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Subdivision Schemes for Positive Definite Matrices

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Abstract The class of symmetric positive definite matrices is an important class both in theory and application. Although this class is well studied, little is known about how to efficiently interpolate such data within the class.

We extend the 4-point interpolatory subdivision scheme, as a method of interpolation, to data consisting of symmetric positive definite matrices. This extension is based on an explicit formula for calculating a binary "geodetic average". Our method generates a smooth curve of matrices, which retain many important properties of the interpolated matrices. Furthermore, the scheme is robust and easy to implement.

Keywords Nonlinear subdivision scheme · Interpolation · Positive definite matrix

Mathematics Subject Classification 41A05 · 65D05 · 65D10

1 Introduction

Positive definite matrices are a well-known tool in pure mathematics, engineering, scientific computation, physics, statistics and more. As a numerical tool they have

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several strengths, such as orthogonal decomposition, real positive eigenvalues, etc. However, no standard approach exists for interpolating positive definite matrices.

Naturally, positive definite matrices take a key role in the representation of data in many applications. A classic example in materials science is deformation tensors where measurements of strain, stress and deformation are modeled as positive definite matrices [12, 25, 27]. The interest in the field of deformation has continued, e.g. see [14]. Another example are diffusion tensors (DMRI) used in medical imaging. In DMRI, the spatial scans are also positive definite matrices [4, 20]. In addition, there are several applications where sequences of positive definite matrices are in use at the core of a calculation process, e.g., in the field of optimization processes [7]. Such applications encourage us to further search for new analysis tools.

Subdivision schemes are common interpolation methods in approximation theory. They are easy to implement, computationally efficient and have high approximation order [1, 23, 31]. In this paper we suggest adapting an interpolatory subdivision scheme to positive definite matrices; our method is based on an extension of the geometric mean of two positive numbers to positive definite matrices.

Goldman et al. [26] defined non-linear spline subdivision schemes for positive numbers, by replacing the arithmetic mean $\frac{a+b}{2}$, by the geometric mean $\sqrt{ab}$. We extend this work to interpolatory schemes, by using $a^t b^{1-t}$ instead of $ta + (1-t)b$, for any $t \in \mathbb{R}$. Furthermore, we use an extension of this non-symmetrical geometric average [2] for positive definite matrices, to define an interpolatory subdivision scheme for such matrices.

The set of positive definite real matrices is an open set and closed under matrix addition and multiplication with positive scalar, yet it is not a vector space. Thus any standard linear interpolatory scheme defined by matrix addition does not necessarily generate positive definite matrices since any such $\mathbf{C}^1$ interpolatory scheme has negative coefficients. So, a different adaptation of subdivision schemes is required.

Donoho et al. [24] proposed a solution, further studied by Yu [22]. It is based on the operation $A \oplus B = \exp(\log(A) + \log(B))$ for positive definite matrices. Under this operation, the set of positive definite matrices is a Lie group. The scheme is defined on the tangent plane, and retains convergence and smoothness by the analyticity of the exp-log map. However, this scheme is not robust for small eigenvalues. Furthermore, the scheme geometric interpretation holds only for dense sampling.

In this paper we suggest a new approach using the geometric mean, suggested in [2]. This geometric mean is defined by $G_{\frac{1}{2}}(A, B) = A(A^{-1}B)^{\frac{1}{2}}$, for any invertible matrices A, B. It turns out that on the manifold of positive definite matrices, the geometric mean $G_{\frac{1}{2}}(A, B)$ is the mid-point of the geodesic joining A and B, defined by the Riemannian metric $d(A, B) = \|\log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})\|$ (for more details see [18]).

Our subdivision scheme is based on the 4-point scheme [11], expressed in terms of repeated binary averages and converges to a "smooth matrix curve". We bound the spectral radii of the matrices along the matrix curve. Moreover, the subdivision operator commutes with common operators from linear algebra, and preserves the spectral properties of the initial data. The scheme is robust and can be extended to positive semidefinite matrices, under additional conditions.

This paper is organized as follows: in Sect. 2 we define interpolatory subdivision schemes for the scalar case based on geometric mean for positive numbers. Next, in Sect. 3, we introduce a new subdivision scheme for positive definite matrix data, based on a geometric mean for matrices. In Sect. 4 we prove the convergence of this process to a smooth "matrix curve". In Sect. 5 we describe algebraic and spectral properties of the generated "matrix curve". An extension to positive semidefinite matrices is made in Sect. 6 along with the proof of robustness.

2 The Scalar Case

Goldman et al. [26] studied a class of non-linear subdivision schemes. These schemes are based on replacing any binary linear average, e.g. the arithmetic mean, in linear subdivision algorithms by binary non-linear averages, e.g., the geometric mean, $x^t y^{1-t}$, for $t \in [0,1]$. We extend this approach in order to define a class of interpolatory subdivision schemes.

A subdivision scheme generates a sequence of refined points from a given sequence of points $\mathbf{p}_0 = \{p_{0,i}\}_{i\in\mathbb{Z}}$ according to the refinement rule,

$$p_{k+1,i} = \sum_{j\in\mathbb{Z}} s_j p_{k,i+j}, \qquad \sum_{j\in\mathbb{Z}} s_j = 1. \tag{2.1}$$

We assume that s_j is nonzero for only a finite set of indices. The refinement operator that generates $\{p_{k+1,i}\}_{i\in\mathbb{Z}}$, $k \in \mathbb{Z}_+$ from $\{p_{k,i}\}_{i\in\mathbb{Z}}$ is called the *refinement step*, and is denoted by $\mathcal{S}$. In the case of convergence of the process, we denote the limit operator by $\mathcal{S}^\infty$.

We modify the linear scheme (2.1), using the geometric mean $x^t y^{1-t}$ for $t \in \mathbb{R}$. The refinement rules have the following form:

$$a_{k+1,i} = \prod_{j\in\mathbb{Z}} a_{k,i+j}^{s_j}, \qquad \sum_{j\in\mathbb{Z}} s_j = 1. \tag{2.2}$$

We denote by $\mathcal{S}_l$ the refinement rule (2.1) of a linear scheme and by $\mathcal{S}_g$ the refinement operator of the geometric counterpart (2.2). We refer to this scheme (2.2) as the *geometric averaging scheme*.

Example 1 The simplest non-trivial interpolatory subdivision schemes is the 4-point scheme [11],

$$\begin{aligned} p_{k+1,2i} &= p_{k,i}, \\ p_{k+1,2i+1} &= \frac{9}{16}(p_{k,i} + p_{k,i+1}) - \frac{1}{16}(p_{k,i-1} + p_{k,i+2}), \end{aligned} \tag{2.3}$$

with the geometric modification

$$\begin{aligned} a_{k+1,2i} &= a_{k,i}, \\ a_{k+1,2i+1} &= \frac{(a_{k,i}a_{k,i+1})^{\frac{9}{16}}}{(a_{k,i-1}a_{k,i+2})^{\frac{1}{16}}}. \end{aligned} \tag{2.4}$$

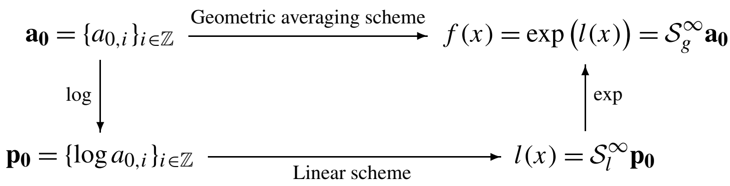


Fig. 1 The log–exp diagram

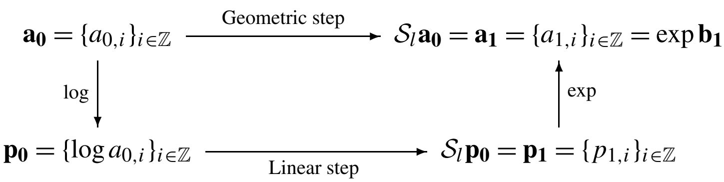


Fig. 2 The single step log–exp diagram

For a full survey on scalar subdivision methods and their analysis see [10]. Next we analyze the properties of the geometric averaging scheme.

2.1 Convergence and Smoothness of the Modified Schemes

All of the results in this section are derived by the commutativity of the diagram presented in Fig. 1.

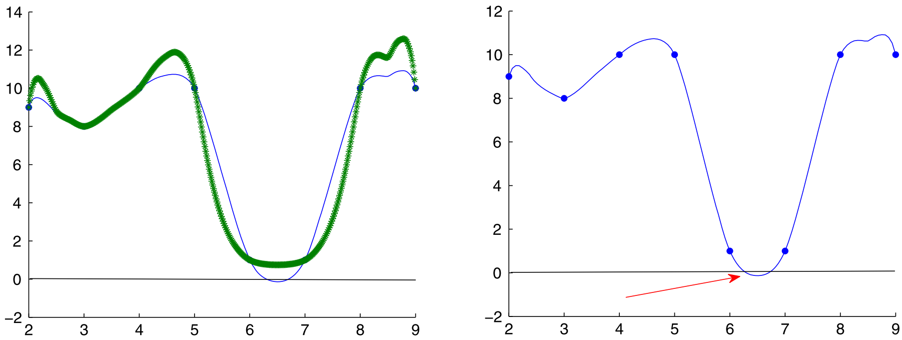
Theorem 2.1 *The* exp–log *diagram given in Fig.* 1 *is commutative for any positive initial data* $\mathbf{a_0}$.

Proof We prove it for a single step, i.e.,

$$\begin{aligned}\exp(p_{1,2i+1}) &= \exp\left\{\sum_j s_j p_{0,i+j}\right\}\\ &= \exp\left\{\sum_j s_j \log a_{0,i+j}\right\}\\ &= \prod_j a_{0,i+j}^{s_j} = a_{2i+1,1}.\end{aligned}$$

The single step illustrated in Fig. 2. Inductively, one can prove the diagram in Fig. 1 for all dyadic points, due to the analytic property of the exponential map. Thus the diagram is commutative for all points in the interval. □

The geometric averaging scheme inherits many of the properties of the linear scheme; see, e.g., the following.



(A) Geometric 4-point (B) Linear 4-point

Fig. 3 Positive and non-positive schemes

Theorem 2.2 *Given a linear scheme in the form* (2.1) *and its modified geometric version* (2.2), *then we have the following*.

(1) *If the linear scheme is* $\mathbf{C}^n$ *then so is the geometric averaging scheme.*
(2) *If the linear scheme is of approximation order of* h^d *then so is the geometric averaging scheme.*

The proof follows straightforwardly from Theorem 2.1, and thus is omitted.
For the next property, we present a new definition.

Definition 2.3 A convergent subdivision scheme, with a refinement step $\mathcal{S}$, is called a *positive scheme* if for any given sequence of positive numbers $\mathbf{a}_0 = \{a_i\}_{i\in\mathbb{Z}}$, the limit function

$$f(x) = \mathcal{S}^\infty\big(\mathbf{a}^0\big)(x) > 0, \quad x \in \mathbb{R}.$$

Another conclusion from Theorem 2.1 is

Corollary 2.4 *The schemes defined by* (2.2) *is a positive scheme.*

Corollary 2.4 is illustrated in Fig. 3 for the 4-point scheme (see Example 1). We compare the linear 4-point scheme (2.3) with the geometric 4-point scheme (2.4) for positive data (in Figs. 3(b) and 3(a), respectively). While the geometric averaging scheme is clearly positive one can verify that the linear scheme is not positive.

2.2 The Differences Between the Linear and Geometric Averaging Schemes

An interesting question is the difference between the linear schemes and their geometric counterparts. Due to convergence it suffices to consider only the first refinement step.

Theorem 2.5 *Let* $\mathbf{a_0} = \{a_{0,i}\}_{i\in\mathbb{Z}}$ *be a sequence of positive numbers, bounded away from zero, namely* $\inf_i\{a_{0,i}\} > c > 0$. *Denote* $\Delta_0 = \sup_i |a_{0,i} - a_{0,i+1}|$. *If* $\Delta_0 < \infty$

then

$$\sup_i \left| \left(\mathcal{S}_l(\mathbf{a_0})\right)_i - \left(\mathcal{S}_g(\mathbf{a_0})\right)_i \right| < C\Delta_0^2,$$

where C is a constant that depends on $\mathbf{a_0}$.

Proof By Taylor expansion of $f_a(x) = a^t x^{1-t}$ we get

$$a^t b^{1-t} = ta + (1-t)b - t(1-t)\frac{(b-a)^2}{a} + O\big((b-a)^3\big),$$

or equivalently $|a^t b^{1-t} - (ta + (1-t)b)| < \frac{1}{c}\Delta_0^2$. Generalizing to n variables,

$$f_{a_n}(x_1, \ldots, x_{n-1}) = a_n^{1-\sum_{i=1}^{n-1} s_i} \prod_{i=1}^{n-1} x_i^{s_i}.$$

The Taylor expansion around $(a_n, \ldots, a_n)$ yields

$$a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n} = f_{a_n}(a_1, \ldots, a_{n-1}) = s_n a_n + \sum_{i=1}^{n-1} s_i a_i + O\big(\Delta^2\big).$$

Denote by $\mathcal{I}$ the finite set of nonzero indices in (2.1) and (2.4), and recall that $\sum_{j \in \mathcal{I}} s_j = 1$. Then

$$\begin{aligned}
\big(S_g\big(\{a_{0,i}\}_{i\in\mathbb{Z}}\big)\big)_i &= \prod_{j\in\mathcal{I}} a_{0,i+j}^{s_j} = f_{a_{j_{\max}}}(a_{0,i+j_{\min}}, \ldots, a_{0,i+j_{\max}-1}) \\
&= s_{0,j_{\max}} a_{0,i+j_{\max}} + \sum_{j=j_{\min}}^{j_{\max}-1} s_j a_{0,i+j} + O\big(\Delta^2\big) \\
&= \big(S_l\big(\{a_{0,i}\}_{i\in\mathbb{Z}}\big)\big)_i + O\big(\Delta_0^2\big),
\end{aligned}$$

where $j_{\max}$, $j_{\min}$ are the maximum and minimum indices in $\mathcal{I}$, respectively. □

3 Subdivision Scheme for Positive Definite Matrices

One generalization of positive numbers is the positive definite matrices. A linear scheme, e.g. the 4-point scheme (2.3), using matrix operations, does not necessary retain the positive definite property. For example one can choose as initial data, the diagonal matrices with the data in Fig. 3 at the diagonal values.

In this section we define the scheme on data consisting of positive definite matrices. We start by updating our notation for the geometric mean of matrices. Next we prove basic properties, which are important for the definition of the scheme.

Our construction is based on one common interpretation of the geometric mean for matrices. This geometric mean [2, 6] is defined for any invertible matrices A, B

and $t \in \mathbb{R}$ as follows:

$$G_t(A,B) = A\big(A^{-1}B\big)^t = A^{\frac{1}{2}}\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^t A^{\frac{1}{2}}. \tag{3.1}$$

When $0 \le t \le 1$, $G_t(A,B)$ is the geodesic curve between A and B in the metric space $(\mathcal{P}, d)$, where $\mathcal{P}$ is the collection of all positive definite matrices with the Riemann metric [18],

$$d(A,B) = \big\|\log\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)\big\|, \tag{3.2}$$

where $\|M\| = \sqrt{\operatorname{tr}(MM^*)}$ is the Frobenius norm [13, Chap. 2], $\operatorname{tr}(\cdot)$ is the standard trace operator and X^* is the standard transpose operator. Henceforth, we use this norm, unless otherwise mentioned. Note that all norms are equivalent in the finite space of $n \times n$ matrices. The next lemma presents fundamental properties of (3.1).

Lemma 3.1 *Let A, B be positive definite matrices. Then for all $t \in \mathbb{R}$*

(1) *$G_t(A,B)$ of (3.1) is positive definite*;
(2) *$G_t(A,B) = G_{1-t}(B,A)$.*

Proof Let $X = (A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) = (A^{-\frac{1}{2}}B^{\frac{1}{2}}B^{\frac{1}{2}}A^{-\frac{1}{2}})$. First, we prove that X is a positive definite matrix. Using the positive definiteness of A, B,

$$\begin{aligned} X = \big(A^{-\frac{1}{2}}B^{\frac{1}{2}}\big)\big(B^{\frac{1}{2}}A^{-\frac{1}{2}}\big) &= \big(A^{-\frac{1}{2}}B^{\frac{1}{2}}\big)\big(\big(B^{\frac{1}{2}}A^{-\frac{1}{2}}\big)^*\big)^* \\ &= \big(A^{-\frac{1}{2}}B^{\frac{1}{2}}\big)\big(A^{-\frac{1}{2}}B^{\frac{1}{2}}\big)^*. \end{aligned}$$

Since the product of a full rank matrices is a full rank matrix we see that X is positive definite. By the spectral decomposition of X it is clear that X^t is positive definite for every $t \in \mathbb{R}$. Secondly, we prove that $G_t(A,B)$ is positive definite,

$$\begin{aligned} G_t(A,B) &= A^{\frac{1}{2}}\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^t A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}X^t A^{\frac{1}{2}} \\ &= \big(A^{\frac{1}{2}}X^{\frac{t}{2}}X^{\frac{t}{2}}A^{\frac{1}{2}}\big) \\ &= \big(A^{\frac{1}{2}}X^{\frac{t}{2}}\big)\big(A^{\frac{1}{2}}X^{\frac{t}{2}}\big)^*. \end{aligned}$$

Thus, $G_t(A,B)$ is positive definite for every $t \in \mathbb{R}$. To prove the second part of the lemma,

$$\begin{aligned} G_t(A,B)G_{1-t}^{-1}(B,A) &= A\big(A^{-1}B\big)^t\big(B\big(B^{-1}A\big)^{1-t}\big)^{-1} \\ &= A\big(A^{-1}B\big)^t\big(B^{-1}A\big)^{t-1}B^{-1} \\ &= A\big(A^{-1}B\big)^t\big(B^{-1}A\big)^t\big(B^{-1}A\big)^{-1}B^{-1} \\ &= A\big(A^{-1}B\big)^t\big(B^{-1}A\big)^t A^{-1} \\ &= A\big(A^{-1}B\big)^t\big(A\big(B^{-1}A\big)^{-t}\big)^{-1} = I. \end{aligned}$$

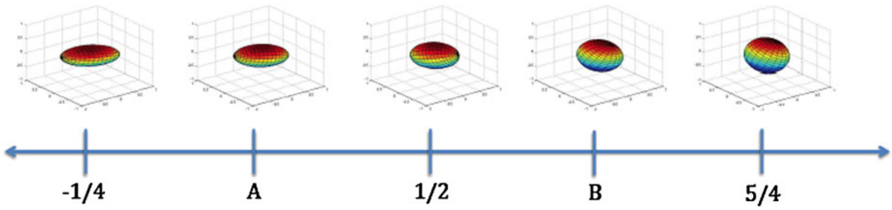


Fig. 4 Geometric means of two matrices, at $t = -\frac{1}{4}, 0, \frac{1}{2}, 1, \frac{5}{4}$

Thus, $G_t(A, B) = G_{1-t}(B, A)$. □

Real symmetric positive definite matrices have a geometrical interpretation. They represent (hyper) ellipsoids in $\mathbb{R}^n$. The eigenvectors of the matrix represent the directions of the ellipsoid radii, while the corresponding eigenvalues represent their lengths. We use this interpretation to demonstrate our definitions, for a 3×3 matrices as ellipsoids in $\mathbb{R}^3$.

In Fig. 4 we demonstrate the geometric mean (3.1), for two positive definite matrices, represented by their correspondence ellipsoids. For

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} 8 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 3 \end{pmatrix},$$

we show $G_t(A, B)$ for $t = -\frac{1}{4}, 0, \frac{1}{2}, 1, \frac{5}{4}$.

Now we can define our subdivision scheme, in the form of (2.4).

Definition 3.2 For a sequence of positive definite matrices $\{A_{0,i}\}_{i \in \mathbb{Z}}$, the refinement step is defined as follows:

$$\begin{aligned} A_{k+1,2i} &= A_{k,i}, \\ A_{k+1,2i+1} &= G_{-\frac{1}{8}}\left(G_{\frac{1}{2}}(A_{k,i}, A_{k,i+1}), G_{\frac{1}{2}}(A_{k,i-1} A_{k,i+2})\right). \end{aligned} \tag{3.3}$$

The subdivision scheme consist of repeated application of the refinement step, which we denote by $\mathcal{S}_G$. Denote by $\{A_{j,i}\}_{i \in \mathbb{Z}}$ the set of matrices after j refinement steps.

By Lemma 3.1 the subdivision scheme is well defined. In order to avoid ill-posed problems from our settings, we assume the following.

Assumption 1 *Let* $\{A_{0,i}\}_{i \in \mathbb{Z}}$ *be initial data. Then*

$$\sup_{l \in \mathbb{Z}} d(A_{0,l}, A_{0,l+1}) < C,$$

where C *is a positive constant.*

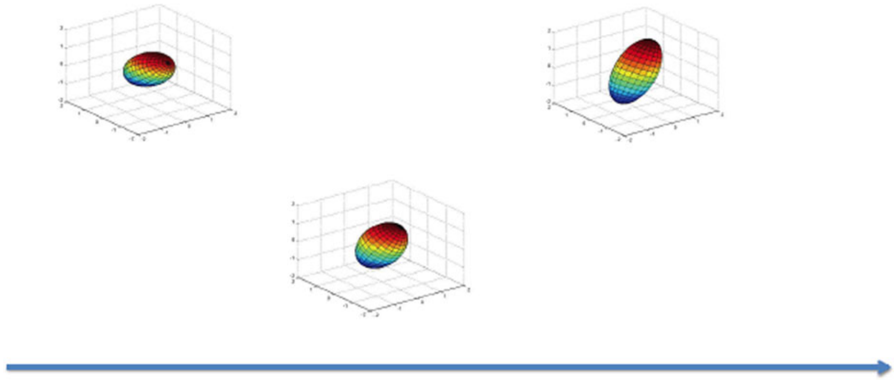


Fig. 5 The three central positive definite matrices, out of seven initial matrices

Remark 3.3 Note that a general modification to the 4-point scheme (2.3), can be done by different refinement steps, using different repeated averages. For example consider

$$A_{k+1,2i+1} = G_{\frac{1}{2}}\big(G_{\frac{9}{8}}(A_{k,i-1}, A_{k,i}), G_{-\frac{1}{8}}(A_{k,i+1}A_{k,i+2})\big).$$

The analysis of each of those different schemes is similar.

Remark 3.4 A broader family of the 4-point scheme consists of the refinement rules (2.3) with a varying parameter w parameter. In this paper we used $w = \frac{1}{16}$, which is well known in the classical scalar setup, because it reconstructs cubic polynomials. This parameter, also known as a "tension" parameter, has been studied for the linear (scalar) 4-point scheme [8]. In the matrix setting, we refer to

$$\begin{aligned} A_{k+1,2i} &= A_{k,i}, \\ A_{k+1,2i+1} &= G_{-2w}\big(G_{\frac{1}{2}}(A_{k,i}, A_{k,i+1}), G_{\frac{1}{2}}(A_{k,i-1}, A_{k,i+2})\big), \end{aligned} \tag{3.4}$$

for $w \in (0, w^*)$, with $w^* \approx 0.19273$ (the unique real solution of the cubic equation $32w^3 + 4w - 1 = 0$), see [17]. This linear 4-point scheme has $\mathbf{C}^1$ continuity and thus the smoothness as well as the rest of the results of this paper hold.

We demonstrate the scheme by three figures, representing data and the refinement process. The data, in Fig. 5, are sampled on the integers $x = 4, 5, 6$ from the (smooth) function $f(x) = \sin^2(x)A + \cos^2(x)B$, with positive definite matrices

$$A = \begin{pmatrix} 19 & -3 & -10 \\ -3 & 10 & 0 \\ -10 & 0 & 8 \end{pmatrix}, \qquad B = \begin{pmatrix} 18 & 8 & 18 \\ 8 & 10 & 12 \\ 18 & 12 & 22 \end{pmatrix}.$$

In Figs. 6 and 7 we present the first and second refinement, done by the geometric averaging scheme (3.3), respectively.

In the next section we analyze the convergence and smoothness of this scheme.

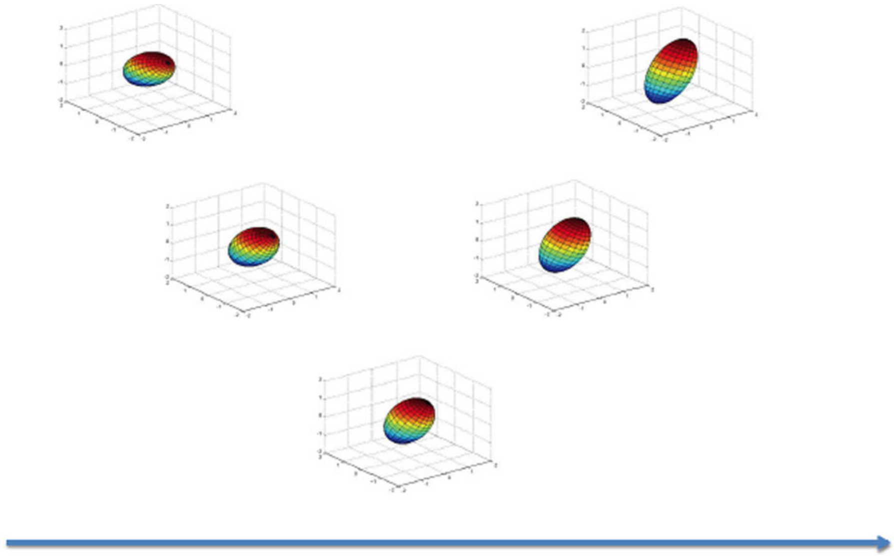


Fig. 6 First refinement step

4 Analysis of the Matrix Scheme

In this section we analyze the convergence and smoothness of the scheme defined by (3.3). Furthermore we show a class of matrix functions which our scheme fully reproduces. We begin by introducing the concept of a "matrix curve" and its derivative.

Definition 4.1 Let $A(x)$ be a matrix valued function satisfying

$$\lim_{\Delta x \to 0} \| A(x + \Delta x) - A(x) \| = 0, \quad x \in I.$$

Then $A(x)$ is called a *continuous matrix curve*. If in addition the limit

$$\lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x}, \quad x \in I,$$

exists for every x, then $A(x)$ is called a *smooth matrix curve*.

Henceforth, unless otherwise stated, we denote the initial data of positive definite matrices by $\mathbf{A}_0 = \{A_{0,i}\}_{i \in \mathbb{Z}}$, and the limit matrix curve by $A(x) = \mathcal{S}_G^\infty(\mathbf{A}_0)(x)$, $x \in \mathbb{R}$.

4.1 Convergence Analysis

We start the convergence analysis with a lemma.

Lemma 4.2 *Let* $\Delta_k = \sup_i d(A_{k,i}, A_{k,i+1})$. *Then* $\Delta_{k+1} < \frac{7}{8}\Delta_k$.

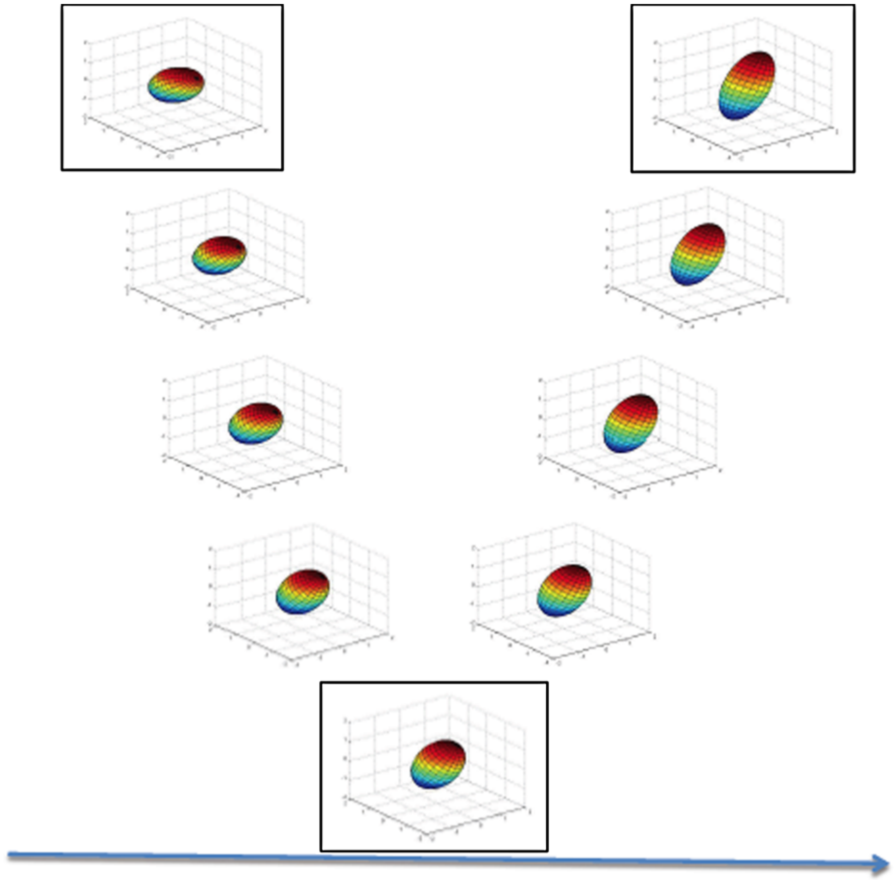


Fig. 7 Second refinement step. The initial matrices are framed

Proof Denote $B_{k,i} = G_{\frac{1}{2}}(A_{k,i}, A_{k,i+1})$ as the mid-point of the geodesic curve, between $A_{k,i}$ and $A_{k,i+1}$. Then, by (3.1) and (3.2) $d(B_{k,i}, A_{k,i}) = \frac{1}{2} d(A_{k,i}, A_{k+1,i})$. By the definition of Δ_k we deduce that $d(A_{k,i-1}, A_{k,i+2}) \leq 3\Delta_k$, which suggests $d(A_{k,i-1}, G_{\frac{1}{2}}(A_{k,i-1}, A_{k,i+2})) \leq \frac{3}{2}\Delta_k$. Then

$$\begin{aligned}
&d(A_{k+1,2i}, A_{k+1,2i+1}) \\
&\quad \leq d(A_{k,i}, B_{k,i}) + d(B_{k,i}, A_{k+1,2i+1}) \\
&\quad = \frac{1}{2}\Delta_k + \left\|\log\left(B_{k,i}^{-\frac{1}{2}}\left(B_{k,i}^{\frac{1}{2}}\left(B_{k,i}^{-\frac{1}{2}} G_{\frac{1}{2}}(A_{k,i-1}, A_{k,i+2}) B_{k,i}^{-\frac{1}{2}}\right)^{-\frac{1}{8}} B_{k,i}^{\frac{1}{2}}\right) B_{k,i}^{-\frac{1}{2}}\right)\right\| \\
&\quad = \frac{1}{2}\Delta_k + \frac{1}{8}\left\|\log\left(B_{k,i}^{-\frac{1}{2}} G_{\frac{1}{2}}(A_{k,i-1}, A_{k,i+2}) B_{k,i}^{-\frac{1}{2}}\right)\right\| \\
&\quad \leq \frac{1}{2}\Delta_k + \frac{1}{8} d\left(B_{k,i}, G_{\frac{1}{2}}(A_{k,i-1}, A_{k,i+2})\right)
\end{aligned}$$

$$\leq \frac{1}{2}\Delta_k + \frac{1}{8}\Big(d(B_{k,i}, A_{k,i}) + d(A_{k,i}, A_{k,i-1}) + d\big(A_{k,i-1}, G_{\frac{1}{2}}(A_{k,i-1}, A_{k,i+2})\big)\Big)$$
$$\leq \frac{1}{2}\Delta_k + \frac{1}{8}\left(\frac{3\Delta_k}{2} + \frac{3\Delta_k}{2}\right) \leq \frac{7}{8}\Delta_k.$$

Since the product inside the logarithm operator is a positive definite matrix, we can use a regular logarithm power rule (see [28]).

The other symmetrical case is

$$d(A_{k+1,2i+2}, A_{k+1,2i+1}) \leq d(A_{k,i+1}, B_{k,i}) + d(B_{k,i}, A_{k+1,2i+1}).$$

Due to Assumption 1, taking the supremum is well defined and the lemma follows. □

Theorem 4.3 *For any initial data* $\mathbf{A}_0$ *the scheme defined by* (3.3) *converges to a continuous matrix curve* $A(x)$, $x \in \mathbb{R}$.

Proof Let $x_{j,i} = 2^{-j}i$, $j \in \mathbb{Z}_+$, $i \in \mathbb{Z}$. Then define the continuous matrix curves,

$$F_j(x) = G_{\frac{x - x_{j,i}}{x_{j,i} - x_{j,i+1}}}(A_{j,i}, A_{j,i+1}), \quad x \in [x_{j,i}, x_{j,i+1}).$$

By the interpolatory property and Lemma 4.2 we have

$$\sup_x d\big(F_j(x), F_{j+1}(x)\big) \leq \sup_x d\big(F_j(x), A_{j,k_x}\big) + \sup_x d\big(A_{j+1,2k_x}, F_{j+1}(x)\big)$$
$$\leq \frac{1}{2}\Delta_j + \Delta_{j+1} \leq \frac{1}{2}\Delta_j + \frac{7}{8}\Delta_j = \frac{11}{8}\Delta_j,$$

with $k_x = \arg\min_{k=i,i+1}\{d(F_j(x), A_{j,k})\}$, $x \in [x_{j,i}, x_{j,i+1})$, the index of the closest matrix in the jth refinement level to $F_j(x)$. Using Lemma 4.2 we have

$$\sum_{m=1}^{\infty} \Delta_m \leq 7\Delta_0. \tag{4.1}$$

Thus, for any $p < l$, $p, l \in \mathbb{N}$

$$\sup_x d\big(F_{j+p}(x), F_{j+l}(x)\big) \leq \frac{11}{8}\sum_{m=j+p}^{j+l} \Delta_m \leq \frac{77}{8}\Delta_j.$$

Since Δ_j tends to zero we see that $\{F_j(x)\}_{j\in\mathbb{N}}$ is a Cauchy series with the uniform norm. The completeness of $\mathbb{R}^{n\times n}$ implies that there exists a limit function $A(x)$ such that the sequence $\{F_j(x)\}_{j\in\mathbb{N}}$ converges uniformly to $A(x)$. □

Another conclusion from the last results is the boundedness of the generated curve $A(x)$. Denote by $\delta_j = \sup_l \min_i d(A_{j,l}, A_{0,i})$. According to (4.1) and the fact that the scheme is interpolatory we get $\delta_j \leq 7\Delta_0$, for all $j \in \mathbb{Z}_+$. We infer the following corollary.

Corollary 4.4 *The matrix curve $A(x)$ is bounded*,

$$\min_i d\big(A(x), A_{0,i}\big) \le 7\Delta_0, \quad x \in \mathbb{R}. \tag{4.2}$$

Remark 4.5 Since $\mathbf{A}_0$ is a sequence of symmetric positive definite matrices we note that the generated matrix curve $A(x)$ consists of symmetric semidefinite matrices, which is the closure of the set of positive definite matrices. Later we prove that $A(x)$ is positive definite for all values of x.

4.2 Smoothness Analysis

In this subsection, we prove the $\mathbf{C}^1$ smoothness of our geometric averaging scheme. We follow Wallner and Dyn [30], and Grohs [15, 16], who studied the proximity of a non-linear subdivision to the corresponding linear subdivision scheme. By the proximity condition the smoothness analysis and approximation order are obtained.

In the context of our scheme, we denote by $\mathcal{S}_L$ the refinement operator of the linear 4-point scheme, defined using matrix operations. The scheme is well defined and is equal to an entry-wise linear 4-point subdivision for scalars. Thus, this subdivision clearly converges to a $\mathbf{C}^1$ smooth parametric matrix curve.

Here we present the matrix analogue to Theorem 2.5, which guarantees the proximity condition for our scheme.

Theorem 4.6 *The scheme*, *defined by* (3.3), *satisfies a proximity condition with the linear* 4*-point scheme of the form*

$$\big\|\mathcal{S}_L(\mathbf{a}_0) - \mathcal{S}_G(\mathbf{a}_0)\big\| < O\big(\Delta_0^2\big),$$

where $\Delta_0 = \sup_i d(A_{0,i}, A_{0,i+1})$.

Proof Let $\{M_i\}_{i\in\mathbb{Z}}$ be a matrix sequence of differences, $M_i = A_{0,i+1} - A_{0,i}$. Clearly M_i is a symmetric matrix.

By the equivalence of norms over a finite dimensional space, $\|M_i\| = O(\Delta_0)$, $i \in \mathbb{Z}$. Furthermore, using $\mathcal{S}_L$, for the linear scheme we get

$$\mathcal{S}_L(\mathbf{A_0})_{2i+1} = A_{0,i} + \frac{1}{2}M_i + \frac{1}{16}M_{i-1} - \frac{1}{16}M_{i+1}. \tag{4.3}$$

We use the big $-O$ notation in the norm sense, i.e., we state $X = Y + O(h)$ when there exist universal positive constants $h_0 > 0$ and C such that for each $0 < h < h_0$, $\|X - Y\| < Ch$. For the geometric averaging scheme using the fact that $(I + X)^t = I + tX + O(\|X^2\|)$, we get the auxiliary results

$$\begin{aligned} G_t(A_{0,i}, A_{0,i+1}) &= G_t(A_{0,i}, A_{0,i} + M_i) \\ &= A_{0,i}\big(I + A_{0,i+1}^{-1} M_i\big)^t \\ &= A_{0,i} + tM_i + O\big(\Delta_0^2\big). \end{aligned} \tag{4.4}$$

Thus $G_{\frac{1}{2}}(A_{0,i}, A_{0,i+1}) = A_{0,i} + \frac{M_i}{2} + O(\Delta_0^2)$ and $G_{\frac{1}{2}}(A_{0,i-1}, A_{i+2,0}) = A_{0,i} + \frac{M_i}{2} + \frac{M_{i+1}-M_{i-1}}{2} + O(\Delta_0^2)$. By (3.3),

$$\begin{aligned} A_{1,2i+1} &= G_{-\frac{1}{8}}\big(G_{\frac{1}{2}}(A_{0,i}, A_{0,i+1}), G_{\frac{1}{2}}(A_{0,i-1}, A_{0,i+2})\big) \\ &= A_{0,i} + \frac{1}{2}M_i + \frac{1}{16}M_{i-1} - \frac{1}{16}M_{i+1} + O\big(\Delta_0^2\big). \end{aligned} \tag{4.5}$$

Therefore, (4.3) together with (4.5) completes the proof. □

Based on Theorems 6 and 9 from [30] and [9], respectively, and the proximity condition of Lemma 4.6 we conclude to the following.

Corollary 4.7 *The geometric averaging scheme defined by* (3.3) *satisfies*

(1) *Convergence to a smooth* $\mathbf{C}^1$ *matrix curve.*
(2) *Approximation order of* h^2.

The next corollary relates the last result with Remark 3.3.

Corollary 4.8 *Any rearrangement of the repeated averages of* (3.3) *determines a subdivision scheme which generates a smooth matrix curve for any* $\mathbf{A}_0$ *with sufficiently small* Δ_0. *Furthermore, the distance of any two different limits, from the same initial data, is* $O(\Delta_0^2)$.

4.3 Functions Reproduced by the Scheme

A great interest in approximation theory is the reproduction of classes of functions [3]—i.e., functions that are perfectly reconstructed. We present two classes of matrix functions which are reproduced by the scheme, under uniform sampling.

Trivially, when given a data set of constant positive definite matrices, namely $A_{0,i} \equiv A$, $i \in \mathbb{Z}$, we generate a constant matrix curve, namely $A(x) \equiv A$ for all $x \in \mathbb{R}$. Here we present a broader class of matrix functions which our scheme perfectly reconstructs.

Theorem 4.9 *Let* $F(x)$ *be a matrix function such that*

$$F(x) = G_{\frac{x-y_1}{y_2-y_1}}\big(F(y_1), F(y_2)\big), \quad x \in [y_1, y_2), \; y_1, y_2 \in \mathbb{R}.$$

Let $\{A_{0,i}\}_{i\in\mathbb{Z}}$ *be a sequence of positive definite matrices sampled uniformly from the matrix function* $F(x)$, *namely*

$$d(A_{0,i}, A_{0,i+1}) = d(A_{0,j}, A_{0,j+1}), \quad i, j \in \mathbb{Z}.$$

Then the generated curve $A(x)$ *is identical to* $F(x)$.

Proof By the definition of $\{A_{0,i}\}_{i\in\mathbb{Z}}$, it is clear that

$$G_{\frac{1}{2}}(A_{0,k}, A_{0,k+1}) = G_{\frac{1}{2}}(A_{0,k-1}, A_{0,k+2}) = F\left(i + \frac{1}{2}\right).$$

Therefore, by (3.3)

$$A_{1,2k+1} = G_{\frac{1}{2}}(A_{0,k}, A_{0,k+1}).$$

Thus F coincide with $\mathcal{S}_G(\{A_{0,i}\}_{i\in\mathbb{Z}})$ at all dyadic points. By the continuity of both functions the claim follows. □

Trivially, $\alpha A \beta B = \alpha\beta AB$ with α, β scalars. Thus by Theorem 2.2, we expect the geometric matrix scheme to reproduce any function of the form $f_A(x) = \exp(p(x))A$ where A is positive definite matrix and $p(x)$ is a function in the space reproduced by the scalar linear scheme. We now summarize.

Corollary 4.10 *Let $F_A(x) = \exp(p(x))A$, with A a positive definite matrix. Assume $F_A(i) = A_{0,i}$, $i \in \mathbb{Z}$. Then the generated curve $A(x)$ is identical with $F(x)$.*

Due to locality of the 4-point scheme, the result above is true also for a local set of indices sampled from a local interval.

5 Properties of the Matrix Scheme

In this section we present properties that the scheme retains. These results indicate that the geometric setting of the subdivision is suitable for positive definite matrices.

First we consider the algebraic properties, which concludes the determinant, inverse, and adjoint matrix [29, Chap. 5]. Then we consider the class of spectral properties. It includes bounds on the eigenvalues and condition number of the generated matrix curve, and the properties of preserving local eigenvectors and invariant spaces.

5.1 Algebraic Properties

We start with a primary connection between the scalar scheme (2.4), which we denote in this section by $\mathcal{S}_g$, and the matrix scheme.

Theorem 5.1 *For all real x,*

$$\det\bigl(\mathcal{S}_G^\infty(\{A_{0,i}\}_{i\in\mathbb{Z}})(x)\bigr) = \mathcal{S}_g^\infty\bigl(\bigl\{\det(A_{0,i})\bigr\}_{i\in\mathbb{Z}}\bigr)(x).$$

Proof It is sufficient to prove the claim for a single refinement step, since by the continuity of the limits of $\mathcal{S}_G$ and $\mathcal{S}_g$, if $\det(\mathcal{S}_G^k(\{A_{0,i}\}_{i\in\mathbb{Z}})_j) = \mathcal{S}_g^k(\{\det(A_{0,i})\}_{i\in\mathbb{Z}})_j$ for all $k \in \mathbb{N}$, the claim of the theorem follows. Note that

$$\det\bigl(G_t(A, B)\bigr) = \det(A)^{1-t} \det(B)^t.$$

Thus

$$\det\big((\mathcal{S}_G(\{A_{k,j}\}_{j\in\mathbb{Z}}))_{2i+1}\big) = \det(A_{k+1,2i+1}) = \frac{\det(A_{k,i})^{\frac{9}{16}}\det(A_{k,i+1})^{\frac{9}{16}}}{\det(A_{k,i-1})^{\frac{1}{16}}\det(A_{k,i+2})^{\frac{1}{16}}} = \big(\mathcal{S}_g\big(\{\det(A_{k,j})\}_{j\in\mathbb{Z}}\big)\big)_{2i+1}. \qquad \square$$

Since the geometric scalar scheme is positive (see Corollary 2.4) Theorem 5.1 implies the following.

Corollary 5.2 *The matrix curve $A(x)$ of (3.3) is positive definite for any $x \in \mathbb{R}$.*

Now we show the commutativity of the scheme with inversion.

Theorem 5.3 *Denote $\mathbf{A}^{-1} = \{A_{0,i}^{-1}\}_{i\in\mathbb{Z}}$, Then for all $x \in \mathbb{R}$,*

$$A^{-1}(x) = \mathcal{S}_G^{\infty}(\mathbf{A}^{-1})(x).$$

Proof As in Theorem 5.1, it is sufficient to show the result for one refinement step, namely $\mathcal{S}_G(\mathbf{A}^{-1}) = (\mathcal{S}_G(\mathbf{A}_0))^{-1}$. Thus, we show that $G_t^{-1}(A,B) = G_t(A^{-1},B^{-1})$, for any positive definite matrices A, B and $t \in \mathbb{R}$.

From Lemma 3.1, $X = (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t$ is a positive definite matrix, and, thus,

$$G_t^{-1}(A,B) = \big(A^{\frac{1}{2}}\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^t A^{\frac{1}{2}}\big)^{-1} = A^{-\frac{1}{2}}X^{-1}A^{-\frac{1}{2}} = A^{-\frac{1}{2}}\big(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\big)A^{-\frac{1}{2}} = G_t\big(A^{-1},B^{-1}\big).$$

Now, following the refinement rule (3.3), with initial data $\{(A_{i,0})^{-1}\}_{i\in\mathbb{Z}}$ and the last result, we have

$$G_{-\frac{1}{8}}\big(G_{\frac{1}{2}}\big(A_{k,i}^{-1},A_{k,i+1}^{-1}\big),G_{\frac{1}{2}}\big(A_{k,i-1}^{-1},A_{k,i+2}^{-1}\big)\big) = G_{-\frac{1}{8}}\big(G_{\frac{1}{2}}^{-1}(A_{k,i},A_{k,i+1}),G_{\frac{1}{2}}^{-1}(A_{k,i-1},A_{k,i+2})\big) = G_{-\frac{1}{8}}^{-1}\big(G_{\frac{1}{2}}(A_{k,i},A_{k,i+1}),G_{\frac{1}{2}}(A_{k,i-1},A_{k,i+2})\big).$$

Namely, after each refinement with the inverse data, we have an inverse result. Thus the lemma follows. $\square$

By (3.1) we can conclude that $G_t(\alpha A,\beta B) = \alpha^{1-t}\beta^t G_t(A,B)$, $t \in \mathbb{R}$. Therefore, by Theorems 5.1 and 5.3, we have for the standard adjoint operator, $X\,\mathrm{adj}(X) = \det(X)I$, the following.

Corollary 5.4 *For all $x \in \mathbb{R}$,*

$$\mathrm{adj}\big(\mathcal{S}_G^{\infty}(\{A_{0,i}\}_{i\in\mathbb{Z}})\big)(x) = \mathcal{S}_G^{\infty}\big(\{\mathrm{adj}(A_{0,i})\}_{i\in\mathbb{Z}}\big)(x).$$

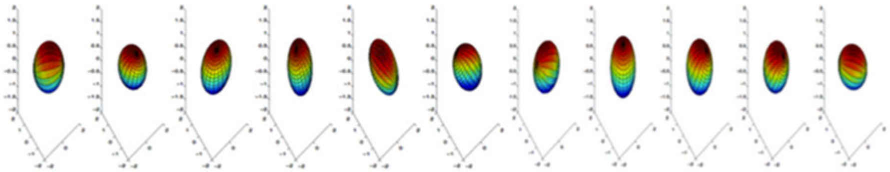


Fig. 8 Random matrices data

5.2 Spectral Properties

We begin with bounds on the spectral radius of the generated matrix curve. We update our notation, for any positive definite matrix X we denote by $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ the maximal and minimal eigenvalues of X, respectively.

The next theorem bounds from above and below, the eigenvalues of the matrices on the limit matrix curve.

Theorem 5.5 *Let* $i(x) = \arg\min_{i\in\mathbb{Z}} d(A(x), A_{0,i})$. *Then for all real* x

(1) $\lambda_{\max}(A(x)) \le \lambda_{\max}(A_{0,i(x)}) \exp(7\Delta_0)$;
(2) $\lambda_{\min}(A(x)) \ge \lambda_{\min}(A_{0,i(x)}) \exp(-7\Delta_0)$.

Proof By Corollary 4.4 we have $\min_i d(A(x), A_{0,i}) \le 7\Delta_0$. Since $\lambda_{\max}(A) \le \|A\|$ we have

$$\lambda_{\max}(B) = \lambda_{\max}\bigl(A^{\frac{1}{2}} A^{-\frac{1}{2}} B A^{-\frac{1}{2}} A^{\frac{1}{2}}\bigr) \le \lambda_{\max}(A) \exp\bigl(d(A, B)\bigr).$$

As for the second claim, by Definition 3.2 and the log rules on matrices,

$$\begin{aligned} d\bigl(A^{-1}, B^{-1}\bigr) &= \bigl\|\log\bigl(A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}}\bigr)\bigr\| \\ &= \bigl\|\log\bigl(\bigl(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\bigr)^{-1}\bigr)\bigr\| = d(A, B). \end{aligned}$$

Therefore,

$$\arg\min_i d\bigl(A(x), A_{0,i}\bigr) = \arg\min_i d\bigl(A^{-1}(x), A_{0,i}^{-1}\bigr). \tag{5.1}$$

The second claim follows from the first claim, (5.1) and Theorem 5.3. □

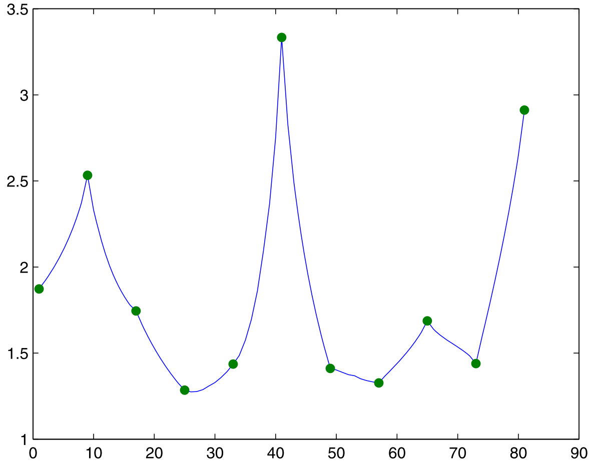
In Fig. 9 we plot the largest eigenvalue $\lambda_{\max}$ of each matrix along the matrix curve. Each value of an interpolated matrix is marked. Clearly the curve is bounded by the maximal element of the set of largest eigenvalues from the initial data. Thus the result of Theorem 5.5 holds. The data taken from a random set of positive definite matrices are presented in Fig. 8.

From Theorem 5.5 we bound the condition number $(\frac{\lambda_{\max}}{\lambda_{\min}})$ of the matrix curve generated by the geometric averaging scheme.

Corollary 5.6 *Let* $\kappa(A(x))$ *be the condition number of* $A(x)$ *for* $x \in \mathbb{R}$. *Using the notation of Theorem* 5.5 *we have*

$$\kappa\bigl(A(x)\bigr) \le \kappa(A_{0,i(x)}) \exp(14\Delta_0).$$

Fig. 9 Largest eigenvalues of the generated matrix curve. We mark the values of the interpolated matrices which are presented in Fig. 8



For the next property we use the following definition.

Definition 5.7 Let $\{A_j\}_{j\in\mathbb{Z}}$ be a sequence of positive definite matrices. A vector v is called a *common eigenvector* in the locality $[m_1, m_2]$ if for all $j \in \mathbb{Z}$, $m_1 \le j \le m_2$, we have $A_j v = \lambda_j v$.

The 4-point scheme is a local interpolation scheme such that the matrix $A(x)$ on the generated matrix curve depends only on data inside the parametric segment $[x - 3, x + 3]$. The next theorem shows that the scheme preserved common eigenspaces of the initial matrices.

Theorem 5.8 *Let v be a common eigenvector in the locality $[m_1, m_2]$ of $\mathbf{A_0}$, with $m_2 - m_1 > 6$, and with the associated eigenvalues $\{\lambda_{0,i}\}_{i\in I}$, $I = [m_1, m_2] \cap \mathbb{Z}$. Then*

$$\mathcal{S}_G^\infty\big(\{A_{0,i}\}_{i\in\mathbb{Z}}\big)(x)v = \mathcal{S}_g^\infty\big(\{\lambda_{0,i}\}_{i\in I}\big)(x)v, \quad x \in [m_1 + 3, m_2 - 3].$$

Proof As in the proof of Theorem 5.1, it is sufficient to focus on a single refinement step. It is easy to verify that for any positive definite matrices A, B, if $Av = \alpha v$ and $Bv = \beta v$, then $G_t(A, B)v = \alpha^{1-t}\beta^t v$. For a single refinement step we get

$$\begin{aligned} A_{1,2i+1}v &= G_{-\frac{1}{8}}\big(G_{\frac{1}{2}}(A_{0,i}, A_{0,i+1}), G_{\frac{1}{2}}(A_{0,i-1}, A_{0,i+2})\big)v \\ &= A_{0,i}\big(A_{0,i}^{-1}A_{0,i+1}\big)^{\frac{1}{2}}\Big(\big[A_{0,i}\big(A_{0,i}^{-1}A_{0,i+1}\big)^{\frac{1}{2}}\big]^{-1}A_{0,i-1}\big(A_{0,i-1}^{-1}A_{0,i+2}\big)^{\frac{1}{2}}\Big)^{-\frac{1}{8}}v \\ &= \frac{(\lambda_{0,i}\lambda_{0,i+1})^{\frac{9}{16}}}{(\lambda_{0,i-1}\lambda_{0,i+2})^{\frac{1}{16}}}v. \end{aligned}$$

□

A basic result from linear algebra states that the sum of all the columns is equal if and only if the vector $(1, 1, \dots, 1)^*$ is an eigenvector and the sum is the associated eigenvalue [19, Chap. 2]. This and Theorem 5.8 yield the following.

Corollary 5.9 *If for all initial matrices the sum of any column is the same then this property is shared by all matrices in the limit matrix curves $A(x)$.*

For the next result we use the following lemma.

Lemma 5.10 *Let M be an invertible matrix such that $M^{-1}A_{0,i}M$ is a positive definite matrix for all i. Then for every real x*

$$\mathcal{S}_G^{\infty}\big(\{M^{-1}A_{0,i}M\}_{i\in\mathbb{Z}}\big)(x) = M^{-1}\mathcal{S}_G^{\infty}(\mathbf{A}_0)(x)M.$$

Proof For every positive definite matrix X we have $(M^{-1}XM)^t = M^{-1}X^tM$ for all $t\in\mathbb{R}$. Therefore, for every pair of positive definite matrices X and Y,

$$G_t\big(M^{-1}XM, M^{-1}YM\big) = M^{-1}G_t(X,Y)M, \quad \forall t\in\mathbb{R}.$$

Combining this with the scheme definition and convergence completes the proof. □

The results of Theorem 5.8 can easily be extended to local common eigen-subspaces. One generalization of eigen-subspace is the invariant subspace. Namely, a subspace V, such that for any $v\in V$, $A_{0,i}v\in V$.

Theorem 5.11 *Let V be a common invariant subspace in the locality $\mathcal{J}=[m_1,m_2]$ with $m_2-m_1>6$. Then V is an invariant subspace of the matrices*

$$A(x) = \mathcal{S}_G^{\infty}\big(\{A_{0,i}\}_{i\in\mathbb{Z}}\big)(x), \quad x\in[m_1+3, m_2-3].$$

Proof By the assumption, we can find a matrix Q such that $A_{0,i}=Q^*B_{0,i}Q$, $Q^*Q=I$, where $B_{0,i}$ is the block matrix

$$B_{0,i} = \left[\begin{array}{c|c} \overline{A}_{0,i}^1 & 0 \\ \hline 0 & \overline{A}_{0,i}^2 \end{array}\right], \tag{5.2}$$

for any $i\in\mathcal{J}$, with $\{\overline{A}_i^1\}_{i\in\mathbb{Z}}$ a set of positive definite $k\times k$ matrices, with $k=\dim(V)$. By Lemma 5.10,

$$\mathcal{S}_G\big(\{A_{0,i}\}_{i\in\mathcal{J}}\big) = \left[\begin{array}{c|c} \mathcal{S}_G(\{\overline{A}_{0,i}^1\}_{i\in\mathcal{J}}) & 0 \\ \hline 0 & \mathcal{S}_G(\{\overline{A}_{0,i}^2\}_{i\in\mathcal{J}}) \end{array}\right].$$

This observation, when used repeatedly leads to the claim of the theorem. □

The last theorem leads to the diagram of Fig. 10.

6 Extension to Positive Semidefinite Matrices

The results of Theorems 5.8 and 5.11 allow us to weaken our assumptions in order to fit a broader range of applications and data. For a general sequence of semidefinite

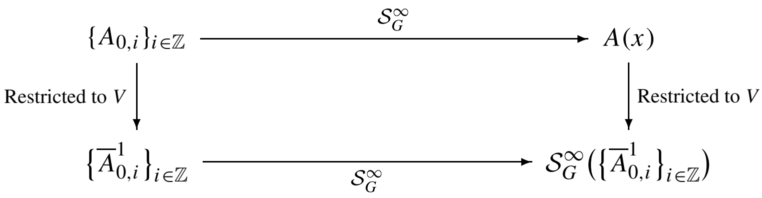


Fig. 10 Geometric averaging scheme restricted to invariant subspace V

matrices the geometric averaging scheme is not well defined. However, in special cases the scheme can be modified in order for it to be well defined and retain most of the properties of the scheme for the positive definite matrices. The last result of this section guarantees the robustness of this extension.

We use the following generalized inverse or "pseudo" inverse matrix definition, also known as Moore–Penrose inverse and denote it by $A^\dagger$. For further information see [5].

Using the notion of the Moore–Penrose inverse, we see for any semidefinite matrix A with a spectral decomposition

$$A = Q^* D Q, \quad Q^* Q = I, \quad D = \operatorname{diag}\{\lambda_1, \dots, \lambda_k, 0, \dots, 0\},\ \lambda_1 \geq \cdots \geq \lambda_k > 0$$

that

$$A^\dagger = Q^* D^\dagger Q, \tag{6.1}$$

where $D^\dagger = \operatorname{diag}\{\lambda_1^{-1}, \dots, \lambda_k^{-1}, 0, \dots, 0\}$. Note that (6.1) defines a unique pseudo inverse matrix $A^\dagger$. In the spirit of (6.1) we define $A^0 = AA^\dagger$ and $A^t = (A^\dagger)^{-t}$ for $t < 0$.

Now we can extend the analysis given in Sect. 4, namely convergence and smoothness, for this case of semidefinite matrices as well.

In this section $\mathbf{A_0}$ consists of positive semidefinite matrices. Furthermore, we assume

$$\operatorname{Ker}(A_{0,i}) = \operatorname{Ker}(A_{0,j}) \quad i, j \in \mathbb{Z}. \tag{6.2}$$

Many classes of matrices have a common kernel, e.g. the class of discrete Laplacian matrices [21], and for these classes we can use the following modified geometric averaging scheme.

Definition 6.1 For A, B positive semidefinite matrices, we define

$$\left(G^\dagger\right)_t(A, B) = A\left(A^\dagger B\right)^t = A^{\frac{1}{2}}\left(\left(A^\dagger\right)^{\frac{1}{2}} B \left(A^\dagger\right)^{\frac{1}{2}}\right)^t A^{\frac{1}{2}}.$$

Moreover, we denoted by $\mathcal{S}_{G^\dagger}$ the geometric refinement step (3.3), using $(G^\dagger)_t$ instead of G_t.

Theorem 6.2 *The refinement step $\mathcal{S}_{G^\dagger}$ is well defined. Furthermore $\mathcal{S}_{G^\dagger}^\infty$ converges to a smooth matrix curve $A(x)$, such that $A(x)$ is a semidefinite matrix for any x, and* $\operatorname{Ker}(A(x)) = \operatorname{Ker}(A_{0,i})$, $i \in \mathbb{Z}$.

Proof Due to the symmetry of the matrices, the orthogonal complement of the kernel $\mathrm{Ker}(A_{0,i})$ is $\mathrm{Image}(A_{0,i})$, which is an invariant subspace, for any $i \in \mathbb{Z}$. Thus we combine (6.1) and arguments similar to those in the proof of Theorem 5.11 to get

$$\mathcal{S}_{G^\dagger}\big(\{A_{0,i}\}_{i\in\mathbb{Z}}\big) = \left[\begin{array}{c|c} \mathcal{S}_G(\{\mathrm{Image}\, A_{0,i}\}_{i\in\mathbb{Z}}) & 0 \\ \hline 0 & 0 \end{array}\right]. \tag{6.3}$$

□

Remark 6.3 Note that in view of (6.3),

$$\mathcal{S}^\infty_{G^\dagger}\big(\{A_{0,i}\}_{i\in\mathbb{Z}}\big) = \left[\begin{array}{c|c} \mathcal{S}^\infty_G(\{\mathrm{Image}\, A_{0,i}\}_{i\in\mathbb{Z}}) & 0 \\ \hline 0 & 0 \end{array}\right].$$

The extension of all the results in the previous sections is straightforward. In particular we mention this:

$$A^\dagger(x) = \mathcal{S}^\infty_G\big(\{A^\dagger_{0,i}\}_{i\in\mathbb{Z}}\big).$$

It is well known that the pseudo inverse in the general case does not preserve continuity. However, by the factorization of the form of (6.3), it is clear that our scheme preserves continuity. The next theorem proves the robustness of our extension to positive semidefinite matrices.

Theorem 6.4 *Let* $\mathbf{A_0}$ *be a sequence of positive semidefinite matrices satisfies* (6.2). *Then for any* $x \in \mathbb{R}$,

$$\lim_{\epsilon\to 0} \mathcal{S}^\infty_G\big(\{A_{0,i} + \epsilon I\}_{i\in\mathbb{Z}}\big)(x) = \mathcal{S}^\infty_{G^\dagger}\big(\{A_{0,i}\}_{i\in\mathbb{Z}}\big)(x).$$

The theorem follows immediately from the next lemma about matrix geometric means.

Lemma 6.5 *Let* A *and* B *be positive semidefinite matrices such that* $\mathrm{Ker}(A) = \mathrm{Ker}(B)$. *Then*

$$\big(G^\dagger\big)_t(A, B) = \lim_{\epsilon\to 0} G_t(A + \epsilon I, B + \epsilon I).$$

Proof Since A and B are symmetric with $\mathrm{Ker}(A) = \mathrm{Ker}(B)$, every vector v has a unique decomposition,

$$v = v_1 + v_2, \quad v_1 \in \mathrm{Image}(A),\ v_2 \in \mathrm{Ker}(A),\ v_1 \perp v_2.$$

Thus it is enough to show that

$$\big(G^\dagger\big)_t(A, B)v_1 = \lim_{\epsilon\to 0} G_t(A + \epsilon I, B + \epsilon I)v_1 \quad \text{and} \quad \lim_{\epsilon\to 0} G_t(A + \epsilon I, B + \epsilon I)v_2 = \vec{0}.$$

For $v_1 \in \mathrm{Image}(A)$ we have

$$\lim_{\epsilon\to 0} G_t(A + \epsilon I, B + \epsilon I)v_1 = \lim_{\epsilon\to 0} (A + \epsilon I)\big((A + \epsilon I)^{-1}(B + \epsilon I)\big)^t v_1.$$

Using the representation as in (6.3) and the continuity of matrix operators we have

$$\lim_{\epsilon\to 0}(A+\epsilon I)\big((A+\epsilon I)^{-1}(B+\epsilon I)\big)^t v_1 = A\big(A^\dagger B\big)^t v_1 = \big(G^\dagger\big)_t(A,B)v_1.$$

For $v_2 \in \mathrm{Ker}(A)$ we get

$$\begin{aligned}\lim_{\epsilon\to 0} G_t(A+\epsilon I, B+\epsilon I)v_2 &= \lim_{\epsilon\to 0}(A+\epsilon I)\big((A+\epsilon I)^{-1}(B+\epsilon I)\big)^t v_2\\ &= \lim_{\epsilon\to 0}\epsilon I\big((\epsilon I)^{-1}(\epsilon I)\big)^t v_2\\ &= \lim_{\epsilon\to 0}\epsilon v_2 = \overrightarrow{0}.\end{aligned}$$

Thus, the proof is complete. □

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