Overtwisted contact structures IV

Goal: layout proof of

Prop 3.1: $\exists$ $K_{univ} : \Delta^{2n-1} \longrightarrow \mathbb{R}$ s.t.

any almost contact (M, ξ) has $\xi \simeq \xi'$ where for finite # of balls $B_i \subseteq M$, we have

1) ξ' genuine on $M \setminus \bigcup_i B_i$

2) (B_i, ξ') equiv. to $(B_{K_{univ}}, \eta_{K_{univ}})$

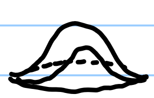
Introduced semicontact regular contact shells (B, σ_ϕ) that are well-adapted to proving this prop.

Recall: D^{2n} disk w/ $\phi : D \to \mathbb{R}$
$\phi = 0$ w/ ∞-jet on ∂D
$\phi > 0$ near ∂D

Define the saucer

$$B = \{(w, v) \in D \times \mathbb{R} \mid 0 \leq F(w) \leq v\}$$

where $F: D \to \mathbb{R}_{\geq 0}$ with
$F > 0$ on $\mathrm{int}(D)$
$F = \phi$ near ∂D



$D_s = \{v = s\phi(w)\}$ give germ of graph $s\phi \subseteq (D \times \mathbb{R}, \xi_{st})$

Prop: Any semicontact regular (B, σ_ϕ) dominates some (B_K, η_K) for $K: \Delta_{cyl} \to \mathbb{R}$

Idea: Regularity lets one view $\phi: D/_{\mathcal{F}} \times [0,1] \to \mathbb{R}$ as a time-dependent contact form on

$\Delta = D/\mathcal{F}$.

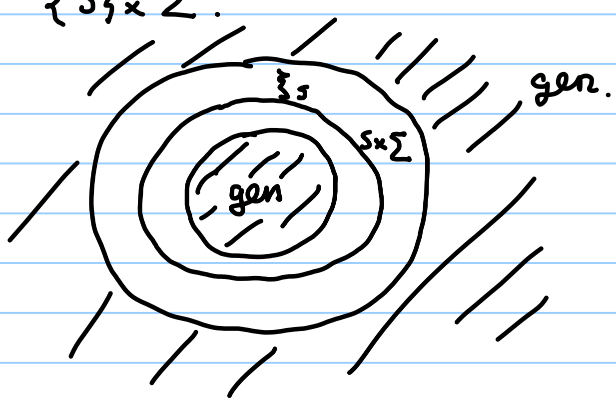
Prop 6.2: Up to homotopy any almost contact (M, ξ) is such that

1) ξ is genuine on complement of finite # of saucers B_i

2) (B_i, ξ) is equivalent to a semicontact regular saucer

Step 1: Let $\Sigma = S^{2n} \subseteq M$ embedded. Apply Gromov's h-principle: we can homotope ξ so that it is genuine away from $\mathcal{O}p\,\Sigma$ and ξ is a semicontact str. $\{\xi_s\}$ on $[0,1] \times \Sigma = \mathcal{O}p\,\Sigma$.

I.e. each ξ_s is a germ of contact str. on $\{s\} \times \Sigma$.



Step 2: Regularize semicontact structure:

There exists a smooth $\psi : \Sigma \to \mathbb{R}$ and contact str. μ on $\Sigma \times \mathbb{R}$ s.t.

1) $\xi_s = \Psi_s^* \mu$ near $\Sigma \times \{s\}$

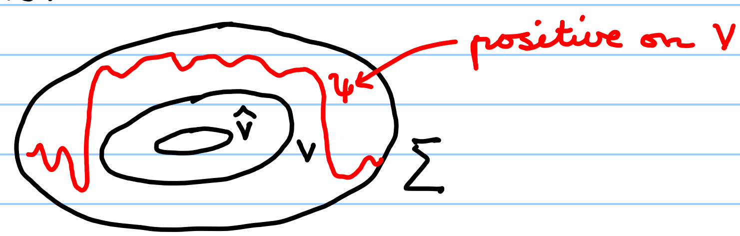
where $\Psi_s(x, s+t) = (x, s\psi(x) + t)$

2) Closed domains $\hat{V} \Subset V \subset \Sigma$ s.t.

$\psi|_V > 0$ and $\mu \pitchfork \mathrm{graph}(s\psi)$ on

$\hat{W} = \Sigma \setminus \hat{V}$.

Picture:



Idea: by decomposing $[0,1] = \bigcup_{i=1}^{\ell} [a_i, a_{i+1}]$,

can assume (1) hold on each $[a_i, a_{i+1}] \times \Sigma$.

by applying Moser to the family of contact germs ξ_s on $[s_0 - \varepsilon, s_0 + \varepsilon] \times \Sigma$

For (2), places where Reeb very transverse to $T(\Sigma \times \{s\})$ and places where μ is transverse to the graph of ψ.

Applying the Reeb flow lets one make $\psi > 0$ away from where $\mathrm{graph}(\psi) \pitchfork \mu$.

<u>Step 3</u>: Since $\psi > 0$ on V one has that $\{\xi_s\}$ is genuine on $V \times [0,1]$.
Need saucers away from V where ψ could be negative.

Let $\{U_i\}_{i=1}^{N}$ be open cover by balls of

$W = \Sigma \setminus V$.

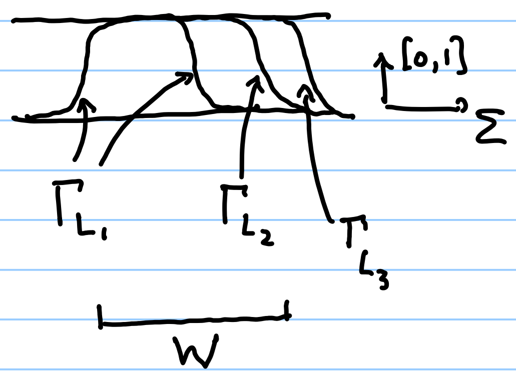
Let $\lambda_i : \Sigma \longrightarrow [0,1]$ have $\mathrm{supp}(\lambda_i) \subseteq U_i$

and $\sum \lambda_i|_W = 1$

Define $L_k = \sum_{i=1}^{k} \lambda_i$ their graphs Γ_{L_k}

partition $\Sigma \times [0,1]$.

Picture :



The region bounded by $\Gamma_{L_{i-1}}$ and Γ_{L_i} are saucers B_i.

Since the graph (S4) ⋔ u, can make cover fine enough s.t. each $(B_i, \{\xi_s\})$ is equivalent to a semi-contact regular saucer.

Prop 8.1: For fixed dimension, $\exists$ finite list of saucers $\{(B_p, \xi_p)\}_{p=1,\dots,L}$

that can be used to fill any (B_K, η_K) for $K : \Delta \to \mathbb{R}$.

Together prove Prop 3.1: Find $K_{univ} : \Delta_{cyl} \to \mathbb{R}$

s.t. $(B_{K_{univ}}, \eta_{K_{univ}})$ is dominated by each of the saucers (B_p, ξ_p) in Prop 8.1.

Now apply Prop 6.2 to get (M, ξ) to be genuine outside a finite # of saucers.

Apply Prop 8.1 to replace each saucer with a finite # of saucers (B_p, ξ_p) from list.

Finally, fill each (B_p, ξ_p) with $(B_{K_{univ}}, \eta_{K_{univ}})$.

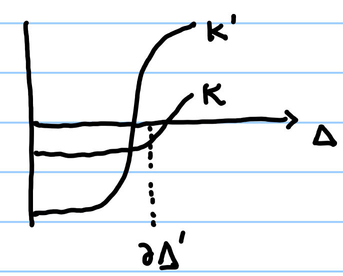
Without Prop 8.1: Can fill each (M, ξ) with a collection of (B_K, η_K) but we have no control on the size of K we need to use. I.e., not clear that $\exists\ (B_K, \eta_K)$ that is dominated by every saucer provided by Prop 6.2.

We can only use an overtwisted disk to fill (B_K, η_K) if disk is defined by $K_0 \leq K$.

(Note: in dim = 3, any somewhere-negative K dominates anything up to conjugation $\Rightarrow$ don't have this problem)

But, from lecture #2: Prop 4.9 basically says that, up to conjugation, domination only cares about $K|_{K \geq 0}$.

In particular:

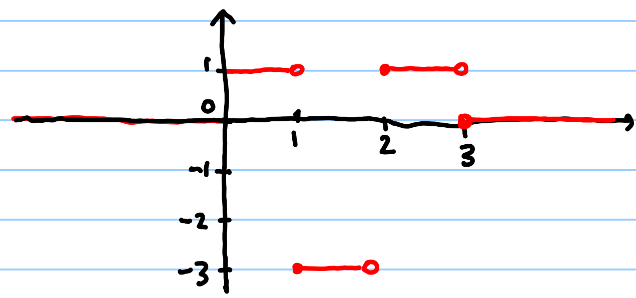


Since $K' > K$ outside Δ', B_K is dominated by $B_{K'}$.

<u>Danger</u>: Prop 6.2 might kick out (B_K, η_K)

where the positive part region $K : \Delta_{cyl} \to \mathbb{R}$ gets arbitrarily small.

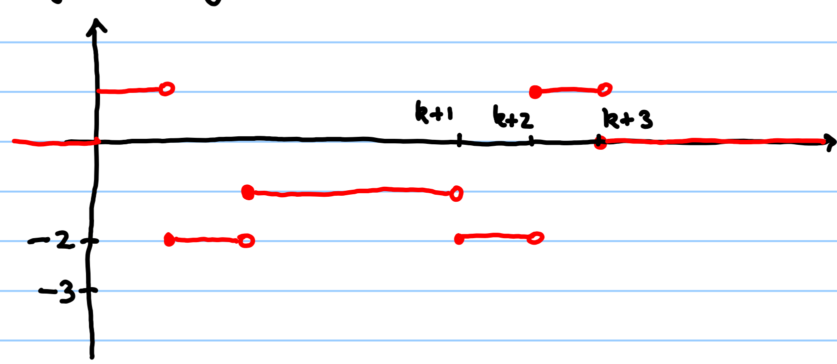
Example behind proof of Prop 8.1.



Consider

$$f_k = \sum_{j=0}^{k} \phi \circ j$$

$$j(x) := x - j$$



f_k whose $\dfrac{\text{pos. region}}{\text{neg. region}} \longrightarrow 0$ as $k \to \infty$

is just a sum of ϕ and its translates.

In higher dimensions:

$$\left(\mathbb{R}^{2n+1}, \ \alpha = dz + \sum_{i=1}^{n-1} (x_i dy_i - y_i dx_i) - y_n dx_n\right)$$

Let $\textcircled{H} \leqslant \mathrm{Cont}(\mathbb{R}^{2n+1})$ generated by

$$\left.\begin{array}{l} T_z : z \mapsto z+1, \ S_{y_j} : (y_j, z) \mapsto (y_j + 1, \ z + x_j) \\ T_{x_n} : x_n \mapsto x_n + 1, \ S_{x_j} : (x_j, z) \mapsto (x_j + 1, \ z - y_j) \end{array}\right\} j = 1, \dots, n-1$$

Everything commutes except

$$[S_{y_j}, S_{x_j}] = T_z^2$$

So everything in Θ can be written as

$$S_{x_1}^{k_1} \cdots S_{x_{n-1}}^{k_{n-1}} S_{y_1}^{l_1} \cdots S_{y_{n-1}}^{l_{n-1}} T_{x_n}^{k_n} T_z^{l_n}$$

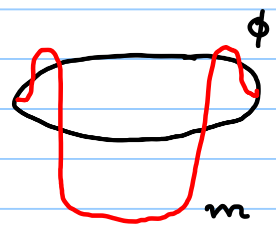
Θ preserves $\Pi = \{y_n = 0\}$, and acts properly discontinuously i.e. $Q \subseteq \Pi$ compact $\Rightarrow$

$$S(Q) = \{g \in \Theta \mid g(Q) \cap Q \neq \phi\}$$

is finite.

Pick $Q \subseteq \Pi$ s.t. $\Theta \cdot Q = \Pi$ (e.g. Q = unit cube).

Enumerate $\Theta = \{g_1, \dots\}$
pick a function



where $m < -(|S(Q)|+1) \max \phi$

why we took -3 in the 1-dim'l example from before.

Define saucers:

$$\tilde{B}_k := \{ \Phi_{k-1} \leq y_n \leq \Phi_k \} \subseteq g_k(Q) \times \mathbb{R}$$

where $\Phi_k = \max(\phi) + \sum_{j=1}^{k} \phi \circ g_j^{-1}$

<u>Lem</u>: Up to equivalence there are only

$2^{|S(Q)|}$ different saucers.

<u>Pf</u>: Look at $\Phi_k \circ g_k : Q \longrightarrow \mathbb{R}$

Have that $= \max(\phi) + \sum_{j=1}^{k} \phi \circ (g_j^{-1} g_k)|_Q$

a summand only contributes when

$g_j g_k^{-1}(Q) \cap Q$

$\underbrace{\text{finite number}}_{|S(Q)|}$ of such j.

For the proof of Prop 8.1: Given $K : \Delta \to \mathbb{R}$ defining B_K, can get finite subset

$\Lambda \subseteq \textcircled{\text{re}}$ and $\Delta' \subseteq \Delta$ w/ $K|_{\Delta \setminus \Delta'} > 0$.

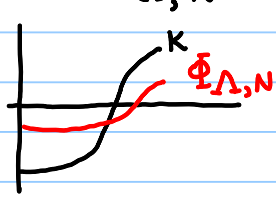
Have

$$\Phi_{\Delta, N} = \frac{\max}{N} + \sum_{g \in \Lambda} \frac{1}{N} \phi \cdot C_N \circ g^{-1}$$

where $C_N(x, y, z) = (Nx, Ny, N^2 z)$

with $\Phi_\Lambda \leq K$ on $\Delta \setminus \Delta'$

and $\Phi_{\Lambda, N} < 0$ on Δ'.



Apply prop to get $B_{\Phi_{\Delta,N}}$ dominated by B_K, and $B_{\phi_{\Lambda,N}}$ decomposes into the saucers $\widetilde{B}_K$ for K coming from list of $2^{|S(Q)|}$ saucers.