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Bivariate interpolation based on univariate subdivision schemes

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Abstract

The paper presents a bivariate subdivision scheme interpolating data consisting of univariate functions along equidistant parallel lines by repeated refinements. This method can be applied to the construction of a surface passing through a given set of parametric curves. Following the methodology of polysplines and tension surfaces, we define a local interpolator of four consecutive univariate functions, from which we sample a univariate function at the mid-point. This refinement step is the basis to an extension of the 4-point subdivision scheme to our setting. The bivariate subdivision scheme can be reduced to a countable number of univariate, interpolatory, non-stationary subdivision schemes. Properties of the generated interpolant are derived, such as continuity, smoothness and approximation order.

Keywords: Bivariate interpolation; Non-stationary subdivision scheme; Approximation order

1. Introduction

This paper is motivated by the search for a subdivision-based solution to the problem of bivariate interpolation in a rectangular domain of data consisting of univariate functions given along parallel lines.

For an interval $[a, b]$ and $N \in \mathbb{N}$, let $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ denote a partition of $[a, b]$. The rectangular domain is $\Omega = [a, b] \times [-\pi, \pi] \subset \mathbb{R}^2$, and $f_i(y)$ is a

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data function given along the partition line $x = x_i, i = 0, 1, \ldots, N$. The problem is to find a smooth bi-variate interpolant $u(x, y)$ over Ω satisfying $u(x_i, y) = f_i(y)$, for $y \in [-\pi, \pi]$, $i = 0, 1, \ldots, N$. We refer to this bivariate interpolation problem as BIUD problem (Bivariate Interpolation of Univariate Data). To fit into the framework of subdivision schemes we consider the BIUD problem with an equally-spaced partition Δ.

Subdivision schemes are used for the construction of smooth curves and surfaces in Geometric Modelling (see e.g. [8,17]), and for the construction of wavelets (see e.g. [4,5]). These schemes can generate/approximate efficiently curves and surfaces by relatively simple iterative procedures.

The naive approach for solving the BIUD problem is to use a univariate interpolation method and interpolate the values $\{f_i(y_0)\}_{i=0}^N$ for every $y_0 \in [-\pi, \pi]$. The main disadvantage of this approach is that at a given point (x, y), the interpolant depends on data values $\{f_i(y)\}_{i=0}^N$, however it does not depend on available data in the neighbourhood of (x, y), such as $f_i(y+\epsilon)$, where $y+\epsilon \in [-\pi, \pi]$. This geometric drawback leads to more sophisticated solutions such as polysplines in [13, Chapters 1 and 3] and tension surfaces in [1]. A brief review of these methods and their relation to our construction is given in Section 2.2.

We introduce a subdivision based approach to the solution of the BIUD problem for 2π-periodic data functions, which yields interpolants depending on six local data functions at each point. Our bivariate subdivision scheme extends the well known 4-point scheme [9,6], replacing cubic polynomial interpolation by a solution of a PDE, obtained by the method of separation of variables. The limit function of this subdivision scheme is a real Fourier series in the variable y, with coefficients which are functions of x, obtained as the limits of univariate, interpolatory, non-stationary subdivision schemes, applied to the Fourier coefficients of the data functions. These univariate schemes reproduce functions from the tension spaces

$$V_m = Span\{1, x, e^{mx}, e^{-mx}\}, \quad m \in \mathbb{Z}_+. \tag{1}$$

We show that the subdivision based solution and its first partial derivatives are well defined and continuous due to the decay rate of the coefficient functions, inherited from the decay rate of the Fourier coefficients of the data functions. The proof of the decay rate of the coefficient functions is based on a uniform bound on the basic limit functions $\{\phi_m\}_{m\in\mathbb{Z}_+}$ of the family of univariate subdivision schemes, and on a slowly growing (with m) bound on $\left\{\frac{d}{dx}\phi_m\right\}_{m\in\mathbb{Z}_+}$.

The outline of this paper is as follows: Section 2 gives the scientific background, which includes relevant notation, definitions and results on subdivision schemes and a brief view of three known methods for the solution of the BIUD problem. We present our subdivision-based solution to the BIUD problem in Section 3. Its existence and continuity is proven in Section 4. In Section 5 we show the C^1 smoothness of our interpolant, as a bivariate function. Most of the proofs of the results in this section are postponed to Appendix A. The approximation order of our method is investigated in Section 6. The proofs of the lemmas in this section are given in Appendix B. We conclude the paper in Section 7 with a brief overall view of our method and discuss several possible applications.

2. Preliminaries

In this section we introduce some definitions and notation, and review relevant material to our problem and its solution. We start with univariate interpolatory subdivision schemes on which our bivariate subdivision scheme is based.

2.1. The 4-point non-stationary schemes and the V_m spaces

Non-stationary versions of the interpolatory 4-point subdivision scheme [9], are obtained when its tension parameter depends on the refinement level k, namely with refinement steps for $k \in \mathbb{Z}_+$ of the form

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= -w^k (f_{i-1}^k + f_{i+2}^k) + \left(\frac{1}{2} + w^k\right)(f_i^k + f_{i+1}^k). \end{aligned} \tag{2}$$

For a special choice of the tension parameters $\{w^k\}_{k \in \mathbb{Z}_+}$, these schemes are the exponentials reproducing subdivision schemes studied in [11]. In [2] it is shown that the space reproduced by (2) is $V_m = Sp\{1, x, e^{mx}, e^{-mx}\}$, $m \in \mathbb{Z}_+$, when

$$w^k := w_m^k = \frac{1}{2\left(e^{\frac{m}{2^k}} + e^{\frac{-m}{2^k}}\right)\left(e^{\frac{m}{2^k}} + e^{\frac{-m}{2^k}} + 2\right)}. \tag{3}$$

In fact, by (2) and (3) $f_{2i+1}^{k+1} = g\left(\left(i + \frac{1}{2}\right)2^{-k}\right)$, where g is the unique function in V_m satisfying $g((i + j)2^{-k}) = f_{i+j}^k$, $j = -1, 0, 1, 2$.

We denote the refinement rule (2) with $w^k = w_m^k$, as in (3), by S_m^k. The subdivision scheme, defined by $f^{k+1} = S_m^k f^k$, for all $k \in \mathbb{Z}_+$ is $\mathbf{S}_m = \{S_m^k\}_{k \in \mathbb{Z}_+}$, and its limit, when applied to the data $f^0 = \{f_i^0\}_{i \in \mathbb{Z}}$ is denoted by $S_m^\infty f^0$. Note that $0 < w_m^k \le \frac{1}{16}$ for any $m \in \mathbb{Z}_+$.

The case $m = 0$ is special due to the fact that $w^k = \frac{1}{16}$ for every k, and (2) becomes stationary. This scheme is the 4-point Dubuc Deslauriers (DD) scheme [6,7].

2.2. Three known methods for solving the BIUD problem

We review shortly three known methods for the solution of the BIUD problem. The first method, so-called the 'naive' approach, defines $u(x, y_0)$ by using a univariate interpolation to the values $\{(x_i, f_i(y_0))\}_{i=0}^N$. In this way we can evaluate the interpolant at every point of Ω. The interpolation property is straightforward and the approximation order can be easily derived from that of the univariate method. The main advantage, in addition to its simplicity, is the use of a well studied univariate interpolation method. Yet, any value of the interpolant $u(x^*, y^*)$ for $(x^*, y^*) \in \Omega$ is determined solely by a subset of the data $\{f_j(y^*)\}_{j=0}^N$ (depending on the univariate interpolation method), and there is no influence of other data values in the neighbourhood of (x^*, y^*). The last observation is a major geometric drawback of the naive method.

In the second method, suggested in [13], for 2π-periodic data functions, the interpolant $u(x, y)$ satisfies,

$$\Delta^2 u(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 u(x, y) = 0, \quad (x, y) \in \Omega_j,\ j = 1, 2, \ldots, N, \tag{4}$$

where Ω_j is the subdomain $(x_{j-1}, x_j) \times [-\pi, \pi]$. The solution is termed the bi-harmonic polyspline and is a special case of the n-polyharmonic solution, corresponding to Δ^n in (4). The bi-harmonic polyspline solution is unique and defines a smooth interpolant for smooth data functions. Moreover it minimizes $I(f) = \int_\Omega (f_{xx}^2 + 2f_{xy}^2 + f_{yy}^2)$ and reproduces bi-harmonic functions. Contrary to the naive approach, the bi-harmonic polyspline $u(x, y)$ at $(x, y) \in \Omega$, $x \neq x_0, \dots, x_N$ depends on the entire data. Yet it has a major geometric disadvantage, since due to its high smoothness in each subdomain it oscillates in a very unnatural way. A good example of such oscillations is observed when $f_j(y) = g(y),\ j = 0, \dots, N$ (see [1]).

Tension surfaces are proposed by Aylon in [1]. This method generates interpolants $u(x, y)$ satisfying

$$\mathcal{L}u(x, y) = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(x, y) = 0, \quad (x, y) \in \Omega_j,\ j = 1, 2, \dots, N. \tag{5}$$

The tension surface solution is unique and minimizes $I(f) = \int_\Omega f_{xx}^2 + 2f_{xy}^2$, which is a nonsymmetric functional in x and y, more appropriate for our nonsymmetric data. The value of the interpolant at $(x, y) \in \Omega$, $x \neq x_0, \dots, x_N$ depends on the entire data, and the interpolant reproduces harmonic functions as well as functions of the form $F(x, y) = g_1(y) + xg_2(y)$. The operator $\mathcal{L}$ in (5) plays a significant role in our method of interpolation.

The solution of (4) and (5) for 2π-periodic data functions, is done by the method of separation of variables. The solution is presented as a Fourier series in the variable y with coefficients which are functions of x, and the differential operator is applied to each term in the series. This construction influences heavily ours.

3. The bivariate interpolatory subdivision scheme for data along parallel lines

To solve the BIUD problem for an equally-spaced partition Δ, with $h = \frac{(b-a)}{n}$, and for 2π-periodic data functions, we extend to this setting the 4-point interpolatory subdivision scheme with $w = \frac{1}{16}$ [9,6]. This scheme refines point values $f_i = f(ih)$, $i = 0, \dots, N$ by inserting the estimated values at $\left(i + \frac{1}{2}\right)h$ as $P_{3,i}\left(\left(i + \frac{1}{2}\right)h\right)$ for $i = 1, \dots, N - 2$. Here $P_{3,i}(x)$ is a cubic polynomial $\left(\text{in the null space of } \frac{d^4}{dx^4}\right)$ satisfying the interpolation conditions $P_{3,i}((i + j)h) = f((i + j)h)$, $j = -1, 0, 1, 2$. By repeating the refinement step again and again a C^1 limit function is generated.

In analogy to this refinement, the bivariate scheme we construct, refines the data $f_i(y)$, $i = 0, \dots, N$ to $\tilde{f}_j(y)$, $j = 2, \dots, 2N - 2$ according to

$$\begin{aligned} &\tilde{f}_{2i}(y) = f_i(y), \quad i = 1, \dots, N - 1, \\ &\tilde{f}_{2i+1}(y) = Q_i\left(\left(i + \frac{1}{2}\right)h, y\right), \quad i = 1, \dots, N - 2, \end{aligned} \tag{6}$$

Here $Q_i(x, y)$ is a bivariate function in the null-space of the differential operator $\mathcal{L}$ in (5), satisfying the interpolation conditions

$$Q_i((i + j)h, y) = f_{i+j}(y), \quad j = -1, 0, 1, 2. \tag{7}$$

We construct Q_i by separation of variables. Assume

$$Q_i(x, y) = \sum_{m=0}^{\infty}\{\tilde{c}_{i,m}(x)\cos(my) + \tilde{s}_{i,m}(x)\sin(my)\}. \tag{8}$$

If $\mathcal{L}$ can be applied to Q_i term by term, then the coefficients must satisfy the ODEs

$$\tilde{c}_{i,m}^{(4)}(x) - m^2\tilde{c}_{i,m}^{(2)}(x) = 0, \qquad \tilde{s}_{i,m}^{(4)}(x) - m^2\tilde{s}_{i,m}^{(2)}(x) = 0, \quad m \in \mathbb{Z}_+. \tag{9}$$

To impose the interpolation condition (7), the 2π-periodic data functions are expanded into Fourier series

$$f_i(y) = \sum_{m=0}^{\infty} \{c_{i,m} \cos(my) + s_{i,m} \sin(my)\}, \quad i = 0, \ldots, N, \tag{10}$$

and the coefficients of Q_i are required to satisfy

$$\tilde{c}_{i,m}^{(4)}((i+j)h) = c_{i+j,m}, \qquad \tilde{s}_{i,m}^{(4)}((i+j)h) = s_{i+j,m}, \quad j = -1, 0, 1, 2. \tag{11}$$

From (9) it is clear that $\tilde{c}_{i,m}$ and $\tilde{s}_{i,m}$ are in $V_m = span\{1, x, e^{mx}, e^{-mx}\}$, where the solution to (11) is unique.

Now we can write explicitly the refined functions. By (6) and (8) we get for $i = 1, \ldots, N-2$

$$\tilde{f}_{2i+1}(y) = \sum_{m=0}^{\infty} \left\{ \tilde{c}_{i,m} \left(\left(i + \frac{1}{2} \right) h \right) \cos(my) + \tilde{s}_{i,m} \left(\left(i + \frac{1}{2} \right) h \right) \sin(my) \right\}. \tag{12}$$

Repeating this refinement step we get in the limit an interpolant given by a Fourier series in y, with coefficients which are functions of x. The coefficients of $\cos(my)$ and $\sin(my)$ are limits of the non-stationary 4-point subdivision scheme $\mathbf{S}_m$, obtained from the initial data

$$\mathbf{c}_m^0 = \{c_{i,m}\}_{i=0}^N, \qquad \mathbf{s}_m^0 = \{s_{i,m}\}_{i=0}^N, \quad m \in \mathbb{Z}_+, \tag{13}$$

respectively. Thus the interpolant is of the form,

$$u(x, y) = \sum_{m=0}^{\infty} \{c_m(x) \cos(my) + s_m(x) \sin(my)\}, \tag{14}$$

with

$$c_m(x) = (S_m^\infty \mathbf{c}_m^0)(x), \qquad s_m(x) = (S_m^\infty \mathbf{s}_m^0)(x). \tag{15}$$

Remark 1. Due to the finiteness of the data in (13), $c_m(x)$ and $s_m(x)$, for any m, are defined on $[x_2, x_{N-2}]$ only, and interpolate the data at the points x_i, $i = 2, \ldots, N-2$. Thus our subdivision-based method solves the interpolation problem in $\Omega^* = [x_2, x_{N-2}] \times [-\pi, \pi]$.

In order to solve the original interpolation problem with our subdivision approach, either extra data outside $[a, b]$ should be given [2], or special refinement rules have to be employed near the boundaries of $[a, b]$ [18]. We assume from now on that $N \geq 5$.

For the analysis of our method we further assume that the data functions satisfy

$$f_i \in \widetilde{W}_2^1, \quad \widetilde{W}_2^1 = \{g : g' \in L_2[-\pi, \pi], g(-\pi) = g(\pi)\}, \quad i = 0, \ldots, N. \tag{16}$$

In the rest of the paper we prove the continuity, smoothness and approximation order the of the solution (14), (15), (13) of the BIUD problem, under the condition (16).

4. Existence and continuity of the interpolant

A key result in the proof of the summability of (14) is a uniform bound on the basic limit functions of the schemes $\{\mathbf{S}_m\}_{m \in \mathbb{Z}_+}$.

4.1. Uniform bound on the basic limit functions

The basic limit function of the subdivision scheme $\mathbf{S}_m$ is

$$\phi_m = S_m^{\infty}\delta^{[0]} = \lim_{k\to\infty}\sum_{i\in\mathbb{Z}} f_i^k H(2^k\cdot -i),$$

here $f_i^k = \left(S_m^{k-1}\cdots S_m^0(\delta^{[0]})\right)_i$, with $\delta^{[0]} = \{\delta_{i,0}\}_{i\in\mathbb{Z}}$, and H is the hat function supported on $[-1,1]$ and given by $H(x) = 1-|x|$ there. By the linearity of $\mathbf{S}_m$ and its uniformity we have for any initial data $f^0 = \{f_i^0\}_{i\in\mathbb{Z}}$,

$$(S_m^{\infty} f^0)(x) = \sum_{j\in\mathbb{Z}} f_j^0\cdot\phi_m(x-j). \tag{17}$$

Our first goal is to establish a uniform bound on $\{\phi_m\}_{m\in\mathbb{Z}_+}$. As for the classical 4-point scheme [10], it is straightforward to show

Lemma 1. *For a fixed $m\in\mathbb{Z}_+$, let $f^k = \{f_i^k\}_{i\in\mathbb{Z}}$ be generated by the refinement rule (2) with $w^k = w_m^k$ as in (3). Then*

$$\Delta\mathbf{f}^{k+1} := \max_{j\in\mathbb{Z}}\left|f_{j+1}^{k+1} - f_j^{k+1}\right| \le \left(2w_m^k + \frac{1}{2}\right)\Delta\mathbf{f}^k \le \prod_{j=0}^{k}\left(2w_m^j + \frac{1}{2}\right)\Delta\mathbf{f}^0.$$

Moreover, if $\mathbf{f}^k$ denotes the polygonal line connecting the points $\{(i2^{-k}, f_i^k)\}_{i\in\mathbb{Z}}$, then

$$\left\|\mathbf{f}^{k+1} - \mathbf{f}^k\right\|_{\infty} \le (2w^k)\Delta\mathbf{f}^k.$$

The next theorem is based on the last lemma,

Theorem 2. *In the notation of Lemma 1*

$$\left\|\mathbf{f}^k\right\|_{\infty} \le \left\|\mathbf{f}^0\right\|_{\infty} + \frac{1}{3}\Delta\mathbf{f}^0.$$

Proof. By Lemma 1,

$$\begin{aligned}\left\|\mathbf{f}^k\right\|_{\infty} &\le \left\|\mathbf{f}^0\right\|_{\infty} + \sum_{n=1}^{k}\left\|\mathbf{f}^n - \mathbf{f}^{n-1}\right\|_{\infty} \le \left\|\mathbf{f}^0\right\|_{\infty} + \sum_{n=1}^{k}(2w_m^{n-1})\Delta\mathbf{f}^{n-1}\\ &\le \left\|\mathbf{f}^0\right\|_{\infty} + \sum_{n=1}^{k}2w_m^{n-1}\prod_{j=1}^{n-1}(2w_m^{j-1}+1/2)\cdot\Delta\mathbf{f}^0.\end{aligned}$$

Since $0 < w_m^k \le 1/16$ we have $\left\|\mathbf{f}^k\right\|_{\infty} \le \left\|\mathbf{f}^0\right\|_{\infty} + \frac{1}{8}(\Delta\mathbf{f}^0)\sum_{n=1}^{k}\left(\frac{5}{8}\right)^{n-1}$, which proves the claim of the theorem. $\square$

For the basic limit function $f_i^0 = \delta_{i,0}$, and $\left\|\mathbf{f}^0\right\|_{\infty} = \Delta\mathbf{f}^0 = 1$. This leads to,

Corollary 3. *For every $m\in\mathbb{Z}_+$,*

$$\|\phi_m\|_{\infty} \le \frac{4}{3}. \tag{18}$$

By the uniform bound on the basic limit functions and by (17) we can show that the subdivision-based interpolant exists and inherits important properties of the data functions.

Theorem 4. *The subdivision-based interpolant of the form (14), (15), (13) is absolutely and uniformly convergent whenever the data functions in (10) are.*

Proof. By (17) the coefficients in (14) have the form

$$c_m(x) = \sum_{i \in I_x} c_{i,m} \phi_m(x-i), \qquad s_m(x) = \sum_{i \in I_x} s_{i,m} \phi_m(x-i), \tag{19}$$

where $I_x = \{i \in \mathbb{Z} \mid |x-i| \le 3\}$, and thus $|I_x| \le 6$. Using (19) and (18) we can obtain a bound on $|u(x,y)|$, by the uniform and absolute convergence of the series of the data functions,

$$|u(x,y)| = \left| \sum_{m=0}^{\infty} c_m(x)\cos(my) + s_m(x)\sin(my) \right| \tag{20}$$

$$\le \frac{4}{3} \sum_{i \in I_x} \sum_{m=0}^{\infty} \left(\left|c_{i,m}\right| + \left|s_{i,m}\right| \right)$$

$$\le 8 \sum_{m=0}^{\infty} \left(\left|c_{i^*,m}\right| + \left|s_{i^*,m}\right| \right) < \infty, \tag{21}$$

where i^* is the index of a data function with maximal sum of the absolute values of the coefficients. Note that (20) is independent of (x,y), and is valid for every $(x,y) \in \Omega^*$. Thus we can use *Weierstrass M-test* which guarantees the absolute and uniform convergence of the Fourier series in (20) [12, Chapter 2]. □

A direct consequence of the last theorem is

Corollary 5. *The interpolant $u(x,y)$ in (14), (15), (13) is continuous.*

5. Smoothness of the interpolant

We cite here two results on Fourier series (see e.g. [12, Chapter 2]), which are needed in our analysis.

Result 1. *Let a function $f \in C^n$ be 2π-periodic with an absolute continuous, 2π-periodic nth derivative, then there exists a positive constant M such that*

$$\widehat{f(k)} \le \frac{M}{|k|^{n+1}},$$

for k large enough.

Result 2. *Let the coefficients c_k satisfy*

$$|c_k| \le \frac{M}{|k|^{n+1+\epsilon}}$$

for k large enough, with constants $M > 0, \epsilon > 0$, and $n \in \mathbb{Z}_+$. Then $\sum_{k \in \mathbb{Z}} c_k e^{ikt}$ is a 2π-periodic function with a continuous nth derivative.

5.1. The first partial derivative in the direction of the subdivision

Consider a series of the form (14), with coefficients given by (15), (13) and its term-by term differentiated series,

$$U(x,y)=\sum_{m=0}^{\infty}\left\{\frac{d}{dx}c_m(x)\cos(my)+\frac{d}{dx}s_m(x)\sin(my)\right\}. \tag{22}$$

Each of the coefficients in (22) has the form

$$\frac{d}{dx}c_m(x)=\sum_{i\in I_x}c_{i,m}\frac{d}{dx}\phi_m(x-i),\qquad \frac{d}{dx}s_m(x)=\sum_{i\in I_x}s_{i,m}\frac{d}{dx}\phi_m(x-i),$$

with $|I_x|=6$, and with $\{c_{i,m},s_{i,m}\}_{i=0}^{N}$ the Fourier coefficients of the data functions as given in (10).

Note that the 4-point subdivision scheme $\mathbf{S}_m$, reproducing V_m, generates C^1-limit functions from any initial set of data [14], and thus the first derivative of ϕ_m exists for all $m\in\mathbb{Z}_+$.

Our first goal is to find a bound of the form $\left\|\frac{d}{dx}\phi_m\right\|_\infty\le g(m)$, where g has a relatively slow growth rate with m. Using this bound, we show that the coefficients of (22) have a decay rate, inherited from the decay rate of the Fourier coefficients of the data functions, sufficient for the series in (22) to converge. The next theorem presents our result for such a bound on $\left\|\frac{d}{dx}\phi_m\right\|_\infty$,

Theorem 6. *For every $m\in\mathbb{Z}_+$, the derivative of the basic limit function satisfies*

$$\frac{d}{dx}\phi_m=O(\log_2(m)).$$

We prove this theorem by a series of lemmas.

We begin by several important notation and identities. For a fixed m consider the divided differences of $\mathbf{f}^k=S_m^{k-1}\cdots S_m^0\mathbf{f}^0=\{f_i^k\mid i\in\mathbb{Z}\}$

$$d_i^k=\frac{f_{i+1}^k-f_i^k}{2^{-k}},\quad i\in\mathbb{Z},\ k\in\mathbb{Z}_+. \tag{23}$$

As in the case of the stationary 4-point scheme, we have,

$$\begin{aligned} d_{2i}^{k+1}&=d_i^k+2w_m^kd_{i-1}^k-2w_m^kd_{i+1}^k,\\ d_{2i+1}^{k+1}&=d_i^k-2w_m^kd_{i-1}^k+2w_m^kd_{i+1}^k, \end{aligned} \tag{24}$$

for each $i\in\mathbb{Z},k=0,1,2,\ldots$. Let $\mathbf{d}_m^k$ be the polygonal line through the points $\{(i2^{-k},d_i^k)\}$, and let $\sigma_m^{k+1}=\left\|\mathbf{d}_m^{k+1}-\mathbf{d}_m^k\right\|_\infty$. Then

$$\sigma_m^{k+1}=\max_{i\in\mathbb{Z}}\left\{\left|d_{2i}^{k+1}-d_i^k\right|,\left|d_{2i+1}^{k+1}-\frac{d_i^k+d_{i+1}^k}{2}\right|\right\}.$$

By (24),

$$\sigma_m^{k+1}=\max_{i\in\mathbb{Z}}\left\{\left|2w_m^k(d_{i+1}^k-d_i^k)+2w_m^k(d_i^k-d_{i-1}^k)\right|,\right.$$

$$\left|\left(2w_m^k - \frac{1}{2}\right)(d_{i+1}^k - d_i^k) + 2w_m^k(d_i^k - d_{i-1}^k)\right|\Bigg\}. \tag{25}$$

In the following lemma we derive a relation between $\mathbf{d}_m^k$ and σ_m^k.

Lemma 7. *Let* $\{d_i^k \mid i \in \mathbb{Z}\}$ *be defined by* (23)*, and let*

$$\Delta\mathbf{d}_m^k = \max_{i\in\mathbb{Z}}\left\{\left|d_{i+1}^k - d_i^k\right|\right\} = \left\|d_{i+1}^k - d_i^k\right\|_\infty.$$

Then

$$\sigma_m^{k+1} \le \frac{1}{2}\Delta\mathbf{d}_m^k.$$

Proof. Since $w_m^k \in \left(0, \frac{1}{16}\right]$, we have $\left|2w_m^k - 1/2\right| = 1/2 - 2w_m^k$. This combined with (25) suggests that $\sigma_m^{k+1} \le \max\left\{4w_m^k\Delta\mathbf{d}_m^k, \frac{1}{2}\Delta\mathbf{d}_m^k\right\}$, which proves the lemma. □

To prove the uniform convergence of $\mathbf{d}_m^k$ we bound σ_m^{k+1} by an element of a convergent series. For that we prove in the next lemmas a relation of the form $\Delta\mathbf{d}_m^{k+1} \le \mu\Delta\mathbf{d}_m^{k-l}$ with $l \ge 0$ and $\mu \in (0, 1)$. The proofs of these lemmas are deferred to Appendix A.

Lemma 8. *In the notation of Lemma 7,*

1. $|d_{2i+1}^{k+1} - d_{2i}^{k+1}| \le 8w^k\Delta\mathbf{d}_m^k$.
2. *For i even,* $|d_{2i+2}^{k+1} - d_{2i+1}^{k+1}| \le 3/4\Delta\mathbf{d}_m^{k-1}$.
3. *For i odd,* $|d_{2i+2}^{k+1} - d_{2i+1}^{k+1}| \le |32w^k w^{k-1} - 4w^k + 1|\Delta\mathbf{d}_m^{k-1}$.

From Lemma 8 we conclude that the only problematic case is 3, since by (3) $\lim_{m\to\infty} w_m^k = 0$, which indicates that for fixed k, $4w_m^k(8w_m^{k-1} - 1)$ is negative and close to 0 for large enough m. Thus for every $m \in \mathbb{Z}_+$, $32w_m^k w_m^{k-1} - 4w_m^k + 1 > 0$, but it cannot be bounded from above by a constant less than 1. To improve the bound in 3 of Lemma 8, we denote

$$B_m^k = 32w_m^k w_m^{k-1} - 4w_m^k + 1 = 4w_m^k(8w_m^{k-1} - 1) + 1. \tag{26}$$

A conclusion from Lemma 8 is

$$\Delta\mathbf{d}_m^k \le \max\left\{8w_m^{k-1}\Delta\mathbf{d}_m^{k-1}, \frac{3}{4}\Delta\mathbf{d}_m^{k-2}, B_m^{k-1}\Delta\mathbf{d}_m^{k-2}\right\} \le B_m^{k-1}\Delta\mathbf{d}_m^{k-2}. \tag{27}$$

Lemma 9. *Let* B_m^k *be defined by* (26)*. Then*

$$B_m^k \le \begin{cases} C_B, & k \ge \log_2(m), \\ 1, & \textit{otherwise}, \end{cases}$$

where $C_B = 4\alpha(8\alpha - 1) + 1 \approx 0.905$, *with* $\alpha = [8\cosh(1)(\cosh(1) + 1)]^{-1} \approx 0.0318$.

Lemmas 8 and 9 lead to

$$\Delta\mathbf{d}_m^k \le C_B^{\left\lceil\frac{k-\log_2(m)}{2}\right\rceil}\underbrace{1\cdot 1\cdot 1\cdot\ldots\cdot 1}_{\frac{\log_2(m)}{2}\text{ times}}\Delta\mathbf{d}_m^0,$$

which together with Lemma 7 yields

$$\sigma_m^{k+1} \le \begin{cases} (1/2)\Delta \mathbf{d}_m^0 (C_B)^{\left\lceil \frac{k-\log_2(m)}{2} \right\rceil}, & k > \log_2(m), \\ (1/2)\Delta \mathbf{d}_m^0, & \text{otherwise}. \end{cases} \tag{28}$$

The next step is to obtain the rate of growth of the bound on the first divided differences as m increases.

Lemma 10. *The divided difference polygon* $\mathbf{d}^k$ *satisfies*

$$\left\| \mathbf{d}^k \right\|_\infty \le \frac{1}{2}(M + \log_2(m)),$$

where M is a constant, independent of m and k.

Since $\frac{d}{dx}\phi_m$ is the limit of $\mathbf{d}^k$ as $k \to \infty$, Theorem 6 follows from Lemma 10.

In view of Result 1, the decay of the Fourier coefficients of the data functions (see (10)) is a consequence of the smoothness of the data functions, which leads to the smoothness of the interpolant, as stated in the next theorem.

Theorem 11. *For C^1 data functions (10), the coefficients of $U(x, y)$ in (22) satisfy*

$$\left| \frac{d}{dx} c_m(x) \right| \le \frac{K_1}{m^{2-\epsilon}}, \qquad \left| \frac{d}{dx} s_m(x) \right| \le \frac{K_2}{m^{2-\epsilon}},$$

for any $\epsilon > 0$, with K_1, K_2 constants independent of m. Moreover $U(x, y)$ is a continuous Fourier series.

Proof. We prove the result for $\frac{d}{dx}c_m(x)$, the case of $\frac{d}{dx}s_m(x)$ is similar. From Result 1 and the assumption on the data functions, there exist $C_i > 0$ and $m_i > 0$, $i = 0, \ldots, N$ such that,

$$\left| c_{i,m} \right| \le \frac{C_i}{m^2}, \quad m > m_i.$$

Since $\frac{d}{dx}c_m(x) = \sum_{i \in I_x} c_{i,m} \frac{d}{dx}\phi_m(x - i)$, where $I_x \subset \{0, 1, \ldots, N\}$,

$$\begin{aligned} \left| \frac{d}{dx} c_m(x) \right| &\le (C \cdot \log_2(m)) \max_{i \in I_x} \frac{C_i}{m^2} \\ &\le (C \cdot \log_2(m)) \max_{i \in \{0,\ldots,N\}} \frac{C_i}{m^2} \le \frac{K}{m^{2-\epsilon}}, \end{aligned}$$

for every $m > \max_{i \in \{0,\ldots,N\}}\{m_i\}$ and for all small $\epsilon > 0$. $\square$

The next theorem summarizes the main result of this section.

Theorem 12. *For C^1 data functions (10), satisfying the conditions of Result 1 with $n = 1$, the subdivision-based interpolant has a continuous first derivative in the subdivision direction.*

Proof. The data functions are in $\widetilde{W}_2^1 \cap C^1$ and thus converge absolutely and uniformly. Hence Theorem 4 asserts the convergence of the subdivision-based interpolant, $u(x, y)$, given in (20). Obviously each term of $u(x, y)$ is in C^1. Result 2 and Theorem 11 imply that the series (22) converges to a continuous function. The convergence is independent of the point $(x, y) \in \Omega^*$ and hence is uniform. This guarantees the convergence of the term by term differentiated series (22) to $\frac{d}{dx}u(x, y)$. $\square$

5.2. The first partial derivative in the direction of the Fourier expansion

Let the term-by-term differentiated series of (14), in the direction of the Fourier expansion be

$$V(x, y) = \sum_{m=1}^{\infty} \{m \cdot s_m(x) \cos(my) - m \cdot c_m(x) \sin(my)\}.$$

As in the proof of Theorem 4, we get

$$|V(x, y)| \leq \sum_{m=1}^{\infty} |m \cdot s_m(x)| + |m \cdot c_m(x)| \leq 8 \sum_{m=1}^{\infty} m(|s_{i,m}| + |c_{i,m}|).$$

Since each data function is in $\widetilde{W}_2^1 \cap C^1$, then $\{ms_{i,m}, -mc_{i,m}\}_{m=1}^{\infty}$ are the Fourier coefficients of $\frac{d}{dy} f_i(y)$, $i = 0, \ldots, N$. Thus

$$|V(x, y)| \leq 8 \sum_{m=1}^{\infty} m(|s_{j^*,m}| + |c_{j^*,m}|),$$

where j^* is the index of the data function with maximal sum of the moduli of Fourier coefficients of its derivative in y.

The above discussion leads to,

Theorem 13. *For C^1 data functions (10), where each function in $\left\{\frac{d}{dy} f_i(y)\right\}_{i=0}^{N}$ has an absolutely and uniformly convergent Fourier expansion, the subdivision-based interpolant has a first continuous derivative in the direction of the Fourier expansion. This derivative has an absolutely and uniformly convergent Fourier series.*

6. Approximation order

6.1. Approximation of infinite Fourier series

We start with the main result,

Theorem 14. *Let the two Fourier series*

$$\sum_{m=0}^{\infty} \{a_m(x) \cos(my) + b_m(x) \sin(my)\} \tag{29}$$

and

$$\sum_{m=0}^{\infty} \left\{ \frac{d^2}{dx^2} a_m(x) \cos(my) + \frac{d^2}{dx^2} b_m(x) \sin(my) \right\} \tag{30}$$

be uniformly convergent in $\Omega = [a, b] \times [-\pi, \pi]$. Given a partition Δ, $x_i = a + ih$, $i = 0, \ldots, N$, with $h = \frac{b-a}{N}$, let the data functions be

$$f_i(y) = F(x_i, y), \quad i = 0, \ldots, N,$$

where $F(x, y)$ is the sum in (29). Then the subdivision-based interpolant of this data, $u(x, y)$ of (14), satisfies

$$\|F(x, \cdot) - u(x, \cdot)\|_{L^2([-\pi,\pi])} \leq C \cdot h^{\frac{3}{2}}, \quad x \in [a^*, b^*]$$

with C a constant depending on F but not on h. Here $a^* = a + 2h$, $b^* = b - 2h$.

Before proving the theorem we state a lemma about the uniform approximation order of the family of subdivision schemes $\{\mathbf{S}_m\}_{m\in\mathbb{Z}_+}$, presented in Section 2.1. The proof of the lemma is given in Appendix B.

Lemma 15. *Let Δ, h be as in Theorem 14. For f twice differentiable s.t. $\frac{d^2}{dx^2}f \in L^p([a,b])$ for some $p \geq 2$,*

$$|f(x) - (S_m^\infty f|_\Delta)(x)| < c \cdot h^{1+\frac{1}{q}}, \quad x \in [a^*, b^*]$$

with $q \in (1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and $c = 36 \left\| \frac{d^2}{dx^2} f \right\|_{L^p([a,b])}$.

Proof of Theorem 14. By the assumptions on the Fourier series (29) and (30), the latter is the second partial derivative in x of the first, which we denote by F_{xx}. The uniform convergence of (29) and Theorem 4 imply that the interpolant

$$u(x,y) = \sum_{m=0}^{\infty} \{c_m(x)\cos(my) + s_m(x)\sin(my)\},$$

is uniformly convergent. This together with Parseval equality leads to

$$\begin{aligned} \|F(x,\cdot) - u(x,\cdot)\|^2_{L^2([-\pi,\pi])} &= \int_{-\pi}^{\pi} (F(x,y) - u(x,y))^2 dy \\ &= \pi \left(\sum_{m=0}^{\infty} [(a_m(x) - c_m(x))^2 + (b_m(x) - s_m(x))^2] \right). \end{aligned} \tag{31}$$

Recall that $c_m(x)$ and $s_m(x)$ are limits of S_m applied to $\{a_m(x_i)\}_{i=0}^N$ and $\{b_m(x_i)\}_{i=0}^N$, respectively. By the assumptions on F_{xx}, we get in view of the last lemma,

$$\begin{aligned} (a_m(x) - c_m(x))^2 &\leq 36^2 h^{2\left(1+\frac{1}{2}\right)} \left(\int_a^b \left| \frac{d^2}{dx^2} a_m(t) \right|^2 dt \right), \quad x \in [a^*, b^*], \\ (b_m(x) - s_m(x))^2 &\leq 36^2 h^{2\left(1+\frac{1}{2}\right)} \left(\int_a^b \left| \frac{d^2}{dx^2} b_m(t) \right|^2 dt \right), \quad x \in [a^*, b^*]. \end{aligned} \tag{32}$$

Thus by (31) and (32)

$$\|F(x,\cdot) - u(x,\cdot)\|^2_{L^2([-\pi,\pi])} \leq \pi 36^2 h^3 \sum_{m=0}^{\infty} \left\{ \left\| \frac{d^2}{dx^2} a_m \right\|^2_{L^2([a,b])} + \left\| \frac{d^2}{dx^2} b_m \right\|^2_{L^2([a,b])} \right\}.$$

Again, by the assumptions on F_{xx} we have for $x \in [a,b]$

$$\begin{aligned} \|F_{xx}\|^2_{L^2(\Omega)} &= \int_a^b \int_{-\pi}^{\pi} (F_{xx}(x,y))^2 dy dx \\ &= \pi \int_a^b \left(\sum_{m=0}^{\infty} \left(\frac{d^2}{dx^2} a_m(x) \right)^2 + \left(\frac{d^2}{dx^2} b_m(x) \right)^2 \right) dx < \infty. \end{aligned} \tag{33}$$

All of the terms in the series above are non-negative and hence, we can apply Lebesgue monotone convergence theorem to obtain

$$\int_a^b \sum_{m=0}^{\infty} \left[\left(\frac{d^2}{dx^2} a_m(x) \right)^2 + \left(\frac{d^2}{dx^2} b_m(x) \right)^2 \right] dx$$
$$= \sum_{m=0}^{\infty} \int_a^b \left[\left(\frac{d^2}{dx^2} a_m(x) \right)^2 + \left(\frac{d^2}{dx^2} b_m(x) \right)^2 \right] dx$$
$$= \sum_{m=0}^{\infty} \left\{ \left\| \frac{d^2}{dx^2} a_m \right\|^2_{L^2([-\pi,\pi])} + \left\| \frac{d^2}{dx^2} b_m \right\|^2_{L^2([-\pi,\pi])} \right\}.$$

Thus by (31)–(33)

$$\|F(x,\cdot) - u(x,\cdot)\|_{L^2([-\pi,\pi])} \le C \cdot h^{\frac{3}{2}}, \quad x \in [a^*, b^*],$$

with $C = C(F) = 36\pi^{\frac{1}{2}} \|F_{xx}\|_{L^2(\Omega)}$. □

6.2. Approximation of finite Fourier sums

In this section we consider the case when the approximated function can be represented by a finite Fourier sum.

$$F(x, y) = \sum_{m=0}^{M} \{a_m(x)\cos(my) + b_m(x)\sin(my)\}, \quad M < \infty. \tag{34}$$

Thus, our data functions are trigonometric polynomials,

$$f_i(y) = \sum_{m=0}^{M} \{a_m \cos(my) + b_m \sin(my)\}, \quad i = 0, \ldots, N. \tag{35}$$

For this case we can give pointwise error bounds.

Theorem 16. *Let $F(\cdot, y) \in C^4[a, b]$ for any $y \in [-\pi, \pi]$ be of the form (34). Given a partition Δ, $x_i = a + ih$, $i = 0, \ldots, N$ with $h = \frac{b-a}{N}$, let the data functions be*

$$f_i(y) = F(x_i, y), i = 0, \ldots, N.$$

Then the subdivision-based interpolant of this data, $u(x, y)$ of (14) satisfies

$$\|F(x, y) - u(x, y)\|_{\infty,\Omega^*} \le C \cdot h^4, \quad (x, y) \in \Omega^*$$

with C a constant depending on F but not on h. Here $\Omega^* = [a^*, b^*] \times [-\pi, \pi]$, with $a^* = a + 2h, b^* = b - 2h$.

Again, we start the proof of the theorem with a lemma. This lemma is about the approximation order of the subdivision scheme $\mathbf{S}_m$, $m \in \mathbb{Z}_+$, of Section 2.1. The proof of the lemma is given in Appendix B.

Lemma 17. *Let Δ, $0 < h < 1$ be as in Theorem 16. Then for $f \in C^4([a, b])$,*

$$|f(x) - (S_m^{\infty} f|_{\Delta})(x)| < C_{f,m} \cdot h^4, \quad x \in [a^*, b^*]$$

with $C_{f,m}$, a constant independent of h but dependent on m and f.

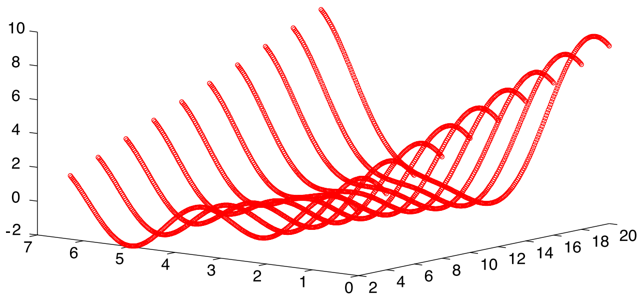


Fig. 1. Data for the BIUD problem.

Proof of Theorem 16. Recall that $F(\cdot, y) \in C^4[a, b]$ for any $y \in [-\pi, \pi]$, and has the form (34). We apply the last lemma to the coefficients and obtain for every $(x, y) \in \Omega^*$,

$$
\begin{aligned}
|F(x, y) - u(x, y)| &= \left| \sum_{m=0}^{M} \{(a_m(x) - c_m(x)) \cos(my) + (b_m(x) - s_m(x)) \sin(my)\} \right| \\
&\leq \sum_{m=0}^{M} \{(|a_m(x) - c_m(x)| + |b_m(x) - s_m(x)|)\} \\
&\leq \sum_{m=0}^{M} (C_{a_m,m} \cdot h^4 + C_{b_m,m} \cdot h^4) \\
&= \left(\sum_{m=0}^{M} \{C_{a_m,m} + C_{b_m,m}\} \right) h^4. \quad \square
\end{aligned}
$$

7. Numerical examples

We present two examples. The first compares the effect of the differential operators; (5) in the subdivision based interpolant and (4) in the biharmonic polyspline. The second example illustrates the locality of the subdivision based interpolant in comparison with the methods suggested in [1,13].

7.1. *The differential operators*

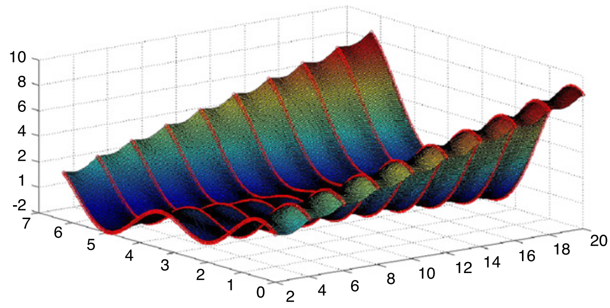
As explained in Section 2.2, the differential operator plays a significant role. We demonstrate this by interpolating a function which is linear in the x axis. In this example the problem defined on the domain $\Omega = [2, 20] \times [0, 2\pi]$ and the data functions are given over the partition

$$\Delta : x_0 = 2 < 4 < \cdots < 20 = x_{10}.$$

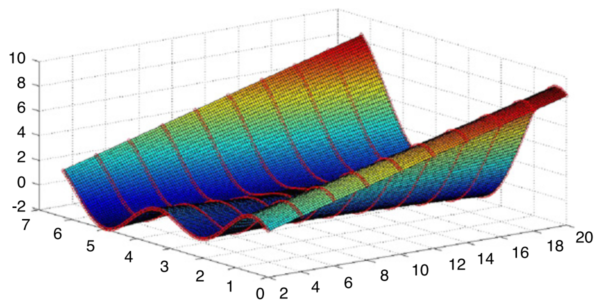
The data functions consist of samples from the bivariate function

$$F(x, y) = \frac{x}{8} + \frac{x}{4} \cos(y) + \cos(2y) + \sin(y) + \sin(2y).$$

The samples are given in Fig. 1.



(a) The biharmonic polyspline.



(b) The subdivision based interpolant.

Fig. 2. Comparison of the biharmonic polyspline and the bivariate subdivision surface for the data in Fig. 1.

The biharmonic polyspline interpolant of [13] is given in Fig. 2(a) and one can clearly see the wiggles of this interpolant. On the other hand, the subdivision based interpolant fully reproduces the function, given in Fig. 2(b). Similar behaviour is observed in the tension surface interpolant [1] due to the use of the same differential operator (5).

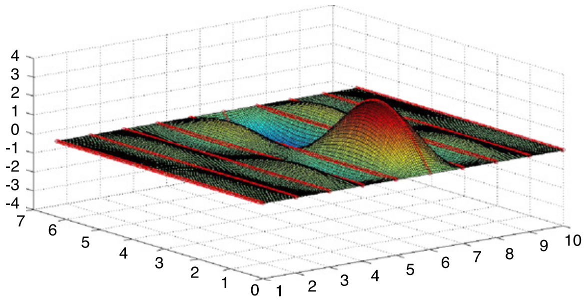
7.2. Locality of the subdivision based interpolant

In the next example we compare the three (non-trivial) solutions of the BIUD problem for locality. Obviously, for data consisting of zero functions, all three solutions produce the zero bivariate function. Here we change one data function to be different from zero.

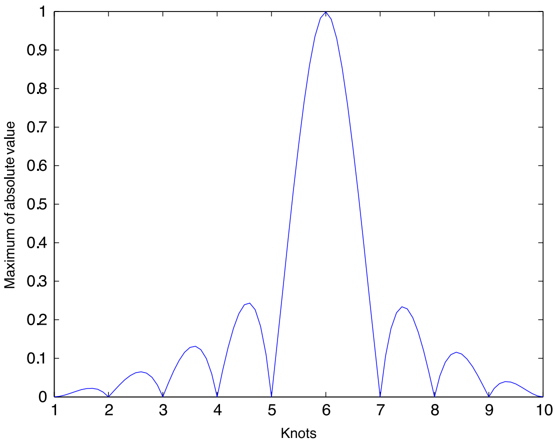
Two important issues must be considered. First the computational effort required for obtaining the new interpolant. The biharmonic polyspline solution [13] and the tension surface [1] require the solution of a new BIUD problem, while for the subdivision based solution we only need to update a small segment which contains three data functions from each side of the new sampled function.

A second issue is the geometric effect of such a function. To demonstrate this issue we introduce the next figures. The problem defined on the domain $\Omega = [1, 10] \times [0, 2\pi]$. The data functions,

$$f_i(y) = \begin{cases} 0, & i \neq 6, \\ \sin(y), & i = 6, \end{cases}$$



(a) The biharmonic polyspline.



(b) Maximum over y.

Fig. 3. The biharmonic polyspline interpolant.

are given over the partition

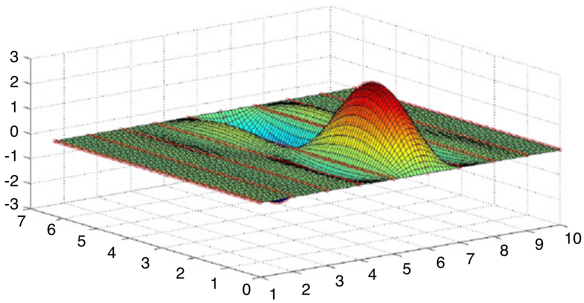
$$\Delta : x_0 = 1 < 2 < \cdots < 10 = x_{10}.$$

We depict the three solutions together with the graph of $\max_{y\in[0,2\pi]} u(\cdot, y)$ for each interpolant u.

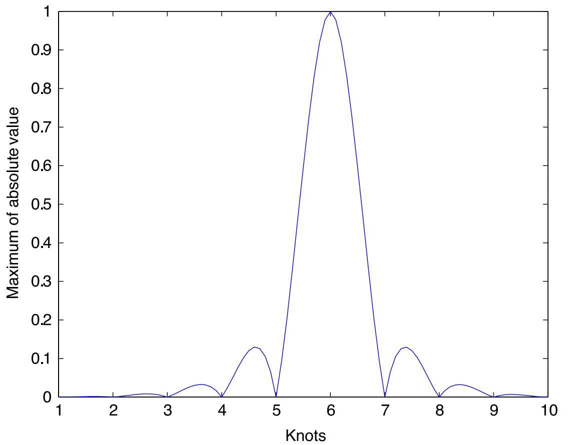
The biharmonic polyspline interpolant is given in Fig. 3. Note the oscillations which occur along the whole domain. The tension surface is presented in Fig. 4, and as expected from spline-based interpolation, the changes decay very fast. The subdivision based interpolant is given in Fig. 5, where we easily observed that the effect of $f_6(y)$ is local.

8. Conclusions and possible applications

This paper presents a solution of the BIUD problem on equidistant partitions for 2π-periodic data functions, based on a bivariate subdivision scheme refining univariate functions. The bivariate subdivision scheme is reduced to a family of univariate, non-stationary interpolatory subdivision schemes reproducing certain spaces of exponentials. Our interpolant inherits the decay rate of its Fourier coefficients from that of the data functions and thus is smooth when the data functions are. We derive its approximation order, for functions which are either finite or



(a) The tension surface.



(b) Maximum over y.

Fig. 4. The tension surface interpolant.

infinite Fourier series, using the approximation properties of the family of univariate subdivision schemes. Note that the main result for periodic data functions can be extended to non-periodic data (see [16]).

Several real life problems, where the data is given as functions along lines, are listed in [13, Chapter 6]. Our method can be applied to these problems as well as to discrete data collected along parallel lines. Such data can be first interpolated along the lines by a univariate method to provide data functions.

Another possible application is the approximation of a "simple" closed surface from sampled closed curves on it, for example from a set of its parallel cross-sections. Since each component of a closed curve is a periodic function, our method can approximate each component of the surface separately, yielding a parametric representation of the surface. Obviously such an application cannot cope with general topologies. Moreover, the method depends on the parametrization of the sampled curves. Nevertheless, it will benefit from all the good properties of our method such as localization, approximation order and easy implementation.

Acknowledgments

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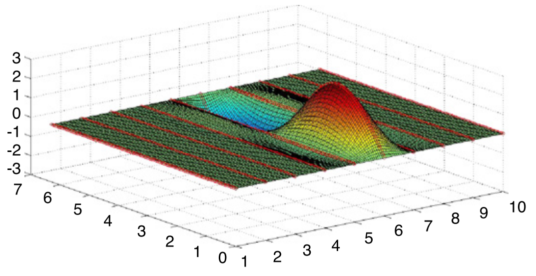
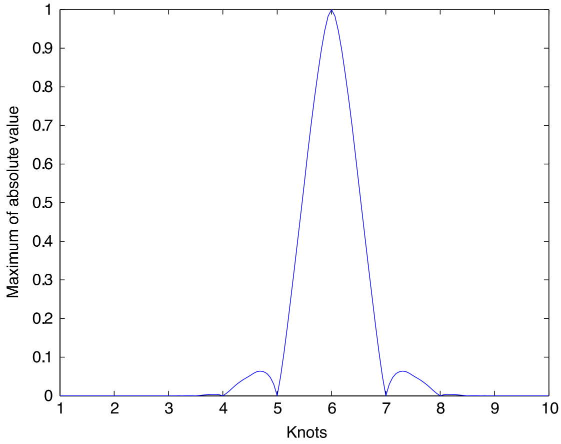


(a) The bivariate subdivision surface.



(b) Maximum over y.

Fig. 5. The bivariate subdivision surface interpolant.

Appendix A. Proofs of results in Section 5

A.1. Proof of Lemma 8

Proof. By (24) we get

$$d_{2i+1}^{k+1} - d_{2i}^{k+1} = 4w_m^k(d_{i+1}^k - d_{i-1}^k),$$

and claim 1 follows. The second and third claims are proved together. Again by (24)

$$d_{2i+2}^{k+1} - d_{2i+1}^{k+1} = -2w_m^k(d_{i+2}^k - d_{i+1}^k) + (1 - 4w_m^k)(d_{i+1}^k - d_i^k) - 2w_m^k(d_i^k - d_{i-1}^k). \quad \text{(A.1)}$$

Thus

$$|d_{2i+2}^{k+1} - d_{2i+1}^{k+1}| \le (4w_m^k + (1 - 4w_m^k))\Delta\mathbf{d}_m^k \le \Delta\mathbf{d}_m^k,$$

and

$$\Delta\mathbf{d}_m^{k+1} \le \Delta\mathbf{d}_m^k. \quad \text{(A.2)}$$

This bound is not good enough and a second recursion on k is required. For i even we obtain from (A.1), (A.2) and claim 1.

$$\begin{aligned}|d_{2i+2}^{k+1} - d_{2i+1}^{k+1}| &\le (4w_m^k)\Delta\mathbf{d}_m^{k-1} + \left|(1-4w_m^k)(8w_m^{k-1})\right| \Delta\mathbf{d}_m^{k-1} \\ &= \left|8w_m^{k-1} - 32w_m^{k-1}w_m^k + 4w_m^k\right| \Delta\mathbf{d}_m^{k-1}.\end{aligned}$$

Since $8w_m^{k-1} - 32w_m^{k-1}w_m^k > 0$,

$$|d_{2i+2}^{k+1} - d_{2i+1}^{k+1}| \le \left|8w_m^{k-1} + 4w_m^k\right| \Delta\mathbf{d}_m^{k-1} \le 12\frac{1}{16}\Delta\mathbf{d}_m^{k-1} = \frac{3}{4}\Delta\mathbf{d}_m^{k-1}.$$

Similarly for i odd, we get by (A.1) and (A.2)

$$|d_{2i+2}^{k+1} - d_{2i+1}^{k+1}| \le (32w_m^k w_m^{k-1} + 1 - 4w_m^k)\Delta\mathbf{d}_m^{k-1}. \quad \square \tag{A.3}$$

A.2. Proof of Lemma 9

Denote by $x = \frac{m}{2^k}$, and consider $0 < x \le 1$. Then w_m^k defined in (3) has the form $w_m^k = h(x)$ where $h(x) = \frac{1}{2}((e^x + e^{-x})(e^x + e^{-x} + 2))^{-1}$, which can be rewritten as

$$h(x) = [8\cosh(x)(\cosh(x) + 1)]^{-1}, \tag{A.4}$$

with $h(0) = \frac{1}{16}$. Since the hyperbolic cosine is monotone increasing for $0 < x \le 1$, $h(x)$ is monotone decreasing for $0 < x \le 1$, and satisfies $[8\cosh(1)(\cosh(1) + 1)]^{-1} \le h(x) < \frac{1}{16}$. For a fixed m, let $k \ge \lceil \log_2(m) \rceil$, namely $2^k \ge m$. Then $0 < x < 1$ and

$$w_m^k \in \left(\alpha, \frac{1}{16}\right), \tag{A.5}$$

with $\alpha = [8\cosh(1)(\cosh(1)+1)]^{-1} \approx 0.0318$. Notice that for a fixed m we have $\lim_{k\to\infty} w_m^k = \frac{1}{16}$. From the above analysis we deduce that the sequence $\{w_m^k\}$ with $k \ge \lceil \log_2(m) \rceil$, increases monotonically with k to $\frac{1}{16}$, namely $w_m^{k-1} \le w_m^k$. Therefore, in case of $k \ge \lceil \log_2(m) \rceil$ we have,

$$B_m^k = 32w_m^k w_m^{k-1} - 4w_m^k + 1 \le 32(w_m^k)^2 - 4w_m^k + 1.$$

The bound above is a quadratic polynomial in w_m^k which attains its maximum in the interval (A.5) at the end point α. Thus

$$B_m^k \le 32\alpha^2 - 4\alpha + 1 \approx 0.905.$$

The other case is trivial due to $0 < w_m^k \le \frac{1}{16}$.

A.3. Proof of Lemma 10

Proof. It is obvious from the definition of σ_m^k that $\|\mathbf{d}_m^k\| \le \sigma_m^k + \cdots + \sigma_m^0 + \|\mathbf{d}_m^0\|$, which for $k > \log_2(m)$ becomes, in view of (28),

$$\|\mathbf{d}_m^k\| \le \sum_{j=\lceil \log_2(m) \rceil+1}^{k} (1/2)\Delta\mathbf{d}_m^0 (C_B)^{\left\lceil \frac{j-\log_2(m)}{2} \right\rceil} + \sum_{j=0}^{\lceil \log_2(m) \rceil} (1/2)\Delta\mathbf{d}_m^0 + \|\mathbf{d}_m^0\|. \tag{A.6}$$

We start by bounding the first sum in (A.6).

$$\sum_{j=\lceil \log_2(m)\rceil+1}^{k} (1/2)\Delta \mathbf{d}_m^0 (C_B)^{\left\lceil \frac{j-\log_2(m)}{2}\right\rceil} \le \left(\frac{1}{2}\Delta \mathbf{d}_m^0\right) \sum_{j=0}^{k-\lceil \log_2(m)\rceil-1} (C_B)^{\left\lceil \frac{j+1}{2}\right\rceil}$$
$$\le \left(\frac{1}{2}\Delta \mathbf{d}_m^0\right) \sum_{j=0}^{k-\lceil \log_2(m)\rceil-1} \sqrt{C_B^{j+1}}.$$

By Lemma 9, $0 < C_B < 1$ hence $0 < \gamma = \sqrt{C_B} < 1$, and

$$\sum_{j=0}^{k-\lceil \log_2(m)\rceil-1} \gamma^{j+1} \le \sum_{j=0}^{\infty} \gamma^{j+1} = \frac{\gamma}{1-\gamma} = M_1.$$

Note that M_1 is a constant independent of m and k. The second sum in (A.6) can be bounded straightforwardly

$$\sum_{j=0}^{\lceil \log_2(m)\rceil} (1/2)\Delta \mathbf{d}_m^0 \le \left(\frac{1}{2}\Delta \mathbf{d}_m^0\right) \cdot (\log_2(m)+1).$$

Recall that we consider here the basic limit function, namely using the initial data $\delta^{[0]} = \{\delta_{i,0}\}_{i\in\mathbb{Z}}$ for which $\|\mathbf{d}_m^0\| = 1$ and $\Delta \mathbf{d}_m^0 = 1$. Thus we get

$$\|\mathbf{d}_m^k\| \le \left(\frac{1}{2}\Delta \mathbf{d}_m^0\right)(M_1 + (\log_2(m)+1)) + \|\mathbf{d}_m^0\| \le \frac{1}{2}(M + \log_2(m)),$$

with $M = M_1 + 3$. $\square$

Appendix B. Proofs of results in Section 6

B.1. Proof of Lemma 15

We use the notation of Theorem 14.

Proof. Let $T_{1,\xi}$ be the linear Taylor expansion of f about the point ξ and let $R_{1,\xi}$ be the corresponding remainder. By the reproduction property of $\mathbf{S}_m$ in $[a^*, b^*]$, we get for $x \in [a^*, b^*]$

$$\begin{aligned} |f(x) - (S_m^\infty f|_\Delta)(x)| &= |f(x) - T_{1,\xi}(x) + T_{1,\xi}(x) - (S_m^\infty f|_\Delta)(x)| \\ &= |R_{1,\xi}(x) + (S_m^\infty T_{1,\xi}|_\Delta)(x) - (S_m^\infty f|_\Delta)(x)|, \end{aligned}$$

choosing $\xi = x$ and using the linearity of the subdivision operator, we get

$$\begin{aligned} |f(x) - (S_m^\infty f|_\Delta)(x)| &= |(S_m^\infty T_{1,x}|_\Delta)(x) - (S_m^\infty f|_\Delta)(x)| \\ &= |(S_m^\infty R_{1,x}|_\Delta)(x)| \\ &= \left|\sum_{j\in I_x} R_{1,x}(x_j)\cdot \phi_m(x - x_j)\right| \\ &\le \|\phi_m\|_\infty \cdot \sum_{j\in I_x} |R_{1,x}(x_j)|. \end{aligned}$$

By the compact support of ϕ_m the set I_x is finite with at most 6 elements, and $|x - x_j| \leq 3h$. Consider the integral formula for the remainder,

$$R_{1,x}(x_j) = \int_x^{x_j} \left(\frac{d^2}{dx^2} f(t)(x_j - t)dt \right).$$

Then by Hölder inequality,

$$\left| \int_x^{x_j} \left(\frac{d^2}{dx^2} f \right)(t)(x_j - t)dt \right| \leq \left(\int_{t_1}^{t_2} \left| \frac{d^2}{dx^2} f(t) \right|^p dt \right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |x_j - t|^q dt \right)^{\frac{1}{q}},$$

with $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in (1, \infty)$ and $t_1 = \min\{x, x_j\}$, $t_2 = \max\{x, x_j\}$. Since $\int_{t_1}^{t_2} |x_j - t|^q dt \leq \frac{(3h)^{q+1}}{q+1}$,

$$\int_x^{x_j} \left| \left(\frac{d^2}{dx^2} f \right)(t)(x_j - t)dt \right| \leq \left(\int_a^b \left| \left(\frac{d^2}{dx^2} f \right)(t) \right|^p dt \right)^{\frac{1}{p}} \cdot c_q h^{\frac{q+1}{q}},$$

with $c_q = \left(\frac{3^{q+1}}{q+1} \right)^{\frac{1}{q}}$. Evidently $1 < q \leq 2$, since $p \geq 2$. It is easy to check that c_q is monotonically decreasing over $[1, 2]$ and therefore $c_q \leq c_1 = \frac{9}{2}$. The claim follows from the assumption $\frac{d^2}{dx^2} f \in L^p([a, b])$. □

B.2. Proof of Lemma 17

Proof. Let $\tilde{x} \in (x_n, x_{n+1})$, for $2 \leq n \leq N - 3$.

Since V_m is an ECT-space (for more details see [15]) one can find $g \in V_m$, for any $f \in C^4[a, b]$, such that [3],

$$f^{(j)}(\tilde{x}) = g^{(j)}(\tilde{x}), \quad j = 0, 1, 2, 3. \tag{B.1}$$

The subdivision operator $\mathbf{S}_m$ is a linear operator reproducing V_m. Hence

$$\begin{aligned} |f(\tilde{x}) - (S_m^\infty f|_\Delta)(\tilde{x})| &= |f(\tilde{x}) - g(\tilde{x}) + g(\tilde{x}) - (S_m^\infty f|_\Delta)(\tilde{x})| \\ &= |(S_m^\infty g|_\Delta)(\tilde{x}) - (S_m^\infty f|_\Delta)(\tilde{x})| \\ &= |(S_m^\infty (g - f)|_\Delta)(\tilde{x})| \\ &\leq \|\phi_m\|_\infty \Sigma_{i \in I_{\tilde{x}}} |(g - f)(x_i)|. \end{aligned} \tag{B.2}$$

Note that $|I_{\tilde{x}}| \leq 6$. Writing $R = g - f$ as a Taylor expansion at $\tilde{x}$, and using (B.1), we obtain

$$R(x) = R^{(4)}(\xi) \frac{(\tilde{x} - x)^4}{4!},$$

with ξ inside the segment between $\tilde{x}$ and x. The explicit form of $R^{(4)} = g^{(4)} - f^{(4)}$ can be obtained from the solution of the following linear system, implied by (B.1),

$$\begin{pmatrix} 1 & \tilde{x} & e^{m\tilde{x}} & e^{-m\tilde{x}} \\ 0 & 1 & me^{m\tilde{x}} & -me^{-m\tilde{x}} \\ 0 & 0 & m^2 e^{m\tilde{x}} & m^2 e^{-m\tilde{x}} \\ 0 & 0 & m^3 e^{m\tilde{x}} & -m^3 e^{-m\tilde{x}} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{B.3}$$

Here $f_j = f^{(j)}(\tilde{x})$ for $j = 0, 1, 2, 3$ and $g(x) = a + bx + ce^{mx} + de^{-mx} \in V_m$. Since $g^{(4)}(x) = m^4 ce^{mx} + m^4 de^{-mx}$, we obtain from (B.3)

$$g^{(4)}(\xi) = m \sinh(m(\tilde{x} - \xi)) f^{(3)}(\tilde{x}) + m^2 \cosh(m(\tilde{x} - \xi)) f^{(2)}(\tilde{x}).$$

Due to the finite support of the basic limit function ϕ_m, $|\tilde{x} - \xi| \leq 3h$, and for $h < 1$,

$$|g^{(4)}(\xi)| \leq m \sinh(3m) \| f^{(3)} \|_{\infty,[a,b]} + m^2 \cosh(3m) \| f^{(2)} \|_{\infty,[a,b]} = K_{f,m}.$$

This leads to the bound

$$\| R^{(4)} \|_\infty \leq K_{f,m} + \| f^{(4)} \|_{\infty,[a,b]},$$

and thus by (B.2)

$$\begin{aligned} |f(\tilde{x}) - (S_m^\infty f|_\Delta)(\tilde{x})| &\leq \|\phi_m\|_\infty |\Sigma_{i \in I_{\tilde{x}}} R(x_i)| \\ &\leq \|\phi_m\|_\infty \cdot 6(K_{f,m} + \| f^{(4)} \|_{\infty,[a,b]}) \frac{(3h)^4}{4!} \\ &\leq C_{f,m} \cdot h^4. \quad \square \end{aligned}$$

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