Tropical Underwear

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1 Algebraic Hypersurfaces in $(\mathbb{C}^*)^n$

Let $\Delta \subset \mathbb{Z}^n$ be a convex lattice polytope. Let

$$X_\Delta := \left\{ \sum_{\alpha \in \Delta} c_\alpha \boldsymbol{z}^\alpha = 0 \right\} \subset (\mathbb{C}^*)^n$$

be a smooth hypersurface, for some coefficients $c_\alpha \in \mathbb{C}^*$. We use the notation

$$\boldsymbol{z}^\alpha \ := \ z_1^{\alpha_1} \dots z_n^{\alpha_n}.$$

The diffeomorphism type of X_Δ is independent of the coefficients c_α. We can study it by making a cunning choice of coefficients:

$$c_\alpha := t^{v(\alpha)}$$

for some $v : \Delta \to \mathbb{Z}$. Call the resulting hypersurface X_Δ^t, and consider the **tropical limit** $t \to \infty$.

We define affine functions (**tropical monomials**)

$$\begin{aligned} f_\alpha : \mathbb{R}^n &\rightarrow \mathbb{R}. \\ f_\alpha(\boldsymbol{r}) &:= v(\alpha) + \alpha \cdot \boldsymbol{r}, \end{aligned}$$

and a Log map

$$\begin{aligned} \mathrm{Log}_t : (\mathbb{C}^*)^n &\rightarrow \mathbb{R}^n, \\ \mathrm{Log}_t(\boldsymbol{z}) &:= \frac{1}{\log(t)} \left(\log|z_1|, \ldots, \log|z_n| \right). \end{aligned}$$

Observe that

$$\log\left| t^{v(\alpha)} \boldsymbol{z}^\alpha \right| = \log(t) f_\alpha(\mathrm{Log}_t(\boldsymbol{z})).$$

If one of the functions $f_\alpha(\mathrm{Log}_t(\boldsymbol{z}))$ dominates the others, then for sufficiently large t the corresponding term dominates the sum

$$\sum_{\alpha\in\Delta} t^{v(\alpha)}\boldsymbol{z}^\alpha,$$

and in particular the sum is not zero. Thus, the image $\mathrm{Log}_t(X^t_\Delta)$ (the **amoeba**) must converge, as $t \to \infty$, to the locus where two monomials f_α are tied for largest. This is a polyhedral complex, called the **tropical amoeba** of the degeneration.

Consider the function (a **tropical polynomial**)

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R}, \\ f &:= \max_{\alpha\in\Delta} f_\alpha. \end{aligned}$$

The tropical amoeba is the non-smooth locus of f.

Let v_c denote the concave hull of v, defined on the whole polytope $\Delta_{\mathbb{R}} \subset \mathbb{R}^n$. f is the **Legendre dual** of v_c. It follows that the tropical amoeba is dual to the decomposition of Δ induced by the non-smooth locus of v_c. We choose v so that it induces a decomposition of Δ into primitive simplices (i.e., ones of minimal volume).

Example: $\Delta = \{(0,0),(1,0),(0,1)\}$ and $v \equiv 0$. Then $X_\Delta = \mathbb{CP}^1 \setminus \{3 \text{ points}\}$, the 1-dimensional **pair of pants**. We have

$$f(x,y) = \max(0,x,y)$$

and the tropical amoeba looks like this:

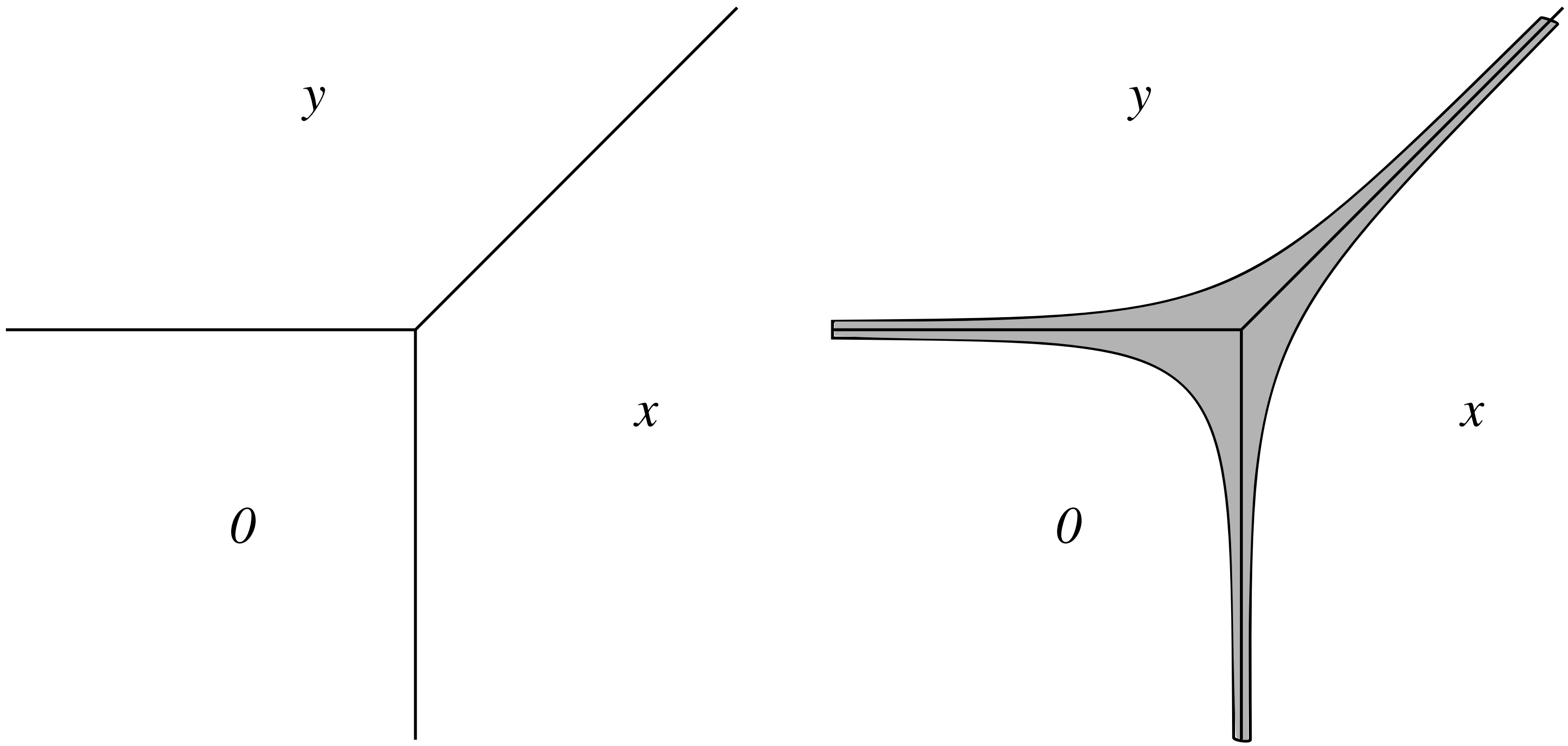

Figure 1: On the left, the tropical amoeba of the pair of pants $\{1+x+y=0\} \subset (\mathbb{C}^*)^2$; on the right, the amoeba for some finite t together with the tropical amoeba.

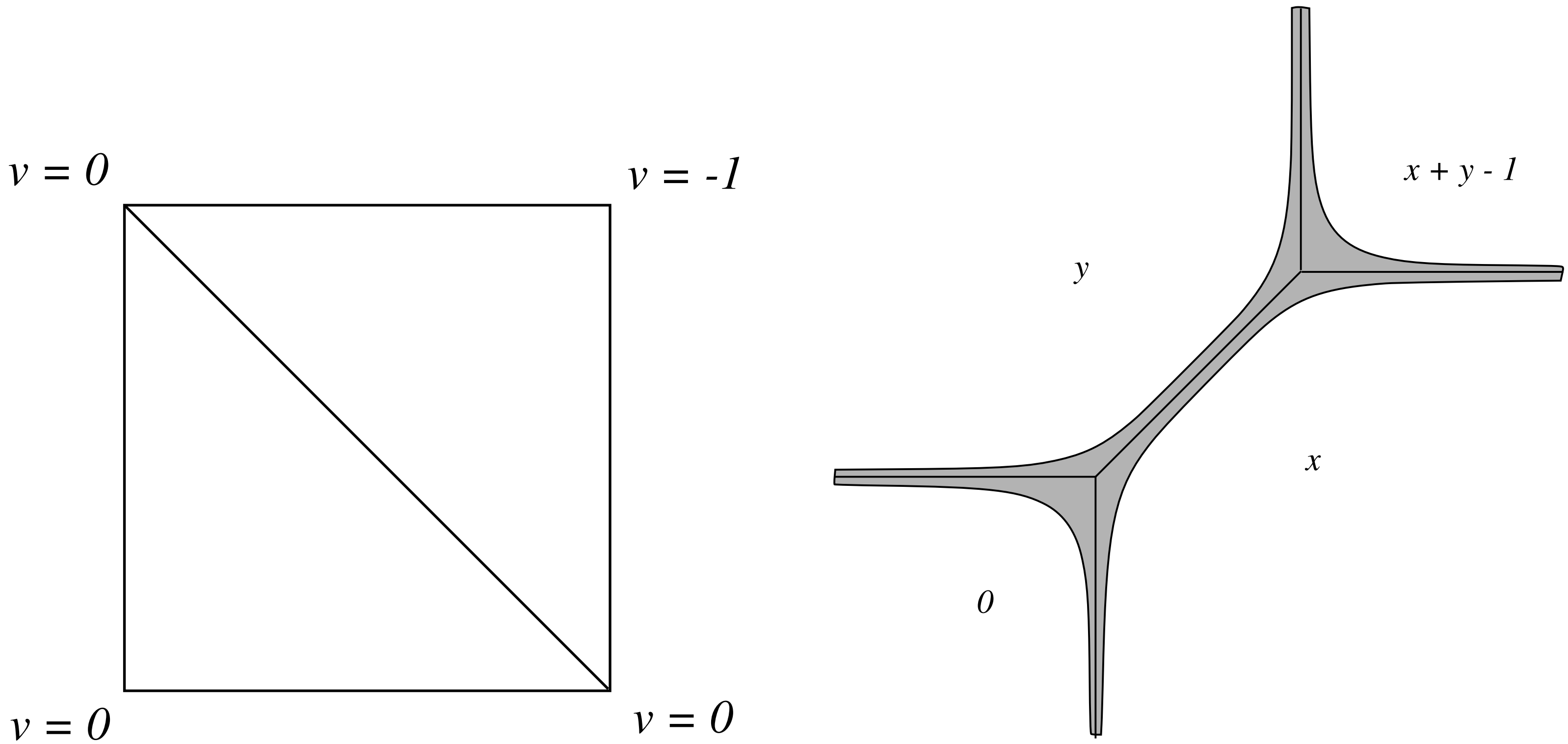

Figure 2: On the left, $\Delta = \{(0,0),(1,0),(0,1),(1,1)\}$, with v as shown. On the right, the corresponding amoeba and tropical amoeba.

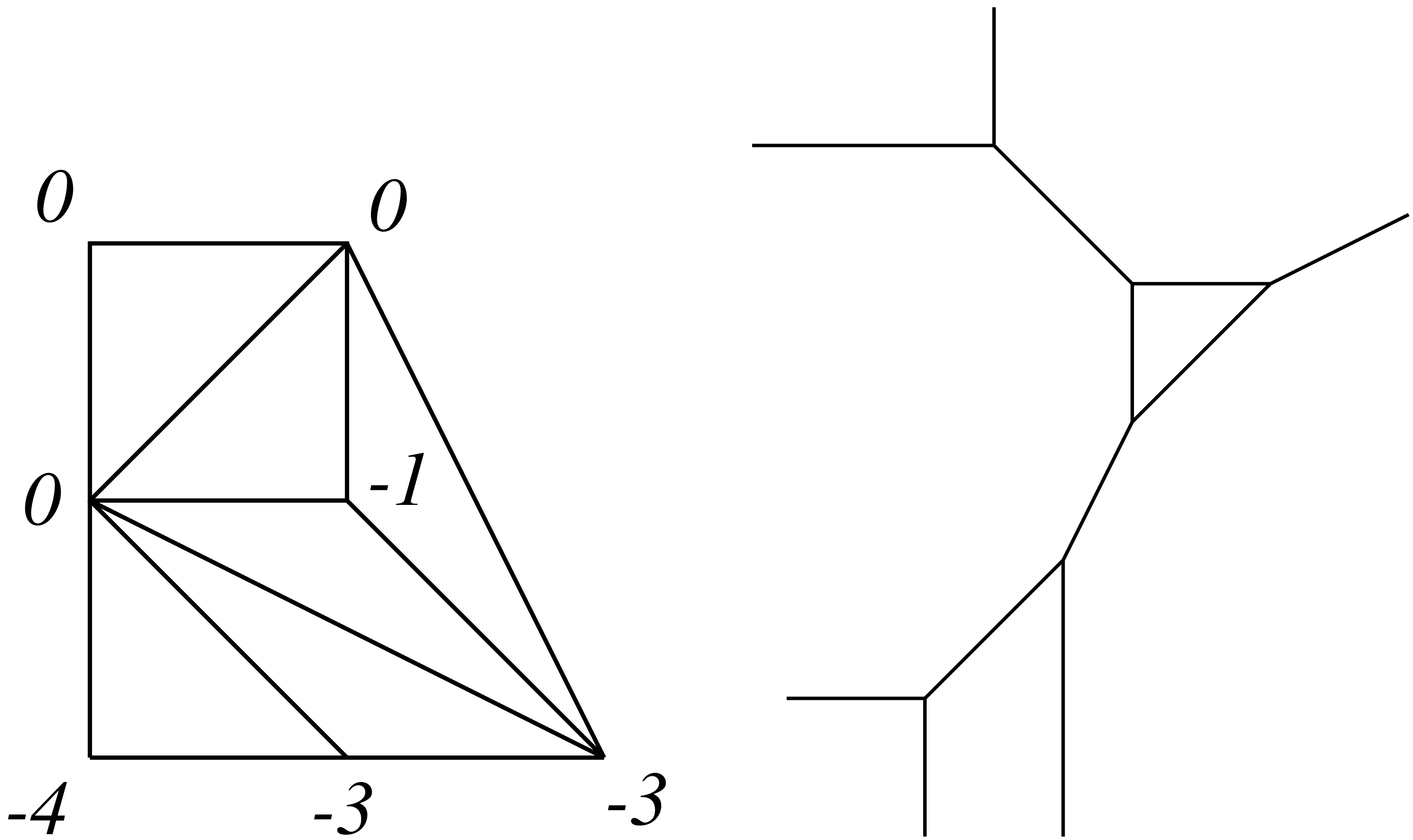

Figure 3: On the left, a more complicated Δ and v. On the right, the corresponding tropical amoeba.

Example: $\Delta = \{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ and $v \equiv 0$. Then

$$X_\Delta = \{1 + z_1 + \ldots + z_n = 0\} \subset (\mathbb{C}^*)^n,$$

which is isomorphic to $\mathbb{CP}^n \setminus \{n + 2 \text{ hyperplanes}\}$. This manifold is called the $(n-1)$-dimensional **pair of pants**.

We have

$$f(r_1, \ldots, r_n) = \max(0, r_1, \ldots, r_n).$$

When $n = 2$, the tropical amoeba looks like:

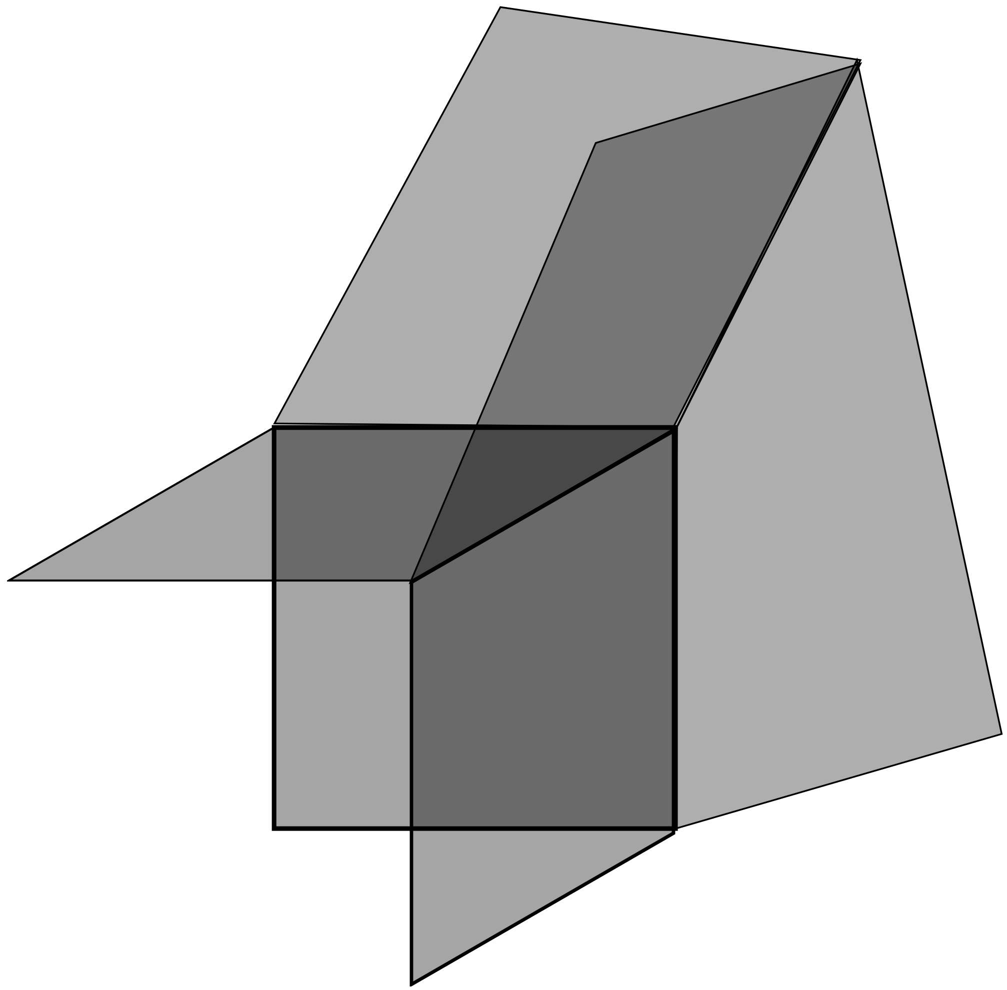

Figure 4: The tropical amoeba of the 2-dimensional pair of pants, $\{1 + x + y + z = 0\} \subset (\mathbb{C}^*)^3$.

Because v induces a maximal triangulation, there are at most $n+1$ dominant tropical monomials f_α at any point. So for sufficiently large t, all of X_Δ^t looks locally like a pair of pants. We obtain a decomposition of X_Δ^t into pairs-of-pants, encoded by the tropical amoeba.

This gives useful topological data already: when $n = 2$, it becomes clear that the genus of a curve X_Δ is equal to the number of internal points of Δ. For example, a degree-d curve in $\mathbb{CP}^2$ has genus $(d-1)(d-2)/2$.

Consider the argument map

$$\text{Arg} : (\mathbb{C}^*)^n \to (S^1)^n.$$

In the same way that the amoeba

$$\text{Log}_t(X_\Delta^t) \subset \mathbb{R}^n$$

Hausdorff converges to the tropical amoeba, the image

$$(\text{Log}_t, \text{Arg})(X_\Delta^t) \subset (\mathbb{C}^*)^n$$

Hausdorff converges to a certain subset X_Δ^{trop}, which projects to the tropical amoeba. The fibre over a point $\boldsymbol{r}$ is the set of all $\boldsymbol{\theta} \in (S^1)^n$ such that

$$\sum_{\alpha : f_\alpha(\boldsymbol{r}) \text{ maximal}} \boldsymbol{z}^\alpha = 0$$

for some $\boldsymbol{z}$ such that $\text{Arg}(\boldsymbol{z}) = \boldsymbol{\theta}$.

Since everything locally looks like a pair of pants, it suffices to understand what X_Δ^{trop} looks like for the pair of pants. Consider the case $n = 2$ first. The fibre over a point on the leg $0 = x > y$, for example, is the set

$$\left\{(\theta_1, \theta_2) \in (S^1)^2 : 1 + e^{r_1 + i\theta_1} = 0 \text{ for some } r_1\right\},$$

which is obviously the circle $\{\theta_1 = \pi\}$. Thus there is a cylinder lying above this leg, and similarly for the others.

Over the central point $(0, 0)$, where all three monomials tie for largest, the fibre is

$$\left\{(\theta_1, \theta_2) \in (S^1)^2 : 1 + e^{r_1 + i\theta_1} + e^{r_2 + i\theta_2} = 0 \text{ for some } r_1, r_2\right\}.$$

It's called the **tighty-whiteys** of the pair of pants, and it looks like this:

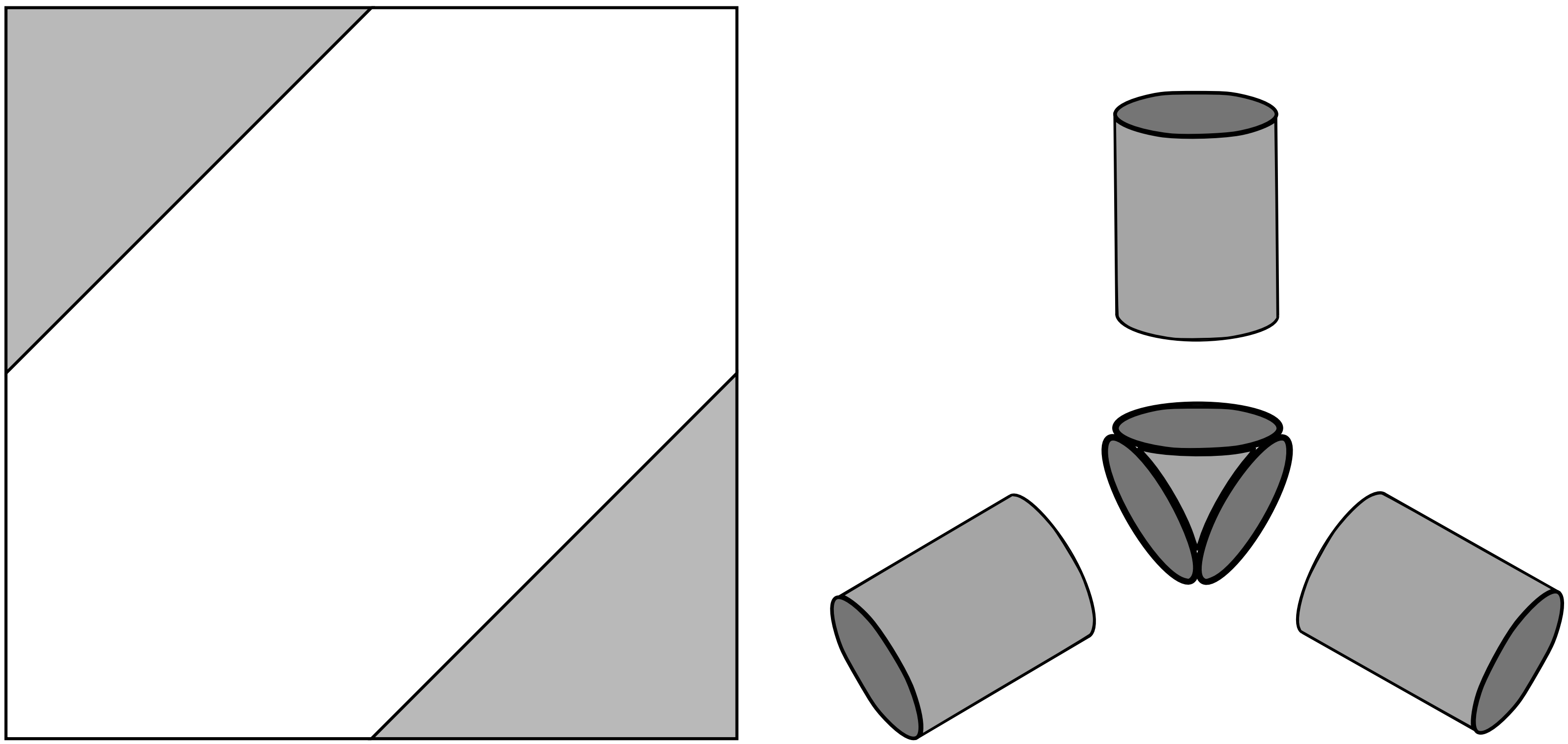

Figure 5: On the left, the tighty-whiteys of the 1-dimensional pair of pants, sitting in the torus. On the right, the corresponding X_Δ^{trop}.

Now let's look at $n = 3$. The tighty-whiteys of the 2-dimensional pair of pants look like this:

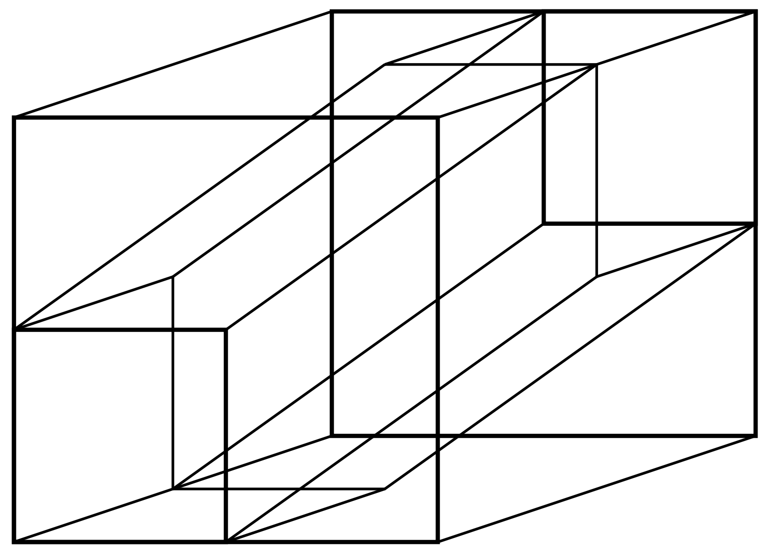

Figure 6: The tighty-whiteys of the 2-dimensional pair of pants. They are the complement in the 3-torus of the 'crystal' shape shown.

In general, the tighty-whiteys of the n-dimensional pair of pants are the complement in $\mathbb{R}^{n+1}/(2\pi\mathbb{Z})^{n+1}$ of the interior of the 'zonotope' centred at $(0, 0, \ldots, 0)$ generated by the vectors

$$\pi e_1, \pi e_2, \ldots, \pi e_{n+1}, \text{ and } -\pi e_1 - \ldots - \pi e_{n+1}.$$

The Arg projection is a homotopy equivalence from the pants onto the tighty-whiteys.

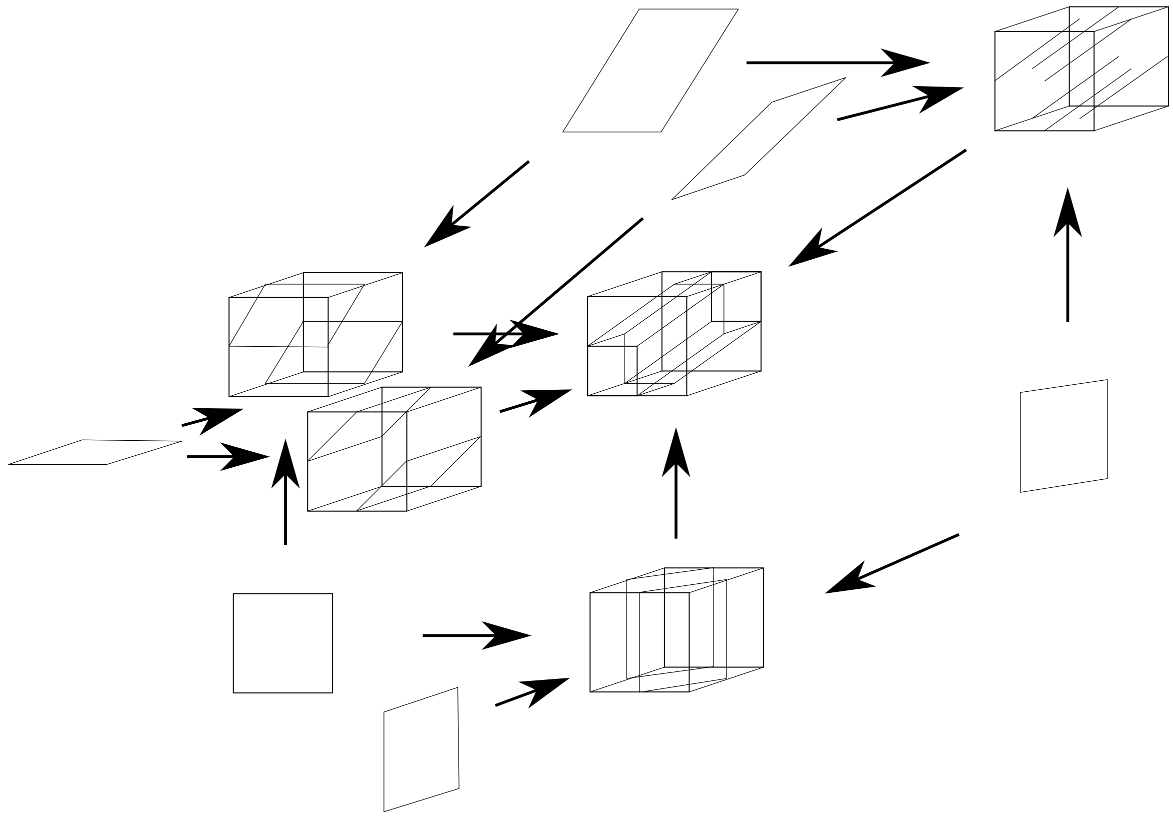

Figure 7: Diagram of X_Δ^{trop} for the 2-dimensional pair of pants, as a fibration over its tropical amoeba. The fibres over the 2-cells of the amoeba are 2-tori, parallel to the cell they lie over (drawn as squares on an angle). The fibres over the 1-cells look like the tighty-whiteys of the 1-dimensional pair of pants crossed with a circle (the one on the upper right is tricky to draw, but is equivalent to the others). The fibre over the 0-cell is the tighty-whiteys of the 2-dimensional pair of pants. The arrows denote gluing maps.

The tighty-whiteys are homotopy equivalent to the pair of pants, and have the homotopy type of a torus with a point removed. Hence n-dimensional pants are homotopy equivalent to a cell complex of dimension n. Can we obtain a more symmetric-looking cell complex?

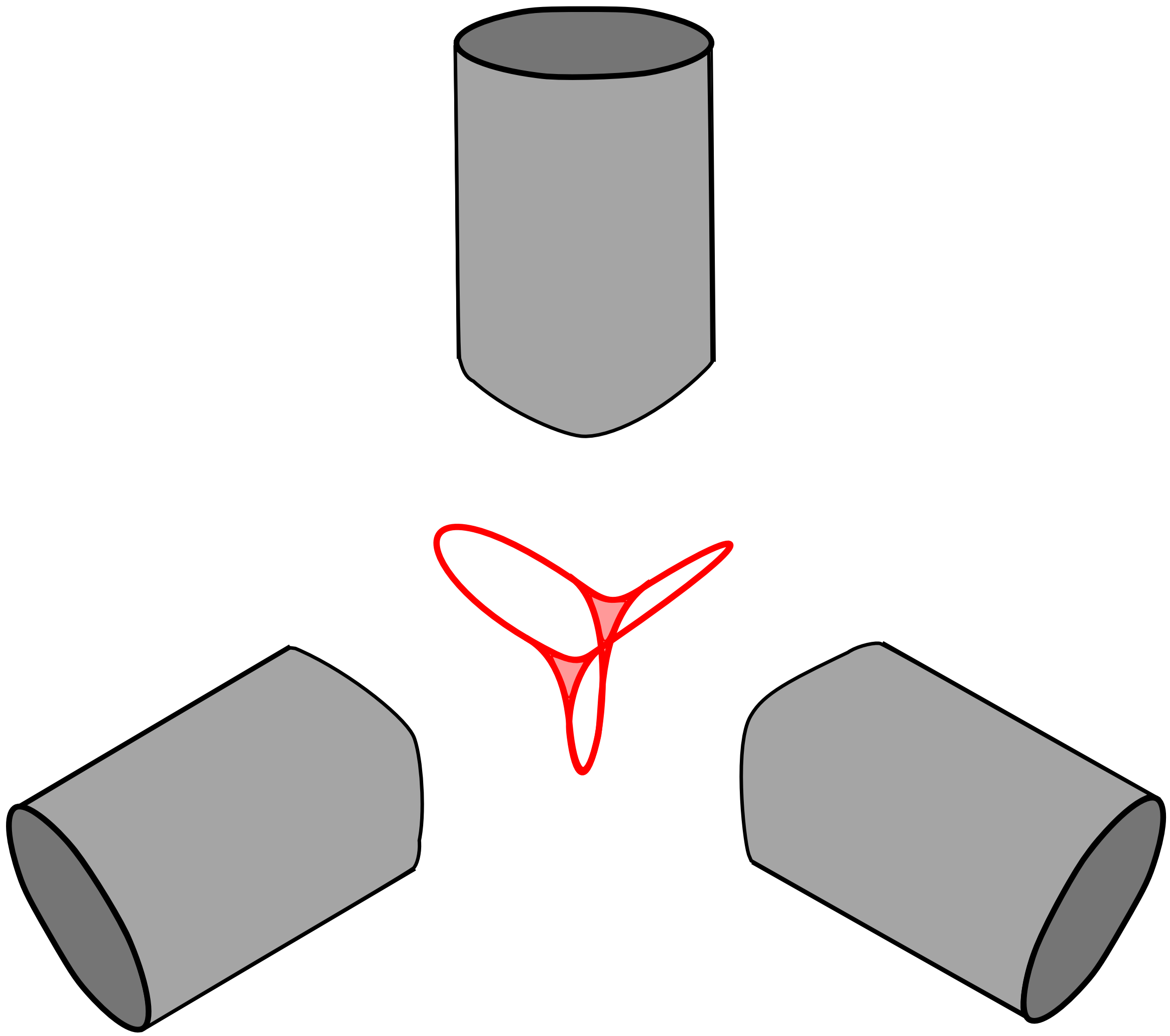

Figure 8: The tropical thong for $n = 2$: three segments attached to two points.

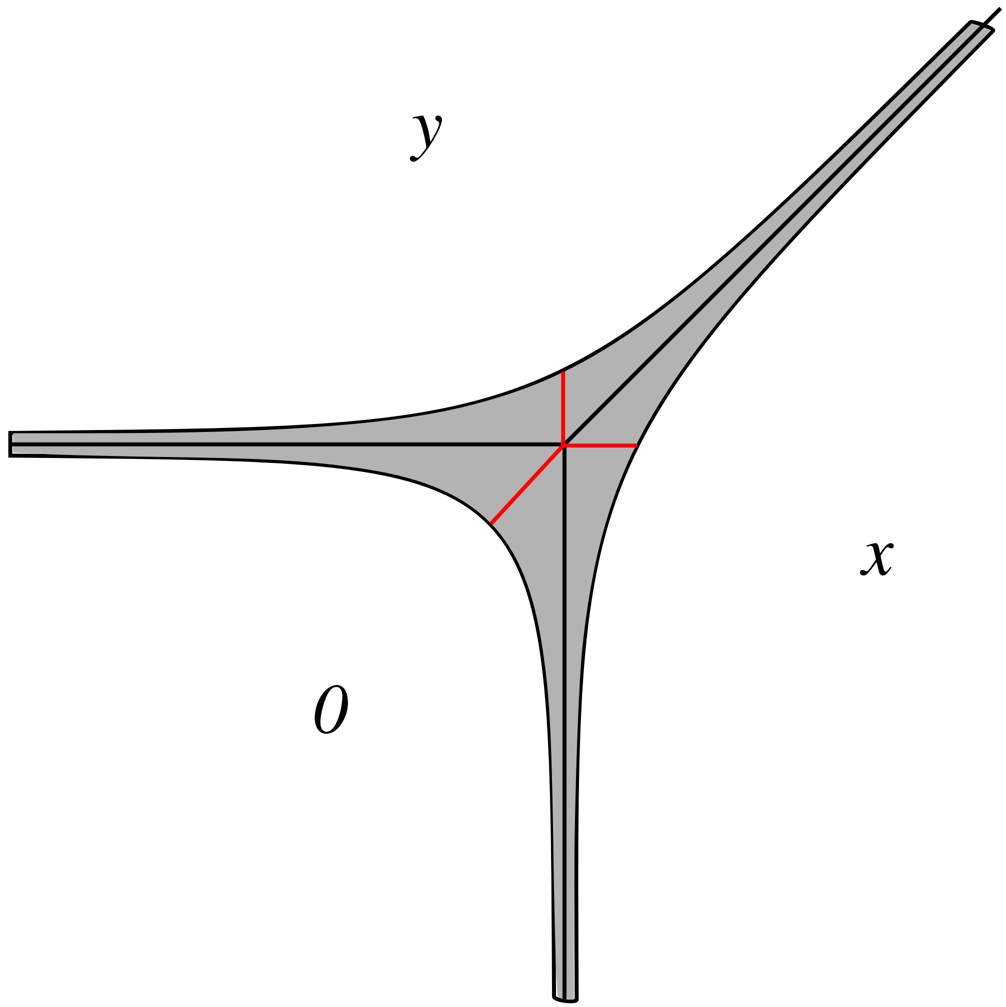

Figure 9: The tropical thong for $n = 2$ is the preimage of the red curve.

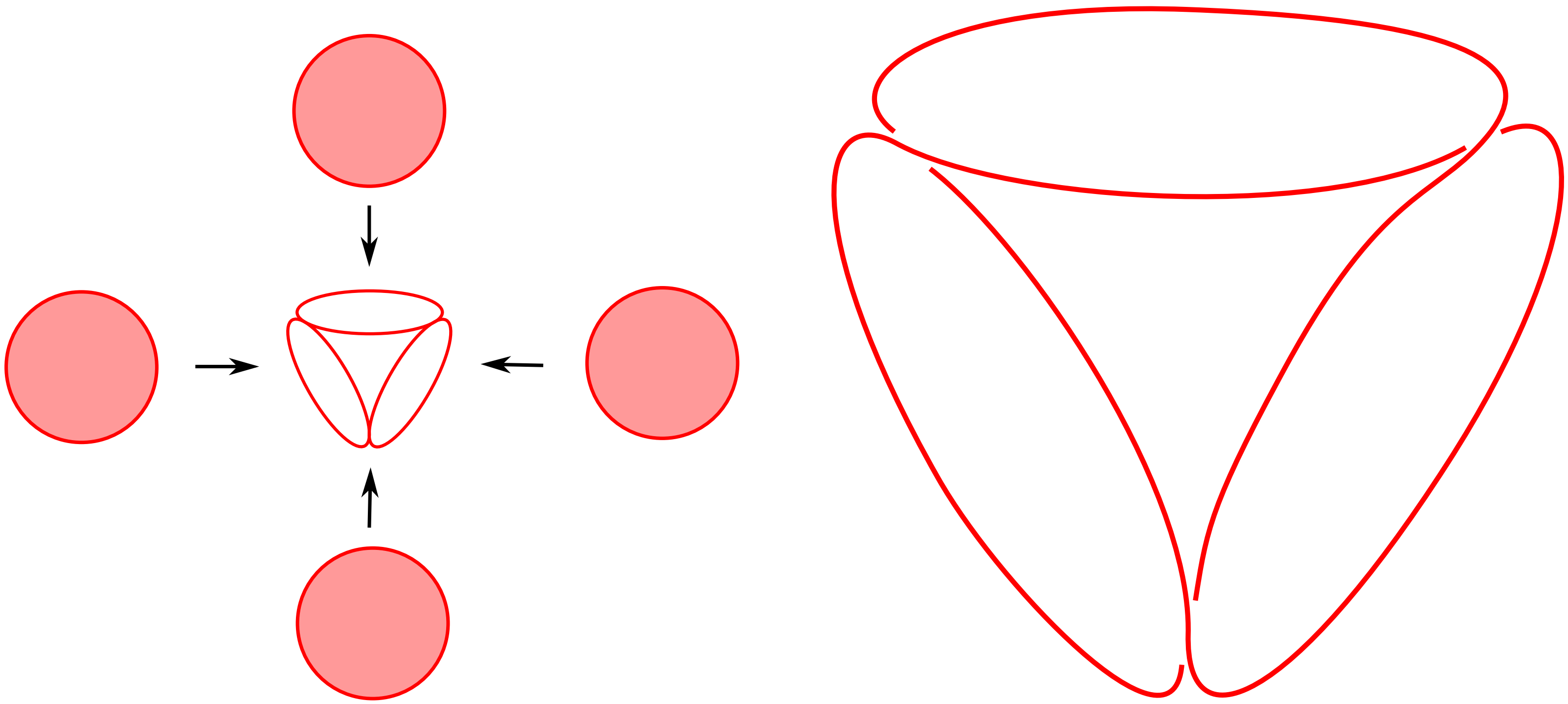

Figure 10: The tropical thong for $n = 3$ is obtained by stitching four disks onto three mutually intersecting circles, using all possible stitching maps that cover the circles exactly once and move clockwise (one example is shown on the right).