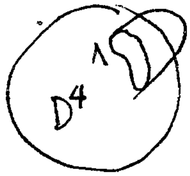
Combinatorics in 3d



$\Lambda \subset (S^3, \xi_{std})$ Legendrian knot

X = Legendrian surgery along Λ, filling contact mfd Y.

$S^3_\Lambda = Y = \partial X$

What is (depends on the filling X).
$\hookrightarrow$ $LH(X)$ and $SH(X)$?

To what extent does this depend on X? It depends on the augmentation you get.

There should be a spectral sequence

$$\mathrm{Sym}^\cdot(CH(X)) \Rightarrow \text{full contact homology (of } Y)$$

$\uparrow$ doesn't necessarily converge

How many augmentations can there be? We just know there <u>can</u> be more than 1.

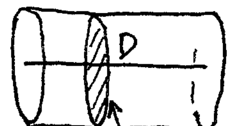
~~What is $LH(X)$ (linearised contact homology of Y w.r.t. filling X),~~

~~and $SH(X)$ (symplectic homology of X)?~~

More examples (enter Yasha stage right) to show how $LH(X)$ depends on the filling X of Y.

$L(4,1)$ has 2 fillings, (McDuff) $\mathbb{CP}^2 - Q$ = quadric $= E = T^*\mathbb{RP}^2$

$\Rightarrow SH = H^{-*}(\mathcal{L}\mathbb{RP}^2)$



one filling is $\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \{Q\}$

$[Q] = (2,1) \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1)$

take $\mathbb{CP}^1$ tangent to Q.
$\Rightarrow$ get $\underline{\underline{1}}$ disc.

augmentation comes from disc at minimum (Morse-Bott)

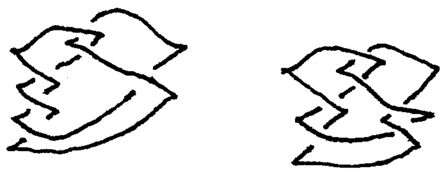
Get $\underline{\underline{2}}$ discs

$SH = 0$

Or (enter Luís): cheaply, S^1 can be filled by $\mathbb{D}^2$, in which the Reeb orbit bounds 1 disc, or by a higher-genus surface, in which case it bounds 0 discs $\Rightarrow$ get different augmentations.

Computations: SH $\Lambda =$  unknot

SH $\Lambda =$  stabilisations

LH  Chekanov

Setup: $\Lambda \subset \mathbb{R}^3$ Reeb chords $c_1, \dots, c_k$

$LH(\Lambda) =$ free noncomm. unital algebra / $\mathbb{Q}$ gen. by $c_1, \dots, c_k$

$d_{LH} : LH \to LH$ $\quad d_{LH}(c) = \sum_{\dim \mathcal{M}/\mathbb{R} = 0} \# \mathcal{M}(c; c_{j_1} \dots c_{j_\ell})/\mathbb{R} \; c_{j_1} \dots c_{j_\ell}$

$H(LH, d_{LH}) = L\mathbb{H}(\Lambda)$.

This can be computed combinatorially in the x-y plane (Chekanov)

E.g.  $\Rightarrow d(c) = 1 - 1 = 0$

Why can we pass from S^3 to $\mathbb{R}^3$?
Draw S^3 as $\frac{|z_1|^2}{1} + \frac{|z_2|^2}{a_2} = 1$

Put ~~Λ~~ our knot near the long Reeb orbit.
$\to$ by index argument nothing goes near ∞.
(Reeb chords escaping a Darboux chart must be very long, hence have large index by monotonicity, so we can discard them).

$LH(\Lambda) = \mathcal{A}$

Consider $\mathcal{A} \big/ \langle [\mathcal{A}, \mathcal{A}], \langle 1 \rangle \rangle$ ← not the ideal generated by this $\Rightarrow$ cyclic words.

$$c_1 (c_2 \dots c_k) = (c_2 \dots c_k) c_1$$

d_{LH} descends to $d_{cyc} : LH^{cyc} \to LH^{cyc}$

$$LH^{cyc}(\Lambda) = H(LH^{cyc}(\Lambda), d_{cyc})$$

If you do $\mathbb{R}^3_\Lambda$, one term in the exact sequence is $CH(\mathbb{R}^3) = 0 \Rightarrow$ you get an isomorphism.

$$\begin{tikzcd} CH(\mathbb{R}^3_\Lambda) \arrow[rr] & & CH(\mathbb{R}^3) = 0 \arrow[dl] \\ & LH^{cyc}(\Lambda) \arrow[ul, "\cong"] & \end{tikzcd}$$

If you do it for S^3, $CH(S^3)$ need not be 0.

You need to compute the map β_φ:

$$\begin{tikzcd} CH(S^3_\Lambda) \arrow[rr] & & CH(S^3) \arrow[dl, "\beta"] \\ & LH^{cyc}(\Lambda) \arrow[ul] & \end{tikzcd}$$

$CH(S^3)$: (for the example below lives in degrees $3, 5, 7, 9, \dots$)

β: deg $= -1$

$LH^{cyc}(\Lambda)$: (lives in degrees $1, 3, 5, 7, \dots$ for the example below)

E.g.  Λ

$LH(\Lambda)$ generated by $c, c^3, c^5, \dots$ because $c^2 = -c^2 \Rightarrow c^2 = 0$
(and $d_{LH} = 0 \Rightarrow$ all these survive).

$\Rightarrow$ by index reasons, β must be 0.

~~Project $\Lambda \subset S^3$ to S^2 via Hopf fibration~~

~~Symplectic homology:~~

Surgery of B^4 along trefoil gives a torus bundle - could one compute CH or SH?

Symplectic homology:

$$LH^{\widetilde{Hoch}}(\Lambda) = \cancel{LH} \; \widehat{\text{Words}} \oplus \left(\widecheck{\text{Words}} \oplus \langle \tau \rangle\right)$$

empty word with $\vee$

words of Reeb chords

$\hat{c}_1 c_2 \ldots c_m$ $\qquad$ $\check{c}_1 c_2 \ldots c_m$ $\qquad$ $\deg = 0$

degree $= |c_1| + \ldots + |c_m| + 1$ $\qquad$ $|c_1| + \ldots + |c_m|$

$$d^{\widetilde{Hoch}} = \begin{pmatrix} \hat{\partial} & 0 \\ d_{Morse} & \check{\partial} \end{pmatrix}$$

$\check{\partial}(\check{c}_1 \ldots c_m) = {}^{\vee}(\partial c_1 \ldots c_m)$ (i.e. put check on first word)

$\check{\partial}\tau = 0$ (τ = empty word)

One way to think of it is as a cyclic word with a placeholder

$$\hat{\partial}(\hat{c}_1 \ldots c_m) = \widehat{(\partial c_1)}\, c_2 \ldots c_m + (-1)^{|\hat{c}_1|}\, \hat{c}_1 (\partial(c_2 \ldots c_m))$$

where we define $\widehat{b_1 \ldots b_s} = \hat{b}_1 b_2 \ldots b_s + b_1 \hat{b}_2 \ldots b_s + \ldots + b_1 \ldots \hat{b}_s$.

$$\partial_{Morse}(\hat{c}_1 \ldots c_m) = \check{c}_1 \ldots c_m - c_1 \check{c}_2 \ldots \check{c}_m$$

You should think of $\wedge$ as maximum, $\vee$ as minimum in Morse Bott.

Thm: $\left(d^{\widetilde{Hoch}}\right)^2 = 0$, and its homology is $SH(X)$.

$$\underset{\substack{\uparrow \\ =0 \text{ if } X_0 \text{ is subcritical.}}}{SH(X_0)} \oplus LH^{\widetilde{Hoch}}(\Lambda)$$
$$\cong$$
$$SH(X) \qquad (\tau \mapsto \text{crit. point in handle})$$

E.g. Unknot 

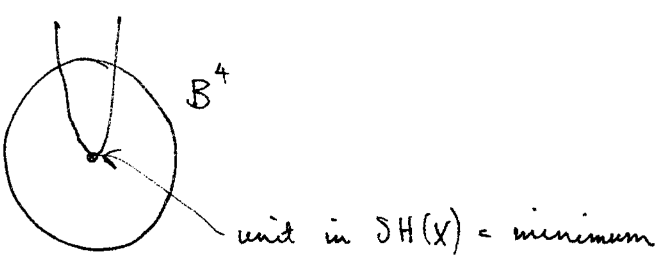
	τ	0
	$\check{a}$	1
$\check{a}^{*}a$	$\hat{a}$	2
$\hat{a}a \xrightarrow{\times 2} \check{a}^{*}a$	$\check{a}a^2$	3
	$\vdots$	
	$\check{}a^k$	k
	$\hat{}a^k$	$k+1$

d_{Morse} (all others 0)

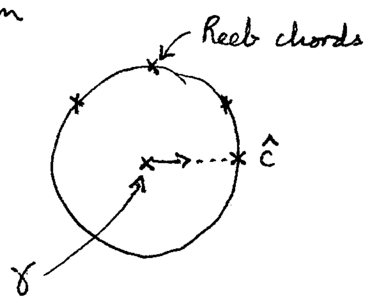
We should be able to identify the unit in the ring structure of $SH(X)$. In this case it is easy for degree reasons.

$\hat{a} \longleftarrow$ unit of $SH(X)$ (in degree 2).

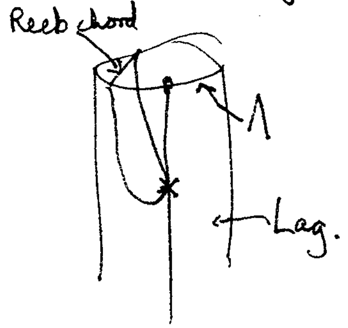
unit $= \hat{a}$.



The map is given by counting



The unit is given (morally) by counting



$\mathcal{H}$ corresponds to the fundamental class of Λ. (conjecturally).

Comment (Paul): count polygons in Chekanov homology with a power of ϵ when it hits a marked point on Λ $\Rightarrow$ get a deformation of $LCH \Rightarrow$ an element of $HH^*(LCH(\Lambda)) \Rightarrow \ldots$?

Reformulation: suppose Λ bounds $L \subset B^4$. Then you can use the augmentation.

<u>Claim</u>: If $\underbrace{\Lambda \text{ is stabilised}}$ then $SH(X) = 0$.

~~version~~ (i.e. if each component is stabilised. If you just stabilise one, then it's as if you erased it).

X = 4-ball with handle attached along Λ.

$$\begin{array}{ccc} LH(\Lambda) & \rightsquigarrow & \widetilde{LH}^{Hoch}(\Lambda) \\ \uparrow & & \uparrow \\ \text{call this } (\mathcal{A}, d) & & \cancel{\text{call this } (\tilde{\mathcal{A}}, d)} \\ c_1, \ldots, c_m & \rightarrow & c_1, \ldots, c_m, \hat{c}_1, \ldots, \hat{c}_m, x \end{array}$$

$|x| = 0$

$|\hat{c}_i| = |c_i| \mp 1$.

define complex $(\tilde{\mathcal{A}}, d)$

$$dx = 0$$

$$d(c_i) = \text{as before}$$

$$d(\hat{c}_i) = \widehat{(dc_i)} + \underbrace{xc_i - c_i x}_{d_{Morse}}$$

$\rightsquigarrow$ complex $(\tilde{\mathcal{C}}, d)$ = cyclic words in $\tilde{\mathcal{A}}$ of total order = 1 in $\wedge + x$.

$\tau \longleftrightarrow x \qquad \check{c}_1 \ldots c_k \longleftrightarrow c_1 x c_2 \ldots c_k$

Legendrian isotopy of Λ changes $(\mathcal{A}, d)$ by stable tame isomorphism
$\Rightarrow$ changes $(\hat{\mathcal{A}}, d)$ " " " "
$\Rightarrow$ changes $\tilde{\mathcal{E}}, d$ by quasi-isomorphism.

We postpone losing the algebra structure to as late as possible.

If we stabilise Λ, $(\mathcal{A}, d)$ for $LH(\Lambda)$ is tamely isomorphic to

$$\partial(c_1) = 1$$
$$\partial(c_2) = 0$$
$$\vdots$$
$$\partial(c_m) = 0$$

$\Rightarrow$ if you calculate $(\tilde{\mathcal{E}}, d)$ you get something trivial.

$\Big(d(c_1 a) = a + c_1 \underbrace{da}_{=0}$ but only after quotienting by cyclic permutation $\Rightarrow$ be careful, but works $\Big)$

3d combinatorics II

Plan: $\Lambda \subset S^3$ or $\mathbb{R}^3$

$\rightsquigarrow LH(\Lambda)$ combinatorially

$\rightsquigarrow SH(X)$

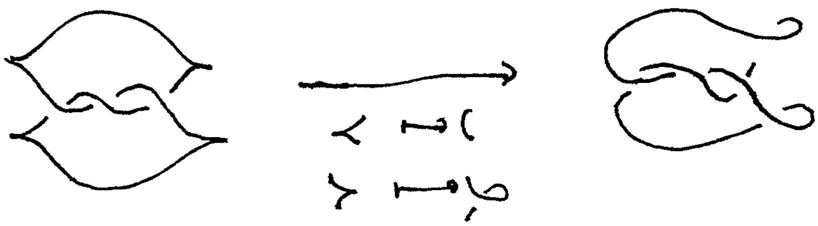
Now:

$\Lambda \subset \#(S^1 \times S^2)$'s

$\rightsquigarrow LH(\Lambda)$ – how to do this combinatorially?

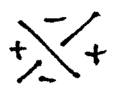
$\rightsquigarrow SH(X)$

In $\mathbb{R}^3$: $\alpha = dz - y\,dx$

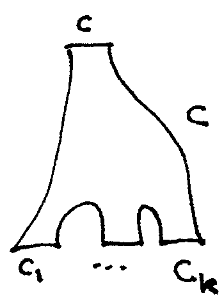


x-y projection of Λ

$\Pi_{xy}(\Lambda)$

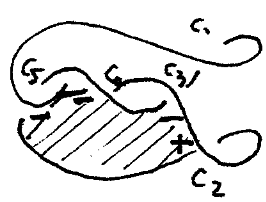
decorate crossings 

$$\mathcal{M}(\underset{+}{c}; \underbrace{c_1, \dots, c_k}_{-}) =$$



$\subset \mathbb{R} \times \Lambda$

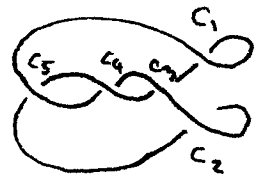
= immersed disks with ∂ on $\Pi_{xy}(\Lambda)$ and convex corners + at c, – at $c_1, \dots$

E.g. 

contributes a term $c_3 c_4 c_5$ to ∂c_2.

$$\partial c = \sum \# \mathcal{M}(c; c_1 \dots c_k)/_{\mathbb{R}} \; c_1 \dots c_k$$

In our example



$$dc_1 = 1 + c_3 + c_5 + c_5 c_4 c_3$$

$$dc_2 = 1 + c_3 + c_5 + c_3 c_4 c_5$$

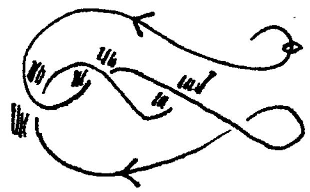
$$dc_3 = dc_4 = dc_5 = 0$$

(Morse differential kills all bad orbits)

Signs:   (orient knot)

no shading
($\mp$ crossing)

So in our case



($tb = 1$, $r = 0$)

$$\text{Sign} = (-1)^{\#\text{ coloured corners in your polygon}}$$

$\Rightarrow$ $dc_1 = t - c_3 - c_5 - c_5 c_4 c_3$ $\qquad |c_3| = |c_4| = |c_5| = 0$

$dc_2 = 1 + c_3 + c_5 + c_3 c_4 c_5$ $\qquad |c_1| = |c_2| = 1$

$dc_3 = dc_4 = dc_5 = 0$

$\mathbb{Z}[t, t^{-1}]$ coefficients, count polygons with t when they go through marked point then set $t = -1$.
When you do surgery you need to take a spin structure coming from the filling.

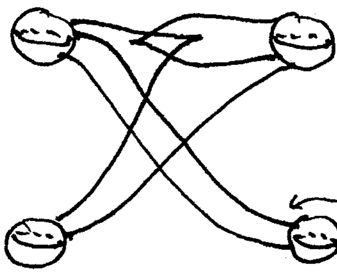
Going from $\mathbb{R}^3 \hookrightarrow S^3$: you need to keep track of twisted coefficients (or assume $c_1 = 0$).

$$LH_0(\Lambda) = \mathbb{Q}\langle c_3, c_4, c_5\rangle \Big/ \langle\!\langle 1 + c_3 + c_5 + c_3c_4c_5,\ 1 + c_3 + c_5 + c_5c_4c_3 \rangle\!\rangle$$

Chekanov knots have diff't LH by degree reasons $\Rightarrow$ surgery is non-contactomorphic (distinguished by set of linearisations).

Knots in $\#(S^1 \times S^2)$'s

E.g.

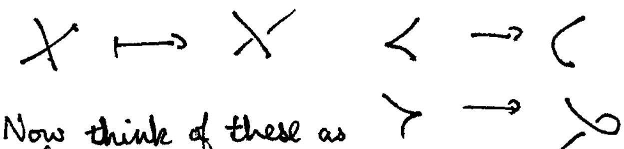


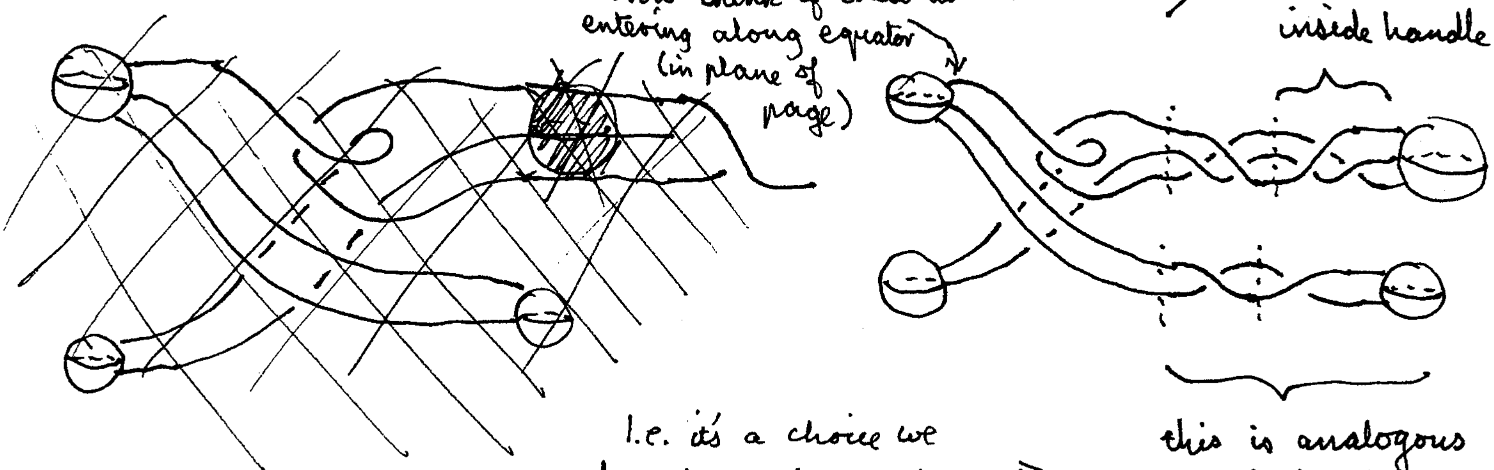
think of these as entering S^2 along the equator  ← horizontal

$\#^2(S^1 \times S^2)$ (remove 4 balls and glue in $S^2 \times I$ twice)

Any knot/link can be represented like this and there is a nice set of moves

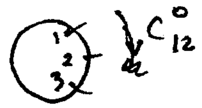
Go to Lag. projection:





I.e. it's a choice we make when going to the Lagrangian projection. ⟶ this is analogous to our choice that

(I think)
Reeb chords inside the handle: (there is one Reeb orbit, along the equator of the S^2 that is the core of the handle, which all link components hit)



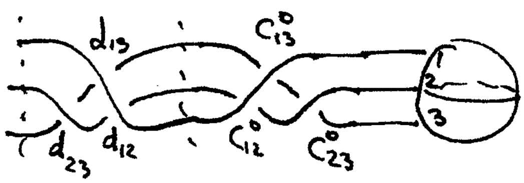
(like the Reeb chords from Sheel & Mays's talk), C^k_{ij} runs from i to j with 'multiplicity' k)

$$C^k_{ij} \quad \begin{array}{l} (\text{if } i<j \quad k \geq 0) \\ (\text{if } i \geq j \quad k>0) \end{array}$$

$$d\,C^k_{ij} = \begin{cases} \sum C^m_{il}\, C^{k-m}_{lj} \\ \left(\sum \quad\right) + 1 & \text{if } k=1 \text{ and } i=j \end{cases}$$

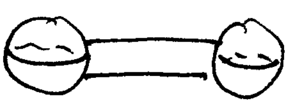
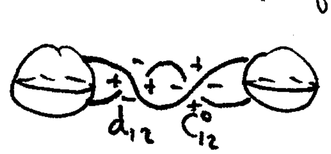
E.g. $d\,C^1_{11} = C^0_{12}\,C^1_{21} + C^0_{13}\,C^1_{31} + 1$

Picture of C^0_{ij}: (see previous diagram)



For all the other crossings (i.e. not near handles) you do the same as before.

The sense in which this is invariant under Legendrian isotopy is no longer stable tame isomorphism - you need to allow ∞ stabilisations as long as their degrees (of the generators you add) go to ∞.

E.g.  $\rightsquigarrow$ 

2 discs: one in middle, one going through handle;

$d(d_{12}) = 0$

$d(C^0_{12}) = 0$

$d(C^1_{11}) = 1 + C^0_{12}\,C^1_{21}$

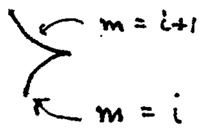
$d(C^1_{12}) = C^1_{11}\,C^0_{12} + C^0_{12}\,C^1_{22}$

$d(C^1_{21}) = 0 \qquad d(C^1_{22}) = 1 + C^1_{21}\,C^0_{12}$

Grading as usual for d_{ij} and crossings outside handles

$$|c_{ij}^{k}| = 2k-1 + m(i) - m(j)$$

Maslov number



In our example, $|c_{12}^{0}| = |c_{21}^{1}| = 0$

$$|c_{22}^{k}| = |c_{11}^{k}| = 2k-1$$

$$|c_{12}^{k}| = 2k$$

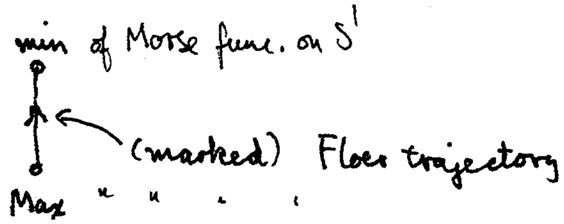
$$|c_{21}^{k}| = 2k-2.$$

Product on $SH(X)$: First review d_{SH}:

$SH(X) = \check{CH}(X) \oplus \hat{CH}(X) \oplus \mathrm{Morse}(-H)[-n]$

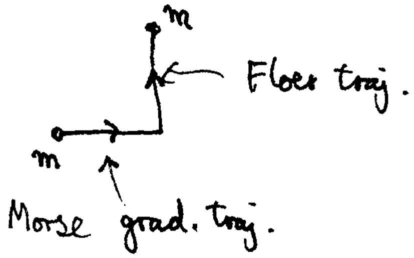
This comes from Morse-Bott picture of SH (Bourgeois).

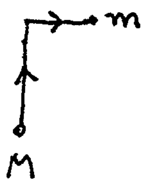
Rigid configurations contributing to d_{SH} (explanation of formula from day 1)



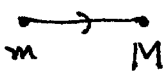
configuration

(We imagine making m and M very close to each other)

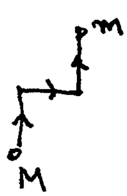




$d_{\hat{CH}}$, $d_{\check{CH}}$



d_{Morse}



configuration

we require the correct cyclic ordering so there is a gradient flow line in the correct direction.

$\underline{SH = 0}$

$SH^*(W) = 0 \quad (\Rightarrow SH^{*,S^1}(W) = 0).$

$\Rightarrow CH(W) \xrightarrow{\cong} H^{S^1}_{n-*}(W, \partial W)$ (trivial S^1-action)

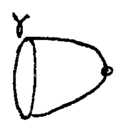
linearised contact homology (has Lie bracket)

trivial bracket

Q: is the bracket on $CH(W)$ trivial?

Spectral sequence from $S^{\cdot}(CH)$ to .....
bracket gives first differential.

$$CH(W) \xrightarrow{\phi = (\phi_k)} H^{S^1}_{n-*}(W, \partial W)$$

$\mathcal{M} =$ 

$\mathcal{M} = \tilde{\mathcal{M}}/S^1$

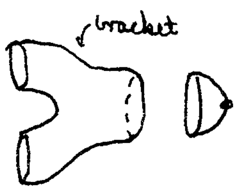
$\uparrow$

$\tilde{\mathcal{M}}$

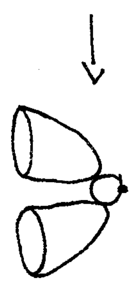
$\phi_k = \int_{\mathcal{M}} c_1(\tilde{\mathcal{M}})^k \cdot ev^*(\ldots)$

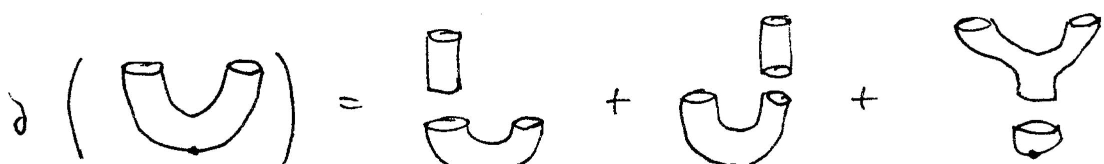
descendant.

Now what happens if



$\phi([x,y]) \stackrel{?}{=} 0$





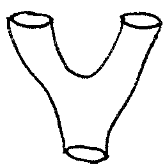
Thus if $SH^*(W) = 0$, Φ gives an L_∞ quasi-isomorphism to the trivial L_∞ algebra $H^{S^1}_{n-*}(W, \partial W)$. ~~(perturbation lemma)~~.

Consider GW potential

$$\sum \underline{\underline{f_1(t)}}\, p + f_2(t) p_i p_j + \dots$$

vanishing of SH^*
tells us something
about this but not higher f_i.

Can we get information out of moduli spaces



<u>without</u> markers?

Kenji: in this situation the L_∞ structure can be pushed into the homology (<u>involutive</u> Lie algebra?)

$\Rightarrow$ operations like

which might be defined if  gives 0, do not give new information.

(i.e. empty moduli space)

Can we get information from torsion?

Potential invariant:
Weinstein mfd with homology = 0 (n handles, $n-1$ handles cancel etc.)
Write intersection matrix of the stable Lagrangians, then try to do handle slides to make it diagonal, the obstruction is Whitehead torsion (to diagonalisati
What is the Lagrangian intersection Floer theory analogue of this?
Some sort of K-theory comes in.

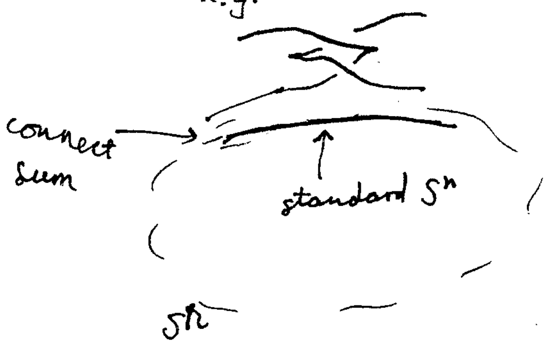
<u>Q</u>: What is the effect of handle slides (of vanishing cycles) on the Fukaya category.

Kenji: should be an exact triangle.

~~E.g. Legendrian in D^*S^n hitting fibre some number of times~~

E.g. Legendrian in $J^1 S^n$ which projects to S^n as a cover of degree 1, but intersects fibre >1 time.

E.g.

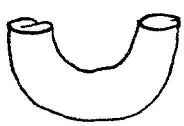


In S^1-case it is topologically knotted, but in higher dimension it may not be. Can it be Legendrian <u>non</u>-isotopic to the standard S^n?

—

Other invariants (Latscher + Wendell, explained by Michael)

T^3, $\lambda = \cos(2\theta)\,dx + \sin(2\theta)\,dy$



$\Rightarrow \hbar = 0$ in algebra (or $\hbar^n = 0$ in general).

Idea: a filling gives an $\mathbb{R}[[\hbar]]$-module map

$$SFT(\partial X) \longrightarrow \mathbb{R}[[\hbar]]$$

Thus if $\hbar^n = 0$ in $SFT(\partial X)$, this is an obstruction to filling because it has to survive the filling map.

Idea (Paul): Take standard ball, forget it's the standard ball, boundary is S^1-bundle over $\mathbb{CP}^n$.
Use $\mathcal{F}uk(\mathbb{CP}^n)$, GW invariants of $\mathbb{CP}^n$ over $\mathbb{Z}$.

~~Q (Ioan): Does the time~~

<u>Q</u>: If SH(affine algebraic surface which is topologically contractible) = 0 need the surface be $D^2 \times$ (punc. Riem. surf.)?

<u>Thm</u>: X = contractible affine alg. surface

Either $X \cong \mathbb{C}^2$ or there are at most 2 different $\mathbb{C} \xhookrightarrow{\text{alg.}} X$, nonconstant.

So, in this class, does $SH^* = 0$ characterise $\mathbb{C}^2$? (Not true in dim 3 – Ioan)

There are many Stein mfds homeomorphic to $\mathbb{R}^4$, can get $SH = 0$, then cross with $T^2 - pt$ (not D^2 as subcritical).

Stein structures on $\mathbb{R}^{16}$:
$\mathbb{R}^{16}$ $SH^* = 0$
affine alg. ones, $SH^* \neq 0$
Stein ones, $\Gamma = \infty$ } McLean
∞ handle ones

(Paul)
Q: X finite type, Weinstein, Stein, $X \not\cong \mathbb{R}^{2n}$, is it possible that

$$X \#_{\text{end}} X \cong X$$

Yasha: "virtually overtwisted" structures (covered by overtwisted ones).

E.g. $p-1$ tight contact structures on $L(p,1)$, ~~but~~ all fillable, but only 1 lifts to filling of S^3.

Q: do covers of boundary say something about symplectic topology of filling?

How does CH of covers behave?

What about ~~lifts of LSH~~ SH of lifts? Lifting contractible loops gives similar to normal homology, what about lifting non-contractible loops to contractible ones?

$$\tilde{M} \longrightarrow M \longrightarrow K(\pi, 1)$$

$$\mathcal{L}\tilde{M} \longrightarrow \mathcal{L}M \longrightarrow \mathcal{L}K(\pi, 1)$$

need to compute local coefficient systems on components of $\mathcal{L}M$.