Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space

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Outline

Context

- Mirror symmetry first came to the attention of mathematicians when Candelas-de la Ossa-Green-Parkes used it to predict counts of rational curves on the quintic three-fold (1991).
- Their predictions were mathematically verified by Givental (1996).
- Kontsevich introduced his ‘Homological Mirror Symmetry’ conjecture in 1994: mirror symmetry is an equivalence of triangulated categories.

Context

- Our main result is to prove homological mirror symmetry for a smooth hypersurface in $\mathbb{CP}^{n-1}$ of degree n.
- For $n = 3$ this is the elliptic curve (related to the result of Polishchuk-Zaslow).
- For $n = 4$ this is the quartic $K3$ (reproduces the result of Seidel).
- For $n = 5$ this is the quintic three-fold (also considered by Nohara-Ueda using my results).

The A-model

- Let $X^n \subset \mathbb{CP}^{n-1}$ be a smooth hypersurface of degree n.
- The **Fukaya category**, $\mathcal{F}(X^n)$, is a $\mathbb{Z}$-graded A_∞ category defined over the Novikov field Λ (elements of which are formal sums

$$\sum_{j=1}^{\infty} c_j r^{\lambda_j}$$

where $\{\lambda_j\} \subset \mathbb{R}$ is an increasing sequence, $\lambda_j \to \infty$).
- Scaling r corresponds to scaling the symplectic form.

The *B*-model

- Define

$$\widetilde{Y}^n := \left\{ u_1 \ldots u_n + r \sum_j u_j^n = 0 \right\} \subset \mathbb{P}^{n-1}_{\Lambda}.$$

- $G_n \cong (\mathbb{Z}_n)^{n-2}$ acts on $\widetilde{Y}^n$ (multiplying coordinates by nth roots of unity), and we define $Y^n := \widetilde{Y}^n / G_n$.
- Consider the category of coherent sheaves on Y^n:

$$Coh(Y^n) \cong Coh^{G_n}\left(\widetilde{Y}^n\right).$$

Main result

Theorem (S.)

There is an equivalence of Λ-linear triangulated categories

$$D^\pi \mathcal{F}(X^n) \cong \Psi \cdot D^b Coh(Y^n),$$

where Ψ is an automorphism (the 'mirror map')

$$\begin{aligned} \Psi : \Lambda &\rightarrow \Lambda, \text{ sending} \\ r &\mapsto \psi(r) r, \end{aligned}$$

where $\psi(r) \in \mathbb{C}[[r]]$ satisfies $\psi(0) = 1$. We are not yet able to determine the higher-order terms in $\psi(r)$.

Fukaya categories ‘relative to’ divisors

- (X, ω) is a compact Kähler manifold.
- $D := D_1 \cup \ldots \cup D_k \subset X$, where each D_j is an ample divisor.

Main example: Fermat hypersurfaces

$$X_a^n := \left\{ \sum_j z_j^a = 0 \right\} \subset \mathbb{CP}^{n-1},$$

with the n coordinate divisors $D_j := \{z_j = 0\}$.

G-graded A_∞ categories

- A **grading datum G** is the data of a morphism of abelian groups:

$$\mathbf{G} = \left\{ \mathbb{Z} \xrightarrow{\iota} Y \right\}.$$

- A **G**-graded A_∞ category has Y-graded morphism spaces, and A_∞ structure map μ^s has degree $\iota(2-s)$.

Equivariant Lagrangian branes

- Let M be a symplectic manifold. An **equivariant Lagrangian brane** in M is a Lagrangian immersion $L \to M$, together with a lift of the map $L \to \mathcal{G}(M)$ to the Lagrangian Grassmannian of M to the universal cover of the total space of $\mathcal{G}(M)$.
- The Fukaya category with these objects is naturally $\mathbf{G}(M)$-graded, where

$$\mathbf{G}(M) := \left\{ \pi_1 \left(\mathcal{G}_p(M) \right) \xrightarrow{\iota} \pi_1(\mathcal{G}(M)) \right\}.$$

Affine and relative Fukaya categories

Notation	Affine: $\mathcal{F}(X \setminus D)$	Relative: $\mathcal{F}(X, D)$
Objects	Equivariant Lagrangian branes in $X \setminus D$	same
Coefficients	$\mathbb{C}$	$\mathbb{C}[[r_1, \ldots, r_k]]$
Morphisms $CF^*(L_0, L_1)$	$\mathbb{C}\langle L_0 \cap L_1 \rangle$	$\mathbb{C}[[r_1, \ldots, r_k]]\langle L_0 \cap L_1 \rangle$
Structure maps μ^s	counts of holomorphic disks $u : (\mathbb{D}, \partial\mathbb{D}) \to (X \setminus D, L_i)$	counts of holomorphic disks $u : (\mathbb{D}, \partial\mathbb{D}) \to (X, L_i)$, with coefficient $r_1^{u \cdot D_1} \ldots r_k^{u \cdot D_k}$
Grading	$\mathbf{G}(X \setminus D)$	$\mathbf{G}(X \setminus D)$

Full Fukaya category

- The 'full' Fukaya category (see FOOO) is denoted $\mathcal{F}(X)$.
- A_∞ structure maps count holomorphic disks $u : \mathbb{D} \to X$, with coefficient $r^{\omega(u)}$.
- There's a fully faithful embedding

$$\mathcal{F}(X, D) \otimes_{\mathbb{C}[[r_1,\ldots,r_k]]} \Lambda \to \mathcal{F}(X)$$

(send each $r_j \mapsto r \in \Lambda$).

Mirror to affine, relative, full Fukaya categories

Mirror to (X^n, D) is the variety $\widetilde{\mathcal{Y}}^n / G_n$, where

$$\widetilde{\mathcal{Y}}^n := \left\{ u_1 \ldots u_n + \sum_{j=1}^{n} r_j u_j^n = 0 \right\} \subset \mathbb{P}^{n-1}_{\mathbb{C}[[r_1,\ldots,r_n]]},$$

which admits an action of $\mathbb{T} := (\mathbb{C}^*)^{n-1}$. $\widetilde{\mathcal{Y}}^n$ is related to $\tilde{Y}^n$ via a base change (think localization):

$$\mathbb{C}[[r_1, \ldots, r_n]] \to \Lambda \text{ sending } r_j \mapsto r.$$

Mirror correspondences

Special fibre (over $\mathrm{Spec}(\mathbb{C})$)	Total space (over $\mathrm{Spec}(\mathbb{C}[[r_1, \ldots, r_n]])$)	Generic fibre (over $\mathrm{Spec}(\Lambda)$)
$\mathrm{Perf}(\mathcal{Y}^n_0)$, $\mathbb{T}$-equivariant sheaves	$Coh(\mathcal{Y}^n)$, $\mathbb{T}$-equivariant sheaves	$Coh(Y^n)$
$\mathcal{F}(X^n \setminus D)$, equivariant Lag. branes	$\mathcal{F}(X^n, D)$, equivariant Lag. branes	$\mathcal{F}(X^n)$

- $\mathbb{T}$-equivariant sheaves have $\mathbb{Z}^{n-1}$-graded morphism spaces.
- Equivariant Lagrangian branes have $H_1(X^n \setminus D) \cong \mathbb{Z}^{n-1}$-graded morphism spaces
- These gradings correspond under mirror symmetry.

Plan for computing $\mathcal{F}(X^n, D)$

- Understand how the relative Fukaya category behaves with respect to branched covers ramified about the divisors D_j.
- Apply this to the branched cover

$$\begin{array}{rcl} X^n = \left\{ \sum_j z_j^n = 0 \right\} & \to & \left\{ \sum_j z_j = 0 \right\} = \mathbb{CP}^{n-2} \subset \mathbb{CP}^{n-1}, \\ {[z_1 : \ldots : z_n]} & \mapsto & [z_1^n : \ldots : z_n^n]. \end{array}$$

Branched covers

- Let $\rho : (Y, D) \to (X, D)$ be a cover with branching of degree a_j about divisor D_j.
- Holomorphic disks in $X \setminus D$ lift to $Y \setminus D$

$$\Rightarrow \mathcal{F}(Y \setminus D) \cong \mathcal{F}(X \setminus D) \# G^*$$

  where G is the covering group.
- How is $\mathcal{F}(Y, D)$ related to $\mathcal{F}(X, D)$?

Smooth orbifold relative Fukaya category

- Define $\mathcal{F}(X, D, a)$: same as $\mathcal{F}(X, D)$, but disks are required to have ramification of order a_j along D_j wherever they intersect.
- Such disks in X lift to disks in Y. Therefore,

$$\mathcal{F}(Y, D) \cong \mathcal{F}(X, D, a) \# G^*.$$

- Now we want to relate $\mathcal{F}(X, D, a)$ to $\mathcal{F}(X, D)$.

First-order deformation classes

- Write the A_∞ structure maps in $\mathcal{F}(X, D)$ as

$$\mu^* = \mu_0^* + \sum_j r_j \alpha_j^* + \mathcal{O}(r^2).$$

- μ_0^* gives $\mathcal{F}(X \setminus D)$.
- A_∞ equation for $\mu^* \Rightarrow \alpha_j^*$ is a Hochschild cochain.
- We call $[\alpha_j^*] \in HH^*(\mathcal{F}(X \setminus D))$ the **first-order deformation classes.**

Behaviour of deformation classes under ramified covers

Proposition (S.)

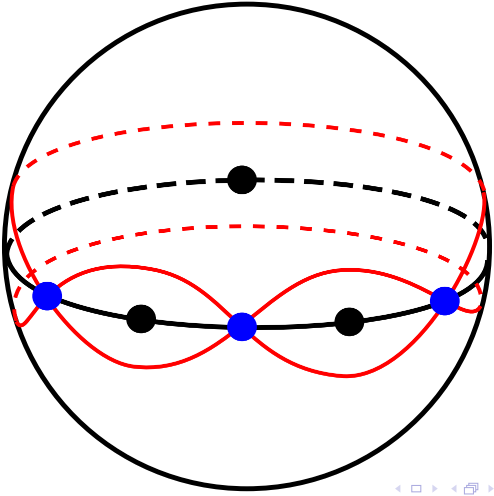
If the first-order deformation classes of $\mathcal{F}(X, D)$ are α_j, then the first-order deformation classes of $\mathcal{F}(X, D, a)$ are $\alpha_j^{a_j}$, where the power is taken with respect to the Yoneda product on Hochschild cohomology.

- Apply this to the branched cover

$$(X^n, D) \to (\mathbb{CP}^{n-2}, D).$$

- $\mathbb{CP}^{n-2} \setminus D$ = ‘pair of pants’.
- We construct an (immersed) equivariant Lagrangian brane $L \subset \mathbb{CP}^{n-2} \setminus D$.
- Compute $CF^*(L, L)$ in $\mathcal{F}(\mathbb{CP}^{n-2}, D)$ to first order.

$L^1 \subset \mathbb{CP}^1 \setminus D$ as a 'pushoff' of $S^1 \to \mathbb{RP}^1 \subset \mathbb{CP}^1$



L^n as a 'pushoff' of $S^{n-2} \to \mathbb{RP}^{n-2} \subset \mathbb{CP}^{n-2}$

- More generally, construct L^n as a pushoff of the immersion

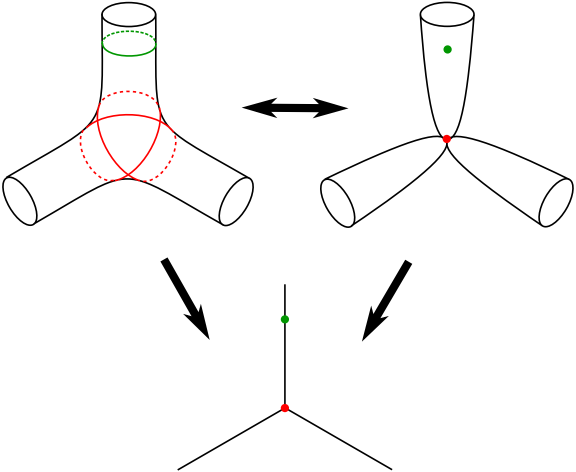
$$S^{n-2} \xrightarrow{2:1} \mathbb{RP}^{n-2} \hookrightarrow \mathbb{CP}^{n-2}$$

  by some Morse function $f : S^{n-2} \to \mathbb{R}$.
- ∇f must be transverse to the hypersurfaces

$$D_j^{\mathbb{R}} := D_j \cap \mathbb{RP}^{n-2}$$

  if L^n is to avoid the divisors D_j.
- See case $n = 4$.

L^1 as a 'fibre' in an SYZ fibration



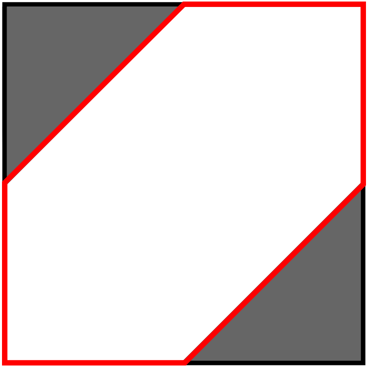
Amoeba and Coamoeba

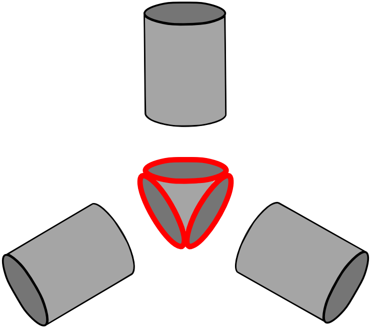
- We have

$$\begin{array}{ccc} \mathbb{CP}^{n-2} \setminus D \hookrightarrow \mathbb{CP}^{n-1} \setminus \{z_j = 0\} & \xrightarrow{\text{Log}} & \mathbb{R}^{n-1} \\ \downarrow{\scriptstyle \text{Arg}} & & \\ (S^1)^{n-1} & & \end{array}$$

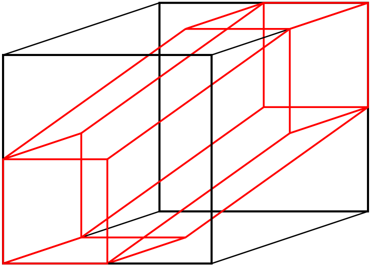
- $\text{Log}(\mathbb{CP}^{n-2} \setminus D)$ is called the 'amoeba'.
- $\text{Arg}(\mathbb{CP}^{n-2} \setminus D)$ is called the 'coamoeba'.

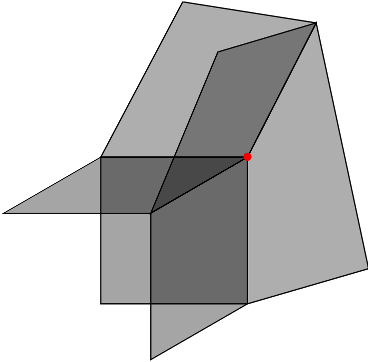
Coamoeba and ‘fibration’ of pants over the amoeba





Coamoeba and amoeba of two-dimensional pants

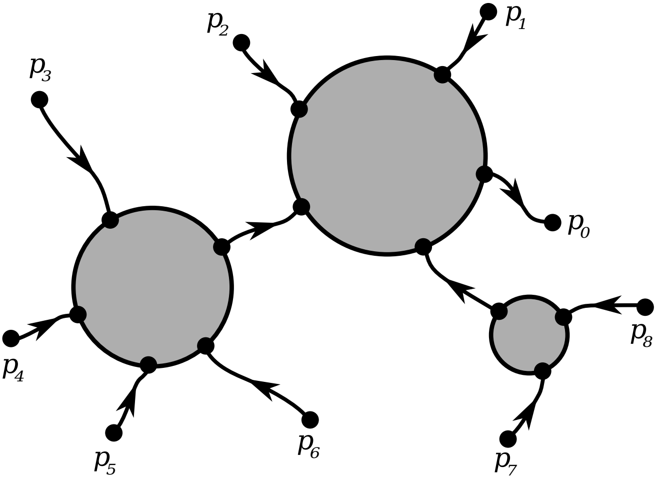




Morse-Bott model for $CF^*(L, L)$

- We constructed L as a pushoff of the immersion $S^{n-2} \to \mathbb{RP}^{n-2} \to \mathbb{CP}^{n-2}$ by a Morse function f.
- Pushoff by ϵf, and consider the limit $\epsilon \to 0$.
- Holomorphic disks with boundary on L degenerate to 'pearly trees', built out of Morse flowlines of f and holomorphic disks with boundary on $\mathbb{RP}^n$ (halves of real algebraic curves).

A pearly tree contributing to $\mu^8(p_8, \ldots, p_1)$



Generators of $CF^*(L, L)$

- $CF^*(L, L) \cong \Lambda^* \mathbb{C}^n$ as a vector space.
- $H^*(S^{n-2})$ gives generators 1 and $\theta_1 \wedge \ldots \wedge \theta_n$.
- Other generators come from self-intersections of L = critical points of f.

Self-intersections of L

- Think of the sphere as
$$S^{n-2} = \left\{ \sum x_j^2 = 1, \sum x_j = 0 \right\} \subset \mathbb{R}^n$$
covering
$$\mathbb{RP}^{n-2} = \left\{ \sum x_j = 0 \right\} \subset \mathbb{RP}^{n-1}.$$
- The hypersurfaces $\{x_j = 0\}$ split S^{n-2} into regions indexed by the set $K \subset \{1, \ldots, n\}$ of positive coordinates, for all K except ϕ and $\{1, \ldots, n\}$.
- f has one critical point in each region, corresponding to the other generators
$$\bigwedge_{j \in K} \theta_j \in \Lambda^* \mathbb{C}^n.$$

$CF^*(L, L)$ in $\mathcal{F}(\mathbb{CP}^{n-2} \setminus D)$

- The differential is 0, and as an algebra,

$$CF^*(L, L) \cong \Lambda^* \mathbb{C}^n =: A.$$

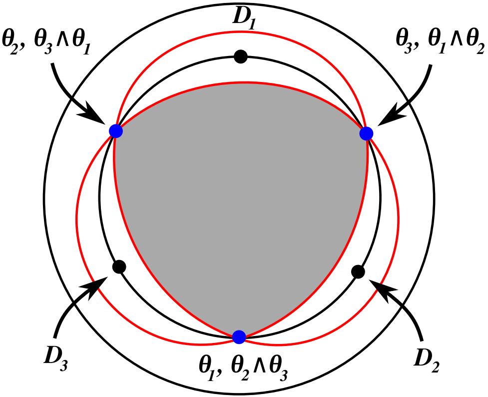
- The A_∞ structure of $\mathcal{A} := CF^*(L, L)$ in $\mathcal{F}(\mathbb{CP}^{n-2} \setminus D)$ is determined by one higher product:

$$\mu^n(\theta_1, \theta_2, \ldots, \theta_n) = 1.$$

- This corresponds to the class

$$z_1 \ldots z_n \in HH^*(A) \cong \mathbb{C}[[z_1, \ldots, z_n]] \otimes A \text{ (HKR)}.$$

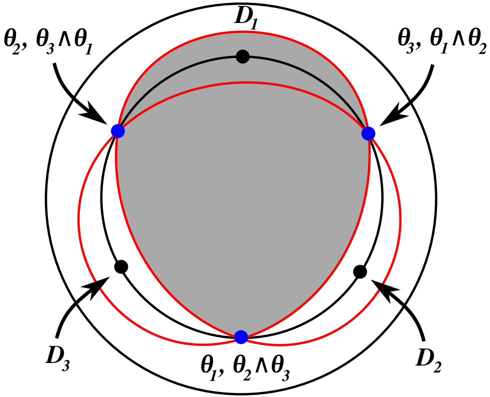
Computing $CF^*(L, L)$ in $\mathcal{F}(\mathbb{CP}^1 \setminus D)$



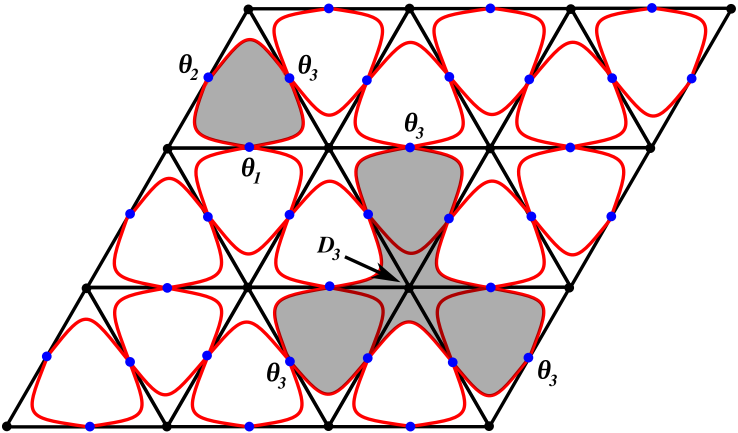
$CF^*(L, L)$ in $\mathcal{F}(\mathbb{CP}^n, D)$

- The first-order deformation class of $CF^*(L, L)$ in $\mathcal{F}(\mathbb{CP}^n, D)$ is
$$\sum_j r_j z_j \in HH^*(\mathcal{A}).$$
- Therefore, the first-order deformation class of $CF^*(L, L)$ in $\mathcal{F}(\mathbb{CP}^n, D, (n, \ldots, n))$ is
$$\sum_j r_j z_j^n.$$
- Thus we can determine the subcategory of $\mathcal{F}(X^n, D)$ generated by lifts of L, to first order.

Calculating $CF^*(L, L)$ in $\mathcal{F}(\mathbb{CP}^1, D)$



Lifts of L to X^1 = elliptic curve



The *B*-model: matrix factorizations

- We make computations in $D^b Coh(\mathcal{Y}^n)$ using **G**-graded matrix factorizations of the superpotential

$$W = u_1 \ldots u_n + \sum_{j=1}^{n} r_j u_j^n.$$

- The object corresponding to L is the structure sheaf of the origin $\mathcal{O}_0$, hence $\mathbb{T}$-equivariant.
- $\mathrm{End}_{MF(W)}(\mathcal{O}_0)$ is an A_∞ deformation of

$$\mathrm{End}_{Coh(\mathbb{C}^n)}(\mathcal{O}_0) \cong \Lambda^* \mathbb{C}^n.$$

The *B*-model: equivalence with the *A*-model

- Homological perturbation lemma $\Rightarrow$ the deformation classes of $\mathrm{End}_{MF(W)}(\mathcal{O}_0)$ are given by the coefficients of

$$W = u_1 \ldots u_n + \sum_{j=1}^{n} r_j u_j^n$$

- $\Rightarrow$ coincides with those of $\mathrm{End}_{\mathcal{F}(\mathbb{CP}^{n-2},D,n)}(L)$.
- Prove that this, combined with $\mathbb{T}$-equivariance, means they're quasi-isomorphic up to a formal change of variables.

The B-model: equivariance and coherent sheaves

- Now $\mathcal{Y}^n := \{W = 0\} \subset \mathbb{P}^n_{\mathbb{C}((r_1,\ldots,r_n))}$.
- $D^b Coh(\mathcal{Y}^n) \cong \mathrm{GrMF}(W) \cong MF^{\mathbb{Z}_n}(W)$ (Orlov).
- $\Rightarrow D^b Coh^G(\mathcal{Y}^n) \cong MF^{\widetilde{G}}(W)$ where $\widetilde{G}$ is an extension of G by $\mathbb{Z}_n$.
- Pass from $MF \rightsquigarrow MF^{\widetilde{G}} \leftrightarrow$ pass to the $\widetilde{G}^*$-cover $\mathcal{F}(\mathbb{CP}^{n-2}, D, n) \rightsquigarrow \mathcal{F}(X^n, D)$

Split-generation (Calabi-Yau case)

- We have shown that the subcategories of $\mathcal{F}(X^n)$ generated by lifts of L, and of $D^b Coh(Y^n)$ generated by the images of twists of $\mathcal{O}_0$ by characters of $\widetilde{G}$, are quasi-equivalent.
- So, if these subcategories generate, we're done.
- Twists of $\mathcal{O}_0$ correspond to twists of the restrictions of the Beilinson exceptional collection $\Rightarrow$ generate.

Split-generation of the Fukaya category (AFOOO)

Theorem* (Abouzaid-Fukaya-Oh-Ohta-Ono, in preparation)

Suppose that X^d is Calabi-Yau, and $\mathcal{C} \subset \mathcal{F}(X^d)$ is a full subcategory. Then, if the top-degree part of the closed-open string map

$$\mathcal{CO} : QH^{2d}(X^d) \to HH^{2d}(\mathcal{C})$$

is non-zero, then $\mathcal{C}$ split-generates $\mathcal{F}(X^d)$.

- Show that
$$\mathcal{CO}([D_j]) = r_j \frac{\partial \mu^*}{\partial r_j}.$$
- Show class of μ^* in $HH^*(A)$ is equal to
$$[\mu^n] = f_1(T) u_1 \ldots u_n + f_2(T) \sum_{j=1}^n r_j u_j^n,$$
where $T = r_1 \ldots r_n$ and $f_j \in \mathbb{C}[[T]]$ satisfy $f_j(0) = 1$.
- Now $\mathcal{CO}$ is a ring homomorphism, so we can compute
$$[D_j]^{n-2} \mapsto \left(r_j \frac{\partial \mu^n}{\partial r_j} \right)^{n-2} \neq 0.$$
- $\Rightarrow$ our collection of Lagrangians split-generates.