Overtwisted contact structures

10/20

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M^{2n+1} smooth mfld

Contact: ξ hyperplane field conformal symp. struc.

$\xi = \ker \alpha$ css def. by $d\alpha$.

Almost contact: same but now $\xi = \ker \alpha$, css defined by ω, ω nondeg 2-form on ξ

convention: $X \subseteq M$
$\mathcal{O}_p X$ open nbhd of X

Thm 1 (existence): Let ξ be an almost contact structure, A closed subset of M, and ξ genuine contact structure on $\mathcal{O}_p A$.
Then ξ is homotopic rel A to a genuine contact structure.

Later we will define overtwisted contact structures

$$\mathrm{Cont}_{OT}(M; A, \xi_0) = \{\text{overtwisted contact structures that agree with } \xi_0 \text{ on } \mathcal{O}_p A\}$$

$$\mathrm{cont}(M; A, \xi_0) = \{\text{almost contact structures agree with } \xi_0 \text{ on } \mathcal{O}_p A\}$$

$A \subseteq M$ closed ξ_0 is almost contact, genuine on $Op\, A$.

<u>Thm 2</u> (parametric):

The inclusion $\mathcal{C}ont_{OT}(M, A, \xi_0) \hookrightarrow cont(M, A, \xi_0)$

is an isomorphism on π_0.

(More can be said about π_j if you fix more data).

<u>Gromov's h-principle</u>:

If V is an open manifold, then

$$\mathcal{C}ont(V) \hookrightarrow cont(V)$$

is a homotopy equivalence.

<u>Corollary</u>: Take a triangulation of M, a closed manifold w/ a.c.s. ξ. Then we can deform ξ so that it is genuine on $Op(\partial B)$ for each B^{2n+1} top-dim'l simplex, by applying to $V = Op(2n\text{-skeleton})$.

Left to understand how to extend ξ to int B as a genuine contact structure.

<u>Defn</u>: Let B be a $(2n+1)$-dim'l domain with piecewise smooth ∂. A <u>contact shell</u> (B, ξ) is an a.c.s. ξ s.t. ξ is genuine on $Op(\partial B)$.

An equivalence of contact shells
$g: (B_1, \xi_1) \longrightarrow (B_2, \xi_2)$

is a diffeo s.t. $g_* \xi_1 = \xi_2$ on $Op(\partial B)$ and $g_* \xi_1$ homotopic to ξ_2 rel $Op(\partial B)$.

So, proving Thm 1 reduces to:

find a way to reduce ξ a.c.s. to a collection of contact shells, each equivalent to a genuine contact structure.

<u>Hamiltonian contact shells:</u>

$$(\mathbb{R}^{2n-1}, \xi_{st} = \ker \lambda_{st}), \quad \lambda_{st} = dz + \sum_{i=1}^{n-1} u_i \, d\phi_i$$

(here $u_i = r_i^2$, (r_i, ϕ_i) polar coords).

A domain $\Delta^{2n-1} \subset (\mathbb{R}^{2n-1}, \xi_{st})$ is <u>star-shaped</u> if it is contactomorphic to a domain where

$$Z = z\partial_z + \sum u_i \partial_{u_i}$$

is transverse to ∂.

Given $K : \Delta \times S^1 \to \mathbb{R}$ s.t. $K|_{\partial\Delta \times S^1} > 0$

we will build a contact shell (B_K, η_K) of dim $2n+1$.

<u>Defn</u>: Let $C \in \mathbb{R}$ s.t. $K + C > 0$

$$B_{K,C} := \{ (x, v, t) \in \underset{x}{\Delta} \times \underset{v,t}{\mathbb{R}^2} : v \leq K(x,t) + C \}$$

polar coords, $v = r^2$

Pick a family of functions

$$\rho_{(x,t)} : \mathbb{R}_{\geq 0} \to \mathbb{R} \quad \text{for } (x,t) \in \Delta \times S^1$$

1) $\rho(0) = 0$

2) $\rho_{(x,t)}(v) = v - C$ on $\mathcal{O}p\{v = K(x,t)\}$

3) $\partial_v \rho(x,t) > 0$ on $\mathcal{O}p\{x \in \partial\Delta\}$

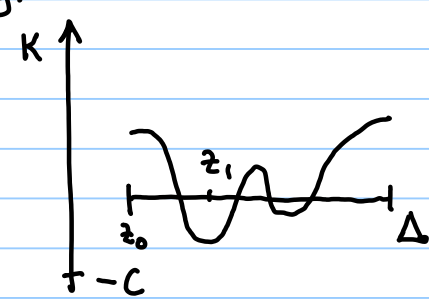
<u>Lemma:</u> $\eta_{K,\rho} := \alpha_\rho - \lambda_{st} + \rho\, dt$ (1-form)

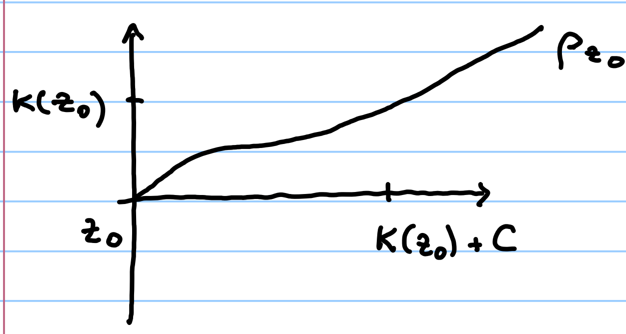
$\omega = d\lambda_{st} + dv\, dt$ (c.s.s.)

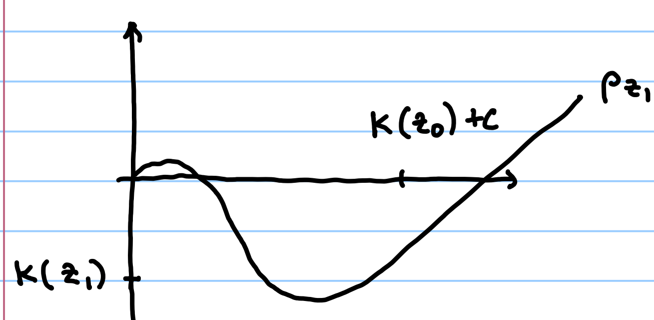
defines a contact shell on $B_{K,C}$.

Moreover, $(B_{K,C}, \eta_{K,\rho})$ up to equivalence is indep't of C and ρ.

E.g.







Note: $\alpha_\rho \wedge (d\alpha_\rho)^n > 0$

iff $\partial_v \rho > 0$

So α_ρ is only contact when

$\partial_v \rho > 0.$

The contact germ of $(\partial B_K, \eta_K)$ (indep't of ρ)

$\widetilde{\Sigma}_K = \widetilde{\Sigma}_{1,K} \cup \widetilde{\Sigma}_{2,K}$ where

$$\widetilde{\Sigma}_{1,K} = \{(x,v,t) : v = K(x,t)\}$$
$$\subseteq \left(\underset{x}{\Delta} \times \underset{v,t}{T^*S^1},\ \ker(\lambda_{st} + v\,dt)\right)$$

$$\widetilde{\Sigma}_{2,K} = \{(x,v,t) : x \in \partial\Delta,\ v \leq K(x,t)\}$$
$$\subseteq \left(\Delta \times \mathbb{R}^2,\ \ker(\lambda_{st} + v\,dt)\right)$$

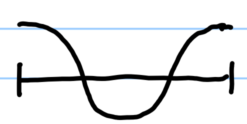
Note the characteristic foliation on $\widetilde{\Sigma}_K$ is given by

$-\partial_t + X_K$ on $\widetilde{\Sigma}_{1,K}$

$v\partial_v$ on $\widetilde{\Sigma}_{2,K}$

X_K is the contact vector field on Δ generated by $K : \Delta \times S^1 \to \mathbb{R}$.

E.g. $\Delta =$ interval, $\lambda_{st} = dz$ $(X_K = K\partial_z)$

 Δ

piecewise
smooth
2-dim'l
overtwisted
disk.

<u>Lemma</u>: the contact germ on $\tilde{\Sigma}_K$ is contactomorphic to the contact germ on $(\partial B_K, \eta_K)$.

Usually: take

$$\Delta_{cyl} = D^{2n-2} \times [-1,1] = \{u \le 1, |z| \le 1\}$$
$$u = \sum_{i=1}^{n-1} u_i \qquad \subseteq \mathbb{R}^{2n-1}$$

<u>Outline of existence proof</u>:

<u>Step 1</u>: Reduction to universal contact shells.

<u>Step 2</u>: Use OT-disk to fill these universal contact shells.

<u>Prop 3.1</u>: For each dimension, there is a universal function $K: \Delta_{cyl} \to \mathbb{R}$ s.t.: any almost contact ξ that is genuine on $\mathcal{O}p\, A$ is homotopic rel A to a structure ξ' with the following properties:

There are a finite number of disjoint

$B_i^{2n+1} \subseteq M \setminus A$

for which

1) ξ is genuine on $M \setminus \bigcup_{i=1}^{L} B_i$
2) (B_i, ξ') equivalent to

$$(B_{K_{univ}}, \eta_{K_{univ}}).$$

Cartoon summary: after homotopy ξ looks like

failure to be contact isolated to fixed model $B_{K_{univ}}$

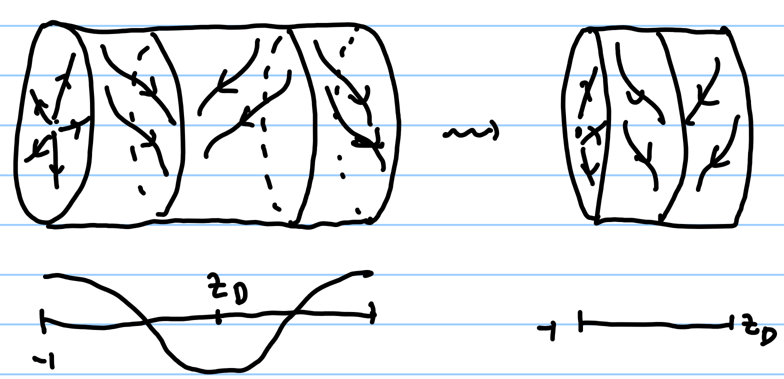
<u>Overtwisted disks</u>

Given $K : \Delta_{cyl} \to \mathbb{R}$ with

$K = K(u, z) \qquad (u = \sum u_i)$

and $K(0, z_D) < 0$, let

$$D_K = \{ (x, v, t) \in \partial B_K \mid z(x) \in [-1, z_D] \}$$



There is a special class of functions

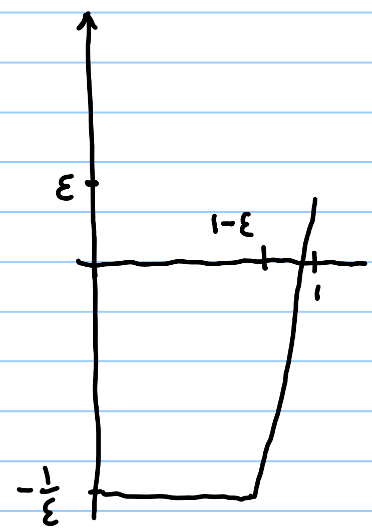
$$\mathcal{K} = \{ K : \Delta_{cyl} \to \mathbb{R} \}$$

<u>Defn</u>: An overtwisted disk

$$(D^{2n}_{OT}, \eta_{OT}) \cong (D_K, \eta_K)$$

where $K \in \mathcal{K}$ and $K \lesssim K_{univ}$

E.g. $K_\varepsilon : [0,1] \to \mathbb{R}$



Define: $K_\varepsilon(u,z) = \max\{ K_\varepsilon(u), K_\varepsilon(|z|) \}$

this is in $\mathcal{K}$ special Hamiltonians.

<u>Prop 3.9.</u> Let (B, ξ) be a contact ball with $(D_K, \eta_K) \subseteq (\partial B, \xi)$ for $K \in \mathcal{K}$ special. Then for any

$$K_0 : \Delta_{cyl} \to \mathbb{R}$$

with $K_0 > K$ have the boundary connected sum

$$(B_{K_0}, \eta_{K_0}) \# (B, \xi)$$

is equivalent to a genuine contact

structure.

Now, proof of Thm 1 (Existence):

Goal: homotope an a.c.s. to a genuine contact structure.

First apply Gromov's h-principle to homotope ξ s.t. exists $B \subseteq M$ with

$$(B, \xi) \text{ contact and } (D_{OT}, \eta_{OT}) \subseteq (\partial B, \xi).$$

Applying Prop 3.1 with $A = B$, we have that ξ is only not contact on finite # of $B_i \subset M \setminus B$ with

$$(B_i, \xi) \cong (B_{K\,univ}, \eta_{K\,univ}).$$

Apply Prop 3.9. to homotope a.c.s. on

$$(B_{K\,univ}, \eta_{K\,univ}) \# \cdots \# (B_{K\,univ}, \eta_{K\,univ}) \# (B, \xi)$$

rel its boundary to a genuine contact structure.