The University of Melbourne,
Department of Mathematics and Statistics

Hamilton's Ricci Flow

Nick Sheridan

Supervisor: Associate Professor Craig Hodgson
Second Reader: Professor Hyam Rubinstein

Honours Thesis, November 2006.

Abstract

The aim of this project is to introduce the basics of Hamilton's Ricci Flow. The Ricci flow is a PDE for evolving the metric tensor in a Riemannian manifold to make it "rounder", in the hope that one may draw topological conclusions from the existence of such "round" metrics. Indeed, the Ricci flow has recently been used to prove two very deep theorems in topology, namely the Geometrization and Poincaré Conjectures. We begin with a brief survey of the differential geometry that is needed in the Ricci flow, then proceed to introduce its basic properties and the basic techniques used to understand it, for example, proving existence and uniqueness and bounds on derivatives of curvature under the Ricci flow using the maximum principle. We use these results to prove the "original" Ricci flow theorem – the 1982 theorem of Richard Hamilton that closed 3-manifolds which admit metrics of strictly positive Ricci curvature are diffeomorphic to quotients of the round 3-sphere by finite groups of isometries acting freely. We conclude with a qualitative discussion of the ideas behind the proof of the Geometrization Conjecture using the Ricci flow.

Most of the project is based on the book by Chow and Knopf [6], the notes by Peter Topping [28] (which have recently been made into a book, see [29]), the papers of Richard Hamilton (in particular [9]) and the lecture course on Geometric Evolution Equations presented by Ben Andrews at the 2006 ICE-EM Graduate School held at the University of Queensland. We have reformulated and expanded the arguments contained in these references in some places. In particular, the proof of Theorem 7.19 is original, based on a suggestion by Gerhard Huisken. We also diverge from the existing references by emphasising the analogy between the techniques applied to the Ricci flow and those applied to the curve-shortening flow, which we feel helps clarify the important ideas behind the technical details of the Ricci flow. Chapter 6 is based on [6, Chap. 6, 7], but we have significantly reformulated the material and elaborated on the proofs. We feel that our organization is easier to follow than Chow and Knopf's book. The attempt to motivate the compactness result in Section 8.1 is also original.

Acknowledgements: Thanks to my supervisor Craig Hodgson for being so generous with his time and for always explaining things in the best possible way. Thanks also to Ben Andrews and Gerhard Huisken for the help they gave me at the ICE-EM Graduate School in July of this year and afterwards, and thank you to ICE-EM and the University of Queensland for running this excellent program.

Contents

Chapter 1

Riemannian Geometry

We will sweep through the basics of Riemannian geometry in this chapter, with a focus on the concepts that will be important for the Ricci flow later. Most proofs will be neglected for brevity, as the main point of much of the chapter is to establish conventions. The particularly useful formulae have been collected in Appendix B. We also give a flavour in Section 1.5 of how geometry can be used to draw topological conclusions. The material in this chapter is presented in much more detail in many texts – for example, [14] presents the basics of manifolds, tangent vectors, tensors and the Lie derivative. The book [18] is an excellent place to learn the theory of curvature.

1.1 Vectors, Tensors and Metrics

A topological space $\mathcal{M}^n$ is an n**-manifold** if it looks like Euclidean space ($\mathbb{R}^n$) near each point. The formal definition has some other technical conditions as well, to avoid certain pathologies that may arise:

Definition 1.1. *A topological space $\mathcal{M}^n$ is a **topological n-manifold** if:*

1. *For each $p \in \mathcal{M}^n$ there is an open neighbourhood U of p and a function $\varphi : U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$. The pair (U, φ) is called a **coordinate chart**. We will frequently write $\varphi(q) = (x^1(q), x^2(q), \ldots, x^n(q))$. These $x^i(q)$ are referred to as **local coordinates** for $\mathcal{M}^n$.*
2. *$\mathcal{M}^n$ is Hausdorff.*
3. *$\mathcal{M}^n$ is paracompact (see [26, Vol. I, App. A, Chap. 1] for a discussion of this condition).*

We will usually write $\mathcal{M}$ for a generic manifold, and $\mathcal{M}^n$ for an n-manifold if the dimension is of particular relevance. In this project we will deal with smooth manifolds, which have more structure than topological manifolds. In order to define what a smooth manifold is, we must first define the concept of a smooth function between subsets of Euclidean space.

Definition 1.2. *A function $f : U \to \mathbb{R}^m$, where U is an open subset of $\mathbb{R}^n$, is called **smooth** or C^∞ if all of its partial derivatives exist and are continuous on U.*

Now we can define the concept of a smooth manifold.

Definition 1.3. *Given two coordinate charts (U, φ) and (V, ψ) on a manifold $\mathcal{M}$, with $U \cap V \neq \emptyset$, we call the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ a **transition map**. Note that each transition map is a homeomorphism from an open set in $\mathbb{R}^n$ to another open set in $\mathbb{R}^n$. We make the definitions:*

1. *$\mathcal{M}$ is called **smooth** or C^∞ if all the transition maps are smooth.*
2. *$\mathcal{M}$ is **orientable** if all the transition maps are orientation-preserving.*

It is now possible to define the the concept of a smooth map between manifolds.

Definition 1.4. *Let $f : \mathcal{M} \to \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are smooth manifolds. f is called* ***smooth*** *if, for every pair of coordinate charts (U, φ) of $\mathcal{M}$ and (V, ψ) of $\mathcal{N}$, the function*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(f(U) \cap V)$$

is smooth.

As a special case we can set $\mathcal{N} = \mathbb{R}$, which has a natural smooth manifold structure. The set of all smooth real-valued functions $f : \mathcal{M} \to \mathbb{R}$ is denoted $C^\infty(M)$.

Two topological manifolds are equivalent if they are homeomorphic. The notion of equivalence on smooth manifolds is a bit more subtle.

Definition 1.5. *Two smooth manifolds $\mathcal{M}, \mathcal{N}$ are equivalent if there exists a smooth function $f : \mathcal{M} \to \mathcal{N}$ which has a smooth inverse. We will call such a function f a* ***diffeomorphism***[1] *and say that $\mathcal{M}$ and $\mathcal{N}$ are* ***diffeomorphic****.*

In 3 dimensions, topological and smooth manifolds are essentially equivalent. That is, any topological manifold is homeomorphic to a unique smooth manifold, and vice versa. This result is not true for higher dimensions. However most of the results in this project will deal with 3-manifolds, so it is of interest to note that we do not lose any generality by assuming that our 3-manifolds are smooth.

Now that we know what a manifold is, we would like to define a tangent vector to our manifold $\mathcal{M}$ at a point $p \in \mathcal{M}$.

Definition 1.6. *A* ***tangent vector*** *to a smooth manifold $\mathcal{M}$ at a point $p \in \mathcal{M}$ is a* ***derivation****, that is, an $\mathbb{R}$-linear function $X : C^\infty(\mathcal{M}) \to \mathbb{R}$ satisfying the product rule:*

$$X(fg) = X(f)g(p) + f(p)X(g).$$

The set of all tangent vectors to an n-manifold $\mathcal{M}^n$ at p forms an n-dimensional vector space $T_p\mathcal{M}^n$.

We note that this definition is related to the more intuitive notion of a tangent vector as a velocity vector of a curve $\gamma : (-\epsilon, \epsilon) \to \mathcal{M}$ in the manifold. If $\gamma(0) = p$ then we associate to the velocity vector of γ at p the derivation $X \in T_p\mathcal{M}$, where

$$X(f) = \frac{d}{dt} f(\gamma(t))|_{t=0}.$$

We then write $X = \dot{\gamma}(0)$.

If (x^i) is a local coordinate system about p on an n-manifold $\mathcal{M}^n$, then the set of derivations $\{\partial/\partial x^i, i = 1, 2, \ldots, n\}$ forms a basis for $T_p\mathcal{M}^n$. To avoid unseemly typesetting nightmares we will often write ∂_i for $\partial/\partial x^i$ if there can be no confusion about the coordinate system being used. The set of all tangent vectors at all points of $\mathcal{M}^n$ forms a $(2n)$-manifold known as the **tangent bundle**, and denoted $T\mathcal{M}^n$.

A **vector field** on a manifold $\mathcal{M}$ is a smoothly-varying choice of tangent vector at each point $p \in \mathcal{M}$. Here "smoothly-varying" means $X(f) \in C^\infty(\mathcal{M})$ for any $f \in C^\infty(\mathcal{M})$.

We note in passing that there is some additional structure to $T\mathcal{M}$ on top of the vector space structure on $T_p\mathcal{M}$: given two vector fields X, Y on $\mathcal{M}$, we can form their **Lie bracket** $[X, Y]$, defined by

$$[X, Y]f = X(Y(f)) - Y(X(f))$$

(it turns out that the $[X, Y]$ defined in this way is a tangent vector in the sense of being a derivation as described above, although this is not immediately obvious).

The example of tangent vectors is a specific case of a more general construction on a manifold, known as a **vector bundle**. The idea is that one associates a vector space to each point of the manifold $\mathcal{M}$, then glues these vector spaces together so as to get a new, higher-dimensional manifold.

[1] A diffeomorphism is usually defined to be a differentiable map with a differentiable inverse. The distinction is important when dealing with manifolds of varying levels of differentiability, but in this project we will deal exclusively with smooth manifolds. Thus the slight abuse of terminology will not cause any confusion.

Definition 1.7. *A k-dimensional vector bundle is a manifold $\mathcal{E}$ (the **total space**) together with a manifold $\mathcal{M}$ (the **base space**) and a surjective map $\pi : \mathcal{E} \to \mathcal{M}$ (the **projection**) such that*

1. *For each $p \in \mathcal{M}$, the set $\mathcal{E}_p := \pi^{-1}(p)$ (the **fibre** of $\mathcal{E}$ over p) has a k-dimensional vector space structure.*
2. *For each $p \in \mathcal{M}$ there is an open neighbourhood U of p and a smooth diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (a **local trivialization**) such that φ takes each fibre $\mathcal{E}_p$ to the corresponding fibre $\{p\} \times \mathbb{R}^k$ by a linear isomorphism.*

*A **section** of $\mathcal{E}$ is a map $F : \mathcal{M} \to \mathcal{E}$ such that $\pi \circ F = \mathrm{Id}_{\mathcal{M}}$. The space of sections of $\mathcal{E}$ is denoted $C^\infty(\mathcal{E})$.*

The tangent bundle is an n-dimensional vector bundle with base space $\mathcal{M}$ and projection defined by $\pi(X) = p$ if $X \in T_p\mathcal{M}$. A vector field is a section of the tangent bundle.

Another important example of a vector bundle is the dual bundle to the tangent bundle, known as the **cotangent bundle**, $T^*\mathcal{M}$. The fibre $T_p^*\mathcal{M} = (T_p\mathcal{M})^*$ consists of all linear functionals acting on the vector space $T_p\mathcal{M}$ (the **covectors** or **1-forms** at p). Given a local coordinate system (x^i), $i = 1, \ldots, n$ about p on an n-manifold $\mathcal{M}^n$, the set of covectors $\{dx^i, i = 1, 2, \ldots, n\}$ (where $dx^i(X) := X(x^i)$) forms a basis for $T_p^*\mathcal{M}^n$.

This method of constructing new vector bundles from old can be generalized. Let $\mathcal{V}$ be the category of finite-dimensional real vector spaces. Given vector bundles $\mathcal{E}^1, \mathcal{E}^2, \ldots, \mathcal{E}^k$ over $\mathcal{M}$ and a covariant functor $T : \mathcal{V} \times \mathcal{V} \times \ldots \times \mathcal{V} \to \mathcal{V}$, it is possible to form a unique vector bundle $\mathcal{E} = T(\mathcal{E}^1, \mathcal{E}^2, \ldots, \mathcal{E}^k)$ over $\mathcal{M}$ having fibres $\mathcal{E}_p = T(\mathcal{E}_p^1, \mathcal{E}_p^2, \ldots, \mathcal{E}_p^k)$ (see [20], Chapter 3). The cotangent bundle arises in the case $k = 1$, $\mathcal{E}^1 = T\mathcal{M}$, and $T(V) = V^*$.

In this way we can form the tensor product of vector bundles, by making T the tensor product functor on vector spaces. We define a $\binom{k}{l}$**-tensor field** to be a section of

$$T_l^k(\mathcal{M}) := \overbrace{T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes \ldots \otimes T^*\mathcal{M}}^{k \text{ copies}} \otimes \overbrace{T\mathcal{M} \otimes T\mathcal{M} \otimes \ldots \otimes T\mathcal{M}}^{l \text{ copies}}.$$

Given a local coordinate system (x^i) about $p \in \mathcal{M}$, we can express any $\binom{k}{l}$-tensor field F in the coordinate system as

$$F = F_{i_1 \ldots i_k}^{j_1 \ldots j_l}(p) \partial_{j_1} \otimes \ldots \otimes \partial_{j_l} \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_k}. \tag{1.1}$$

In this equation we sum over each index j_p, i_q that is repeated twice, once raised and once lowered – this is known as the **Einstein summation convention**. We will almost always use this coordinate representation of tensors because it makes technical calculations easier a lot of the time, and we will often write $F_{i_1 \ldots i_k}^{j_1 \ldots j_l}$ when we mean F.

Definition 1.8. *Given a map between manifolds $\phi : \mathcal{M} \to \mathcal{N}$, we define the **derivative map** between the corresponding tangent spaces, $\phi_* : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ by*

$$(\phi_* V)(f) = V(f \circ \phi)$$

for $V \in T_p\mathcal{M}$ and $f \in C^\infty(\mathcal{N})$. Defining $\phi_*(A \otimes B) := \phi_*(A) \otimes \phi_*(B)$, we can extend this definition to apply to all $\binom{0}{k}$-tensors.

In a similar way we can define the derivative map between the corresponding cotangent spaces, $\phi^* : T_{f(p)}^*\mathcal{N} \to T_p^*\mathcal{M}$, by

$$(\phi^* \omega)(V) = \omega(\phi_* V)$$

for $V \in T_p\mathcal{M}$, $\omega \in T_{f(p)}^*\mathcal{N}$. By a similar method to above we can extend ϕ^* to apply to all $\binom{k}{0}$-tensors.

Given a tensor $F_{i_1 \ldots i_k}^{j_1 \ldots j_l} \in T_l^k(\mathcal{M})$ we can take the **trace** over one raised and one lowered index as follows:

$$(\mathrm{tr} F)_{i_2 \ldots i_k}^{j_2 \ldots j_l} = F_{p i_2 \ldots i_k}^{p j_2 \ldots j_l}$$

to get an element of $T^{k-1}_{l-1}(\mathcal{M})$. Note the use of the Einstein summation convention: the index p is summed over. Obviously the trace depends on which indices you choose to trace over – here we traced over the j_1 and i_1 indices. Although it is not immediately obvious, the resulting tensor does **not** depend on the local coordinate system you are working in.

A k**-form** on $\mathcal{M}$ is a section of $\wedge^k T^*\mathcal{M}$, i.e. a $\binom{k}{0}$-tensor field that is completely antisymmetric in all its indices. A k**-vector** field on $\mathcal{M}$ is a section of $\wedge^k T\mathcal{M}$.

Definition 1.9. *For a $\binom{2}{0}$-tensor A we write $A > 0$ ($A \geq 0$) if*

$$A(V,V) > 0 \ (A(V,V) \geq 0)$$

for all $V \in T\mathcal{M}, V \neq 0$. We can similarly write $A > B$ ($A \geq B$) if $A - B > 0$ ($A - B \geq 0$).

Definition 1.10. *A* ***Riemannian metric*** *on a smooth manifold $\mathcal{M}$ is a smoothly-varying inner product on the tangent space at each point of $\mathcal{M}$, i.e. a $\binom{2}{0}$-tensor field which is symmetric and positive definite at each point of $\mathcal{M}$. We will usually write g for a Riemannian metric, and g_{ij} for its coordinate representation. Given such a g, there is an induced norm on each $T_p\mathcal{M}$ which we write*

$$|X|_g := \sqrt{g(X,X)} \tag{1.2}$$

for $X \in T_p\mathcal{M}$. A manifold together with a Riemannian metric, $(\mathcal{M}, g)$, is called a ***Riemannian manifold****.*

When there is only one metric under consideration, we will usually neglect the subscript g in equation (1.2), but there will be some situations in the study of the Ricci flow where we will need to distinguish between the norms induced by different metrics. We will sometimes use the notation $\langle X, Y\rangle$ for $g(X,Y)$.

Note that a Riemannian metric is not actually a metric (although we will frequently say "metric" rather than "Riemannian metric" for brevity, in contexts where no confusion is possible). It can be thought of as an "infinitesimal metric". In fact any Riemannian metric g on a manifold $\mathcal{M}$ induces a bona fide metric on $\mathcal{M}$, as we will now see:

Definition 1.11. *Given a Riemannian metric g we can define the* ***length*** *of a piecewise C^1 curve $\gamma : [0,1] \to \mathcal{M}$ by*

$$\ell(\gamma) := \int_0^1 \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt$$

where $\dot\gamma(t) := d\gamma/dt$.

This allows us to define a metric d on $\mathcal{M}$ ***induced*** *by the metric g:*

$$d(p,q) := \inf\{\ell(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ curve in } \mathcal{M} \text{ starting at } p \text{ and ending at } q\}.$$

We will sometimes use the metric space notation for a ball: if $(\mathcal{M}, g)$ is a Riemannian manifold, $p \in \mathcal{M}$ and $r > 0$, then

$$B(p,r) := \{q \in \mathcal{M} : d(p,q) < r\}$$

where d is the metric induced by g.

Finally, we say that a map $\phi : \mathcal{M} \to \mathcal{N}$ between Riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ is an ***isometry*** *if it is a diffeomorphism and $\phi^* h = g$. In this case we say that the two Riemannian manifolds are* ***isometric****.*

It is not hard to see that an isometry between Riemannian manifolds in the sense just defined is also a metric-space isometry between the manifolds, if we view them as metric spaces with the metrics induced by the corresponding Riemannian metrics.

We note that any smooth manifold, by virtue of its paracompactness, admits a smooth Riemannian metric. Thus we do not lose any generality by tackling topological properties of smooth manifolds from the point of view of Riemannian geometry.

Given a Riemannian manifold $(\mathcal{M}, g)$ and a manifold $\mathcal{N}$ embedded in $\mathcal{M}$ (a **submanifold** of $\mathcal{M}$), there is an induced Riemannian metric $\bar g$ on $\mathcal{N}$ defined by restricting g to $T_p\mathcal{N}$ at each point $p \in \mathcal{N}$.

Given the metric g_{ij}, which is a positive-definite symmetric matrix at each point of $\mathcal{M}$, we define the **metric inverse** g^{ij} to be the inverse matrix at each point, satisfying $g^{ij}g_{jk} = \delta^i_k$ where δ^i_k is the Kronecker delta. Any inner product on a vector space gives a natural isomorphism $V \cong V^*$, via $X \mapsto X^\flat$, where $X^\flat(Y) = \langle X, Y \rangle$. In coordinates, $(X^\flat)_i = g_{ij}X^j$. In general, we can **lower** an index i on a tensor F^{ijk}_{pq}, for example, by setting

$$F^{jk}_{ipq} := g_{im}F^{mjk}_{pq}.$$

This takes us from an element of $T^k_l(\mathcal{M})$ to something in $T^{k+1}_{l-1}(\mathcal{M})$. We can similarly **raise** an index using g^{ij}. Using the metric inverse we can also define a norm on the space of tensors, for example

$$|F^{ijk}_{pq}|^2_g := g_{i_1i_2}g_{j_1j_2}g_{k_1k_2}g^{p_1p_2}g^{q_1q_2}F^{i_1j_1k_1}_{p_1q_1}F^{i_2j_2k_2}_{p_2q_2}.$$

We will make frequent use of the very convenient **$*$-notation** in later chapters. Given two tensors A, B, the expression $A * B$ means "some linear combination of traces of $A \otimes B$ with coefficients that do not depend on A or B". For example if we have $A = A^{klm}_{ij}$, $B = B^s_{pqr}$, then $A * B$ might represent

$$17A^{kil}_{ij}B^j_{lqr} - n!A^{lrs}_{qi}B^k_{isr}$$

where n is the dimension of the manifold, or

$$A^{ijk}_{ij}B^l_{klm}.$$

The meaning is obviously very broad, and is of most use when we want to obtain bounds on complicated combinations of tensor quantities (as we will in later chapters). The most useful property of this notation is that, for any given expression of the form $A * B$, there is a constant C which does not depend on A or B such that

$$|A * B| \leq C|A||B|$$

by the Cauchy-Schwarz inequality. As a particular case that we will use frequently, we have

Lemma 1.1. *If A is an $n \times n$ matrix then*

$$|A|^2 \geq \frac{1}{n}(\mathrm{tr}A)^2.$$

The $*$-notation can obviously be extended to multiple $*$-products like $A * B * \ldots * Z$ or powers $A^{*n} := A * A * \ldots * A$. We also define $*(A, B, \ldots, Z)$ to mean any combination of $*$-products of any powers of $A, B, \ldots, Z$, for example

$$B^{*3} * Z + A^{*2} * Z^{*4}.$$

In later chapters we will use the notation

$$\mathop{\Huge *}_{1 \leq i \leq n}(A_i) := *(A_1, A_2, \ldots, A_n).$$

1.2 The Covariant Derivative

We can differentiate scalar functions on a manifold $\mathcal{M}$ without any problem: to find the rate of change of a function f in the direction of the tangent vector X, we simply calculate $X(f)$. We ought also to be able to differentiate vector fields, or more generally a section Y of an arbitrary vector bundle, in the direction of a given tangent vector X.

Definition 1.12. *Given a vector bundle $\mathcal{E}$ over $\mathcal{M}$, a* **connection** *in $\mathcal{E}$ is a map*

$$\nabla : C^\infty(T\mathcal{M}) \times C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$$

with the following properties:

1. $\nabla_X Y$ is linear over $C^\infty(\mathcal{M})$ in X.

2. $\nabla_X Y$ is linear over $\mathbb{R}$ in Y.

3. ∇ satisfies the product rule:

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y.$$

*We call $\nabla(X,Y)$ the **covariant derivative** of Y in the direction X. We usually write $\nabla_X Y$ rather than $\nabla(X,Y)$.*

*It is possible to calculate $\nabla_X Y(p)$ if we are given X_p and the values of Y along a curve $\gamma : (-\epsilon,\epsilon) \to \mathcal{M}$ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. A section Y of $\mathcal{E}$ defined along a curve γ in $\mathcal{M}$ is said to be **parallel** along γ if $\nabla_{\dot\gamma(t)} Y = 0$ along γ.*

A connection on the vector bundle $\mathcal{E}$ is completely specified by its **Christoffel symbols** Γ_{ij}^k in a local coordinate system (x^i) with a local basis (E_j) for $\mathcal{E}$, defined by:

$$\nabla_{\partial_i} E_j = \Gamma_{ij}^k E_k.$$

As a very important special case, we can consider connections on the bundles $T_l^k(\mathcal{M})$.

Lemma 1.2. *Given a connection ∇ on the tangent bundle $T\mathcal{M}$, we can define connections on all of the tensor bundles $T_l^k(\mathcal{M})$ (which we will also denote ∇) satisfying:*

1. ∇ is the given connection on $T\mathcal{M}$.

2. For a scalar function f, $\nabla_X f = X(f)$.

3. $\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$.

4. ∇_X commutes with all traces:

$$\nabla_X(\mathrm{tr}Y) = \mathrm{tr}(\nabla_X Y)$$

for all traces (over any indices) of the tensor Y.

If F is a $\binom{k}{l}$-tensor field on $\mathcal{M}$ which is given in local coordinates by equation (1.1), we will write the coordinate form of the covariant derivative ∇F as

$$(\nabla_X F) := (\nabla_p F_{i_1\ldots i_k}^{j_1\ldots j_l})\partial_{j_1} \otimes \ldots \otimes \partial_{j_l} \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_k} X^p.$$

We can write down the coordinate form of ∇ explicitly:

$$\nabla_p F_{i_1\ldots i_k}^{j_1\ldots j_l} = \partial_p F_{i_1\ldots i_k}^{j_1\ldots j_l} + \sum_{s=1}^{l} F_{i_1\ldots i_k}^{j_1\ldots q\ldots j_l}\Gamma_{pq}^{j_s} - \sum_{s=1}^{k} F_{i_1\ldots q\ldots i_k}^{j_1\ldots j_l}\Gamma_{pi_s}^{q}. \tag{1.3}$$

In the second term of the above equation, the upper q index occupies the position normally occupied by j_s, and in the third term the lower q index occupies that position normally occupied by i_s.

Note that equation (1.3) can be expressed using the $*$-notation as

$$\nabla F = \partial F + \Gamma * F. \tag{1.4}$$

We can generalize this observation:

Lemma 1.3. *Let $\nabla^m F$ denote the mth iterated covariant derivative of F and $\partial^m F$ denote the coordinate expression*

$$(\partial^m F)_{i_1\ldots i_m} := \partial_{i_1\ldots i_m} F$$

in some local coordinate system[2] (x^i) *defined in a coordinate patch* U *on the manifold* $\mathcal{M}$*. Then*

$$\nabla^m F \quad = \quad \partial^m F + \sum_{i=0}^{m-1} \left(\underset{j \leq m-1}{\ast} (\partial^j \Gamma) \right) \ast \nabla^i F$$

in U*.*

Proof. The result follows from equation (1.4) by induction. □

Although there are many possible connections on the tangent bundle $T\mathcal{M}$, if $\mathcal{M}$ is equipped with a Riemannian metric then there is one in particular that has more geometric significance.

Lemma 1.4. *Given a Riemannian metric* g_{ij} *on* $\mathcal{M}$*, there is a unique connection* ∇ *on* $T\mathcal{M}$ *that satisfies*

1. $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ *(*∇ *is **compatible** with* g*). This is equivalent to* $\nabla g = 0$ *where* ∇g *is defined by Lemma 1.2.*

2. *The **torsion tensor** of* ∇*,*

$$\tau(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

   is identically 0.

*This connection is known as the **Levi-Civita connection** of the metric* g*. Its Christoffel symbols are given in local coordinates by*

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{1.5}$$

Using the Levi-Civita connection we can define the Laplacian:

Definition 1.13. *The **Laplacian***[3] *is a family of operators*

$$\triangle : C^\infty(T_l^k \mathcal{M}) \to C^\infty(T_l^k \mathcal{M})$$

(where $(\mathcal{M}, g)$ *is a Riemannian manifold) defined by*

$$\triangle F := g^{ij} \nabla_i \nabla_j F,$$

where ∇ *is the Levi-Civita connection of the metric* g*. If* F *has the coordinate form given in equation (1.1), we will write*

$$\triangle F_{i_1 \ldots i_k}^{j_1 \ldots j_l}$$

for

$$(\triangle F)_{i_1 \ldots i_k}^{j_1 \ldots j_l}.$$

If we are given a connection on $T\mathcal{M}$ (we will usually be using the Levi-Civita connection for some metric g_{ij}) we can define **geodesics** to be the paths that you could move along in the manifold "without feeling any force". That is, a path $\gamma : [0,1] \to \mathcal{M}$ is a geodesic if $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$ at each point. Given an initial point and velocity (or equivalently an element of $T\mathcal{M}$), there is a unique geodesic in $\mathcal{M}$ setting off from that point with the given initial velocity. If $\mathcal{M}$ is complete (in particular if it is closed, as it will be for all of our applications), the geodesic will exist for all time.

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[2] The coordinate expression $\partial^m F$ does not represent any coordinate-independent tensor field in the way that $\nabla^m F$ does. We regard $\partial^m F$ as representing the tensor field

$$\partial_{i_1 \ldots i_m} F dx^{i_1} \otimes \ldots \otimes dx^{i_m},$$

defined **only** in U. We similarly represent by Γ the tensor field

$$\Gamma_{ij}^k dx^i \otimes dx^j \otimes \partial_k,$$

also defined only in U.

[3] There exist other, non-equivalent definitions of the Laplacian in different settings. For that reason the Laplacian defined here is sometimes called the **rough Laplacian**. We will only ever use the rough Laplacian in this project, so we simply call it the "Laplacian".

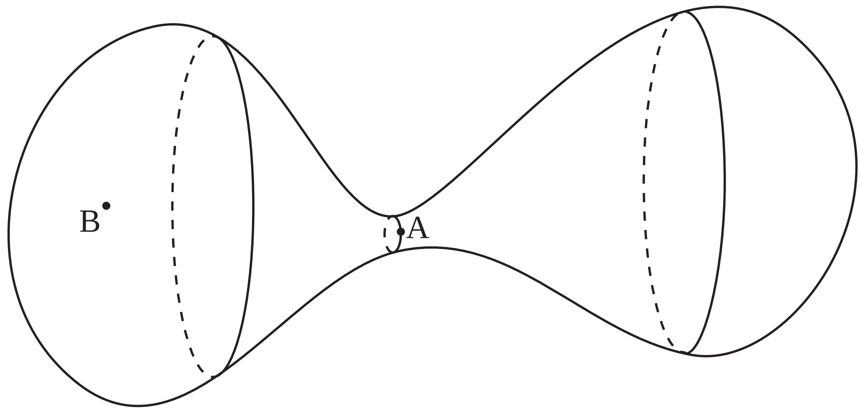

Figure 1.1: The injectivity radius: $\text{inj}(A)$ is small, $\text{inj}(B)$ is large.

Definition 1.14. *Suppose $\mathcal{M}$ is complete and has no boundary. Given $v \in T_p\mathcal{M}$, let $\gamma_v : [0,1] \to \mathcal{M}$ be the unique geodesic in $\mathcal{M}$ that starts at p with initial velocity vector v. That is, $\gamma_v(0) = p$ and $\gamma_p'(0) = v$. We define the* ***exponential map*** $\exp : T\mathcal{M} \to \mathcal{M}$ *by* $\exp(v) = \gamma_v(1)$*. We denote by* $\exp_p$ *the exponential map restricted to $T_p\mathcal{M}$. The exponential map is smooth.*

Because $\exp_p$ is locally a diffeomorphism at the origin of $T_p\mathcal{M}$, it is possible to choose an open neighbourhood U of p that is diffeomorphic via $\exp_p$ to an open set in $T_p\mathcal{M} = \mathbb{R}^n$, by the inverse function theorem. We choose coordinates $(x_i), i = 1, \ldots, n$ on $T_p\mathcal{M}$ so that the basis vectors $\{\partial_i : i = 1, \ldots, n\}$ are orthonormal with respect to the metric g at p. We then have a coordinate chart $(U, (\exp_p)^{-1})$. These coordinates are called **normal coordinates** about p and have some properties which make them very convenient for doing calculations:

Lemma 1.5. *In normal coordinates about p, we have*

1. $g_{ij} = \delta_{ij}$ *at p.*
2. *If $v \in \mathbb{R}^n$ then the curve $\gamma_v(t) = tv$ is a geodesic for as long as it exists.*
3. $\partial_k g_{ij} = 0$ *and* $\Gamma_{ij}^k = 0$ *at p. Thus*

$$\nabla_k F_{i_1 \ldots i_k}^{j_1 \ldots j_l} = \partial_k F_{i_1 \ldots i_k}^{j_1 \ldots j_l}$$

at p, by the formula (1.3).

Definition 1.15. *The* ***injectivity radius*** $\text{inj}(p)$ *at a point $p \in \mathcal{M}$ is defined by*

$$\text{inj}(p) := \sup\{r > 0 : \exp_p : B(0,r) \to \mathcal{M} \text{ is injective}\}.$$

The injectivity radius of a manifold $\mathcal{M}$ with metric g is defined by

$$\text{inj}(\mathcal{M}, g) := \inf\{\text{inj}(p) : p \in \mathcal{M}\}.$$

One can interpret the injectivity radius at $p \in \mathcal{M}$ as follows: imagine there is a flash of light at p at some time. Light rays propagate in all directions – the injectivity radius is the smallest distance one of these rays has to travel before it collides with another light ray. The injectivity radius is a crucial concept in the study of the convergence of manifolds, as we shall see in Chapter 8. Where the injectivity radius is small (as at A in Figure 1.1), the manifold is close to "pinching off" as described in Section 2.4.

1.3 The Lie Derivative

The covariant derivative gives one way of differentiating tensor fields. We will now look at a different way, which can be defined purely from the manifold structure of $\mathcal{M}$, without any reference to a Riemannian metric or the extra structure of a connection.

Given a vector field X on a manifold $\mathcal{M}$, we define a time-dependent family of diffeomorphisms of $\mathcal{M}$ to itself, $\varphi_t : \mathcal{M} \to \mathcal{M}$ for $t \in (-\epsilon, \epsilon)$, such that φ_0 is the identity and

$$\frac{d}{dt}\varphi_t = X$$

at each point (the existence of such φ_t follows from basic existence theorems for differential equations – see [5, Sec. 6-2] for this argument and other details relating to the Lie derivative). This is to be interpreted as a "flow" of the manifold in the direction of the vector field X. We now define the derivative of some $\binom{k}{l}$-tensor field F in the direction of X as the change in F when we move a small step in the direction of X – but we need to have some way of comparing the value of F at the point a little step away with that at the original point. Rather than making this comparison using a "connection" as before, we make it by pushing the value of F at the translated point back to the original point using the diffeomorphism φ_t.

We define

$$\begin{aligned}(*\varphi_t)F_p(X_1, \ldots, X_k, \omega^1, \ldots, \omega^l) \quad := \quad & F_{\varphi_t(p)}(\varphi_{t*}(X_{1(p)}), \ldots, \varphi_{t*}(X_{k(p)}), \\ & (\varphi_t^{-1})^*(\omega^1_{(p)}), \ldots, (\varphi_t^{-1})^*(\omega^l_{(p)})).\end{aligned}$$

Note that $(*\varphi_t)F_p \in T^k_l\mathcal{M}_p$ for all t. It is now possible to define the **Lie derivative** of F in the direction X as

$$\mathcal{L}_X F = \left(\frac{d}{dt}(*\varphi_t)F\right)_{t=0}$$

Lemma 1.6. *The Lie derivative is well-defined. It obeys similar conditions to those satisfied by the covariant derivative as outlined in Lemma 1.2, with one important exception:*

1. *For a scalar function f, $\mathcal{L}_X f = X(f)$.*
2. *If Y is a vector field then $\mathcal{L}_X Y = [X, Y]$.*
3. *$\mathcal{L}_X(F \otimes G) = (\mathcal{L}_X F) \otimes G + F \otimes (\mathcal{L}_X G)$.*
4. *$\mathcal{L}_X$ commutes with all traces:*
$$\mathcal{L}_X(\mathrm{tr}Y) = \mathrm{tr}(\mathcal{L}_X Y)$$
for all traces (over any indices) of the tensor Y.

Proof. See [5, Sec. 6-2]. □

Lemma 1.7. *On a Riemannian manifold $(\mathcal{M}, g)$, we have*

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i,$$

where ∇ denotes the Levi-Civita connection of the metric g, for any vector field X.

Proof. Let ω be the 1-form dual to the vector field X, $\omega(Y) = \langle X, Y\rangle$. Using the product rule (from Lemma 1.6) and the metric compatibility and torsion-free conditions on the Levi-Civita connection we have

$$\begin{aligned}
\mathcal{L}_X g(Y, Z) &= X(g(Y, Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z) \\
&= \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle - \langle[X, Y], Z\rangle - \langle Y, [X, Z]\rangle \\
&= \langle\nabla_X Y - [X, Y], Z\rangle + \langle Y, \nabla_X Z - [X, Z]\rangle \\
&= \langle\nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle \\
&= Y\langle X, Z\rangle - \langle X, \nabla_Y Z\rangle + Z\langle Y, X\rangle - \langle\nabla_Z Y, X\rangle \\
&= Y(\omega(Z)) - \omega(\nabla_Y Z) + Z(\omega(Y)) - \omega(\nabla_Z Y) \\
&= (\nabla_Y \omega)(Z) + (\nabla_Z \omega)(Y)
\end{aligned}$$

which is the coordinate-free way of expressing the identity we wanted. Note that we used the product rule again to get the last line. $\square$

1.4 Curvature

We start with a description of curvature for 2-manifolds. Given a Riemannian 2-manifold $(\mathcal{M}^2, g)$, we would like to define some quantity that describes "how curved" the surface is at a certain point $p \in \mathcal{M}^2$. Consider two tangent vectors $v_1, v_2 \in T_p\mathcal{M}^2$, with $|v_1| = |v_2| = 1$ (with respect to g). Let $\gamma_i : [0,1] \to \mathcal{M}^2, i = 1,2$ be two geodesics in $\mathcal{M}^2$ that start at p, with $\dot{\gamma_i}(0) = v_i$. Let us define $f : \mathbb{R}^+ \to \mathbb{R}^+$ so that $f(r)$ is the distance from $\gamma_1(r)$ to $\gamma_2(r)$ along the circle with centre p and radius r (where the radius is measured with respect to the metric d on $\mathcal{M}^2$ induced by the Riemannian metric g). If θ is the angle between v_1 and v_2 then it is clear that for small r, $f(r) \sim \theta r$, i.e. $f'(0) = \theta$ (see Figure 1.2).

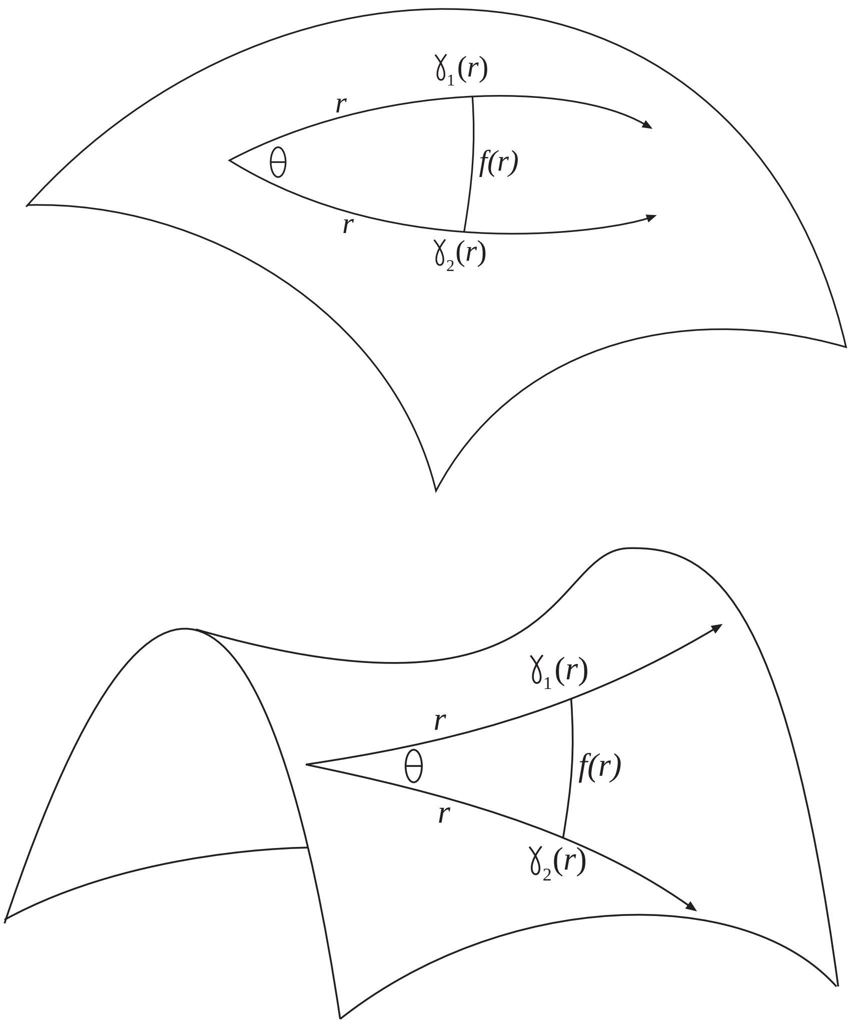

Figure 1.2: Top: geodesics in a 2-manifold of positive curvature converge. Bottom: geodesics in a 2-manifold of negative curvature diverge.

Curvature deals with second-order effects. We define the **Gaussian curvature**, K, of the surface at p so that

$$\frac{f(r)}{r} \sim \theta\left(1 - \frac{K}{6}r^2\right)$$

for small r. See Figure 1.2 for pictures of manifolds with positive curvature ($K > 0$) and negative curvature ($K < 0$). Euclidean space $\mathbb{R}^2$ is flat ($K = 0$).

The generalization to higher-dimensional manifolds of this concept of curvature is not at all obvious, and was first achieved by Riemann. His idea was that the curvature, intuitively, is the obstruction to the "flatness" of a manifold. In other words, non-zero curvature at p is what stops us from choosing coordinates in which the metric is the Euclidean, flat metric $g_{ij} = \delta_{ij}$ in a neighbourhood of p. Normal coordinates about p are, in some sense, "as close as we can get" to a Euclidean metric in a neighbourhood of p: we know from Lemma 1.5 that in normal coordinates about p, $g_{ij}(p) = \delta_{ij}(p)$ and $\partial_k g_{ij}(p) = 0$. However, in general, the second- and higher-order derivatives of the metric at p may be non-zero.

This motivates the definition of the $\binom{4}{0}$ **Riemann curvature tensor**, which holds all of the information about the second-order derivatives of g. It is defined to be the $\binom{4}{0}$-tensor with coordinates R_{ipqj} such that, in normal coordinates about p,

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}R_{ipqj}x^p x^q + (\text{third- and higher-order terms in } x). \tag{1.6}$$

The factor of 1/3 is introduced so that this definition agrees with the alternative one that we introduce next.

Since Riemann's original work, it has emerged that the Riemann curvature tensor can be defined in another way, using the Levi-Civita connection ∇. The $\binom{3}{1}$ **Riemann curvature tensor** is a $\binom{3}{1}$-tensor which can be defined for vector fields X, Y, Z and a 1-form ω by

$$\mathrm{Rm}(X, Y, Z, \omega) \quad := \quad \omega(R(X, Y)Z) \tag{1.7}$$

where

$$\begin{aligned} R(X, Y)Z &= \nabla^2 Z(X, Y) - \nabla^2 Z(Y, X) \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z. \end{aligned}$$

Note here that

$$(\nabla^2 Z)(X, Y) \neq \nabla_X(\nabla_Y Z).$$

In fact,

$$(\nabla^2 Z)(X, Y) = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y}Z.$$

There is a distinction because, by $\nabla^2 Z$, we really mean the covariant derivative of the $\binom{1}{1}$-tensor ∇Z. Thus $\nabla^2 Z$ is a $\binom{2}{1}$-tensor.

It is definitely not obvious (but can easily be checked) that this is in fact a $\binom{3}{1}$-tensor (in particular that, for example, $\mathrm{Rm}(X, Y, fZ, \omega) = f\mathrm{Rm}(X, Y, Z, \omega)$ for any $f \in C^\infty(\mathcal{M})$). We represent it in terms of local coordinates as R^l_{ijk}. The $\binom{3}{1}$ curvature tensor R^l_{ijk} defined by equation (1.7) is obtained from the $\binom{4}{0}$ curvature tensor R_{ijkl} defined by equation (1.6) by raising the final index.

The coordinates of Rm can be explicitly calculated by applying equation (1.3) with the Levi-Civita connection ∇:

Lemma 1.8. *The Riemann curvature tensor has the explicit form*

$$R^l_{ijk} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp}.$$

Rm has many symmetries, which can be proven by manipulating the defining equation (1.7):

Lemma 1.9. ***(Symmetries of the curvature tensor)*** *The Riemann curvature tensor has the following properties:*

1. $R_{ijkl} = R_{klij} = -R_{jikl} = -R_{ijlk}$

2. The first Bianchi identity:

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

3. The second Bianchi identity:

$$\nabla_p R_{ijkl} + \nabla_i R_{jpkl} + \nabla_j R_{pikl} = 0 \tag{1.8}$$

Proof. See [18, Prop. 7.4]. □

The first of these symmetries allows us to view Rm as a section of the bundle

$$\wedge^2 T^*\mathcal{M} \otimes_S \wedge^2 T^*\mathcal{M}$$

of symmetric bilinear forms on the space of 2-vectors. If $\phi = \phi^{ij}\partial_i \wedge \partial_j$ and $\psi = \psi^{ij}\partial_i \wedge \partial_j$ are 2-vectors then we define the action of the **curvature operator** at p, $\mathcal{R}_p : \wedge^2 T_p\mathcal{M} \otimes \wedge^2 T_p\mathcal{M} \to \mathbb{R}$, by

$$\mathcal{R}(\phi, \psi) = R_{ijkl}\phi^{ij}\psi^{lk}$$

(all evaluated at p). Note that the curvature operator is defined on 2-vectors by the antisymmetry in the first two and in the last two indices of Rm, and is symmetric because of the symmetry $R_{ijkl} = R_{klij}$.

We can now explain the relationship with Gaussian curvature. If there is a 2-plane element $\phi \in \wedge^2 T_p\mathcal{M}$ representing a 2-dimensional subspace Π of $T_p\mathcal{M}$, we can imagine the 2-dimensional submanifold of $\mathcal{M}$ that is the image under the exponential map $\exp_p$ of Π (near p). The Gaussian curvature of this 2-dimensional submanifold at p is called the **sectional curvature** of $\mathcal{M}$ associated with Π, and denoted $K(\Pi)$.

Lemma 1.10. *If Π is a 2-plane in $T_p\mathcal{M}$ spanned by the vectors $X, Y \in T_p\mathcal{M}$, and $\phi = X \wedge Y$, then*

$$K(\Pi) = \frac{\mathcal{R}(\phi, \phi)}{|\phi|^2},$$

where $|\phi|^2 = g_{ik}g_{jl}\phi^{ij}\phi^{kl}$.

Proof. See [18, Prop. 8.8]. □

Because a rank 4 tensor is a bit awkward to work with, we define some simpler quantities by taking the trace of R^l_{ijk}:

Definition 1.16. *The **Ricci curvature tensor**, Rc, is the $\binom{2}{0}$-tensor with coordinate expression*

$$R_{ij} := R^p_{pij}.$$

It is symmetric: $R_{ij} = R_{ji}$.

*The **scalar curvature** is the trace of the Ricci tensor,*

$$R := g^{ij}R_{ij}.$$

On a 2-manifold, it is equal to twice the Gaussian curvature.

The Ricci and scalar curvatures can be interpreted in terms of sectional curvatures:

Lemma 1.11. *Let $X \in T_p\mathcal{M}$ be a unit vector. Suppose that X is contained in some orthonormal basis for $T_p\mathcal{M}$. $\mathrm{Rc}(X, X)$ is then the sum of the sectional curvatures of planes spanned by X and other elements of the basis.*

Given an orthonormal basis for $T_p\mathcal{M}$, the scalar curvature at p is the sum of all sectional curvatures of planes spanned by pairs of basis elements.

Proof. See [18, end of Chap. 8]. □

The Ricci and scalar curvatures can also be expressed in terms of the curvature operator.

Lemma 1.12. *On a 3-dimensional Riemannian manifold $(\mathcal{M}^3, g)$, let us diagonalize the curvature operator $\mathcal{R}$ with respect to a basis $\{e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2\}$ of $\wedge^2 T\mathcal{M}^3$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of $T\mathcal{M}^3$ (this is possible because $\mathcal{R}$ is symmetric). Suppose that, with respect to this basis, $\mathcal{R}$ is a diagonal matrix with entries $\lambda_1, \lambda_2, \lambda_3$ down the diagonal. Then with respect to the basis $\{e_1, e_2, e_3\}$, the Ricci tensor takes the form*

$$\mathrm{Rc} = \frac{1}{2}\begin{pmatrix} \lambda_2 + \lambda_3 & 0 & 0 \\ 0 & \lambda_3 + \lambda_1 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 \end{pmatrix} \tag{1.9}$$

and the scalar curvature is

$$R = \lambda_1 + \lambda_2 + \lambda_3. \tag{1.10}$$

Proof. Note that for $i \neq j$, $|e_i \wedge e_j|^2 = |e_i \otimes e_j - e_j \otimes e_i|^2 = 1^2 + 1^2 = 2$ (if we chose a different norm on $\wedge^2 T\mathcal{M}^3$ the result of Lemma 1.10 would change). Thus, by the result of Lemma 1.11, using Lemma 1.10 to calculate the sectional curvatures, we have

$$\begin{aligned} \mathrm{Rc}(e_1, e_1) &= \frac{1}{2}\left(\mathcal{R}(e_1 \wedge e_2, e_1 \wedge e_2) + \mathcal{R}(e_1 \wedge e_3, e_1 \wedge e_3)\right) \\ &= \frac{1}{2}\left(\lambda_3 + \lambda_2\right). \end{aligned} \tag{1.11}$$

Similarly, $\mathrm{Rc}(e_2, e_2) = (\lambda_1 + \lambda_3)/2$ and $\mathrm{Rc}(e_3, e_3) = (\lambda_1 + \lambda_2)/2$. This accounts for the diagonal entries of the matrix of Rc.

We now need to show that the off-diagonal matrix elements of Rc are zero, for example that $\mathrm{Rc}(e_1, e_2) = 0$. We have

$$\mathrm{Rc}(e_1, e_2) = \frac{\mathrm{Rc}(e_1 + e_2, e_1 + e_2) - \mathrm{Rc}(e_1, e_1) - \mathrm{Rc}(e_2, e_2)}{2},$$

hence it suffices to show that

$$\mathrm{Rc}(e_1 + e_2, e_1 + e_2) = \mathrm{Rc}(e_1, e_1) + \mathrm{Rc}(e_2, e_2). \tag{1.12}$$

To do this we note that

$$\left\{\frac{e_1 + e_2}{\sqrt{2}}, \frac{e_1 - e_2}{\sqrt{2}}, e_3\right\}$$

is also an orthonormal basis for $T_p\mathcal{M}^3$. We can re-apply formula (1.11) for this new basis to calculate $\mathrm{Rc}(e_1 + e_2, e_1 + e_2)$. The result is as in equation (1.12), hence Rc has the stated form. The formula for R follows by taking the trace of Rc. □

Definition 1.17. *The **Einstein tensor** on a Riemannian n-manifold $(\mathcal{M}^n, g)$ is the tensor*

$$E_{ij} := R_{ij} - \frac{1}{n} R g_{ij}.$$

*It is also known as the traceless part of the Ricci tensor. A metric g is called **Einstein** if its Einstein tensor is identically 0.*

In dimension 3, the Einstein tensor has particular significance. We record here a result which shows that, for 3-dimensional manifolds, the Riemann curvature tensor can be expressed quite simply in terms of the Einstein tensor and the scalar curvature.

Lemma 1.13. *On a 3-manifold,*

$$\mathrm{Rm} = \frac{R}{4}(g \odot g) + E \odot g,$$

where $\odot$ denotes the Kulkarni-Nomizu product of symmetric tensors:

$$(P \odot Q)_{ijkl} = P_{il}Q_{jk} + P_{jk}Q_{il} - P_{ik}Q_{jl} - P_{jl}Q_{ik}.$$

Proof. This result follows from the decomposition of Rm, which holds on any n-manifold:

$$\mathrm{Rm} = \frac{R}{2(n-1)(n-2)}(g \odot g) + \frac{1}{n-2}(E \odot g) + W, \tag{1.13}$$

where W is the so-called "Weyl tensor", which is defined by equation (1.13). The Weyl tensor vanishes in dimension 3 (see [9, Sec. 8] for a simple proof using the symmetries of W), from which the result follows. □

In dimension 3, Einstein metrics are very special.

Lemma 1.14. *An Einstein metric on a manifold of dimension $n \geq 3$ has constant scalar curvature. If $n = 3$, the metric has constant sectional curvature.*

Proof. Consider the second Bianchi identity (1.8). If we raise the indices k, l we obtain

$$\nabla_p R_{ij}{}^{kl} + \nabla_i R_{jp}{}^{kl} + \nabla_j R_{pi}{}^{kl} = 0.$$

Taking the trace over the indices i, l and using the symmetries of the curvature tensor we obtain

$$\nabla_p R_j^k - \nabla_i R_{jp}{}^{ik} - \nabla_j R_p^k = 0.$$

Taking the trace over the indices j, k now gives us

$$\nabla_p R = \nabla_i R_p^i + \nabla_i R_p^i,$$

or equivalently

$$g^{ij} \nabla_i R_{jp} = \frac{1}{2} \nabla_p R.$$

Thus we have

$$g^{ij} \nabla_i E_{jp} = g^{ij} \nabla_i \left(R_{jp} - \frac{1}{n} R g_{jp} \right) = \frac{1}{2} \nabla_p R - \frac{1}{n} \nabla_p R = \left(\frac{1}{2} - \frac{1}{n} \right) \nabla_p R \tag{1.14}$$

(using $\nabla g = 0$). If the metric is Einstein then $E_{jp} = 0$, so

$$\left(\frac{1}{2} - \frac{1}{n} \right) \nabla_p R = 0.$$

Because $n \neq 2$ this means $\nabla R = 0$, hence R is constant.

If $n = 3$, we can apply formula (1.9) from Lemma 1.12 and deduce that

$$\mathrm{Rc} = \frac{1}{2} \begin{pmatrix} \lambda_2 + \lambda_3 & 0 & 0 \\ 0 & \lambda_3 + \lambda_1 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 \end{pmatrix} = \frac{R}{3} g = \begin{pmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix}, \tag{1.15}$$

where $C = \frac{1}{3}R$ is a constant over the whole manifold. Therefore the curvature eigenvalues λ_i are all equal to the constant value C, from which it follows that the manifold has constant sectional curvature. □

If a metric has constant sectional curvature, we call it a **constant curvature metric**. There are three essentially distinct possibilities: the value of the sectional curvatures can be positive, zero or negative. We have the following examples:

Definition 1.18. ***Constant Curvature Metrics***

Euclidean n-space, *$\mathbb{E}^n := \mathbb{R}^n$ with the standard metric, has constant sectional curvature 0.*

*The **n-dimensional sphere of radius** R,*

$$\mathbb{S}^n_R = \{x \in \mathbb{R}^{n+1} : |x| = R\}$$

with the metric induced as a submanifold of $\mathbb{E}^{n+1}$, has constant sectional curvature $1/R^2$.

*The **hyperbolic n-space of radius** R, $\mathbb{H}^n_R$, is the open ball of radius R in $\mathbb{R}^n$ with the metric*

$$g_{ij}(x) = \frac{4R^4 \delta_{ij}}{(R^2 - |x|^2)^2}.$$

It has constant sectional curvature $-1/R^2$ (there are other equivalent ways of representing this Riemannian manifold).

See [18, end of Chap. 8] for more details, including a proof that these spaces have constant curvature.

Using the theory of Jacobi fields (see [18, Chap. 10]), one can prove many comparison theorems, as well as results about constant curvature metrics. For example one can prove that the local structure of a metric with constant sectional curvature C is unique (see [18, Prop. 10.9]) – we will see a more general result in Theorem 1.16. One can also prove the following, which will come in handy in Chapter 7:

Theorem 1.15. ***The Bishop-Günther Volume Comparison Theorem.*** *Let us denote by $V_n^k(r)$ the volume of the ball of radius r in the complete, simply-connected n-dimensional space of constant sectional curvature k (this will be either $\mathbb{S}^n_R$, $\mathbb{E}^n$ or $\mathbb{H}^n_R$). Suppose that $(\mathcal{M}^n, g)$ is a Riemannian n-manifold, and $p \in \mathcal{M}^n$. Then*

1. *If there is a constant $a > 0$ such that $\mathrm{Rc} \geq (n-1)ag$ then*

$$\mathrm{Vol}(B(p,r)) \leq V_n^a(r).$$

2. *If there is a constant b such that all sectional curvatures of $(\mathcal{M}^n, g)$ are bounded above by b, and the exponential map is injective on the ball $B(p,r)$, then*

$$\mathrm{Vol}(B(p,r)) \geq V_n^b(r).$$

Proof. See [8, Theorem 3.101]. □

1.5 Topology from Geometry

The central idea of the Geometrization Conjecture is that the topology of a manifold and the type of geometry that the manifold can have are intimately related. Here we will record some of the theorems that relate the geometric structure of a manifold to its topology.

Theorem 1.16. *Let $(\mathcal{M}^n, g)$ be a complete, simply-connected Riemannian n-manifold with constant sectional curvature C. Then $\mathcal{M}^n$ is isometric to one of $\mathbb{E}^n$ (if $C = 0$), $\mathbb{S}^n_R$ (if $C = 1/R^2$) or $\mathbb{H}^n_R$ (if $C = -1/R^2$).*

Proof. See [18, Theorem 11.12]. □

In particular, any simply-connected manifold with constant non-positive sectional curvature is diffeomorphic to $\mathbb{R}^n$, and any simply-connected manifold with constant positive sectional curvature is diffeomorphic to $\mathbb{S}^n$.

This also allows us to characterize non-simply-connected manifolds of constant curvature – by applying Theorem 1.16 to the universal covering space, we see that any of these manifolds must be a quotient of $\mathbb{R}^n, \mathbb{S}^n$ or $\mathbb{H}^n$ by a discrete group of isometries acting freely.

We can also derive topological information from bounds on the curvatures, as the following theorems show:

Theorem 1.17. ***(Myers' Theorem)*** *Suppose $(\mathcal{M}^n, g)$ is a complete, connected Riemannian n-manifold whose Ricci tensor satisfies*

$$\mathrm{Rc} \geq (n-1)Hg$$

for some constant H. Then $\mathcal{M}^n$ is compact with finite fundamental group and diameter at most $\pi H^{-\frac{1}{2}}$.

Proof. See [18, Theorem 11.8]. □

Theorem 1.18. ***(The Sphere Theorem)*** *We say that the Riemannian manifold $(\mathcal{M}, g)$ is* ***strictly δ-pinched*** *(for some $\delta > 0$) if there is a constant $K > 0$ such that all of the sectional curvatures of $\mathcal{M}$ lie in the interval $(\delta K, K]$.*

If $\mathcal{M}^n$ is a complete, simply-connected and strictly $\frac{1}{4}$-pinched n-manifold then $\mathcal{M}^n$ is homeomorphic to $\mathbb{S}^n$.

Proof. See [3]. □

1.6 Scaling

We will be interested later on (when we deal with the normalized Ricci flow in Chapter 7) in how various geometric quantities scale when the metric is scaled by a constant factor C.

Lemma 1.19. *If $\tilde{g} = Cg$ are two Riemannian metrics on an n-manifold $\mathcal{M}^n$, related by a scaling factor C, then the various geometric quantities scale as follows:*

1. $\tilde{g}^{ij} = C^{-1} g^{ij}$.
2. $\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k$.
3. $\tilde{R}_{ijk}^l = R_{ijk}^l$.
4. $\tilde{R}_{ijkl} = C R_{ijkl}$.
5. $\tilde{R}_{ij} = R_{ij}$.
6. $\tilde{R} = C^{-1} R$.
7. *The volume elements:* $d\tilde{\mu} = C^{n/2} d\mu$.

1.7 Time-evolving metrics

When our metrics depend on time, as in the Ricci flow, we will want to know how the various geometric quantities evolve when the metric evolves.

Lemma 1.20. *Suppose that $g_{ij}(t)$ is a time-dependent Riemannian metric, and*

$$\frac{\partial}{\partial t} g_{ij}(t) = h_{ij}(t).$$

Then the various geometric quantities evolve according to the following equations:

1. *Metric inverse:*
$$\frac{\partial}{\partial t} g^{ij} = -h^{ij} = -g^{ik} g^{jl} h_{kl}. \tag{1.16}$$
2. *Christoffel symbols:*
$$\frac{\partial}{\partial t} \Gamma_{ij}^k = \frac{1}{2} g^{kl} (\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}). \tag{1.17}$$
3. *Riemann curvature tensor:*
$$\frac{\partial}{\partial t} R_{ijk}^l = \frac{1}{2} g^{lp} \left\{ \begin{array}{c} \nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} \\ -\nabla_j \nabla_i h_{kp} - \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik} \end{array} \right\}. \tag{1.18}$$
4. *Ricci tensor:*
$$\frac{\partial}{\partial t} R_{ij} = \frac{1}{2} g^{pq} (\nabla_q \nabla_i h_{jp} + \nabla_q \nabla_j h_{ip} - \nabla_q \nabla_p h_{ij} - \nabla_i \nabla_j h_{qp}). \tag{1.19}$$
5. *Scalar curvature:*
$$\frac{\partial}{\partial t} R = -\triangle H + \nabla^p \nabla^q h_{pq} - h^{pq} R_{pq} \tag{1.20}$$
where $H = g^{pq} h_{pq}$.
6. *Volume element:*
$$\frac{\partial}{\partial t} d\mu = \frac{H}{2} d\mu. \tag{1.21}$$

7. *Volume of manifold:*

$$\frac{d}{dt}\int_{\mathcal{M}} d\mu = \int_{\mathcal{M}} \frac{H}{2} d\mu. \tag{1.22}$$

8. *Total scalar curvature on a closed manifold $\mathcal{M}$:*

$$\frac{d}{dt}\int_{\mathcal{M}} R d\mu = \int_{\mathcal{M}} \left(\frac{1}{2}RH - h^{ij}R_{ij}\right) d\mu. \tag{1.23}$$

Proof. **Proof of (1.16):** We have $g^{ij}g_{jk} = \delta^i_k = \text{constant}$. Differentiating,

$$\begin{aligned} \partial_t(g^{ij}g_{jk}) &= 0 \\ \Rightarrow \quad (\partial_t g^{ij})g_{jk} + g^{ij}(\partial_t g_{jk}) &= 0 \\ \Rightarrow \quad \partial_t g^{ij} &= -h^{ij}. \end{aligned}$$

Proof of (1.17): We use formula (1.5), which yields

$$\begin{aligned} \partial_t \Gamma^k_{ij} &= \frac{1}{2}(\partial_t g^{kl})(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \\ &\quad + \frac{1}{2} g^{kl}(\partial_i \partial_t g_{jl} + \partial_j \partial_t g_{il} - \partial_l \partial_t g_{ij}). \end{aligned}$$

Now we work in normal coordinates about a point p. By Lemma 1.5 we have $\partial_i g_{jk} = 0$ at p, and $\partial_i A = \nabla_i A$ at p for any tensor A. Hence

$$\partial_t \Gamma^k_{ij}(p) = \frac{1}{2} g^{kl}\left(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\right)(p).$$

Now although the Christoffel symbols are not the coordinates of a tensor quantity, their derivative is.[4] Hence both sides of this equation are the coordinates of tensorial quantities, so it does not matter what coordinates we evaluate them in. In particular, the equation is true for any coordinates, not just normal coordinates, and about any point p.

Proof of (1.18): We use the result of Lemma 1.8. Each term can be expressed using the result of formula (1.17):

$$\begin{aligned} R^l_{ijk} &= \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp} \\ \Rightarrow \quad \partial_t R^l_{ijk} &= \partial_i(\partial_t \Gamma^l_{jk}) - \partial_j(\partial_t \Gamma^l_{ik}) \\ &\quad + (\partial_t \Gamma^p_{jk})\Gamma^l_{ip} + \Gamma^p_{jk}(\partial_t \Gamma^l_{ip}) - (\partial_t \Gamma^p_{ik})\Gamma^l_{jp} - \Gamma^p_{ik}(\partial_t \Gamma^l_{jp}). \end{aligned}$$

Once again we work in normal coordinates so that $\Gamma^k_{ij}(p) = 0$. This gives us

$$\partial_t R^l_{ijk}(p) = \nabla_i\left(\partial_t \Gamma^l_{jk}\right)(p) - \nabla_j\left(\partial_t \Gamma^l_{ik}\right)(p).$$

Once again, both sides are tensors, so the equation holds in any coordinates. Plugging the result of formula (1.17) in on the RHS yields the result.

Proof of (1.19): This follows from formula (1.18) by taking the trace over the indices i, l.

Proof of (1.20): This follows from the formulae (1.19) and (1.16):

$$\begin{aligned} \partial_t R &= \partial_t\left(g^{ij}R_{ij}\right) \\ &= (\partial_t g^{ij})R_{ij} + g^{ij}(\partial_t R_{ij}) \\ &= -h^{ij}R_{ij} + g^{ij}\left(\frac{1}{2}g^{pq}(\nabla_q\nabla_i h_{jp} + \nabla_q\nabla_j h_{ip} - \nabla_q\nabla_p h_{ij} - \nabla_i\nabla_j h_{qp})\right) \\ &= -\triangle H + \nabla^p\nabla^q h_{pq} - h^{pq}R_{pq} \end{aligned}$$

[4] This is true because the difference between the Christoffel symbols of two connections is a tensor. Thus, by taking a fixed connection with Christoffel symbols $\tilde{\Gamma}^k_{ij}$, we have

$$\partial_t \Gamma^k_{ij} = \partial_t\left(\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}\right)$$

and the right hand side is clearly a tensor.

(recall that $\nabla g = 0$ and $\triangle = g^{ij}\nabla_i\nabla_j$).

Proof of (1.21): We will use the formula

$$d\mu = \sqrt{\det g_{ij}}dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

(see [14, Sec. 7.5] on the volume form).

First we need to calculate the variation in the determinant of a matrix $\det A$, when the matrix A itself varies. Because A will end up being a metric, we may assume that it is symmetric and hence we can choose a basis in which A is diagonalized with eigenvalues $\lambda_i \neq 0$. Then $A_{ij} = \lambda_i\delta_{ij}$, and $\det A = \prod_i \lambda_i$. If we then vary the entry A_{ij}, the determinant will not change unless $i = j$. If $i = j$, we have

$$\frac{\partial \det A}{\partial A_{ii}} = \frac{\partial\left(\prod_j \lambda_j\right)}{\partial \lambda_i} = \frac{1}{\lambda_i}\prod_j \lambda_j = \frac{1}{\lambda_i}\det A.$$

Therefore by the chain rule,

$$\begin{aligned}
\frac{d}{dt}\det A &= \sum_{i,j=0}^{n}\left(\frac{\partial \det A}{\partial A_{ij}}\right)\frac{dA_{ij}}{dt} \\
&= \sum_{i,j=0}^{n}\delta_{ij}\frac{1}{\lambda_i}\det A\frac{dA_{ij}}{dt} \\
&= \left(A^{-1}\right)^{ij}\left(\frac{dA_{ij}}{dt}\right)\det A
\end{aligned}$$

where we have observed that $(A^{-1})^{ij} = \delta^{ij}/\lambda_i$ and we are now using the Einstein summation convention. This formula manifestly does not depend on the basis we choose as traces are basis-independent.

It now follows by the chain rule that

$$\begin{aligned}
\partial_t d\mu &= \partial_t\sqrt{\det g_{ij}}dx^1 \wedge \ldots \wedge dx^n \\
&= \frac{1}{2\sqrt{\det g_{ij}}}g^{ij}h_{ij}\det g_{ij}dx^1 \wedge \ldots \wedge dx^n \\
&= \frac{H}{2}d\mu
\end{aligned}$$

where $H = g^{ij}h_{ij}$.

Proof of (1.22): This follows from formula (1.21) by taking the derivative under the integral sign.

Proof of (1.23): This follows from formulae (1.21) and (1.20):

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathcal{M}} Rd\mu &= \int_{\mathcal{M}}(\partial_t R)d\mu + R(\partial_t d\mu) \\
&= \int_{\mathcal{M}}\left(-\triangle H + \nabla^p\nabla^q h_{pq} - h^{pq}R_{pq} + \frac{1}{2}RH\right)d\mu \\
&= \int_{\mathcal{M}}\left(\frac{1}{2}RH - h^{pq}R_{pq}\right)d\mu.
\end{aligned}$$

Here we have used Stokes' Theorem (see [14, Sec. 7.5]) to get rid of the first two terms in the integral because they were expressible as the divergence of a vector field on $\mathcal{M}$, and $\mathcal{M}$ has no boundary. □

Chapter 2

Introduction to the Ricci Flow

The main aim of this project is to introduce the basics of Hamilton's Ricci flow program, which is aimed at proving Thurston's Geometrization Conjecture and consequently the Poincaré Conjecture. In this chapter we will present the context for the Ricci flow: what is it, what are the problems that it is intended to solve, and why might it be expected to solve them? In the process we will also see some simple solutions to the Ricci flow and try to gain a bit of intuition about its behaviour. A lot of space in this introduction is devoted to material that is not elaborated (or is elaborated very little) in the main body of the thesis – namely the description of the Geometrization Conjecture and the section on pinching. That is because the introduction is intended to serve as a "big picture" guide to the reasons we study the Ricci flow, so that the reader understands, when encountering the formidable technical details of Chapter 7, why it is all worthwhile.

2.1 The Prehistory of the Ricci Flow – Geometrization

The Poincaré Conjecture was one of the iconic unsolved problems of 20th century mathematics. Around 1900, Poincaré asked if a simply-connected closed 3-manifold is necessarily the 3-sphere $\mathbb{S}^3$. After many years of topological difficulties, William Thurston made promising progress in the 1970s. He proposed, not only a way of approaching the Poincaré Conjecture, but a far more general conjecture that, if proven, would lead to a classification of all compact 3-manifolds. We will outline, extremely vaguely, Thurston's conjecture.

It was already known by the time Thurston came onto the scene that compact orientable 3-manifolds could be decomposed into simpler manifolds using the "connected sum" decomposition. Figure 2.1 shows the connect sum operation # for 2-manifolds. For 3-manifolds, the definition is analogous: one cuts out a 3-ball from each manifold then glues them together along the $\mathbb{S}^2$ boundaries created.

If a manifold can not be nontrivially decomposed using the connected sum decomposition then it is called **prime**. Hellmuth Kneser showed in 1929 that every compact orientable 3-manifold can be decomposed into a finite number of prime "factors" (see [16]), and John Milnor showed in 1962 that this decomposition is unique (see [19]). Thus, to classify 3-manifolds it suffices to classify prime 3-manifolds.

The connected sum decomposition involves cutting 3-manifolds along 2-spheres; the next step is to cut them along 2-tori. Thurston conjectured that any compact, orientable, prime 3-manifold can be decomposed by some finite number of embedded tori into pieces so that each piece has one of eight fundamental, highly symmetric geometric structures (of which the constant-curvature metrics on $\mathbb{S}^3, \mathbb{H}^3, \mathbb{R}^3$ are three). Note the analogy with the 2-dimensional case: any compact, orientable 2-manifold is either $\mathbb{S}^2$ (which can be given a metric of constant positive curvature), a torus T^2 (which can be given a metric with zero curvature), or can be decomposed using the connected sum into tori with holes (which can be given metrics of constant negative curvature) (see [22, Theorem 77.5] for the topological classification and [27, Chap. 1] for a discussion about the existence of geometric structures).

It might not be obvious that the geometric structure will give us useful topological information, but we saw in Chapter 1 how it is possible to make progress towards classifying constant curvature

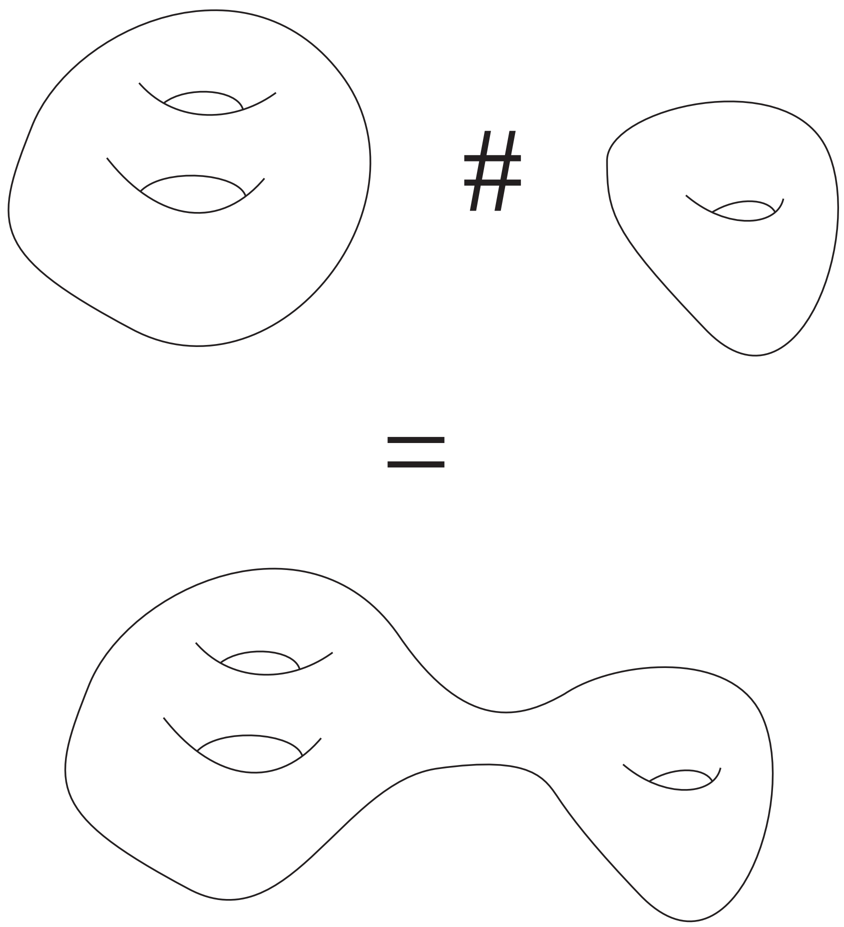

Figure 2.1: Connect sum for 2-manifolds: cut a hole in each of M and N, then glue them together along the boundary circles created to get the connected sum $M\#N$.

metrics in any dimension – an example of the way that a knowledge of the geometric structure of a manifold can lead to knowledge of the topological structure. Thurston proved that this decomposition worked for a certain class of 3-manifolds (the so-called Haken manifolds), but not in full generality. If his conjectured classification of 3-manifolds, known as "Thurston's Geometrization Conjecture", were proven for all 3-manifolds, then the Poincaré Conjecture would follow (see [2, Sec. 7.7]).

2.2 Hamilton's Ricci Flow

Enter Richard Hamilton. Hamilton published a groundbreaking paper ([9]) in 1982, introducing the concept of the Ricci flow. If you have a Riemannian manifold $\mathcal{M}$ with metric g_0, the Ricci flow is a PDE that evolves the metric tensor:

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2\mathrm{Rc}(g(t)) \\ g(0) &= g_0 \end{aligned}$$

where $\mathrm{Rc}(g(t))$ denotes the Ricci curvature of the metric $g(t)$.

The idea is to try to evolve the metric in some way that will make the manifold "rounder and rounder". We hope that the metric will evolve towards one of Thurston's eight fundamental geometric structures, and that the decomposition by spheres and tori will somehow emerge naturally. In choosing what should go on the right hand side of the equation of the Ricci flow, we know that it should be a rank-2 tensor, symmetric (so that the metric g remains symmetric), and it should involve the curvature somehow – the Ricci curvature tensor is the obvious choice. The minus sign makes the Ricci flow a heat-type (parabolic) equation (as we shall see in Chapter 5), so it is expected to "average out" the curvature. This should make the metric rounder in the way that we want.

A characteristic property of heat-type equations is the maximum principle, which we will see in Chapter 3. We will use the maximum principle in Chapter 7 to prove quantitatively that this rounding of the metric does indeed happen in one specific case. The main theorem we will prove is the one proved by Hamilton in the paper that introduced the Ricci flow:

Theorem 2.1. *(Hamilton, 1982) Let $\mathcal{M}^3$ be a closed 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature. Then $\mathcal{M}^3$ also admits a metric of constant positive curvature.*

In particular, by Theorem 1.16, any **simply-connected** closed 3-manifold which admits a metric of strictly positive Ricci curvature is diffeomorphic to the 3-sphere. We are certainly starting to get into the territory of the Poincaré Conjecture with this result!

More specifically, we will see that if the initial metric g_0 has strictly positive Ricci curvature then the manifold $\mathcal{M}^3$ will shrink to a point in finite time under the Ricci flow. But if we dilate the metric by a time-dependent factor so that the volume remains constant, the problem of shrinking to a point is removed. Furthermore, we can show that the rescaled metric converges uniformly to the desired metric of constant positive curvature on $\mathcal{M}^3$. This process of "blowing up" the manifold when it is becoming singular is a crucial one in the Ricci flow program. Because it is rather complicated and can get lost in the technical details, we have provided an analogous but much simpler argument for the "curve-shortening flow" in Chapter 4 in the hope that this will make it easier to understand what is going on in later chapters.

We note that the Uniformization Theorem (any Riemannian metric on a closed 2-manifold is conformal to one of constant curvature) can be proved using the Ricci flow as well (see [6, Chap. 5] for the bulk of the argument, and [4] for the remainder). Once again we encounter problems with singularities: the manifold will shrink to a point in some cases and become arbitrarily large in others. As before, the solution is to rescale so that the volume is constant. When we do this, the rescaled metric converges smoothly to a smooth metric of constant curvature. Note that the Uniformization Theorem is the closest thing to a 2-dimensional analogue of Thurston's Geometrization Conjecture, so the fact that the Ricci flow solves it in this way is another strong indicator that we're heading in the right direction.

In 2002 and 2003, Grisha Perelman posted three papers ([23, 25, 24]) on arXiv.org which claimed to have completed Hamilton's work towards using the Ricci flow to prove the Geometrization Conjecture. More recently several authors have posted papers elaborating on the technical details of Perelman's papers ([2, 21]). The proof seems to have been accepted by the mathematical community, wrapping up the 100-year history of the Poincaré Conjecture (and the younger but no less important Geometrization Conjecture).

2.3 Special Solutions of the Ricci Flow

In this section we will exhibit some of the special solutions of the Ricci flow. The first thing, of course, is to know what they do on the spaces of constant curvature.

On an n-dimensional sphere of radius r (where $n > 1$), the metric is given by $g = r^2\bar{g}$ where $\bar{g}$ is the metric on the unit sphere. The sectional curvatures are all $1/r^2$. Thus for any unit vector v, the result of Lemma 1.11 tells us that $\mathrm{Rc}(v, v) = (n-1)/r^2$. Therefore

$$\mathrm{Rc} = \frac{n-1}{r^2} g = (n-1)\bar{g},$$

so the Ricci flow equation becomes an ODE:

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2\mathrm{Rc} \\ \Rightarrow \quad \frac{\partial}{\partial t}(r^2 \bar{g}) &= -2(n-1)\bar{g} \\ \Rightarrow \quad \frac{d(r^2)}{dt} &= -2(n-1). \end{aligned}$$

We have the solution

$$r(t) = \sqrt{R_0^2 - 2(n-1)t},$$

where R_0 is the initial radius of the sphere. The manifold shrinks to a point as $t \to R_0^2/(2(n-1))$.

Similarly, for hyperbolic n-space $\mathbb{H}^n$ (where $n > 1$), the Ricci flow reduces to the ODE

$$\frac{d(r^2)}{dt} = 2(n-1)$$

which has the solution

$$r(t) = \sqrt{R_0^2 + 2(n-1)t}.$$

So the solution expands out to infinity.

Of course the flat metric on $\mathbb{E}^n$ has zero Ricci curvature, so it does not evolve at all under the Ricci flow. There are other non-trivial Riemannian manifolds with vanishing Ricci curvature (the metric is flat, i.e. locally isometric to Euclidean space, if and only if the Riemann curvature tensor vanishes). These metrics can be regarded as the "fixed points" of the Ricci flow. However, we ought really to regard $\mathbb{S}^n$ and $\mathbb{H}^n$ as honorary fixed points of the flow – even though the metric was changing under the flow, it only ever changed by a rescaling of the metric.

Even more generally, one can regard as "generalized fixed points" of the Ricci flow those manifolds which change only by a diffeomorphism and a rescaling under the Ricci flow. Let $(\mathcal{M}^n, g(t))$ be a solution of the Ricci flow, and suppose that $\varphi_t : \mathcal{M}^n \to \mathcal{M}^n$ is a time-dependent family of diffeomorphisms (with $\varphi_0 = id$) and $\sigma(t)$ is a time-dependent scale factor (with $\sigma(0) = 1$). If we then have

$$g(t) = \sigma(t)\varphi_t^* g(0)$$

then the solution $(\mathcal{M}^n, g(t))$ is called a **Ricci soliton**. Taking the derivative of this equation and evaluating at $t = 0$ yields

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= \frac{d\sigma(t)}{dt}\varphi_t^* g(0) + \sigma(t)\frac{\partial}{\partial t}\varphi_t^* g(0) \\ -2Rc(g(0)) &= \sigma'(0)g(0) + \mathcal{L}_V g(0), \end{aligned}$$

where $V = d\varphi_t/dt$, by the definition of the Lie derivative given in Section 1.3. Let us set $\sigma'(0) = 2\lambda$. We can now use the result of Lemma 1.7 to write this in coordinates as

$$-2R_{ij} = 2\lambda g_{ij} + \nabla_i V_j + \nabla_j V_i.$$

As a special case we can consider the case that V is the gradient vector field of some scalar function f on $\mathcal{M}^n$, i.e. $V_i = \nabla_i f$. The equation then becomes

$$R_{ij} + \lambda g_{ij} + \nabla_i \nabla_j f = 0. \tag{2.1}$$

Such solutions are known as **gradient Ricci solitons**. A gradient Ricci soliton is called **shrinking** if $\lambda < 0$, **static** if $\lambda = 0$, and **expanding** if $\lambda > 0$. The gradient Ricci solitons play a role in motivating the definition of Perelman's $\mathcal{F}$- and $\mathcal{W}$-functionals which we will see in Section 8.4.

One example of a gradient Ricci soliton is Hamilton's **cigar soliton**: Let $\Sigma^2 = \mathbb{R}^2$ with the standard coordinates and

$$g_{ij} = \frac{\delta_{ij}}{1 + x^2 + y^2}.$$

Because the metric is rotationally symmetric, the calculation of the Ricci curvature tensor and the covariant derivative is particularly simple (by the rotational symmetry we can embed Σ^2 in $\mathbb{R}^3$ as a manifold of revolution – see [18, Ex. 8.2] for the calculation of the Gaussian curvature of manifolds of revolution). We can show that

$$R_{ij} + \nabla_i \nabla_j f = 0$$

where $f(x, y) = (x^2 + y^2)/2$, hence (Σ^2, g) is a static gradient Ricci soliton.

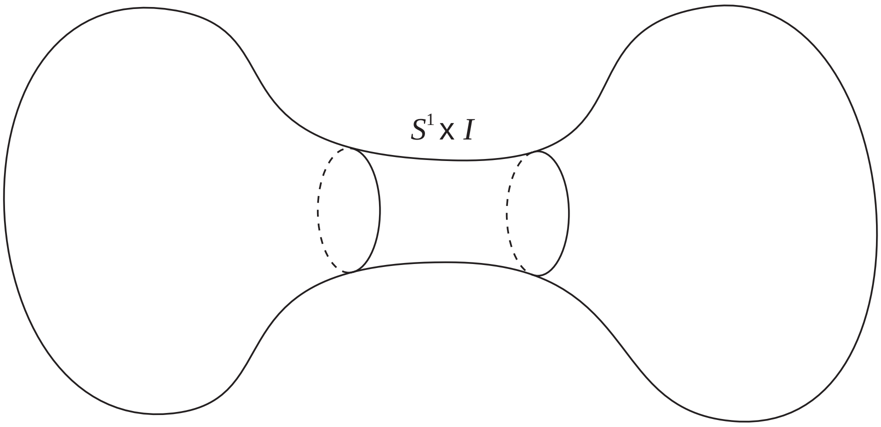

Figure 2.2: A neck in a 2-manifold.

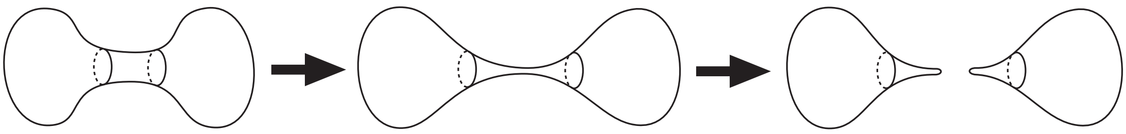

Figure 2.3: A neck "pinching off" in a 2-manifold. This diagram is intended to illustrate by lower-dimensional analogy what a neckpinch in a 3-manifold is like – the Ricci flow on 2-manifolds does not give rise to neckpinches.

2.4 Pinching

We now describe how the connected sum decomposition arises out of the Ricci flow. Firstly we note that, if we have a product Riemannian manifold with a product Riemannian metric, then under the Ricci flow each factor in the product evolves independently.

Now consider a "neck" shaped like $\mathbb{S}^1 \times I$ (I is an interval) between two parts of a 2-manifold, as shown in Figure 2.2. We expect the metric on the neck to be close to a product metric; the I factor has no curvature so it will not change, and the $\mathbb{S}^1$ factor also has no intrinsic curvature so it will not change. In contrast, when we have an analogous neck shaped like $\mathbb{S}^2 \times I$ in a 3-manifold, we have seen in Section 2.3 that the 2-sphere will shrink to a point in finite time. Therefore we expect that in some situations, the manifold will "pinch off" at such a neck (see Figure 2.3 for the 2-dimensional analogue of such a neckpinch). Thus the Ricci flow can actually perform the connected sum decomposition for us! The details of how this pinching off actually happens (how one ought to perform "surgery" on one's manifold) are very tricky.

We will discuss the pinching process in more detail in Chapter 8.

The torus decomposition arises in a different way. After we have have performed the connected-sum decomposition by surgery on the neckpinches, we consider the evolution of the remaining pieces. Some will be quotients of $\mathbb{S}^3$ (which shrink to a point) or products $\mathbb{S}^2 \times \mathbb{S}^1$ (which shrink down to the $\mathbb{S}^1$ factor). Both of these cases satisfy the Geometrization Conjecture, so it is the remaining pieces that concern us. They will exist without singularities as $t \to \infty$.

We expect these pieces to be made up of pieces with the fundamental geometries of Thurston, glued together along torus boundaries (if the Geometrization Conjecture is true). Some pieces will be hyperbolic, and we have seen in Section 2.3 that hyperbolic metrics will tend to expand. The metric on the torus necks, on the other hand, will be close to a product metric on $T^2 \times I$. Recall that under the Ricci flow on a product metric, each factor will evolve independently. Both factors have zero curvature, so we expect the neck to remain static, with the hyperbolic pieces of the manifold expanding around it. If we rescale the flow so that the volume remains constant, the torus neck will become very thin, in the same way that the $\mathbb{S}^2 \times I$ neck becomes very thin before a neckpinch. This is how the torus decomposition arises.

Under the rescaling, the manifold splits along tori into hyperbolic pieces with cusps and "collapsing" pieces (with injectivity radius shrinking to 0). The collapsing pieces can be identified as so-called "graph manifolds", which are known to satisfy the Geometrization Conjecture. For a more detailed description of how this decomposition arises, see [1]. For the full story on the proof of the Geometrization Conjecture see [2].

Chapter 3

The Maximum Principle

The **maximum principle** is the key tool in understanding many parabolic partial differential equations. It appears in many guises, but it always essentially expresses the fact that parabolic or "heat-type" PDEs will "average out" the values of whatever quantity is evolving. It is crucial to understanding the Ricci flow (which will be shown in Chapter 5 to be parabolic modulo reparametrization), where it can be used to put bounds on the curvature of the metric. In some situations we will need more refined estimates than can be obtained by applying the maximum principle to scalar quantities related to curvature, so we must apply the maximum principle to tensor quantities like the curvature operator. The question of what it means for a tensor quantity to "average out" naturally arises.

In this chapter we first motivate the idea of the maximum principle using the example of the heat equation on the real line. We generalize this situation to a scalar function obeying a heat-type equation on an arbitrary Riemannian manifold (rather than just $\mathbb{R}$), then finally generalize to an arbitrary section of a vector bundle over a manifold (rather than just the trivial $\mathbb{R}$-bundle whose sections are the scalar functions).

3.1 The Heat Equation

The simplest parabolic PDE is the heat equation, describing the time evolution of the temperature distribution $u(x,t)$ on an infinite 1-dimensional bar:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{3.1}$$

The maximum principle says that the temperature of the hottest point on the bar is a non-increasing function of time, and the temperature of the coldest point on the bar is a non-decreasing function of time.

This can be seen intuitively from the fundamental or Green's function solution of the heat equation, which is the temperature distribution due to a delta-function distribution at time $t = 0$. That is, it solves the PDE

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \\ u(x,0) &= \delta(x). \end{aligned}$$

The solution is

$$u(x,t) = \frac{\exp(-\frac{x^2}{4t})}{\sqrt{4\pi t}}.$$

It can be seen in Figure 3.1 that at each time the temperature distribution is Gaussian, and the distribution becomes flatter and wider as time evolves. This is what we expect of heat flow: the heat spreads out as time evolves, any relatively hot region becomes cooler, and the maximum temperature on the bar drops over time. Any solution of the heat equation can be expressed as

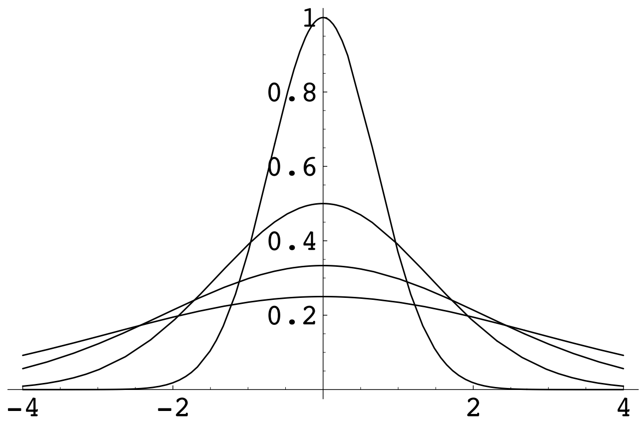

Figure 3.1: The evolution of the temperature distribution due to a pointlike source at time $t = 0$. The solution becomes flatter and wider as time increases.

a weighted integral of such fundamental solutions, by the method of Green's functions. Thus it is plausible that the "averaging out" behaviour of the fundamental solution is characteristic of all solutions of the heat equation.

One can see why the maximum principle is true in a different way. At a point of maximum temperature (at a given time), the temperature distribution u must have a local maximum. Thus $\partial^2 u/\partial x^2 \leq 0$. It follows by the heat equation (3.1) that, at the hottest point of the bar,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \leq 0,$$

so the temperature is nonincreasing. Therefore one would expect the maximum temperature on the bar to drop over time.

3.2 A Scalar Maximum Principle on Manifolds

We now generalize to scalar functions on an arbitrary manifold, and we also take reaction terms into account. This and the subsequent section are based on [6, Chap. 4].

Let us work on a closed manifold $\mathcal{M}$ with a Riemannian metric $g(t)$ that varies with time. In this section we will consider PDEs of the form

$$\frac{\partial u}{\partial t} = \triangle_{g(t)} u + \langle X(t), \nabla u \rangle + F(u), \tag{3.2}$$

where $u : \mathcal{M} \times [0, T) \to \mathbb{R}$ is a time-dependent real-valued function on $\mathcal{M}$, $X(t)$ is a time-dependent vector field on $\mathcal{M}$ and $F : \mathbb{R} \to \mathbb{R}$. We will see many PDEs of this broad type – they consist of a Laplacian term $\triangle_{g(t)} u$ and the **reaction terms** $\langle X(t), \nabla u \rangle + F(u)$. We call such PDEs **heat-type equations**, because of the analogy with the heat equation (3.1).

In practice, we will apply the maximum principle to heat-type equations with such fearsomely complicated reaction terms that it is impossible to keep track of them all. The best we can hope to do, in most cases, is bound them. Incorporating bounds on the reaction terms into the heat-type equation will give rise to differential inequalities, rather than equalities. For this reason, the results stated and proved in this chapter relate to differential inequalities, rather than equalities as in equation (3.2).

Before showing how to deal with the reaction term $F(v)$ we will prove the maximum principle in the simpler case that $F = 0$.

Lemma 3.1. *Let $(\mathcal{M}, g(t))$ be a closed manifold with a time-dependent Riemannian metric $g(t)$. Suppose that $u : \mathcal{M} \times [0, T) \to \mathbb{R}$ is initially nonpositive (i.e. $u(x, 0) \leq 0$ for all $x \in \mathcal{M}$) and that* *it satisfies the differential inequality*

$$\frac{\partial u}{\partial t} \leq \triangle_{g(t)} u + \langle X(t), \nabla u \rangle$$

at all points $(x,t) \in \mathcal{M} \times [0,T)$ *where* $u(x,t) > 0$*. Then* $u(x,t) \leq 0$ *for all* $x \in \mathcal{M}$ *and* $t \in [0,T)$*.*

Note the rather strange condition that the inequality is satisfied only at points and times where something has "gone wrong" (in the sense that $u(x,t)$ has become positive). The condition is designed so that we can apply this lemma in the proof of the more general scalar maximum principle (Theorem 3.2).

Proof. The proof will be based on the idea outlined at the end of Section 3.1.

Given $\epsilon > 0$, let us set $v_\epsilon = u - \epsilon(1+t)$. Note that $v_\epsilon(x,0) \leq -\epsilon < 0$. We will show that for any $\epsilon > 0$, $v_\epsilon < 0$ on $\mathcal{M} \times [0,T)$. From the fact that $\epsilon > 0$ is arbitrary it will follow that u is non-positive.

First we compute the evolution equation (or evolution inequality) of v_ϵ, which follows from that of u:

$$\begin{aligned} \frac{\partial}{\partial t} v_\epsilon &\leq \triangle u + \langle X, \nabla u \rangle - \epsilon \\ &= \triangle v_\epsilon + \langle X, \nabla v_\epsilon \rangle - \epsilon \end{aligned} \tag{3.3}$$

at any point x and time t where $u(x,t) > 0$. In particular, because $v_\epsilon < u$, the inequality holds at any (x,t) such that $v_\epsilon(x,t) = 0$. Note that $\triangle v_\epsilon = \triangle u$ and $\nabla v_\epsilon = \nabla u$ because $-\epsilon(1+t)$ has no spatial dependence.

Now suppose, for a contradiction, that $v_\epsilon(x,t) \geq 0$ for some $(x,t) \in \mathcal{M} \times [0,T)$. By compactness, there exists a first time $t_0 \in [0,T)$ and point $x_0 \in \mathcal{M}$ at which v_ϵ hits 0. That is, $v_\epsilon(x_0,t_0) = 0$ but $v_\epsilon(x,t) < 0$ for any $t < t_0$. It follows that

$$\frac{\partial v_\epsilon}{\partial t}(x_0,t_0) \geq 0.$$

Furthermore, x_0 must be a local maximum of $v_\epsilon(\cdot, t_0)$, so

$$\begin{aligned} \nabla v_\epsilon(x_0,t_0) &= 0 \\ \triangle v_\epsilon(x_0,t_0) &\leq 0. \end{aligned}$$

We can plug these inequalities into equation (3.3), which holds at (x_0,t_0) because $v_\epsilon(x_0,t_0) = 0$. We obtain:

$$0 \leq \frac{\partial v_\epsilon}{\partial t}(x_0,t_0) \leq \triangle v_\epsilon(x_0,t_0) + \langle X(t_0), \nabla v_\epsilon(x_0,t_0) \rangle - \epsilon \leq -\epsilon < 0,$$

which is a contradiction.

Therefore, $v_\epsilon(x,t) < 0$ for all $(x,t) \in \mathcal{M} \times [0,T)$ and for all $\epsilon > 0$. Hence $u(x,t) \leq 0$ for all $(x,t) \in \mathcal{M} \times [0,T)$, as required. ☐

We can now prove the main result of this section, which shows how to deal with the reaction term $F(u)$ in equation (3.2).

Theorem 3.2. ***The Scalar Maximum Principle.*** *Let* $(\mathcal{M}, g(t))$ *be a closed manifold with a time-dependent Riemannian metric* $g(t)$*. Suppose that* $u : \mathcal{M} \times [0,T) \to \mathbb{R}$ *satisfies*

$$\begin{aligned} \frac{\partial u}{\partial t} &\leq \triangle_{g(t)} u + \langle X(t), \nabla u \rangle + F(u) \\ u(x,0) &\leq C \text{ for all } x \in \mathcal{M}, \end{aligned}$$

for some constant C*, where* $X(t)$ *is a time-dependent vector field on* $\mathcal{M}$ *and* $F : \mathbb{R} \to \mathbb{R}$ *is locally Lipschitz. Suppose that* $\phi : \mathbb{R} \to \mathbb{R}$ *is the solution of the* ***associated*** *ODE, which is formed by neglecting the Laplacian and gradient terms:*

$$\begin{aligned} \frac{d\phi}{dt} &= F(\phi) \\ \phi(0) &= C. \end{aligned}$$

Then

$$u(x,t) \leq \phi(t)$$

for all $x \in \mathcal{M}$ and $t \in [0,T)$ such that $\phi(t)$ exists.

The theorem essentially tells us that our upper bound grows no faster than we would expect from the reaction term $F(u)$.

Proof. Let us set $v = u - \phi$. We know that $v(x,0) \leq 0$ for all $x \in \mathcal{M}$, and we desire to show that $v(x,t) \leq 0$ for all $(x,t) \in \mathcal{M} \times [0,T)$. To do this, we fix an arbitrary $\tau \in [0,T)$ and show that $v \leq 0$ on $[0,\tau]$, for any $\tau \in [0,T)$. First note that

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial (u-\phi)}{\partial t} \\ &\leq \triangle u + \langle X, \nabla u \rangle + F(u) - F(\phi) \\ &= \triangle v + \langle X, \nabla v \rangle + (F(u) - F(\phi)). \end{aligned} \quad (3.4)$$

Note that $\nabla u = \nabla v$ and $\triangle u = \triangle v$ because ϕ depends only on t. We now want to deal with the last term on the RHS.

Because $\mathcal{M} \times [0,\tau]$ is compact, there exists a constant C (dependent on τ) such that $|u(x,t)| \leq C$ and $|\phi(t)| \leq C$ on $\mathcal{M} \times [0,\tau]$. Because F is locally Lipschitz and the interval $[-C,C]$ is compact, there exists C_1 (also dependent on τ) such that $|F(x) - F(y)| \leq C_1|x-y|$ for all $x,y \in [-C,C]$. Therefore, because $u, \phi \in [-C,C]$, $|F(u) - F(\phi)| \leq C_1|u-\phi| = C_1|v|$ on $\mathcal{M} \times [0,\tau]$. Plugging this into the evolution equation (3.4) for v, we obtain

$$\frac{\partial v}{\partial t} \leq \triangle v + \langle X, \nabla v \rangle + C_1|v|.$$

Now let $w = e^{-C_1 t} v$. Then we have

$$\begin{aligned} \frac{\partial w}{\partial t} &\leq e^{-C_1 t}\left(\triangle v + \langle X, \nabla v \rangle + C_1|v| - C_1 v\right) \\ &= \triangle w + \langle X, \nabla w \rangle + C_1(|w| - w). \end{aligned} \quad (3.5)$$

We are going to apply Lemma 3.1 to the function w. Because $v(x,0) \leq 0$ for all $x \in \mathcal{M}$, we have $w(x,0) \leq 0$ for all $x \in \mathcal{M}$. Furthermore, if $w \geq 0$ then $|w| = w$, so the differential inequality (3.5) gives us

$$\frac{\partial w}{\partial t} \leq \triangle w + \langle X, \nabla w \rangle$$

at any point $(x,t) \in \mathcal{M} \times [0,\tau]$ such that $w(x,t) \geq 0$. Hence, by Lemma 3.1, $w(x,t) \leq 0$ for all $(x,t) \in \mathcal{M} \times [0,\tau]$.

It follows that $v(x,t) \leq 0$, and hence $u(x,t) \leq \phi(t)$ for all $(x,t) \in \mathcal{M} \times [0,\tau]$, for any $\tau \in [0,T)$. Therefore $u(x,t) \leq \phi(t)$ for any $(x,t) \in \mathcal{M} \times [0,T)$. □

3.3 A Maximum Principle for Vector Bundles

In [10], Hamilton showed how to generalize the maximum principle to apply to sections of a vector bundle (of which scalar functions are a special case). There is a bit of a subtlety in this case though – a heat-type equation has a Laplacian term in it, but what exactly do we mean by a Laplacian in this situation? Let $\pi : \mathcal{E} \to \mathcal{M}$ be a vector bundle over $\mathcal{M}$ with a fixed bundle metric h, $\bar{\nabla}(t)$ a smooth time-dependent family of connections on $\mathcal{E}$ compatible with h, and $g(t)$ a time-dependent Riemannian metric on $\mathcal{M}$. Now to define the Laplacian of a section φ we need to take two covariant derivatives of φ then take the trace. However the first covariant derivative is $\bar{\nabla}\varphi \in C^\infty(T^*\mathcal{M} \otimes \mathcal{E})$ (compare Definition 1.12).

There is a problem because $\bar{\nabla}\varphi$ is not a section of $\mathcal{E}$, so we cannot simply take the second covariant derivative using $\bar{\nabla}(t)$. To resolve this, we define a connection $\hat{\nabla}(t)$ on the vector bundle $\mathcal{E} \otimes T^*\mathcal{M}$ by the product rule:

$$\hat{\nabla}_X(\varphi \otimes \xi) \equiv \bar{\nabla}_X \varphi \otimes \xi + \varphi \otimes \nabla_X \xi$$

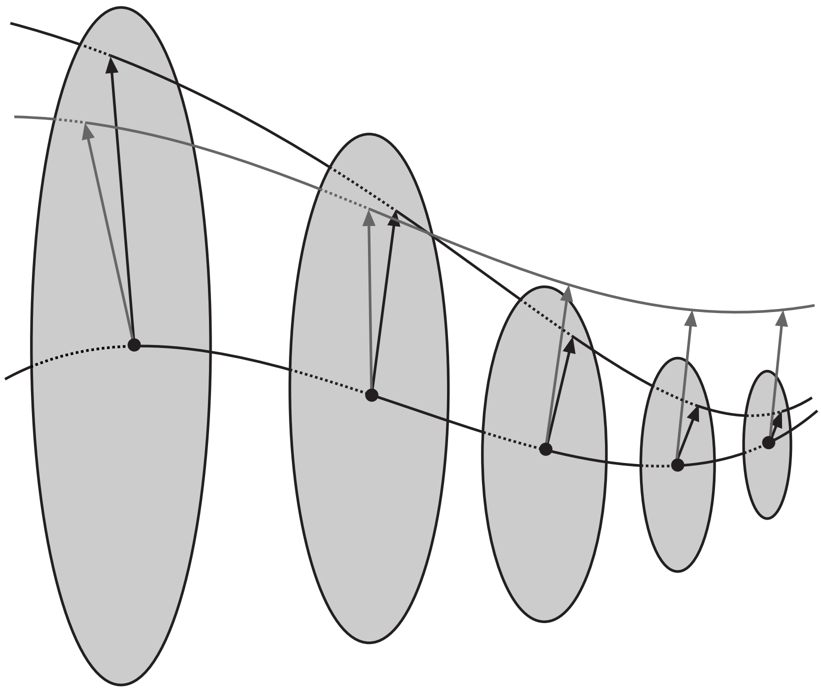

Figure 3.2: A 2-dimensional vector bundle over a 1-manifold. The set $\mathcal{K}$ (shaded) is not invariant under parallel translation. The initial section (in black) is contained in $\mathcal{K}$, but averages out to to a section (shown in grey) that is not contained in $\mathcal{K}$.

where $\varphi \in C^\infty(\mathcal{E})$, $\xi \in C^\infty(T^*\mathcal{M})$, and ∇ denotes the Levi-Civita connection of the metric $g(t)$. We can now define the Laplacian: if we use the index a for sections of $\mathcal{E}$ and indices i, j for sections of $T^*\mathcal{M}$ (so that an element H of $\mathcal{E} \otimes T^*\mathcal{M}$ might have coordinates H_{ia} with respect to some basis) then

$$(\hat{\triangle}\varphi)_a := g^{ij}\hat{\nabla}_i\bar{\nabla}_j\varphi_a$$

(compare this with Definition 1.13).

According to our earlier discussion, the maximum principle should somehow express the tendency of solutions of heat-type equations to "average out". What exactly does this mean for a section of a vector bundle? We would expect a vector in $\mathbb{R}^k$ that is getting averaged out to remain inside any convex set that initially contains it. So rather than proving that the lower bound of our solution is preserved, as we did for a scalar function, we prove that our solution stays inside convex sets. The set should also be closed and invariant under parallel translation. The reason for the latter can be seen from Figure 3.2 which shows a section of a 2-dimensional vector bundle over a 1-dimensional manifold getting averaged out. The section is initially contained inside the set $\mathcal{K}$, which is closed and convex but *not* invariant under parallel translation, but the averaging out takes it outside $\mathcal{K}$.

Theorem 3.3. *Let $\pi : \mathcal{E} \to \mathcal{M}$ be a vector bundle over $\mathcal{M}$ with a fixed bundle metric h, $\bar{\nabla}(t)$ a smooth time-dependent family of connections on $\mathcal{E}$ compatible with h, and $g(t)$ a time-dependent Riemannian metric on $\mathcal{M}$. Let $\mathcal{K}$ be a subset of $\mathcal{E}$ that is closed and convex in each fibre, and invariant under parallel translation. Let $F : \mathcal{E} \times [0, T) \to \mathcal{E}$ be a continuous map that is fibre-preserving, and Lipschitz in each fibre. Let $\alpha(t)$ be a time-dependent section of $\mathcal{E}$ that satisfies the* *conditions*

$$\begin{aligned}\frac{\partial}{\partial t}\alpha &= \hat{\triangle}\alpha + F(\alpha)\\ \alpha(0) &\in \mathcal{K}.\end{aligned}$$

Let $\mathcal{K}_x = \pi^{-1}(x) \cap \mathcal{K}$. Suppose that every solution of the ODE

$$\begin{aligned}\frac{da}{dt} &= F(a)\\ a(0) &\in \mathcal{K}_x\end{aligned}$$

remains in $\mathcal{K}_x$. Then the solution $\alpha(t)$ of the PDE *remains in $\mathcal{K}$.*

That is, if the reaction term does not force it out, then the solution will remain in $\mathcal{K}$. Note that the behaviour of the ODE is much simpler than the behaviour of the original PDE because we need only consider each fibre individually – the solution of the ODE is a time-dependent vector in some vector space rather than a time-dependent section of some vector bundle.

Chapter 4

Curve-Shortening Flow

This chapter is based on the lectures on Geometric Evolution Equations given by Ben Andrews at the ICE-EM Graduate School in July 2006.

When Hamilton introduced the Ricci flow and used it to prove Theorem 2.1, he introduced quite a general method of dealing with geometric evolution equations. This method has since been applied to other flows, such as the mean curvature flow (MCF) and the curve-shortening flow (CSF) (see [30] for a description of both). Although these settings for Hamilton's method of proof emerged after Hamilton's original Ricci flow paper, the CSF in particular is much simpler than the Ricci flow. Many of the concepts central to the theory of the Ricci flow presented in Chapters 5, 6, 7 and 8 find simpler analogies in the theory of the CSF.

In this chapter we will outline the basic theory of the CSF, with the aim of providing a "blueprint" for subsequent development of the theory of the Ricci flow. We will not prove all of the results that we state – those proofs and those results that assist in building intuition for the methods of proof employed in later chapters are emphasised.

4.1 Steepest Descent Flow for Length

Given a smooth immersion $X_0 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ (i.e. a smooth immersion of a circle into the plane), we can evolve it by taking $X : \mathbb{R}/\mathbb{Z} \times [0, T) \to \mathbb{R}^2$ such that

$$\begin{aligned} \frac{\partial X}{\partial t}(u,t) &= -\kappa \mathbf{N}(u,t) \\ X(u,0) &= X_0(u). \end{aligned} \tag{4.1}$$

Here, κ is the curvature of our curve, and $\mathbf{N}$ is the unit normal vector to it, defined by

$$-\kappa \mathbf{N} = \frac{\partial^2 X}{\partial s^2}$$

where s is the arclength parameter (see Figure 4.1). The book [15, Chap. 1] is a good reference for the theory of curves in space.

One might wonder where this definition comes from. The idea is to look for the way to move a curve in the plane that gives the greatest decrease in total length for the smallest total "movement" – we say that the CSF is the "steepest descent flow for length".

Lemma 4.1. *Denote the length of the curve $u \mapsto X(u,t)$ at time t by*

$$L(t) = \int_0^1 |X_u| du.$$

Consider an arbitrary variation of the curve $X_0(u) = X(u,0)$ with

$$\frac{\partial X}{\partial t}(u,t) = V(u,t),$$

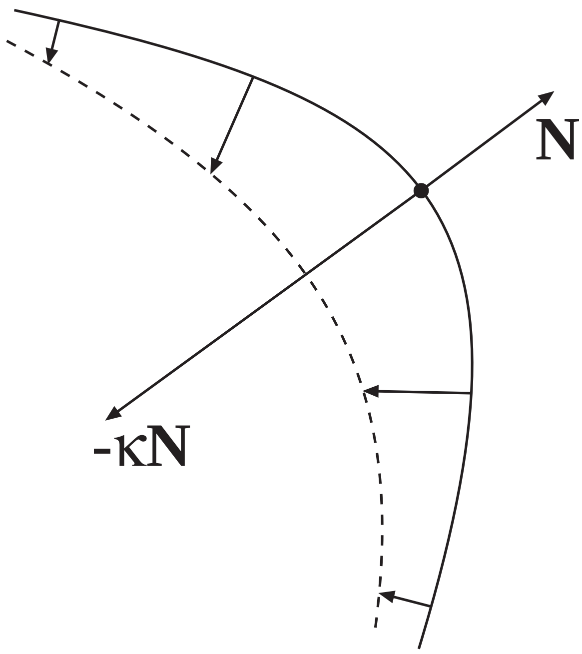

Figure 4.1: The curve-shortening flow moves a curve in the direction of its curvature vector.

which is normalized so that

$$\int_0^L |V|^2 ds = 1$$

where s is the arclength along X_0 and $L = L(0)$. Then the rate at which the length decreases, $-dL/dt$, is maximised for the CSF, *i.e.*

$$V \propto -\kappa \boldsymbol{N}.$$

Proof. We note that

$$\begin{aligned}
\partial_t |X_u| &= \partial_t \left(\langle X_u, X_u \rangle^{\frac{1}{2}} \right) \\
&= \frac{1}{|X_u|} \langle X_u, \partial_t X_u \rangle \\
&= \frac{1}{|X_u|} \langle X_u, \partial_u \partial_t X \rangle \\
&= \left\langle X_u, \frac{\partial_u V}{|X_u|} \right\rangle \\
&= \langle X_u, \partial_s V \rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{dL(t)}{dt} &= \int_0^1 \partial_t |X_u| du \\
&= \int_0^L \langle X_u, \partial_s V\rangle \frac{ds}{|X_u|} \\
&= \int_0^L \langle \partial_s X, \partial_s V\rangle ds \\
&= [\langle \partial_s X, V\rangle]_0^L - \int_0^L \langle \partial_{ss} X, V\rangle ds \\
&= \int_0^L \langle \kappa \mathbf{N}, V\rangle \\
&\geq -\left(\int_0^L \kappa^2 ds\right)^{\frac{1}{2}} \left(\int_0^L |V|^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

We have integrated by parts, and the boundary terms vanished by periodicity. The last line follows from the Cauchy-Schwarz inequality, so equality holds if and only if $V \propto -\kappa\mathbf{N}$. Thus the CSF is exactly the variation that gives the fastest decrease in length. □

Example: The shrinking circle. If our initial curve X_0 is a circle of radius r_0 centred at the origin, then the solution will take the form $X(u,t) = r(t)(\cos(u), \sin(u))$. We have $\mathbf{N} = X/r, \kappa = 1/r$, so the CSF equation becomes

$$\frac{dr}{dt} = -\frac{1}{r}$$

which has the solution

$$X(u,t) = (r_0^2 - 2t)^{\frac{1}{2}}(\cos(u), \sin(u)). \tag{4.2}$$

So the circle shrinks to a point at a finite time $t = \sqrt{r_0^2/2} = r_0/\sqrt{2}$.

4.2 Short Time Existence

Before anything else we must check that this system has a solution for a short time. To do this we will use the existence and uniqueness theorem for parabolic PDEs (Theorem A.1). We need to check if our system is strongly parabolic before we can apply this theorem.

If we set $X(u) = (x(u), y(u))$ and use the notation $x_u = \frac{\partial x}{\partial u}$, then the CSF equation becomes

$$\frac{\partial}{\partial t}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(x_u^2 + y_u^2)^2}\begin{bmatrix} y_u^2 & -x_u y_u \\ -x_u y_u & x_u^2 \end{bmatrix}\begin{bmatrix} x_{uu} \\ y_{uu} \end{bmatrix}$$

For the system to be strongly parabolic, the matrix on the RHS ought to be positive-definite. It clearly isn't (it is even singular), so we seem to be in trouble.

Although it appears that our system is not parabolic, all that is going on is that the parabolic nature of the system is being hidden from us by a poor parametrization of our solution. Suppose, for the moment, that we have a solution of the CSF given by $X : \mathbb{R}/\mathbb{Z} \times [0,T) \to \mathbb{R}^2$. Consider a time-dependent family of diffeomorphisms from the circle to itself:

$$\varphi : \mathbb{R}/\mathbb{Z} \times [0,T) \to \mathbb{R}/\mathbb{Z}.$$

Then we can define $\tilde{X}(u,t) = X(\varphi(u,t),t)$. $\tilde{X}$ is just a **time-dependent reparametrization** of X. In particular, it won't change how the curve "looks", i.e. its image in $\mathbb{R}^2$.

The evolution equation satisfied by $\tilde{X}$ is then

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{X}(u,t) &= \frac{\partial}{\partial t}X(\varphi(u,t),t) + X_u \frac{\partial}{\partial t}\varphi(u,t) \\
&= -\kappa\mathbf{N} + X_u V \\
&= \frac{1}{(x_u^2 + y_u^2)^2}\begin{bmatrix} y_u \\ -x_u \end{bmatrix}\begin{bmatrix} y_u & -x_u \end{bmatrix}\begin{bmatrix} x_{uu} \\ y_{uu} \end{bmatrix} + \begin{bmatrix} x_u \\ y_u \end{bmatrix} V
\end{aligned}$$

where we define

$$\frac{\partial}{\partial t}\varphi(u,t) = V(\varphi(u,t),t). \tag{4.3}$$

We are free to choose how to reparametrize our curve – that is, we may choose V and then solve the differential equation (4.3) for $\varphi(u,t)$. We choose V so that our system of CSF with time-dependent reparametrization is strongly parabolic:

$$V = \frac{1}{(x_u^2+y_u^2)^2}\begin{bmatrix} x_u & y_u \end{bmatrix}\begin{bmatrix} x_{uu} \\ y_{uu} \end{bmatrix}. \tag{4.4}$$

Then we have, setting $\tilde{X} = (\tilde{x},\tilde{y})$,

$$\begin{aligned}
\frac{\partial}{\partial t}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} &= \frac{1}{(x_u^2+y_u^2)^2}\left\{\begin{bmatrix} y_u \\ -x_u \end{bmatrix}\begin{bmatrix} y_u & -x_u \end{bmatrix} + \begin{bmatrix} x_u \\ y_u \end{bmatrix}\begin{bmatrix} x_u & y_u \end{bmatrix}\right\}\begin{bmatrix} x_{uu} \\ y_{uu} \end{bmatrix} \\
&= \frac{1}{(x_u^2+y_u^2)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_{uu} \\ y_{uu} \end{bmatrix}.
\end{aligned}$$

We can now express x_u, x_{uu}, y_u, y_{uu} in terms of $\tilde{x}_u, \tilde{x}_{uu}, \tilde{y}_u, \tilde{y}_{uu}$ using the chain rule. This gives us a second-order differential equation for $(\tilde{x},\tilde{y})$, which *is* strongly parabolic because the matrix multiplying $(\tilde{x}_{uu},\tilde{y}_{uu})$ is positive definite. By Theorem A.1 there exists a solution $\tilde{X} = (\tilde{x},\tilde{y})$ on some time interval $[0,T)$, and the solution is unique for as long as it exists. Using this solution it is possible to reconstruct $\varphi(u,t)$ from (4.3) – because we know $\tilde{x},\tilde{y}$ this is a simple ODE. Once we know the reparametrization used, we can find $X(u,t) = \tilde{X}(\varphi^{-1}(u,t),t)$ and show that it is a solution of the CSF. Therefore, although the CSF is not a strongly parabolic system, by reparametrizing it we can show that it has a solution on some time interval $[0,T)$, and it is not difficult to further prove that the solution is unique for as long as it exists. Because of this uniqueness we may without ambiguity choose

$$T = \sup\{\tau : X(u,t) \text{ exists and is smooth for } t \in [0,\tau)\}.$$

In this situation, we call $[0,T)$ a **maximal** time interval – it is the largest time interval on which the solution exists (note that T could be ∞).

Theorem 4.2. *For any smooth immersion $X_0 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$, there is a unique smooth solution*

$$X : \mathbb{R}/\mathbb{Z} \times [0,T) \to \mathbb{R}^2$$

satisfying the CSF *and with*

$$X(u,0) = X_0(u)$$

on a maximal time interval $[0,T)$.

4.3 Finite Time Singularity

Theorem 4.3. *The Avoidance Principle.* *If X, Y are solutions of the* CSF *on $[0,T)$, and $X(u,0) \neq Y(v,0)$ for all u,v, then $X(u,t) \neq Y(v,t)$ for all u,v, for all $t \in [0,T)$. That is, if the curves do not intersect initially, they will not intersect at any later time.*

This theorem can be proven using a version of the maximum principle to show that the smallest distance between the two curves is a nondecreasing function of time.

Theorem 4.4. *The* CSF *exists for only a finite time before becoming singular. That is, the maximal time T mentioned in Theorem 4.2 is finite.*

Proof. Given any X_0, we can take a circle that surrounds it. Running the CSF on the circle and the curve X_0, the Avoidance Principle shows that the solution X can never move outside the evolving circle. As we showed in (4.2), the circle shrinks to a point in a finite time, "crushing" our solution X and forcing it to become singular in finite time. Therefore the maximal time T mentioned in Theorem 4.2 must be finite. □

4.4 Curvature Explodes

Theorem 4.5. *Suppose we have a solution of the* CSF *on a* ***maximal*** *time interval* $[0, T)$*, where* $T < \infty$ *by Theorem 4.4. Then*

$$\lim_{t \to T} \sup\{|\kappa(u,t)| : u \in \mathbb{R}/\mathbb{Z}\} = \infty.$$

We will give an outline of the proof. The basic idea is to assume the curvature *were* bounded as $t \to T$, and show that the maps $X_t(\cdot) = X(\cdot, t)$ must then converge so nicely as $t \to T$ that it is possible to extend the solution past T, contradicting the assumption that $[0, T)$ was the maximal time interval on which the solution existed.

By differentiating equation (4.1) one can obtain an evolution equation for the curvature:

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 \kappa}{\partial s^2} + \kappa^3.$$

This is a heat-type equation. From it we can derive evolution equations for the derivatives of the curvature $\partial^n \kappa / \partial s^n$, which will also be heat-type equations. Assuming an upper bound on κ, one can use the maximum principle (in the form of Theorem 3.2) to prove upper bounds on the derivatives of κ. These bounds are analogous to the bounds obtained in Chapter 6 for the Ricci flow.

From the bounds on the derivatives of the curvature, it is possible to prove bounds on the derivatives of X. Once can show that all of the quantities

$$\left| \frac{\partial^k}{\partial u^k} \frac{\partial^l}{\partial t^l} X \right|$$

are bounded above. This means that all derivatives of X are equicontinuous, so by the Ascoli-Arzela theorem, for any $k, l > 0$ there is a sequence of times (t_i) such that the maps

$$\frac{\partial^k}{\partial u^k} \frac{\partial^l}{\partial t^l} X(\cdot, t_i)$$

converge uniformly. Because the time-derivatives are also bounded, this means that the maps

$$\frac{\partial k}{\partial u^k} \frac{\partial^l}{\partial t^l} X(\cdot, t)$$

converge uniformly. Therefore the map $X(\cdot, t)$ and all its derivatives converge uniformly to some smooth map $X(\cdot, T)$. We can then use Theorem 4.2 to find a solution $\bar{X}$ to the CSF with the initial curve $\bar{X}_0(\cdot) = X(\cdot, T)$, defined on some time interval $[0, \epsilon)$. The solution $\bar{X}$ can now be "glued onto the end of X" to extend the solution X past T. Defining $X(\cdot, t) = \bar{X}(\cdot, t - T)$ for $t \in [T, T + \epsilon)$ (and leaving X unchanged for $t \in [0, T)$), we obtain a smooth solution[1] X to the CSF with the given initial data, which is defined on the time interval $[0, T + \epsilon)$. This contradicts the maximality of T. Hence the original assumption that the curvature was bounded was incorrect.

4.5 Grayson's Theorem

We have seen that our curve becomes singular in finite time; in fact it can be shown that an embedded circle will shrink to a point and become round as it approaches the maximal time T. What does "become round" mean though, when our solution is shrinking to a point? To make sense of this, we "blow up" our solution – we dilate it by a time-dependent factor so that the area enclosed by the curve is constant. Then the solution will not shrink to a point, but will stay of the same size, and "becomes round" simply means that the curve converges to a circle.

[1]The solution has smooth u-derivatives at $t = T$ because of the uniform convergence of all derivatives of $X(\cdot, t)$ to $X(\cdot, T)$ as $t \to T$. The time-derivatives can be expressed in terms of the space derivatives using the equation of the CSF, so they too are smooth.

Theorem 4.6. *(Grayson) Given any embedded circle X_0, the solution to the* CSF *$X(u,t)$ with $X(u,0) = X_0(u)$ will remain embedded and will shrink to a point x as t approaches the maximal time of existence T. Furthermore, if $A(t)$ denotes the area enclosed by the curve $X(\cdot,t)$ at time t, then the curve*

$$\tilde{X}(u,t) = \sqrt{\frac{\pi}{A(t)}}(X(u,t) - x)$$

converges exponentially to a unit circle as $t \to T$.

Proof. Once again, we will not give a complete proof but will outline one of the main concepts, because it is analogous to the blowup technique that we will introduce for the Ricci flow in Chapter 8. See [30, Chap. 1 and Sec. 5.2] for the full proof of this theorem.

If the initial curve X_0 is convex, one can prove using derivative estimates that the curvature converges uniformly to 1, and all derivatives of curvature are bounded. Once again the Ascoli-Arzela theorem can be employed to show that a limit exists, and that the limit is a smooth embedded curve with curvature 1, hence the unit circle (see [30, Chap. 1] for the details). This result is known as the Gage-Hamilton Theorem (a corresponding result for the mean curvature flow was proven by Gerhard Huisken).

Grayson proved that any initial curve will eventually become convex under the CSF, from which the result follows by the Gage-Hamilton Theorem. We will outline the proof presented in [30, Sec. 5.2]. We use the result of Theorem 4.5, which says that the curvature explodes as we approach some singular time $T < \infty$. This means we may choose points $p_n \in \mathbb{R}/\mathbb{Z}$ and times t_n that converge to T such that

$$|\kappa(p_n, t_n)| = \sup_{p \in \mathbb{R}/\mathbb{Z}, t \leq t_n} |\kappa(p,t)|,$$

and define $M_n := |\kappa(p_n, t_n)|$, so that $\lim_{n\to\infty} M_n = \infty$. We then define rescaled flows X_n by

$$X_n(p,t) := M_n \left(X\left(p_n + p, t_n + \frac{t}{M_n^2}\right) - X(p_n, t_n)\right).$$

It is not hard to see that these rescaled flows are solutions of the CSF defined for $t \in [-M_n^2 t_n, M_n^2(T - t_n))$, with $X_n(0,0) = 0$ and $|\kappa_n(p,t)| \leq |\kappa_n(0,0)| = 1$ for $t \in [-M_n t_n, 0]$. This technique is known as "blowing up" at a singularity; we rescale the flow so that the curvature no longer diverges.

We saw in the discussion of Theorem 4.5 that it is possible to deduce bounds on derivatives of the curvature from a bound on the curvature, using the maximum principle. Moreover, one can get bounds on the derivatives of the maps $X_n : \mathbb{R} \times [-M_n t_n, 0]$ (considering X_n as a periodic map on the real line) from the bounds on derivatives of the curvature. Hence, by the Ascoli-Arzela theorem there is a subsequence of X_n that converges uniformly in all C^k norms on all compact subsets of $\mathbb{R} \times (-\infty, 0]$ to a limit solution $X_\infty : \mathbb{R} \times (-\infty, 0] \to \mathbb{R}^2$. This limit solution satisfies $X_\infty(0,0) = 0$ and $\kappa_\infty(p,t) = \kappa_\infty(0,0) = 1$ for all $t \leq 0$.

After considerable work, one can show that this limit of rescaled solutions is convex with compact image for all $t \in (-\infty, 0]$, from which it follows that some X_n is eventually convex and hence the CSF $X(\cdot,t)$ is eventually convex for large enough t, as required. □

Chapter 5

Short Time Existence and Uniqueness of the Ricci Flow

Before we can do anything with the Ricci flow, we must show that a solution exists for a short time. We would like to apply the short-time existence and uniqueness theorem for parabolic PDEs (Theorem A.1) to the system

$$\begin{aligned}\frac{\partial}{\partial t}g(t) &= -2\mathrm{Rc}(g(t))\\ g(0) &= g_0.\end{aligned}$$

We need to check if the system is strongly parabolic. This chapter is based on [6, Chap. 3] and [28, Chap. 5].

5.1 The Linearization of the Ricci Tensor

The first thing to do is to work out the linearization of the Ricci tensor in the sense described in Appendix A. Recall that, if we have a time-dependent metric tensor $g_{ij}(t)$ (we are *not* assuming that this time-dependent metric satisfies the Ricci flow or any other specific equation), we define the linearization of the Ricci tensor, $D[\mathrm{Rc}] : C^\infty(T^*\mathcal{M} \otimes_S T^*\mathcal{M}) \to C^\infty(T^*\mathcal{M} \otimes_S T^*\mathcal{M})$ so that

$$D[\mathrm{Rc}]\left(\frac{\partial g_{ij}}{\partial t}\right) = \frac{\partial}{\partial t}\mathrm{Rc}(g_{ij}(t)).$$

Lemma 5.1. *The linearization of the Ricci tensor is given by*

$$D[\mathrm{Rc}](h)_{ij} = \frac{1}{2}g^{pq}(-\nabla_p\nabla_q h_{ij} - \nabla_i\nabla_j h_{pq} + \nabla_q\nabla_i h_{jp} + \nabla_q\nabla_j h_{ip}).$$

Proof. This follows from equation (1.19). We have interchanged the indices p and q in places, which we can do because the metric inverse g^{pq} is symmetric. $\square$

To check if the system is strongly parabolic we must compute the principal symbol (see Appendix A for the definition). The symbol can be computed easily from the linearization:

Corollary 5.2. *The principal symbol of the differential operator* $-2\mathrm{Rc}$ *(as a function of the metric* g*) is:*

$$\hat{\sigma}[-2\mathrm{Rc}](\varphi)(h)_{ij} = g^{pq}(\varphi_p\varphi_q h_{ij} + \varphi_i\varphi_j h_{pq} - \varphi_q\varphi_i h_{jp} - \varphi_q\varphi_j h_{ip}).$$

Now we recall that the Ricci flow is strongly parabolic if there exists $\delta > 0$ such that for all covectors $\varphi \neq 0$ and all (symmetric) $h_{ij} \neq 0$,

$$\langle\hat{\sigma}[-2\mathrm{Rc}](\varphi)(h), h\rangle > \delta|\varphi|^2|h|^2$$

or, by Corollary 5.2,

$$g^{pq}(\varphi_p\varphi_q h_{ij} + \varphi_i\varphi_j h_{pq} - \varphi_q\varphi_i h_{jp} - \varphi_q\varphi_j h_{ip})h^{ij} > \delta\varphi_k\varphi^k h_{rs}h^{rs}.$$

However, if we choose $h_{ij} = \varphi_i\varphi_j$, it is clear that the LHS of this equation is 0, so the inequality can not hold. Therefore the Ricci flow is *not* strongly parabolic.

5.2 The DeTurck Trick

Because the Ricci flow is not strongly parabolic, we can not immediately apply Theorem A.1. Hamilton, in his original paper [9], used the Nash-Moser inverse function theorem to prove short-time existence and uniqueness. Shortly afterwards, Dennis DeTurck introduced (in [7]) a much simpler way of proving the short-time existence of the Ricci flow. In this section we show how one does the "DeTurck Trick".

We first rewrite the linearization of the Ricci tensor so that it is possible to see which terms are making it non-parabolic.

Lemma 5.3. *The linearization of* $-2\mathrm{Rc}$ *is equal to*

$$D[-2\mathrm{Rc}](h)_{ij} = g^{pq}\nabla_p\nabla_q h_{ij} + \nabla_i V_j + \nabla_j V_i + \text{ lower-order terms in } h \tag{5.1}$$

where

$$V_i = g^{pq}(\frac{1}{2}\nabla_i h_{pq} - \nabla_q h_{pi}).$$

Proof. From Lemma 5.1 we have

$$D[-2\mathrm{Rc}](h)_{ij} = g^{pq}(\nabla_p\nabla_q h_{ij} + \nabla_i\nabla_j h_{pq} - \nabla_q\nabla_i h_{jp} - \nabla_q\nabla_j h_{ip}). \tag{5.2}$$

Applying the formula (B.3) for commuting covariant derivatives, we obtain

$$\begin{aligned}\nabla_q\nabla_i h_{jp} &= \nabla_i\nabla_q h_{jp} - R^r_{qij}h_{rp} - R^r_{qip}h_{jm} \\ &= \nabla_i\nabla_q h_{jp} + \text{ lower-order terms in } h.\end{aligned}$$

Lower-order terms in h make no contribution to the principal symbol. Thus, as far as the principal symbol is concerned, covariant derivatives commute. We can therefore rearrange equation (5.2) by commuting covariant derivatives to give

$$\begin{aligned}D[-2\mathrm{Rc}](h)_{ij} &= g^{pq}\nabla_p\nabla_q h_{ij} + \nabla_i(\frac{1}{2}\nabla_j h_{pq} - \nabla_q h_{jp}) + \nabla_j(\frac{1}{2}\nabla_i h_{pq} - \nabla_q h_{ip}) \\ &\quad + \text{ lower-order terms in } h \\ &= g^{pq}\nabla_p\nabla_q h_{ij} + \nabla_i V_j + \nabla_j V_i + \text{ lower-order terms in } h\end{aligned}$$

(recalling that $\nabla g = 0$). □

The lower-order terms in h make no contribution to the principal symbol. The first term in equation (5.1) is a good term (we can identify it as a Laplacian, which has a strictly positive principal symbol), but the terms in V are bad: they make the Ricci flow non-parabolic.

The situation is analogous to that for the curve-shortening flow described in Section 4.2: the flow is not parabolic because we have a bad parametrization of our manifold. To obtain a parabolic system, we must introduce a time-dependent reparametrization.

In analogy with the curve-shortening flow, we seek a system

$$\frac{\partial}{\partial t}\bar{g}_t = P(\bar{g}_t) \tag{5.3}$$

that *is* parabolic, and a time-dependent diffeomorphism φ_t from our manifold $\mathcal{M}$ to itself, with $\varphi_0 = \mathrm{id}$, so that the metric

$$g_t = \varphi_t^*\bar{g}_t$$

is a solution to the Ricci flow. We can compute

$$\begin{aligned}\frac{\partial}{\partial t}g_t &= \frac{\partial\left(\varphi_t^*\bar{g}_t\right)}{\partial t}\\ &= \left(\frac{\partial\left(\varphi_{t+s}^*\bar{g}_{t+s}\right)}{\partial s}\right)_{s=0}\\ &= \left(\varphi_t^*\frac{\partial\bar{g}_{t+s}}{\partial s}\right)_{s=0}+\left(\frac{\partial\left(\varphi_{t+s}^*\bar{g}_t\right)}{\partial s}\right)_{s=0}\\ &= \varphi_t^*P(\bar{g}_t)+\varphi_t^*\mathcal{L}_{\frac{\partial\varphi_t}{\partial t}}\bar{g}_t.\end{aligned}$$

Here we have used the product rule and then the rule for the Lie derivative on tensors (see Section 1.3). We choose φ_t to satisfy

$$\begin{aligned}\frac{\partial}{\partial t}\varphi_t &= W(t)\\ \varphi_0 &= \mathrm{id}\end{aligned}$$

for some time-dependent vector field $W(t)$ (this system will have a solution φ_t for as long as $W(t)$ exists – see [6, Lem. 3.15]). The problem now reduces to finding a differential operator P such that the system (5.3) is strongly parabolic, and a time-dependent vector field $W(t)$ such that, if we define a 1-parameter family of diffeomorphisms that satisfies (5.4), then

$$\varphi_t^*P(\bar{g}_t)+\varphi_t^*\mathcal{L}_{W(t)}\bar{g}_t=-2\mathrm{Rc}(\varphi_t^*\bar{g}_t)=-2\varphi_t^*\mathrm{Rc}(\bar{g}_t)$$

(using the diffeomorphism invariance of the Ricci tensor). This is equivalent to

$$P(\bar{g}_t)=-2\mathrm{Rc}(\bar{g}_t)-\mathcal{L}_{W(t)}\bar{g}_t.$$

Note that the choice of $W(t)$ corresponds to the choice of V in Section 4.2.

Now by Lemma 1.7 we have

$$(\mathcal{L}_W\bar{g}_t)_{ij}=\nabla_iW_j+\nabla_jW_i.$$

Therefore we can use Lemma 5.3 to write the linearization of P as

$$\begin{aligned}D[P](h) &= g^{pq}\nabla_p\nabla_qh_{ij}+\nabla_iV_j+\nabla_jV_i+\text{ lower-order terms in } h \qquad (5.4)\\ &\quad -D[\nabla_iW_j+\nabla_jW_i](h). \qquad (5.5)\end{aligned}$$

Recall that the first term in equation (5.4) is a good term: if the other second-order terms cancelled then the linearization would satisfy the condition for parabolicity. So our aim is to choose W in such a way that the second-order part of the term (5.5) cancels the second-order part of the terms in V. This will happen if the principal part of the linearization of W_i is equal to that of V_i[1].

We defined V by

$$\begin{aligned}V_i &= g^{pq}(\frac{1}{2}\nabla_ih_{pq}-\nabla_ph_{qi})\\ &= -\frac{1}{2}g^{pq}(\nabla_ph_{qi}+\nabla_qh_{pi}-\nabla_ih_{pq}).\end{aligned}$$

We recall from formula (1.17) that

$$D[\Gamma_{ij}^k](h)=\frac{\partial}{\partial t}\left(\Gamma_{ij}^k(g(t))\right)=\frac{1}{2}g^{kl}(\nabla_ih_{jl}+\nabla_jh_{il}-\nabla_lh_{ij}),$$

where $h_{ij}=\frac{\partial}{\partial t}g_{ij}$. This looks pretty similar to the form of V, so we might try

$$W_i=-g^{pq}g_{ij}\Gamma_{pq}^j.$$

[1]Note that, as far as the principal symbol is concerned, a covariant derivative ∇_i is the same as a coordinate derivative ∂_i by formula (1.3). Therefore, if the principal symbol of the linearization of W_i is equal to that of V_i then the principal symbol of the linearization of ∇_jW_i is equal to that of ∇_jV_i.

We need to be careful though – this expression depends on the coordinates we choose, because the Christoffel symbol Γ is not a tensor. However, the difference between two connections *is* a tensor, so we can fix a constant connection with Christoffel symbols $\tilde{\Gamma}^k_{ij}$ and define

$$W_i = -g^{pq} g_{ij} \left(\Gamma^j_{pq} - \tilde{\Gamma}^j_{pq} \right),$$

which *is* the coordinate form of some coordinate-independent vector field. The fixed background connection, because it is independent of the metric, will make no contribution to the symbol of W, so the principal symbol of W_i will be equal to that of V_i and all of the second-order terms other than the first in equation (5.4) will cancel.

Thus, making this choice of W, the principal symbol of the linearization of P is just

$$\hat{\sigma}(D[P])(\varphi)(h)_{ij} = g^{pq}\varphi_p\varphi_q h_{ij},$$

so we have

$$\langle \hat{\sigma}(D[P])(\varphi)(h), h\rangle = |\varphi|^2 |h|^2.$$

Thus the Ricci-DeTurck flow defined by

$$\frac{\partial}{\partial t}\bar{g}_{ij} = P(\bar{g}) = -2\bar{R}_{ij} + \nabla_i W_j + \nabla_j W_i$$

is strongly parabolic, and therefore has a solution for a short time by Theorem A.1.

For as long as this solution exists, the vector field $W(t)$ exists, and the time-dependent diffeomorphisms φ_t can be obtained by solving the ODE (5.4) with initial condition φ_0 =id. Once we know the φ_t exist, the above calculations show that the pullback metrics $g_t = \varphi_t^* \bar{g}_t$ satisfy the Ricci flow equation. Thus, the Ricci flow has a solution for a short time. In fact the solution is also unique (see [6, Sec. 4.4]), so we have the following result:

Theorem 5.4. *Given a smooth Riemannian metric g_0 on a closed manifold $\mathcal{M}$, there exists a maximal time interval $[0, T)$ such that a solution $g(t)$ of the Ricci flow, with $g(0) = g_0$, exists and is smooth on $[0, T)$, and this solution is unique.*

Chapter 6

Derivative Estimates and Curvature Explosion at Singularities

In this chapter we will use the maximum principle in some creative ways to obtain bounds on derivatives of the curvature and metric evolving under the Ricci flow. We will then use our estimates to show that the curvature must explode[1] as we approach a finite-time singularity in the Ricci flow, in analogy with Theorem 4.5 for the curve-shortening flow. This result is a key part of the proof of Theorem 2.1 that we will present in Chapter 7. The derivative estimates themselves are also vital to the proof of the compactness result presented in Chapter 8. Our exposition is based on [6, Chap. 6,7] (with the exception of Corollary 6.13 which is based on results in [28] and Section 6.2 which is based on [9, Sec. 13]), with significant reformulation of many of the arguments.

6.1 Evolution of Geometric Quantities Under the Ricci Flow

To apply maximum principle arguments to the curvature, we need to know what the equations describing the evolution of curvature quantities under the Ricci flow are. We have already done the bulk of the calculations in Lemma 1.20: the evolution equations for the Ricci flow follow by substituting $h_{ij} = -2R_{ij}$ into those formulae.

Lemma 6.1. *Suppose that $g_{ij}(t)$ is a solution of the Ricci flow:*

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

Then the various geometric quantities evolve according to the following equations:

1. *Metric inverse:*
$$\frac{\partial}{\partial t} g^{ij} = 2R^{ij}.$$

2. *Christoffel symbols:*
$$\frac{\partial}{\partial t} \Gamma^k_{ij} = -g^{kl}(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}).$$

3. *Riemann curvature tensor:*
$$\begin{aligned}\frac{\partial}{\partial t} R_{ijkl} &= \triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ &\quad -(R_i^p R_{pjkl} + R_j^p R_{ipkl} + R_k^p R_{ijpl} + R_l^p R_{ijkp}),\end{aligned}$$

[1]Some books on the Ricci flow use the words "blow up" here – unfortunately this expressive nomenclature is already employed to describe our method of examining singularities as in Chapter 8, so we substitute "explode". All it means is that the curvature goes to $+\infty$, as detailed in Theorem 6.7.

where

$$B_{ijkl} \equiv -R^{q}_{pij} R^{p}_{qlk}.$$

See Definition 1.13 for the definition of the Laplacian $\triangle$.

4. *Ricci tensor:*

$$\frac{\partial}{\partial t} R_{ij} = \triangle R_{ij} + 2R_{pijq}R^{pq} - 2R^{p}_{i} R_{pj}.$$

5. *Scalar curvature:*

$$\frac{\partial}{\partial t} R = \triangle R + 2|\mathrm{Rc}|^2.$$

6. *Einstein tensor on a 3-manifold:*

$$\frac{\partial}{\partial t}|E|^2 \quad = \quad \triangle|E|^2 - 2|\nabla E|^2 - 8R^{j}_{i} R^{k}_{j} R^{i}_{k} + \frac{26}{3} R|\mathrm{Rc}|^2 - 2R^3.$$

7. *Volume element:*

$$\frac{\partial}{\partial t} d\mu = -R d\mu.$$

8. *Volume of manifold:*

$$\frac{\partial}{\partial t} \int_{\mathcal{M}} d\mu = -\int_{\mathcal{M}} R d\mu.$$

9. *Total scalar curvature (on a closed manifold):*

$$\frac{\partial}{\partial t} \int_{\mathcal{M}} R d\mu = \int_{\mathcal{M}} \left(-R^2 + 2|\mathrm{Rc}|^2\right) d\mu.$$

Proof. Most of these follow easily from Lemma 1.20, but some require a bit of extra fiddling to put them in a nice form. The proof of the equation for the Riemann curvature tensor in particular is rather lengthy and is contained (along with those for Rc and R) in [6, Sec. 6.1]. The main technique is to use the identity (B.3) to commute covariant derivatives, then use the Bianchi identities.

The proof of the evolution equation for $|E|^2$ in dimension 3 can be found in [6, Cor. 6.39]. It can be proven using the formula

$$|E|^2 = |\mathrm{Rc}|^2 - \frac{1}{3} R^2,$$

the evolution equations for Rc and R given above, and the result of Lemma 1.13 to express the Riemann curvature tensor in terms of the Ricci tensor. We will not use this result in the current chapter, but it comes in handy in Chapter 7. □

6.2 Evolution Equations for Derivatives of Curvature

We aim to obtain bounds on the derivatives of the curvature, i.e. on quantities of the form $|\nabla^k \mathrm{Rm}|^2$ (here the k is not a raised index, it indicates the kth iterated covariant derivative). Our hope is to apply the maximum principle of Theorem 3.2 to get such bounds, but before we can apply this theorem we must have some PDEs that describe the evolution of the quantities.

Lemma 6.2. *If A is a tensor quantity that satisfies a heat-type evolution equation:*

$$\frac{\partial}{\partial t} A = \triangle A + F$$

under the Ricci flow (where F is a tensor of the same type as A), then the square of its norm satisfies the heat-type equation

$$\frac{\partial}{\partial t}|A|^2 = \triangle|A|^2 - 2|\nabla A|^2 + F * A + \mathrm{Rc} * A^{*2}.$$

Proof. We use g_t to denote the metric at time t.

$$\begin{aligned}\frac{\partial}{\partial t} g_t(A,A) &= 2g_t\left(\frac{\partial}{\partial t}A, A\right) + \frac{\partial g_t}{\partial t}(A,A) \\ &= 2g_t(\triangle A + F, A) + \mathrm{Rc} * A^{*2} \\ &= \triangle|A|^2 - 2|\nabla A|^2 + F * A + \mathrm{Rc} * A^{*2}\end{aligned}$$

where the final term in the second line comes from the derivative of the metric (which is $-2\mathrm{Rc}$ for the Ricci flow) and we have used the identity

$$\triangle|A|^2 = 2\langle \triangle A, A\rangle + 2|\nabla A|^2.$$

□

Lemma 6.3. *If A is a tensor quantity that satisfies a heat-type evolution equation*

$$\frac{\partial}{\partial t}A = \triangle A + F$$

under the Ricci flow (where F is a tensor of the same type as A), then its covariant derivative satisfies a heat-type equation

$$\frac{\partial}{\partial t}\nabla A = \triangle(\nabla A) + \nabla F + \mathrm{Rm} * \nabla A + \nabla \mathrm{Rc} * A.$$

Proof. Recall from formula (1.3) that ∇A has the form

$$\nabla A = \partial A + f(\Gamma, A)$$

where $f(\Gamma, A)$ is some expression of the form $\Gamma * A$, which depends on the type of the tensor A. Also, by Lemma 6.1 we have

$$\partial_t \Gamma = \left(g^{-1}\right) * \nabla \mathrm{Rc}.$$

It follows that

$$\begin{aligned}\partial_t \nabla A &= \partial_t \partial A + \partial_t f(\Gamma, A) \\ &= \partial \partial_t A + f(\Gamma, \partial_t A) + f(\partial_t \Gamma, A) \text{ (by the product rule)} \\ &= \nabla(\partial_t A) + f(g^{-1} * \nabla \mathrm{Rc}, A) \text{ (because } \partial_t A \text{ is a tensor of the same type as } A) \\ &= \nabla(\triangle A + F) + \nabla \mathrm{Rc} * A \\ &= (\triangle \nabla A + \mathrm{Rm} * \nabla A + \nabla \mathrm{Rc} * A) + \nabla F + \nabla \mathrm{Rc} * A \\ &= \triangle \nabla A + \nabla F + \mathrm{Rm} * \nabla A + \nabla \mathrm{Rc} * A.\end{aligned}$$

We used the formula

$$[\nabla, \triangle]A := \nabla \triangle A - \triangle \nabla A = \mathrm{Rm} * \nabla A + \nabla \mathrm{Rc} * A,$$

which follows from careful use of formula (B.3) followed by the second Bianchi identity (1.8) (see [6, p. 227]). □

Now we note that the formula for the evolution of the Riemann curvature tensor under the Ricci flow given in Lemma 6.1 translates into $*$-notation as

$$\frac{\partial}{\partial t}\mathrm{Rm} = \triangle \mathrm{Rm} + \mathrm{Rm}^{*2}. \tag{6.1}$$

This allows us to use the preceding lemmas to calculate evolution equations for covariant derivatives of the metric.

Lemma 6.4. *The evolution equation for the kth iterated covariant derivative of the Riemann curvature tensor under the Ricci flow is:*

$$\frac{\partial}{\partial t}\nabla^k \mathrm{Rm} = \triangle \nabla^k \mathrm{Rm} + \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j} \mathrm{Rm}.$$

Proof. We prove the result by induction. The evolution equation for Rm given above in equation (6.1) is the base case $k = 0$. We assume the relation holds true for a given k and apply Lemma 6.3 with $A = \nabla^k \mathrm{Rm}$ and

$$F = \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j} \mathrm{Rm}.$$

This tells us that

$$\frac{\partial}{\partial t} \nabla \nabla^k \mathrm{Rm} = \triangle(\nabla \nabla^k \mathrm{Rm}) + \nabla F + \mathrm{Rm} * \nabla(\nabla^k \mathrm{Rm}) + \nabla \mathrm{Rc} * \nabla^k \mathrm{Rm}. \tag{6.2}$$

It is clear that all of the reaction terms on the RHS are of the form $\nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm}$ where $i + j = k+1$, hence

$$\frac{\partial}{\partial t} \nabla^{k+1} \mathrm{Rm} = \triangle \nabla^{k+1} \mathrm{Rm} + \sum_{j=0}^{k+1} \nabla^j \mathrm{Rm} * \nabla^{k+1-j} \mathrm{Rm},$$

completing the inductive step. □

Corollary 6.5. *The square of the norm of the kth covariant derivative of the Riemann curvature tensor satisfies the heat-type equation*

$$\frac{\partial}{\partial t} |\nabla^k \mathrm{Rm}|^2 = \triangle |\nabla^k \mathrm{Rm}|^2 - 2|\nabla^{k+1} \mathrm{Rm}|^2 + \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j} \mathrm{Rm} * \nabla^k \mathrm{Rm}. \tag{6.3}$$

Proof. We simply apply Lemma 6.2 with $A = \nabla^k \mathrm{Rm}$ and

$$F = \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j} \mathrm{Rm},$$

by the result of Lemma 6.4. The result is

$$\frac{\partial}{\partial t} |\nabla^k \mathrm{Rm}|^2 = \triangle |\nabla^k \mathrm{Rm}|^2 - 2|\nabla^{k+1} \mathrm{Rm}|^2 + F * \nabla^k \mathrm{Rm} + \mathrm{Rc} * (\nabla^k \mathrm{Rm})^{*2}.$$

It is clear that all the terms on the left hand side other than the first two are of the form $\nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^k \mathrm{Rm}$ where $i + j = k$, and hence

$$\frac{\partial}{\partial t} |\nabla^k \mathrm{Rm}|^2 = \triangle |\nabla^k \mathrm{Rm}|^2 - 2|\nabla^{k+1} \mathrm{Rm}|^2 + \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j} \mathrm{Rm} * \nabla^k \mathrm{Rm}$$

as required. □

6.3 The Bernstein-Bando-Shi Estimates

We now apply the maximum principle to the evolution equations derived in the last section to get bounds on the derivatives of the curvature. We aim to obtain these bounds under the assumption that the curvature itself is bounded above by some constant, $|\mathrm{Rm}| < K$. There are two problems we face in trying to apply the maximum principle to the evolution equation we derived for the covariant derivatives of the curvature: firstly, we can not guarantee any initial conditions on the derivatives of the curvature if we are only given bounds on the curvature, and secondly the evolution equation has some terms in it which we are not sure how to control (namely the terms of the summation in equation (6.3)).

We bypass the first problem by proving time-dependent upper bounds that diverge at $t = 0$ (thus they only give us useful information after $t = 0$). We will see how to bypass the second in the process of the proof.

Theorem 6.6. ***The Bernstein-Bando-Shi Estimates.*** *Let $(\mathcal{M}^n, g(t))$ be a solution of the Ricci flow on a closed n-manifold. Then for each $\alpha > 0$ and $m \in \mathbb{N}$, there exists a constant C_m depending only on m, n and $\max\{\alpha, 1\}$ such that if*

$$|\mathrm{Rm}(x,t)|_{g(t)} \leq K \textit{ for all } t \in [0, \tfrac{\alpha}{K}],$$

then

$$|\nabla^m \mathrm{Rm}(x,t)|_{g(t)} \leq \frac{C_m K}{t^{m/2}} \textit{ for all } t \in (0, \tfrac{\alpha}{K}].$$

Proof. We prove the result by induction on m. For $m = 0$ the result is true by hypothesis, with $C_0 = 1$. Assume the result is true for all $p \leq m-1$. The result of Corollary 6.5 tells us that

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^m \mathrm{Rm}|^2 &= \triangle|\nabla^m \mathrm{Rm}|^2 - 2|\nabla^{m+1}\mathrm{Rm}|^2 + \sum_{j=0}^{m} \nabla^j \mathrm{Rm} * \nabla^{m-j}\mathrm{Rm} * \nabla^m \mathrm{Rm} \\
&\leq \triangle|\nabla^m \mathrm{Rm}|^2 - 2|\nabla^{m+1}\mathrm{Rm}|^2 + \sum_{j=0}^{m} c_{mj}|\nabla^j \mathrm{Rm}||\nabla^{m-j}\mathrm{Rm}||\nabla^m \mathrm{Rm}| \\
&\leq \triangle|\nabla^m \mathrm{Rm}|^2 - 2|\nabla^{m+1}\mathrm{Rm}|^2 + \left(\sum_{j=1}^{m-1} c_{mj}\frac{C_j}{t^{j/2}}\frac{C_{m-j}}{t^{(m-j)/2}}\right) K^2|\nabla^m \mathrm{Rm}| \\
&\quad + (c_{m0} + c_{mm})K|\nabla^m \mathrm{Rm}|^2 \\
&\leq \triangle|\nabla^m \mathrm{Rm}|^2 - 2|\nabla^{m+1}\mathrm{Rm}|^2 + C'_m K|\nabla^m \mathrm{Rm}|^2 + \frac{C''_m}{t^{m/2}}K^2|\nabla^m \mathrm{Rm}|
\end{aligned}$$

for $t \in (0, \frac{\alpha}{K}]$, where C'_m, C''_m are constants depending only on m and n. We have used the inductive hypothesis to go from the second to the third line. We can complete the square (in the variable $|\nabla^m \mathrm{Rm}|$) on the right hand side then use the inequality $(a+b)^2/2 \leq a^2 + b^2$ to obtain

$$\frac{\partial}{\partial t}|\nabla^m \mathrm{Rm}|^2 \leq \triangle|\nabla^m \mathrm{Rm}|^2 - 2|\nabla^{m+1}\mathrm{Rm}|^2 + \bar{C}_m K\left(|\nabla^m \mathrm{Rm}|^2 + \frac{K^2}{t^m}\right) \tag{6.4}$$

for some constant $\bar{C}_m$.

To get the desired bound, we need to find an upper bound on $t^m|\nabla^m \mathrm{Rm}|^2$. This quantity clearly has an upper bound at $t = 0$ because it is equal to 0. So to apply the maximum principle and get an upper bound we need only show that the reaction terms in its evolution equation cause it to decrease. The problem we have is that this quantity satisfies the differential inequality

$$\begin{aligned}
\frac{\partial}{\partial t}(t^m|\nabla^m \mathrm{Rm}|^2) &\leq \triangle(t^m|\nabla^m \mathrm{Rm}|^2) - 2t^m|\nabla^{m+1}\mathrm{Rm}|^2 + &(6.5)\\
&\quad (\bar{C}_m K t + m)t^{m-1}|\nabla^m \mathrm{Rm}|^2 + \bar{C}_m K^3, &(6.6)
\end{aligned}$$

and the reaction terms (6.6) are not negative. This is the "second difficulty" outlined at the start of this section.

To fix this problem, we make use of the term $-2|\nabla^{k+1}\mathrm{Rm}|^2$ in the evolution equation of Corollary 6.5. By adding the right amount of $t^{m-1}|\nabla^{m-1}\mathrm{Rm}|^2$ (which we know by our inductive hypothesis is bounded above by a constant) we can cancel off the unruly reactionary terms involving $t^m|\nabla^m \mathrm{Rm}|^2$. In so doing, we will introduce new unruly terms in $t^{m-1}|\nabla^{m-1}\mathrm{Rm}|^2$ – so we will need to add the right amount of the next derivative down and so on. We define

$$G = t^m|\nabla^m \mathrm{Rm}|^2 + \sum_{j=0}^{m-1} \alpha_{mj} t^j |\nabla^j \mathrm{Rm}|^2,$$

where we will determine the constants α_{mj} so that all the terms cancel as we want them to.

By equation (6.5), we have

$$\begin{aligned}
\frac{\partial}{\partial t}G &\leq \triangle G + (\bar{C}_m K t + m)t^{m-1}|\nabla^m \mathrm{Rm}|^2 + \bar{C}_m K^3 \\
&\quad + \sum_{j=0}^{m-1} \alpha_{mj}\left\{-2t^j|\nabla^{j+1}\mathrm{Rm}|^2 + (\bar{C}_j K t + j)t^{j-1}|\nabla^j \mathrm{Rm}|^2 + \bar{C}_j K^3\right\}.
\end{aligned}$$

By the inductive hypothesis there are numbers D_j depending only on j, n for $1 \leq j \leq m-1$ such that

$$\bar{C}_j K t^j |\nabla^j \mathrm{Rm}|^2 + \bar{C}_j K^3 \leq D_j K^3$$

for all $t \in (0, \frac{\alpha}{K}]$. Hence we have

$$\begin{aligned}
\frac{\partial}{\partial t} G &\leq \triangle G + (\bar{C}_m K t + m) t^{m-1} |\nabla^m \mathrm{Rm}|^2 + \bar{C}_m K^3) \\
&\quad + \sum_{j=0}^{m-1} \alpha_{mj} \left\{ -2t^j |\nabla^{j+1} \mathrm{Rm}|^2 + j t^{j-1} |\nabla^j \mathrm{Rm}|^2 + D_j K^3 \right\} \\
&= \triangle G + (\bar{C}_m K t + m - 2\alpha_{m,m-1}) t^{m-1} |\nabla^m \mathrm{Rm}|^2 \\
&\quad + \sum_{j=0}^{m-1} \left\{ j \alpha_{mj} - 2\alpha_{m,j-1} \right\} t^{j-1} |\nabla^j \mathrm{Rm}|^2 \\
&\quad + \bar{C}_m K^3 + \sum_{j=0}^{m-1} \alpha_{mj} D_j K^3.
\end{aligned}$$

Now we choose the α_{mj} so that the terms in this equation cancel: choose $\alpha_{m,m-1}$ such that

$$0 = \bar{C}_m \alpha + m - 2\alpha_{m,m-1} \geq \bar{C}_m K t + m - 2\alpha_{m,m-1},$$

where the second step follows because we are working with $t \in (0, \alpha/K]$. Now define $\alpha_{m,m-2}$, $\alpha_{m,m-3}, \ldots, \alpha_{m0}$ in that order, at each step setting

$$j \alpha_{mj} - 2\alpha_{m,j-1} = 0.$$

If we now define

$$B_m := \bar{C}_m + \sum_{j=0}^{m-1} \alpha_{mj} D_j$$

then our evolution can be written as

$$\frac{\partial}{\partial t} G \leq \triangle G + B_m K^3.$$

The reaction term is simply a constant, so it gives linear growth at worst. Because $G = \alpha_{m0} |\mathrm{Rm}|^2 \leq \alpha_{m0} K^2$ at $t = 0$, the scalar maximum principle (Theorem 3.2) tells us that

$$G \leq \alpha_{m0} K^2 + B_m K^3 t \leq (\alpha_{m0} + B_m \alpha) K^2 := C_m^2 K^2$$

for $t \in (0, \frac{\alpha}{K}]$, where C_m is a constant depending only on m, n and $\max\{\alpha, 1\}$.

Therefore, we have

$$|\nabla^m \mathrm{Rm}| \leq \sqrt{\frac{G}{t^m}} \leq \frac{C_m K}{t^{m/2}} \text{ for } t \in \left(0, \frac{\alpha}{K}\right],$$

as required. □

6.4 Curvature Explodes at Finite-time Singularities

This section will be devoted to proving that, if the Ricci flow becomes singular in finite time, the curvature must explode as we approach the singular time T.

Theorem 6.7. *If g_0 is a smooth metric on a compact manifold $\mathcal{M}$, the Ricci flow with $g(0) = g_0$ has a unique solution $g(t)$ on a maximal time interval $t \in [0, T)$ where $T \leq \infty$. If $T < \infty$ then*

$$\lim_{t \to T} \left(\sup_{x \in \mathcal{M}} |\mathrm{Rm}(x, t)| \right) = \infty.$$

The proof that the curvature must explode as $t \to T$ follows the same outline as the proof of the corresponding result for the curve-shortening flow, Theorem 4.5. That is, we assume that $|\mathrm{Rm}|_g$ is bounded above by a constant K, and show that the metric $g(t)$ converges smoothly to a smooth metric $g(T)$. It is then possible to use the short-time existence result of Theorem 5.4, with initial metric $g(T)$, to extend the solution past T. This contradicts the choice of T as the maximal time such that the Ricci flow exists on $[0, T)$.

A key element of the proof is the following theorem and its corollary, which show that a limit metric $g(T)$ exists and is continuous:

Theorem 6.8. *Let $\mathcal{M}$ be a closed manifold and $g(t)$ a smooth time-dependent metric on $\mathcal{M}$, defined for $t \in [0, T)$. If there exists a constant $C < \infty$ such that*

$$\int_0^T \left| \frac{\partial}{\partial t} g(x,t) \right|_{g(t)} dt \leq C \tag{6.7}$$

for all $x \in \mathcal{M}$, then the metrics $g(t)$ converge uniformly as $t \to T$ to a continuous metric $g(T)$ such that

$$e^{-C} g(x,0) \leq g(x,T) \leq e^{C} g(x,0).$$

Note that this means $g(T)$ is uniformly equivalent to $g(0)$.

Proof. Let $x \in \mathcal{M}, t_0 \in [0, T), V \in T_x\mathcal{M}$ be arbitrary. Then

$$\begin{aligned}
\left|\log\left(\frac{g_{(x,t_0)}(V,V)}{g_{(x,0)}(V,V)}\right)\right| &= \left|\int_0^{t_0} \frac{\partial}{\partial t}\left[\log g_{(x,t)}(V,V)\right] dt\right| \\
&= \left|\int_0^{t_0} \frac{\frac{\partial}{\partial t} g_{(x,t)}(V,V)}{g_{(x,t)}(V,V)} dt\right| \\
&\leq \int_0^{t_0} \left|\frac{\partial}{\partial t} g_{(x,t)}\left(\frac{V}{|V|_{g(t)}}, \frac{V}{|V|_{g(t)}}\right)\right| dt \\
&\leq \int_0^{t_0} \left|\frac{\partial}{\partial t} g(x,t)\right|_{g(t)} dt \\
&\leq C
\end{aligned}$$

where the penultimate step follows as $|A(U,U)| \leq |A|_g$ for any 2-tensor A and unit vector U.

Exponentiating both sides of this inequality gives us:

$$e^{-C} g_{(x,0)}(V,V) \leq g_{(x,t_0)}(V,V) \leq e^{C} g_{(x,0)}(V,V),$$

for any V. Thus

$$e^{-C} g(x,0) \leq g(x,t_0) \leq e^{C} g(x,0), \tag{6.8}$$

so the metrics $g(t)$ are uniformly equivalent.

Hence, we have

$$\int_0^T \left| \frac{\partial}{\partial t} g(x,t) \right|_{g(0)} dt \leq C' \tag{6.9}$$

for some $C' > 0$. Note the difference from equation (6.7): the norm is taken with respect to a constant metric $g(0)$ rather than the time-dependent one $g(t)$.

We now define

$$g(x,T) = g(x,0) + \int_0^T \frac{\partial}{\partial t} g(x,t) dt.$$

This integral exists as the metrics are smooth and the integrand is absolutely integrable with respect to the norm induced by $g(0)$, by equation (6.9). Now

$$|g(x,t) - g(x,T)|_{g(0)} \leq \int_t^T \left| \frac{\partial}{\partial t} g(x,t) \right|_{g(0)} \to 0$$

as $t \to T$ for each $x \in \mathcal{M}$. Because $\mathcal{M}$ is compact, this convergence is uniform on $\mathcal{M}$. Hence $g(t) \to g(T)$ uniformly, so $g(T)$ is continuous. By taking the limit of equation (6.8) we can show that

$$e^{-C} g(x,0) \leq g(x,T) \leq e^{C} g(x,0),$$

hence that $g(T)$ is positive definite.

Therefore, the metrics $g(t)$ converge uniformly to a continuous Riemannian metric $g(T)$ which is uniformly equivalent to $g(0)$. □

Corollary 6.9. *Let $(\mathcal{M}, g(t))$ be a solution of the Ricci flow on a closed manifold. If $|\mathrm{Rm}|_g$ is bounded on $[0,T)$ (where $T < \infty$) then $g(t)$ converges uniformly as $t \to T$ to a continuous metric $g(T)$ which is uniformly equivalent to $g(0)$.*

Proof. Any bound on $|\mathrm{Rm}|_g$ implies one on $|\mathrm{Rc}|_g$, and hence on $|\frac{\partial}{\partial t} g|_g$ (by the equation of the Ricci flow). The integral (6.7) is then an integral of a bounded quantity over a finite interval, and is hence bounded. Thus Theorem 6.8 applies. □

We have now got a foothold: we have shown that there is a limit metric $g(T)$, and it is continuous. We now want to show that this metric is smooth, because we need this if we are to use the short-time existence result of Theorem 5.4 to extend our solution past T. To do this, we need to make sure the spatial derivatives of g near the limit time T are not exploding – we need bounds on them. The first step is to bound the derivatives of the curvature, in analogy with the outlined proof of Theorem 4.5. We do this via the Bernstein-Bando-Shi derivative estimates (Theorem 6.6), which give bounds on the derivatives of the curvature under the assumption of bounded curvature.

We recall that the Bernstein-Bando-Shi estimates are completely useless at $t = 0$ (as one might expect, for bounds on an arbitrary curvature tensor will not tell us anything about its derivatives – it is only after a period of Ricci flowing that the derivatives start to be brought under control). That is fine for us because we are interested in derivative estimates near $t = T$. We can use Theorem 6.6 to get such estimates, by considering our Ricci flow as starting at some time shortly before T. This gives us

Corollary 6.10. *(of Theorem 6.6) Let $(\mathcal{M}^n, g(t))$ be a solution of the Ricci flow on a compact n-manifold. If there exist $\beta, K > 0$ such that*

$$|\mathrm{Rm}(x,t)|_{g(t)} \leq K \text{ for all } t \in [0,T],$$

where $T > \beta/K$, then for each $m \in \mathbb{N}$ there exists a constant B_m depending only on m, n and $\min\{\beta, 1\}$ such that

$$|\nabla^m \mathrm{Rm}(x,t)|_{g(t)} \leq B_m K^{1+\frac{m}{2}} \text{ for all } t \in \left[\tfrac{\min\{\beta,1\}}{K}, T\right].$$

Proof. We use the result of Theorem 6.6. Let $\beta_1 = \min\{\beta, 1\}$. Now, given a time $t_0 \in [\beta_1/K, T]$, we consider the Ricci flow as **starting** at the time $T_0 = t_0 - \beta_1/K$. Applying Theorem 6.6 to this Ricci flow, with $\alpha = \beta_1$, tells us that

$$|\nabla^m \mathrm{Rm}| \leq \frac{C_m K}{(t - T_0)^{m/2}}$$

where C_m depends only on m, n and $\min\{\alpha, 1\}$. Hence at $t = t_0$ we have

$$|\nabla^m \mathrm{Rm}| \leq \frac{C_m K}{(\frac{\beta_1}{K})^{m/2}} = \frac{C_m}{\beta_1^m} K^{1+m/2},$$

from which the result follows. □

Using this bound on the derivatives of curvature near $t = T$, we can now proceed to bound the derivatives of the metric g near $t = T$. The following corollary and its proof are a reformulation of the argument found in [6, Prop. 6.48].

Corollary 6.11. *Let $(\mathcal{M}^n, g(t))$ be a solution of the Ricci flow on a closed n-manifold, and let (x^i), $i = 1, \ldots, n$ be a local coordinate system defined on some coordinate chart $U \subset \mathcal{M}^n$. If there exists $K > 0$ such that*

$$|\mathrm{Rm}(x,t)|_{g(t)} \le K \text{ for all } t \in [0,T)$$

then for each $m \in \mathbb{N}$ there exist constants C_m, C'_m depending only on the chosen coordinate chart such that

$$|\partial^m g(x,t)| \le C_m$$

and

$$|\partial^m \mathrm{Rc}(x,t)| \le C'_m$$

for all $(x,t) \in U \times [0,T)$, where the norms are taken with respect to the Euclidean metric in the coordinate system (x^i).

We should explain some of the notation used. By $\partial^m g$ we mean the $\binom{m+2}{0}$-tensor field, defined **only** in the coordinate chart U, which has coordinates

$$\partial_{i_1} \ldots \partial_{i_m} g_{pq}$$

with respect to the coordinate system (x^i). The Euclidean metric, which is also defined only in U, is the metric which has coordinates δ_{ij} with respect to the coordinate system (x^i).

Proof. Throughout this proof, we will treat the Christoffel symbols Γ^k_{ij} as the coordinates, with respect to the coordinate system (x^i), of a tensor Γ (Γ is defined only in U).

Note that by Corollary 6.10, for each $m \in \mathbb{N}$ there is a uniform upper bound on $|\nabla^m \mathrm{Rc}|$ on a time interval $(\beta/K, T)$. There is also an upper bound on the same quantity on the interval $[0, \beta/K]$ because the interval is compact, so the quantity is bounded for all m:

$$|\nabla^m \mathrm{Rc}| \le D_m \tag{6.10}$$

for all $x \in \mathcal{M}^n$ and $t \in [0,T)$, where D_m is some constant depending on m and the particular Ricci flow $(\mathcal{M}^n, g(t))$.

We will now prove by strong induction that there exist constants P_m, Q_m, R_m for each $m \in \mathbb{N} \cup \{0\}$ such that

1. $|\partial^{m-1}\Gamma| \le P_m$ (we only prove this for $m \ge 1$);
2. $|\partial^m \mathrm{Rc}| \le Q_m$;
3. $|\partial^m g| \le R_m$

for all $t \in [0,T)$.

For the base case $m = 0$, (2) follows from the bound $|\mathrm{Rm}| \le K$ and (3) follows from Corollary 6.9.

Assume (1) – (3) are satisfied for all $m \le p-1$. We will prove they are true for $m = p$, starting with (1). Note that there is some constant C such that $|\partial^{p-1}\Gamma| \le C$ at $t = 0$, because the manifold $\mathcal{M}^n$ is compact.

Bounds on $|\partial^m g|$ imply bounds on $|\partial^m(g^{-1})|$ by differentiating the formula $g^{ij}g_{jk} = \delta^i_k$, m times. Recalling the formula for $\partial_t \Gamma$ from Lemma 6.1, we can prove:

$$\begin{aligned} \partial_t \partial^{p-1}\Gamma &= \partial^{p-1}(\partial_t \Gamma) \\ &= \partial^{p-1}(g^{-1} * \nabla \mathrm{Rc}) \\ &= \sum_{i=0}^{p-1} \partial^{p-i-1}(g^{-1}) * \partial^i \nabla \mathrm{Rc}. \end{aligned}$$

We have bounds on the derivatives of g^{-1}, so we just need to bound the other terms. For $i \le p-1$ we have, by Lemma 1.3 (recalling the notation from the end of Section 1.1),

$$\partial^i \nabla \mathrm{Rc} = \nabla^{i+1}\mathrm{Rc} + \mathop{\Large *}_{\substack{0 \le j \le i-1 \\ k \le i}} \left(\partial^j \Gamma, \partial^k \mathrm{Rc}\right).$$

Note that we have a bound on $\nabla^{i+1}\mathrm{Rc}$ by equation (6.10), and an inductive upper bound on all of the other terms on the right hand side, because $j \le i-1 \le p-2$ and $k \le i \le p-1$. Hence $|\partial^i \nabla \mathrm{Rc}| \le C$ for $i \le p-1$, therefore

$$|\partial_t \partial^{p-1}\Gamma| \le C$$

for some constant C (this C is different from the previous one – there are so many constant uniform bounds at play in this and subsequent arguments that we will refer to most of them as 'C' for convenience, and hope that this does not cause too much confusion). Thus $|\partial^{p-1}\Gamma|$ is bounded at $t = 0$ and experiences, at worst, linear growth on the finite time interval $[0, T)$. Therefore $|\partial^{p-1}\Gamma| \le P_p$ for some constant P_p. This completes the proof of (1).

We can now compute (again by Lemma 1.3)

$$\partial^p \mathrm{Rc} = \nabla^p \mathrm{Rc} + \mathop{\ast}_{\substack{j \le p-1 \\ k \le p-1}} \left(\partial^j \Gamma, \partial^k \mathrm{Rc}\right).$$

All terms on the right hand side are bounded inductively (with the exception of $\partial^{p-1}\Gamma$, which we just showed was bounded as part of the inductive step), so we have $|\partial^p \mathrm{Rc}| \le Q_p$ for some constant Q_p. This completes the proof of (2).

Finally,

$$|\partial_t \partial^p g| = |-2\partial^p \mathrm{Rc}| \le C$$

by the second part of the inductive step, so because we are on a finite time interval, $|\partial^p g| \le R_p$ for some constant R_p. This proves (3), completing the inductive step.

Therefore the result is true by induction. □

Corollary 6.12. *The metric $g(T)$ of Corollary 6.9 is smooth, and the metrics $g(t)$ converge uniformly in every C^k norm to $g(T)$ as $t \to T$.*

Proof. To show that $g(T)$ is smooth we must take derivatives with respect to some system of coordinates which we can only choose arbitrarily, so take some coordinate patch U of $\mathcal{M}^n$. We have, from the Ricci flow equation,

$$g_{ij}(x, T) = g_{ij}(x, t) - 2\int_t^T R_{ij}(x, \tau)d\tau$$

for any $t \in [0, T)$.

Now if α is any multi-index (see Definition A.1), then Corollary 6.11 tells us that $\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}$ and $\frac{\partial^{|\alpha|}}{\partial x^\alpha} R_{ij}$ are uniformly bounded on $U \times [0, T)$, and hence that we may take the derivative under the integral sign:

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}(x, T) = \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}\right)(x, t) - 2\int_t^T \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} R_{ij}\right)(x, \tau)d\tau \tag{6.11}$$

for any $x \in U$. In particular, the LHS of the above equation exists for all α, so $g(T)$ is smooth.

Now we show that the convergence is uniform in every C^m norm, in the following sense: we can choose coordinate charts covering $\mathcal{M}$, such that for any multi-index α, and any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}(x, T) - \frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}(x, t)\right|_{g(x,T)} < \epsilon \tag{6.12}$$

in any of the chosen coordinate charts, for any $t \in [T - \delta, T)$ and $x \in \mathcal{M}^n$.

Because $\mathcal{M}^n$ is compact, we can choose a finite set of coordinate charts such that the closed unit balls of the coordinate charts cover $\mathcal{M}^n$. Because the closed unit balls are compact, the Euclidean metric on each is equivalent to $g(T)$. Because there are finitely many coordinate charts, the Euclidean metrics are uniformly equivalent to $g(T)$. Thus it suffices to prove that equation (6.12) holds if we take the norm with respect to one of the Euclidean metrics at each point x, rather than with respect to $g(x, T)$.

By Corollary 6.11, for each of the coordinate charts we have chosen there exists C'_m such that $|\partial^m \mathrm{Rc}| \le C'_m$ with respect to the Euclidean norm in that chart. If we choose C to be the largest of these C'_m where $m = |\alpha|$ (C finite because there are only finitely many coordinate charts), then equation (6.11) gives us

$$\begin{aligned}\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}(x,T) - \frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}(x,t)\right| &= \left|2\int_t^T \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} R_{ij}\right)(x,\tau)d\tau\right| \\ &\leq 2\int_t^T |\partial^m \mathrm{Rc}(x,\tau)|\, d\tau \\ &\leq 2C(T-t).\end{aligned}$$

It follows that $g(t) \to g(T)$ uniformly in any C^m norm as $t \to T$. □

We now prove Theorem 6.7. We assume, for a contradiction, that $|\mathrm{Rm}(x,t)|_g$ is bounded above by K. It follows from Corollaries 6.9 and 6.12 that the metrics $g(t)$ converge uniformly in any C^k norm to a smooth metric $g(T)$.

Because $g(T)$ is smooth, it is possible to find a solution to the Ricci flow with initial metric $g(T)$, by the result of Theorem 5.4. Thus our solution to the Ricci flow can be extended past $t = T$, as we did for the curve-shortening flow in the proof of Theorem 4.5. This extension is smooth, because all spatial derivatives are continuous at $t = T$ (by the convergence of $g(t)$ in any C^k norm). It follows that all space-time derivatives are continuous at $t = T$ because the Ricci flow equation allows us to write time derivatives of quantities related to the metric in terms of space derivatives of those quantities, and the space-derivatives have been shown to be continuous. Therefore, the solution can be extended past the time $t = T$, so the time T could not have been maximal. This is a contradiction, so the original assumption that $|\mathrm{Rm}(x,t)|_g$ is bounded must be incorrect. This completes the proof of Theorem 6.7. □

Corollary 6.13. *Suppose that $(\mathcal{M}, g(t))$ is a solution of the Ricci flow on a closed manifold, such that $|\mathrm{Rm}| \leq K$ at $t = 0$. Then there exists a positive constant b depending only on K and the dimension n of the manifold such that the Ricci flow exists for all $t \in [0, b)$.*

Furthermore, there exists a constant C depending only on n such that

$$|\mathrm{Rm}| \leq \frac{K}{1 - \frac{1}{2}CKt}$$

for as long as the RHS *exists.*

Proof. We will first prove the second statement, then deduce the first.

Let $u = |\mathrm{Rm}|^2$. By Corollary 6.5 with $k = 0$, we have

$$\frac{\partial u}{\partial t} \leq \triangle u + Cu^{3/2}$$

for some $C > 0$ depending only on n. By hypothesis, $|\mathrm{Rm}|^2 \leq K^2$ at $t = 0$. The situation is perfect for applying the maximum principle (Theorem 3.2). We need only solve the associated ODE:

$$\frac{d\phi}{dt} = C\phi^{3/2}\,, \;\; \phi(0) = K^2.$$

The solution is

$$\phi(t) = \left(\frac{K}{1 - \frac{1}{2}CKt}\right)^2.$$

It follows by Theorem 3.2 that

$$u \leq \left(\frac{K}{1 - \frac{1}{2}CKt}\right)^2,$$

from which it follows that

$$|\mathrm{Rm}| \leq \frac{K}{1 - \frac{1}{2}CKt}$$

as required.

Now let $b = 1/(CK)$. We then have

$$|\mathrm{Rm}(x,t)| \leq 2K$$

for all $t \in [0, b)$ and $x \in \mathcal{M}$, and b depends only on K and the dimension of the manifold. Because

$$\lim_{t \to T} \sup_{x \in \mathcal{M}} |\mathrm{Rm}(x,t)| = \infty,$$

where T is the maximal time of existence of the Ricci flow, we must have $b < T$, hence the Ricci flow exists for all $t \in [0, b)$. $\square$

Chapter 7

3-Manifolds With Positive Ricci Curvature

Hamilton's first major achievement using the Ricci flow was the following, proved in [9].

Theorem 7.1. *Let $\mathcal{M}^3$ be a closed 3-manifold which admits a smooth Riemannian metric with strictly positive Ricci curvature. Then $\mathcal{M}^3$ also admits a smooth metric of constant positive curvature.*

In particular, if $\mathcal{M}^3$ is simply-connected, it follows from Theorem 1.16 that $\mathcal{M}^3$ is diffeomorphic to $\mathbb{S}^3$.

The proof in [9] was substantially improved by the introduction of the maximum principle that we have called Theorem 3.3, in [10]. This improved proof is also presented in Chapter 6 of [6], and it is this proof that we present in this chapter (we base our exposition on Chow and Knopf's with the exception of the proof of Theorem 7.19, the outline of which was suggested to us by Gerhard Huisken). We will present a more "modern" proof (as presented in [28, 2, 21]) in Section 8.3, based on Perelman's recent work.

Because the proofs of some of the results we will use require lengthy calculations which distract from the main course of the argument, we will relegate these details to Section 7.8.

7.1 The Plan of Attack

To prove Theorem 7.1 we will consider the Ricci flow on the closed 3-manifold $\mathcal{M}^3$, starting from a metric with strictly positive Ricci curvature. In this situation, the metric will become rounder and rounder under the Ricci flow, but also smaller and smaller. The manifold shrinks to a point in finite time, and its shape (locally) approaches that of a 3-sphere as we get closer to this time. We want to take the limit as this finite time is approached, and show that the limit has constant positive sectional curvature, but we are thwarted by the manifold shrinking to a point as the limit time is approached. To overcome this problem, we rescale the manifold (and also time) so that the volume of the manifold is constant, in analogy with the rescaling of the curve-shortening flow in Theorem 4.6 to keep the enclosed area constant.

This rescaled metric will not shrink to a point. It will exist for all time, and in fact we can show that it will converge smoothly to a smooth limit metric. While it is converging, its sectional curvatures are getting exponentially closer and closer together, so that the limit metric has constant positive sectional curvature. Thus the limit metric is exactly the one we need to prove Theorem 7.1.

Here is a brief outline of the proof of Hamilton's theorem that we will present:

1. Prove the Ricci flow has a solution for a short time, and this solution is unique. Thus we can consider a solution on a maximal time interval $[0, T)$. Note the analogy with Theorem 4.2 for the CSF. In addition, $T < \infty$, in analogy with Theorem 4.4 for the CSF. It follows by Theorem 6.7 that the curvature explodes as $t \to T$, in analogy with Theorem 4.5 for the CSF.

2. Show that the sectional curvatures must get close to each other as the curvature explodes.
3. Rescale time and the metric to get a solution to the normalized (volume-preserving) Ricci flow,

$$\frac{\partial}{\partial t}g = -2\mathrm{Rc} + \frac{2}{n}\frac{\int_{\mathcal{M}^n} R dV}{\int_{\mathcal{M}^n} dV}g$$

(in this case we have a 3-manifold, so $n = 3$). Show that this solution exists for all time, and converges as $t \to \infty$ to a metric of constant curvature.

7.2 Existence and Finite-time Explosion of Curvature

By Theorem 5.4, the Ricci flow has a unique solution on a maximal time interval $[0, T)$. We now show that in the case under consideration, namely when the initial Ricci curvature is strictly positive, T is finite.

Theorem 7.2. *Let $(\mathcal{M}, g(t))$ be a solution of the Ricci flow on a compact manifold, defined for $t \in [0, T)$. If the metric $g(0)$ has strictly positive scalar curvature (in particular, if it has strictly positive Ricci curvature) then $g(t)$ becomes singular in finite time, i.e. $T < \infty$.*

Proof. Because $\mathcal{M}$ is compact and the scalar curvature at time 0 is strictly positive, it is bounded below by some $\rho > 0$. Using the result for the evolution of the scalar curvature in Lemma 6.1 we have

$$\frac{\partial}{\partial t}R = \triangle R + 2|\mathrm{Rc}|^2 \geq \triangle R + \frac{2}{n}R^2$$

(here we have used Lemma 1.1). We can now apply the scalar maximum principle (Theorem 3.2). The solution of the ODE

$$\frac{d\phi}{dt} = \frac{2}{n}\phi^2$$

with $\phi(0) = \rho$ is

$$\phi(t) = \frac{1}{\frac{1}{\rho} - \frac{2t}{n}}.$$

Thus by Theorem 3.2,

$$R(x, t) \geq \phi(t).$$

But $\phi(t)$ clearly diverges to $+\infty$ in finite time, hence $R(x, t)$ becomes singular and so the solution $g(t)$ becomes singular in finite time. □

Corollary 7.3. *The curvature explodes as $t \to T$:*

$$\lim_{t \to T}\left(\sup_{x \in \mathcal{M}^3} |\mathrm{Rm}(x, t)|\right) = \infty \tag{7.1}$$

$$\lim_{t \to T}\left(\sup_{x \in \mathcal{M}^3} |\mathrm{Rc}(x, t)|\right) = \infty \tag{7.2}$$

Proof. By Theorem 7.2, the maximal time T is finite; by Theorem 6.7 the curvature must explode as we approach this singularity. This proves (7.1). It now follows from Lemma 1.13 (which applies because the manifold is 3-dimensional) that there exists a constant C such that $|\mathrm{Rm}| \leq C|\mathrm{Rc}|$, from which (7.2) follows. □

7.3 Setting the Scene for the Maximum Principle – The Uhlenbeck Trick

Having seen that the curvature explodes as the maximal time T is approached, we now show that the metric $g(t)$ becomes “rounder” when the curvature becomes large. To do this, we will apply the maximum principle for vector bundles (Theorem 3.3) to the Riemann curvature tensor.

Recall, from Lemma 6.1, that under the Ricci flow the Riemann curvature tensor evolves according to

$$\frac{\partial}{\partial t}R_{ijkl} = \triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \tag{7.3}$$

$$-(R_i^p R_{pjkl} + R_j^p R_{ipkl} + R_k^p R_{ijpl} + R_l^p R_{ijkp}), \tag{7.4}$$

where

$$B_{ijkl} \equiv -R_{pij}^q R_{qlk}^p. \tag{7.5}$$

Naïvely we might try to apply Theorem 3.3 to R_{ijkl} as a section of the vector bundle of 4-tensors. However, the reaction terms are so hideous that we will not be able to say anything useful about the ODE that we need to solve to apply Theorem 3.3. Furthermore, Theorem 3.3 can't deal with bundle metrics that depend on time. The way forward is what is known as the **Uhlenbeck trick**.

The idea is to fix the initial tangent bundle with its initial metric, then to evolve the isometry between this fixed bundle and the tangent bundle with its time-evolving metric. Suppose that we have a solution $g(t)$ of the Ricci flow on $\mathcal{M}$. Let (V, h) be a vector bundle over $\mathcal{M}$ with metric h such that

$$u_0 : (V, h) \to (T\mathcal{M}, g_0)$$

is a bundle isometry. We evolve the isometry $u(t)$ by

$$\frac{\partial}{\partial t}u_a^i = R_l^i u_a^l \tag{7.6}$$

where the u_a^i are the components of the isometry with respect to some local bases of V and $T\mathcal{M}$. We will use indices $a, b, \ldots$ on the vector bundle V and $i, j, \ldots$ on $T\mathcal{M}$ to distinguish them more clearly.

Lemma 7.4. *$u(t)$ remains an isometry.*

Proof. $u(t) : (V, h) \to (T\mathcal{M}, g(t))$ is an isometry as long as h is the pullback of $g(t)$ via $u(t)$, i.e. $h = u(t)^* g(t)$. Because h is constant and $u(0)$ is an isometry by definition, it suffices to show that $u(t)^* g(t)$ does not change in time.

$$\begin{aligned}\frac{\partial}{\partial t}(u(t)^* g(t))_{ab} &= \frac{\partial}{\partial t}(u_a^i u_b^j g_{ij}) \\ &= R_l^i u_a^l u_b^j g_{ij} + u_a^i R_l^j u_b^l g_{ij} + u_a^i u_b^j (-2R_{ij}) \\ &= 0\end{aligned}$$

as required, so $u(t)$ remains an isometry. □

Therefore we can consider the behaviour of the pullback of Rm to V, $u^*\mathrm{Rm}$, rather than Rm itself.

The equation describing the evolution of $(u^*\mathrm{Rm})_{abcd} \equiv R_{abcd}$ is then the same as that for R_{ijkl} but without the irritating terms (7.4) (the fact that the Uhlenbeck trick gives the evolution equation in such a nice form, as well as providing the framework that allows us to use the tensor maximum principle, is a little mysterious – Hamilton refers to it as "magic"):

$$\begin{aligned}
\frac{\partial}{\partial t} R_{abcd} &= \frac{\partial}{\partial t}(u_a^i u_b^j u_c^k u_d^l R_{ijkl}) \\
&= (R_m^i u_a^m) u_b^j u_c^k u_d^l R_{ijkl} + u_a^i (R_m^j u_b^m) u_c^k u_d^l R_{ijkl} \\
&\quad + u_a^i u_b^j (R_m^k u_c^m) u_d^l R_{ijkl} + u_a^i u_b^j u_c^k (R_m^l u_d^m) R_{ijkl} \\
&\quad + u_a^i u_b^j u_c^k u_d^l [\triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\
&\quad - (R_i^p R_{pjkl} + R_j^p R_{ipkl} + R_k^p R_{ijpl} + R_l^p R_{ijkp})] \\
&= \triangle R_{abcd} + 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}) \qquad (7.7)
\end{aligned}$$

where B_{abcd} is defined by equation (7.5).

Recall that we can view Rm as the curvature operator $\mathcal{R}$, which is a section of the bundle

$$\mathcal{E} = \wedge^2 T^* \mathcal{M}^n \otimes_S \wedge^2 T^* \mathcal{M}^n$$

of symmetric bilinear forms on the space $\wedge^2 T\mathcal{M}^3$ of 2-vectors. This viewpoint was outlined in Section 1.4.

In 3 dimensions, $\wedge^2 T\mathcal{M}^3$ has dimension 3, so each fibre of this bundle $\mathcal{E}$ is naturally isomorphic to the vector space of 3×3 self-adjoint (i.e. symmetric) matrices. We can diagonalize these with respect to some orthonormal basis $\{e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1\}$ of $\wedge^2 T\mathcal{M}^3$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of $T\mathcal{M}^3$.

Now we would like to apply the vector bundle maximum principle (Theorem 3.3) to $\mathcal{R}$, viewed as a section of $\mathcal{E}$. We must consider the ODE corresponding to the PDE describing the evolution of Rm (which is given by equation (7.7)), namely

$$\frac{d}{dt} Q_{abcd} = 2(B_{abcd}(Q) - B_{abdc}(Q) + B_{acbd}(Q) - B_{adbc}(Q))$$

where $B_{abcd}(Q)$ is defined by analogy with equation (7.5).

In 3 dimensions this ODE has a particularly convenient form. Choose a basis $\{e_i\}$ of $T\mathcal{M}_x^3$ so that Q is diagonal with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ down the diagonal, then the equation is (see [6, Sec. 6.3, 6.4]):

$$\frac{d}{dt}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 + \lambda_2\lambda_3 & 0 & 0 \\ 0 & \lambda_2^2 + \lambda_3\lambda_1 & 0 \\ 0 & 0 & \lambda_3^2 + \lambda_1\lambda_2 \end{pmatrix}.$$

In particular, the matrix Q will remain diagonal – this does not happen for higher dimensions, because the matrix on the RHS will not be diagonal (there are other useful decompositions in the $n = 4$ case though – see [10]). Therefore, in three dimensions the three parameters $\lambda_1, \lambda_2, \lambda_3$ completely describe Q, and we may represent Q as a point $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ moving in $\mathbb{R}^3$ according to the ODE

$$\frac{d}{dt}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 + \lambda_2\lambda_3 \\ \lambda_2^2 + \lambda_3\lambda_1 \\ \lambda_3^2 + \lambda_1\lambda_2 \end{pmatrix} = \nabla\left(\frac{(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)}{3} + \lambda_1\lambda_2\lambda_3\right). \qquad (7.8)$$

We note that the initial value of λ also tells us all about the initial Ricci and scalar curvatures, by the formulae (1.9) and (1.10):

$$\mathrm{Rc} = \frac{1}{2}\begin{pmatrix} \lambda_2 + \lambda_3 & 0 & 0 \\ 0 & \lambda_3 + \lambda_1 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 \end{pmatrix} \qquad (7.9)$$

and

$$R = \lambda_1 + \lambda_2 + \lambda_3. \qquad (7.10)$$

We can visualize the behaviour of the ODE (7.8) by visualizing the evolution of $\lambda(t)$. The ODE is homogeneous, so we can scale any solution to get another solution. We consider "projecting" the solution $\lambda(t)$ onto the unit sphere to get a path $\mu(t) = \lambda(t)/|\lambda(t)|$. Given $\mu(0)$, the evolution of $\mu(t)$ of course depends on which solution $\lambda(t)$ we choose, satisfying $\lambda(0) = |\lambda(0)|\mu(0)$. However,

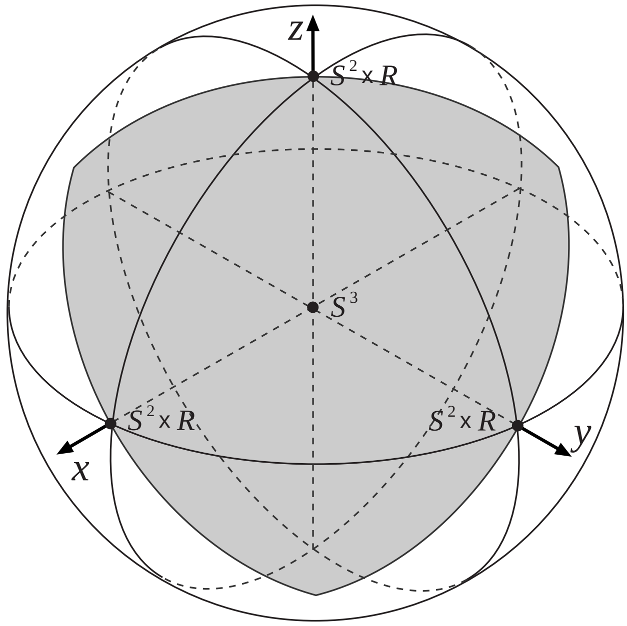

Figure 7.1: The axes and the region Rc > 0 (shaded) on the unit sphere in $\mathbb{R}^3$. The points corresponding to $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{R}$ are labelled. Note that $\mathbb{S}^3$ corresponds to the point $(1, 1, 1)/\sqrt{3}$, not the origin.

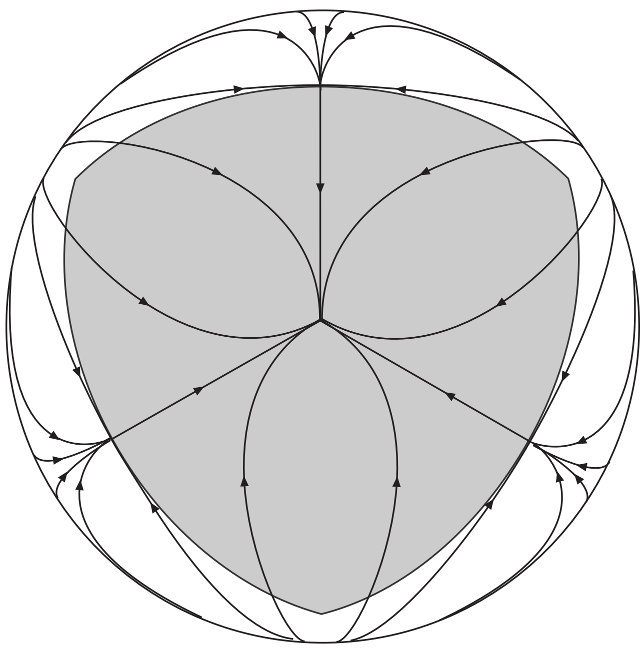

Figure 7.2: The paths $\mu(t)$ followed by the solutions to equation (7.8) on the 2-sphere.

the different paths $\mu(t)$ will only differ by a rescaling of time, by the homogeneity of the ODE. Hence we can unambiguously consider the paths that the solution follows on the unit 2-sphere, if we do not take into account the speed (this procedure corresponds to our freedom to scale the manifold without changing its shape).

Figure 7.2 shows the paths $\mu(t)$ followed on the sphere by the solutions to the ODE (7.8). Figure 7.1 shows the axes, and the region corresponding to $\mathrm{Rc} > 0$ is shaded grey (the diagram becomes incoherent when these two are superimposed so we've split them up for clarity).

The solution is stationary at the points $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1), (1, 1, 1)/\sqrt{3}$ and $(-1, -1, -1)/\sqrt{3}$. We recall that the eigenvalues of the curvature operator $\mathcal{R}$ correspond to sectional curvatures. Thus the points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ correspond to a manifold with positive curvature in one 2-plane and zero curvature in orthogonal 2-planes – we can identify these points with the manifold $\mathbb{S}^2 \times \mathbb{R}$, or quotients thereof if $\mathcal{M}$ is not simply connected. Similarly, we identify the points $(-1, 0, 0), (0, -1, 0), (0, 0, -1)$ with the manifold $\mathbb{H}^2 \times \mathbb{R}$ or quotients thereof. The point $(1, 1, 1)/\sqrt{3}$ corresponds to a manifold of constant positive sectional curvature, which we identify with $\mathbb{S}^3$ or quotients thereof, and $(-1, -1, -1)/\sqrt{3}$ corresponds to a manifold with constant negative sectional curvature, which we identify with $\mathbb{H}^3$ or quotients thereof.

We can see from Figure 7.2 that the paths are flowing towards $\mathbb{S}^3$, but that some of the ones flowing from outside the set of metrics with positive Ricci curvature get "caught" on the naughty $\mathbb{S}^2 \times \mathbb{R}$s. The set of metrics of strictly positive Ricci curvature is the largest convex set (recall that we need convexity for Theorem 3.3 to apply) that is symmetric in $\lambda_1, \lambda_2, \lambda_3$ and does not contain the $\mathbb{S}^2 \times \mathbb{R}$ points. Thus we can see that the Ricci flow is expected to converge towards $\mathbb{S}^3$ whenever it starts off with strictly positive Ricci curvature, but no stronger condition can be imposed.

It is interesting to note that, if we look at the sphere from the opposite direction, the arrows are all reversed, the $\mathbb{S}^3$ is replaced by $\mathbb{H}^3$ and the $\mathbb{S}^2 \times \mathbb{R}$s are replaced by $\mathbb{H}^2 \times \mathbb{R}$s. Of course the region shown in Figure 7.2 is all that is important to the proof of Theorem 7.1.

If one chooses $\lambda_1(0) \geq \lambda_2(0) \geq \lambda_3(0)$ then it is easy to show that this condition is preserved under the evolution equation (7.8):

$$\frac{d}{dt} \log(\lambda_1 - \lambda_2) = (\lambda_1 + \lambda_2 - \lambda_3)$$

and

$$\frac{d}{dt} \log(\lambda_2 - \lambda_3) = (\lambda_2 + \lambda_3 - \lambda_1).$$

7.4 Local Curvature Pinching from the Maximum Principle

We would now like to obtain quantitative results showing that this convergence towards constant sectional curvature does actually happen. In this section we will assume that $(\mathcal{M}^3, g(t))$ is a solution of the Ricci flow on a closed Riemannian 3-manifold with initially strictly positive Ricci curvature.

We first use Theorem 3.3 to prove that, if the eigenvalues of $\mathcal{R}$ are initially close together, they will remain so.

Lemma 7.5. *There exist constants $C < \infty$ and $\epsilon > 0$ depending only on the initial metric such that*

$$\frac{\lambda_1}{\lambda_3 + \lambda_2} \leq C$$

(where $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are the eigenvalues of the curvature operator) and

$$\mathrm{Rc} \geq \epsilon g.$$

We have diverged slightly from Chow and Knopf's formulation of this lemma (namely by including the second condition) to avoid a technicality that arises in the proof.

Proof. See Section 7.8. □

Corollary 7.6. *The scalar curvature explodes as $t \to T$. That is, if we define $R_{\max}(t) := \sup_{x \in \mathcal{M}} R(x,t)$, then*

$$\lim_{t \to T} R_{\max}(t) = \infty.$$

Proof. By the result of Lemma 7.5, Rc remains strictly positive under the Ricci flow. Therefore, if we diagonalize Rc with respect to some orthonormal basis, with eigenvalues $a, b, c > 0$, then

$$|\mathrm{Rc}|^2 = a^2 + b^2 + c^2 < (a + b + c)^2 = R^2,$$

from which the result follows by Corollary 7.3. □

Corollary 7.7. *There is a constant $\beta > 0$ such that*

$$\mathrm{Rc} \geq 2\beta^2 R g.$$

Proof. We use Lemma 7.5 and formulae (1.9) and (1.10) to deduce that

$$\mathrm{Rc} \geq \frac{\lambda_2 + \lambda_3}{2} g \geq 2\beta^2(\lambda_1 + \lambda_2 + \lambda_3) g = 2\beta^2 R g,$$

for some $\beta > 0$. □

We now show that the pinching together of the eigenvalues actually gets better as the scalar curvature goes to $+\infty$.

Theorem 7.8. *There exist positive constants $\delta < 1$ and $\bar{C}$ (depending only on g_0) such that*

$$\frac{\lambda_1 - \lambda_3}{R} \leq \frac{\bar{C}}{R^{\delta}},$$

where $R = \lambda_1 + \lambda_2 + \lambda_3$ is the scalar curvature.

Proof. See Section 7.8. □

Theorem 7.8 shows that, at each point, the sectional curvatures get "pinched" together as the curvature explodes ($\lambda_1 - \lambda_3$ is the greatest difference between any two eigenvalues). Furthermore the left hand side of this estimate is scale-invariant – so even when we rescale the metric by some factor, this bound tells us that the eigenvalues (i.e. the sectional curvatures) will be close together.

Corollary 7.9. *There exist positive constants $B, \bar{\delta}$ such that*

$$\frac{|E|^2}{R^2} \leq B R^{-\bar{\delta}}$$

where E is the Einstein tensor, $E_{ij} = R_{ij} - \frac{1}{3} R g_{ij}$.

Proof. See Section 7.8. □

This quantity $|E|^2$ measures how far away the metric is from being an Einstein metric. By Lemma 1.14, when $|E|^2 = 0$ we have $R_{ij} = C g_{ij}$ where C is constant over the whole manifold. In the case of 3 dimensions, it follows that the metric has constant sectional curvature (see Lemma 1.14). Thus bounds on $|E|^2$ are very useful.

7.5 Global Curvature Pinching

The previous section dealt with the pinching of sectional curvatures at a point. We know they pinch together if the scalar curvature explodes at that point, but we only know that the curvature explodes somewhere on our manifold as we approach the singular time – that is not enough to conclude that the sectional curvatures pinch together everywhere.

However we recall that, by Corollary 7.9,

$$\frac{|E|^2}{R^2} \leq B R^{-\bar{\delta}}$$

for some $B, \bar{\delta} > 0$. We recall the result of Lemma 1.14, which says that if $E = 0$ then R is constant. Thus we would expect that, when we have a bound on $|E|^2$ (as we do, by Corollary 7.9) everywhere on the manifold, the scalar curvature R might be close to being constant. So it is reasonable to expect that we will be able to obtain a bound on $|\nabla R|$ from our pinching result. This bound would be very useful to have because it will allow us to compare values of R at different places in $\mathcal{M}$. Because we already know that R is exploding somewhere on $\mathcal{M}$ we will be able to show that it is getting large everywhere, and hence that the sectional curvatures are getting close everywhere. One way of bounding $|\nabla R|$ is:

Theorem 7.10. *Let $(\mathcal{M}^3, g(t))$ be a solution of the Ricci flow on a closed 3-manifold with initially strictly positive Ricci curvature. There exist $\bar{\beta}, \gamma > 0$ depending only on the initial metric such that for any $\beta \in [0, \bar{\beta}]$ there exists C such that*

$$\frac{|\nabla R|^2}{R^3} \leq \beta R^{-\gamma} + CR^{-2}.$$

Proof. See Section 7.8. □

Using this estimate we can get global bounds on the variation of the curvatures:

Theorem 7.11. *Let $(\mathcal{M}^3, g(t))$ be a solution of the Ricci flow on a closed 3-manifold with initially strictly positive Ricci curvature, defined for $t \in [0, T)$. Then there exist constants $C, \gamma > 0$ depending only on initial data such that*

$$\frac{R_{\min}}{R_{\max}} \geq 1 - CR_{\max}^{-\gamma}.$$

Note that this means $R_{\min}/R_{\max} \to 1$ as $t \to T$, because $R_{\max} \to \infty$ as $t \to T$ by Corollary 7.6. It follows that $R \to \infty$ uniformly as $t \to T$.

Proof. See Section 7.8. □

Now recall the discussion at the start of this section. Because $R \to \infty$ as $t \to T$, the curvature should be getting uniformly pinched.

Corollary 7.12. *Let $\lambda_1(x,t) \geq \lambda_2(x,t) \geq \lambda_3(x,t)$ denote the eigenvalues of the curvature operator at (x,t), then for any $\epsilon \in (0,1)$ there exists $T_\epsilon \in [0,T)$ such that*

$$\min_{x \in \mathcal{M}^3} \lambda_3(x,t) \geq (1-\epsilon) \max_{y \in \mathcal{M}^3} \lambda_1(y,t) > 0$$

for all $t \in [T_\epsilon, T)$. Note that this means the metric will eventually attain positive sectional curvature everywhere.

Proof. See Section 7.8. □

7.6 Normalized Ricci Flow

We now know that the Ricci flow becomes singular in finite time T, that the curvature explodes as we approach time T, and that the sectional curvatures get pinched together as the curvature explodes. We want a metric of constant sectional curvature on our manifold, so the idea is to take the limit of our flow as we approach the time T. The problem is that the manifold is shrinking down to a point at time T, just as was the case for the CSF. In analogy with the CSF, where we rescaled the flow so that the area enclosed by the curve was constant, we consider a rescaling of our manifold so that its volume is constant.

Let $\tilde{g}(t) = \psi(t) g(t)$ be a rescaling of our Ricci flow metric $g(t)$ on $\mathcal{M}^n$, with $\psi(0) = 1$. We use a tilde to distinguish quantities that refer to the metric $\tilde{g}$, for example $\tilde{R} := R(\tilde{g})$ denotes the scalar curvature of the metric $\tilde{g}$. Let us choose $\psi(t)$ so that the volume (in fact the volume element) of the manifold with respect to $\tilde{g}$ is constant. Using the result of Lemma 1.19 for the scaling of the volume elements we have

$$\begin{aligned} \mathrm{Vol}(\tilde{g}(t)) &= \mathrm{Vol}(\tilde{g}(0)) \\ \Rightarrow \quad \psi(t)^{n/2}\mathrm{Vol}(g(t)) &= \mathrm{Vol}(g(0)) \\ \Rightarrow \quad \psi(t) &= \left(\frac{\int_{\mathcal{M}^n} d\mu(t)}{\int_{\mathcal{M}^n} d\mu(0)}\right)^{-\frac{2}{n}} \end{aligned}$$

We then have, from Lemma 6.1,

$$\begin{aligned} \frac{d}{dt}\psi(t) &= -\frac{2}{n}\left(\frac{\int_{\mathcal{M}^n} d\mu(t)}{\int_{\mathcal{M}^n} d\mu(0)}\right)^{-\frac{2}{n}-1} \frac{\frac{d}{dt}\int_{\mathcal{M}^n} d\mu(t)}{\int_{\mathcal{M}^n} d\mu(0)} \\ &= \frac{2}{n}\frac{\psi(t)}{\left(\frac{\int_{\mathcal{M}^n} d\mu(t)}{\int_{\mathcal{M}^n} d\mu(0)}\right)}\frac{\int_{\mathcal{M}^n} R d\mu(t)}{\int_{\mathcal{M}^n} d\mu(0)} \\ &= \frac{2r}{n}\psi(t) \\ &= \frac{2\tilde{r}}{n}\left(\psi(t)\right)^2 \end{aligned} \tag{7.11}$$

where the average scalar curvature, r, is defined by

$$r := \frac{\int_{\mathcal{M}^n} R d\mu}{\int_{\mathcal{M}^n} d\mu},$$

and the normalized average scalar curvature, $\tilde{r}$, is defined by

$$\tilde{r} := \frac{\int_{\mathcal{M}^n} \tilde{R} d\mu}{\int_{\mathcal{M}^n} d\mu} = \frac{r}{\psi(t)}$$

by the result of Lemma 1.19.

From this we can compute the evolution equation for $\tilde{g}$:

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{g} &= \left(\frac{\partial}{\partial t} g\right)\psi(t) + g\frac{d}{dt}\psi(t) \\ &= -2\psi(t)\mathrm{Rc}(g(t)) + \frac{2\tilde{r}}{n}\left(\psi(t)\right)^2 g \\ &= \psi(t)\left(-2\mathrm{Rc}(\tilde{g}(t)) + \frac{2\tilde{r}}{n}\tilde{g}\right) \end{aligned}$$

We define a rescaling of time to get rid of the $\psi(t)$ terms in this evolution equation:

$$\tau = \int_0^t \psi(u)du.$$

The evolution equation is then

$$\frac{\partial}{\partial \tau}\tilde{g} = -2\tilde{\mathrm{Rc}} + \frac{2\tilde{r}}{n}\tilde{g}. \tag{7.12}$$

This is the equation of the **normalized Ricci flow**. It differs from the ordinary Ricci flow by a rescaling of time and the metric, so that the metric has constant volume and the problem of the manifold shrinking to a point as we approach the singular time is eliminated. Because we have only rescaled the Ricci flow solution, the results that we have proven so far for the unnormalized Ricci flow translate without too much pain to results for the normalized Ricci flow using the results of Lemma 1.19. The issue of the curvature exploding is also dealt with by the following lemma:

Lemma 7.13. $\tilde{R}_{\max}$ *is bounded, for the normalized Ricci flow on a closed 3-manifold with initially strictly positive Ricci curvature.*

Proof. We use Corollary 7.7 to deduce that

$$\tilde{\mathrm{Rc}} = \mathrm{Rc} \geq 2\beta^2 R g \geq 2\beta^2 R_{\min} g = 2\beta^2 \tilde{R}_{\min} \tilde{g}$$

for some $\beta > 0$. It follows from Myers' Theorem (Theorem 1.17) that $\tilde{L}$, the diameter of $\mathcal{M}$ with respect to the metric $\tilde{g}$, satisfies

$$\tilde{L} \leq \frac{\pi}{\beta\sqrt{\tilde{R}_{\min}}}.$$

Now we can also get a lower bound for $\tilde{L}$ from the fact that the volume of our manifold is constant with respect to $\tilde{g}$, by the first part of the Bishop-Günther volume comparison theorem (Theorem 1.15). Because $\tilde{\mathrm{Rc}} = \mathrm{Rc} \geq 0$, we have (in the notation of Theorem 1.15):

$$\tilde{V} = \mathrm{Vol}(B(p, \tilde{L})) \leq V_3^0(\tilde{L}) = \frac{4\pi}{3}\tilde{L}^3$$

where $\tilde{V}$ is the volume of $\mathcal{M}$ with respect to $\tilde{g}$, which we know is equal to a constant, which we call $\tilde{V}_0$. Combining these gives us

$$\left(\frac{3\tilde{V}_0}{4\pi}\right)^{\frac{1}{3}} \leq \tilde{L} \leq \frac{\pi}{\beta\sqrt{\tilde{R}_{\min}}}.$$

This gives us an upper bound for $\tilde{R}_{\min}$. Finally, it follows from the first part of Theorem 7.11 and scaling invariance that there exists some $C > 0$ such that

$$\frac{\tilde{R}_{\min}}{\tilde{R}_{\max}} = \frac{R_{\min}}{R_{\max}} \geq \frac{1}{C}.$$

Hence the upper bound for $\tilde{R}_{\min}$ implies one for $\tilde{R}_{\max}$, and the proof is complete. □

We can now show that our rescaling of time has in fact taken us to an infinite time interval.

Theorem 7.14. *If $(\mathcal{M}^3, g(t))$ is a solution of the Ricci flow on a closed 3-manifold with initially strictly positive Ricci curvature then the corresponding normalized Ricci flow exists for all time.*

Proof. First we show that

$$\int_0^T R_{\max}(t)dt = \infty,$$

where $R_{\max}(t)$ is the maximum scalar curvature of the metric $g(t)$ and $[0, T)$ is the maximal time interval on which the unnormalized Ricci flow exists. This can be done by setting

$$f(t) = \exp\left[2\int_0^t R_{\max}(u)du\right] R_{\max}(0),$$

which then satisfies

$$\frac{df}{dt} = 2R_{\max}(t)f$$

and hence, using the evolution equation for R given in Lemma 6.1,

$$\begin{aligned} \frac{\partial}{\partial t}(R - f) &= \triangle R + 2|\mathrm{Rc}|^2 - 2R_{\max} f \\ &\leq \triangle(R - f) + 2R_{\max}(R - f) \end{aligned}$$

(using $|\mathrm{Rc}|^2 \leq R^2$, which follows as Rc is positive). Now we have $(R - f) \leq 0$ initially, as $f(0) = R_{\max}(0)$. This condition is preserved by the ODE obtained by ignoring the Laplacian term in the above equation, as

$$\frac{d}{dt}\phi(t) = 2R_{\max}(t)\phi(t)$$

is equivalent to

$$\frac{d}{dt}\log(|\phi(t)|) = 2R_{\max}(t)$$

from which we can see that ϕ never changes sign. Hence the condition $(R - f) \leq 0$ is preserved, by the scalar maximum principle. By Corollary 7.6, $R_{\max} \to \infty$ as $t \to T$, so $f \to \infty$ as $t \to T$. Recalling the definition of f, this means that

$$\lim_{t\to T}\int_0^t R_{\max}(u)du = \infty.$$

Now the corresponding integral for the normalized flow will be the same:

$$\int_0^{\tilde{T}} \tilde{R}_{\max}(\tau)d\tau = \int_0^T r(t)dt = \infty$$

(if we define $[0, \tilde{T})$ to be the maximal time interval on which the normalized Ricci flow exists). This follows easily from the scaling results of Lemma 1.19 as $\tilde{R}_{\max}d\tau = (\psi^{-1}R_{\max})(\psi dt) = rdt$. So the integral for the normalized flow must also diverge – but the integrand is bounded by Lemma 7.13. Hence the region of integration must be infinite, so $\tilde{T} = \infty$, i.e. the normalized Ricci flow exists for all time. □

7.7 Convergence of the Normalized Flow

Now we will prove that the normalized Ricci flow converges as $\tau \to \infty$ to a smooth metric $\tilde{g}_\infty$ of constant positive sectional curvature. We will use the notations

$$\tilde{g}_\infty = \tilde{g}(\infty) := \lim_{\tau\to\infty}\tilde{g}(\tau).$$

The proof will be very similar to that of Theorem 6.7, and the key element will again be Theorem 6.8. To show that $\tilde{g}(\infty)$ exists and is continuous, we must show that there exists some $C < \infty$ such that

$$\int_0^\infty \left|\frac{\partial}{\partial\tau}\tilde{g}\right|_{\tilde{g}} d\tau < C.$$

Using the normalized Ricci flow equation (7.12), that is equivalent to showing that the integral

$$\int_0^\infty \left|\tilde{\mathrm{Rc}} - \frac{\tilde{r}}{3}\tilde{g}\right| d\tau \tag{7.13}$$

is bounded. We know from Corollary 7.9 that

$$\frac{\left|\mathrm{Rc} - \frac{1}{3}Rg\right|^2}{R^2} \leq CR^{-\delta}.$$

This is a bound on exactly the right sort of quantity for the unnormalized Ricci flow (Theorem 7.11 tells us that as $t \to \infty$, r and R approach one another uniformly). We need a bound that holds for the normalized Ricci flow though, and one that will force the integral (7.13) to be bounded too. The easiest way to do this is to prove the integrand is bounded by a decaying exponential (a constant bound will not suffice as it did for Corollary 6.12, as the region of integration is infinite). We will need a few preliminary results.

Lemma 7.15. *If $(\mathcal{M}^3, \tilde{g}(\tau))$ is a solution of the normalized Ricci flow on a closed 3-manifold with initially strictly positive Ricci curvature then there exists $\epsilon > 0$ such that $\tilde{R} \geq \epsilon$ for all $\tau > 0$.*

Proof. See Section 7.8. □

We are now ready to prove the key estimate that allows us to prove the integral (7.13) is bounded.

Theorem 7.16. *If $(\mathcal{M}^3, \tilde{g}(\tau))$ is a solution of the normalized Ricci flow on a closed 3-manifold with initially strictly positive Ricci curvature then there exist constants $C, \delta > 0$ such that*

$$\left|\tilde{E}\right| \leq Ce^{-\delta\tau}.$$

Proof. See Section 7.8. □

To apply Theorem 6.8 we must prove an exponential bound on $|\tilde{\mathrm{Rc}} - \tilde{r}/3\tilde{g}|$. The previous theorem gives us a bound on $|\tilde{E}| = |\tilde{\mathrm{Rc}} - \tilde{R}/3\tilde{g}|$ which is almost what we want. We just need to go from $\tilde{R}$ to $\tilde{r}$. So we need to show that the difference $|\tilde{R} - \tilde{r}|$ is exponentially bounded. We will actually prove something a bit stronger:

Lemma 7.17. *There exist constants $C, \delta > 0$ such that*

$$\tilde{R}_{\max} - \tilde{R}_{\min} < Ce^{-\delta\tau}.$$

This Lemma comes from [9, Lemma 17.4].

Proof. See 7.8. □

This allows us to prove the following:

Theorem 7.18. *If $(\mathcal{M}^3, \tilde{g}(\tau))$ is a solution of the normalized Ricci flow with initially strictly positive Ricci curvature, then $\tilde{g}(\tau)$ exists for all $\tau \in [0, \infty)$ and converges uniformly as $\tau \to \infty$ to a continuous metric $\tilde{g}(\infty)$.*

Proof. By Theorem 7.16 and Lemma 7.17 we have

$$\begin{aligned}
\int_0^\infty \left|\frac{\partial \tilde{g}}{\partial \tau}\right| d\tau &= \int_0^\infty \left|\tilde{\mathrm{Rc}} - \frac{\tilde{r}}{3}\tilde{g}\right| d\tau \\
&\leq \int_0^\infty \left|\tilde{\mathrm{Rc}} - \frac{\tilde{R}}{3}\tilde{g}\right| + \left|\frac{\tilde{R} - \tilde{r}}{3}\tilde{g}\right| d\tau \\
&< \int_0^\infty Ce^{-\delta\tau} d\tau < \infty
\end{aligned}$$

where we have amalgamated the two exponential bounds into one. It follows by Theorem 6.8 that $\tilde{g}(\tau)$ converges uniformly to a continuous metric $\tilde{g}(\infty)$ as $\tau \to \infty$. □

The next step is to show that this convergence is smooth. This is important because we want our limit metric to be smooth, and also because we want the curvatures of the normalized flow to converge to the corresponding curvatures of the limit metric (as curvature quantities are all essentially second-order derivatives of the metric). That will allow us to conclude that the curvature pinching results we have proven for the flow lead to similar results for the limit metric, and hence that the limit metric has constant curvature.

Theorem 7.19. *The limit metric $\tilde{g}_\infty := \tilde{g}(\infty)$ of Theorem 7.18 is smooth, and the convergence of $\tilde{g}(\tau)$ to $\tilde{g}_\infty$ as $\tau \to \infty$ is uniform in every C^m norm.*

Proof. See Section 7.8. □

We are now finally in a position to prove Theorem 7.1. We show that the limit metric $\tilde{g}_\infty$ has the properties required.

Theorem 7.20. *The limit metric $\tilde{g}_\infty$ is a smooth metric with constant positive sectional curvature.*

Proof. By Theorem 7.19, $\tilde{g}(\tau)$ converges to $\tilde{g}_\infty$ in the C^0, C^1 and C^2 norms. Because all curvature quantities are combinations of "0th-order", 1st-order and 2nd-order derivatives of the metric, this means that we can take the limit to show that the Einstein tensor of the metric $\tilde{g}_\infty$ vanishes:

$$\left|\tilde{E}_\infty\right| = \lim_{\tau\to\infty} \left|\tilde{E}(\tau)\right| \leq \lim_{\tau\to\infty} Ce^{-\delta\tau} = 0$$

by the result of Theorem 7.16. Therefore $\tilde{g}_\infty$ is an Einstein metric, so by Lemma 1.14 $\tilde{g}_\infty$ has constant (positive) sectional curvature. □

7.8 Details

Proof. (of Lemma 7.5) Note that such a C and ϵ will exist at time $t = 0$ by the compactness of the manifold $\mathcal{M}^3$: $\lambda_1/(\lambda_2 + \lambda_3)$ is a continuous, positive function at $t = 0$ so it has an upper bound, and the eigenvalues of Rc are $(\lambda_2+\lambda_3)/2, (\lambda_3+\lambda_1)/2, (\lambda_1+\lambda_2)/2$, all of which are strictly positive at time $t = 0$ so they have a strictly positive lower bound. Thus it suffices to show that these conditions are preserved under evolution by the Ricci flow.

We use the maximum principle in the form of Theorem 3.3, on the vector bundle $\mathcal{E}$, with the set

$$\mathcal{K} \equiv \{Q \in \mathcal{E} : \lambda_1(Q) - C(\lambda_3(Q) + \lambda_2(Q)) \leq 0 \text{ and } \lambda_2 + \lambda_3 \geq 2\epsilon\}.$$

Note that because $\lambda_1 \geq \lambda_2 \geq \lambda_3$, the last condition states exactly that $\mathrm{Rc} \geq \epsilon g$ by formula (1.9). This set is invariant under parallel translation and convex in each fibre. Convexity follows because the functions

$$\begin{aligned} f(Q) &= \lambda_1(Q) \\ &= \max_{|U|=1}(Q(U,U)) \end{aligned}$$

and

$$\begin{aligned} h(Q) &= -(\lambda_2(Q) + \lambda_3(Q)) \\ &= \max_{\substack{|V| = |W| = 1 \\ \langle V, W\rangle = 0}} (-Q(V,V) - Q(W,W)) \end{aligned}$$

are convex, so the function $j(Q) = f(Q) + Ch(Q)$ is convex, therefore the set

$$\mathcal{K} = h^{-1}((-\infty, -\epsilon]) \cap j^{-1}((-\infty, 0])$$

is convex.

We just need to show that the solution of the associated ODE stays in $\mathcal{K}$. That is, if

$$\frac{\lambda_1}{\lambda_2 + \lambda_3} \leq C$$

and

$$\lambda_2 + \lambda_3 \geq 2\epsilon$$

initially, then this condition remains true under the evolution (7.8). This can be shown by proving first that

$$\frac{d}{dt}(\lambda_2 + \lambda_3) = \lambda_2^2 + \lambda_3^2 + \lambda_1(\lambda_2 + \lambda_3) \geq 0,$$

from which it follows that the condition $\lambda_2 + \lambda_3 \geq 2\epsilon$ is preserved, then

$$\frac{d}{dt} \log\left(\frac{\lambda_1}{\lambda_3 + \lambda_2}\right) = \frac{\lambda_2^2(\lambda_3 - \lambda_1) + \lambda_3^2(\lambda_2 - \lambda_1)}{\lambda_1(\lambda_3 + \lambda_2)} \leq 0$$

by the evolution equation (7.8), because $\lambda_2 + \lambda_3 \geq \epsilon > 0$ and hence $\lambda_1 > 0$ also. This is the sole reason for including the condition $\mathrm{Rc} \geq \epsilon g$ – to show that the RHS of this equation does not become singular.

Thus all the hypotheses of Theorem 3.3 are satisfied, so the solution to the PDE stays in $\mathcal{K}$, i.e. the conditions

$$\frac{\lambda_1}{\lambda_2 + \lambda_3} \leq C$$

and

$$\mathrm{Rc} \geq \epsilon g$$

are preserved. □

Proof. (of Theorem 7.8) The proof is very similar to that of Lemma 7.5. It suffices to show that there exist $\bar{C}, \delta$ such that

$$\frac{\lambda_1 - \lambda_3}{\lambda_2 + \lambda_3} \leq \frac{\bar{C}}{(\lambda_2 + \lambda_3)^{\delta}}. \tag{7.14}$$

By compactness, given $\delta \in (0, 1)$ we can choose such a $\bar{C}$ at time $t = 0$, because $\lambda_2 + \lambda_3 > 0$ everywhere by the positive Ricci curvature condition. We show that this condition is preserved using the maximum principle (Theorem 3.3) with the same vector bundle $\mathcal{E}$ as we used in the proof of Lemma 7.5, but a different $\mathcal{K}$:

$$\mathcal{K} \equiv \{Q \in \mathcal{E} : [\lambda_1(Q) - \lambda_3(Q)] - \bar{C}[\lambda_2(Q) + \lambda_3(Q)]^{1-\delta} \leq 0\}$$

Once again, the set $\mathcal{K}$ is invariant under parallel translation and convex in each fiber by a similar argument to that in the proof of Lemma 7.5.

To show that the solution of the ODE stays inside $\mathcal{K}$ we compute, from equation 7.8:

$$\frac{d}{dt} \log \left(\frac{\lambda_1 - \lambda_3}{(\lambda_2 + \lambda_3)^{1-\delta}} \right) \leq \delta\lambda_1 - \frac{1}{2}(1 - \delta)(\lambda_3 + \lambda_2).$$

Now, by Lemma 7.5, it is possible to choose δ small enough that this is always non-positive, so

$$\frac{\lambda_1 - \lambda_3}{(\lambda_2 + \lambda_3)^{1-\delta}}$$

is non-increasing, hence the inequality (7.14) is preserved by the ODE. Thus by the maximum principle it is also preserved by the PDE. □

Proof. (of Corollary 7.9) Using the formulae (1.9) and (1.10) we have

$$\mathrm{E} = \frac{1}{6} \begin{pmatrix} \lambda_2 + \lambda_3 - 2\lambda_1 & 0 & 0 \\ 0 & \lambda_3 + \lambda_1 - 2\lambda_2 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 - 2\lambda_3 \end{pmatrix}, \tag{7.15}$$

hence

$$\begin{aligned} \frac{|E|^2}{R^2} &= \frac{(\lambda_2 + \lambda_3 - 2\lambda_1)^2 + (\lambda_3 + \lambda_1 - 2\lambda_2)^2 + (\lambda_1 + \lambda_2 - 2\lambda_3)^2}{36(\lambda_1 + \lambda_2 + \lambda_3)^2} \\ &= \frac{(\lambda_1 - \lambda_2)^2 + (\lambda_2 - \lambda_3)^2 + (\lambda_3 - \lambda_1)^2}{12(\lambda_1 + \lambda_2 + \lambda_3)^2} \\ &\leq \frac{3(\lambda_3 - \lambda_1)^2}{12(\lambda_1 + \lambda_2 + \lambda_3)^2} \\ &\leq BR^{-\bar{\delta}} \end{aligned}$$

by the result of Theorem 7.8, with $\bar{\delta} = 2\delta$ (the penultimate step follows as $\lambda_1 - \lambda_3$ is the greatest difference between two eigenvalues). □

Proof. (of Theorem 7.10) We provide an outline of the proof without showing all of the excruciating details. A more complete outline can be found in [6, Section (6.6)].

Using the evolution equation for R under the Ricci flow, given in Lemma 6.1, it is possible to deduce that

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{|\nabla R|^2}{R}\right) &= \triangle\left(\frac{|\nabla R|^2}{R}\right) - 2R\left|\nabla\left(\frac{\nabla R}{R}\right)\right|^2 \\ &\quad -2\frac{|\nabla R|^2}{R^2}|\mathrm{Rc}|^2 + \frac{4}{R}\langle \nabla R, \nabla |\mathrm{Rc}|^2\rangle \\ &\leq \triangle\left(\frac{|\nabla R|^2}{R}\right) - 2R\left|\nabla\left(\frac{\nabla R}{R}\right)\right|^2 \\ &\quad -2\frac{|\nabla R|^2}{R^2}|\mathrm{Rc}|^2 + 8\sqrt{3}|\nabla \mathrm{Rc}|^2 \end{aligned}$$

where the last line follows from the Cauchy-Schwarz inequality and Lemma 1.1 (to prove $|\nabla R|^2 \leq 3|\nabla \mathrm{Rc}|^2$):

$$\begin{aligned}\frac{4}{R}\langle \nabla R, \nabla |\mathrm{Rc}|^2\rangle &\leq \frac{4}{R}|\nabla R|\,\left|\nabla |\mathrm{Rc}|^2\right| \\ &\leq 8|\nabla R||\nabla \mathrm{Rc}|\frac{|\mathrm{Rc}|}{R} \\ &\leq 8\sqrt{3}|\nabla \mathrm{Rc}|^2\end{aligned}$$

(using $|\mathrm{Rc}| < R$ because $\mathrm{Rc} > 0$, as in Corollary 7.6). We would like to use the maximum principle (Theorem 3.2) to show that the quantity $|\nabla R|^2/R$ does not increase. All reaction terms on the right hand side of our evolution equation are negative with the exception of the last one in $|\nabla \mathrm{Rc}|^2$. We need to deal with this meddlesome customer. We note that

$$|\nabla E|^2 \geq \frac{1}{37}|\nabla \mathrm{Rc}|^2 \tag{7.16}$$

where E is the Einstein tensor (which we know ought to be falling away nicely as our manifold gets rounder). To see why this is true, we combine Lemma 1.1 with equation (1.14) to get

$$|\nabla_k E_{ij}|^2 \geq \frac{1}{3}|\nabla^j E_{ij}|^2 = \frac{1}{3}\cdot\frac{1}{36}|\nabla R|^2.$$

Equation (7.16) now follows easily:

$$\begin{aligned}|\nabla \mathrm{Rc}|^2 - \frac{1}{3}|\nabla R|^2 &= |\nabla E|^2 \\ &\geq \frac{1}{3}\frac{1}{36}|\nabla R|^2\end{aligned}$$

which can be rearranged to obtain

$$|\nabla \mathrm{Rc}|^2 - \frac{1}{3}|\nabla R|^2 \geq \frac{1}{37}|\nabla \mathrm{Rc}|^2$$

where $|\nabla E|^2$ is exactly equal to the LHS.

Therefore our evolution equation becomes

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla R|^2}{R}\right) &\leq \triangle\left(\frac{|\nabla R|^2}{R}\right) - 2R\left|\nabla\left(\frac{\nabla R}{R}\right)\right|^2 \\ &\quad -2\frac{|\nabla R|^2}{R^2}|\mathrm{Rc}|^2 + 8\sqrt{3}\cdot 37|\nabla E|^2.\end{aligned}$$

We are going to try to get rid of the bad final term of this equation by adding on some multiple of $|E|^2$. The evolution equation for $|E|^2$ under the Ricci flow is given in Lemma 6.1. It is

$$\frac{\partial |E|^2}{\partial t} = \triangle |E|^2 - 2|\nabla E|^2 + W \tag{7.17}$$

where

$$W = \frac{26}{3}R|\mathrm{Rc}|^2 - 8R_i^j R_j^k R_k^i - 2R^3. \tag{7.18}$$

The second term on the right hand side of equation (7.17) interests us: we can use it to cancel off the recalcitrant $|\nabla E|^2$ term in the evolution equation for $|\nabla R|^2/R$. We set

$$V := \frac{|\nabla R|^2}{R} + \frac{37}{2}(8\sqrt{3}+1)|E|^2$$

then calculate

$$\frac{\partial}{\partial t}V \leq \triangle V - |\nabla \mathrm{Rc}|^2 + C_1 W$$

(reusing the inequality (7.16)) where C_1 is a positive constant.

It is possible to prove that

$$W \leq \frac{50}{3} R|E|^2 \tag{7.19}$$

purely algebraically; as Rc is positive and symmetric we can diagonalize it with positive real eigenvalues a, b, c. Each term in the inequality can be expressed in terms of a, b and c, and we are left with an easy homogeneous inequality in three real variables. Equality occurs when $a = b = c$, i.e. for an Einstein metric (this proof is significantly simpler than that presented in [6]). We also have

$$|\nabla R|^2 \leq 3|\nabla \mathrm{Rc}|^2$$

by Lemma 1.1.

Hence we have

$$\begin{aligned} \frac{\partial}{\partial t} V &\leq \triangle V - |\nabla \mathrm{Rc}|^2 + C_2 R|E|^2 \\ &\leq \triangle V - \frac{1}{3}|\nabla R|^2 + C_3 R^{3-2\gamma} \end{aligned}$$

by the result of Corollary 7.9, for some constants C and $\gamma > 0$. Now we set

$$U := V - \beta R^{2-\gamma}$$

where β is to be chosen. It is not difficult to calculate from the result of Lemma 6.1 that

$$\frac{\partial}{\partial t} R^{2-\gamma} = \triangle(R^{2-\gamma}) - (2-\gamma)(1-\gamma)R^{-\gamma}|\nabla R|^2 + 2(2-\gamma)R^{1-\gamma}|\mathrm{Rc}|^2,$$

hence that

$$\begin{aligned} \frac{\partial}{\partial t} U &\leq \triangle U + \left[\beta(2-\gamma)(1-\gamma)R^{-\gamma} - \frac{1}{3}\right]|\nabla R|^2 \\ &\quad + C_3 R^{3-2\gamma} - 2\beta(2-\gamma)R^{1-\gamma}|\mathrm{Rc}|^2. \end{aligned} \tag{7.20}$$

Now by Theorem 7.8 we have $\mathrm{Rc} \geq \epsilon g$, hence $R \geq 3\epsilon$ for some $\epsilon > 0$. Thus we may choose $\bar{\beta} > 0$ such that

$$\bar{\beta} < \frac{(3\epsilon)^\gamma}{3(2-\gamma)(1-\gamma)},$$

so that for any $\beta \in [0, \bar{\beta}]$ the second term in the evolution equation (7.20) is non-positive. For the rest of the terms, we use the inequality $3|\mathrm{Rc}|^2 \geq |R|^2$ to obtain

$$C_3 R^{3-2\gamma} - 2\beta(2-\gamma)R^{1-\gamma}|\mathrm{Rc}|^2 \leq C_3 R^{3-2\gamma} - C_4 R^{3-\gamma}$$

where C_3, C_4 are constants. For large R this term is dominated by the $R^{3-\gamma}$ term, which is negative. As it does not diverge at $R = 0$, there is a uniform upper bound C_5 for this term, hence we have

$$\frac{\partial}{\partial t} U \leq \triangle U + C_5.$$

Thus the growth in U is at worst linear, by the maximum principle (Theorem 3.2). By Theorem 7.2 the time interval we are considering is finite, hence we have a uniform upper bound C for U. Therefore

$$\frac{|\nabla R|^2}{R} \leq V = U + \beta R^{2-\gamma} \leq C + \beta R^{2-\gamma}$$

from which the result follows. □

Proof. (of Theorem 7.11) Our main tool here will be Theorem 7.10, which allows us to compare the curvature at different points on the manifold. By Corollary 7.6, we also have $R_{\max} \to \infty$ as

$t \to T$. Thus for t sufficiently close to T, Theorem 7.10 tells us that there exist positive constants A, α such that

$$|\nabla R| \le A R_{\max}^{3/2-\alpha}.$$

Thus we have (for t sufficiently close to T)

$$|R(x) - R(y)| \le \int_\gamma |\nabla R| ds \le A R_{\max}^{3/2-\alpha} d(x,y) \tag{7.21}$$

(here γ is a minimizing geodesic connecting x and y). Let us choose $x(t)$ such that $R_{\max}(t) = R(x,t)$ (we can do this as $\mathcal{M}$ is compact), and define

$$L(t) := \frac{1}{\epsilon\sqrt{R_{\max}(t)}}$$

where $\epsilon > 0$. Then for all $y \in B(x(t), L(t))$ we have, by equation (7.21):

$$R(y) \ge R(x) - A R_{\max}^{3/2-\alpha} L \ge R_{\max}\left(1 - \frac{A}{\epsilon} R_{\max}^{-\alpha}\right). \tag{7.22}$$

Recall that $R_{\max} \to \infty$ as $t \to T$. Therefore, given $\delta > 0$, for t sufficiently close to T we have

$$R(y) \ge (1-\delta) R_{\max} \tag{7.23}$$

for all $y \in B(x(t), L(t))$.

This will prove the theorem if we can show that $B(x(t), L(t))$ is all of $\mathcal{M}$ for t sufficiently close to T. We will prove this using Myers' Theorem (Theorem 1.17). Define $\mathcal{N} = B(x(t), L(t))$. We have the estimate $\mathrm{Rc} \ge 2\beta^2 R g$ for some $\beta > 0$ from Corollary 7.7. Hence, by equation (7.23), for any $\delta > 0$

$$\mathrm{Rc} \ge 2\beta^2 (1-\delta) R_{\max} g$$

in $\mathcal{N}$ for t close enough to T. By the proof of Myers' Theorem (see [18, Theorem 11.8]), we have

$$\mathrm{diam}(\mathcal{N}) \le \frac{\pi}{\beta\sqrt{(1-\delta)R_{\max}}} < \frac{1}{\delta\sqrt{R_{\max}}} < \frac{1}{\epsilon\sqrt{R_{\max}}} = L$$

for δ chosen sufficiently small. Because $\mathcal{N}$ is defined to be $B(x(t), L(t))$ but has diameter $< L$, if there are any points in $\mathcal{M}$ that lie outside $\mathcal{N}$ then $\mathcal{M}$ must be disconnected. We know that it isn't, hence $\mathcal{N}$ must be all of $\mathcal{M}$, from which the result follows by equation (7.22). ☐

Proof. (of Corollary 7.12) We apply Theorem 7.8:

$$\lambda_3 \ge \lambda_1 - \bar{C}(\lambda_1 + \lambda_2 + \lambda_3)^{1-\delta} \ge \lambda_1(1 - 3\bar{C}R^{-\delta}) \tag{7.24}$$

at each point $x \in \mathcal{M}$. Now it follows from Theorem 7.11 that $R \to \infty$ uniformly as $t \to T$. Thus, if we are given $\eta > 0$, for times t close enough to T we have

$$\begin{aligned}
\lambda_3(x,t) &\ge (1-\eta)\lambda_1(x,t) \qquad (7.25)\\
&\quad \text{(by equation (7.24), as } R^{-\delta} \to 0)\\
&\ge \frac{1-\eta}{3} R(x,t)\\
&\quad \text{(as } 3\lambda_1 \ge \lambda_1 + \lambda_2 + \lambda_3 = R)\\
&\ge \frac{(1-\eta)^2}{3} R(y,t)\\
&\quad \text{(as } R_{\max}/R_{\min} \to 1)\\
&\ge \frac{(1-\eta)^2}{3}(\lambda_1(y,t) + 2(1-\eta)\lambda_1(y,t))\\
&\quad \text{(as } \lambda_2 + \lambda_3 \ge 2\lambda_3 \ge 2(1-\eta)\lambda_1\text{, reusing inequality (7.25))}\\
&\ge (1-\eta)^3 \lambda_1(y,t).
\end{aligned}$$

The result now follows by taking the supremum over all $x, y \in M$. ☐

Proof. *(of Lemma 7.15)* We note that the proof of this Lemma presented in [6] is incorrect. The proof we present is based on [9, Lemma 16.7].

We use the following lemma, which is used to prove the Sphere Theorem (Theorem 1.18):

Lemma 7.21. *If $\mathcal{M}$ is a simply connected manifold of dimension ≥ 3 whose sectional curvatures are pinched between K and $\frac{1}{4}K$ for some constant K ($\mathcal{M}$ is "$\frac{1}{4}$-pinched"), then the injectivity radius of $\mathcal{M}$ is at least $\pi/\sqrt{K}$.*

Proof. See [3], Theorem 5.10. □

Now by Corollary 7.12,

$$\frac{\lambda_3(x,t)}{\lambda_1(y,t)} \to 1$$

as $t \to T$, uniformly for all $x, y \in \mathcal{M}$. It follows by scaling invariance and the fact that $\lambda_1 \geq \lambda_2 \geq \lambda_3$ that

$$\frac{\tilde{\lambda}_i}{\tilde{\lambda}_j} \to 1$$

uniformly as $\tau \to \infty$ for all i, j.

Therefore our manifold is eventually as pinched as we care to make it – in particular, it will eventually be $\frac{1}{4}$-pinched, with the constant K equal to some multiple of $\tilde{R}_{\min}$ by the result of Theorem 7.11. We may now apply Lemma 7.21 to the universal cover $\mathcal{N}$ of $\mathcal{M}$. The volume of $\mathcal{N}$ is at least some multiple of the cube of the injectivity radius $\rho(\mathcal{N})$ by the second part of the Bishop-Günther Volume Comparison Theorem (Theorem 1.15), because we have a uniform upper bound on sectional curvatures by Lemma 7.13. Hence we have

$$\mathrm{Vol}(\mathcal{N}) \geq C'\rho(\mathcal{N})^3 \geq C'\left(\frac{\pi}{\sqrt{K}}\right)^3 \geq C\tilde{R}_{\max}^{-\frac{3}{2}}. \tag{7.26}$$

Note that, because the Ricci tensor of $\mathcal{M}$ is bounded below by Corollary 7.7, the fundamental group of $\mathcal{M}$ is finite by Myers' theorem (Theorem 1.17). Furthermore the volume of $\mathcal{M}$ is constant under the normalized Ricci flow. Hence

$$\mathrm{Vol}(\mathcal{N}) = |\pi_1(\mathcal{M})|\mathrm{Vol}(\mathcal{M}) = \text{constant}.$$

Combining this with equation (7.26) gives us a lower bound on $\tilde{R}_{\max}$. The result of Theorem 7.11 now gives a uniform lower bound on $\tilde{R}_{\min}$, as required. □

We will need, in the proofs of many of the remaining theorems, to use maximum principle arguments similar to those employed in Chapter 6 and early in this chapter. For this we must know the evolution equation satisfied by the quantity we are considering under the *normalized* Ricci flow – luckily there is a very simple way of going from the unnormalized evolution equation to the normalized one. We need to introduce the concept of the degree of a tensor. If P is some tensor involving the metric and the curvature for the unnormalized flow, then by the results of Lemma 1.19, the same quantity calculated for the normalized flow (which is just a dilation of the unnormalized flow by a factor of ψ, as in Section 7.6) will be related to P by a rule of the form $\tilde{P} = \psi^n P$. We call n the **degree** of P. For example, by Lemma 1.19, the scalar curvature R has degree -1.

Lemma 7.22. *If P satisfies*

$$\frac{\partial P}{\partial t} = \triangle P + Q$$

for the unnormalized flow on a 3-manifold, and P has degree n, then Q has degree $n-1$ and for the normalized equation,

$$\frac{\partial \tilde{P}}{\partial \tau} = \tilde{\triangle}\tilde{P} + \tilde{Q} + \frac{2\tilde{r}n}{3}\tilde{P}.$$

Proof. Note that Q has degree $n-1$ because $\partial\tau/\partial t = \psi$ so $\partial P/\partial t$ has degree $n-1$. We also have $\triangle = \psi\tilde{\triangle}$, so $\triangle P$ has degree $n-1$, thus Q must have degree $n-1$ too.

We now have $\tilde{P} = \psi^n P$ and the formula (7.11) for $d\psi/dt$, so

$$\begin{aligned}
\frac{\partial \tilde{P}}{\partial \tau} &= \psi^{-1}\frac{\partial(\psi^n P)}{\partial t} \\
&= \psi^{n-1}\frac{\partial P}{\partial t} + n\psi^{n-2}\frac{d\psi}{dt}P \\
&= \psi^{n-1}(\triangle P + Q) + n\psi^{n-2}\left(\frac{2\tilde{r}}{3}\psi^2\right)P \\
&= \tilde{\triangle}\tilde{P} + \tilde{Q} + \frac{2\tilde{r}n}{3}\tilde{P}
\end{aligned}$$

as required. □

Proof. (of Theorem 7.16) This proof is a simplification of that presented in [6].

Because $|\tilde{E}|^2 \leq (\tilde{\lambda}_1 - \tilde{\lambda}_3)^2/4$ as shown in the proof of Corollary 7.9, it suffices for us to prove a decaying exponential bound on $(\tilde{\lambda_1} - \tilde{\lambda_3})$. In fact we will prove that there exist constants $C, \delta > 0$ such that

$$\tilde{\lambda_1} - \tilde{\lambda_3} \leq Ce^{-\delta\tau}(\tilde{\lambda_2} + \tilde{\lambda_3}).$$

The result will then follow from Lemma 7.13, which gives a uniform upper bound on $(\tilde{\lambda_2} + \tilde{\lambda_3})$.

We again use the Uhlenbeck trick, this time for the normalized Ricci flow equation. We replace the isometry evolution equation (7.6) by

$$\frac{\partial}{\partial\tau}u^i_a = \tilde{R}^i_l u^l_a - \frac{\tilde{r}}{2}u^i_a.$$

The evolution equation for $u^*\tilde{\mathrm{Rm}}$ is then (using Lemmas 6.1 and 7.22)

$$\frac{\partial}{\partial t}\tilde{R}_{abcd} = \tilde{\triangle}\tilde{R}_{abcd} + 2(\tilde{B}_{abcd} - \tilde{B}_{abdc} + \tilde{B}_{acbd} - \tilde{B}_{adbc}) - \tilde{r}\tilde{R}_{abcd}.$$

This leads to the analogue of equation (7.8) for the behaviour of the ODE corresponding to the Ricci flow PDE:

$$\frac{d}{d\tau}\begin{pmatrix}\tilde{\lambda_1}\\ \tilde{\lambda_2}\\ \tilde{\lambda_3}\end{pmatrix} = \begin{pmatrix}\tilde{\lambda_1}^2 + \tilde{\lambda_2}\tilde{\lambda_3} - \tilde{r}\tilde{\lambda_1}\\ \tilde{\lambda_2}^2 + \tilde{\lambda_3}\tilde{\lambda_1} - \tilde{r}\tilde{\lambda_2}\\ \tilde{\lambda_3}^2 + \tilde{\lambda_1}\tilde{\lambda_2} - \tilde{r}\tilde{\lambda_3}\end{pmatrix}. \quad (7.27)$$

We now apply the vector bundle maximum principle (Theorem 3.3) exactly as we did in the proofs of Lemmas 7.5 and 7.8, with the time-dependent pinching set[1]

$$\mathcal{K}(\tau) \equiv \{Q \in \mathcal{E} : e^{\delta\tau}(\tilde{\lambda_1}(Q) - \tilde{\lambda_3}(Q)) - C(\tilde{\lambda_3}(Q) + \tilde{\lambda_2}(Q)) \leq 0\}$$

for some C, α, δ to be chosen. From equation (7.27) we obtain

$$\begin{aligned}
\frac{d}{dt}\log\left(e^{\delta t}\frac{\tilde{\lambda_1} - \tilde{\lambda_3}}{(\tilde{\lambda_2} + \tilde{\lambda_3})}\right) &= \delta - (\tilde{\lambda_2} - \tilde{\lambda_3}) - \frac{\tilde{\lambda_2}^2 + \tilde{\lambda_3}^2}{\tilde{\lambda_2} + \tilde{\lambda_3}} \\
&\leq \delta - \frac{1}{2}(\tilde{\lambda_2} + \tilde{\lambda_3})
\end{aligned}$$

where we have used the fact that $\tilde{\lambda_2} - \tilde{\lambda_3}$ remains non-negative under the normalized Ricci flow, which follows from equation (7.27) in the same way as it did from equation (7.8) for the unnormalized flow.

By Lemma 7.15, we can also choose $\epsilon > 0$ such that

$$2\epsilon \leq \tilde{\lambda_1} + \tilde{\lambda_2} + \tilde{\lambda_3} \leq (1 + B)(\tilde{\lambda_2} + \tilde{\lambda_3})$$

[1] Our Theorem 3.3 does not cater for time-dependent pinching sets, but the extension is simple – see [6, Theorem 4.9].

for some $B > 0$, where the last step follows by Lemma 7.5 as, by scale-invariance,

$$\frac{\tilde{\lambda_1}}{\tilde{\lambda_2} + \tilde{\lambda_3}} = \frac{\lambda_1}{\lambda_2 + \lambda_3} \leq B.$$

We can now choose $\delta > 0$ small enough that

$$\delta - \frac{\epsilon}{1+B} < 0.$$

Combining these choices yields

$$\begin{aligned} \frac{d}{dt} \log \left(e^{\delta\tau} \frac{\tilde{\lambda_1} - \tilde{\lambda_3}}{(\tilde{\lambda_2} + \tilde{\lambda_3})} \right) &\leq \delta - \frac{1}{2}(\tilde{\lambda_2} + \tilde{\lambda_3}) \\ &\leq \delta - \frac{\epsilon}{1+B} \\ &\leq 0. \end{aligned}$$

Hence the ODE remains inside $\mathcal{K}(\tau)$ for appropriately chosen $C, \delta > 0$. $\mathcal{K}(\tau)$ is closed, convex and invariant under parallel translation by analogous methods to those used in the proofs of Lemma 7.5 and Theorem 7.8, so the result follows from the analogue of Theorem 3.3 for time-dependent pinching sets ([6, Theorem 4.9]). □

Proof. (of Lemma 7.17) Because we have a uniform upper bound on the diameter of our manifold from Myers' Theorem (as noted in the proof of Lemma 7.13), it suffices to prove an exponential bound on $|\tilde{\nabla}\tilde{R}|$. This is achieved by a maximum principle argument similar to that of Theorem 7.10. Let

$$G = \frac{|\nabla R|^2}{R^2} + \alpha|E|^2,$$

for some $\alpha > 0$ chosen later. G then has degree -2, and using similar methods to Theorem 7.10 can be shown, for appropriately chosen $\alpha > 0$, to satisfy an evolution equation of the form

$$\frac{\partial}{\partial t} G \leq \triangle G + \beta R|E|^2$$

for some $\beta > 0$ (see [9, Lemma 17.4]). Thus by Lemma 7.22 we have

$$\begin{aligned} \frac{\partial}{\partial \tau}\tilde{G} &\leq \tilde{\triangle}\tilde{G} + \beta\tilde{R}|\tilde{E}|^2 - \frac{4\tilde{r}}{3}\tilde{G} \\ &\leq \tilde{\triangle}\tilde{G} + Ce^{-\delta\tau} - \delta\tilde{G} \end{aligned}$$

for some $C, \delta > 0$, by the result of Theorem 7.16, and also using the fact that $\tilde{R}$ is bounded below by Lemma 7.15 to bound $\tilde{r} > 3\delta/4$ for some $\delta > 0$. Therefore

$$\frac{\partial}{\partial \tau}\left(e^{\delta\tau}\tilde{G} - C\tau\right) \leq \tilde{\triangle}\left(e^{\delta\tau}\tilde{G} - C\tau\right),$$

so by the maximum principle $\left(e^{\delta\tau}\tilde{G} - C\tau\right) \leq C$ from which it follows that $\tilde{G}$ is exponentially decaying. Hence, as $\tilde{R}$ is bounded above by Lemma 7.13, $|\tilde{\nabla}\tilde{R}|^2$ is also exponentially decaying, and the result follows. □

Proof. (of Theorem 7.19) The proof we present is different from Hamilton's original proof in [9], which relied on integral estimates to bound derivatives of the metric. We instead use the maximum principle to do the same thing.

As we did in the proof of Corollary 6.12, we must show that in any coordinate patch U, we have

$$\int_0^\infty |\partial_\tau \partial^k g| d\tau < \infty$$

for any $k \in \mathbb{N}$. From the equation for the normalized Ricci flow, we see that it suffices to prove

$$|\partial^k F| \le Ce^{-\delta\tau}$$

where

$$F = \tilde{\mathrm{Rc}} - \frac{1}{3}\tilde{r}\tilde{g}.$$

Combining Theorem 7.16 with Lemma 7.17 (as we did in the proof of Theorem 7.18) we see that $|F|$ is exponentially bounded. The most important step in proving that its derivatives are also exponentially bounded will be to prove that its covariant derivatives are exponentially bounded. Because $\tilde{r}$ is constant (in space, not time) and $\tilde{\nabla}\tilde{g} = 0$, this is equivalent to proving that the covariant derivatives of $\tilde{\mathrm{Rc}}$ are exponentially bounded with respect to $\tilde{g}(\tau)$ (recall that the metrics $\tilde{g}(\tau)$ are uniformly equivalent to $\tilde{g}(0)$, so this is equivalent to proving that the derivatives are bounded with respect to a fixed metric):

Lemma 7.23. *There exist positive constants C_k, δ_k such that*

$$|\tilde{\nabla}^k \tilde{\mathrm{Rc}}| \le C_k e^{-\delta_k \tau}$$

for all $k \ge 1$.

Proof. The proof will follow very similar lines to the proof of Theorem 6.6, and will also use some of the ideas from the proof of Theorem 7.10. In particular, the proof will be by strong induction on k, the Einstein tensor E will again be very useful, and C, δ will denote different constants in just about every equation – they are just generic upper bounds.

We note first of all that $|\tilde{\mathrm{Rc}}| \le |\tilde{\mathrm{Rc}} - 1/3\tilde{R}\tilde{g}| + |\tilde{R}|/3 \le C$ by the results of Theorem 7.16 and Lemma 7.13.

Now let us begin the induction. We do not need a base case, because there is a bit of monkey business going on at $k = 0$ – which is expected because we know that $|\tilde{\mathrm{Rc}}|$ is not actually exponentially bounded. Assume that $|\tilde{\nabla}^j \tilde{\mathrm{Rc}}| \le Ce^{-\delta\tau}$ for all $1 \le j \le k-1$, for some positive constants C, δ. In the $k = 1$ case this is a null assumption, hence true.

By Lemma 1.13, Rm can be expressed as a linear combination of traces of Rc (because we are in dimension 3). It now follows by Lemma 6.4 that $\nabla^k \mathrm{Rc}$ satisfies a heat-type equation which is basically that given in Lemma 6.4 but with Rc substituted for Rm. It then follows by Lemma 6.2 that $|\nabla^k \mathrm{Rc}|^2$ satisfies a heat-type evolution equation of the form

$$\frac{\partial}{\partial t}|\nabla^k \mathrm{Rc}|^2 = \triangle|\nabla^k \mathrm{Rc}|^2 - 2|\nabla^{k+1}\mathrm{Rc}|^2 + \sum_{j=0}^{k} \nabla^j \mathrm{Rc} * \nabla^{k-j}\mathrm{Rc} * \nabla^k \mathrm{Rc}.$$

This is the evolution equation for the unnormalized flow; the equation for the normalized flow will be the same thing except for an extra term of the form $(2n\tilde{r}/3)|\tilde{\nabla}^k \tilde{\mathrm{Rc}}|^2$, by Lemma 7.22. This term will be absorbed into the $j = 0$ term of the summation in the evolution equation though, as it is bounded by a term of the form $\tilde{\mathrm{Rc}} * \tilde{\nabla}^k \tilde{\mathrm{Rc}} * \tilde{\nabla}^k \tilde{\mathrm{Rc}}$ (note that by the result of Theorem 7.11, $\tilde{r}$ is uniformly bounded above by CR for some constant $C > 0$).

Therefore we have the heat-type equation satisfied by $|\tilde{\nabla}^k \tilde{\mathrm{Rc}}|^2$ under the normalized Ricci flow, and we can simplify it using the inductive hypothesis:

$$\begin{aligned}
\partial_t |\tilde{\nabla}^k \tilde{\mathrm{Rc}}|^2 &\le \tilde{\triangle}|\tilde{\nabla}^k \tilde{\mathrm{Rc}}|^2 - 2|\tilde{\nabla}^{k+1}\tilde{\mathrm{Rc}}|^2 + \sum_{j=0}^{k} \tilde{\nabla}^j \tilde{\mathrm{Rc}} * \tilde{\nabla}^{k-j}\tilde{\mathrm{Rc}} * \tilde{\nabla}^k \tilde{\mathrm{Rc}} \\
&\le \tilde{\triangle}|\tilde{\nabla}^k \tilde{\mathrm{Rc}}|^2 - 2|\tilde{\nabla}^{k+1}\tilde{\mathrm{Rc}}|^2 + Ce^{-\delta\tau} + \tilde{\mathrm{Rc}} * (\tilde{\nabla}^k \tilde{\mathrm{Rc}})^{*2} \\
&\le \tilde{\triangle}|\tilde{\nabla}^k \tilde{\mathrm{Rc}}|^2 - 2|\tilde{\nabla}^{k+1}\tilde{\mathrm{Rc}}|^2 + Ce^{-\delta\tau} + B_k|\tilde{\nabla}^k \tilde{\mathrm{Rc}}|^2.
\end{aligned}$$

We have amalgamated the exponential bounds that came from the inductive hypothesis, and used the uniform upper bound on $|\tilde{\nabla}\tilde{\mathrm{Rc}}|$. This is not quite enough though – the reaction terms here would lead to exponential growth. We use the same trick as we did in the proof of Theorem 6.6: we add on multiples of the lower-order terms to get rid of the bad reaction terms. However we must be careful: if we add on a multiple of $|\tilde{\mathrm{Rc}}|^2$ we will get something that is not exponentially bounded. We must instead add on a multiple of $|\tilde{E}|^2$, which **is** exponentially bounded.

Recall the formula (7.17), and combine it with the inequality (7.19) to get

$$\partial_t |E|^2 \quad \leq \quad \triangle |E|^2 - 2|\nabla E|^2 + \frac{50}{3} R|E|^2.$$

Now apply equation (7.16) and Lemma 7.22 to obtain

$$\begin{aligned} \partial_\tau |\tilde{E}|^2 \quad &\leq \quad \tilde{\triangle} |\tilde{E}|^2 - \frac{2}{37} |\tilde{\nabla} \tilde{\mathrm{Rc}}|^2 + \frac{50}{3} \tilde{R} |\tilde{E}|^2 - \frac{4\tilde{r}}{3} |\tilde{E}|^2 \\ &\leq \quad \tilde{\triangle} |\tilde{E}|^2 - \frac{2}{37} |\tilde{\nabla} \tilde{\mathrm{Rc}}|^2 + Ce^{-\delta\tau} \end{aligned}$$

(we used the bounds $\tilde{R} \leq C$ and $|\tilde{E}| \leq Ce^{-\delta\tau}$ to get the last term).

Now, let us define

$$V := |\tilde{\nabla}^k \tilde{\mathrm{Rc}}|^2 + \alpha_{k0} |\tilde{E}|^2 + \sum_{j=1}^{k-1} \alpha_{kj} |\tilde{\nabla}^j \tilde{\mathrm{Rc}}|^2,$$

where the α_{kj} are constants that we will choose carefully. V satisfies the evolution equation:

$$\begin{aligned} \partial_\tau V \quad \leq \quad & \tilde{\triangle} V + Ce^{-\delta\tau} + (B_k - 2\alpha_{k,k-1}) |\tilde{\nabla}^k \tilde{\mathrm{Rc}}|^2 + \sum_{j=1}^{k-1} (\alpha_{kj} B_j - 2\alpha_{k,j-1}) |\tilde{\nabla}^j \tilde{\mathrm{Rc}}|^2 \\ & + \left(\alpha_{k1} B_1 - \frac{2}{37} \alpha_{k0} \right) |\tilde{\nabla} \tilde{\mathrm{Rc}}|^2 \end{aligned}$$

where we have grouped all of the exponentially decaying terms together.

Let us choose the α_{kj} such that:

$$\begin{aligned} B_k - 2\alpha_{k,k-1} \quad &\leq \quad -1 \\ \alpha_{kj} B_j - 2\alpha_{k,j-1} \quad &\leq \quad -\alpha_{kj} \quad \text{for } 2 \leq j \leq k-1 \\ \alpha_{k1} B_1 - \frac{2}{37} \alpha_{k0} \quad &\leq \quad -\alpha_{k1}. \end{aligned}$$

The evolution equation for V then becomes

$$\partial_\tau V \leq \tilde{\triangle} V + Ce^{-\delta\tau} - V.$$

We have an upper bound on V at time $\tau = 0$ by the compactness of $\mathcal{M}$. The ODE associated with this PDE will lead to exponential decay:

$$\frac{d\phi}{d\tau} = Ce^{-\delta\tau} - \phi$$

has the solution

$$\phi(\tau) = Be^{-\tau} + \frac{C}{1-\delta} e^{-\delta\tau}.$$

Hence, by the maximum principle (Theorem 3.2), V is exponentially bounded. So

$$|\tilde{\nabla}^k \tilde{\mathrm{Rc}}|^2 \leq V \leq C_k e^{-\delta_k \tau}$$

as required. □

Now we have to go from bounds on $|\tilde{\nabla}^k \tilde{\mathrm{Rc}}|^2$ to bounds on $|\partial^k \tilde{\mathrm{Rc}}|^2$. As we did in Corollary 6.11 we treat the Christoffel symbols as a tensor in some coordinate chart U. To go from covariant derivatives to coordinate derivatives we will need bounds on the derivatives of the Christoffel symbols.

Lemma 7.24. *In a fixed coordinate chart U, for all $k \in \mathbb{N}$ we have a uniform upper bound*

$$|\partial^{k-1} \tilde{\Gamma}| \leq C_k.$$

Proof. We will prove the result by strong induction. Assume the result is true for all $1 \le k \le m-1$. As in the previous lemma, no base case is required.

We can now do the inductive step. We can calculate $\partial_\tau \tilde{\Gamma}$ from formula (1.17) in Lemma 1.20, with $h_{ij} = -2\tilde{R}_{ij} + (2\tilde{r}/3)\tilde{g}_{ij}$ for the normalized Ricci flow. The result is

$$\partial_\tau \tilde{\Gamma}^k_{ij} = -\tilde{g}^{kl}(\tilde{\nabla}_i \tilde{R}_{jl} + \tilde{\nabla}_j \tilde{R}_{il} - \tilde{\nabla}_l \tilde{R}_{ij}) \tag{7.28}$$

(recalling that $\tilde{\nabla}\tilde{g} = 0$ and $\tilde{\nabla}\tilde{r} = 0$). It now follows by Lemma 1.3 and equation (7.28) that

$$\begin{aligned}\partial^{m-1}\partial_\tau \tilde{\Gamma} &= \sum_{i=0}^{m-1} \left(\underset{j \le m-2}{\ast} \left(\partial^j \tilde{\Gamma}\right) \right) * \tilde{\nabla}^i (\partial_\tau \tilde{\Gamma}) \\ &= \sum_{i=1}^{m} \left(\underset{j \le m-2}{\ast} \left(\partial^j \tilde{\Gamma}\right) \right) * \tilde{\nabla}^i \tilde{\mathrm{Rc}}\end{aligned}$$

By the inductive hypothesis we have a uniform bound on each of the terms $\partial^j \tilde{\Gamma}$ and by Lemma 7.23 we have a decaying exponential bound on the terms $\tilde{\nabla}^i \tilde{\mathrm{Rc}}$. Thus we have

$$|\partial_\tau \partial^{m-1} \tilde{\Gamma}| \le Ce^{-\delta\tau}.$$

It follows that

$$|\partial^{m-1}\tilde{\Gamma}| \le |\partial^{m-1}\tilde{\Gamma}|_{t=0} + \int_0^\infty Ce^{-\delta\tau} d\tau \le C_m$$

for some positive constant C_m. This completes the inductive step and hence the proof. □

We now have, by Lemma 1.3,

$$|\partial^k F| = \left| \sum_{i=0}^{k} \left(\underset{j \le k-1}{\ast} (\partial^j \Gamma) \right) * \tilde{\nabla}^i F \right| \le Ce^{-\delta\tau}$$

for all $k \in \mathbb{N}$. We have used the bounds $|\partial^j \tilde{\Gamma}| \le C$ for $j \le m-1$ from Lemma 7.24 and the bounds $|\tilde{\nabla}^i F|^2 \le Ce^{-\delta\tau}$ from Lemma 7.23 and Theorem 7.16.

Hence, because we have

$$\begin{aligned}\tilde{g}_\infty(x) &= \tilde{g}(x,0) - 2\int_0^\infty \frac{\partial \tilde{g}}{\partial \tau} d\tau \\ &= \tilde{g}(x,0) - 2\int_0^\infty F(x,\tau) d\tau,\end{aligned}$$

we may conclude that for any multi-index α, we can take the derivative under the integral sign:

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}\tilde{g}_\infty(x) = \frac{\partial^{|\alpha|}}{\partial x^\alpha}\tilde{g}(x,0) - 2\int_0^\infty \frac{\partial^{|\alpha|}}{\partial x^\alpha} F(x,\tau) d\tau,$$

where the integral is finite by the result proved above:

$$\int_0^\infty \frac{\partial^{|\alpha|}}{\partial x^\alpha} F(x,\tau) d\tau \le \int_0^\infty |\partial^{|\alpha|} F(x,\tau)| d\tau \le \int_0^\infty Ce^{-\delta\tau} d\tau < \infty.$$

Therefore $\tilde{g}_\infty$ has derivatives of all orders (it is smooth) and it is clear that the convergence is uniform in any C^k norm, in the same way that it followed for Corollary 6.12. □

Chapter 8

Singularities in the Ricci Flow

This chapter is based on [28, Chap. 6, 7, 8].

8.1 Blowing Up at Singularities – Heuristics

It is time to step back and try to see where the achievements of Chapter 7 take us, in the context of an attack on Thurston's Geometrization Conjecture. We saw that, in the specific case of strictly positive initial Ricci curvature, the Ricci flow has a singularity in finite time, as $R_{\max} \to \infty$ as $t \to T$. In fact we proved more than this; we showed that there is a singularity occurring simultaneously everywhere in the manifold (this is a consequence of Theorem 7.11). We were able to rescale the manifold by a time-dependent factor and show that the rescaled flow converged smoothly to the geometric structure that we desired (namely a constant-curvature metric). We also rescaled time, but that was more a matter of convenience because it made the evolution equation (the normalized Ricci flow) easier to deal with.

The Uniformization Theorem for 2-manifolds can be proven using the Ricci flow in a similar way. Once more one introduces the normalized Ricci flow, and shows that it converges smoothly to a constant-curvature metric (see [6, Chap. 5]).

The normalized Ricci flow can not bypass all of our singularity problems however. The two cases we have mentioned so far are special, because the singularity in the Ricci flow happens everywhere simultaneously. We saw heuristically in Section 2.4 that singularities can, in general, be expected to develop along some subset of our manifold (for example in neckpinches). In this case, a rescaling to make the volume constant would not be expected to remove the singularity, it would make the manifold unbounded. We need a new technique. We would like to somehow blow our manifold up **at the singularity**. The aim of this chapter is to describe how this is done in broad terms, with very little detail. There is a strong analogy with the blowup idea introduced in the proof of Grayson's theorem (Theorem 4.6) for the CSF.

Consider a Ricci flow $(\mathcal{M}, g(t))$ with a localised singularity at time T. We might have something like the neckpinch discussed in Section 2.4 (see Figure 8.1). The curvature is exploding near some subset of $\mathcal{M}$ as $t \to T$. Let us choose times t_i such that $t_i \to T$, and points $x_i \in \mathcal{M}$ such that $|\mathrm{Rm}(g(t_i))|_{(x_i)} = \sup_{\mathcal{M}} |\mathrm{Rm}(g(t_i))|$.

We then have $|\mathrm{Rm}(g(t_i))|_{(x_i)} \to \infty$. If we rescale our manifold by a factor of $|\mathrm{Rm}(g(t_i))|_{(x_i)}$ at each time t_i then (by the result of Lemma 1.19) the curvature will no longer explode as we approach T, it will always have a maximum of 1. We would like to know what the limit of these manifolds as $i \to \infty$ is. That will tell us about the nature of the singularity, and what it means topologically. This is the notion of "blowing up" at a singularity. We now need to know what it means to take a "limit of manifolds" – this is a subtle question.

Consider, for example, the case of the neckpinch that we have already described. As we blow the manifold up, the neck itself will remain roughly the same size, but the rest of the manifold (which was not shrinking as we approached the singularity but is now getting dilated by the large factors needed to keep the neck "open") will get pushed out to infinity. Thus we expect the limit of these blowups to be an infinite cylinder $\mathbb{S}^2 \times \mathbb{R}$ – we completely lose the original topological

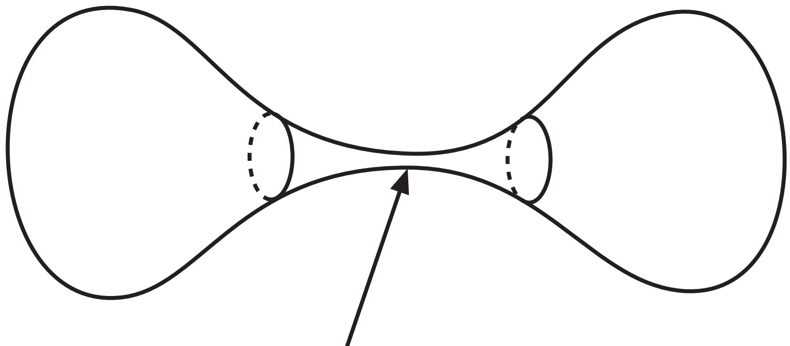

Figure 8.1: Curvature becomes large as the manifold gets close to pinching.

information. Despite all of the manifolds in the flow being compact and homeomorphic to each other, the limit manifold is neither compact nor homeomorphic to the manifolds in the flow. Note also that the limit depends on "where we are looking". For example, suppose we had two neckpinches happening simultaneously in different places. The limit should be well-defined, i.e. it can't be **both** the blowup of the first neckpinch **and** the blowup of the second neckpinch. Thus when we blow up, we need to blow up **about** a specified point. We define the notion of a **pointed** Riemannian manifold $(\mathcal{M}, g, p)$ as a Riemannian manifold $(\mathcal{M}, g)$ together with a distinguished point $p \in \mathcal{M}$.

8.2 Convergence of Manifolds

The theory of limits and convergence of manifolds was developed by Gromov, Cheeger, Peters, Greene and Wu among others, building on the idea of Hausdorff convergence in metric spaces. The definition we use is as follows:

Definition 8.1. *A sequence $(\mathcal{M}_i, g_i, p_i)$ of smooth, complete, pointed Riemannian manifolds is said to **converge** smoothly to the smooth, complete, pointed manifold $(\mathcal{M}, g, p)$ as $i \to \infty$ if the following are true:*

1. *There is a sequence of compact sets $K_i \subset \mathcal{M}$ such that any compact subset of $\mathcal{M}$ is contained in K_i for sufficiently large i, and such that $p \in int(K_i)$ for all i (we say that the sequence of sets (K_i) **exhausts** $\mathcal{M}$).*

2. *There exist smooth maps $\phi_i : K_i \to \mathcal{M}_i$ that are diffeomorphic onto their image and such that $\phi_i(p) = p_i$ for all i.*

3. *$\phi_i^* g_i \to g$ smoothly as $i \to \infty$, in the sense that the convergence is uniform in any C^k norm on compact subsets of $\mathcal{M}$.*

Note that the points p_i tell us "where to look", as explained at the end of Section 8.1.

Hamilton proved in [11] the following compactness theorem for pointed Riemannian manifolds:

Theorem 8.1. *Suppose that $(\mathcal{M}_i, g_i, p_i)$ is a sequence of complete, smooth, pointed Riemannian manifolds of dimension n such that*

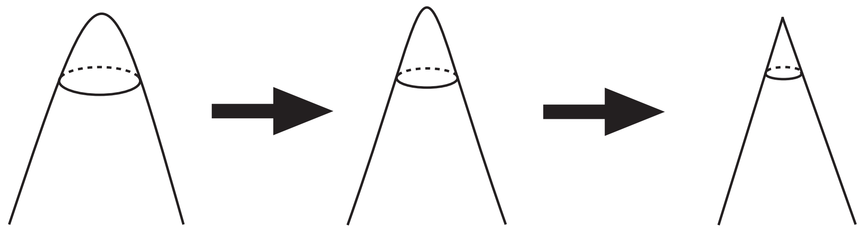

Figure 8.2: Non-smooth points can develop if curvature is not controlled.

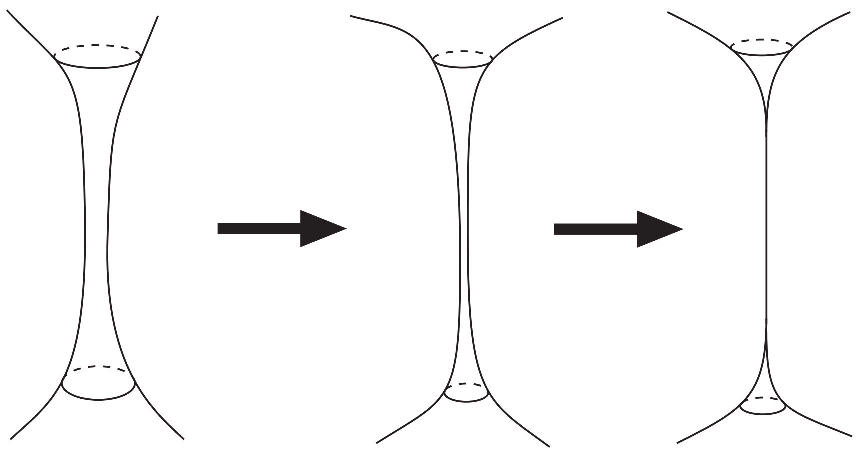

Figure 8.3: A neck losing a dimension because its injectivity radius was not controlled.

1. *For all* $r > 0$ *and* $k \in \mathbb{N}$*, we have*

$$\sup_{i\in\mathbb{N}} \sup_{B_{g_i}(p_i,r)} |\nabla^k \mathrm{Rm}(g_i)| < \infty;$$

2. *The injectivity radius at the points* p_i *is uniformly bounded away from zero:*

$$\inf_i inj(\mathcal{M}_i, g_i, p_i) > 0.$$

Then some subsequence of the sequence $(\mathcal{M}_i, g_i, p_i)$ *converges smoothly to a complete, smooth, pointed Riemannian manifold of dimension* n*.*

Note that both conditions are necessary – the first to avoid situations where the limit manifold is not smooth, as in Figure 8.2. The second avoids such monstrosities as dimensional reduction, see Figure 8.3.

Hamilton extended this compactness result to deal with limits of Ricci flows. In this way, the limit when we blow up our singularity will not only be a pointed Riemannian manifold, it will be a pointed solution of the Ricci flow.

Definition 8.2. *Let* $(\mathcal{M}_i, g_i(t), p_i)$ *be a sequence of pointed smooth manifolds with time-evolving metrics defined for on some time interval* $t \in (a, b)$*. If* $(\mathcal{M}, g(t), p)$ *is a pointed smooth manifold with a time-evolving metric for* $t \in (a, b)$*, we say that* $(\mathcal{M}_i, g_i(t), p_i)$ *converges smoothly to* $(\mathcal{M}, g(t), p)$ *as* $i \to \infty$ *if the following conditions are satisfied:*

1. *There is a sequence of compact sets* $K_i \subset \mathcal{M}$ *that exhausts* $\mathcal{M}$*, such that* $p \in int(K_i)$ *for all* i*.*

2. *There is a sequence of smooth maps $\phi_i : K_i \to \mathcal{M}_i$ which are diffeomorphisms onto their image, such that $\phi_i(p) = p_i$.*

3. *$\phi_i^* g_i(t) \to g(t)$ smoothly in both space* ***and*** *time, in the sense that the convergence is uniform in any C^k norm on any compact subset of $\mathcal{M} \times (a, b)$.*

The compactness theorem for manifolds (Theorem 8.1), can be extended to a compactness theorem for Ricci flows (this also first appeared in [11])

Theorem 8.2. *Let $(\mathcal{M}_i, g_i(t), p_i)$ be a sequence of n-dimensional pointed solutions of the Ricci flow defined for $t \in (a, b)$, where $0 \in (a, b)$. Suppose that*

1. $$\sup_{i\in\mathbb{N},(x,t)\in\mathcal{M}_i\times(a,b)} |\mathrm{Rm}(g_i(t))|_{(x)} < \infty;$$

2. $$\inf_i inj(\mathcal{M}_i, g_i(0), p_i) > 0.$$

That is, the curvature is uniformly bounded above and the injectivity radius at p_i is uniformly bounded away from zero at $t = 0$. Then there exists a marked solution of the Ricci flow $(\mathcal{M}, g(t), p)$ defined for $t \in (a, b)$ such that a subsequence of $(\mathcal{M}_i, g_i(t), p_i)$ converges smoothly to $(\mathcal{M}, g(t), p)$.

Note that we do not need control over all derivatives of the curvature as we did for Theorem 8.1 because of the Bernstein-Bando-Shi estimates (Theorem 6.6) which give us bounds on all derivatives of the curvature in terms of a bound on the curvature. Of course these estimates rely crucially on the fact that the metrics $g_i(t)$ satisfy the Ricci flow equation.

8.3 Blowing Up at Singularities – Results

Let us now apply these compactness results to the blowup of finite-time singularities as described in Section 8.1. Suppose that a solution of the Ricci flow $(\mathcal{M}, g(t))$ exists on a maximal time interval $t \in [0, T)$. By the result of Theorem 6.7, the maximum value of $|\mathrm{Rm}|$ on $\mathcal{M}$ explodes to $+\infty$ as $t \to T$. Thus we may choose points $p_i \in \mathcal{M}$ and times $t_i \in [0, T)$ such that $t_i \to T$ and

$$|\mathrm{Rm}|(p_i, t_i) = \sup_{x\in\mathcal{M},t\in[0,t_i]} |\mathrm{Rm}|(x, t) \to \infty. \tag{8.1}$$

Let us define $M_i := |\mathrm{Rm}|(p_i, t_i)$. Note the analogy with the proof of Grayson's Theorem (Theorem 4.6).

We now blow up the Ricci flows as described in Section 8.1: define

$$g_i(t) = M_i g\left(t_i + \frac{t}{M_i}\right).$$

This is merely a parabolic[1] rescaling of time and the metric in addition to a translation in time, and it is not difficult to see that $g_i(t)$ is a solution of the Ricci flow defined on the time interval $[-Mt_i, M(T - t_i))$. Furthermore, for $t \le 0$ we have

$$|\mathrm{Rm}(g_i(t))| = \frac{|\mathrm{Rm}\left(g\left(t_i + \frac{t}{M_i}\right)\right)|}{M_i} \le \frac{M_i}{M_i} = 1$$

by the way we defined M_i in equation (8.1). By Corollary 6.13, there exists $b > 0$ (depending only on the dimension of the manifold) such that, for all i, $g_i(t)$ is defined for $t \in (-Mt_i, b)$, and

$$\sup_{i\in\mathbb{N},(x,t)\in\mathcal{M}_i\times(-Mt_i,b)} |\mathrm{Rm}(g_i(t))|_{(x)} < \infty. \tag{8.2}$$

[1]A **parabolic** rescaling of a system scales time by a factor α and length by a factor $\sqrt{\alpha}$ – it is a characteristic of heat-type evolution equations that a parabolic rescaling of any solution is another solution.

Because the lower limits $-Mt_i$ of the time intervals tend to $-\infty$, we seek to define a limit flow for $t \in (-\infty, b)$ (for any $-B < 0$ arbitrarily large, the flows $g_i(t)$ will eventually be defined on $(-B, b)$ so the limit can be defined). Now equation (8.2) shows that the first condition of Theorem 8.2 is satisfied, so the only thing we need in order to pass to a limit flow is to uniform bound the injectivity radii of the manifolds $(\mathcal{M}, g_i(t))$ away from zero, for all $i \in \mathbb{N}$ and $t \in (-Mt_i, b)$. This turned out to be a major stumbling block for quite a while, but was resolved by Perelman in [23]. We will briefly discuss some of Perelman's ideas in Section 8.4. An accessible exposition of Perelman's proof of the injectivity radius bounds is contained in [28, Chap. 8]. Using Perelman's injectivity radius bounds, we can apply Theorem 8.2. The result is (passing to a subsequence of $(\mathcal{M}, g_i(t), p_i)$):

Theorem 8.3. *Suppose that $(\mathcal{M}, g(t))$ is a solution of the Ricci flow defined on a maximal time interval $[0, T)$, where $T < \infty$. Then there exist points $p_i \in \mathcal{M}$ and times $t_i \in [0, T)$, $t_i \to T$ such that*

$$M_i := |\mathrm{Rm}|(p_i, t_i) = \sup_{x \in \mathcal{M}, t \in [0, t_i]} |\mathrm{Rm}|(x, t) \to \infty.$$

If we now define

$$g_i(t) = M_i g\left(t_i + \frac{t}{M_i}\right)$$

then there exists $b > 0$ such that $(\mathcal{M}, g_i(t), p_i)$ converges to a Ricci flow $(\mathcal{N}, g_\infty(t), p_\infty)$ defined for $t \in (-\infty, b)$. Furthermore $|\mathrm{Rm}(g_\infty(0))|(p_\infty) = 1$ and $|\mathrm{Rm}(g_\infty(t))| \leq 1$ for $t \leq 0$.

One can now seek to classify the possible singularities that one can obtain from such a limiting argument. Note that the limit flow is defined for $t \in (-\infty, b)$ – such solutions are called **ancient**. Ancient solutions are very special. If we run the Ricci flow equation in reverse we get a non-parabolic system that tends to develop singularities. Thus we do not expect to be able to extend an arbitrary Ricci flow on some finite interval (a, b) to an ancient one defined on $(-\infty, b)$. Because ancient solutions are so special, we can hope to classify them despite the fact that classification of arbitrary Ricci flows is a hopelessly complicated task.

Hamilton went a long way towards performing this classification in [12], showing that (if injectivity radius bounds were satisfied) the limit of the singularity has to be either a 3-sphere $\mathbb{S}^3$ or quotient thereof (which shrinks to a point in finite time) or a cylinder $\mathbb{S}^2 \times \mathbb{R}$, which corresponds to a neckpinch (see Figure 8.4 for the 2-dimensional analogue). He was unable to rule out a third possible limit, namely $\Sigma^2 \times \mathbb{R}$ where Σ^2 is Hamilton's "cigar soliton" (described in Section 2.3), but Perelman has done away with this difficulty. "Degenerate neckpinches", in which one of the halves of the manifold shrinks to a point at the same time as the neck becomes singular (producing a sort of cusp in the manifold) are also possible.

The neckpinches, as described in Section 2.4, correspond to connected sum decompositions of the manifold (although some trivial decompositions may occur). The idea is to identify when a neckpinch is happening, and to perform the connected sum decomposition "manually" – that is, we cut out the neck part and seal the ends off with discs, glued on smoothly (compare Figure 2.3). This is the process known as "surgery". The flow can then continue. It might have any number of singularities that are either quotients of 3-spheres shrinking to a point in finite time, or products $\mathbb{S}^2 \times \mathbb{S}^1$ that shrink down to the $\mathbb{S}^1$ factor in finite time, or neckpinches which we must deal with by surgery. Perelman showed that the surgery times are locally finite (i.e., in any finite time interval there are only a finite number of times at which surgery occurs). We are left over with the solutions that have no finite-time singularity, which Perelman showed (building on work by Hamilton in [13]) must satisfy the Geometrization Conjecture (under rescaling the manifold will split along tori into expanding hyperbolic pieces with cusps and collapsing "graph manifolds", as discussed in Section 2.4).

To show the power of Theorem 8.3 we provide an alternative proof of Theorem 7.1.

Proof. (of Theorem 7.1) We consider a Ricci flow on a closed 3-manifold with initially strictly positive Ricci curvature. We showed in Theorem 7.2 that the flow must have a singularity in finite time, hence Theorem 8.3 applies. Let us say that the maximal time interval of existence is $[0, T)$. Thus there exist points $p_i \in \mathcal{M}$, times $t_i \to T$ and rescaled flows $g_i(t)$ defined for $t \in (-a_i, b)$

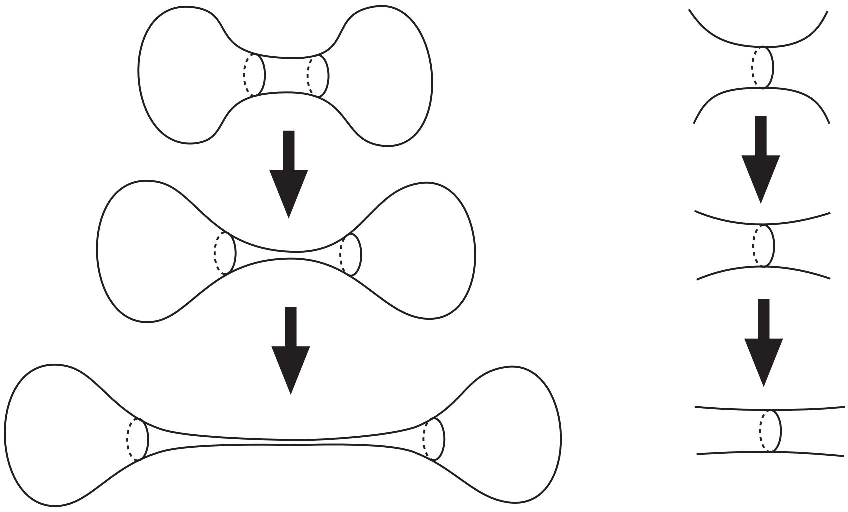

Figure 8.4: The limit of a manifold that gets close to pinching is a cylinder.

(where $b > 0$ and $a_i \to +\infty$) such that the pointed, rescaled flows $(\mathcal{M}, g_i(t), p_i)$ converge smoothly to a Ricci flow $(\mathcal{N}, g_\infty(t), p_\infty)$ defined for $t \in (-\infty, b)$.

We use the result of Corollary 7.9, which says that for the Ricci flow $(\mathcal{M}, g(t))$,

$$|E(g)|^2 \leq CR(g)^{2-\delta}$$

(where E is the Einstein tensor) for some positive constants C, δ. Now the flows $g_i(t)$ are just rescalings of $g(t)$ by the respective factors M_i (defined in Theorem 8.3, with the property that $M_i \to \infty$), hence one can deduce from the results of Lemma 1.19 that

$$\begin{aligned} M_i^2 |E(g_i)|^2 &\leq CM_i^{2-\delta} R(g_i)^{2-\delta} \\ \Rightarrow \qquad |E(g_i)|^2 &\leq CR(g_i)^{2-\delta} M_i^{-\delta}. \end{aligned}$$

Taking the limit as $i \to \infty$ and evaluating at $t = 0$, we have

$$|E(g_\infty(0))|^2 \leq CR(g_\infty(0))^{2-\delta} \lim_{i\to\infty} M_i^{-\delta} = 0$$

because $M_i \to \infty$.

Therefore $g_\infty(0)$ is an Einstein metric. By Lemma 1.14, it has constant sectional curvature. Because $R(g_i(t)) > 0$ for all i, in the limit we must have $R(g_\infty(0)) \geq 0$ hence the sectional curvatures are non-negative. Furthermore $|\mathrm{Rm}(g_\infty(0))| = 1$ so the sectional curvatures are strictly positive.

It follows that the Ricci curvature tensor of $g_\infty(0)$ on $\mathcal{N}$ is bounded below and hence, by Myers' Theorem (Theorem 1.17) that $\mathcal{N}$ is compact. Therefore, by the definition of convergence of manifolds, $\mathcal{N}$ must in fact be diffeomorphic to $\mathcal{M}$. Thus the metric $g_\infty(0)$ induces a metric of constant, strictly positive sectional curvature on $\mathcal{M}$, as required.

Note that, although we needed the pinching results of Section 7.4 for this proof, we did not need to prove any of the subsequent global pinching or convergence results – so significantly less work is required than in Chapter 7, if we assume Theorem 8.3! □

8.4 Perelman's $\mathcal{F}$- and $\mathcal{W}$-functionals

After people had tried unsuccessfully for some time to find a way of bounding the injectivity radius so that Theorem 8.3 could be obtained, Perelman came along and cleared things up (in [23]). We will not give much of an outline of his argument (an accessible exposition is contained in [28, Chap.

6,8]) but will present one of the key ideas behind his proof – that the Ricci flow can be formulated as a gradient flow.

We return to the analogy of the CSF. Recall Lemma 4.1, which showed that the CSF was a steepest-descent flow for length. Can the Ricci flow be similarly formulated as the steepest-descent flow for some functional? The simplest approaches do not work. In analogy with the length functional for the CSF the simplest functional we might consider is the volume of the manifold with the metric g – but we know that some manifolds increase in volume and some decrease in volume under the Ricci flow (specifically, hyperbolic manifolds increase in volume while manifolds of positive sectional curvature decrease in volume – see Section 2.3), so that will not work. The second-simplest functional we might try is the total scalar curvature, $\int R d\mu$. The corresponding gradient flow is, by formula (1.23) in Lemma 1.20,

$$\frac{\partial g}{\partial t} = -\mathrm{Rc} + \frac{1}{2} R g.$$

This is similar to the Ricci flow and looks a bit like the unnormalized Ricci flow but is not parabolic (and cannot be made parabolic using methods like the DeTurck trick) so it is completely useless as an evolution equation.

What clues do we have to the nature of the correct functional? One property it ought to have is that it should obviously be constant on fixed points of the Ricci flow. But recall from Section 2.3 that there exist "generalized fixed points" of the Ricci flow, the so-called Ricci solitons on which the Ricci flow is only changing the manifold by reparametrization and scaling. We hypothesize that the functional we seek should also be constant on gradient Ricci solitons (gradient solitons are much easier to deal with than general solitons). For the moment we will consider gradient solitons that do not involve any rescaling of the metric, so the constant λ that appears in equation (2.1) is 0. Thus we expect our functional to be constant if there exists a scalar function $f : \mathcal{M} \to \mathbb{R}$ such that $R_{ij} + \nabla_i \nabla_j f = 0$.

This gives us a clue that our functional should not depend on g alone, but should also depend on a scalar function f. In fact we define Perelman's $\mathcal{F}$-functional:

$$\mathcal{F}(g, f) := \int_{\mathcal{M}} \left(R + |\nabla f|^2\right) e^{-f} d\mu,$$

defined for Riemannian metrics g and scalar functions f. To work out what its gradient flow is we must calculate how it evolves under an arbitrary variation in the metric g and the function f: $\partial g/\partial t = h$ and $\partial f/\partial t = k$.

Lemma 8.4. *The variation of $\mathcal{F}(g, f)$ is given by:*

$$\frac{d}{dt}\mathcal{F}(g, f) = \int_{\mathcal{M}} \left[h_{ij}(-R^{ij} - \nabla^i \nabla^j f) + (2\triangle f - |\nabla f|^2 + R) \left(\frac{1}{2} h^{ij} g_{ij} - k\right)\right] e^{-f} d\mu.$$

We will not prove this result, but it follows from the results and methods used in Lemma 1.20.

Perelman now defined f so that the volume form $e^{-f} d\mu$ is preserved. By formula (1.21) of Lemma 1.20, we have

$$\frac{\partial}{\partial t} e^{-f} d\mu = \left(\frac{1}{2} g^{ij} h_{ij} - k\right) e^{-f} d\mu.$$

So, if we determine the evolution of f by

$$\frac{\partial f}{\partial t} = \frac{1}{2} g^{ij} \frac{\partial}{\partial t} g_{ij}$$

and define the (now constant) "modified volume form" $d\xi := e^{-f} d\mu$, then the result of Lemma 8.4 is:

$$\frac{d}{dt}\mathcal{F}(g, f) = \int_{\mathcal{M}} h^{ij}(-R_{ij} - \nabla_i \nabla_j f) d\xi.$$

Hence the steepest descent flow for this functional is the one with $h_{ij} = -R_{ij} - \nabla_i \nabla_j f$, i.e.

$$\frac{\partial}{\partial t} g_{ij} = -R_{ij} - \nabla_i \nabla_j f.$$

This is exactly the Ricci flow, with a rescaling of time (as we have $-R_{ij}$ rather than $-2R_{ij}$ on the RHS) and a time-dependent reparametrisation moving along the gradient vector field of f (as outlined in the discussion of Ricci solitons in Section 2.3). So the gradient flow of this functional is equivalent to the Ricci flow. We also note that if the solution is stationary under this flow, it satisfies the gradient Ricci soliton equation (2.1) with no rescaling term, as we have argued it should.

We would like to extend the $\mathcal{F}$-functional in some way so that the functional is also constant on gradient solitons that include rescaling. This is the idea behind the $\mathcal{W}$-functional. We introduce a positive real-valued scaling parameter $\tau > 0$ and define

$$\mathcal{W}(g, f, \tau) := \int_{\mathcal{M}^n} \big(\tau(R + |\nabla f|^2) + f - n\big)\, u d\mu$$

where n is the dimension of the manifold and

$$u := (4\pi\tau)^{-n/2} e^{-f}.$$

The arguments g, f, τ are said to be **compatible** if

$$\int_{\mathcal{M}^n} u d\mu = 1.$$

We would like an evolution equation for $\mathcal{W}$ in the style of Lemma 8.4. The analogous result is

Lemma 8.5. *If g, f, τ evolve according to*

$$\begin{aligned} \frac{\partial g}{\partial t} &= -2\mathrm{Rc} \\ \frac{\partial \tau}{\partial t} &= -1 \\ \frac{\partial f}{\partial t} &= -\triangle f + |\nabla f|^2 - R + \frac{n}{2\tau} \end{aligned}$$

then

$$\frac{d}{dt}\mathcal{W}(g, f, \tau) = 2\tau \int_{\mathcal{M}^n} |R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}|^2 u d\mu.$$

In particular, $\mathcal{W}$ is nondecreasing in time and remains constant if and only if

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} = 0,$$

that is, if and only if g is a shrinking gradient Ricci soliton. The condition that g, f, τ are compatible is preserved under this evolution. Note that the evolution equation for f is a backwards heat equation (because of the $-$ sign in front of the Laplacian). Backwards heat equations are not parabolic, and generally can not be expected to have a solution even for a short time. However, if we reverse time we get a normal heat-type equation, which does have a short-time solution. Thus, rather than specifying initial conditions on f, we can only specify final conditions.

Perelman was able to use the $\mathcal{W}$-functional to prove bounds on the volume of balls of given radius as the metric evolves under the Ricci flow, by defining f so that e^{-f} is a bump function with support inside the ball ($\mathcal{W}$ then has the form of an integral over the ball, which can be related to the ball's volume). He showed that the volume of such balls will not collapse, from which it is possible to show that their injectivity radius is bounded below, which is exactly the sort of result we needed to prove Theorem 8.3. See [28, Chap. 8] for the full argument.

Appendix A

Existence Theory for Parabolic PDEs

In this Appendix we will outline the main existence results for PDEs that we will use. This material is based on [6, Sec 3.2] and [28, Chap. 4].

Consider a vector bundle $\pi : \mathcal{E} \to \mathcal{M}$, where $\mathcal{M}$ is some Riemannian manifold, with some bundle metric h. We will be interested in PDEs describing the evolution of some time-dependent section of $\mathcal{E}$, $u : \mathcal{M} \times [0, T) \to \mathcal{E}$, having the form

$$\frac{\partial u}{\partial t} = L(u) \tag{A.1}$$
$$u(x, 0) = u_0(x) \tag{A.2}$$

where $L : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ is some differential operator.

A.1 Linear Theory

In this section we will use the multi-index notation for partial derivatives.

Definition A.1. *A* ***multi-index*** α *is an* n*-tuple of non-negative integers* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$*. We define*

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

If f *is a function of* n *variables* $x_1, x_2, \ldots, x_n$ *then we define*

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

We first consider the case that L is a linear differential operator. That is, using multi-index notation,

$$L(u) = \sum_{|\alpha| \leq k} L_\alpha \partial^\alpha u \tag{A.3}$$

where k is the **order** of L and $L_\alpha \in \mathrm{Hom}(\mathcal{E}, \mathcal{E})$.

For example, we might consider a second-order linear differential operator acting on scalar functions on $\mathbb{R}^n$ (so $\mathcal{M} = \mathbb{R}^n, \mathcal{E} = \mathcal{M} \times \mathbb{R}, k = 2$). This will have the form

$$\bar{L}(u) = \sum_{i,j} a_{ij} \partial^i \partial^j u + \sum_i b_i \partial^i u + cu.$$

If φ is some covector field (i.e. $\varphi \in C^\infty(T^*\mathcal{M})$) then we define the **total symbol** of L in the direction φ to be the bundle homomorphism $\sigma[L](\varphi) : \mathcal{E} \to \mathcal{E}$

$$\sigma[L](\varphi)(u) = \sum_{|\alpha|\leq k} L_\alpha(u) \prod_j \varphi^{\alpha_j}.$$

In the case of the example $\bar{L}$ we gave above, the total symbol in the direction φ is

$$\sigma[\bar{L}](\varphi)(u) = \sum_{i,j} a_{ij}\varphi^i\varphi^j u + \sum_i b_i\varphi^i u + cu.$$

We define the **principal symbol** of L in the direction φ to be the bundle homomorphism $\hat{\sigma}[L](\varphi) : \mathcal{E} \to \mathcal{E}$ that comes only from the highest order derivative terms in L:

$$\hat{\sigma}[L](\varphi)(u) = \sum_{|\alpha|=k} L_\alpha(u) \prod_j \varphi^{\alpha_j}.$$

The principal symbol of the example $\bar{L}$ in the direction φ is thus:

$$\hat{\sigma}[\bar{L}](\varphi)(u) = \sum_{i,j} a_{ij}\varphi^i\varphi^j u.$$

The operator L is said to be **elliptic** if $\hat{\sigma}[L](\varphi)$ is a vector bundle isomorphism whenever $\varphi \neq 0$. For our example $\bar{L}$ this means that

$$\sum_{i,j} a_{ij}\varphi^i\varphi^j \neq 0$$

whenever $\varphi \neq 0$, i.e. that $[a_{ij}]$ is a non-singular matrix.

The system A.1 is said to be **strongly parabolic**[1] if there exists $\delta > 0$ such that at each point of the manifold $\mathcal{M}$, for all covectors $\varphi \neq 0$ and elements $u \neq 0$ of $\mathcal{E}$,

$$\langle\hat{\sigma}[L](\varphi)(u), u\rangle > \delta|\varphi|^2|u|^2.$$

In the order-2 example we considered above, this is saying that

$$\sum_{i,j} a_{ij}\varphi^i\varphi^j > \delta|\varphi|^2$$

for all $\varphi \neq 0$, i.e. that the matrix $[a_{ij}]$ is positive-definite.

A.2 Linearization of Nonlinear PDEs

We will mainly deal with **non**-linear PDE's, where L is not given by the convenient form of A.3. What does it mean for a non-linear PDE to be parabolic? We need to define the **linearization** of the non-linear operator L. The linearization is defined in analogy with the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$, which is the linear map $Df : \mathbb{R}^n \to \mathbb{R}^m$ defined to "best approximate" f at each point. That is, if we have curve $x : [0, 1] \to \mathbb{R}^n$ satisfying

$$\begin{aligned} x(0) &= p \\ x'(0) &= v \end{aligned}$$

then

$$Df(v) = \frac{d}{dt} f(x(t))\bigg|_{t=0}$$

In a similar way we define the linearization of a non-linear operator L on a vector bundle $\mathcal{E}$. If $u : [0, 1] \to C^\infty(\mathcal{E})$ is a time-dependent section of $\mathcal{E}$ such that

$$\begin{aligned} u(0) &= u_0 \\ u'(0) &= v. \end{aligned}$$

[1] We will often just say "parabolic" when we mean "strongly parabolic"

We define the linearization of L at u_0 to be the linear map $D[L] : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ so that

$$D[L](v) = \frac{d}{dt} L(u(t))\Big|_{t=0}$$

(if it exists).

We say that the system A.1 is **strongly parabolic** at u_0 if the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= D[L](u) &\quad (A.4)\\ u(x,0) &= u_0(x) &\quad (A.5) \end{aligned}$$

is strongly parabolic in the sense described previously. We will only be interested in the case where $D[L]$ has the form

$$D[L] = \sum_{|\alpha| \leq k} \tilde{L}_\alpha \partial^\alpha$$

with some finite k.

Theorem A.1. *If the system A.1 is strongly parabolic at u_0 then there exists a solution on some time interval $[0, T)$, and the solution is unique for as long as it exists.*

Practically all existence proofs in this project will rely on this theorem. See [17] for the proof.

Appendix B

Differential Geometry Formulae

1. Christoffel symbols, for the Levi-Civita connection of the metric g_{ij}:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

2. In normal coordinates about the point p:
   (a) $\gamma_V(t) = (tV^1, tV^2, \ldots, tV^n)$ is a geodesic
   (b) $g_{ij}(p) = \delta_{ij}$
   (c) $\Gamma_{ij}^k(p) = 0$, $\partial_i g_{jk}(p) = 0$

3. Covariant derivative:

$$\nabla_p F_{i_1 \ldots i_k}^{j_1 \ldots j_l} = \partial_p F_{i_1 \ldots i_k}^{j_1 \ldots j_l} + \sum_{s=1}^{l} F_{i_1 \ldots i_k}^{j_1 \ldots q \ldots j_l} \Gamma_{pq}^{j_s} - \sum_{s=1}^{k} F_{i_1 \ldots q \ldots i_k}^{j_1 \ldots j_l} \Gamma_{p i_s}^{q} \tag{B.1}$$

4. Riemann curvature tensor:

$$[\nabla_i, \nabla_j] X^l \equiv R_{ijk}^l X^k$$

5. Symmetries of the Riemann curvature tensor:

$$R_{ijkl} = R_{klij} = -R_{jikl} = -R_{ijlk}$$

6. First Bianchi identity:

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

7. Second Bianchi identity:

$$\nabla_p R_{ijkl} + \nabla_i R_{jpkl} + \nabla_j R_{pikl} = 0 \tag{B.2}$$

8. Ricci and scalar curvatures:

$$R_{ij} \equiv R_{mjk}^m \ , \ \ R \equiv R_i^i$$

9. Contracted second Bianchi identity:

$$\nabla^j R_{ij} = \frac{1}{2} \nabla_i R$$

10. Commuting covariant derivatives:

$$[\nabla_p, \nabla_q] F^{j_1 \dots j_l}_{i_1 \dots i_k} = \sum_{s=1}^{l} R^{j_s}_{pqm} F^{j_1 \dots m \dots j_l}_{i_1 \dots i_k} - \sum_{s=1}^{k} R^{m}_{pqi_s} F^{j_1 \dots j_l}_{i_1 \dots m \dots i_k} \tag{B.3}$$

11. Coordinate form of the Riemann curvature tensor:

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^p_{jk} \Gamma^l_{ip} - \Gamma^p_{ik} \Gamma^l_{jp}$$

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