# Triangle-free graphs with no six-vertex induced path

Maria Chudnovsky [1], Paul Seymour [2], Sophie Spirkl
Princeton University, Princeton, NJ 08544

Mingxian Zhong
Columbia University, New York, NY 10027

June 12, 2017; revised April 12, 2018

[1]This material is based upon work supported by, or in part by, the U. S. Army Research Laboratory and the U. S. Army Research Office under contract/grant number W911NF-16-1-0404, and supported by NSF Grant DMS-1550991.

[2]Supported by ONR grant N00014-14-1-0084 and NSF grant DMS-1265563.

**Abstract**

The graphs with no five-vertex induced path are still not understood. But in the triangle-free case, we can do this and one better; we give an explicit construction for all triangle-free graphs with no six-vertex induced path. Here are three examples: the 16-vertex Clebsch graph, the graph obtained from an 8-cycle by making opposite vertices adjacent, and the graph obtained from a complete bipartite graph by subdividing a perfect matching. We show that every connected triangle-free graph with no six-vertex induced path is an induced subgraph of one of these three (modulo some twinning and duplication).

# 1 Introduction

Graphs in this paper are without loops and multiple edges, and finite unless we say otherwise. If $G$ is a graph and $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced on $X$; and we say that $G$ *contains* $H$ if some induced subgraph of $G$ is isomorphic to $H$. If $G$ has no induced subgraph isomorphic to $H$, we say $G$ is *$H$-free*, and if $\mathcal{C}$ is a set of graphs, $G$ is *$\mathcal{C}$-free* if $G$ is $H$-free for all $H \in \mathcal{C}$. We denote the $k$-vertex path graph by $P_k$. Our objective in this paper is to find a construction for all $\{P_6, K_3\}$-free graphs.

Constructing all $P_5$-free graphs remains open, although it has been heavily investigated, mostly because $P_5$ is one of the minimal graphs for which the Erdős-Hajnal conjecture [2, 3] is unsolved. It is also not clear whether we know how to construct all $K_3$-free graphs (it depends on what exactly counts as a "construction"). But we can construct all $\{P_5, K_3\}$-free graphs, and indeed all $\{P_6, K_3\}$-free graphs, and the answer is surprisingly pretty.

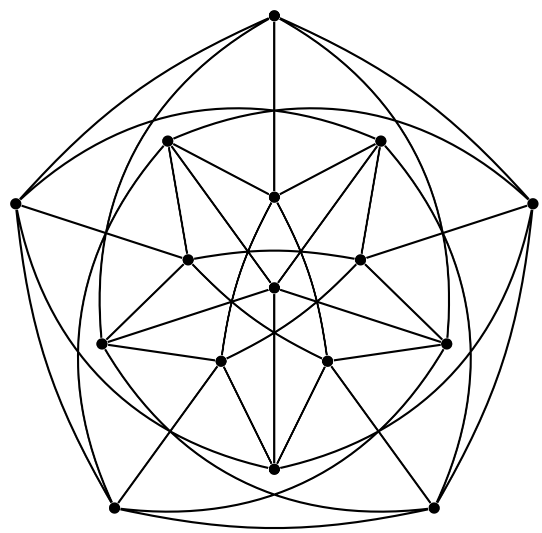

Figure 1: The Clebsch graph

The Clebsch graph, shown in figure 1, is the most interesting of the $\{P_6, K_3\}$-free graphs. Here are three alternative definitions for it.

- Take the five-dimensional cube graph, and identify all pairs of opposite vertices.
- Take the elements of the field $GF(16)$, and say two of them are adjacent if their difference is a cube.
- Start with the Petersen graph; for each stable subset $X$ of cardinality four (there are five such subsets) add a new vertex adjacent to the vertices in $X$; and then add one more vertex adjacent to the five new vertices. (This third definition is the least symmetric but in practise we found it the most helpful).

For every edge $uv$ of the Clebsch graph, the subgraph induced on the set of vertices nonadjacent to $u, v$ is a three-edge matching, and from this it follows that the graph is $P_6$-free. We say $G$ is *Clebschian* if $G$ is contained in the Clebsch graph.

There is another kind of $\{P_6, K_3\}$-free graph that we need to discuss, the following. Take a complete bipartite graph $K_{n,n}$ with bipartition $\{a_1, \dots, a_n\}$, $\{b_1, \dots, b_n\}$, and subdivide each edge $a_ib_i$; that is, for $1 \le i \le n$ we delete the edge $a_ib_i$, and add a new vertex $c_i$ adjacent to $a_i, b_i$. This graph, $H$ say, is $\{P_6, K_3\}$-free, and we say a graph $G$ is *climbable* if it is isomorphic to an induced subgraph of $H$ for some $n$.

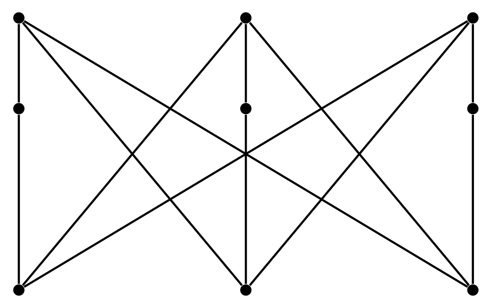

Figure 2: A climbable graph

Our aim is to prove something like "every $\{P_6, K_3\}$-free $G$ is either climbable or Clebschian", but by itself this is not true. For instance, we have to assume $G$ is connected, because otherwise the disjoint union of two Clebsch graphs would be a counterexample. But just assuming connectivity is not enough. For instance, if $v$ is a vertex of a $\{P_6, K_3\}$-free graph, then we could add a new vertex with the same neighbours as $v$, and the enlarged graph would still be $\{P_6, K_3\}$-free. Let us say two vertices are *twins* if they are nonadjacent and have the same neighbour sets. Thus we need to assume that $G$ has no twins. There are two other "thickening" operations of this kind that we define later.

Our main result is the following (although some definitions have not yet been given):

**1.1** *Let $G$ be a connected $\{P_6, K_3\}$-free graph without twins. Then either*

- *$G$ is Clebschian, climbable, or a $V_8$-expansion; or*
- *$G$ admits a nontrivial simplicial homogeneous pair.*

There has already been work on this and similar questions:

- In [7], Randerath, Schiermeyer and Tewes proved that every connected $\{P_6, K_3\}$-free graph which is not 3-colourable, and in which no vertex dominates another, is an induced subgraph of the Clebsch graph.
- Brandstädt, Klembt and Mahfud [1] proved that every $\{P_6, K_3\}$-free graph can be decomposed in a certain way that implies that all such graphs have bounded clique-width.
- Lozin [6] gave a construction for all bipartite graphs not containing what he called a "skew star", obtained from $P_6$ by adding one more vertex adjacent to its third vertex.

- Lokshantov, Vatshelle and Villanger [5] found a polynomial-time algorithm to find a stable set of maximum weight in a $P_5$-free graph; and more recently Grzesik, Klimošová, Pilipczuk and Pilipczuk [4] claim to have the same for $P_6$-free graphs.

## 2 An overview of the proof

The proof of 1.1 divides into a number of cases, and here we give a high-level survey of how it all works. We define some "thickening" operations, that when applied to $\{P_6, K_3\}$-free graphs, will produce larger graphs that are still $\{P_6, K_3\}$-free. One of the thickening operations involves substituting a bipartite $P_6$-free graph for an edge, so we need to understand the bipartite $P_6$-free graphs, and it is convenient to postpone this until 5.3. Except for that, it suffices to understand the $\{P_6, K_3\}$-free graphs that cannot be built from a smaller graph by means of one of the thickening operations (let us temporarily call such graphs "unthickened").

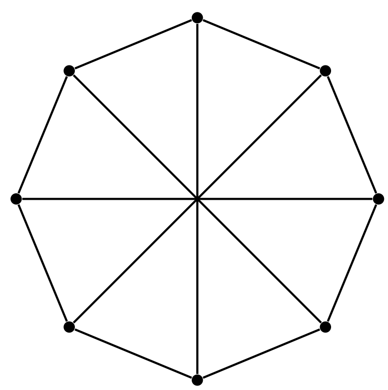

Figure 3: The graph $V_8$

Let $Q_5$ be the graph obtained from $P_6$ by deleting one of its two middle vertices, that is, the disjoint union of a $P_3$ and a $P_2$; and let $V_8$ be the graph obtained from a cycle of length eight by making opposite pairs of vertices adjacent. The proof of 1.1 is in two stages. First, we figure out the connected unthickened triangle-free graphs that contain $Q_5$ but not $P_6$. One such graph is $V_8$; and it turns out every connected unthickened triangle-free graph that contain $Q_5$ but not $P_6$ is contained in $V_8$.

Then, second, we find a construction for all unthickened connected $\{Q_5, K_3\}$-free graphs. We prove that every such graph is either Clebschian, or climbable, or bipartite. (We figure out directly how to construct all bipartite $Q_5$-free graphs in section 5.)

Our main approach is via the size of the largest induced matching in $G$, denoted by $\mu(G)$. If $M$ is an induced matching, let $V_M$ be the vertices incident with edges in $M$, and let $N_M$ be the set of vertices not in $V_M$ but adjacent to a vertex in $V_M$. There are three easy but very useful observations (5.1 and 6.2): if $G$ is $\{Q_5, K_3\}$-free and $M$ is a matching in $G$, then

- every vertex in $N_M$ has exactly one neighbour in *every* edge of $M$;
- if $u, v$ in $N_M$ are adjacent, then they are joined to opposite ends of each edge of $M$;

- if $u, v$ in $N_M$ are nonadjacent, they are joined to the same end of all edges of $M$ except at most two.

This tells us most when $M$ is large and $N_M$ is not stable; for instance, if $G$ is unthickened, it tells us (shown in 9.1) that either $|M| \leq 4$ or $N_M$ is stable. So two special cases need attention, when $\mu(G)$ is small, and when $N_M$ is stable. We do the case when $N_M$ is stable first, because we need some of those lemmas to handle the case $\mu(G) = 2$.

When $N_M$ is stable, the proof breaks into three subcases corresponding to the ways to choose a set of vertices of an $n$-cube pairwise with Hamming distance at most 2. There are three essentially different ways to choose such a set (namely, choose vertices all with distance at most one from some given vertex; choose the vertices of a square; or choose the even vertices of a 3-cube). The first subcase more-or-less implies that $G$ is climbable, the second implies $G$ is Clebschian, and the third implies that $G$ contains the Petersen graph, which makes it easy to handle. This is all done in section 7.

Now we turn to the cases when $\mu(G)$ is small. The case $\mu(G) \leq 1$ is easy, and it turns out that the $\mu(G) = 2$ case is the most difficult, and we devote section 8 to it. The *bipartite complement* of a graph $G$ with a given bipartition $(A, B)$ is the graph $H$ with the same bipartition $(A, B)$, in which for $a \in A$ and $b \in B$, $a, b$ are adjacent in exactly one of $G, H$. To analyze the graphs $G$ with $\mu(G) = 2$, we need to look at another parameter, $\nu(G)$, the size of the largest antimatching in $G$ (this is a bipartite induced subgraph whose bipartite complement is a perfect matching); and handle the cases $\mu(G) = 2, \nu(G) > 2$ and $\mu(G) = 2, \nu(G) \leq 2$ separately (in 8.1, 8.2 respectively). From now on we can assume $\mu(G) \geq 3$, and the proof gets easier.

If $N_M$ is not stable, and $G$ is unthickened, then $|M| \leq 4$, and since we have already handled the cases when $|M| \leq 2$, we just have to figure out what happens when $3 \leq |M| \leq 4$. In both cases $G$ turns out to be Clebschian (shown in 9.2, 9.3). That will complete the proof of the main theorem.

One annoyance is that the thickening operations are designed to not introduce $P_6$, but they might introduce $Q_5$. Thus the construction that we give for all unthickened $\{Q_5, K_3\}$-free graphs does not directly give a construction for all $\{Q_5, K_3\}$-free graphs. This can easily be remedied, and we do so in section 10.

## 3 Thickening

A *k-path* in $G$ means an induced subgraph isomorphic to $P_k$. Two induced subgraphs $H, J$ of $G$ are *anticomplete* to each other if $V(H) \cap V(J) = \emptyset$, and there is no edge of $G$ between $V(H)$ and $V(J)$. They are *complete* to each other if $V(H) \cap V(J) = \emptyset$ and every vertex in $H$ is adjacent to every vertex in $J$. We use the same terminology for subsets of $V(G)$ instead of induced subgraphs, and for single vertices instead of subsets. We also say $H$ *misses* $J$ if $H$ is anticomplete to $J$.

As we mentioned, adding twins to a graph with no 6-path will not introduce a 6-path. Let us say a subset $X \subseteq V(G)$ is a *homogeneous set* if $X$ is stable and every vertex of $G$ not in $X$ is either complete or anticomplete to $X$. (The requirement that $X$ is stable is not the standard definition, but we add it here to make the concept useful for constructing triangle-free graphs.) If $G$ has a homogeneous set $X$ of cardinality at least two, then $G$ is obtained from a smaller graph by adding twins; so let us say a homogeneous set $X$ of $G$ is *nontrivial* if $|X| \geq 2$.

There is another, similar concept, "homogeneous pairs". A pair $(A, B)$ of disjoint stable subsets of $V(G)$ is a *homogeneous pair* if every vertex not in $A \cup B$ is either complete or anticomplete to $A$, and either complete or anticomplete to $B$. (Again, the requirement that $A, B$ are stable is not standard.) It is *nontrivial* if $|A| + |B| \geq 3$ and there is an edge between $A$ and $B$, and $A \cup B \neq V(G)$ (this again is slightly nonstandard; the requirement of an edge between $A, B$ is not usual, but convenient for us).

Let $(A, B)$ be a nontrivial homogeneous pair in $G$, let $J = G[A \cup B]$, and let us identify the vertices of $A$ into one vertex $a$, and identify those in $B$ to one vertex $b$ (and thus $a, b$ will be adjacent). The graph we produce, $H$ say, is contained in $G$ (because there is an edge of $G$ between $A$ and $B$), and so is $J$, and from the hypotheses they both have fewer vertices than $G$. We can regard $G$ as being constructed from $H$ by "substituting $J$ for the edge $ab$".

This operation, applied to two triangle-free graphs $H, J$, produces a triangle-free graph $G$, but it might well introduce 6-paths; we need to restrict it further to make it safe. A homogeneous pair $(A, B)$ of $G$ is *simplicial* if every vertex in $V(G) \setminus (A \cup B)$ with a neighbour in $A$ is adjacent to every vertex in $V(G) \setminus (A \cup B)$ with a neighbour in $B$. Then, with $H, J$ as above, if both $H, J$ are $\{P_6, K_3\}$-free, then $G$ is also $\{P_6, K_3\}$-free (proved below); and so nontrivial simplicial homogeneous pairs can be used safely to construct larger $\{P_6, K_3\}$-free graphs from smaller ones. A special case is when there are no vertices in $V(G) \setminus (A \cup B)$ with a neighbour in $B$; in this case we say the homogeneous pair is *pendant*.

To regard this as a construction, we need to understand how to construct the building blocks, and in particular, how to build all bipartite $P_6$-free graphs. That issue is addressed in section 5.

Let us prove the claim above.

**3.1** *Let $(A, B)$ be a simplicial homogeneous pair in a graph $G$, and let $H$ be obtained from $G$ by identifying the vertices of $A$ into one vertex $a$, and identifying those in $B$ to one vertex $b$, where $a, b$ are adjacent. Let $J = G[A \cup B]$. If $H, J$ are both $\{P_6, K_3\}$-free then so is $G$.*

**Proof.** Since $A, B$ are stable, and the edge $ab$ is not in a triangle of $H$, it follows that $G$ is $K_3$-free. Let $C$ be the set of vertices in $V(G) \setminus (A \cup B)$ that are complete to $A$ and anticomplete to $B$, and define $D$ similarly with $A, B$ exchanged. Suppose that there is a 6-path $P$ in $G$. Define $A' = A \cap V(P)$, and define $B', C', D'$ similarly. Since there is no 6-path in $H$, it follows that $|A' \cup B'| \geq 2$; and since no two vertices of $P$ are twins in $P$, it follows that $A', B' \neq \emptyset$.

Suppose that $A', B'$ are anticomplete. Each vertex of $A'$ has a neighbour in $P$ and hence in $C'$; and so $|D'| \leq 1$ (since $C'$ is complete to $D'$, because $(A, B)$ is simplicial). Since each vertex of $B'$ has a neighbour in $P$, it follows that $|D'| = 1$, and similarly $|C'| = 1$. But then $|A'| = |B'| = 1$, and since the vertices in $C' \cup D'$ have no more neighbours in $P$, it follows that $|V(P)| = 4$, a contradiction.

Consequently there exist adjacent vertices $u \in A'$ and $v \in B'$. Hence not both $C', D'$ are nonempty (because $C'$ is complete to $D'$), so we may assume that $D' = \emptyset$. Since $J$ is $P_6$-free, $C' \neq \emptyset$; and so $v$ has only one neighbour in $A$. Hence $v$ is an end of $P$, and indeed, so is every vertex in $B'$, since each has a neighbour in $A'$. In particular $|B'| \leq 2$. If $A' = \{u\}$ then $B' = \{v\}$, contradicting that $H$ is $P_6$-free; so there exists $u' \neq u \in A'$. So $|C'| = 1$, and the vertex in $C'$, $w$ say, has no neighbours in $P$ except $u, u'$. Consequently $V(P) \subseteq A' \cup B' \cup \{w\}$. Since $|B'| \leq 2$, it follows that $|A'| \geq 3$, contradicting that $w$ has degree two in $P$. Thus there is no such $P$. This proves 3.1. ∎

The second way we use homogeneous pairs is to construct a class of $\{P_6, K_3\}$-free graphs, as follows. A bipartite graph $J$ with bipartition $(A, B)$ is an *antisubmatching* (relative to the given bipartition) if every vertex in $A$ has at most one nonneighbour in $B$, and vice versa. It is an *antimatching* if every vertex in $A$ has exactly one nonneighbour in $B$ and vice versa. Let $V_8$ be the graph with eight vertices $a_1, \dots, a_8$, in which distinct $a_i, a_j$ are adjacent if $j - i = 1, 4$ or $7$ modulo 8. For $i = 1, 2$, let $J_i$ be an antisubmatching relative to $(A_i, B_i)$, with at least one edge; and let $G$ be obtained from $V_8$ (with vertices $a_1, \dots, a_8$ as above) by substituting $(A_1, B_1)$ for $a_1a_5$, substituting $(A_2, B_2)$ for $a_3a_7$, and possibly deleting some of $a_2, a_4, a_6, a_8$. We say $G$ is a *$V_8$-expansion*. Again, it is easy to check that every $V_8$-expansion is $\{P_6, K_3\}$-free.

Together, these two will suffice to account for the $\{P_6, K_3\}$-free graphs that contain $Q_5$. We will show the following (in the next section).

**3.2** *Let $G$ be a connected $\{P_6, K_3\}$-free graph without twins that contains $Q_5$. Then either*

- *$G$ is a $V_8$-expansion; or*
- *$G$ admits a nontrivial simplicial homogeneous pair.*

# 4 Graphs that contain $Q_5$

We recall that $Q_5$ is the disjoint union of a $P_3$ and a $P_2$, that is, the graph obtained from $P_6$ by deleting one of its two middle vertices. Certainly $Q_5$-free graphs are $P_6$-free, and it will turn out that the converse is "nearly" true; and in particular the two classes described earlier (Clebschian graphs and climbable graphs) are $Q_5$-free. In this section we describe all triangle-free graphs that are $P_6$-free and not $Q_5$-free. Some notation: we write $p_1$-$p_2$-$\cdots$-$p_k$ to denote a path with vertices $p_1, \dots, p_k$ in order.

Let $Q_7$ be the graph obtained from $P_7$ by adding an edge between the second and fifth vertex.

**4.1** *Let $G$ be a connected $\{P_6, K_3\}$-free graph without twins that contains $Q_5$. Then $G$ contains $Q_7$.*

**Proof.** Let $R_6$ be obtained from $P_5$ by adding a new vertex adjacent to the second vertex of $P_5$; and let $S_6$ be obtained from $P_5$ by adding a new vertex adjacent to the first and third vertices of $P_5$.

(1) *$G$ contains one of $R_6, S_6$.*

Since $G$ contains $Q_5$, there is a 3-path $P$ and a 2-path $Q$ in $G$ that are anticomplete. Since $G$ is connected, there is a path $R$ of $G \setminus V(P \cup Q)$ that misses neither of $P, Q$. By choosing $R$ minimal, it follows that $G[V(Q) \cup V(R)]$ is a path, and contains a 3-path $S$ of $G$ such that one of its ends, $s$ say, has a neighbour in $P$, and its other vertices do not. If $s$ is adjacent to the middle vertex of $P$ then $G$ contains $R_6$; if $s$ is adjacent to both ends of $P$ then $G$ contains $S_6$, and if $s$ is adjacent to just one end of $P$, then $G$ contains $P_6$, which is impossible. This proves (1).

(2) *If $G$ contains $R_6$ then $G$ contains $Q_7$.*

Since $G$ contains $R_6$, there is a 4-path $p_1$-$\cdots$-$p_4$, and two vertices $a_1, a_2$ both adjacent to $p_1$ and not to $p_2, p_3, p_4$. Since $a_1, a_2$ are not twins, there is a vertex $v$ adjacent to one of them and not the other, say $v$ is adjacent to $a_1$ and not to $a_2$. If $v$ is adjacent to either of $p_3, p_4$, then the path $a_2$-$p_1$-$a_1$-$v$ can be extended by $v$-$p_3$-$p_4$ or $v$-$p_4$-$p_3$, giving $P_6$, a contradiction. Thus $v$ is nonadjacent to both $p_3, p_4$. If it is also nonadjacent to $p_2$, then $v$-$a_1$-$p_1$-$p_2$-$p_3$-$p_4$ is a 6-path, a contradiction. So $v$ is adjacent to $p_2$, and so $G$ contains $Q_7$. This proves (2).

(3) *If $G$ contains $S_6$ then $G$ contains $Q_7$.*

Since $G$ contains $S_6$, there is a 3-path $p_1$-$p_2$-$p_3$, two more vertices $a_1, a_2$ adjacent to $p_1$ and not to $p_2, p_3$, and a vertex $p_0$ adjacent to $a_1, a_2$ and not to $p_1, p_2, p_3$. Since $a_1, a_2$ are not twins, we may assume there is a vertex $v$ adjacent to $a_1$ and not to $a_2$. Thus $v$ is nonadjacent to $p_0, p_1$. If $v$ is adjacent to one of $p_2, p_3$, then the path $a_2$-$p_0$-$a_1$-$v$ can be extended to a 6-path via one of $v$-$p_2$-$p_3$ or $v$-$p_3$-$p_2$, a contradiction; so $v$ is nonadjacent to $p_2, p_3$, and $G$ contains $Q_7$. This proves (3).

From (1), (2) and (3), the result follows. ∎

Now we can prove 3.2, which we restate:

**4.2** *Let $G$ be a connected $\{P_6, K_3\}$-free graph without twins that contains $Q_5$. Then either*

- *$G$ is a $V_8$-expansion; or*
- *$G$ admits a nontrivial simplicial homogeneous pair.*

**Proof.** By 4.1, $G$ contains $Q_7$, and so there is a pair of disjoint sets $A_1, A_5 \subseteq V(G)$, both stable, with $|A_1|, |A_5| \geq 2$, such that $G[A_1 \cup A_5]$ is a connected bipartite graph and $A_1$ is not complete to $A_5$; and a 3-path with vertices $b_6$-$b_7$-$b_3$, such that $b_6$ is complete to $A_5$ (and consequently has no neighbours in $A_1$, since $G$ is triangle-free), and $b_7, b_3$ have no neighbours in $A_1, A_5$. (The reason for the subscript numbering will emerge later. Throughout we read subscripts modulo 8.) Choose $A_1, A_5$ maximal with this property. Define $A_2, A_4, A_6, A_8$ by:

- let $A_2$ be the set of all vertices of $G$ that are complete to $A_1 \cup \{b_3, b_6\}$;
- let $A_4$ be the set complete to $A_5 \cup \{b_3\}$;
- let $A_6$ be the set complete to $A_5 \cup \{b_7\}$ (thus, $b_6 \in A_6$); and
- let $A_8$ be the set complete to $A_1 \cup \{b_7\}$.

(1) *Every vertex of $G$ with a neighbour in $A_1 \cup A_5$ belongs to one of $A_1, A_2, A_4, A_5, A_6, A_8$.*

Let $v \in V(G) \setminus (A_1 \cup A_5)$ with a neighbour in $A_1 \cup A_5$. For $i = 1, 5$ let $A'_i$ be the set of neighbours of $v$ in $A_i$. Suppose first that $A_1 \neq A'_1 \neq \emptyset$. Since $G[A_1 \cup A_5]$ is connected, there is a vertex $a_5 \in A_5$ with a neighbour $a'_1 \in A'_1$ and a neighbour $a_1 \in A_1 \setminus A'_1$. Since the path $a_1$-$a_5$-$a'_1$-$v$ cannot be extended to a 6-path, it follows that $v$ is nonadjacent to $b_3, b_7$; and since $v$-$a'_1$-$a_5$-$b_6$-$b_7$-$b_3$ is not a 6-path, $v$ is adjacent to $b_6$. Consequently $A'_5 = \emptyset$, and $v$ can be added to $A_5$, contradicting the maximality of $A_5$. This proves that either $A'_1 = A_1$ or $A'_1 = \emptyset$.

Now suppose that $A_5 \neq A'_5 \neq \emptyset$, and consequently $A'_1 = \emptyset$ and $v$ is nonadjacent to $b_6$. From the maximality of $A_1$, $v$ is adjacent to one of $b_3, b_7$; but there exists $a_1 \in A_1$ with a neighbour $a'_5 \in A'_5$ and a neighbour $a_5 \in A_5 \setminus A'_5$, and the path $a_5$-$a_1$-$a'_5$-$v$ can be extended to a 6-path via $v$-$b_3$-$b_7$ or $v$-$b_7$-$b_3$, a contradiction. This proves that either $A'_5 = A_5$ or $A'_5 = \emptyset$.

Suppose that $A'_1 \neq \emptyset$, and so $A'_1 = A_1$ and $A'_5 = \emptyset$. If $v$ is adjacent to $b_7$ then $v \in A_8$, so we assume not. Now we must decide its adjacency to $b_6$ and $b_3$. If $v$ is nonadjacent to both $b_6, b_3$, there is a 6-path $v$-$a_1$-$a_5$-$b_6$-$b_7$-$b_3$ (where $a_1 \in A_1$ and $a_5 \in A_5$ are adjacent), a contradiction. If $v$ is adjacent to $b_6$ and not to $b_3$, then $v$ can be added to $A_5$, contrary to the maximality of $A_5$. If $v$ is adjacent to $b_3$ and not to $b_6$, choose $a_1 \in A_1$ and $a_5 \in A_5$ nonadjacent; then there is a 6-path $a_1$-$v$-$b_3$-$b_7$-$b_6$-$a_5$, a contradiction. Thus $v$ is adjacent to both $b_3, b_6$, and so $v \in A_2$, as required.

Finally suppose that $A'_5 \neq \emptyset$, and so $A'_5 = A_5$ and $A'_1 = \emptyset$, and $v$ is nonadjacent to $b_6$. From the maximality of $A_1$, $v$ is adjacent to one of $b_3, b_7$, and hence belongs to one of $A_4, A_6$. This proves (1).

Choose $A_3, A_7 \subseteq V(G)$ disjoint from $A_1, A_2, A_4, A_5, A_6, A_8$ and from each other, with $b_3 \in A_3$ and $b_7 \in A_7$, maximal such that

- $A_3 \cup A_7$ is anticomplete to $A_1 \cup A_5$;
- $A_7$ is complete to $b_6$; and
- the graph $G[A_3 \cup A_7]$ is connected.

(2) *For $1 \leq i \leq 8$, $A_i$ is stable, complete to $A_{i+1}$, and anticomplete to $A_{i+2}, A_{i+3}$.*

There are many pairs to check, and first we check the sets and pairs not involving $A_3, A_7$. $A_1$ is stable and complete to $A_2$ by definition of $A_2$, and anticomplete to $A_4$ since every vertex in $A_1$ has a neighbour in $A_5$ which is complete to $A_4$. $A_2$ is stable and anticomplete to $A_4$ since $A_2 \cup A_4$ is complete to $b_3$, and anticomplete to $A_5$ since $A_2$ is complete to $b_6$. $A_4$ is complete to $A_5$ by definition, and stable and anticomplete to $A_6$ since $A_4 \cup A_6$ is complete to $A_5 \neq \emptyset$. $A_5$ is stable and complete to $A_6$ by definition, and anticomplete to $A_8$ since $A_8$ is complete to $A_1$ and every vertex in $A_5$ has a neighbour in $A_1$. $A_6$ is stable and anticomplete to $A_8$ since $A_6 \cup A_8$ is complete to $b_7$; and $A_6$ is anticomplete to $A_1$ since every vertex in $A_1$ has a neighbour in $A_5$. $A_8$ is complete to $A_1$ by definition, and so stable and anticomplete to $A_2$. That does the sets and pairs not involving $A_3, A_7$.

Now $A_7$ is anticomplete to $A_5, A_1$ by definition, and stable since $A_7$ is complete to $b_6$. Also $A_3$ is anticomplete to $A_1, A_5$ by definition. Suppose that $A_3$ is not stable, let $u, v \in A_3$ be adjacent, and choose $a_7 \in A_7$ adjacent to $u$ (and hence not to $v$). Since $u, v$ have neighbours in $A_7$, they are nonadjacent to $b_6$. Choose $a_5 \in A_5$ and $a_1 \in A_1$ adjacent; then $v$-$u$-$a_7$-$b_6$-$a_5$-$a_1$ is a 6-path, a contradiction. Thus $A_3$ is stable.

Suppose that $A_6$ is not complete to $A_7$, and choose $a_6 \in A_6$ with a nonneighbour in $A_7$. Since $a_6$ is adjacent to $b_7$, and the graph $G[A_3 \cup A_7]$ is connected and $A_3$ is stable, there exists $a_3 \in A_3$ adjacent to a neighbour $a_7$ and a nonneighbour $a'_7$ of $a_6$ in $A_7$. Choose $a_1 \in A_1$ and $a_5 \in A_5$, adjacent; then $a_1$-$a_5$-$a_6$-$a_7$-$a_3$-$a'_7$ is a 6-path, a contradiction. Thus $A_6$ is complete to $A_7$. Similarly, since $A_8$ is complete to $b_7$, it follows that $A_8$ is complete to $A_7$; and similarly $A_3$ is complete to $A_2, A_4$. Since every vertex in $A_3$ has a neighbour in $A_7$, it follows that $A_3$ is anticomplete to $A_6 \cup A_8$, and similarly $A_7$ is anticomplete to $A_2 \cup A_4$. This proves (2).

(3) *$A_2$ is complete to $A_6$, and $A_4$ is complete to $A_8$.*

Choose $a_1 \in A_1$ and $a_5 \in A_5$, nonadjacent. If $a_2 \in A_2$ is nonadjacent to $a_6 \in A_6$, then $a_1$-$a_2$-$b_3$-$b_7$-$a_6$-$a_5$ is a 6-path, a contradiction. If $a_4 \in A_4$ is nonadjacent to $a_8 \in A_8$, $a_1$-$a_8$-$b_7$-$b_3$-$a_4$-$a_5$ is a 6-path, a contradiction. This proves (3).

Thus we have a certain symmetry between the sets $A_1, \dots, A_8$. It is not perfect, because $A_2, A_4, A_8$ might be empty (the other five sets are nonempty); and the bipartite graph between the pair $A_1, A_5$ contains a $P_4$, and between $A_2, A_6$ and $A_4, A_8$ it is complete. Thus for instance, if we prove something about $A_8$, we cannot deduce the same for $A_2$ (using the symmetry that exchanges $A_i$ with $A_{10-i}$ for $i = 2, 3, 4$) until we have proved that $A_4 \neq \emptyset$. Let $A_0$ be the set of vertices of $G$ not in any of the sets $A_1, \dots, A_8$.

(4) *$A_0$ is anticomplete to $A_1, A_3, A_5, A_7$.*

Let $v \in A_0$. It is anticomplete to $A_1 \cup A_5$ by (1); and anticomplete to $A_7$ since if not it could be added to $A_3$, contrary to the maximality of $A_3$. If $v$ has a neighbour $a_3 \in A_3$, choose $a_7 \in A_7$ adjacent to $a_3$, and choose $a_1 \in A_1$ and $a_5 \in A_5$, adjacent. From the maximality of $A_7$, $v$ is nonadjacent to $b_6$; and so $v$-$a_3$-$a_7$-$b_6$-$a_5$-$a_1$ is a 6-path, a contradiction. This proves (4).

(5) *We may assume that $A_8 \neq \emptyset$, and consequently the graph $G[A_1 \cup A_5]$ is an antisubmatching.*

If $A_2, A_8$ are both empty, then $(A_5, A_1)$ is a nontrivial pendant homogeneous pair. If $A_4, A_8$ are both empty, then $(A_1 \cup A_3, A_5 \cup A_7)$ is a nontrivial simplicial homogeneous pair. Thus we may assume that either $A_8 \neq \emptyset$ or both $A_2, A_4 \neq \emptyset$.

Suppose that $A_8 = \emptyset$; then $A_2, A_4 \neq \emptyset$, and we may relabel the sets, exchanging $A_i$ with $A_{10-i}$ for $i = 2, 3, 4$. Thus we may assume that $A_8 \neq \emptyset$. Suppose that there exist $a_1, a_1' \in A_1$ and $a_5, a_5' \in A_5$ such that $a_1, a_5$ are adjacent, but the other three pairs are nonadjacent. Choose $a_6 \in A_6$ and $a_8 \in A_8$. Then $a_1'$-$a_8$-$a_1$-$a_5$-$a_6$-$a_5'$ is a 6-path, a contradiction. Now suppose that some $a_1' \in A_1$ is nonadjacent to two vertices $a_5, a_5' \in A_5$. Since $a_5, a_5'$ are not twins, there is a vertex adjacent to exactly one of them, and by (1) it belongs to $A_1$; so we may assume there exists $a_1 \in A_1$ adjacent to $a_5$ and not to $a_5'$, contrary to what we just proved. Similarly every vertex in $A_5$ is nonadjacent to at most one vertex in $A_1$, and so $G[A_1 \cup A_5]$ is an antisubmatching. This proves (5).

(6) *For $i = 1, 3, 5, 7$, if $A_i$ is not complete to $A_{i+4}$, and $A_{i+1}, A_{i+3} \neq \emptyset$, then every vertex in $A_0$ is complete or anticomplete to $A_{i+1} \cup A_{i+3}$, and consequently $|A_{i+1}|, |A_{i+3}| = 1$.*

Suppose that $a_{i+1} \in A_{i+1}$ and $a_{i+3} \in A_{i+3}$, and $v \in A_0$ is adjacent to $a_{i+1}$ and not to $a_{i+3}$. Choose $a_{i+4}, a'_{i+4} \in A_{i+4}$ and $a_i \in A_i$ such that $a_i$ is adjacent to $a_{i+4}$ and not to $a'_{i+4}$. (This is possible since $A_i$ is not complete to $A_{i+4}$ and $G[A_i \cup A_{i+4}]$ is connected.) Then $v$-$a_{i+1}$-$a_i$-$a_{i+4}$-$a_{i+3}$-$a'_{i+4}$ is a 6-path, a contradiction. Similarly if $v \in A_0$ has a neighbour in $A_{i+3}$ then it is complete to $A_{i+1}$. Thus $v$ is complete or anticomplete to $A_{i+1} \cup A_{i+3}$. In particular, $A_{i+1}$ is a homogeneous set, and so $|A_{i+1}| = 1$, and the same for $A_{i+3}$. This proves (6).

In particular, from (6) it follows that $|A_6| = |A_8| = 1$, and every vertex in $A_0$ is complete or anticomplete to $A_6 \cup A_8$. Let $A_8 = \{b_8\}$. Let $C_7$ be the set of vertices in $A_0$ that are complete to $A_6 \cup A_8$, let $C_3$ be the set of vertices in $A_0$ with a neighbour in $A_2 \cup A_4$, and let $C_0 = A_0 \setminus (C_3 \cup C_7)$. (It follows that $C_3 \cap C_7 = \emptyset$, since every vertex in $A_2 \cup A_4$ has a neighbour in $A_6 \cup A_8$.) Once again, we mention that $A_6, A_8$ are nonempty, but $A_2, A_4$ may be empty.

(7) *We may assume that $A_4 \neq \emptyset$, and consequently $C_0 = \emptyset$, and $|A_3| = |A_7| = 1$.*

For suppose that $A_2, A_4 = \emptyset$, and so $C_3 = \emptyset$. We claim that $C_0$ is stable; for if not, there is a 3-path $p_1$-$p_2$-$p_3$ where $p_3 \in C_7$ and $p_1, p_2 \in C_0$, and this extends to a 6-path via $p_3$-$b_6$-$a_5$-$a_1$, choosing $a_5 \in A_5$ and $a_1 \in A_1$ adjacent. Thus $C_0$ is stable, and so $(C_7 \cup A_7, C_0 \cup A_3)$ is a pendant homogeneous pair; we may assume it is not nontrivial, and hence $A_0 = \emptyset$, and $|A_3| = |A_7| = 1$. But then $G$ is a $V_8$-expansion.

Thus we may assume that one of $A_2, A_4 \neq \emptyset$, and from the symmetry (exchanging $A_8$ with $A_6$, and exchanging $A_i$ with $A_{6-i}$ for $i = 1, 2$) we may assume that $A_4 \neq \emptyset$. If there is a vertex in $C_0$, then since $G$ is connected, there is an edge $uv$ with $u \in C_0$ and $v \in C_3 \cup C_7$. But this extends to a 6-path, as follows. Choose $w \in A_2 \cup A_4 \cup A_6 \cup A_8$ adjacent to $v$, with $w = b_6$ if possible. It follows that $w \neq b_8$, by (6). If $w = b_6$, choose $a_4 \in A_4$; then $u$-$v$-$w$-$b_7$-$b_3$-$a_4$ is a 6-path. Similarly if $w \in A_4$, $u$-$v$-$w$-$b_3$-$b_7$-$b_6$ is a 6-path; and if $w \in A_2$, $u$-$v$-$w$-$b_3$-$b_7$-$b_8$ is a 6-path. Thus $C_0 = \emptyset$.

Now suppose that $A_3$ is not complete to $A_7$. Then by (6), $|A_4| = 1$, and every vertex in $A_0$ is complete or anticomplete to $A_4 \cup A_6$, and therefore to $A_4 \cup A_6 \cup A_8$. No vertex is complete to $A_4 \cup A_8$ since this set is not stable, and therefore every vertex in $A_0$ is anticomplete to $A_4 \cup A_6 \cup A_8$. Similarly they are all anticomplete to $A_2$, so $A_0 = \emptyset$. Moreover $|A_2| \leq 1$ (because there are no twins), and so $G$ is a $V_8$-expansion. We may therefore assume that $A_3$ is complete to $A_7$. Consequently $A_3$ is a homogeneous set, so $|A_3| = 1$, and the same for $A_7$. This proves (7).

(8) *$C_3$ is complete to $C_7$, and $|C_7| \leq 1$.*

Suppose that $c_3 \in C_3$ and $c_7 \in C_7$ are nonadjacent. If there exists $a_4 \in A_4$ adjacent to $c_3$, then $c_3$-$a_4$-$b_3$-$b_7$-$b_6$-$c_7$ is a 6-path; and similarly if there exists $a_2 \in A_2$ adjacent to $c_3$ then $c_3$-$a_2$-$b_3$-$b_7$-$b_8$-$c_7$ is a 6-path, in both cases a contradiction. Thus $C_7$ is complete to $C_3$, so $C_7$ is a homogeneous set, and therefore $|C_7| \leq 1$. This proves (8).

If $C_3$ is not complete to $A_4$, then $A_2 = \emptyset$ by (6); choose $c_3 \in C_3$ and $a_4 \in A_4$, nonadjacent. Since we may assume that $(A_4, C_3)$ is not a nontrivial pendant homogeneous pair, it follows that $C_7 \neq \emptyset$; let $C_7 = \{c_7\}$. But then $c_3$-$c_7$-$b_6$-$b_7$-$b_3$-$a_4$ is a 6-path. Thus $C_3$ is complete to $A_4$. Hence $|C_3| \leq 1$, and $|A_4| = 1$, and similarly $|A_2| \leq 1$. But then the graph $G[C_3 \cup C_7 \cup \{b_3, b_7\}]$ is an antisubmatching, relative to the bipartition $(C_7 \cup \{b_7\}, A_3 \cup C_3)$; and so $G$ is a $V_8$-expansion. This proves 4.2. ∎

# 5 Bipartite $Q_5$-free graphs

In view of 4.2 and 4.1, it suffices to understand $\{Q_5, K_3\}$-free graphs, and that is the content of the remainder of the paper. In this section we handle the bipartite case, which is very different from the non-bipartite case. In particular, $Q_5$ is isomorphic to its own bipartite complement, and so we should expect to prove something invariant under taking bipartite complements. (We recall that the *bipartite complement* of a graph $G$ with a given bipartition $(A, B)$ is the graph $H$ with the same bipartition $(A, B)$, in which for $a \in A$ and $b \in B$, $a, b$ are adjacent in exactly one of $G, H$.)

An *induced matching* in $G$ is a set $M$ of edges of $G$, pairwise vertex-disjoint, such that for all distinct $e, f \in M$, no end of $e$ is adjacent in $G$ to an end of $f$. We recall that $V_M$ denotes the set of vertices incident with edges in $M$, and $\mu(G)$ denotes the cardinality of the largest induced matching in $G$. We remark, and leave the reader to prove, that:

**5.1** *If $G$ is $\{Q_5, K_3\}$-free, and $M$ is an induced matching in $G$, then every vertex in $V(G) \setminus V_M$ with a neighbour in $V_M$ has a neighbour in each edge of $M$.*

Let $(A, B)$ be a bipartition of a graph $G$, and let $A' \subseteq A$ and $B' \subseteq B$. We say the pair $(A', B')$ is *matched* if every vertex in $A'$ has exactly one neighbour in $B'$ and vice versa; and *antimatched* if every vertex in $A'$ has exactly one nonneighbour in $B'$ and vice versa. If $(A', B')$ is matched or antimatched, it follows that $|A'| = |B'|$.

A *half-graph* is a bipartite graph with no induced two-edge matching; or equivalently, a graph that admits a bipartition $(\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\})$ such that for all $i, i'$ with $1 \leq i \leq i' \leq m$ and all $j, j'$ with $1 \leq j \leq j' \leq n$, if $a_i, b_j$ are adjacent then $a_{i'}, b_{j'}$ are adjacent. (This equivalence is well-known, but here is a proof that the first implies the second. Let $(A, B)$ be a bipartition of a bipartite graph with no induced two-edge matching, and choose $a \in A$ with maximum degree; then all its nonneighbours in $B$ have degree zero. This proves that if $A \neq \emptyset$ then either some vertex in $B$ has degree zero, or some vertex in $A$ is adjacent to all of $B$; and the claim follows by induction on $|V(G)|$.) We need an extension of this.

We say a graph $G$ is a *half-graph expansion* if there is a bipartition $(A, B)$ of $G$, and partitions $(A_1, \ldots, A_n)$ of $A$ and $(B_1, \ldots, B_n)$ of $B$ (where some of $A_1, \ldots, A_n, B_1, \ldots, B_n$ may be empty), with the following properties:

- for $1 \leq i < j \leq n$, $A_i$ is complete to $B_j$, and $A_j$ is anticomplete to $B_i$;
- for $1 \leq i \leq n$, either $|A_i \cup B_i| = 1$ or $(A_i, B_i)$ is matched or antimatched.

It is easy to see that every half-graph expansion is $Q_5$-free. We prove the converse:

**5.2** *Every $Q_5$-free bipartite graph is a half-graph expansion.*

**Proof.** Let $G$ be $Q_5$-free, and let $(A, B)$ be a bipartition of $G$. Let us say a *box* is a maximal subset $C \subseteq V(G)$ such that either $|C| = 1$, or $|C| \geq 4$ and $(A \cap C, B \cap C)$ is matched or antimatched.

(1) *Every two distinct boxes are disjoint.*

Let $C, C'$ be distinct boxes, and suppose that $C \cap C' \neq \emptyset$. If $|C| = 1$ then $C \subseteq C'$, and so $C = C'$ from the maximality of $C$, a contradiction. Consequently $|C|, |C'| \geq 4$, and we may assume that $|C| \leq |C'|$. Since $G$ is $Q_5$-free, so is its bipartite complement, and since what we want to prove is invariant under taking bipartite complements, we may assume (taking bipartite complements if necessary) that $(A \cap C, B \cap C)$ is matched. Let $A \cap C = \{a_1, \ldots, a_k\}$ and $B \cap C = \{b_1, \ldots, b_k\}$ where $a_i, b_j$ are adjacent if and only if $i = j$. It follows that $k \geq 2$.

We may assume that $C' \cap C \cap B \neq \emptyset$; let $b_k \in C'$ say. Suppose that $(C' \setminus C) \cap B$ is also nonempty, and choose $d \in (C' \setminus C) \cap B$.

Suppose first that $d$ is nonadjacent to $a_k$. Since $d \notin C$, 5.1 implies that $d$ is anticomplete to $\{a_1, \ldots, a_k\}$. Since $(A \cap C', B \cap C')$ is matched or antimatched, there exists $c \in A \cap C'$ adjacent to $d$ and not to $b_k$. Consequently $c \notin \{a_1, \ldots, a_k\}$, and so again 5.1 implies that $c$ is anticomplete to $\{b_1, \ldots, b_k\}$. But then $((A \cap C) \cup \{c\}, (B \cap C) \cup \{d\})$ is matched, contradicting that $C$ is a box.

This proves that $d, a_k$ are adjacent. By 5.1 $d$ is complete to $\{a_1, \ldots, a_k\}$. Since $(A \cap C', B \cap C')$ is matched or antimatched, there exists $c \in A \cap C'$ adjacent to $b_k$ and not to $d$. Consequently

$c \notin \{a_1, \dots, a_k\}$, and 5.1 implies that $c$ is complete to $\{b_1, \dots, b_k\}$. If $k \geq 3$, the 3-path $a_1$-$d$-$a_2$ misses the edge $cb_3$, a contradiction. So $k = 2$, and $(\{a_1, a_2, c\}, \{b_1, b_2, d\})$ is antimatched, contradicting that $C$ is a box.

We have shown that $(C' \setminus C) \cap B = \emptyset$; and since $|C| \leq |C'|$, it follows that $C \cap B = C' \cap B$. Since $C \neq C'$, some vertex $v \in C' \cap A$ does not belong to $C$; but then by 5.1 $v$ is complete or anticomplete to $C \cap B$, contradicting that $(A \cap C', B \cap C')$ is matched or antimatched and $|B \cap C'| = |C'|/2 \geq 2$. This proves (1).

(2) *If $C, C'$ are distinct boxes, then either*

- *$C \cap A$ is complete to $C' \cap B$ and $C \cap B$ is anticomplete to $C' \cap A$, or*
- *$C \cap A$ is anticomplete to $C' \cap B$ and $C \cap B$ is complete to $C' \cap A$.*

By 5.1, applied to the bipartite complement of $G$ if necessary (and trivially if $|C'| = 1$), it follows that every vertex in $A \cap C$ is complete or anticomplete to $B \cap C'$; and similarly every vertex in $B \cap C'$ is complete or anticomplete to $A \cap C$. Consequently $A \cap C$ is complete or anticomplete to $B \cap C'$, and similarly $B \cap C$ is complete or anticomplete to $A \cap C'$. Thus, by taking bipartite complements if necessary, we may assume that $A \cap C$ is complete to $B \cap C'$, and $|C|, |C'| \geq 4$. Suppose that $B \cap C$ is complete to $A \cap C'$. If both $(A \cap C, B \cap C)$ and $(A \cap C', B \cap C')$ are antimatched, then $(A \cap (C \cup C'), B \cap (C \cup C'))$ is antimatched, contradicting that $C$ is a box. Thus, exchanging $C, C'$ if necessary, we may assume that $(A \cap C, B \cap C)$ is not antimatched, and so $|C| \geq 6$ and $(A \cap C, B \cap C)$ is matched. Choose $a' \in A \cap C'$ and $b' \in B \cap C'$, nonadjacent. Then there is a 3-path in $G[(A \cap C) \cup \{b'\}]$ that misses an edge of $G[(B \cap C) \cup \{a'\}]$, a contradiction. This proves (2).

Let $\mathcal{C}$ be the set of all boxes. Let $H$ be the digraph with vertex set $\mathcal{C}$, in which for all distinct $C, C' \in \mathcal{C}$, $C$ is adjacent from $C'$ (that is, $C'C \in E(H)$) if $C \cap A$ and $C' \cap B$ are complete to each other, and $C \cap B$ and $C' \cap A$ are anticomplete to each other, and either

- $C \cap A$ and $C' \cap B$ are both nonempty, or
- $C \cap B$ and $C' \cap A$ are both nonempty.

From (2), if $C, C' \in \mathcal{C}$ are distinct, and $C \cup C'$ intersects both $A, B$, then either $CC' \in E(H)$ or $C'C \in E(H)$.

(3) *$H$ has no directed cycles.*

Suppose it has, and take the shortest directed cycle $D$, with vertices $C_1$-$C_2$-$\cdots$-$C_k$-$C_1$ say, where $C_iC_{i+1} \in E(D)$ for $1 \leq i \leq k$ (reading subscripts modulo $k$). It follows from the minimality of $k$ that for all distinct $u, v \in V(D)$, if $uv, vu \notin E(D)$, then $uv, vu \notin E(H)$. Suppose first that $C_1 \cap A, C_1 \cap B \neq \emptyset$. Then by (2), one of $C_1C_3, C_3C_1 \in E(H)$, and so $k = 3$. Since $C_2C_3 \in E(H)$, either $C_2 \cap A$ and $C_3 \cap B$ are both nonempty, or $C_2 \cap B$ and $C_3 \cap A$ are both nonempty.

Suppose the first. Now $C_2 \cap A$ is complete to $C_3 \cap B$, since $C_2C_3 \in E(H)$; $C_1 \cap B$ is anticomplete to $C_2 \cap A$, since $C_1C_2 \in E(H)$; and $C_1 \cap A$ is anticomplete to $C_3 \cap B$, since $C_3C_1 \in E(H)$. But then an edge of $G$ between $C_1 \cap A$ and $C_1 \cap B$ misses an edge between $C_2 \cap A$ and $C_3 \cap B$, and so these two edges make an induced matching in $G$ whose vertex set is not included in a box, a contradiction.

Now suppose that $C_2 \cap B$ and $C_3 \cap A$ are both nonempty. Then $C_2 \cap B$ is anticomplete to $C_3 \cap A$, since $C_2C_3 \in E(H)$; $C_1 \cap A$ is complete to $C_2 \cap B$, since $C_1C_2 \in E(H)$; and $C_1 \cap B$ is complete to $C_3 \cap A$, since $C_3C_1 \in E(H)$. But then an edge of $G$ between $C_1 \cap A$ and $C_2 \cap B$ misses a (carefully chosen) edge between $C_3 \cap A$ and $C_1 \cap B$, again a contradiction. This proves that each of $C_1, \dots, C_k$ is a subset of one of $A, B$.

Consequently $k$ is even, and we may assume that $C_i \subseteq A$ for all odd $i$, and $C_i \subseteq B$ for all even $i$. Hence one of $C_1C_4, C_4C_1 \in E(H)$, from (2), and so $k = 4$. Since $C_1C_2 \in E(H)$, $C_1$ is complete to $C_2$, and similarly $C_3$ is complete to $C_4$. Since $C_2C_3 \in E(H)$, $C_2$ is anticomplete to $C_3$, and similarly $C_4$ is anticomplete to $C_1$. But then an edge of $G$ between $C_1, C_2$ misses an edge between $C_3, C_4$, a contradiction. This proves (3).

From (3), we may number $\mathcal{C}$ as $\{C_1, \dots, C_n\}$ such that for all distinct $i, j \in \{1, \dots, n\}$ with $i < j$, $C_i \cap A$ is complete to $C_j \cap B$, and $C_i \cap B$ is anticomplete to $C_j \cap A$. Setting $A_i = C_i \cap A$ and $B_i = C_i \cap B$ for each $i$, we deduce that $G$ is a half-graph expansion. This proves 5.2. ∎

What about bipartite $P_6$-free graphs? Let $G$ be a bipartite graph. We say $G$ is an *antimatching recursion* if $G$ has no twins, and either

- $G$ is an antimatching; or
- $G$ is disconnected, and each of its components is an antimatching recursion; or
- there is a vertex $v$ such that the set of nonneighbours of $v$ is stable, and the graph obtained by deleting $v$ is an antimatching recursion; or
- $G$ admits a nontrivial simplicial homogeneous pair $(A, B)$, such that if $J$ denotes $G[A \cup B]$ and $H$ is the graph obtained from $G$ by identifying the vertices in $A$ and identifying the vertices in $B$, then both $H, J$ are antimatching recursions.

We have:

**5.3** *Let $G$ be a bipartite graph without twins. Then $G$ is $P_6$-free if and only if $G$ is an antimatching recursion.*

**Proof.** It is easy to check that all antimatching recursions are $P_6$-free, and so we just need to prove the converse. Let $G$ be a $P_6$-free bipartite graph without twins. We prove by induction on $|V(G)|$ that $G$ is an antimatching recursion. We may assume that $|V(G)| \geq 3$, and $G$ is connected.

If there is a vertex $v$ such that set of nonneighbours of $v$ is stable, then the graph obtained by deleting $v$ has no twins and so is an antimatching recursion from the inductive hypothesis, and hence so is $G$. We assume then that there is no such vertex $v$.

Suppose that $G$ admits a nontrivial simplicial homogeneous pair $(A, B)$, and choose $(A, B)$ with $|A \cup B|$ maximal. Let $J$ denote $G[A \cup B]$ and let $H$ be the graph obtained from $G$ by identifying the vertices in $A$ into one vertex $a$ say, and identifying the vertices in $B$ into one vertex $b$. Let $C, D$ be respectively the sets of vertices in $G$ that are not in $A \cup B$ but are complete to $A$, and complete to $B$, respectively. Then $J$ has no twins, and so is an antimatching recursion from the inductive hypothesis. If the same holds for $H$ then the result follows, so we may assume that $H$ has twins, and consequently we may assume that there is some vertex $d \in D$, with neighbour set $B \cup C$ (so $d$

and $a$ are twins in $H$). From the maximality of $A \cup B$, $(A \cup \{d\}, B)$ is not a nontrivial simplicial homogeneous pair, and so $V(G) = A \cup B \cup \{d\}$; but then the set of nonneighbours of $d$ in $G$ is stable, a contradiction.

Thus $G$ has no nontrivial simplicial homogeneous pair. Let $(A, B)$ be a bipartition. We have shown that every vertex in $A$ has a nonneighbour in $B$, and vice versa.

Suppose that $G$ is a $V_8$-expansion. With $a_2, a_4, a_6, a_8$ as in the definition of $V_8$-expansion, it follows since $G$ is bipartite that either $a_2, a_6$ are both deleted, or $a_4, a_8$ are both deleted; and we assume the first without loss of generality. Since $G$ is connected, not both $a_4, a_8$ are deleted; so we may assume that $a_4 \in A$ say. But then $a_4$ is complete to $B$, a contradiction.

Thus $G$ is not a $V_8$-expansion, and so $G$ is $Q_5$-free, by 4.2. By 5.2, there are partitions $(A_1, \dots, A_n)$ of $A$ and $(B_1, \dots, B_n)$ of $B$ (where some of $A_1, \dots, A_n, B_1, \dots, B_n$ may be empty), such that:

- for $1 \le i < j \le n$, $A_i$ is complete to $B_j$, and $A_j$ is anticomplete to $B_i$;
- for $1 \le i \le n$, either $|A_i \cup B_i| = 1$ or $(A_i, B_i)$ is matched or antimatched.

We may assume that $A_i \cup B_i \neq \emptyset$ for $1 \le i \le n$.

If $B_1 = \emptyset$ then $A_1 \neq \emptyset$, and every vertex in $A_1$ is complete to $B$, a contradiction. Thus $B_1 \neq \emptyset$. Since $|V(G)| \ge 2$ and $G$ is connected, it follows that $A_1 \neq \emptyset$. Every vertex in $A_1$ has a nonneighbour in $B_1$, and every vertex in $B_1$ has a neighbour in $A_1$; and so $|A_1|, |B_1| \ge 2$. If $A_1 \cup B_1 \neq V(G)$, then $(A_1, B_1)$ is a nontrivial simplicial homogeneous pair, a contradiction; so $A_1 \cup B_1 = V(G)$. Now $(A_1, B_1)$ is matched or antimatched, and not matched since $G$ is connected. Consequently $G$ is an antimatching, and hence an antimatching recursion. This proves 5.3. ∎

# 6 Using a maximum induced matching

The goal of the remainder of the paper is to prove the following:

**6.1** *Let $G$ be connected and $\{Q_5, K_3\}$-free, with no twins. Then either:*

- *$G$ is Clebschian, climbable, or bipartite; or*
- *$G$ admits a nontrivial simplicial homogeneous pair.*

Incidentally, in the remainder of the paper, there are a number of places where we claim that an explicitly-given graph is Clebschian, and we leave the reader to verify this. Here is the method we used, which we found quite practical. To verify that some triangle-free graph $G$ (with a 5-cycle) is Clebschian, it suffices to do the following:

- select any 5-cycle $C$ of $G$; label its vertices $v_1, \dots, v_5$ in order (we read subscripts modulo 5);
- for each $v \in V(G) \setminus V(C)$, if $v$ has a unique neighbour $v_i \in V(C)$, label $v = v_i'$;
- if some vertex of $G$ has no neighbour in $C$, label it $\infty$;
- check that at most one vertex receives any label;
- check that if $v_i', v_j'$ both exist then they are adjacent if and only if $j - i \in \{2, 3\}$ (modulo 5);

- check that the unlabelled vertices are pairwise nonadjacent, and each is adjacent to the vertex labelled $\infty$ if it exists;
- check that every unlabelled vertex is adjacent to two nonconsecutive vertices $v_{i-1}, v_{i+1}$ of $C$ and to $v'_{i-2}, v'_{i+2}$ if they exist, and not to $v'_i$ if that exists; and that no two unlabelled vertices are adjacent to the same pair of vertices of $C$.

If all these checks succeed then $G$ is Clebschian, and otherwise it is not. The point is, because of the symmetry of the Clebsch graph, it is not necessary to repeat this for every 5-cycle; doing it for one 5-cycle is already enough.

Let us set up some notation before we go on. Let $M = \{a_1b_1, \dots, a_nb_n\}$ be an induced matching in a graph $G$. We recall the key observation that, by 5.1, if $G$ is $\{Q_5, K_3\}$-free, then every vertex in $V(G) \setminus V_M$ with a neighbour in $V_M$ has a neighbour in each edge of $M$. If $u, v \in N_M$, $\delta_M(u, v)$ denotes the number of $i \in \{1, \dots, n\}$ such that $u$ is adjacent to one of $a_i, b_i$ and $v$ is adjacent to the other. There is a second key observation:

**6.2** *Let $G$ be $\{Q_5, K_3\}$-free, let $M$ be an induced matching in $G$, and let $u, v \in N_M$ be distinct.*

- *If $u, v$ are adjacent then $\delta_M(u, v) = |M|$.*
- *If $u, v$ are nonadjacent then $\delta_M(u, v) \leq 2$.*

**Proof.** Each of $u, v$ has a neighbour in every edge of $M$, by 5.1. If $u, v$ are adjacent then they have no common neighbour in $V_M$ since $G$ is triangle-free, so the first statement is true. If $u, v$ are nonadjacent, let $M = \{a_1b_1, \dots, a_nb_n\}$, and suppose that $u$ is adjacent to $a_1, a_2, a_3$ and $v$ to $b_1, b_2, b_3$. Then $a_1$-$u$-$a_2$ misses $vb_3$, a contradiction. This proves 6.2. ∎

The case when $\mu(G) \leq 1$ is exceptional (and easy) so let us get that out of the way.

**6.3** *Let $G$ be connected and $K_3$-free, with no twins and with $\mu(G) \leq 1$. Then either $G$ is a half-graph, or $G$ is a cycle of length five.*

**Proof.** If $G$ is bipartite, it is a half-graph and the theorem holds, so we assume not; and consequently $G$ has an induced 5-cycle $C$, with vertices $v_1$-$v_2$-$\cdots$-$v_5$-$v_1$ in order. Let $W = V(G) \setminus V(C)$. Let $P$ be the set of vertices in $W$ with a neighbour in $V(C)$, and $Q = W \setminus P$. Let $v \in P$; then $v$ has at most two neighbours in $V(C)$, since $G$ is triangle-free; and $v$ does not have just one neighbour in $V(C)$ since $\mu(G) = 1$. Consequently $v$ has exactly two neighbours in $V(C)$, and they are nonadjacent. For $1 \leq i \leq 5$ let $P_i$ be the set of vertices in $P$ adjacent to $v_{i-1}$ and $v_{i+1}$ (reading subscripts modulo 5). For $1 \leq i \leq 5$, $P_i$ is complete to $P_{i+1}$ (since $\mu(G) = 1$), and $P_i$ is anticomplete to $P_{i+2}$ (since $G$ is triangle-free). Moreover, no vertex in $Q$ has a neighbour in $P$ (since $\mu(G) = 1$), and so $Q = \emptyset$ (since $G$ is connected). Since $G$ has no twins, it follows that $P = \emptyset$. This proves 6.3. ∎

Incidentally, this yields a result we mentioned before – how to construct all $\{P_5, K_3\}$-free graphs.

**6.4** *A graph is $\{P_5, K_3\}$-free if and only if each of its components is either a half-graph or obtained from a cycle of length five by adding twins.*

**Proof.** It is easy to see that a graph $G$ is $\{P_5, K_3\}$-free if and only if each of its components $G'$ is $K_3$-free and satisfies $\mu(G') \leq 1$; and by 6.3, this is true if and only if every component is either a half-graph or obtained from a cycle of length five by adding twins. ∎

## 7 Matchings with stable neighbour sets

The goal of this section is to handle the case when there is a maximum induced matching $M$ such that $N_M$ is stable.

**7.1** *Let $G$ be $\{Q_5, K_3\}$-free, and let $M$ be a maximum induced matching in $G$. If $N_M$ is stable, then $M$ can be labelled $\{a_1b_1, \ldots, a_nb_n\}$ such that either:*

- *every member of $N_M$ is adjacent to at least $n-1$ of $a_1, \ldots, a_n$; or*
- *$n \geq 2$, and $N_M$ is complete to $\{a_3, \ldots, a_n\}$, and each of the pairs $(a_1, a_2), (a_1, b_2), (b_1, a_2)$, $(b_1, b_2)$ has a common neighbour in $N_M$; or*
- *$n \geq 3$, and $N_M$ is complete to $\{a_4, \ldots, a_n\}$, and every member of $N_M$ is adjacent to an odd number of $a_1, a_2, a_3$; and each of the triples $(a_1, a_2, a_3), (a_1, b_2, b_3), (b_1, a_2, b_3), (b_1, b_2, a_3)$ has a common neighbour in $N_M$.*

**Proof.** By 6.2, $\delta_M(u, v) \leq 2$ for all $u, v \in N_M$. If $N_M = \emptyset$ the claim is trivial, so we assume there exists $w \in N_M$. Choose $w \in N_M$ such that $\max_{v \in N_M} \delta_M(w, v)$ is as small as possible. Let $M = \{a_1b_1, \ldots, a_nb_n\}$, labelled such that $w$ is adjacent to $a_i$ for each $i$.

(1) *We may assume that for each $u \in N_M$ there exists $v \in N_M$ with $\delta_M(u, v) = 2$.*

If $\max_{v \in N_M} \delta_M(w, v) \leq 1$ then the first bullet holds, so we assume this maximum equals two. Hence, from the choice of $w$, this proves (1).

Suppose that there exists $u \in N_M$ with $\delta_M(u, w) = 1$. From (1), there exists $v \in N_M$ with $\delta_M(u, v) = 2$, and hence with $\delta_M(v, w)$ odd, and therefore with $\delta_M(v, w) = 1$; and by (1) again, there exists $x \in N_M$ with $\delta_M(w, x) = 2$ and hence with $\delta_M(u, x) = \delta_M(v, x) = 1$. If $y \in N_M$, then since $\delta_M(y, z) \leq 2$ for each $z \in \{u, v, w, x\}$, it follows that $\delta_M(y, z) = 0$ for some $z \in \{u, v, w, x\}$ and the second bullet holds.

So we may assume that $\delta_M(u, w) = 0$ or 2, for each $u \in N_M$. Let $H$ be the graph with vertex set $\{1, \ldots, n\}$, in which distinct $i, j$ are adjacent if there exists $u \in N_M$ adjacent to $b_i$ and to $b_j$. Every two edges of $H$ have a common end in $H$. If some vertex $i$ of $H$ belongs to all edges of $H$, then the first bullet holds (with $a_i, b_i$ exchanged), so we may assume that $H$ has a triangle, and hence $H$ has only three edges. But then the third bullet holds. This proves 7.1. ∎

Let us say a *leaf* is a vertex of degree one; and an edge $e$ such that neither end of $e$ is a leaf is *internal*. In the remainder of this section we handle separately the three cases of the output of 7.1. The first case itself devolves into three subcases: when $\mu(G) \leq 1$ (which we have already handled), when $\mu(G) \geq 2$ but at most one edge of $M$ is internal, and when $\mu(G) \geq 2$ and at least two of its edges are internal.

**7.2** *Let $G$ be connected and $\{Q_5, K_3\}$-free, with $\mu(G) \geq 2$, and let $M$ be a maximum induced matching in $G$; moreover, if $\mu(G) = 2$, assume that $G$ has no twins and $M$ is chosen such that, if possible, both its edges are internal. If there is at most one edge in $M$ that is internal, then either*

- *$G$ is Clebschian; or*
- *$G$ admits a nontrivial pendant homogeneous pair.*

**Proof.** Let $M = \{a_1b_1, \ldots, a_nb_n\}$, where $n = \mu(G) \geq 2$ and $b_2, \ldots, b_n$ are leaves of $G$. If either $n \geq 3$ or one of $a_1, b_1$ is a leaf then $G$ admits a nontrivial pendant homogeneous pair. Consequently we may assume that $n = 2$ and $a_1b_1$ is internal; and from the choice of $M$, there is no induced matching $M'$ in $G$ of cardinality two such that both its edges are internal. Let $A$, $B$ be the sets of all vertices in $N_M$ adjacent to $a_1$ and to $b_1$ respectively. Thus $A, B \neq \emptyset$, and by 5.1 $N_M = A \cup B$ and $A \cap B = \emptyset$, and $a_2$ is complete to $A \cup B$, and so $A \cup B$ is stable. Let $R = V(G) \setminus (A \cup B \cup V_M)$. Thus $R$ is anticomplete to $V_M$, and hence stable, from the maximality of $M$. Since $G$ is connected, every vertex in $R$ has a neighbour in $A \cup B$.

Let $D$ be the set of all vertices in $R$ with neighbours in both $A$ and $B$, and suppose that $v \in D$. If $v$ is not complete to one of $A, B$, say $B$, choose $b \in B$ nonadjacent to $v$, and choose $a \in A$ adjacent to $v$; then the edges $va, b_1b$ form a 2-edge induced matching $M'$ and no vertex in $V(M')$ is a leaf of $G$, a contradiction. So $v$ is complete to $A \cup B$; and so $(\{v, a_2\}, \{b_2\})$ is a nontrivial pendant homogeneous pair. Hence we may assume that $D = \emptyset$.

Let $A'$ be the set of all vertices in $R$ with neighbours in $A$ and with none in $B$, and define $B'$ similarly. If both $A', B' \neq \emptyset$, there is a 3-path of $G[A' \cup A \cup \{a_1\}]$ that is anticomplete to an edge of $G[B' \cup B]$, a contradiction. Thus we may assume that $B' = \emptyset$, and so $|B| = 1$, since $G$ has no twins. If $A' = \emptyset$, then $|A| = 1$ since $G$ has no twins, and so $G$ is Clebschian. Thus we assume that $A' \neq \emptyset$, and so we may assume that $|A| = |A'| = 1$, since $(A, A')$ is a pendant homogeneous pair, and again $G$ is Clebschian. This proves 7.2. ∎

**7.3** *Let $G$ be $\{Q_5, K_3\}$-free, let $M$ be a maximum induced matching in $G$, and suppose that at least two edges of $M$ are internal. Suppose that $N_M$ is stable, and $M$ is labelled $\{a_ic_i : 1 \leq i \leq n\}$ such that every member of $N_M$ is adjacent to at least $n-1$ of $a_1, \ldots, a_n$. Then $G$ is climbable.*

**Proof.** First we prove:

(1) *For $1 \leq i \leq n$, there is at most one vertex in $N_M$ nonadjacent to $a_i$.*

For suppose that $u, v \in N_M$ are both nonadjacent to $a_1$ say. Since some edge of $M$ different from $a_1c_1$, say $a_2c_2$, is internal, there exists $w \in N_M$ adjacent to $c_2$, and hence nonadjacent to $c_1$, since $w$ is adjacent to at least $n-1$ vertices in $\{a_1, \ldots, a_n\}$. For the same reason $u, v$ are nonadjacent to $c_2$. But $u$-$c_1$-$v$ is a 3-path by 5.1, and it misses the edge $wc_2$, a contradiction. This proves (1).

For each $v \in N_M$, if there exists $i \in \{1, \ldots, n\}$ such that $v$ is nonadjacent to $a_i$, define $b_i = v$. (This is well-defined by (1).) Thus $b_i$ is defined if and only if $c_i$ is not a leaf of $G$. We may assume that $b_i$ is defined for $1 \leq i \leq m$ and not defined for $m+1 \leq i \leq n$.

Let $P$ be the set of vertices in $N_M$ that are complete to $\{a_1, \ldots, a_n\}$, that is, $N_M \setminus \{b_1, \ldots, b_m\}$. Number $P$ as $b_{n+1}, \ldots, b_t$ say. Let $Q$ be the set of vertices in $V(G) \setminus (V_M \cup N_M)$ that have a neighbour in $\{b_1, \ldots, b_m\}$, and $R$ the remainder, that is, $R = V(G) \setminus (V_M \cup N_M \cup Q)$.

(2) *$Q \cup R$ is stable, and $Q$ is complete to $\{b_1, \ldots, b_m\}$.*

Certainly $Q \cup R$ is stable, from the maximality of $M$. Suppose that $v \in Q$ is nonadjacent to $b_1$ say. Since $v$ has a neighbour in $\{b_1, \ldots, b_m\}$, we may assume that $v$ is adjacent to $b_2$. But then $v$-$b_2$-$a_1$ is a 3-path that misses the edge $a_2b_1$, a contradiction. This proves (2).

(3) *Every vertex in $Q$ has at most one nonneighbour in $P$, and no two vertices in $Q$ have the same nonneighbour in $P$.*

Suppose that $v \in Q$ is nonadjacent to $x, y \in P$; then $x$-$a_1$-$y$ is a 3-path missing the edge $v$-$b_1$. Now suppose that $u, v \in Q$ are both nonadjacent to $x \in P$; then $u$-$b_1$-$v$ is a 3-path missing the edge $xa_1$. This proves (3).

For each $v \in Q$, if $v$ is nonadjacent to some (unique by (3)) $b_i \in P$, define $a_i = v$, and if $v$ is complete to $P$, choose some (distinct for all $v \in Q$) integer $i > t$, and define $a_i = v$.

(4) *Every vertex in $R$ has at most one neighbour in $P$; and no two vertices in $R$ have the same neighbour in $P$.*

Suppose that $v \in R$ is adjacent to $x, y \in P$, where $x \neq y$. Now $x, y$ are nonadjacent to $c_1$ since they are adjacent to $a_1$, and they are nonadjacent to $b_1$ since $N_M$ is stable. Moreover, $v$ is nonadjacent to $c_1$ since $v \notin N_M$, and $v$ is nonadjacent to $b_1$ since $v \notin Q$. Hence $x$-$v$-$y$ is a 3-path, and misses the edge $b_1c_1$, a contradiction. Now suppose that $u, v \in R$ are adjacent to $x \in P$. Then similarly $u$-$x$-$v$ is a 3-path missing the edge $b_1c_1$, a contradiction. This proves (4).

(5) *For $i > m$, if $v \in R$ is adjacent to $b_i$ then $a_i$ is not defined.*

Suppose that $a_i, b_i$ are defined and $v \in R$ is adjacent to $b_i$. Then $v$ is nonadjacent to $b_1, b_2$, since $v \notin Q$, and so $b_1$-$a_i$-$b_2$ is a 3-path missing the edge $vb_i$. This proves (5).

For each $v \in R$ with a (unique) neighbour in $P$, choose $i$ such that this neighbour is $b_i$, and define $c_i = v$. (This is well-defined, from (4).) For each $v \in R$ with no neighbour in $P$ choose an integer $i > n$ (distinct for all $v \in R$) such that $b_i$ is not defined, and define $c_i = v$. Thus for each $c_i \in R$, $a_i$ is not defined, and for each $j$ such that $b_j$ is defined, $c_i$ is adjacent to $b_j$ if and only if $i = j$. This proves that $G$ is climbable, and so proves 7.3. ∎

For the second outcome of 7.1 we use:

**7.4** *Let $G$ be $\{Q_5, K_3\}$-free with $\mu(G) \geq 2$, and with no twins if $\mu(G) = 2$; and let $M$ be a maximum induced matching in $G$. Suppose that $N_M$ is stable, and $M$ can be labelled $\{c_id_i : 1 \leq i \leq n\}$ such that $N_M$ is complete to $\{c_3, \ldots, c_n\}$, and each of the pairs $(c_1, c_2), (c_1, d_2), (d_1, c_2), (d_1, d_2)$ has a common neighbour in $N_M$. Then either $G$ is Clebschian, or $G$ admits a nontrivial pendant homogeneous pair.*

**Proof.** Define $a_1 = c_1, a_3 = c_2, a_5 = d_1, a_7 = d_2$ and let $A_i = \{a_i\}$ for $i = 1, 3, 5, 7$; and for $i = 2, 4, 6, 8$, let $A_i$ be the set of vertices in $N_M$ complete to $\{a_{i-1}, a_{i+1}\}$ (reading subscripts modulo 8 throughout). Thus $N_M = A_2 \cup A_4 \cup A_6 \cup A_8$, and $A_2$ is anticomplete to $A_6$, and $A_4$ is anticomplete to $A_8$ (since by hypothesis, $N_M$ is stable). Also $A_2, A_4, A_6, A_8$ are all nonempty, by hypothesis. Any 3-path in $G[A_2 \cup \{a_1\}]$ misses an edge of $G[A_6 \cup \{a_7\}]$, so $|A_2| = 1$, and similarly $|A_i| = 1$ for $i = 2, 4, 6, 8$; let $A_i = \{a_i\}$ for $i = 2, 4, 6, 8$. Let $R = V(G) \setminus (A_1 \cup \cdots \cup A_8)$; thus $c_i, d_i \in R$ for $3 \leq i \leq n$. If $v \in R$ is adjacent to $a_2$, then since $v$-$a_2$-$a_3$ does not miss $a_5a_6$, it follows that $v$ is adjacent to $a_6$; and similarly for the other vertices in $N_M$. Define $R_0, \ldots, R_3$ as follows.

- $R_0$ is the set of vertices in $R$ with no neighbour among $a_2, a_4, a_6, a_8$
- $R_1$ is the set adjacent to $a_2, a_6$ and not to $a_4, a_8$;
- $R_2$ is the set adjacent to $a_4, a_8$ and not to $a_2, a_6$; and
- $R_3$ is the set adjacent to all of $a_2, a_4, a_6, a_8$.

It follows that $R = R_0 \cup \cdots \cup R_3$, and $c_3, \ldots, c_n \in R_3$, and $d_3, \ldots, d_n \in R_0$. Now $R_0 \cup R_1$ is stable, since it misses $a_3$-$a_4$-$a_5$, and similarly $R_0 \cup R_2$ is stable. There are no edges between $R_3$ and $R_1 \cup R_2$ since $G$ is triangle-free; and so $R_1 \cup R_2$ misses $R_0 \cup R_3$, and hence does not have any neighbour in $V_M$. Since $M$ is a maximum induced matching it follows that $R_1 \cup R_2$ is stable. So the only edges in $G[R]$ are edges between $R_0$ and $R_3$. Now $|R_1| \leq 1$ since any 3-path in $G[R_1 \cup \{a_2\}]$ misses $a_4a_5$; and similarly $|R_2| \leq 1$. Not both $R_1, R_2$ are nonempty, since otherwise a 3-path in $G[R_1 \cup \{a_2, a_6\}]$ would miss an edge of $G[R_2 \cup \{a_4\}]$; so we may assume that $R_2 = \emptyset$. If $R_0 \neq \emptyset$, then since every vertex in $R_0$ has a neighbour in $R_3$ (because $G$ is connected), it follows that $(R_3, R_0)$ is a pendant homogeneous pair, so we may assume that $|R_3|, |R_0| = 1$, and therefore $G$ is Clebschian. Thus we may assume that $R_0 = \emptyset$. Hence $\mu(G) = n = 2$, and so by hypothesis $G$ has no twins; and again it follows that $|R_3| \leq 1$ and $G$ is Clebschian. This proves 7.4. ∎

For the third outcome of 7.1 we use:

**7.5** *Let $G$ be connected and $\{Q_5, K_3\}$-free, containing the Petersen graph. Then either $G$ is Clebschian, or $G$ admits a nontrivial pendant homogeneous pair.*

**Proof.** Let us label the vertices of the Petersen subgraph $H$ say, as follows: there is an induced 5-cycle $C$ with vertices $v_1$-$v_2$-$\cdots$-$v_5$-$v_1$ in order; for $1 \leq i \leq 5$ there is a vertex $v_i'$ adjacent to $v_i$ and to no other vertex in $C$; and for all distinct $i, j \in \{1, \ldots, 5\}$, $v_i', v_j'$ are adjacent if and only if $j - i = 2$ or 3 modulo 5. (We read subscripts modulo 5 throughout.) Let $R = V(G) \setminus V(H)$. Let $R_0$ be the set of vertices in $R$ with no neighbour in $V(H)$, and for $1 \leq i \leq 5$, let $R_i$ be the set of vertices in $R$ whose neighbour set in $V(H)$ is $\{v_{i-1}, v_{i+1}, v_{i-2}', v_{i+2}'\}$. (The five quadruples of $V(H)$ just listed are the stable subsets of $H$ of cardinality four, and so this list is invariant under the symmetries of the Petersen graph.)

(1) $R = R_0 \cup R_1 \cup \cdots \cup R_5$.

For let $v \in R$, and let $N_M$ be the set of its neighbours in $V(H)$. We may assume that $N_M \neq \emptyset$, and so from the symmetry we may assume that $v_1 \in N_M$. Consequently $v_2, v_5, v_1' \notin N_M$. The path $v$-$v_1$-$v_1'$ hits $v_3v_4$, so from the symmetry we may assume that $v_3 \in N_M$. Hence $v_4, v_3' \notin N_M$. Since $v$-$v_1$-$v_2$ hits $v_4$-$v_4'$, it follows that $v_4' \in N_M$, and so $v_2' \notin N_M$. Since $v$-$v_3$-$v_4$ hits $v_2'v_5'$, it follows that $v_5' \in N_M$; and so $N_M = \{v_1, v_3, v_4', v_5'\}$, and hence $v \in R_2$. This proves (1).

Since every two vertices in $R_1 \cup \cdots \cup R_5$ have a common neighbour, it follows that $R_1 \cup \cdots \cup R_5$ is stable. Thus if $R_0 = \emptyset$, $G$ is Clebschian, so we assume that $R_0 \neq \emptyset$. Also, $R_0$ is stable, since $R_0$ misses $v_1$-$v_2$-$v_3$.

(2) *We may assume that $R_0$ is complete to $R_1 \cup \cdots \cup R_5$.*

Let $v \in R_0$. Since $R_0$ is stable and $G$ is connected, $v$ has a neighbour in $R_1 \cup \cdots \cup R_5$, say $r_1 \in R_1$. We claim that $v$ is complete to $R_2 \cup \cdots \cup R_5$. To see this, let $r_2 \in R_2$ (from the symmetry of the Petersen graph, it is enough to show that a vertex with a neighbour in $R_1$ is complete to $R_2$). Since $v$-$r_1$-$v_3'$ hits $r_2v_1$, it follows that $v, r_2$ are adjacent. So $v$ is complete to $R_2 \cup \cdots \cup R_5$. If one of $R_2, \ldots, R_5$ is nonempty, the same argument shows that $v$ is complete to $R_1$ as required; so we assume that $R_2, \ldots, R_5 = \emptyset$. Now any 3-path in $G[R_0 \cup R_1]$ misses $v_1v_1'$, so every vertex in $R_0$ has at most one neighbour in $R_1$ and vice versa; and since $(R_1, R_0)$ is a pendant homogeneous pair, we may assume that $|R_0| = |R_1| = 1$, and so $R_1$ is complete to $R_0$. This proves (2).

Any 3-path in $G[R_0 \cup R_1]$ misses $v_1v_1'$, and since $R_0$ is complete to $R_1$ and $R_0 \neq \emptyset$ it follows that $|R_1| \leq 1$, and similarly $|R_i| \leq 1$ for $1 \leq i \leq 5$. We may assume that $|R_1| = 1$, since $R_0 \neq \emptyset$, and so by the same argument $|R_0| = 1$. Hence $G$ is Clebschian. This proves 7.5. ∎

We deduce:

**7.6** *Let $G$ be connected and $\{Q_5, K_3\}$-free, with $\mu(G) \geq 2$ and with no twins if $\mu(G) \leq 2$, and let $M$ be a maximum induced matching in $G$. Suppose that $N_M$ is stable. Then either:*

- *$G$ is Clebschian or climbable; or*
- *$G$ admits a nontrivial pendant homogeneous pair.*

**Proof.** By 7.1, $M$ can be labelled $\{a_ib_i : 1 \leq i \leq n\}$ such that one of the following holds.

- Every member of $N_M$ is adjacent to at least $n-1$ of $a_1, \ldots, a_n$. In this case, if at most one of the edges $a_ib_i$ is internal, then the theorem holds by 7.2, and if at least two edges of $M$ are internal, it holds by 7.3.
- $N_M$ is complete to $\{a_3, \ldots, a_n\}$, and each of the pairs $(a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2)$ has a common neighbour in $N_M$. In this case the theorem holds by 7.4.
- $n \geq 3$, $N_M$ is complete to $\{a_4, \ldots, a_n\}$, and every member of $N_M$ is adjacent to an odd number of $a_1, a_2, a_3$; and each of the triples $(a_1, a_2, a_3), (a_1, b_2, b_3), (b_1, a_2, b_3), (b_1, b_2, a_3)$ has a common neighbour in $N_M$. In this case, the six vertices $a_1, b_1, a_2, b_2, a_3, b_3$, together with four common neighbours in $N_M$ of the four triples listed above, induce a subgraph of $G$ isomorphic to the Petersen graph; and so the theorem holds by 7.5.

This proves 7.6. ∎

## 8 Maximum matching of cardinality two

We already handled the case when $\mu(G) \le 1$, in 6.3; now we are ready to handle $\mu(G) = 2$, and this section is devoted to it. We recall that an antimatching is a bipartite graph, the bipartite complement of a perfect matching. We recall that $\nu(G)$ denotes the largest cardinality of an antimatching in $G$ (more precisely, $\nu(G)$ means the largest $k$ such that there is an antimatching with bipartition $(A, B)$ where $|A| = |B| = k$). When $\mu(G) = 2$, the maximum induced matching induces a subgraph that is an antimatching, and it might be possible to extend this to a larger antimatching; and we find it advantageous to work with the largest antimatching.

**8.1** *Let $G$ be a connected $\{Q_5, K_3\}$-free graph with no twins and with $\mu(G) = 2$ and $\nu(G) \ge 3$. Then either*

- *$G$ is Clebschian, climbable, or bipartite; or*
- *$G$ admits a nontrivial simplicial homogeneous pair.*

**Proof.** Let $(A, B) = (\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\})$ be the bipartition of the largest antimatching in $G$, where $a_i, b_i$ are nonadjacent for $1 \le i \le n$. Thus $n = \nu(G) \ge 3$. For $1 \le i \le n$ let $C_i$ be the set of all vertices in $V(G) \setminus (A \cup B)$ adjacent to both $a_i$ and $b_i$ and nonadjacent to every other vertex in $A \cup B$; let $C = C_1 \cup \dots \cup C_n$; let $P$ be the set of vertices in $V(G) \setminus (A \cup B)$ complete to $A$ and anticomplete to $B$, and let $Q$ be the set complete to $B$ and anticomplete to $A$.

(1) *Every vertex in $V(G) \setminus (A \cup B)$ with a neighbour in $A \cup B$ belongs to one of $P, Q, C$. Moreover, $P$ is complete to $Q$, and $P \cup Q$ is anticomplete to $C$.*

Let $v \in V(G) \setminus (A \cup B)$ be adjacent to $a_1$ say. If $v$ is complete to $A$ then it is anticomplete to $B$ and $v \in P$; so we may assume that $v$ is nonadjacent to $a_2$. Since the 3-path $v$-$a_1$-$b_2$ does not miss the edge $a_2b_1$, it follows that $v$ is adjacent to $b_1$; and so $v$ has no more neighbours in $A \cup B$ and $v \in C_1$. This proves the first assertion. Now $P$ is complete to $Q$ by the maximality of the antimatching. Finally, $P \cup Q$ is anticomplete to $C$ since $G$ is triangle-free. This proves (1).

(2) *For all distinct $i, j \in \{1, \dots, n\}$, every vertex in $C_i$ has at most one nonneighbour in $C_j$.*

For otherwise there is a 3-path in $G[C_j \cup \{a_j\}]$ that misses an edge of $G[C_i \cup \{a_i\}]$.

(3) *We may assume that $A \cup B \cup P \cup Q \cup C = V(G)$.*

Let $R = V(G) \setminus (A \cup B \cup P \cup Q \cup C)$, and suppose that there exists $r \in R$. Let $rs$ be an edge incident with $r$. By (1), $s \notin A \cup B$. If $s \notin P \cup Q$, we may assume that $s \notin C_1 \cup C_2$ since $n \ge 3$, and then $\{rs, a_1b_2, a_2b_1\}$ is an induced matching, a contradiction. Thus $s \in P \cup Q$. Since $G$ is triangle-free, it follows that for every vertex in $R$, either all its neighbours belong to $P$ or they all belong to $Q$; let $R_1$ be the set of vertices in $R$ such that all their neighbours belong to $P$, and $R_2 = R \setminus R_1$. With $r, s$ as before, we may assume from the symmetry that $r \in R_1$ and $s \in P$. Suppose that at least two of $C_1, \dots, C_n$ are nonempty, say $C_1, C_2$, and choose $c_i \in C_i$ $(i = 1, 2)$. If $c_1, c_2$ are nonadjacent, then $\{rs, b_1c_1, b_2c_2\}$ is an induced matching, a contradiction. If $c_1, c_2$ are adjacent, the 3-path $r$-$s$-$a_3$

misses $c_1c_2$, a contradiction. Thus at most one of $C_1, \dots, C_n$ is nonempty. We may assume that $G$ is not bipartite, and so we may assume that $C_1 \neq \emptyset$, and $C_2, \dots, C_n = \emptyset$. Since $(P, R_1)$ is a pendant homogeneous pair, we may assume that $R_1 = \{r\}$ and $P = \{s\}$. Also either $R_2 = \emptyset$ (and hence $|Q| = 1$ since $G$ has no twins) or $(Q, R_2)$ is also a pendant homogeneous pair (and so we may assume that $|R_2| = |Q| = 1$). Since any 3-path in $G[C_1 \cup \{b_1\}]$ would miss the edge of $G[P \cup R_1]$, it follows that $|C_1| = 1$; but then $G$ is climbable. This proves (3).

(4) *For $1 \leq i \leq n$, each vertex in $C_i$ has at most one nonneighbour in $C \setminus C_i$.*

Let $i = 1$ say, and suppose that $c_1 \in C_1$ has at least two nonneighbours in $C \setminus C_1$. By (2) we may assume that $c_1$ is nonadjacent to $c_2 \in C_2$ and $c_3 \in C_3$. Since $\{a_1c_1, a_2c_2, a_3c_3\}$ is not an induced matching, it follows that $c_2, c_3$ are adjacent. But then $a_1$-$c_1$-$b_1$ misses $c_2c_3$. This proves (4).

(5) *We may assume that at most two of $C_1, \dots, C_n$ are nonempty.*

Suppose $C_1, C_2, C_3$ are nonempty. If $n \geq 4$, then by (4) there is an edge joining two of $C_1, C_2, C_3$, say $c_1c_2$ where $c_i \in C_i$ for $i = 1, 2$; but then $\{c_1c_2, a_3b_4, a_4b_3\}$ is an induced matching, a contradiction. Thus $n = 3$. Also $|P| \leq 1$, since otherwise for some $i \in \{1, 2, 3\}$ a 3-path in $G[P \cup \{a_i\}]$ would miss an edge of $G[C_1 \cup C_2 \cup C_3]$. Similarly $|Q| \leq 1$.

Suppose that $c_1 \in C_1$ has at least two neighbours in $C_2$, say $c_2, c_2'$. If $c_1$ is adjacent to $c_3 \in C_3$, then $c_3$ has two nonneighbours in $C_2$, namely $c_2, c_2'$, contrary to (2). So $c_1$ is anticomplete to $C_3$. Hence by (4) some vertex (indeed, every vertex) of $C_3$ is adjacent to $c_2, c_2'$, and so by the same argument $c_2, c_2'$ are complete to $C_1$. In particular, there are no edges between $C_1$ and $C_3$, so $C_1, C_3$ are complete to $C_2$ by (4), and $|C_1| = |C_3| = 1$ by (2). If $p \in P$, let $c_3 \in C_3$; then the 3-path $c_2$-$c_3$-$c_2'$ misses $pa_1$, a contradiction. So $P, Q = \emptyset$, and $c_2, c_2'$ are twins, a contradiction.

We may assume therefore that for all distinct $i, j \in \{1, 2, 3\}$, every vertex in $C_i$ has at most one neighbour and at most one nonneighbour in $C_j$. Let $c_2 \in C_2$ be adjacent to $c_1 \in C_1$ and $c_3 \in C_3$. Then no other vertex $c_2' \in C_2$ is adjacent to either of $c_1, c_3$, and therefore $\{c_1, c_2', c_3\}$ is stable, contrary to (4), for each such vertex $c_2'$; and hence $C_2 = \{c_2\}$. Since $c_2$ has at most one neighbour and at most one nonneighbour in $C_1$, it follows that $|C_1| \leq 2$, and similarly $|C_3| \leq 2$; and not both $|C_1|, |C_3| = 2$, since for every $c_1' \in C_1$ and $c_3' \in C_3$, $c_2$ is adjacent to one of them. So we may assume that $|C_3| = 1$, and $C_3 = \{c_3\}$. Also either $C_1 = \{c_1\}$, or $C_1 = \{c_1, c_1'\}$, for some $c_1'$ adjacent to $c_3$. Now $P$ is complete to $Q$ by (1), and so one of $P, Q$ is empty, since the 3-path $c_1$-$c_2$-$c_3$ misses every edge between $P, Q$. Since $|P|, |Q| \leq 1$, it follows that $G$ is Clebschian. This proves (5).

From (5), we may assume that $C_3, \dots, C_n = \emptyset$.

(6) *We may assume that there is no edge between $C_1, C_2$.*

Suppose that $c_1 \in C_1$ is adjacent to $c_2 \in C_2$. Not both $P, Q$ are nonempty; because $P$ is complete to $Q$, by (1), and if $p \in P$ is adjacent to $q \in Q$ then $q$-$p$-$a_3$ misses $c_1c_2$. So we may assume that $Q = \emptyset$. If $|P| \geq 2$, a 3-path in $G[P \cup \{a_3\}]$ misses $c_1c_2$; so $|P| \leq 1$. Suppose that there exists $p \in P$. Then $pa_3$ misses any 3-path in $G[C_1 \cup C_2]$; so every vertex in $C_1$ has at most one neighbour in $C_2$, and vice versa. By (2), $|C_1|, |C_2| \leq 2$. Moreover, not both $|C_1|, |C_2| = 2$, since if so there would be an induced matching consisting of two edges between $C_1, C_2$ and the edge $pa_3$. But then

$G$ is Clebschian. We may therefore assume that $P = Q = \emptyset$. But then $G$ is climbable, by 7.3 applied to the induced matching $\{a_1b_3, a_3b_1\}$. This proves (6).

Since $G$ has no twins, (4), (5) and (6) imply that $|C_i| \le 1$ for $i = 1, 2$, and $|P|, |Q| \le 1$. But then $G$ is climbable. This proves 8.1. ∎

**8.2** *Let $G$ be a connected $\{Q_5, K_3\}$-free graph with no twins and with $\mu(G) = \nu(G) = 2$. Then either*

- *$G$ is Clebschian; or*
- *$G$ admits a nontrivial simplicial homogeneous pair.*

**Proof.** Let $M = \{a_1a_5, a_3a_7\}$ be a maximum induced matching in $G$, chosen such that no vertices in $V_M$ are leaves if possible; and for $i = 2, 4, 6, 8$ let $A_i$ be the set of all vertices that are complete to $\{a_{i-1}, a_{i+1}\}$. (Throughout we read subscript modulo 8.) Thus $N_M = A_2 \cup A_4 \cup A_6 \cup A_8$. For $i = 1, 3, 5, 7$ let $A_i = \{a_i\}$. If there exist $a_2 \in A_2$ and $a_6 \in A_6$, nonadjacent, then $G[\{a_1, a_2, a_3, a_5, a_6, a_7\}]$ is an antimatching, contradicting that $\nu(G) = 2$; so $A_2$ is complete to $A_6$, and similarly $A_4$ to $A_8$. So for all distinct $i, j$ with $1 \le i, j \le 8$, if $j - i = 1, 4$ or $7$ then $A_i$ is complete to $A_j$ and otherwise $A_i$ is anticomplete to $A_j$. If some vertex of $V_M$ is a leaf then the theorem holds by 7.2, so we may assume that both edges of $M$ are internal. Hence either $A_2, A_6$ are both nonempty, or $A_4, A_8$ are both nonempty, and we assume the first without loss of generality.

Let $R$ be the set of vertices of $G$ not in $A_1, \ldots, A_8$. Since $R$ is stable (because $M$ is a maximum induced matching), it follows that every vertex in $R$ has a neighbour in $A_2 \cup A_4 \cup A_6 \cup A_8$.

Suppose first that $A_4, A_8$ are both nonempty. Since any 3-path in $G[A_2 \cup A_6]$ would miss an edge of $G[A_4 \cup A_8]$, it follows that $|A_2| = |A_6| = 1$, and similarly $|A_4| = |A_8| = 1$. Let $A_i = \{a_i\}$ for $i = 2, 4, 6, 8$. If $v \in R$ is adjacent to $a_2$ say, then since the 3-path $v$-$a_2$-$a_6$ does not miss $a_4a_8$, it follows that $v$ is adjacent to one of $a_4, a_8$; so every vertex in $R$ belongs to one of the four disjoint sets $R_1, R_3, R_5, R_7$, where $R_i$ is the set of vertices in $R$ adjacent to $a_{i-1}$ and $a_{i+1}$ for $i = 1, 3, 5, 7$. Now $R_1$ is anticomplete to $R_5$, since $R$ is stable, and so if they are both nonempty, a 3-path of $G[R_1 \cup \{a_2, a_1\}]$ misses an edge of $G[R_5 \cup \{a_4\}]$. Hence one of $R_1, R_5 = \emptyset$, and similarly one of $R_3, R_7 = \emptyset$, so we may assume that $R_5, R_7 = \emptyset$. If $|R_1| > 1$ then a 3-path of $G[R_1 \cup \{a_2\}]$ misses $a_4a_5$, so $|R_1|, |R_3| \le 1$. But then $G$ is Clebschian.

We may therefore assume that $A_8 = \emptyset$. If $A_4 = \emptyset$ then $(\{a_1, a_3\}, \{a_5, a_7\})$ is a nontrivial simplicial homogeneous pair, so we may assume that $A_4 \ne \emptyset$. Define:

- $R_1$ is the set of vertices in $R$ with a neighbour in $A_2$ and no neighbour in $A_4$;
- $R_3$ is the set complete to $A_2$ with a neighbour in $A_4$;
- $R_5$ is the set complete to $A_6$ with a neighbour in $A_4$; and
- $R_7$ is the set with a neighbour in $A_6$ and no neighbour in $A_4$.

(1) *Every vertex in $R$ belongs to one of $R_1, R_3, R_5, R_7$.*

If $v \in R$ has no neighbour in $A_4$ then $v \in R_1 \cup R_7$ since $v$ has a neighbour in $A_2 \cup A_4 \cup A_6 \cup A_8$.

Thus we may assume $v$ is adjacent to $a_4 \in A_4$. Since every 3-path in $G[A_2 \cup A_6 \cup \{a_7\}]$ hits $va_4$, $v$ is complete to one of $A_2, A_6$ and so $v \in R_3 \cup R_5$. This proves (1).

If $R_1 = \emptyset$ then $|A_2| = 1$, since $G$ has no twins; and if $R_1 \neq \emptyset$ then we may assume that $|A_2| = |R_1| = 1$, since otherwise $(A_2, R_1)$ is a nontrivial pendant homogeneous pair. Thus in either case we may assume that $|A_2| = 1$, and $|R_1| \le 1$; and similarly $|A_6| = 1$ and $|R_7| \le 1$. Let $A_2 = \{a_2\}$ and $A_6 = \{a_6\}$.

If $R_3 = R_5 = \emptyset$, then $|A_4| = 1$ since $G$ has no twins, and so $G$ is Clebschian. Thus we may assume that $R_3 \neq \emptyset$. Hence there is a 3-path in $G[R_3 \cup A_4 \cup \{a_3\}]$, and so there is no edge in $G[A_6 \cup R_7]$, and hence $R_7 = \emptyset$. If $|R_3| > 1$, there is either a 3-path or an induced 2-edge matching in $G[R_3 \cup A_4]$, and in either case it misses $a_6a_7$, a contradiction. So $|R_3| = 1$; let $R_3 = \{r_3\}$. By the same argument, $r_3$ has only one neighbour in $A_4$, say $a_4$. Every 3-path in $G[(A_4 \setminus \{a_4\}) \cup \{a_5\}]$ misses the edges of $G[A_2 \cup \{r_3\}]$; so $|A_4 \setminus \{a_4\}| \le 1$.

Suppose that $R_5 \neq \emptyset$. Then by the same argument, $R_1 = \emptyset$, and $|R_5| = 1$, and the vertex $r_5 \in R_5$ has a unique neighbour in $A_4$. Since no 3-path in $G[A_4 \cup \{r_5\} \cup A_6]$ misses $r_3a_4$, it follows that $a_4, r_5$ are adjacent. But then $G$ is Clebschian.

We may assume therefore that $R_5 = \emptyset$. Suppose that $R_1 \neq \emptyset$, and so $|R_1| = 1$; let $R_1 = \{r_1\}$. Any 3-path in $G[A_4 \cup \{a_5\}]$ misses $r_1a_2$, so $A_4 = \{a_4\}$. But then $G$ is Clebschian.

Thus we may assume that $R_1 = \emptyset$. In summary, $R_1 = R_5 = R_7 = \emptyset$ and $R_3 = \{r_3\}$, and $A_4$ contains only one neighbour of $r_3$, and at most one of its non-neighbours. But then $G$ is Clebschian. This proves 8.2. ∎

Combining 8.1 and 8.2, we obtain:

**8.3** *Let $G$ be a connected $\{Q_5, K_3\}$-free graph with $\mu(G) = 2$ and with no twins. Then either*

- *$G$ is Clebschian, climbable, or bipartite; or*
- *$G$ admits a nontrivial simplicial homogeneous pair.*

# 9 Edges in the neighbourhood of a matching

In this section we handle the final case, when $\mu(G) \ge 3$ and $N_M$ is not stable. We divide it into three subcases, when $\mu(G) \ge 5$, when $\mu(G) = 4$, and when $\mu(G) = 3$, in increasing order of difficulty.

**9.1** *Let $G$ be connected and $\{Q_5, K_3\}$-free, with $\mu(G) \ge 5$, and let $M$ be a maximum induced matching in $G$. Suppose that $N_M$ is not stable. Then $G$ admits a nontrivial simplicial homogeneous pair.*

**Proof.** Let $M = \{a_ib_i : 1 \le i \le n\}$, and let $u, v \in N_M$ be adjacent. By 6.2 we may assume that $u$ is complete to $\{a_1, \dots, a_n\}$ and $v$ to $\{b_1, \dots, b_n\}$. Let $N_1, N_2$ be the sets of vertices in $V(G) \setminus V_M$ that are complete to $\{a_1, \dots, a_n\}$ and complete to $\{b_1, \dots, b_n\}$ respectively. Thus $u \in N_1$ and $v \in N_2$. We claim that $N_M = N_1 \cup N_2$; for let $w \in N_M$. If say $w$ is adjacent to $a_1$ and to $b_2$, then it shares a neighbour with each of $u, v$ and so is nonadjacent to them both. There are at most two values of $i$ such that $w$ is adjacent to $b_i$, by 6.2 applied to $w, u$, and also only two such that $w$ is adjacent to $a_i$, by 6.2 applied to $w, v$. But this is impossible since $n \ge 5$. Consequently $N_M = N_1 \cup N_2$. For each

$u' \in N_1$ and $v' \in N_2$, the 3-path $a_1$-$u'$-$a_2$ meets the edge $b_3v'$, and so $u', v'$ are adjacent; so $N_1$ is complete to $N_2$. Hence $(\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\})$ is a nontrivial simplicial homogeneous pair, and the theorem holds. This proves 9.1. ∎

**9.2** *Let $G$ be connected and $\{Q_5, K_3\}$-free, with no twins and with $\mu(G) = 4$, and let $M$ be a maximum induced matching in $G$. Suppose that $N_M$ is not stable. Then either $G$ is Clebschian or $G$ admits a nontrivial simplicial homogeneous pair.*

**Proof.** Let $M = \{a_ib_i : 1 \le i \le 4\}$. By hypothesis, there exist adjacent $p_1, p_2 \in N_M$; and by 6.2, $\delta_M(p_1, p_2) = 4$. If each $v \in N_M$ satisfies $\delta_M(p_1, v) = 0$ or $\delta_M(p_2, v) = 0$ then $G$ admits a nontrivial simplicial homogeneous pair, so we may assume there exists $p_3 \in N_M$ with $\delta_M(p_1, p_3), \delta_M(p_2, p_3) \ge 1$. By 6.2, $p_3$ is nonadjacent to $p_1, p_2$, so $\delta_M(p_1, p_3), \delta_M(p_2, p_3) \le 2$; and since $\delta_M$ satisfies the triangle inequality, it follows that $\delta_M(p_1, q), \delta_M(p_2, q) = 2$, for each $q \in N_M$ with $\delta_M(p_1, q), \delta_M(p_2, q) \neq 0$.

(1) *Every vertex of $G$ belongs to $V_M \cup N_M$.*

Let $R = V(G) \setminus (V_M \cup N_M)$, and suppose that $r \in R$. Thus $R$ is stable, from the maximality of $M$, and so $r$ in $R$ has a neighbour in $N_M$. Let $Q$ be the set of $q \in N_M$ such that $r$ is adjacent to $q$. Not both $p_1, p_2 \in Q$ since $p_1$ is adjacent to $p_2$; and from the existence of $p_1, p_2, p_3$, it follows that there exist nonadjacent $q, q' \in N_M$, with $q \in Q$ and $q' \notin Q$, and with $\delta_M(q, q') = 2$. We may assume that $q$ is adjacent to $a_1, a_2$ and $q'$ is adjacent to $b_1, b_2$. But then the 3-path $b_1$-$q'$-$b_2$ misses the edge $qr$, a contradiction. This proves (1).

From 6.2 and (1), the neighbour set of each $v \in N_M$ is determined by the set of neighbours of $v$ in $V_M$, and since $G$ has no twins, there do not exist distinct $u, v \in N_M$ with $\delta_M(u, v) = 0$. But then $G$ is Clebschian. This proves 9.2. ∎

**9.3** *Let $G$ be connected and $\{Q_5, K_3\}$-free, with no twins and with $\mu(G) = 3$, and let $M$ be a maximum induced matching in $G$. Suppose that $N_M$ is not stable. Then either $G$ is Clebschian, or $G$ admits a nontrivial simplicial homogeneous pair.*

**Proof.** Let $M = \{a_1b_1, a_2b_2, a_3b_3\}$. By hypothesis, there exist $p_1, p_2 \in N_M$ with $\delta_M(p_1, p_2) = 3$. For $i = 1, 2$, let $P_i$ be the set of $v \in N_M$ such that $\delta_M(p_i, v) = 0$. If each $v \in N_M$ satisfies $\delta_M(p_1, v) = 0$ or $\delta_M(p_2, v) = 0$ then $G$ admits a nontrivial simplicial homogeneous pair, so we may assume there exists $p_3 \in N_M$ with $\delta_M(p_1, p_3), \delta_M(p_2, p_3) \ge 1$. Let $N_1$ be the set of $v \in N_M$ adjacent to an odd number of $a_1, a_2, a_3$, and $N_2 = N_M \setminus N_1$. We may assume that $p_i \in N_i$ for $i = 1, 2$.

Let $R = V(G) \setminus (V_M \cup N_M)$. Thus $R$ is stable, from the maximality of $M$, and every vertex in $R$ has a neighbour in $N_M$. For $i = 1, 2$, every two vertices in $N_i$ have a common neighbour in $V_M$ and so $N_i$ is stable.

(1) *If $r \in R$, and $q_1, q_2 \in N_M$ with $\delta_M(q_1, q_2) = 2$, and $r$ is adjacent to $q_1$, then $r$ is adjacent to $q_2$.*

We may assume that $q_1$ is adjacent to $a_1, a_2, a_3$, and $q_2$ is adjacent to $b_1, b_2, a_3$. Since the 3-path

$r$-$q_1$-$a_1$ meets the edge $q_2b_2$, it follows that $r$ is adjacent to $q_2$. This proves (1).

(2) *For $i = 1, 2$, if $r \in R$ has a neighbour in $N_i$, then $r$ has a neighbour in $P_i$.*

Let $r$ be adjacent to $v \in N_1$ say. If $\delta_M(v, p_1) = 0$ the claim is true, and if $\delta_M(v, p_1) = 2$ the claim follows from (1). This proves (2).

For $i = 1, 2$ let $R_i$ be the set of vertices in $R$ with a neighbour in $N_i$.

(3) *Not both $R_1, R_2$ are nonempty.*

If $r \in R_1 \cap R_2$, then by (2) $r$ has neighbours in both $P_1, P_2$, which is impossible since they are complete to each other by 6.2. So $R_1 \cap R_2 = \emptyset$. Suppose that there exist $r_1 \in R_1$ and $r_2 \in R_2$. We may assume that $\delta_M(p_3, p_1) = 1$; and $p_1$ is adjacent to $a_1, a_2, a_3$ and $p_3$ to $a_1, a_2, b_3$. By (2), $r_2$ has a neighbour in $P_2$, and hence is adjacent to $p_3$ by (1). Choose $v_1 \in P_1$ adjacent to $r_1$. Then $r_2$-$p_3$-$b_3$ misses $r_1v_1$, a contradiction. This proves (3).

(4) *We may assume that for $i = 1, 2$, $R_i$ is complete to $N_i$.*

Let $i = 1$ say, and let $r_1 \in R_1$. By (2), $r_1$ has a neighbour in $P_1$, and hence $r_1$ is adjacent to $q$ for every $q \in N_i$ with $\delta_M(p_1, q) = 2$, by (1). Thus if there exists $q \in N_i$ with $\delta_M(p_1, q) = 2$, then by (1) again, $r_1$ is complete to $P_1$ and hence to $N_1$; so we may assume that $N_1 = P_1$. Thus $(R_1, P_1)$ is a pendant homogeneous pair, and we may assume that $|R_1| = |N_1| = 1$, and once again the claim holds. This proves (4).

(5) *For $i = 1, 2$, $|R_i| \leq 1$.*

Suppose that $|R_1| \geq 2$ say. From the existence of $p_3$, there exist $q_1 \in N_1$ and $q_2 \in N_2$ with $\delta_M(q_1, q_2) = 1$. Let $q_1$ be adjacent to $a_1, a_2, a_3$, and $q_2$ to $a_1, a_2, b_3$ say. By (3) and (4), $R_1$ is complete to $q_1$ and anticomplete to $q_2$. Then a 3-path in $G[R_1 \cup \{q_1\}]$ misses the edge $q_2b_3$. This proves (5).

If $R = \emptyset$, $G$ is Clebschian. So we may assume that $|R_1| = \{r_1\}$ say, by (5), and hence $R_2 = \emptyset$, by (3). From 6.2 and (4), the neighbour set of each $v \in N_M$ is determined by the set of neighbours of $v$ in $V_M$, and since $G$ has no twins, there do not exist distinct $u, v \in N_M$ with $\delta_M(u, v) = 0$. Consequently $G$ is Clebschian. This proves 9.3. ∎

In summary, from 9.1, 9.2 and 9.3, we have shown that:

**9.4** *Let $G$ be connected and $\{Q_5, K_3\}$-free, with no twins and with $\mu(G) \geq 3$, and let $M$ be a maximum induced matching in $G$. If $N_M$ is not stable, then either $G$ is Clebschian or $G$ admits a nontrivial simplicial homogeneous pair.*

Combining with 6.3, 8.3, and 7.6, we obtain 6.1, which we restate:

**9.5** *Let $G$ be connected and $\{Q_5, K_3\}$-free, with no twins. Then either:*

- *$G$ is Clebschian, climbable, or bipartite; or*
- *$G$ admits a nontrivial simplicial homogeneous pair.*

And finally we obtain our main result 1.1, which we restate:

**9.6** *Let $G$ be connected and $\{P_6, K_3\}$-free, with no twins. Then either*

- *$G$ is Clebschian, climbable, or a $V_8$-expansion; or*
- *$G$ admits a nontrivial simplicial homogeneous pair.*

**Proof.** From 9.5 and 4.2, it suffices to check that $G$ satisfies the theorem when $G$ is bipartite; and hence, from 5.2, we may assume that $G$ is a half-graph expansion. Let $G$ be a connected half-graph expansion without twins. If $|V(G)| \leq 4$ then $G$ is climbable, so we assume that $|V(G)| \geq 5$. Let $(A, B)$ be a bipartition of $G$, and let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be as in the definition of half-graph expansion, where $A_i \cup B_i \neq \emptyset$ for $1 \leq i \leq n$. Since $A_1 \cup B_1 \neq \emptyset$ and $G$ is connected, it follows that $A_1 \neq \emptyset$.

Suppose that $n = 1$. Since $|A_1 \cup B_1| = |V(G)| \geq 5$, it follows that $(A_1, B_1)$ is matched or antimatched; and it is not matched since $G$ is connected. Thus if $n = 1$ then $G$ is an antimatching and so climbable, and therefore we may assume that $n \geq 2$.

If $B_1 = \emptyset$ then $|A_1| = 1$ since $G$ has no twins; and if $B_1 \neq \emptyset$ then $(A_1, B_1)$ is a simplicial homogeneous pair (since $n > 1$), so we may assume that $|A_1| = 1$ and $|B_1| \leq 1$. Thus in either case we have $|A_1| = 1$ and $|B_1| \leq 1$, and consequently $|V(G) \setminus (A_1 \cup B_1)| \geq 3$. Since $G$ has no twins, it follows that $A_2 \cup \cdots \cup A_n \neq \emptyset$, and so $(A_2 \cup \cdots \cup A_n, B_2 \cup \cdots \cup B_n)$ is a nontrivial simplicial homogeneous pair. This proves 9.6. ∎

## 10 A structure theorem for $\{Q_5, K_3\}$-free graphs

Finally, let us see how to convert 6.1 to a construction for all $\{Q_5, K_3\}$-free graphs. There are two problems with 6.1:

- adding twins without restriction might introduce $Q_5$;
- if $G$ admits a simplicial homogeneous pair $(A, B)$, and the corresponding graphs $H, J$ (defined as usual) are $\{Q_5, K_3\}$-free, that does not guarantee that $G$ is $\{Q_5, K_3\}$-free.

But both can easily be remedied. Let $v$ be a vertex of $G$ such that no edge incident with $v$ belongs to an induced matching with cardinality two. Such a vertex $v$ is called a *twinnable vertex* of $G$. If $G$ is $\{Q_5, K_3\}$-free, then adding twins to a twinnable vertex $v$ results in a larger $\{Q_5, K_3\}$-free graph.

Let $H$ be an antisubmatching, with bipartition $(A, B)$, and let $a \in A$ and $b \in B$ be nonadjacent. Add a new vertex $c$ adjacent to $a, b$, forming a graph $G$; then $c$ is a twinnable vertex of $G$. Any graph obtained from $G$ by adding twins of $c$ is called an *extended antisubmatching*. (This is the only time when we need to add twins to a climbable graph, so we gave such graphs a name.)

Let $(A, B)$ be a nontrivial simplicial homogeneous pair of $G$, and correspondingly let $G$ is obtained from $H$ by substituting $J = G[A \cup B]$ for some edge $ab$ of $H$. Let us say that $(A, B)$ is *submatched* if every vertex in $A$ has at most one neighbour in $B$ and vice versa. This makes it safe again; if $H, J$ are $\{Q_5, K_3\}$-free, and $(A, B)$ is a nontrivial submatched simplicial homogeneous pair of $G$, then $G$ is $\{Q_5, K_3\}$-free. Then we have a construction for all connected $\{Q_5, K_3\}$-free graphs, as follows:

**10.1** *Let $G$ be connected and $\{Q_5, K_3\}$-free. Then either:*

- *$G$ may be obtained from a Clebschian graph by adding twins of some of its twinnable vertices; or*
- *$G$ is climbable, or an extended antisubmatching; or*
- *$G$ is a half-graph expansion; or*
- *$G$ admits a nontrivial submatched simplicial homogeneous pair.*

**Proof.** Let $G$ be connected and $\{Q_5, K_3\}$-free. If $G$ is bipartite, then $G$ is a half-graph expansion by 5.2, and the theorem holds. Thus we may assume that $G$ is not bipartite.

Suppose that $G$ admits a nontrivial simplicial homogeneous pair $(A, B)$. If $(A, B)$ is submatched then the theorem holds, so we assume not; and so we may assume that some vertex $a \in A$ has two neighbours $b_1, b_2 \in B$. Let $C$ be the set of vertices in $V(G) \setminus (A \cup B)$ that are complete to $A$ and anticomplete to $B$, and define $D$ similarly with $A, B$ exchanged. Let $R = V(G) \setminus (A \cup B \cup C \cup D)$. Since $(A, B)$ is simplicial, $C$ is complete to $D$; and since $G$ is $Q_5$-free, the 3-path $b_1$-$a$-$b_2$ meets every edge, and so $R$ is stable. Since no vertex in $R$ has a neighbour in $C$ and another in $D$, it follows that $G$ is bipartite, a contradiction. Thus we may assume that $G$ does not admit a nontrivial simplicial homogeneous pair.

Now $G$ can be obtained from a graph $G'$ by adding twins, where $G'$ has no twins, and $G'$ is also connected and $\{Q_5, K_3\}$-free. It follows that $G'$ has no nontrivial simplicial homogeneous pair. From 9.5 $G'$ is Clebschian, climbable, or bipartite. Since $G$ is not bipartite, neither is $G'$. If $G'$ is Clebschian, then, since adding twins of a non-twinnable vertex will introduce $Q_5$, it follows that $G$ may be obtained from a Clebschian graph by adding twins of some of its twinnable vertices, and the theorem holds. If $G'$ is climbable and not Clebschian then (since $G'$ is not bipartite) it is easy to check that either $G$ is climbable or $G$ is an extended antisubmatching (we omit the details, which are straightforward). This proves 10.1. ∎

One could, if necessary, be more exact about safely adding twins to a Clebschian graph; one could list exactly the subgraphs of the Clebsch graph that have twinnable vertices, and list those vertices. (There are not many.)


## Acknowledgement

We would like to thank both referees for carefully reading the paper and suggesting a number of improvements.

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