Overtwisted contact structures 10/27

Domination + conjugation for contact Hamiltonian shells

A contact shell is p.w. smooth ball (B, η)
η almost contact η genuine in $\mathcal{O}p(\partial B)$

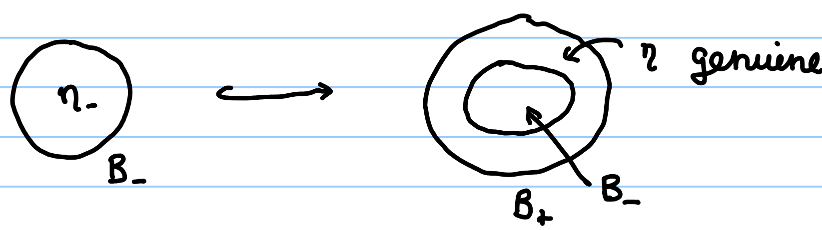
Main issue: when can we homotope η to a genuine contact str.?

Defn: The contact shell (B_-, η_-) is dominated by shell (B_+, η_+) if η_+ can be homotoped rel ∂B_+ to η s.t.

1) $(B_-, \eta_-) \subseteq (B_+, \eta_+)$

as almost contact str.

2) η is genuine contact on $B_+ \setminus B_-$



Lem: If (B_-, η_-) is equivalent to a genuine cont. str., then so is (B_+, η_+).

Recall: $K : \Delta \times S^1 \to \mathbb{R}$ where $\Delta \subseteq \mathbb{R}^{2n-1}$ star-shaped domain

built contact shell (B_K, η_K).

Recall: $(\partial B_K, \eta_K) \cong \tilde{\Sigma}_K = \tilde{\Sigma}_K^{\mathbb{R}^2} \cup \tilde{\Sigma}_K^{T^*S^1}$

$$\tilde{\Sigma}_K^{\mathbb{R}^2} = \{ (x,v,t) : v \leq K(x,t),\ x \in \partial\Delta \}$$

$$\subseteq \partial\Delta \times \mathbb{R}^2$$
$$\lambda + v\, dt$$

$$\tilde{\Sigma}_K^{T^*S^1} = \{ (x,v,t) : v = K(x,t) \}$$

$$\subseteq \Delta \times T^*S^1$$
$$\lambda + v\, dt$$

<u>Order on pairs (K,Δ):</u>

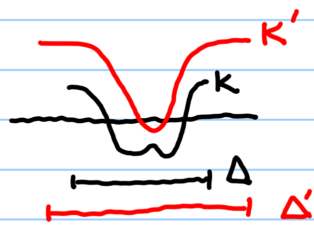
$(K,\Delta) \leq (K',\Delta')$ defined to mean

0) $\Delta \subseteq \Delta'$

1) $K \leq K'$ on $\Delta \times S^1$

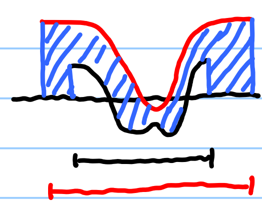
2) $K' > 0$ on $\Delta' \setminus \Delta$

Picture:



<u>Lemma</u>: If $(K,\Delta) \leq (K',\Delta')$ then (B_K, η_K) is dominated by $(B_{K'}, \eta_{K'})$.

<u>Pf</u>: Take the cobordism:



The contact annulus is given by

$$\{K(x,t) \leq v \leq K'(x,t) : x \in \Delta\} \subseteq \Delta \times T^*S^1_{\lambda + v\,dt}$$

$$\cup \{v \leq K'(x,t),\ x \in \Delta' \setminus \Delta\} \subseteq \Delta' \times \mathbb{R}^2_{\lambda + v\,dt}$$

<u>Conjugation for contact Ham. shells</u>

Recall: $(M, \xi = \ker \alpha)$

$$C^\infty(M) \longleftrightarrow \text{contact vector fields}$$

$$K \longmapsto X_K \quad \begin{aligned} &\alpha(X_K) = K \\ &d\alpha(X_K, \cdot) = -dK + dK(R_\alpha)\alpha \end{aligned}$$

$$\alpha(X) \longleftarrow X$$

Claim: Let $\Phi \in \mathrm{Cont}(M,\xi)$ w/ $\Phi^*\alpha = C_\Phi \alpha$

Then $\Phi \varphi^t_{X_K} \Phi^{-1}$ is the flow generated

by $\Phi_* K : M \to \mathbb{R}$ where

$$(\Phi_* K)(\Phi(x)) = C_\Phi(x) \cdot K(x).$$

<u>Lemma</u>: Pushforward of K leads to equivalence of contact shells.

i.e., if $\Phi : \Delta \xrightarrow{\sim} \Delta'$ contactomorphism $(\Delta, \Delta' \subseteq \mathbb{R}^{2n-1})$ then it induces an equivalence

$$\tilde{\Phi} : (B_K, \eta_K) \longrightarrow (B_{\Phi_* K}, \eta_{\Phi_* K})$$

<u>Proof</u>: $\hat{\Phi} : \Delta \times \mathbb{R}^2 \longrightarrow \Delta' \times \mathbb{R}^2$

$$\Delta \times T^*S^1 \longrightarrow \Delta' \times T^*S^1$$

$$(x, v, t) \longmapsto (\Phi(x), C_\Phi(x)v, t)$$

Shows $\hat{\Phi} : (\partial B_K, \eta_K) \xrightarrow{\cong} (\partial B_{\Phi_* K}, \eta_{\Phi_* K})$

So, to show (B_K, η_K) $\quad (K, \Delta)$
dominated by $(B_{K'}, \eta_{K'})$ $\quad (K', \Delta')$

suffices to find contact embedding

$$\Phi : \Delta \hookrightarrow \Delta'$$

with

$$(\Phi_* K, \Phi(\Delta)) \leq (K', \Delta')$$

<u>Highlight in 3-dim case</u>:

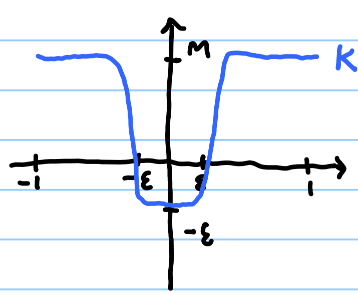
$$\Delta = [-1, 1] \subseteq \mathbb{R} \qquad (n=1)$$

<u>Lemma</u>: If $K : \Delta \to \mathbb{R}$ is negative somewhere and $K' : \Delta \to \mathbb{R}$ is anything, then

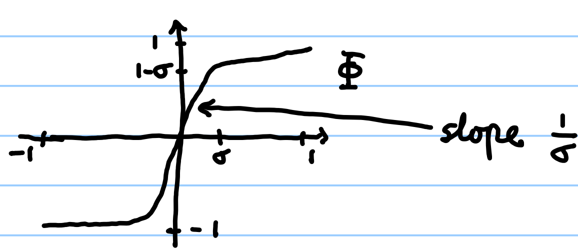
$$(B_K, \eta_K) \text{ is dominated by } (B_{K'}, \eta_{K'}).$$

I.e. in 3-dim case, any somewhere-negative Ham. is minimal w.r.t. the partial order (up to conjugation).

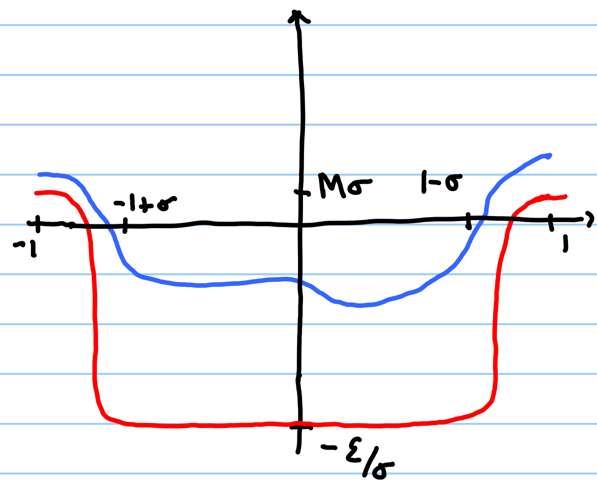
<u>Proof</u>:



WLOG K looks like this for some M and $\varepsilon > 0$.



$(\Phi_* K)(\Phi(x)) = \Phi'(x)\, K(x)$



Rmk. unknown if lemma is true in higher dimensions. Probably answer either way is interesting!

<u>What remains in higher dimensions:</u>

Motto: up to conjugation the partial order $\leq$ only sees $K|_{\{K \geq 0\}}$

i.e. size of positive part.

<u>Prop</u>: For $K_0, K_1 : \Delta \to \mathbb{R}$ we have (B_{K_0}, η_{K_0}) dominated by (B_{K_1}, η_{K_1})

if $\exists$ starshaped domain $\tilde{\Delta} \subseteq \operatorname{int} \Delta$:

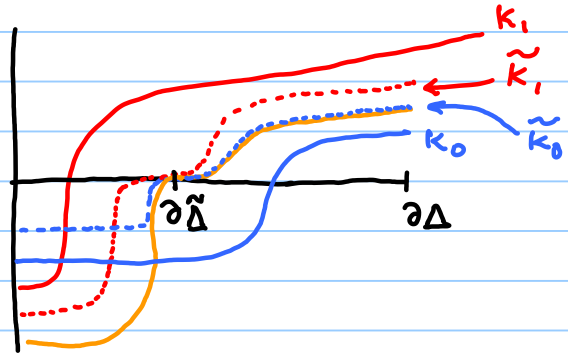
1) $K_0 \leq K_1$ on $\operatorname{Op}(\Delta \setminus \operatorname{int} \tilde{\Delta})$

2) $0 \leq K_1$ on $\operatorname{Op}(\partial \tilde{\Delta})$

3) $K_0 \leq 0$ on $\operatorname{Op}(\partial \tilde{\Delta})$ w/ $K_0|_{\tilde{\Delta}} \neq 0$.

<u>Proof</u>: Take $\tilde{K}_1 \leq K_1$ and $\tilde{K}_0 \geq K_0$ s.t.
1) $\tilde{K}_1 \geq \tilde{K}_0$ outside $\tilde{\Delta}$

2) $\tilde{K}_1$ and $\tilde{K}_0 \equiv 0$ on $\mathcal{O}p\, \partial \tilde{\Delta}$



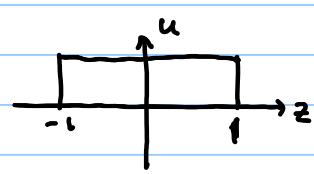
using a radial rescaling contactomorphism Φ supported in $\mathrm{int}(\tilde{\Delta})$ can get

$$\Phi_*(\tilde{K}_0) \leq \tilde{K}_1.$$

Using overtwisted disks to fill contact Hamiltonian holes

$$\Delta = \Delta_{cyl} = \{ |z| \leq 1, \ u \leq 1 \} \subseteq \mathbb{R}^{2n-1}$$

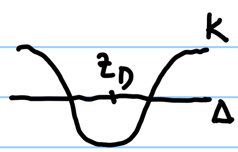
$u = u_1 + \dots + u_{n-1}$

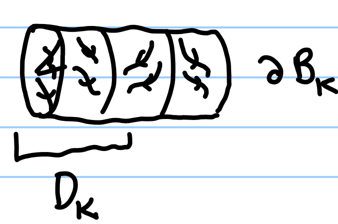


Let $K : \Delta \longrightarrow \mathbb{R}$ with $K = K(u, z)$

$$K(0, z_D) < 0 \quad \text{for } z_D \in (-1, 1)$$

$$D_K = \{ (x, v, t) \in \partial B_K \mid z(x) \in [-1, z_D] \}$$





Prop 3.9: If $K \in \mathcal{K}$ is *special* then for any

$K_0 : \Delta \to \mathbb{R}$ with $K_0 \geq K$ and
(B, ξ) a contact ball with $D_K \subseteq (\partial B, \xi)$

we have $(B_{K_0} \# B, \eta_{K_0} \# \xi)$ is equivalent
to genuine contact structure.

Boundary connected sum for contact shells

Let (W^{2n+1}, η) contact shell with
embedding $i : D^{2n} \hookrightarrow \partial W$ s.t. $i^* \alpha = \theta$,
where

$$D^{2n} \subseteq (\mathbb{R}^{2n}, \theta = \Sigma u_i \, d\phi_i)$$

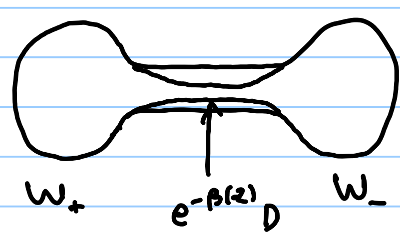
starshaped and α defined η near ∂W
(call this a *gluing place*).

Lemma: Given gluing places $i_\pm \, D^{2n} \hookrightarrow (W_\pm, \eta_\pm)$

can form abstract boundary connect
sum

$$(W_+ \# W_-, \eta_+ \# \eta_-) =$$

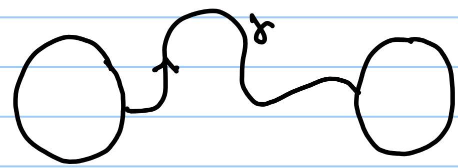
$$W_+ \cup \underbrace{\{ D \times [-1, \ell]_z \}}_{dz + \theta} \cup W_- \Big/ \sim$$



also works

so we can identify this abstract
connect sum with ambient connect sum

as long as we have a path γ transverse to contact hyperplane:

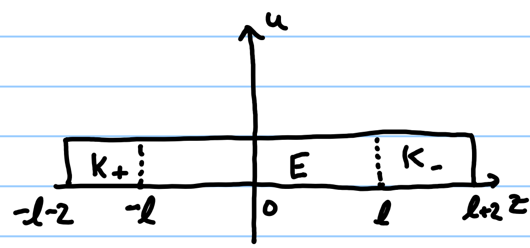


(by making the neck suff. narrow)

Boundary connect sum for Hamiltonian shells

$$K_\pm : \Delta \longrightarrow \mathbb{R} \quad \text{w/} \quad K_+(u,+1) = K_-(u,-1) =: E(u)$$

$$K_+ \# K_- : \Delta \#_\ell \Delta \longrightarrow \mathbb{R}$$



Lemma: Can identify

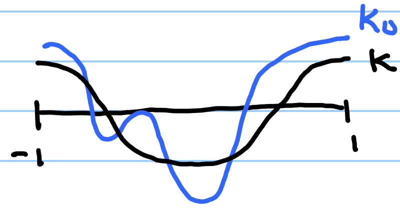
$$(B_{K_+\# K_-}, \eta_{K_+\# K_-}) \cong (B_{K_+} \# B_{K_-}, \eta_{K_+} \# \eta_{K_-})$$

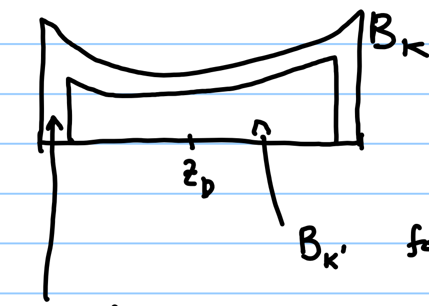
Sketch proof of prop 3.9 in dim 3:

$\Delta = [-1,1]$ and $K : [-1,1] \longrightarrow \mathbb{R}$ neg. at $z_0 \in [-1,1]$. For any other $K_0 : [-1,1] \to \mathbb{R}$ we have

$$(B_{K_0} \# B_{K_1}, \eta_{K_0} \# \xi)$$

equiv. to genuine cont. struc.

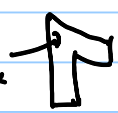




for $K' = K - \varepsilon$ on $[-1+\varepsilon, 1-\varepsilon]$

(A, ξ) genuine cont. str.

Let $(B, \xi) = \{ (z, v, t) \in A : z \in [-1, z_D] \}$

D_K  contact ball

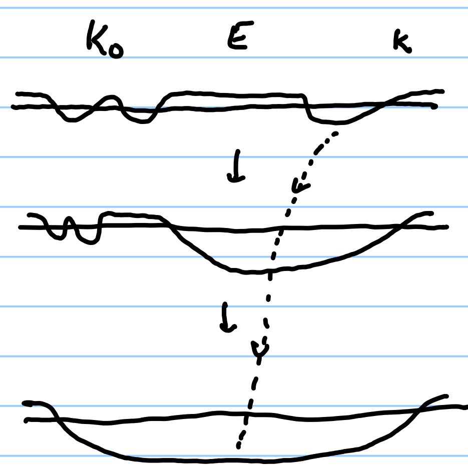
Easy in dim = 3: build a family of embeddings

$\Theta_\sigma : [-1, 1] \longrightarrow [-2-\ell, 2+\ell]$ s.t.

1) $\Theta_0 = (z \mapsto z + 1 + \ell)$

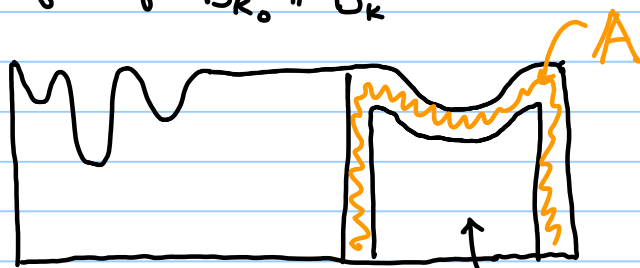
2) $\Theta_\sigma = (z \mapsto z + 1 + \ell)$ if $z \geqslant z_D$

3) $\Theta_{1*} K' \leqslant K_0 \# K$ (hard bit)



$\Delta \# \Delta$

From this family Θ_σ build an isotopy Ψ_σ of $B_{K_0} \# B_K$



so that

1) $\Psi_0 = \mathbb{1}$ on $(z \geq z_D + 1 + \ell)$

2) $\Psi_\sigma \circ \hat{\Theta}_0 = \hat{\Theta}_\sigma \cdot B_K \to B_{K_0} \# B_K$

3) $\Psi_1 (B_{K_0} \# A) = B_{K_0 \# K} \setminus (\mathrm{Int}\ \Theta_{1*} B_{K'})$

<u>Have</u>: $\xi_\sigma = \Psi_\sigma^* (\eta_{K_0 \# K})$

family of almost contact structures on $B_{K_0} \# B$ w/

$$\xi_1 = \Psi_1^* (\eta_{K_0 \# K})$$

a genuine cont. str.