Last time:

Prop 3.9: If $K \in \mathcal{K}$ special and $K \leq K_0 : \Delta^{2n-1}_{cyl} \to \mathbb{R}$, then for any contact ball (B, ξ) w/ $(D_K, \eta_K) \leq (B, \xi)$ we have

$$(B_{K_0} \# B, \eta_{K_0} \# \xi)$$

equiv. to genuine cont. str.

Main Lemma:

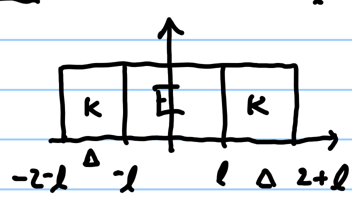
For $K \in \mathcal{K}$ special, $\exists$ a contact isomorphism

$$\textcircled{\omega} : \Delta \longrightarrow \Delta \#_\ell \Delta$$

with 1) $\textcircled{\omega} = (z \mapsto z + 1 + \ell)$ on $z \geq z_D$

2) $\textcircled{\omega}_* K \leq K \# K$

Recall: $K \# K : \Delta \#_\ell \Delta \to \mathbb{R}$



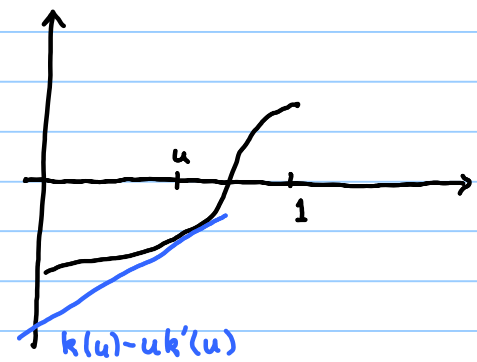
$$E(u) = K(u, \pm 1)$$

Defn: 1) A smooth function $k : [0, \infty) \to \mathbb{R}$ is *special* if $k(1) > 0$ and

$$\textcircled{\star} \qquad a\,k\!\left(\frac{u}{a}\right) < k(u)$$

for all $a > 1$ and $u \geq 0$.

Note ⊛ $\Rightarrow$ $k(u) - u\,k'(u) < 0 \quad \forall\, u$



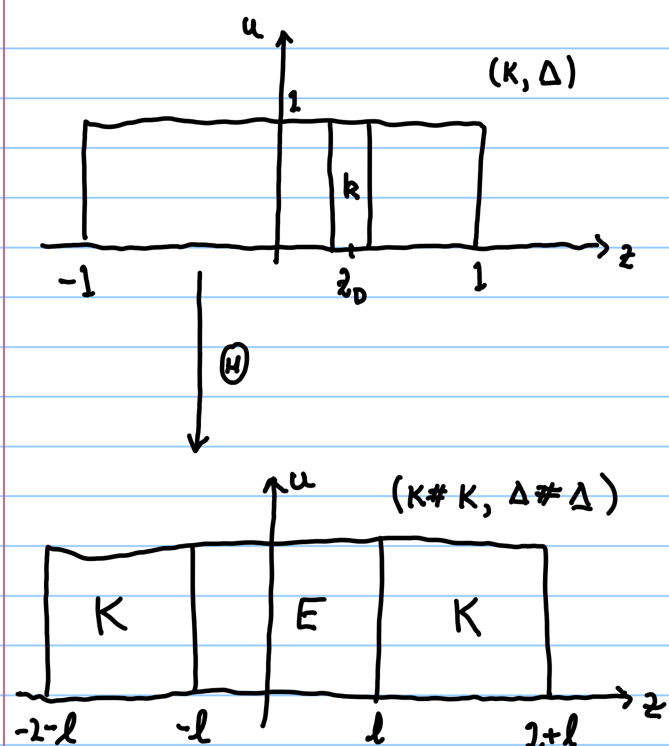
In particular any linear function w/ large slope and $k(1) > 0$ is special.

2) $K = K(u,z) : \Delta_{cyl} \longrightarrow \mathbb{R}$ is <u>special</u> if

$\exists$ a special $k : [0,\infty) \to \mathbb{R}$ where

i) $K(u,z) \geq k(u)$

ii) $K(u,z) = k(u)$ when $z \in Op\{z_0\}$ with $z_0 \in (-1,1)$.



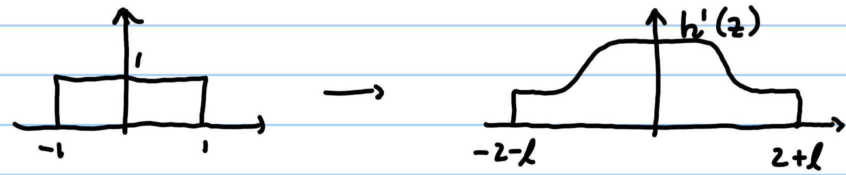
Two contact embeddings

$$\left(\mathbb{R}^{2n-1}, \ \lambda_{st} = dz + \sum u_i \, d\phi_i\right)$$

Scaling: Given $h \in \mathrm{Diff}_+(\mathbb{R})$, define

$$\Phi_h : \mathbb{R}^{2n-1} \circlearrowleft \ \text{by} \ \Phi_h(z, u_i, \phi_i) = (h(z), h'(z)u_i, \phi_i)$$

$\Rightarrow$ can map



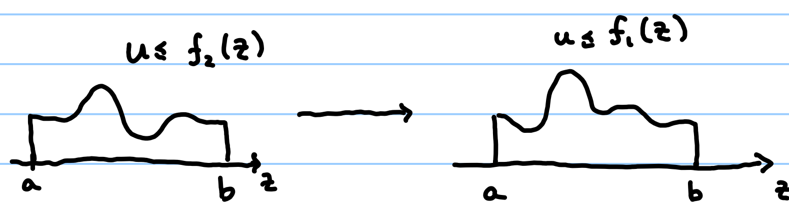
Twisting: $g \in C^\infty(\mathbb{R})$, define

$$\Psi_g(z, u_i, \phi_i) = \left(z, \frac{u_i}{1+g(z)u_i}, \phi_i - \int g(z)\,dz\right)$$

contactomorphism

$$\Psi_g : \{1 + g(z)u > 0\} \xrightarrow{\cong} \{1 - g(z)u > 0\}$$

E.g.

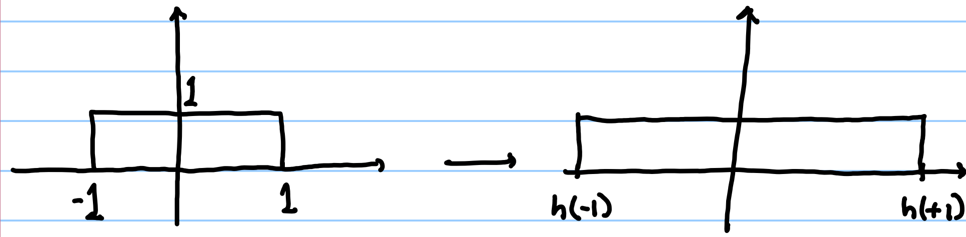


contactomorphic, $g(z) = \dfrac{1}{f_1(z)} - \dfrac{1}{f_2(z)}$.

Together: For $h \in \mathrm{Diff}_+(\mathbb{R})$ let

$$g(z) := 1 - \frac{1}{h'(h^{-1}(z))}$$

$$\Gamma_h = \Psi_g \circ \Phi_h : \{u \leq 1, \ |z| \leq 1\} \to \{u \leq 1, \ z \in [h(-1), h(+1)]\}$$

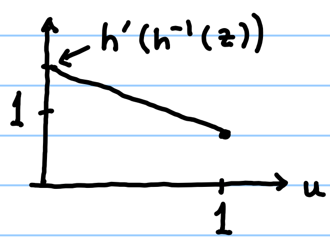


$$\Gamma_h(u,z) = \left(\frac{h'(z)\,u}{1+(h'(z)-1)\,u},\ h(z) \right)$$

$$\left(\Gamma_{h*} K\right)(u,z) = \tilde{h}(u,z)\, K\!\left(\frac{u}{\tilde{h}(u,z)},\ h^{-1}(z) \right)$$

where $\tilde{h}(u,z) = h'(h^{-1}(z)) - \left(h'(h^{-1}(z)) - 1\right)u$

≥ 1 if $h' \geq 1$ and $u \leq 1$

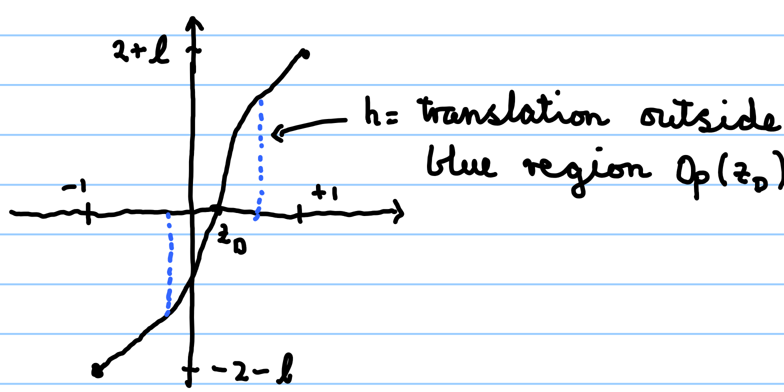


<u>Proof of Lemma</u>

Let $h : [-1, 1] \xrightarrow{\cong} [-2-\ell,\ 2+\ell]$ s.t.

1) $h' \geq 1$ w/ inequality only if

2) $h(1) = 2+\ell$, $h(-1) = -2-\ell$



<u>Compute</u>: where $h'(z) = 1$, have $\tilde{h}(u,z) = 1$

so, have $\Gamma_{h*} K = K(u, h^{-1}(z))$

Near $z = h(z_0)$ have $K = k$ by (ii)

$(\Gamma_{h*} K)(u,z) = \tilde{h}(u,z) \cdot k(u/\tilde{h}(u,z))$

$\leq k(u)$ (as k special)

$\leq K(u,z)$ (by (i))

Altogether, we have that

$$\Gamma_{h*} K \leq \begin{cases} K(u, z \pm (1+\ell)) & \text{at ends} \\ k(u) & \text{in middle} \end{cases}$$

but have $k(u) \leq K(u,z)$

$E(u) = K(u, \pm 1)$

Hence $\Gamma_{h*} K \leq K \# K$.

<u>Completes proof of Prop 3.9</u>: one of the main parts of the theorem.

The other main proposition:

<u>Prop 3.1</u>: For each dimension, exists a universal $K_{univ} : \Delta^{2n-1}_{cyl} \to \mathbb{R}$ s.t.:

any almost contact structure (M^{2n+1}, ξ) is homotopic to another ξ' s.t. $\exists$ disjoint collection of balls $B_i \subseteq M$ $i = 1, \dots, L$ where

1) $\left(M \setminus \bigcup_{i=1}^{L} \operatorname{int} B_i \, , \, \xi\right)$ is genuine

2) (B_i, ξ') equiv't as a contact shell to $(B_{K_{univ}}, \eta_{K_{univ}})$

<u>Outline of proof:</u>

1) Introduce new model of contact shell called contact saucer (B, σ_ϕ)

2) Prove <u>Prop 6.12</u>: Any semi-regular contact saucer (B, σ_ϕ) dominates some (B_K, η_K) for $K : \Delta_{cyl} \to \mathbb{R}$

3) Prove Prop 3.1 with $(B_{K_{univ}}, \eta_{K_{univ}})$

replaced by a finite list (depending only on dimension) of semi-regular contact saucer.

<u>Together prove 3.1</u>: Pick $K_{univ} : \Delta_{cyl} \to \mathbb{R}$

to be such that $(B_{K_{univ}}, \eta_{K_{univ}})$ is

dominated by each saucer in the finite list for (3), using the fact that if $K_1 \leq K_2$ then (B_{K_1}, η_{K_1}) is dominated by (B_{K_2}, η_{K_2}). Then (3) combined with choice of K_{univ} proves Prop 3.1.

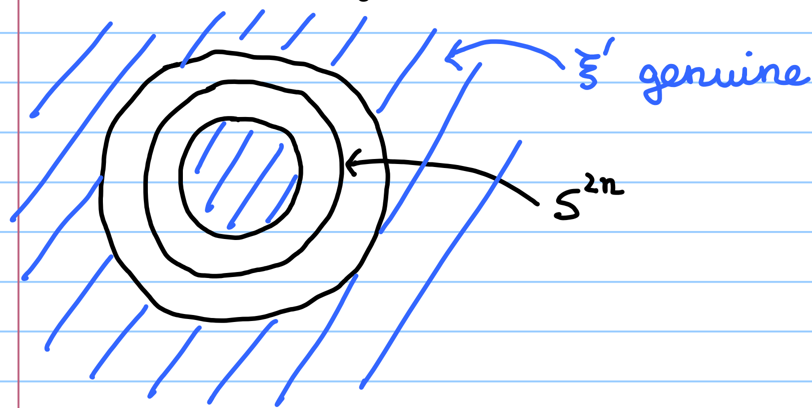
<u>Regular semi-contact structures:</u>

<u>Semi-contact structures</u>: Let $S^{2n} \subseteq (M^{2n+1}, \xi)$
almost-contact

Apply Gromov's h-principle to

$M^{2n+1} \setminus \mathcal{O}p\, S^{2n}$

to get a new almost-contact structure ξ' that is genuine away from S^{2n}.



Consider the annulus $(S^{2n} \times [-1,1], \xi')$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad v$

ξ' is genuine near $r = \pm 1$.

To the family of almost-contact germs on $S^{2n} \times \{v\}$, apply Gromov again to get a smooth family of germs of contact structures on $Op(S^{2n} \times \{v\})$.

Left with a <u>semi-contact structure</u> on the annulus $S^{2n} \times [-1,1]$.

<u>Defn</u>: A semi-contact structure on $\Sigma^{2n} \times [a,b]$ is a smooth family of germs of contact structures on the slices $\Sigma_s = \Sigma \times \{s\}$.

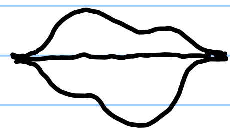
Any semi-contact structure uniquely determines an almost-contact structure on $\Sigma \times [a,b]$.

E.g. Let ξ be a genuine cont. str. on $\Sigma \times \mathbb{R}$. Let $\phi_s : \Sigma \to \mathbb{R}$ be a smooth 1-param. family of functions. Define the germ on Σ_s to be the germ on the graph$(\phi_s) \subset \Sigma \times \mathbb{R}$.

<u>Saucer:</u> Let D^{2n} be a disk

$$B = \{ (w,v) \in D \times \mathbb{R} \mid f_-(w) \leq v \leq f_+(w) \}$$

where $f_- < f_+$ on $\mathrm{int}(D)$, and are equal with ∞-jets on ∂D.



B is foliated by disks

$$D_s = \{ v = (1-s) f_-(w) + s f_+(w) \}$$

A semicontact structure on B is a family of germs ξ_s on D_s, which agree on the common $\partial D_s = \partial D$.

Such a structure gives $(B, \{\xi_s\})$ form of a contact shell.

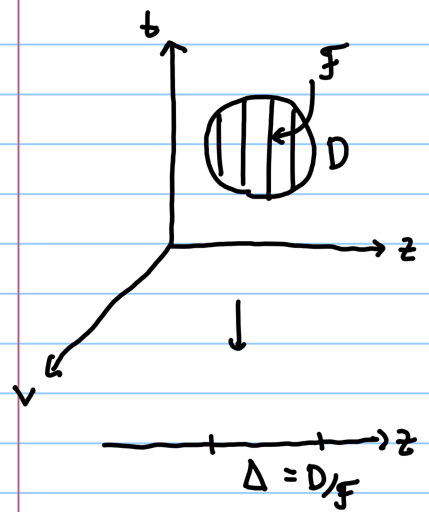
<u>Regular semi-contact saucers</u>

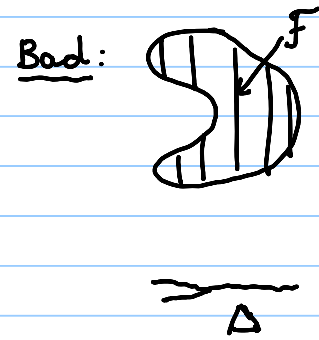
$$\left(\mathbb{R}^{2n+1}, \xi_{st} = \ker(\lambda_{st} + v\,dt)\right) \quad \begin{array}{l} v = -y_n \\ t = x_n \end{array}$$

Let $D^{2n} \subseteq \Pi := \{v = 0\} \subseteq \mathbb{R}^{2n+1}$

be a disk that is regular.

I.e. $\Delta := D/\mathcal{F}$ is a star-shaped contact domain where $\mathcal{F}$ is the characteristic foliation on $\Pi \subseteq \mathbb{R}^{2n+1}$ spanned by ∂_t.

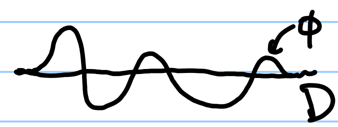




Let $\phi : D \to \mathbb{R}$ smooth w/

1) $\phi > 0$ on $\mathrm{Int}(D) \cap \mathcal{O}p(\partial D)$

2) $\phi = 0$ w/ ∞-jet on ∂D

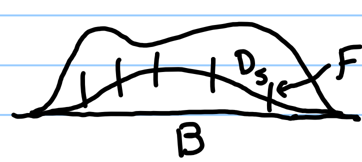


Define the saucer

$$B = \{ (w, v) \in D \times \mathbb{R} \mid 0 \leq v \leq F(w) \}$$

where $F = \phi$ on $\mathcal{O}p(\partial D)$

Give the disk $D_s \subseteq B$ the germ associated to $\mathrm{graph}(s\phi) \subseteq (\mathbb{R}^{2n+1}, \xi_{st})$



The result is a regular semicontact saucer (B, σ_ϕ) up to diffeomorphism depends only on $\phi : D \to \mathbb{R}$.