Hamilton's Ricci Flow and Thurston's Geometrization Conjecture

Nick Sheridan

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1 Topology

- Exercise: Classify closed 1-manifolds.
- 2-manifolds are classified by orientation and number of holes.
- There is no algorithm for classifying n-manifolds for $n \geq 4$.
- What about $n = 3$?
  - In 1904, Poincaré asked if any simply-connected 3-manifold is a 3-sphere.
  - In 1982, Thurston conjectured a classification of 3-manifolds from which the Poincaré conjecture would follow.
  - In 2003, Perelman used the Ricci Flow (introduced by Hamilton in 1982) to prove Thurston's Geometrization Conjecture, and hence the Poincaré Conjecture.

2 Background: 3-manifold topology

- We can "connect sum" two closed 3-manifolds M, N together to get $M \# N$.
- M is **irreducible** if any connect sum decomposition is trivial.
- **Theorem (Kneser, Milnor):** There is a unique minimal decomposition of any closed 3-manifold M into finitely many irreducibles $M = M_1 \# M_2 \# \ldots \# M_k$.
- **Jaco-Shalen-Johannson (JSJ) Decomposition:** Every irreducible closed 3-manifold has a unique minimal collection of disjoint embedded incompressible tori that split it into manifolds of two types: **atoroidal** (no incompressible tori) and **Seifert fibred manifolds** (S^1 bundles over a surface with some singular fibres, where the nearby fibres twist around multiple times before closing).
- Seifert fibred manifolds were classified by Seifert in 1933.

3 Thurston's Geometrization Conjecture

- Classifying topological 3-manifolds $\iff$ classifying differentiable 3-manifolds.
- **The Geometrization Conjecture:** Closed, irreducible 3-manifolds can be split along incompressible tori into components, each of which admits a finite volume **geometric structure**, by which we mean a complete Riemannian metric which is locally homogeneous (i.e., any two points have isometric neighbourhoods).
- A **model geometry** is a universal cover of a compact manifold with a geometric structure. It has a homogeneous Riemannian metric, i.e., the group of isometries acts transitively. Thurston proved that there are essentially only eight model geometries: $\mathbb{S}^3$, $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, PSL, Sol.
- So all the pieces of the above torus decomposition are quotients of one of the model geometries by a discrete group of isometries.
- The Geometrization Conjecture is a 3-dimensional version of the Uniformization Theorem, which says that a compact orientable 2-manifold admits a locally homogeneous metric with universal cover $\mathbb{S}^2, \mathbb{E}^2$ or $\mathbb{H}^2$.

- Six of the eight model geometries correspond to Seifert fibred spaces (classified).
- *Sol*-manifolds can easily be classified (they are all $\mathbb{T}^2$ bundles over $\mathbb{S}^1$).
- The remaining geometry is $\mathbb{H}^3$, the **hyperbolic manifolds**, which are quotients of $\mathbb{H}^3$ by discrete groups of isometries. They account for 'almost everything' and are hardest to classify.
- Morgan: "It is my view that before Thurston's work on hyperbolic 3-manifolds and . . . the Geometrization conjecture there was no consensus among the experts as to whether the Poincaré conjecture was true or false. After Thurston's work, notwithstanding the fact that it had no direct bearing on the Poincaré Conjecture, a consensus developed that the Poincaré conjecture (and the Geometrization conjecture) were true."

4 Geometric Flows

- **Curve Shortening Flow:** Take an embedded circle in $\mathbb{R}^2$ and evolve it in the direction of its curvature vector. It shrinks to a point in finite time. If we rescale it so that the area enclosed is constant, it converges to a round circle (Grayson). I.e., It 'shrinks to a round point'. E.g. Oil drops on water.
- **Gaussian Curvature Flow:** Take a hypersurface in $\mathbb{R}^3$ and evolve it by its Gaussian curvature vector. Strictly convex hypersurfaces shrink to a round point in finite time. This was introduced by Firey, who pointed out it describes the wearing of stones. ??
- **Mean Curvature Flow:** Same thing but evolve by the mean curvature vector rather than Gaussian curvature. This describes the evolution of soap bubbles in a vacuum.

5 Ricci Flow – The Grand Plan

Enter Hamilton, stage left:

- **Idea:** Start with a Riemannian metric g_0 on your closed, irreducible 3-manifold M. Evolve it by a geometric flow that converges to a **geometric structure** on M.
- Hamilton introduced the Ricci Flow in 1982:

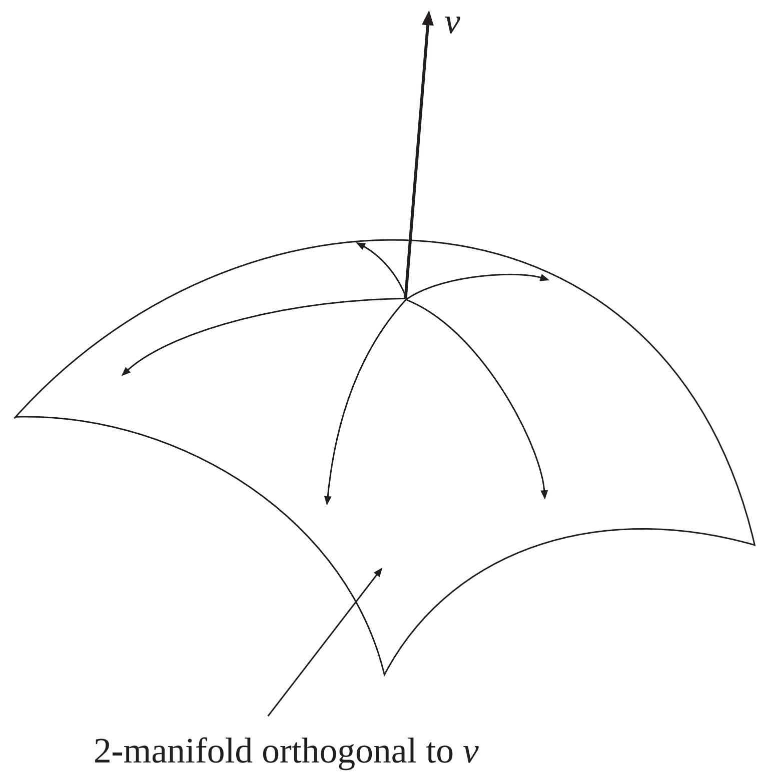
$$\frac{\partial g_t}{\partial t} = -2\mathrm{Rc}(g_t)$$

  where Rc is the **Ricci curvature tensor** of g_t.

- This is a **parabolic** or **heat-type** equation, so we hope it might smooth g_0 to a highly symmetric metric. For example, the scalar curvature R_t of g_t satisfies

$$\frac{\partial R_t}{\partial t} \geq \triangle R_t + \frac{2}{3}R_t^2.$$

For a 3-manifold M, a point $p \in M$ and a 2-dimensional subspace $\Pi \subset T_pM$, define the **sectional curvature** of Π, $K(\Pi)$, to be the Gaussian curvature of the 2-manifold swept out by geodesics starting at p and tangent to Π.



If the sectional curvature is constant and positive then the universal cover is $\mathbb{S}^3$.

- The Riemann curvature tensor $\mathcal{R}$ is a symmetric bilinear form acting on $\wedge^2 TM$, so that for $u, v \in T_pM$, $\mathcal{R}(u \wedge v, u \wedge v) = K(\text{span of } u, v)|u \wedge v|^2$.
- The Ricci curvature tensor Rc is a symmetric bilinear form acting on the tangent space TM, so that for $u \in T_pM$, $\mathrm{Rc}(u, u)$ is a sum of sectional curvatures of two orthogonal planes containing u, multiplied by $|u|^2$.
- The **scalar curvature** R at p is twice the sum of the sectional curvatures of three orthogonal 2-planes in T_pM.
- So we expect: $\mathbb{S}^3$ shrinks to a point in finite time, $\mathbb{H}^3$ grows bigger, while $\mathbb{S}^2 \times \mathbb{R}$ shrinks but only in the $\mathbb{S}^2$ direction – a 'neckpinch'.

6 Hamilton's Theorem

Theorem: *(Hamilton 1982)* If M is a closed 3-manifold and admits a Riemannian metric with positive Ricci curvature, then it admits a Riemannian metric with constant positive sectional curvature.

- In particular, the universal cover of M is $\mathbb{S}^3$.

7 The Proof of Hamilton's Theorem

Under the Ricci Flow, $\mathcal{R}$ evolves according to the equation:

$$\frac{\partial \mathcal{R}}{\partial t} = \triangle \mathcal{R} + Q(\mathcal{R})$$

where

$$Q\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 + \lambda_2\lambda_3 & 0 & 0 \\ 0 & \lambda_2^2 + \lambda_3\lambda_1 & 0 \\ 0 & 0 & \lambda_3^2 + \lambda_1\lambda_2 \end{pmatrix}.$$

The Laplacian just smooths things out. The behaviour of the solution is determined by the reaction terms, which leads us to examine the ODE

$$\frac{dA}{dt} = Q(A).$$

This is quantified by a version of the maximum principle for vector bundles due to Hamilton.

Theorem: Let M be a Riemannian manifold and $\pi : E \to M$ a vector bundle over M (equipped with a metric and a connection). Let K be a subset of E that is closed and convex in each fibre, and invariant under parallel translation. Let $F : E \times [0,T) \to E$ be a continuous map that is fibre-preserving, and Lipschitz in each fibre. Let $\alpha(t)$ be a time-dependent section of E satisfying

$$\begin{aligned} \frac{\partial}{\partial t}\alpha &= \triangle\alpha + F(\alpha) \\ \alpha(0) &\in K. \end{aligned}$$

Let $K_x = \pi^{-1}(x) \cap K$, and suppose that every solution of the ODE

$$\begin{aligned} \frac{da}{dt} &= F(a) \\ a(0) &\in K_x \end{aligned}$$

remains in the K_x. Then the solution $\alpha(t)$ of the PDE remains in K.

Diagonalising, the ODE is equivalent to evolution of a point in $\mathbb{R}^3$, namely

$$\frac{d}{dt}(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1^2 + \lambda_2\lambda_3, \lambda_2^2 + \lambda_3\lambda_1, \lambda_3^2 + \lambda_1\lambda_2).$$

Scaling by a positive constant doesn't change the 'shape' so consider the evolution on $\mathbb{S}^2$.

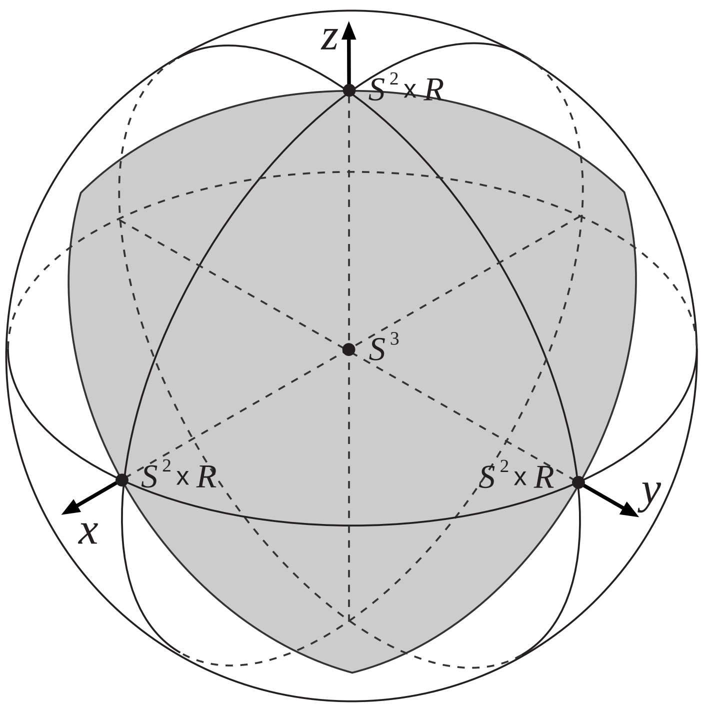


Figure 1: The axes and the region Rc > 0 (shaded) on the unit sphere in $\mathbb{R}^3$.

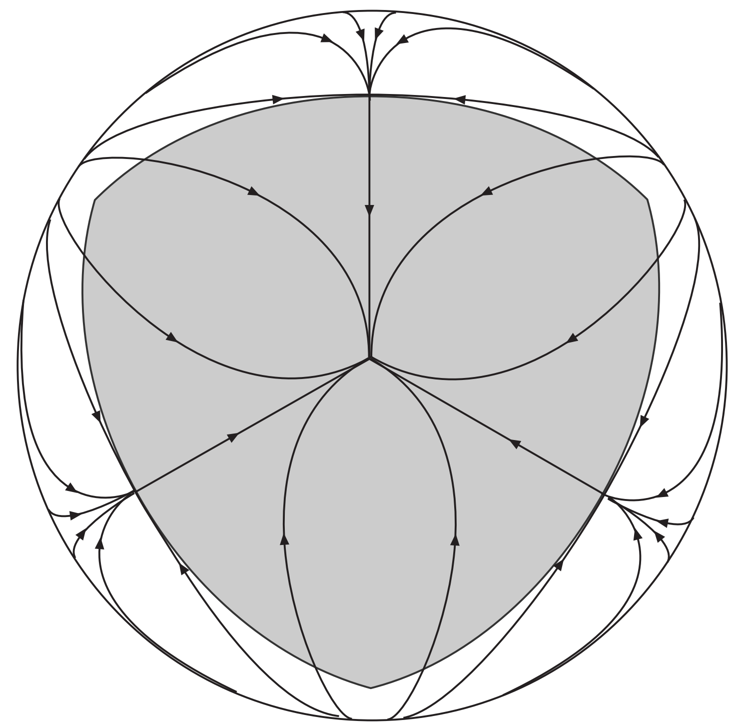


Figure 2: The flowlines of the ODE.

- The solution evolves towards the point with all sectional curvatures positive and equal – we have seen that this is the 3-sphere $\mathbb{S}^3$. Thus the solution is getting 'pushed' towards the 3-sphere, in addition to getting 'smoothed' by the Laplacian.

- Using the maximum principle, show that

$$\frac{(\lambda_1 - \lambda_2)^2 + (\lambda_2 - \lambda_3)^2 + (\lambda_3 - \lambda_1)^2}{R^2} \leq CR^{-\delta}$$

  for some positive constants C, δ, where $R = \lambda_1 + \lambda_2 + \lambda_3$ is the scalar curvature.

- Using the equation for evolution of R,

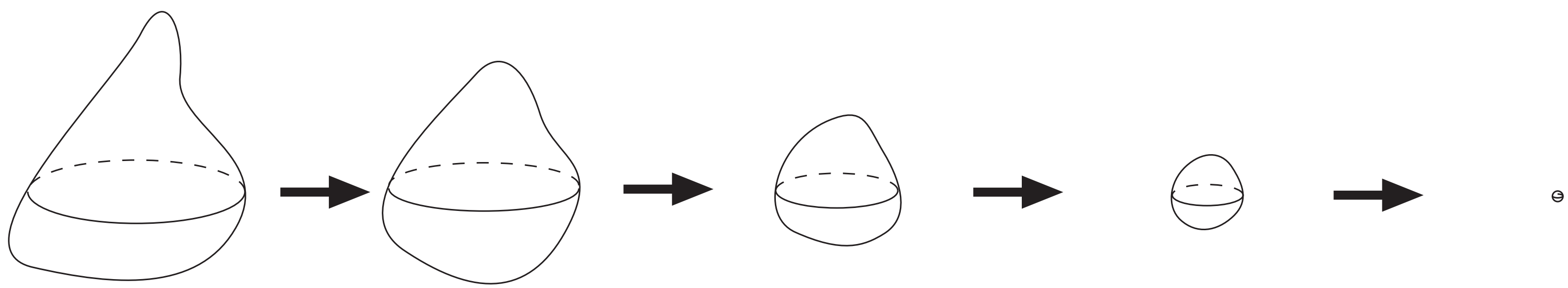
$$\frac{\partial R}{\partial t} \geq \triangle R + \frac{2}{3} R^2,$$

  and the maximum principle we see that $R \to \infty$ in finite time. We can also get a bound on $|\nabla R|$ in terms of R which shows that $R \to \infty$ uniformly.

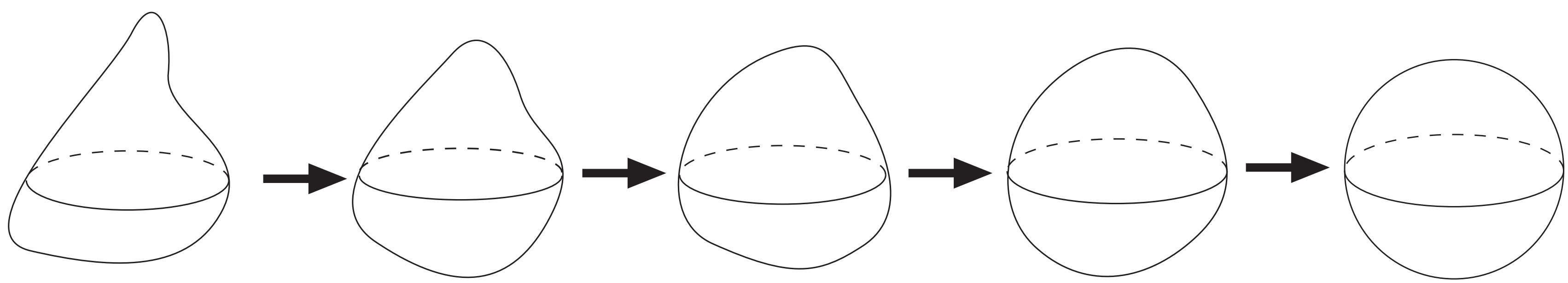
- The manifold shrinks to a point in finite time, but gets uniformly rounder and rounder as it does so.

- Now rescale g_t and t so the volume is constant.

Ricci Flow



Normalized Ricci Flow



- The normalized Ricci Flow exists for all time and converges exponentially quickly to a metric with constant (positive) sectional curvature. This limit metric is exactly the metric required for Hamilton's Theorem.

8 The Proof of the Geometrization Conjecture

- **Hamilton:** If the curvature is bounded as $t \to T$, then the Ricci flow can be extended past time $t = T$.
- **Hamilton:** If the Ricci flow exists for $t \to \infty$ and has bounded curvature, then the limit is a disjoint union of expanding hyperbolic parts bounded by tori, and the remainder is made up of Seifert fibered manifolds and Sol manifolds. Thus it satisfies the Geometrization Conjecture.

  What to do if the curvature blows up?
- **Hamilton:** Rescale the flow by successively larger constants (and translate in time) so that curvature is bounded. If we can control the injectivity radius of these rescalings then there is some sort of Cheeger-Gromov compactness result: a subsequence of these rescalings converges. How to control the injectivity radius?

Exeunt Hamilton stage right, pursued by a bear.

Perelman formulated the Ricci flow as a gradient flow in some sense. The gradient flow of the functional $\int R dV$ involves a scalar curvature term and is not parabolic. Perelman defined a functional on the space of metrics g and smooth functions f, so that fixed points corresponded to 'solitons'. Choosing f to approximate a delta function and observing that the functional increases with Ricci flow allows one to control the way that singularities form.

Perelman showed that curvature blowups come from either manifolds shrinking to a point (like $\mathbb{S}^3$) or $\mathbb{S}^2 \times \mathbb{R}$ 'necks' corresponding to a connected sum. He showed how to cut out a neck and cap off the ends, and continue the flow past it, in such a way that Hamilton's result still holds.

This allows us to prove Hamilton's original theorem a different way: rescale so that curvature does not blow up. Then we can apply the compactness result to extract a convergent subsequence. The limit must have constant positive sectional curvature by the pinching result we proved earlier. Thus it is a quotient of $\mathbb{S}^3$, hence compact, hence actually diffeomorphic to the original manifold.