, \mathbf{S_2}$ hold.

Next suppose that $\mathbf{S_1}$ holds, and hence $\mathbf{S_2}$ is false. Then $V(R) \cup \{q_2, \ldots, q_n\}$ includes the vertex set of a minimal path T between r and some vertex t that has a neighbour in $V(P_2) \setminus \{a_2, b_2\}$. But a_0, b_0 have no neighbours in this path, and a_2, b_2 have no neighbours in this path different from t (since b_2 is nonadjacent to $q_2, \ldots, q_{n-1}$), contrary to 6.2.

Next suppose that $\mathbf{S_2}$ holds, and so $\mathbf{S_1}$ is false. Since r has a neighbour in

$$V(C_1) \cup V(C_2) \setminus \{a_0, b_1, b_2, p_2\},$$

it follows that r has a neighbour in $V(P_2) \setminus \{b_2\}$. Let T be a chordless path between q_1 and some vertex t that has a neighbour in $V(P_2) \setminus \{a_2, b_2\}$, with $V(T) \subseteq V(R) \cup \{q_1\}$, and choose T minimal. Then no vertex of T is adjacent to a_0 or to b_0, and a_2, b_2 both have no neighbours in $V(T) \setminus \{t\}$, contrary to 6.2. This proves (1).

Let T be a chordless path between q_1 and r with interior in $V(R)$. If r has no neighbour in $V(P_1)$, then there are three paths between q_1, b_0 forming an odd theta, namely q_1-T-r-b_0, a path with interior in $V(P_1)$, and q_1-b_2-a_0-b_0, a contradiction. Thus r has a neighbour in $V(P_1)$. If r, b_1 are nonadjacent, then there are three paths joining r, b_1 forming an odd theta, namely r-b_0-a_0-b_1, a path with interior in $V(P_1) \setminus \{a_1\}$, and a path with interior in $V(R) \cup \{q_2, \ldots, q_n\}$, a contradiction. Thus r, b_1 are adjacent. Let b_1 have k neighbours in T; thus $k \geq 2$. Since we can complete T to a hole via a subpath of P_1 that contains no neighbour of b_1, it follows that k is even. But we can also complete T to a hole via r-b_0-a_0-b_2-q_1, and in this hole b_1 has $k+1$ neighbours, contrary to 6.1. This completes the proof of 6.3. ∎

This has the following useful corollary. Let us say a domino (ab, C_1, C_2) is *big* if for $i = 1, 2$, C_i has length at least six, and if C_i has length six then both the vertices of $C_i \setminus \{a, b\}$ that are adjacent to a or b have degree at least three in G.

6.4 *Every balanceable graph that contains a big domino admits either a double star cutset or an internal* 6*-join.*

Proof. Let (a_1a_2, C_1, C_2) be a big domino in a balanceable graph G. We may assume that G does not admit a double star cutset. By 6.3, G admits a 6-join (V_1, V_2) such that $V(C_i) \setminus \{a_1, a_2\} \subseteq V_i$ for $i = 1, 2$, and V_1, V_2 each contain exactly one of a_1, a_2. Let $A_1, \ldots, A_6$ be as in the definition of 6-join. We may assume that $a_1 \in A_1$ and $a_2 \in A_2$, and we suppose for a contradiction that $|V_1| \leq 7$. Let a_6 be the neighbour of a_1 in C_2 different from a_2, and let a_3 be the neighbour of a_2 in C_1 different from a_1. Thus $a_6 \in A_2 \cup A_6$, since $a_6 \in V_2$ and a_6 is adjacent to a_1, and similarly $a_3 \in A_1 \cup A_3$. Since a_3, a_6 are nonadjacent, it follows that $a_3 \in A_3$ and $a_6 \in A_6$. Since a_6 has no neighbour in $V(C_1)$ except a_1, it follows that $V(C_1) \cap (A_1 \cup A_5) = \{a_1\}$. Also, $V(C_1) \cap A_3 = \{a_3\}$ since a_1, a_3 are the only neighbours of a_2 in $V(C_1)$. Consequently all vertices of C_1 except three belong to A_0, where $A_0 = V_1 \setminus (A_1 \cup A_3 \cup A_5)$. Since A_1, A_3, A_5 are nonempty, it follows that $|V_1| \geq |V(C_1)|$. But $|V_1| \leq 7$, and $|V(C_1)|$ is even, and so C_1 is a 6-hole. Let the vertices of C_1 be a_1-a_2-a_3-c_4-c_5-c_6-a_1 in order. Since G is bipartite, c_5 has no neighbour in $A_1 \cup A_3 \cup A_5$.

(1) *If $a_5 \in A_5$, then a_5 is adjacent to both or neither of c_4, c_6.*

For suppose that a_5 is adjacent to c_6 and not to c_4 say. Let $a_4 \in A_4$. Then the paths c_6-a_1-a_2-a_3, c_6-a_5-a_4-a_3 and c_6-c_5-c_4-a_3 form an odd theta, contrary to 6.1. This proves (1).

Suppose first that $|A_0| = 3$, and so $A_0 = \{c_4, c_5, c_6\}$. Since we may assume that $A_1 \cup A_3 \cup \{a_2\}$ is not a double star cutset, one of c_4, c_5, c_6 (and therefore both c_4, c_6, by (1), and not c_5, since c_5, a_5 would have the same biparity) has a neighbour $a_5 \in A_5$. But then a_5 dominates c_5, contrary to 3.2.

Thus $|A_0| > 3$. Consequently $|A_0| = 4$, and $|A_i| = 1$ for $i = 1, 3, 5$. Let $A_5 = \{a_5\}$. Suppose that c_4, c_6 are adjacent to a_5. Since a_5 does not dominate c_5 by 3.2, some neighbour x of c_5 is nonadjacent to c_5 and in particular is different from c_4, c_6. Hence $x \in A_0$. For the same reason, some neighbour of x is nonadjacent to c_4, and so x, a_1 are adjacent; and similarly x, a_3 are adjacent. But then $G|V_1$ is an odd wheel with centre c_5, contrary to 6.1.

Thus not both c_4, c_6 are adjacent to a_5, and hence by (1), c_4, c_6 are both nonadjacent to a_5. But c_6 has degree at least three in G, since (a_1a_2, C_1, C_2) is a big domino; let $x \neq a_1, c_5$ be adjacent to c_6. Thus $x \in A_0$. But none of c_4, c_5, c_6, x are adjacent to a_5 (because c_5, x are nonadjacent to a_5

since they have the same biparity), and therefore $\{a_1, a_2, a_3\}$ is a star cutset, contrary to 3.2. Thus $|V_1| \geq 8$, and similarly $|V_2| \geq 8$. This proves 6.4. ∎

7 Small dominoes

A domino (ab, C_1, C_2) is *small* if C_1, C_2 are both 4-holes. Our next goal is an analogue of 6.4 for small dominoes, but we first need two more lemmas. The first is theorem 6.2 of [3]. (The graph R_{10} consists of a ten-vertex cycle with edges between the five opposite pairs of vertices of the cycle.)

7.1 *Let G be balanceable, with an induced subgraph isomorphic to R_{10}. Then either G is isomorphic to R_{10}, or G admits a double star cutset.*

Let (a_0b_0, C_1, C_2) be a small domino in a bipartite graph G. A *left ear* for (a_0b_0, C_1, C_2) is a hole H_1 such that (a_1b_1, C_1, H_1) is a domino (where $V(C_1) = \{a_0, b_0, a_1, b_1\}$) and $V(C_2) \setminus \{a_0, b_0\}$ is anticomplete to H_1. A *right ear* for (a_0b_0, C_1, C_2) is a left ear for (a_0b_0, C_2, C_1).

7.2 *Let G be balanceable, and let (a_0b_0, C_1, C_2) be a small domino with a left ear and a right ear. Then either G is isomorphic to R_{10}, or G admits a double star cutset or an internal 6-join.*

Proof. We may assume that G admits no double star cutset. For $i = 1, 2$, let C_i have vertices a_0-b_i-a_i-b_0-a_0 in order. Let H_1 be a left ear with vertices a_1-p_1-p_2-$\cdots$-p_m-b_1-a_1 in order, and let H_2 be a right ear with vertices a_2-q_1-q_2-$\cdots$-q_n-b_2-a_2 in order. Thus $\{p_1, \ldots, p_m\}$ is anticomplete to $V(C_2)$, and $\{q_1, \ldots, q_n\}$ is anticomplete to $V(C_1)$. However, the sets $\{p_1, \ldots, p_m\}$ and $\{q_1, \ldots, q_n\}$ may not be anticomplete to each other, and may even not be disjoint. If either $\{p_1, \ldots, p_m\}$ and $\{q_1, \ldots, q_n\}$ are not disjoint, or are disjoint but not anticomplete to each other, let $k(H_1, H_2) = 0$. If $\{p_1, \ldots, p_m\}$ and $\{q_1, \ldots, q_n\}$ are disjoint and anticomplete, define $k(H_1, H_2)$ to be the minimum k such that there is a path r_1-$\cdots$-r_k with r_1 adjacent to one of $p_1, \ldots, p_m$, and r_k adjacent to one of $q_1, \ldots, q_n$, and a_0, b_0 nonadjacent to $r_1, \ldots, r_k$ (such a path exists since G does not admit a double star cutset). We proceed by induction on $k(H_1, H_2)$.

(1) *If $k(H_1, H_2) = 0$ then the theorem holds.*

For suppose first that one of $p_1, \ldots, p_{m-1}$ either equals or is adjacent to one of $q_1, \ldots, q_{n-1}$. Then there is a chordless path R between a_1 and a_2 with interior in $\{p_1, \ldots, p_{m-1}, q_1, \ldots, q_{n-1}\}$, and therefore b_1, b_2 have no neighbours in the interior of R. But then b_0 has three neighbours in the hole a_1-R-a_2-b_2-a_0-b_1-a_1, so G contains an odd wheel, contrary to 6.1. Thus $\{p_1, \ldots, p_{m-1}\}$ is disjoint from and anticomplete to $\{q_1, \ldots, q_{n-1}\}$. Since $p_m \notin \{q_1, \ldots, q_n\}$ since p_m is adjacent to b_1, and similarly $q_n \notin \{p_1, \ldots, p_m\}$, it follows that $\{p_1, \ldots, p_m\}$ is disjoint from $\{q_1, \ldots, q_n\}$. Moreover, for $1 \leq i \leq m$ and $1 \leq j \leq n$, if p_i, q_j are adjacent then either $i = m$ or $j = n$. Similarly either $i = 1$ or $j = 1$. Thus the only pairs p_iq_j that might be adjacent are p_1q_n and p_mq_1. Since $k(H_1, H_2) = 0$ it follows that at least one of these is an edge, so from the symmetry we may assume that p_1, q_n are adjacent. If p_mq_1 is not an edge then

$$p_1\text{-}\cdots\text{-}p_m\text{-}b_1\text{-}a_0\text{-}b_0\text{-}a_2\text{-}q_1\text{-}\cdots\text{-}q_n\text{-}p_1$$

is a hole, containing three neighbours of a_1, contrary to 6.1. Thus $p_m q_1$ is an edge. Since the three paths p_1-$\cdots$-p_m, p_1-a_1-b_1-p_m and p_1-q_n-$\cdots$-q_1-p_m do not form an odd theta, it follows that $m = 2$ and similarly $n = 2$; but then G contains an induced subgraph isomorphic to R_{10} and the theorem holds by 7.1. This proves (1).

Henceforth then we assume that $\{p_1, \ldots, p_m\}$ and $\{q_1, \ldots, q_n\}$ are disjoint and anticomplete, so $k(H_1, H_2) > 0$. Choose a path r_1-$\cdots$-r_k such that r_1 is adjacent to one of $p_1, \ldots, p_m$, and r_k is adjacent to one of $q_1, \ldots, q_n$, and a_0, b_0 are nonadjacent to $r_1, \ldots, r_k$, with $k = k(H_1, H_2)$; then this path is chordless. Hence $\{r_1, \ldots, r_{k-1}\}$ is anticomplete to $\{q_1, \ldots, q_n\}$, and $\{r_2, \ldots, r_k\}$ is anticomplete to $\{p_1, \ldots, p_m\}$. However, there may be edges between $\{a_1, b_1, a_2, b_2\}$ and $\{r_1, \ldots, r_k\}$.

(2) *We may assume that either* $\{a_1, b_2\}$ *is anticomplete to* $\{r_1, \ldots, r_k\}$*, or* $\{a_2, b_1\}$ *is anticomplete to* $\{r_1, \ldots, r_k\}$*.*

For suppose not. If some r_i is adjacent to two of a_1, b_1, a_2, b_2, then $G|\{a_0, b_0, a_1, b_1, a_2, b_2, r_i\}$ is an odd wheel, a contradiction. Thus each r_i is adjacent to at most one of a_1, b_1, a_2, b_2. Choose a chordless path c_2-$\cdots$-c_{t-1} with t minimum such that some $c_1 \in \{a_1, b_2\}$ is adjacent to c_2 and some $c_t \in \{a_2, b_1\}$ is adjacent to c_{t-1}, and $c_2, \ldots, c_{t-1} \in \{r_1, \ldots, r_k\}$. Thus $t \geq 4$. From the minimality of t, none of $c_3, \ldots, c_{t-2}$ is adjacent to any of a_1, b_1, a_2, b_2. From the symmetry we may assume that $c_1 = a_1$. If $c_t = a_2$ then b_0 has three neighbours in the hole c_1-$\cdots$-c_t-b_2-a_0-b_1-c_1, a contradiction. If $c_t = b_1$, let H_3 be the hole c_1-c_2-$\cdots$-c_t-c_1; then $k(H_2, H_3) < k$ and the result follows from the inductive hypothesis. This proves (2).

Thus we may assume that $\{a_2, b_1\}$ is anticomplete to $\{r_1, \ldots, r_k\}$.

(3) *Either* a_1 *is adjacent to one of* $r_2, \ldots, r_k$*, or* b_2 *is adjacent to one of* $r_1, \ldots, r_{k-1}$*, and in particular* $k > 1$*.*

For suppose not. If there is a chordless path P' between a_1 and r_1 with interior in $\{p_1, \ldots, p_{m-1}\}$ and a chordless path Q' between a_2 and r_k with interior in $\{q_1, \ldots, q_{n-1}\}$, then

$$a_0\text{-}b_1\text{-}a_1\text{-}P'\text{-}r_1\text{-}\cdots\text{-}r_k\text{-}Q'\text{-}a_2\text{-}b_2\text{-}a_0$$

is a hole containing three neighbours of b_0, contrary to 6.1. So we may assume that there is no such path P' say, and therefore p_m is the only neighbour of r_1 in $\{a_1, p_1, \ldots, p_m\}$. Let Q' be a chordless path between r_k and b_2 with interior in $\{q_1, \ldots, q_n\}$; then

$$a_0\text{-}b_0\text{-}a_1\text{-}p_1\text{-}\cdots\text{-}p_m\text{-}r_1\text{-}\cdots\text{-}r_k\text{-}Q'\text{-}b_2\text{-}a_0$$

is a hole containing three neighbours of b_1, contrary to 6.1. This proves (3).

From (3) we may assume that a_1 is adjacent to some r_j with $j > 1$. Choose $j \leq k$ maximum with this property.

(4) b_2 *is adjacent to at least one of* $r_1, \ldots, r_{j-1}$*.*

For suppose not. Let Q', Q'' be chordless paths from r_j to b_2 and a_2 respectively with interiors in

$$\{r_{j+1}, \dots, r_k, q_1, \dots, q_n\},$$

and choose h with $1 \le h \le m$ maximum such that r_1, p_h are adjacent. Then

$$a_0\text{-}b_1\text{-}p_m\text{-}\cdots\text{-}p_h\text{-}r_1\text{-}\cdots\text{-}r_j$$

is a chordless path, and it can be completed to a hole via r_j-Q'-b_2-a_0 and via r_j-Q''-a_2-b_0-a_0. The numbers of neighbours of a_1 in these two holes differ by one, and yet b_1, r_j are neighbours of a_1 that belong to both holes, and so G contains an odd wheel, contrary to 6.1. This proves (4).

Choose i with $1 \le i < j$ minimum such that b_2, r_i are adjacent.

(5) $j = i + 1$.

For suppose not; then $i \le j - 2$, and there are three paths between a_1 and b_2 that form an odd theta, namely a path with interior in $\{p_1, \dots, p_m, r_1, \dots, r_i\}$, a path with interior in $\{r_j, \dots, r_k, q_1, \dots, q_n\}$, and the path a_1-b_0-a_0-b_2, contrary to 6.1. This proves (5).

Now a_0-b_2-r_i-r_j-a_1-b_0-a_0 is a 6-hole. Let

$$\begin{aligned} A &= \{b_1, p_1, \dots, p_m, r_1, \dots, r_{i-1}\} \\ B &= \{a_2, q_1, \dots, q_n, r_{j+1}, \dots, r_k\}. \end{aligned}$$

Then $G|A, G|B$ are connected, and the hypotheses of 5.2 are satisfied, and since G admits no double star cutset, it follows that G admits a 6-join (V_1, V_2) with $A \cup \{a_0, a_1, r_i\} \subseteq V_1$ and $B \cup \{b_0, b_2, r_j\} \subseteq V_2$. Suppose that $|V_1| \le 7$. Then

$$|\{b_1, p_1, \dots, p_m, r_1, \dots, r_{i-1}, r_i, a_0, a_1\}| \le 7,$$

and so $m \le 3$; and since m is even it follows that $m = 2$. Also, $i \le 2$. Now a_1, r_i have the same biparity (since a_1, r_{i+1} are adjacent). If r_1 is adjacent to p_2, then it follows that $i = 2$ (since a_1, r_i have the same biparity), and so $V_1 = \{p_1, p_2, a_1, b_1, r_1, r_2, a_0\}$. But then a_1, p_2 are the only neighbours of p_1, and so $\{a_1, p_2, b_1\}$ is a star cutset, contrary to 3.2. Hence r_1 is adjacent to p_1. Since a_1, r_i have the same biparity it follows that $i = 1$, and so a_1, r_2 are adjacent. Since a_1 does not dominate p_2 by 3.2, it follows that p_2 has a neighbour x nonadjacent to a_1, and in particular $x \ne p_1, b_1$; and so $V_1 = \{p_1, p_2, a_1, b_1, r_1, x, a_0\}$. Since x has a neighbour nonadjacent to p_1 by 3.2, it follows that x, a_0 are adjacent. But then x-p_2-p_1-a_1-b_0-a_0-x is a 6-hole and b_1 has three neighbours in it, contrary to 6.1. This completes the proof of 7.2. ∎

7.3 *Every balanceable graph not isomorphic to R_{10} that contains a small domino admits either a double star cutset or an internal 6-join.*

Proof. Let G be a balanceable graph not isomorphic to R_{10}, and let (a_0b_0, C_1, C_2) be a small domino in G. By 7.2 we may assume that G does not contain a right ear for this domino. For $i = 1, 2$ let C_i have vertices a_0-b_i-a_i-b_0-a_0 in order. By 3.2 there is a hole H such that (a_2b_2, C_2, H) is a domino;

let H have vertices a_2-p_1-$\cdots$-p_m-b_2-a_2 in order. Since H is not a right ear, one of a_1, b_1 is adjacent to one of $p_1, \ldots, p_m$. From the symmetry we may assume that b_1 is adjacent to one of $p_1, \ldots, p_m$; choose h, j with $1 \leq h, j \leq m$ minimum and maximum respectively such that b_1 is adjacent to p_h, p_j. If a_1 is nonadjacent to all of $p_{j+1}, \ldots, p_m$, then (since a_1, p_j have the same biparity and are therefore nonadjacent)

$$b_1\text{-}p_j\text{-}\cdots\text{-}p_m\text{-}b_2\text{-}a_2\text{-}b_0\text{-}a_1\text{-}b_1$$

is a hole containing three neighbours of a_0, contrary to 6.1. So a_1 is adjacent to one of $p_{j+1}, \ldots, p_m$, and in particular $j < m$. If $h = j$ then the three paths p_h-b_1-a_0-b_2, p_h-p_{h-1}-$\cdots$-p_1-a_2-b_2 and p_h-p_{h+1}-$\cdots$-p_m-b_2 form an odd theta, contrary to 6.1, so $h < j$. Choose i, k with $1 \leq i, k \leq m$ minimum and maximum respectively such that a_1 is adjacent to p_i, p_k. Thus $k > j$, and from the symmetry it follows that $h < i < k$. If $i \geq j$, then the numbers of neighbours of b_1 in the two holes H and

$$a_2\text{-}p_1\text{-}\cdots\text{-}p_i\text{-}a_1\text{-}b_0\text{-}a_2$$

differ by one, and b_1 has at least three neighbours in the second hole (since $h < j$), contrary to 6.1. Thus $i < j$.

Let us choose the hole H described above such that b_1 has as few neighbours in it as possible. Choose h' with $h < h' \leq m$ minimum such that $b_1, p_{h'}$ are adjacent. We may assume that there is a chordless path p_{h+1}-r_1-$\cdots$-r_n-b_0 such that $r_1, \ldots, r_n$ are nonadjacent to a_0, b_1, for otherwise G admits a double star cutset. From the choice of H it follows that every vertex of H that belongs to $\{r_1, \ldots, r_{n-1}\}$ or has a neighbour in $\{r_1, \ldots, r_{n-1}\}$ belongs to $\{p_h, p_{h+1}, \ldots, p_{h'}\}$.

(1) *r_n is adjacent to one of $p_1, \ldots, p_{h-1}$.*

For suppose not. Let P be a chordless path between p_h and b_0 with interior in $\{p_{h+1}, r_1, \ldots, r_n\}$; then the three paths P, p_h-b_1-a_0-b_0 and p_h-p_{h-1}-$\cdots$-p_1-a_2-b_0 form an odd theta (note that r_n is nonadjacent to a_2, p_h since they have the same biparity), a contradiction. This proves (1).

(2) *Every neighbour of r_n in H belongs to $\{b_2, p_1, \ldots, p_{h-1}, p_{h+1}\}$.*

For b_1-a_0-b_0-r_n is a chordless path, and by (1) there is a chordless path between b_1 and r_n with interior in $\{p_1, \ldots, p_h\}$, so if there is a chordless path between b_1 and r_n with interior in $\{p_{h+2}, p_{h+3}, \ldots, p_m\}$ then these three paths would form an odd prism. This proves (2).

Let P be a chordless path between $p_{h'}$ and b_2 with interior in $\{a_2, p_1, p_2, \ldots, p_{h'-1}, r_1, \ldots, r_n\} \setminus \{p_h\}$. It can be completed to a hole via $p_{h'}$-$p_{h'+1}$-$\cdots$-p_m-b_2. The number of neighbours of b_1 in this hole is exactly one fewer than the number of neighbours of b_1 in H, and so by 6.1, b_1 has exactly two neighbours in H, that is, $h' = j$. But then P and the paths p_j-b_1-a_0-b_2 and p_j-p_{j+1}-$\cdots$-p_m-b_2 form an odd theta, contrary to 6.1. This completes the proof. ∎

8 A proof of conjecture 9.29 of [6]

A bipartite graph G is *strongly balanceable* if it is balanceable and no induced subgraph is a cycle with exactly one chord. For the proof of 3.1 we also need theorem 6.1 of [3], the following:

8.1 *Every connected balanceable bipartite graph that is not strongly balanceable either equals R_{10} or admits a 2-join, a 6-join, or a double star cutset.*

Proof of 3.1. We prove 3.1 by induction on $|V(G)|$. Suppose then that G is a nonregular balanceable graph (and consequently $|V(G)| \geq 6$), and every nonregular balanceable graph with fewer vertices than G admits a double star cutset. Suppose for a contradiction that G does not admit a double star cutset. By 3.2, G is connected.

(1) *G does not admit a 2-join.*

For suppose it does, and let V_i, A_i, B_i $(i = 1, 2)$ be as in the definition of 2-join. Suppose first that there exist $x \in A_1$ and $y \in B_1$, adjacent. Every path between $V_1 \setminus \{x, y\}$ and $V_2 \setminus (A_2 \cup B_2)$ contains a member of $N[xy]$ in its interior, and since $V_1 \setminus \{x, y\} \neq \emptyset$ and G does not admit a double star cutset, it follows that $V_2 = A_2 \cup B_2$. Hence there is an edge between A_2, B_2, and so similarly $V_1 = A_1 \cup B_1$. Not both $|A_1|, |B_1| = 1$ from the definition of a 2-join, and if say $v, w \in A_1$ are distinct then by 3.2, since neither of v, w dominates the other, it follows that $|B_1| > 1$. Moreover, the same argument proves that there exist $a_1, a'_1 \in A_1$ and $b_1, b'_1 \in B_1$ such that a_1b_1 and $a'_1b'_1$ are edges, and a_1, b'_1 are nonadjacent, and a'_1, b_1 are nonadjacent. Similarly there exist $a_2, a'_2 \in A_2$ and $b_2, b'_2 \in B_2$ such that the only edges between $\{a_2, a'_2\}$ and $\{b_2, b'_2\}$ are a_2b_2 and $a'_2b'_2$. But then the subgraph induced on $\{a_2, a_1, b_1, b'_2, b'_1, a'_1\}$ is a cycle, and b_2 has exactly three neighbours in it, contrary to 6.1. Hence there are no edges between A_1 and B_1, and similarly no edges between A_2 and B_2.

Let P_2 be a chordless path of $G|V_2$ between A_2 and B_2 with no internal vertex in $A_2 \cup B_2$. Suppose that $G|(V_1 \cup V(P_2))$ is not regular. Let P_2 have vertices $p_1\text{-}\cdots\text{-}p_k$ say, where $p_1 \in A_2$ and $p_k \in B_2$. Every vertex in A_1 has a neighbour in $V_1 \setminus A_1$ (from the definition of a 2-join if $|A_1| = 1$, and by 3.2 if $|A_1| > 1$); and every component of $G|(V_1 \setminus A_1)$ contains a vertex of B_1, by 3.2 applied to G and the edge p_1p_2. It follows that $G|((V_1 \setminus A_1) \cup \{p_k\})$ is connected, and every vertex in A_1 has a neighbour in $(V_1 \setminus A_1) \cup \{p_k\}$; and an analogous statement holds with A_1, B_1 exchanged and p_1, p_k exchanged.

We claim that if $k = 3$ then $|V_2| \geq 6$. For if $|A_2| = |B_2| = 1$ then $V(P)$ is a double star cutset (since $P_2 \neq G|V_2$ from the definition of a 2-join), so from the symmetry we may assume that $|B_2| > 1$. Choose $b_2 \in B_2$ with $b_2 \neq p_3$. From 3.2, there is a chordless path between b_2 and A_1 containing no neighbour of either of p_2, p_3 (except possibly b_2). In particular, this path is disjoint from B_1, and therefore contains a vertex of $V_2 \setminus (A_2 \cup B_2)$ different from p_2, and a vertex of A_2 different from p_1. Consequently $|V_2| \geq 6$, as claimed.

Let $G_1 = G|(V_1 \cup V(P_2))$. If $k > 3$ let $G' = G_1$ and let $P'_2 = P_2$. If $k = 3$ let G' denote the graph obtained from G_1 by subdividing twice some edge of P_2 (that is, replacing some edge of P_2 by a three-edge path), and let P'_2 be the path obtained from P_2 by this double subdivision. Then in either case no edge of P'_2 is the centre of a double star cutset of G'. Yet G' is balanceable and not regular, and since $|V(G')| < |V(G)|$, the inductive hypothesis implies that there is an edge uv of G' that is the centre of a double star cutset of G'. It follows that at least one of $u, v \in V_1$, and uv is an edge of G_1, and therefore uv is the centre of a double star cutset of G_1.

There is a subset $X \subseteq N[uv] \cap V(G_1)$ with $u, v \in X$ such that $G_1 \setminus X$ is disconnected. Since $X \cup (N[uv] \setminus V(G_1))$ is not a double star cutset of G, there is a chordless path Q of G with ends x, y belonging to different components of $G_1 \setminus X$, and such that no internal vertex of Q belongs to $V(G_1) \cup N[uv]$. Since $Q^* \subseteq V_2$, there is a chordless path P of G_1 between x, y with $P^* \subseteq V(P_2)$; and therefore $N[uv] \cap P^* \neq \emptyset$. Since at least one of $u, v \in V_1$, it follows that $\{u, v\} \cap (A_1 \cup B_1) \neq \emptyset$, say $u \in A_1$. Then $A_2 \cap Q^* = \emptyset$, and since x, y both have neighbours in $Q^* \subseteq V_2 \setminus A_2$, it follows that $x, y \in V_2 \cup B_1$. So $P^* \cap A_2 = \emptyset$, and therefore $N[uv] \cap V_2 \not\subseteq A_2$. Hence $v \in A_2$, and so $v = p_1$, the end of P_2 in A_2. Thus $x, y \notin A_2$, and so $N[uv] \cap P^* = \emptyset$, a contradiction.

This proves that $G_1 = G|(V_1 \cup V(P_2))$ is regular. Similarly, let P_1 be a chordless path of $G|V_1$ between A_1 and B_1 with no internal vertex in $A_1 \cup B_1$; then $G_2 = G|(V_2 \cup V(P_1))$ is regular. But then by 4.1, G is regular, a contradiction. This proves (1).

(2) *If (V_1, V_2) is a 6-join in G, then one of $(V_1, V_2), (V_2, V_1)$ is skeletal.*

For let $A_1, \dots, A_6$ be as in the definition of a 6-join, and choose $a_i \in A_i$ for $1 \le i \le 6$. By 4.3, not both blocks of the 6-join are regular; so we may assume that G_1 is not regular, where G_1 is the block obtained by adding three vertices b_2, b_4, b_6 to $G|V_1$, with adjacency as before. For convenience we assume (as we may) that $b_i = a_i$ for $i = 2, 4, 6$. Since G_1 is therefore an induced subgraph of G, it follows that G_1 is balanceable. Define $A_7 = V_1 \setminus (A_1 \cup A_3 \cup A_5)$. Let H be obtained from G_1 by adding four vertices c_2, c_4, c_6, c_8 to G_1 and the edges $c_i c_8$ and $c_i a_i$ for $i = 2, 4, 6$. We shall show that G, H are isomorphic. Certainly H is balanceable (to see this, take a map $w : E(G_1) \to \{-1, 1\}$ such that $w(C)$ is a multiple of four for every induced cycle C of G_1; by reversing the signs of $w(e)$ on some edge-cutsets if necessary we may assume as usual that $w(a'_i a_{i+1}) = 1$ and $w(a'_i a_{i-1}) = -1$ for $i = 1, 3, 5$ and each $a'_i \in A_i$, where a_0 means a_6; then extend the domain of w to $E(H)$ by defining $w(e) = 1$ for every edge $e \in E(H) \setminus E(G_1)$; and it is easy to check that $w(C)$ is a multiple of four for every induced cycle C of H.)

We recall that G admits no double star cutset. Suppose that H admits a double star cutset X, with centre uv say. Up to symmetry there are five possibilities for uv, namely c_2c_8, c_2a_2, a_1a_2, a_7a_1 for some $a_7 \in A_7$, and $a_7a'_7$ for some $a_7, a'_7 \in A_7$. If $uv = c_2c_8$, then $A_1 \cup A_3 \cup A_5 \cup \{a_4, a_6\}$ is a subset of the vertex set of one component of $H \setminus X$, and so H and hence G is disconnected, a contradiction. If $uv = c_2a_2$, then the members of $A_5 \cup \{a_4, c_4, a_6, c_6\}$ all belong to the same component of $H \setminus X$, as does every vertex of $A_1 \cup A_3$ not in X, and so there is a component C of $H \setminus X$ with $V(C) \subseteq A_7$. Hence C is a component of $G \setminus N[a_2]$, and therefore G admits a star cutset, contrary to 3.2. If $uv = a_1a_2$, then the members of $A_5 \cup \{a_4, c_4, c_6, c_8\}$ all belong to the same component of $H \setminus X$, as does every member of $A_3 \setminus X$, and so there is a second component C say with $V(C) \subseteq A_7 \cup A_1$. But then $(X \setminus \{c_2\}) \cup A_2 \cup A_6$ is a double star cutset of G, a contradiction. If $uv = a_7a_1$ for some $a_7 \in A_7$, then a_4, c_2, c_4, c_6, c_8 all belong to the same component of $H \setminus X$, as does every member of $(A_3 \cup A_5 \cup \{a_2, a_6\}) \setminus X$. Hence there is a component C of $H \setminus X$ with $V(C) \subseteq A_7 \cup A_1$, and so $(X \cap V_1) \cup (A_2 \cup A_6)$ is a double star cutset of G, a contradiction. Finally, if $u, v \in A_7$, then $a_2, a_4, a_6, c_2, c_4, c_6, c_8$ all belong to the same component of $H \setminus X$, as does every member of $(A_1 \cup A_3 \cup A_5) \setminus X$, and so there is a component C of $H \setminus X$ with $V(C) \subseteq A_7$; but then X is a double star cutset of G, a contradiction. It follows that H does not admit a double star cutset.

From the inductive hypothesis, we deduce that $|V(H)| \ge |V(G)|$, and so $|V_2| \le 7$. Let $A_8 = V_2 \setminus (A_2 \cup A_4 \cup A_6)$. If $A_8 = \emptyset$, then since $|V_2| \ge 4$ it follows that two members of V_2 are twins, contradicting 3.2. Thus $A_8 \ne \emptyset$. Suppose that some vertex in A_8 has neighbours in two of A_2, A_4, A_6, say $a_8 \in A_8$ is adjacent to $a_2 \in A_2$ and to $a_4 \in A_4$. Since $N[a_3a_2] \setminus \{a_8\}$ is not a double star cutset, there is a chordless path P between a_8 and A_6 such that $V(P) \setminus \{a_8\}$ is anticomplete to a_2. Choose P minimal; then all its vertices belong to A_8 except for its final vertex a_6 say in A_6. Since the subgraph induced on $\{a_8, a_1, \dots, a_6\}$ is not an odd wheel by 6.1, it follows that a_8, a_6 are nonadjacent, and since a_8, a_6 have opposite biparity it follows that P has odd length, and length at least three. Since $|V_2| \leq 7$ and so $|A_8| \leq 4$, it follows that P has length three; let its vertices be a_8-p_1-p_2-a_6 in order. Now the paths a_8-P-a_6, a_8-a_2-a_1-a_6 and a_8-a_4-a_5-a_6 do not form an odd theta, by 6.1, and so a_4 is adjacent to p_2. But then every neighbour of p_1 is adjacent to a_4, contrary to 3.2. This proves that no vertex of A_8 has neighbours in two of A_2, A_4, A_6.

Let C be a component of $G|A_8$. Since G does not admit a double star cutset, at least one member of A_i has a neighbour in C for $i = 2, 4, 6$, and so we may assume that for $i = 2, 4, 6$, a_i is adjacent to $c_i \in C$. Hence c_2, c_4, c_6 are all distinct, since no vertex of A_8 has neighbours in two of A_2, A_4, A_6. Moreover, since C is connected, there is a vertex $c_8 \in C$ such that a_2, c_8 have the same biparity. Hence $A_8 = \{c_2, c_4, c_6, c_8\}$, and $A_i = \{a_i\}$ for $i = 2, 4, 6$; and since C is connected, it follows that c_8 is adjacent to each of c_2, c_4, c_6. But then (V_1, V_2) is skeletal. This proves (2).

From (2) and 6.4 and 7.3, it follows that G contains no big domino or small domino.

(3) *Let (V_1, V_2) be a 6-join, and let $A_1, \ldots, A_6$ be defined as usual. If $v \in V_1$ has a neighbour in A_1 and a neighbour in A_3, then v is complete to $A_1 \cup A_3$.*

For let v be adjacent to $a_1 \in A_1$ and $a_3 \in A_3$, and suppose it is nonadjacent to some $a_1' \in A_1$ say. Choose $a_2 \in A_2$ and $a_6 \in A_6$; then

$$(a_1a_2, a_1\text{-}a_2\text{-}a_1'\text{-}a_6\text{-}a_1, a_1\text{-}a_2\text{-}a_3\text{-}v\text{-}a_1)$$

is a small domino, a contradiction. This proves (3).

(4) *Let (v_1v_2, C, D) be a domino, where $|V(C)| \geq 8$ and $|V(D)| \geq 6$. For $i = 1, 2$, let c_i, d_i be the neighbours of v_i in $C \setminus \{v_1, v_2\}, D \setminus \{v_1, v_2\}$ respectively, and let d_1 have degree at least three in G. Then every vertex of G adjacent to both c_2 and v_1 is adjacent to every neighbour of v_2 except possibly d_2, and d_2 belongs to no irregularity in G.*

For by 6.3, G admits a 6-join (V_1, V_2) such that $V(C) \setminus \{v_1, v_2\} \subseteq V_1$ and $V(D) \setminus \{v_1, v_2\} \subseteq V_2$, and V_1, V_2 each contain exactly one of v_1, v_2. Let $\{i, j\} = \{1, 2\}$, where $v_j \in V_1$ and $v_i \in V_2$. Since C has length at least eight, it follows that (V_2, V_1) is not skeletal, and so by (2) (V_1, V_2) is skeletal. Hence d_i has degree two, and by 5.1 d_i does not belong to any irregularity in G. Since d_1 has degree at least three, it follows that $i = 2$ and $j = 1$. Let the sets $A_1, \ldots, A_6$ be defined as usual, where $c_2 \in A_1, v_2 \in A_2, v_1 \in A_3$ and $d_1 \in A_4$. Then $N[v_2] = A_1 \cup A_3 \cup \{v_2, d_2\}$, and v_2 is the only vertex in V_2 adjacent to both c_2, v_1, and by (3) every vertex in V_1 adjacent to both c_2, v_1 is complete to $A_1 \cup A_3$. This proves (4).

Since R_{10} is regular, it follows from 8.1 that either G is strongly balanceable, or G admits a 2-join, or G admits a 6-join. The first is impossible since it is a theorem of [5] that every strongly balanceable graph is regular. So by (1) and (2), it follows that G admits a skeletal 6-join (V_1, V_2). Let $A_1, \ldots, A_6$ be as in the definition of a 6-join, and for $1 \leq i \leq 6$ choose $a_i \in A_i$. Thus $A_i = \{a_i\}$ for $i = 2, 4, 6$. Since G is not regular, it follows that there is an irregularity H, and from 5.1 $V(H) \cap V_2 \subseteq \{a_2, a_4, a_6\}$. Choose H such that $|V(H) \cap \{a_2, a_4, a_6\}|$ is as small as possible.

(5) *If $a_2 \in V(H)$ then there is no vertex $v \in V_1$ with a neighbour in A_1 and a neighbour in A_3.*

For suppose that $a_2 \in V(H)$ and such a vertex v exists. By (3) v is complete to $A_1 \cup A_3$. Since G does not admit a double star cutset, there is a chordless path v-p_1-$\cdots$-p_k of G such that $p_1, \ldots, p_{k-1} \in V_1$ and $\{p_1, \ldots, p_k\}$ is anticomplete to $\{a_1, a_2\}$, and $p_k \in A_1 \cup A_3 \cup A_5$ (and therefore $p_k \in A_5$, since p_k is nonadjacent to a_2). By 6.1, v does not have three neighbours in the hole induced on $\{a_1, \ldots, a_6\}$, and so $k > 1$. Since the paths v-p_1-$\cdots$-p_k, v-a_3-a_4-p_k and v-a_1-a_6-p_k do not form an odd theta by 6.1, it follows that a_3 is adjacent to one of $p_1, \ldots, p_k$. Choose i with $1 \leq i \leq k$ minimum such that a_3, p_i are adjacent. Since v, p_i have the same biparity, it follows that $i < k$, and i is even. If $i = 2$ then

$$(va_3, v\text{-}a_1\text{-}a_2\text{-}a_3\text{-}v, v\text{-}p_1\text{-}p_2\text{-}a_3\text{-}v)$$

is a small domino, a contradiction; so $i \geq 4$. Since (V_1, V_2) is skeletal, there is a chordless path Q of length four between a_6 and a_4 with interior in $V_2 \setminus \{a_2, a_4, a_6\}$. Let C be the hole v-a_1-a_6-Q-a_4-a_3-v, and let D be the hole v-p_1-$\cdots$-p_i-a_3-v. Then (va_3, C, D) is a domino, and C has length eight, and D has length at least six. Let $v_1 = a_3$ and $v_2 = v$, and for $i = 1, 2$, let c_i, d_i be the neighbours of v_i in $C \setminus \{v_1, v_2\}, D \setminus \{v_1, v_2\}$ respectively; then $d_1 = p_i$, and therefore d_1 has degree at least three in G. By (4) every vertex of G adjacent to both c_2 and v_1 is adjacent to every neighbour of v_2 except possibly d_2, and d_2 belongs to no irregularity in G. Since $c_2 = a_1$, and $d_2 = p_1$, and a_2 is adjacent to both a_1, a_3, it follows that a_2 is adjacent to every neighbour of v_2 except p_1. But the neighbour set of a_2 is $A_1 \cup A_3 \cup \{u_2\}$ for some $u_2 \in V_2 \setminus \{a_2, a_4, a_6\}$, where no irregularity contains u_2 by 5.1; and since we have already seen that v is complete to $A_1 \cup A_3$, it follows that the neighbour set of v is $A_1 \cup A_3 \cup \{p_1\}$. If $v \in V(H)$ then since $p_1, u_2 \notin V(H)$ it follows that v, a_2 are twins in H, which is impossible. Thus $v \notin V(H)$. Since $p_1, u_2 \notin V(H)$, it follows that the subgraph induced on $(V(H) \setminus \{a_2\}) \cup \{v\}$ is isomorphic to H, and therefore is also an irregularity, contrary to the minimality of $|V(H) \cap \{a_2, a_4, a_6\}|$. This proves (5).

Let I be the set of all $i \in \{2, 4, 6\}$ such that no vertex in V_1 has a neighbour in A_{i-1} and a neighbour in A_{i+1}, where A_7 means A_1. Let J be the subgraph of G induced on $V_1 \cup \{a_i : i \in I\}$.

(6) *J does not admit a double star cutset.*

For suppose there is a double star cutset X in J, with centre uv say, and let C_1, C_2 be distinct components of $J \setminus X$. Let $X' = X \cup (N[uv] \setminus V(J))$. Since X' is not a double star cutset of G, it follows that C_1, C_2 are both subgraphs of the same component C of $G \setminus X'$. In particular, for $j = 1, 2$, some vertex p_j of C_j has a neighbour $q_j \in V(C) \setminus V(C_j)$ that is nonadjacent to both u, v. Thus $q_j \notin V(J)$, and hence $p_j \in A_1 \cup A_3 \cup A_5 \cup \{a_i : i \in I\}$. For $i = 2, 4, 6$, if $i \in I$ let $a'_i = a_i$, and if $i \notin I$ let a'_i be some vertex in V_1 that is complete to $A_{i-1} \cup A_{i+1}$ (this exists, by (3) and the definition of I). We observe that for $i = 2, 4, 6$, a'_i is anticomplete to A_{i+3}; for this is clear if $a'_i = a_i$, and if $a'_i \neq a_i$ then it follows since otherwise a'_i would have three neighbours in a 6-hole contained in $A_1 \cup \cdots \cup A_6$. Let R be the subgraph of G induced on $A_1 \cup A_3 \cup A_5 \cup \{a'_2, a'_4, a'_6\}$). Then R is a subgraph of J, and is connected, and both $p_1, p_2 \in V(R)$.

Suppose first that $u = a_2$ say, and therefore $2 \in I$. From the symmetry we may assume that $v \in A_1$. Hence $X \cap (A_5 \cup \{a'_4\}) = \emptyset$, and so we may assume that $A_5 \cup \{a'_4\} \subseteq V(C_1)$. Thus $p_2 \notin A_3 \cup A_5 \cup \{a'_2, a'_4, a'_6\}$, and so $p_2 \in A_1$. But then $q_2 = a_6$, and so q_2 is adjacent to v, a contradiction. Thus $u \neq a_2$, and similarly $u, v \neq a_2, a_4, a_6$.

Next suppose that $u \in A_1$. Thus v has a neighbour in A_1, and so is anticomplete to one of A_3, A_5, say A_5 without loss of generality. Hence $X \cap (A_5 \cup \{a_4'\}) = \emptyset$, and so we may assume that $A_5 \cup \{a_4'\} \subseteq V(C_1)$. Consequently $p_2 \in A_1 \cup \{a_2'\}$. If $p_2 = a_2'$, then $a_2' = a_2$ and so $2 \in I$, and so v has no neighbour in A_3; but then $A_3 \cap X = \emptyset$, and so $A_3 \subseteq V(C_1)$, a contradiction. Thus $p_2 \in A_1$, and so q_2 is adjacent to u, a contradiction. Thus $u \notin A_1$, and similarly $u, v \notin A_1 \cup A_3 \cup A_5$.

Next suppose that a_2', a_4' belong to the same component of $J \setminus X$, say $a_2', a_4' \in V(C_1)$. Then $V(C_2) \cap (A_1 \cup A_3 \cup A_5) = \emptyset$, and so $p_2 = a_6' = a_6$ and $6 \in I$. But then $A_1 \cup A_5 \subseteq X$, and so one of u, v is complete to $A_1 \cup A_5$, contradicting that $6 \in I$.

Next suppose that $a_2', a_4' \in X$. Thus we may assume that u is adjacent to a_2', a_4'. Since $u \notin A_1 \cup A_3 \cup A_5$, it follows that $a_2', a_4' \notin V_2$. But then

$$(a_2'a_3, a_2'\text{-}u\text{-}a_4'\text{-}a_3\text{-}a_2', a_2'\text{-}a_1\text{-}a_2\text{-}a_3\text{-}a_2')$$

is a small domino in G, a contradiction. Thus at most one of a_2', a_4', a_6' belongs to X, and so we may assume that $a_2', a_4' \notin X$.

Since a_2', a_4' are not both in the same component of $J \setminus X$, we may therefore assume that $a_2' \in V(C_1)$ and $a_4' \in V(C_2)$. Consequently $A_3 \subseteq X$, and so we may assume that u is complete to A_3. Suppose that v is adjacent to a_6', and therefore $a_6' \neq a_6$. If u, a_1 are adjacent then

$$(ua_1, u\text{-}a_3\text{-}a_2\text{-}a_1\text{-}u, u\text{-}v\text{-}a_6'\text{-}a_1\text{-}u)$$

is a small domino, a contradiction. Thus u is nonadjacent to a_1 and similarly to a_5. But then the subgraph induced on $\{u, v, a_1, a_2, a_3, a_4, a_5, a_6'\}$ is an odd theta, contrary to 6.1. Thus v, a_6' are nonadjacent, and so $a_6' \notin X$. Since u is complete to A_3, it follows that u is anticomplete to one of A_1, A_5, say A_1 without loss of generality. It follows that $A_1 \cap X = \emptyset$ (since v has the same biparity as the members of A_1) and so $A_1 \subseteq V(C_1)$; and since $a_6' \notin X$, we deduce that $a_6' \in V(C_1)$. Consequently $A_5 \subseteq X$, and so u is complete to A_5. It follows that $4 \notin I$, and therefore $p_2 \neq a_4'$; but $p_2 \in V(R) \cap V(C_2) \subseteq \{a_4'\}$, a contradiction. This proves (6).

Now $|V(J)| < |V(G)|$, since $|I| \leq 3$ and $|V_2| \geq 4$. From (5), $V(H) \subseteq V(J)$ and so J is not regular. But from (6), J has no double star cutset, contrary to the inductive hypothesis. Thus our assumption that G has no double star cutset is false. This completes the proof of 3.1. ∎

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