CORRECTIONS TO *RIGID LOCAL SYSTEMS* MANUSCRIPT

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Corrections to Introduction, local page numbers

page 6, line -3: should read

$$\lambda(1-\lambda)(d/d\lambda)^2 f + (c - (a+b+1)\lambda)(d/d\lambda)f - abf = 0$$

Corrections to Chapter 2, local page numbers

page 32, line 1 of Remarks 2.10.4: should begin

1) Here is a slightly variant...

page 48, lines 2-3 of Corollary 2.13.3 should end/begin

Then $K = \mathcal{L}_{\chi(x-1)}[1]$ satisfies $K \star_{mid\times} D(inv^\star K) = \delta_1$.

Corrections to Chapter 3, local page numbers

page 10, last line (proof of Lemma 3.3.1) should read

proof This was proven in 2.10.2 and 2.10.8 above. QED

Corrections to Chapter 4, local page numbers

page 7, lines 2-3 of Lemma 4.3.8 should end/begin

For $\mathcal{F}$ lisse on $X - D$ and tame along D, $j : X - D \to X$ and $i : D \to X$ the inclusions, we have

Corrections to Chapter 8, local page numbers

page 4, lines 10-11 of the proof of 2) of Lemma 8.2.2 should end/begin

By proper base change

page 14, line 11 of 8.5.1 should begin

of $\otimes_i \mathcal{L}_{\chi_{2,i}(X_2 - T_i)}$. So essentially...

page 14, line 11 of 8.5.1: This is now correct, but still a bit confusing: the characters $\chi_{2,i}$ occurring in "$\otimes_i \mathcal{L}_{\chi_{2,i}(X_2 - T_i)}$" were defined on line -2 of the previous page as

$$\chi_{a,i} = \chi^{e(a,i)}.$$

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Rigid Local Systems

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Contents

Introduction

It is now nearly 140 years since Riemann introduced [Rie-EDL] the concept of a "local system" on $\mathbb{P}^1$ - {a finite set of points}. His idea was that one could (and should) study the solutions of an n'th order linear differential equation by studying the rank n local system (of its local holomorphic solutions) to which it gave rise. Riemann knew that a rank n local system on $\mathbb{P}^1$ - {m points} was "nothing more" than a collection of m invertible matrices A_i in $GL(n, \mathbb{C})$ which satisfy the matrix equation $A_1A_2...A_m = (id_n)$, such collections taken up to simultaneous conjugation by a single element of $GL(n, \mathbb{C})$. He also knew each individual A_i was, up to $GL(n, \mathbb{C})$ conjugacy, just the effect of analytic continuation along a small loop encircling the i'th missing point.

His first application of these then revolutionary ideas was to study the classical Gauss hypergeometric function [Rie-SG], which he did by studying rank two local systems on $\mathbb{P}^1$ - {three points}. His investigation was a stunning success, in large part because any such (irreducible) local system is **rigid** in the sense that it is determined up to isomorphism as soon as one knows separately the individual conjugacy classes of all its local monodromies. By exploiting this rigidity, Riemann was able to recover Kummer's transformation theory of hypergeometric functions "almost without calculation" [Rie-APM].

It soon became clear that Riemann had been "lucky", in the sense that the most local systems are not rigid. For instance, rank two irreducible local systems on $\mathbb{P}^1$ - {m points}, all of whose local monodromies are non-scalar, are rigid precisely for m=3. And rank n irreducible local systems on $\mathbb{P}^1$ - {three points}, each of whose local monodromies has n distinct eigenvalues, are rigid precisely for n=1 and n=2.

On the other hand, some of the best known classical functions are solutions of differential equations whose local systems are rigid, including both of the standard generalizations of the hypergeometric function, namely ${}_nF_{n-1}$, which gives a rank n local system on $\mathbb{P}^1$ - $\{0,1,\infty\}$, and the Pochhammer hypergeometric functions, which give rank n local systems on $\mathbb{P}^1$ - {n+1 points}.

In the classical literature, rigidity or its lack is expressed in terms of the vanishing or nonvanishing of the "number of accessory parameters". But the object whose rigidity is classically in question is not a rank n local system on $\mathbb{P}^1$ - {m points}, but rather an n'th order linear differential equation with rational function coefficients which has regular singularities at the m missing points, and no other singularities. In practice, one assumes also that each of the local monodromies has n distinct eigenvalues, expressed classically by saying that no two exponents differ by integers. One then looks for the most general n'th order linear differential equation with rational function coefficients which has regular singularities at the m missing points, no other singularities, and whose indicial polynomial at each missing point is the same as for the equation we started with. [This game with indicial polynomials makes sense for any equation with regular singularities, but it is only a meaningful game in the case where each of its local monodromies has n distinct eigenvalues, for only then can we be sure that any equation with the same indicial polynomials automatically has isomorphic local monodromies.] The "number of parameters" upon which such an equation depends is called the "number of accessory parameters", or the "number of constants in excess" [Ince, 20.4]. For example, in the classical literature one finds that for second order equations with m regular singularities, the number of accessory parameters is m-3 [Ince, top of page 506].

From a modern point of view, what corresponds precisely to a rank n local system on $\mathbb{P}^1$ - {m points} is an $n \times n$ first order system of differential equations with regular singularities at the named points (i.e., an algebraic vector bundle with integrable connection on $\mathbb{P}^1$ - {m points} with regular singularities at the m missing points) [De-ED]. So what corresponds to rigidity for a local system is the absence of deformations of the corresponding $n \times n$ system which preserve local monodromy. But in the classical literature, the question of deforming such a system while preserving its local monodromy does not seem to be addressed. Even, or perhaps especially, if we start with an n'th order equation with regular singularities, there is a priori a great difference between deforming it as an equation and deforming it as a system (in both cases preserving local monodromies).

For an irreducible local system $\mathcal{F}$ on $\mathbb{P}^1$ - {m points}, there is a simple cohomological invariant, rig($\mathcal{F}$), the "index of rigidity", which measures the rigidity or lack thereof of $\mathcal{F}$, cf. Chapter 1. One denotes by

$$j: \mathbb{P}^1 - \{m \text{ points}\} \to \mathbb{P}^1$$

the inclusion, one forms the sheaf $j_*\mathrm{End}(\mathcal{F})$ on $\mathbb{P}^1$, and one then defines $\mathrm{rig}(\mathcal{F})$ to be the Euler characteristic $\chi(\mathbb{P}^1, j_*\mathrm{End}(\mathcal{F}))$. Since $\mathcal{F}$ is irreducible, we have

$$\mathrm{rig}(\mathcal{F}) := 2 - h^1(\mathbb{P}^1, j_*\mathrm{End}(\mathcal{F})).$$

One proves (1.1.2) that the irreducible local system $\mathcal{F}$ is rigid if and only if $\mathrm{rig}(\mathcal{F}) = 2$, i.e., if and only if $h^1(\mathbb{P}^1, j_*\mathrm{End}(\mathcal{F}))$ vanishes. So the integer $h^1(\mathbb{P}^1, j_*\mathrm{End}(\mathcal{F}))$ appears as a cohomological analogue of the number of accessory parameters, at least in the cases where that number was defined classically. In terms of the corresponding $n\times n$ system, this h^1 is the "number" of its deformations to systems having the same local monodromies. So in case we start with an n'th order equation, we should expect this h^1 to be larger than the number of accessory parameters, since the h^1 allows deformations as system, while in computing the number of accessory parameters we allow only deformations as equation.

In the case of a rank n irreducible local system $\mathcal{F}$ which arises from an n'th order equation, and each of whose local monodromies has all distinct eigenvalues, one can separately compute both the number, say ρ, of accessory parameters, and the number $h^1(\mathbb{P}^1, j_*\mathrm{End}(\mathcal{F}))$. One finds a doubling:

$$h^1(\mathbb{P}^1, j_*\mathrm{End}(\mathcal{F})) = 2\rho.$$

[Both sides come out to be $2 - [(2-m)n^2 + mn]$), cf. [Forsythe, pp. 127-128] for the calculation of ρ, and Chapter 1 for the calculation of h^1.] That the h^1 turns out to be even is not a surprise, because $H^1(\mathbb{P}^1, j_*\mathrm{End}(\mathcal{F}))$ carries a symplectic autoduality. But why the underlying n'th order equation should have "twice as many" system-deformations as equation-deformations seems entirely mysterious. It is as though the group $H^1(\mathbb{P}^1, j_*\mathrm{End}(\mathcal{F}))$ carried a weight one Hodge structure, in such a way that the "holomorphic part" $H^{1,0}$ corresponded to deformations as equation. But there is almost nothing to back up such speculation.

Let us return to the consideration of a rank n local system on $\mathbb{P}^1 - \{m \text{ points}\}$, in its incarnation as a collection, taken up to simultaneous conjugation, of m elements A_i of $GL(n, \mathbb{C})$ whose product is 1. Such a local system is said to be irreducible if the subgroup of $GL(n, \mathbb{C})$ generated by all the A_i is irreducible, i.e., if there exists no proper nonzero subspace of $\mathbb{C}^n$ which is respected by all the A_i. Given a local system, we extract, at each of the m missing points, the conjugacy class of its local monodromy at that point (concretely, the Jordan normal form of each separate A_i), and call this the numerical data of our local system. There is also the notion of abstract numerical data of rank n on $\mathbb{P}^1$ - {m points}: one specifies at each of the m missing points a conjugacy class in $GL(n, \mathbb{C})$, i.e., a Jordan normal form.

Two basic problems in the subject are:

Irreducible Recognition Problem Given abstract numerical data of rank n on $\mathbb{P}^1$ - {m points}, determine if it is the numerical data attached to an irreducible rank n local system on $\mathbb{P}^1$ - {m points}.

Irreducible Construction Problem Given abstract numerical data of rank n on $\mathbb{P}^1$ - {m points} which one is told arises from an irreducible local system on $\mathbb{P}^1$ - {m points}, construct explicitly at least one such local system. Or construct all such local systems.

Because the index of rigidity $\mathrm{rig}(\mathcal{F})$ can be expressed in terms of the underlying numerical data of $\mathcal{F}$ by universal formulas, one can pose these two problems separately for each of the a priori possible values 2, 0, -2, -4, ... of the index of rigidity. This book is devoted to the solution of these two problems in the special case of rigid local systems, $\mathrm{rig}(\mathcal{F}) = 2$.

The case of more general local systems remains entirely open. Already the next simplest case, $\mathrm{rig}(\mathcal{F}) = 0$, seems out of reach. Also the analogues of these problems for reducible local systems remain entirely open.

Another problem which remains open is this. Suppose we are given a rank n irreducible rigid local system $\mathcal{F}$ on $\mathbb{P}^1$ - {m points}. We know [De-ED] that it is the local system attached to a (unique, by rigidity) $n\times n$ system on $\mathbb{P}^1$ - {m points}, with regular singular points at the m missing points. Is this this $n\times n$ system in fact (the system attached to) an n'th order equation with regular singular points only at the m missing points? By [Ka-CV], any system has, Zariski locally, cyclic vectors. So at the expense of allowing finitely many additional "apparent singularities" in our n'th order equation, we can always get one whose local system is $\mathcal{F}$. The question is whether there exists such an equation **without** apparent singularities. It should be remarked that if we drop the word "rigid" from this question, a "counting constants" argument of Poincaré [Poin, pp. 314-315 of Tome II, where he counts projective representations] suggests that "in general" it should **not** be possible to avoid apparent singularities in this "strong form" of the Riemann-Hilbert problem. Another heuristic argument that one cannot avoid apparent singularities, pointed out to me by Washnitzer, is this. Consider an irreducible n'th order equation with regular singular points on $\mathbb{P}^1$ - {m points}, each of whose local monodromies has n distinct eigenvalues. Suppose the underlying local system, say $\mathcal{F}$, is not rigid. Then the calculation $h^1 = 2\rho$ discussed above suggests that $\mathcal{F}$ has a 2ρ-dimensional deformation space of local systems with the same local monodromies, and that only for deformations $\mathcal{G}$ in a ρ-dimensional subspace will there be an n'th order equation on $\mathbb{P}^1$ - {m points}, i.e, without apparent singularities, with $\mathcal{G}$ as monodromy.

Let us now turn to a more detailed discussion of the contents of this book. Although the Irreducible Recognition Problem is, on its face, an elementary problem about multiplying complex matrices which could be explained to a bright high school student, our solution for rigids is, unfortunately, far from elementary.

We begin with the trivial observation that on $\mathbb{P}^1$ - {m points}, any rank one local system is both irreducible and rigid. Our basic idea is to construct two sorts "operations" (which we call "middle convolution" and "middle tensor product") on a suitable collection of irreducible local systems which preserve the index of rigidity and whose effect on local monodromy we can calculate. The "middle tensor" operation offers no difficulty; all the work is in working out the theory of middle convolution.

What does convolution have to do with rigid local systems? The idea is simple. The earliest known, and still the best known, rigid local system is the local system of solutions of the second order differential equation

$$\lambda(1-\lambda)(d/d\lambda)^2 f + (c - (a+b+1)x)(d/d\lambda)f - abf = 0$$

satisfied by the Gauss hypergeometric function $F(a, b, c, \lambda)$. A solution is given [WW, page 293] by the integral

$$\int (x)^{a-c}(1-x)^{c-b-1}(\lambda - x)^{-a}dx.$$

Our key observation is that formally, this integral is the additive convolution

$$\int f(x)g(\lambda-x)dx$$

of $f(x) := (x)^{a-c}(1-x)^{c-b-1}$ and $g(x) := x^{-a}$. We then view $f(x)$ as incarnating a rigid local system, and $g(x)$ as incarnating a Kummer sheaf on $\mathbb{G}_m$, and think about forming the additive convolution of two such objects. In some sense, our entire book consists of first making sense of this, and then exploiting it.

In Chapter 2 we define the middle convolution operators, and work out their basic properties. The theory of perverse sheaves is the indispensable setting for this theory. The theory we need is that of middle convolution on $\mathbb{A}^1$ as additive group. However, we also devote some attention to the theory on $\mathbb{G}_m$, where it ties in nicely with our previous work [Ka-GKM] and [Ka-ESDE] on Kloosterman and hypergeometric sheaves and differential equations.

Chapters 3 and 4 are devoted to the proof that our middle convolution operators do in fact preserve the index of rigidity, and to calculating their effect on local monodromy. Here the main technical tool is the ℓ-adic Fourier Transform in characteristic $p > 0$. In Chapter 3, we show that Fourier Transform preserves the index of rigidity in characteristic p. Because middle convolution in characteristic $p > 0$ has a simple expression in terms of Fourier Transform, we find that middle convolution in characteristic p also preserves the index of rigidity. Because Laumon has worked out the precise effect of Fourier Transform on local monodromy, we also get the effect of middle convolution on local monodromy, still in characteristic $p > 0$. In Chapter 4, we use a specialization argument to show that these results on middle convolution still hold in characteristic zero (despite the fact that the Fourier Transform no longer exists).

The next step is to show, in Chapter 5, that any rigid irreducible local system can be built up from a rank one local system by applying a finite sequence of middle convolution and middle tensor operations. The proof of this last step gives us an algorithm to calculate, for any given irreducible rigid local system $\mathcal{F}$, exactly what sequence of operations to apply to what rank one local system $\mathcal{L}$ in order to end up with $\mathcal{F}$. This algorithm is thus a solution to the Irreducible Construction Problem for rigids.

This algorithm depends only on the "numerical data" of $\mathcal{F}$.

Roughly speaking, we solve the Irreducible Recognition Problem (in Chapter 6) by showing that if we are given some abstract numerical data which is rigid, then it comes from an irreducible $\mathcal{F}$ if and only if the algorithm, applied "formally", gives a meaningful answer.

In Chapter 7, we explore some of the diophantine aspects of rigidity. In Chapter 8, we reinterpret our construction of rigids in terms of "pieces" of the relative de Rham cohomology of suitable families of varieties. In Chapter 9, we use this cohomological expression, together with an easy but previously overlooked generalization of our earlier work on Grothendieck's p-curvature conjecture, to prove Grothendieck's p-curvature conjecture for all those differential equations on $\mathbb{P}^1$ - {m points} with regular singular points whose underlying local systems are rigid and irreducible.

Let us now discuss what is **not** done in this book. One could try to classify systematically all rigid irreducible local systems, since one has an algorithm to recognize their numerical data. Even a cursory glance at the kinds of local monodromy one can get by starting with a rank one local system and applying a cleverly chosen sequence of middle convolution and middle tensor operations leaves one with the impression that there is a fascinating bestiary waiting to be compiled.

One could also study the identities between special functions which presumably result whenever a rigid irreducible $\mathcal{F}$ can be built in two or more different ways out of rank one local systems by successive middle convolution and middle tensor operations. Already for the case of ${}_nF_{n-1}$, any $n \geq 2$, there are in general a plethora of such building paths. Do we get anything about ${}_nF_{n-1}$ which is not already in the classical literature?

In characteristic zero, we have shown how to construct all rigid irreducible local systems out of rank one local systems, by using the operations of middle convolution and middle tensor. Our arguments in fact begin in characteristic $p > 0$, where we prove the same result for rigid irreducible local systems which are everywhere tamely ramified. But in characteristic p, the everywhere tame local systems are by far the least interesting ones. What can be said about arbitrary rigid irreducible local systems in characteristic p? Is it true that any rigid irreducible local system in characteristic p on $\mathbb{P}^1$ - {m points} is built out of a rank one local system by finitely iterating the operations Fourier Transform, middle tensor with a rank one local system, and pullback by an automorphism of $\mathbb{P}^1$? For example, all the irreducible ℓ-adic hypergeometrics are rigid, and they are all obtained in this way [Ka-ESDE, proof of 8.5.3].

In characteristic zero, the analogue of a not necessarily tame local system is a differential equation which does not necessarily have regular singular points. There is no difficulty in defining the index of rigidity in the holonomic $\mathcal{D}$-module context, cf. [Ka-ESDE, 3.7.3 and 2.9.8.1]. One knows, for example, that the generalized hypergeometric equations studied in [Ka-ESDE] are rigid. One also has the $\mathcal{D}$-module Fourier Transform. It **should** be true that Fourier Transform preserves the index of rigidity in the $\mathcal{D}$-module context, but this is unknown. The main stumbling block to proving this is the absence of an $\mathcal{D}$-module analogue of Laumon's theory of local Fourier Transform and his stationary phase theorem relating the local and global Fourier Transforms: Laumon's theory in the ℓ-adic case was the main technical tool in our proof that Fourier Transform preserves index of rigidity. If the $\mathcal{D}$-module analogue of Laumon's local Fourier Transform exists **and** satisfies stationary phase, one can then use stationary phase to "compute" what the local Fourier Transforms must be, in terms of slope decompositions at ∞ of global Fourier Transforms of suitable "canonical extensions" in the sense of [Ka-DGG, 2.4.11] of various completions of the input $\mathcal{D}$-module. In this sense, one could say that the theory of local Fourier Transform does already exist, and that "all" that one lacks is the $\mathcal{D}$-module analogue of stationary phase.

If one could prove that the $\mathcal{D}$-module Fourier Transform preserves index of rigidity, or even that it preserves rigid objects, one could ask if any irreducible $\mathcal{D}$-module is built out of an irreducible $\mathcal{D}$-module of generic rank one by finitely iterating the operations Fourier Transform, middle tensor with an irreducible object of generic rank one, and pullback by an automorphism of $\mathbb{P}^1$.

Another question we are unable to treat is the following. Given a rigid irreducible local system, say $\mathcal{F}$, of rank n, consider its geometric monodromy group G_{geom}, defined as the algebraic subgroup of $GL(n, \mathbb{C})$ which is the Zariski closure of the monodromy representation which $\mathcal{F}$ "is". This group G_{geom}, and consequently its Lie algebra, is determined up to $GL(n, \mathbb{C})$-conjugacy by $\mathcal{F}$, and hence by the numerical data of $\mathcal{F}$. How can we determine G_{geom}, or even its Lie algebra, as a function of the numerical data. Even in asking when G_{geom} is finite, which we prove is equivalent to (the associated differential equation's) having p-curvature zero for almost all primes p, we do not know how to read this from the numerical data.

There is also a nagging technical point that is not treated in this book. In Chapter I, we show (1.1.2) that for complex irreducible local systems on $\mathbb{P}^1$ - {m points} over $\mathbb{C}$, rigidity is equivalent to having index of rigidity equal to 2. And we show (5.0.2) that in any characteristic $\neq \ell$, having index of rigidity equal to 2 is a sufficient condition for an irreducible ℓ-adic local system to be rigid. However, we do not show, even over $\mathbb{C}$, that for irreducible ℓ-adic local systems, being rigid is in fact equivalent to having index of rigidity equal to 2.

It is perhaps striking that although this book is concerned with problems that go back to Riemann, it depends for its very existence on a great deal of mathematics that did not exist until quite recently: Grothendieck's etale cohomology theory [SGA], Deligne's proof of his far-reaching generalization of the original Weil Conjectures [De-Weil II], the theory of perverse sheaves [BBD], Laumon's work on the ℓ-adic Fourier Transform [Lau-TF], all these are indispensable ingredients.

My interest in rigid local systems was first aroused by a conversation with Ofer Gabber some years ago, who told me about some lectures Deligne had given at I.H.E.S. on them. It was later re-aroused by conversations with Carlos Simpson, who was pursuing them from quite a different point of view than that used here. It is a pleasure to acknowledge helpful discussions with Deligne, Gabber and Simpson, and to thank Beilinson and Faltings for asking some incisive questions. I would also like to thank the referee, whose helpful comments and suggestions led to a number of corrections to and clarifications of the original manuscript.

I gave a series of lectures on some of the material in this book in January, 1991 at the University of Minnesota as an Ordway Visitor. The material on middle convolution was presented in March of 1993 at Johns Hopkins, at a Symposium in honor of Professor Igusa. I also gave lectures on some of this book in May of 1993 in Berkeley as a Miller Visiting Fellow. It is a pleasure to thank all of those institutions for their support and hospitality.

1.0 Generalities concerning rigid local systems over $\mathbb{C}$

(1.0.1) Let X be a projective smooth connected curve over $\mathbb{C}$, of genus g, S a nonempty finite subset of $X(\mathbb{C})$, and $U := X - S$ the open complement. Suppose we are given on the complex manifold U^{an} a local system $\mathcal{F}$, i.e., a locally constant sheaf of finite-dimensional $\mathbb{C}$-vector spaces. As on any connected complex manifold, if we fix a base point u in U^{an}, the functor "fibre at u", $\mathcal{F} \mapsto \mathcal{F}_u$, defines an equivalence of categories

$$\text{local systems on } U^{an} \approx \text{fin.-dim'l. } \mathbb{C}\text{-rep.'s of } \pi_1(U^{an}, u).$$

We say that the local system $\mathcal{F}$ is irreducible if the corresponding representation $\Lambda_{\mathcal{F}}$ of $\pi_1(U^{an}, u)$ is irreducible.

(1.0.2) For every "point at ∞" s in $S := X - U$, the punctured neighborhood

$$D^*(s) := U^{an} \cap (\text{a small disc around s in } X^{an})$$

is a punctured disc, whose fundamental group

$$I(s) := \pi_1(D^*(s), \text{any base point})$$

is canonically $\mathbb{Z}$, with generator γ_s := "turning once around s in the counterclockwise direction", the "local monodromy transformation at s". We say that two local systems $\mathcal{F}$ and $\mathcal{G}$ on U^{an} have isomorphic local monodromy if for every "point at ∞" s in $S := X - U$, there exists an isomorphism of local systems on $D^*(s)$

$$\mathcal{F} \mid D^*(s) \approx \mathcal{G} \mid D^*(s).$$

(1.0.3) We say that a local system $\mathcal{F}$ on U^{an} is **physically rigid** if for every local system $\mathcal{G}$ on U^{an} such that $\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy, there exists an isomorphism $\mathcal{F} \approx \mathcal{G}$ of local systems on U^{an}. Because we have assumed that S is nonempty, if $\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy, they necessarily have the same rank.

(1.0.4) This notion of physical rigidity is reasonable only for genus zero. Indeed, if X has genus $g \geq 1$, there exist local systems $\mathcal{L}$ of rank one on X^{an} no tensor power of which is trivial. [A rank one local system on X^{an} is a homomorphism from $\pi_1(X^{an})^{ab} \approx \mathbb{Z}^{2g}$ to $\mathbb{C}^\times$.] Denote by $j: U^{an} \to X^{an}$ the inclusion. Because the map

$$j_* : \pi_1(U^{an}, u) \to \pi_1(X^{an}, u)$$

is surjective, no tensor power of $j^*\mathcal{L}$ is trivial. But $j^*\mathcal{L}$ has trivial local monodromy, so for any local system $\mathcal{F}$ on U^{an}, $\mathcal{F}$ and $\mathcal{F}\otimes j^*\mathcal{L}$ has isomorphic local monodromy. But $\mathcal{F}$ and $\mathcal{F}\otimes j^*\mathcal{L}$ are not isomorphic unless $\mathcal{F} = 0$, since already their determinants, $\det(\mathcal{F})$ and $\det(\mathcal{F})\otimes(j^*\mathcal{L})^{\otimes \mathrm{rank}(\mathcal{F})}$ have a ratio which is nontrivial, indeed of infinite order. Thus no non-zero local system $\mathcal{F}$ on U^{an} is physically rigid when X has genus $g \geq 1$.

1.1 The case of genus zero

(1.1.1) Let us now explore in greater detail the situation when X is $\mathbb{P}^1$. Even here, the situation is only understood for local systems $\mathcal{F}$ on U^{an} which are irreducible. In this case, there is a numerical criterion for physical rigidity.

Theorem 1.1.2 Let S a nonempty finite subset of $\mathbb{P}^1(\mathbb{C})$, and $U := \mathbb{P}^1 - S$ the open complement, $j : U^{an} \to (\mathbb{P}^1)^{an}$ the inclusion, $\mathcal{F}$ an irreducible local system on U^{an} of rank $n \geq 1$. Then $\mathcal{F}$ is physically rigid if and only if $\chi((\mathbb{P}^1)^{an}, j_*\underline{End}(\mathcal{F})) = 2$.

proof Suppose first that $\chi((\mathbb{P}^1)^{an}, j_*\underline{End}(\mathcal{F})) = 2$, and let $\mathcal{G}$ be a local system on U^{an} such that $\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy. We will show the existence of an isomorphism from $\mathcal{F}$ to $\mathcal{G}$.

For any local system $\mathcal{H}$ on U^{an}, the Euler-Poincaré formula states

$$\chi((\mathbb{P}^1)^{an}, j_*\mathcal{H})= \chi(U^{an}, \mathbb{C})\times\mathrm{rank}(\mathcal{H}) + \Sigma_{s \text{ in } S} \dim_{\mathbb{C}} \mathcal{H}^{I(s)}.$$

Therefore if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two local systems U^{an} with isomorphic local monodromy, we have $\chi((\mathbb{P}^1)^{an}, j_*\mathcal{H}_1) = \chi((\mathbb{P}^1)^{an}, j_*\mathcal{H}_2)$. We apply this to $\mathcal{H}_1 = \underline{End}(\mathcal{F})$ and to $\mathcal{H}_2 = \underline{Hom}(\mathcal{F}, \mathcal{G})$, which have isomorphic local monodromy. Thus we find

$$\chi((\mathbb{P}^1)^{an}, j_*\underline{Hom}(\mathcal{F}, \mathcal{G})) = \chi((\mathbb{P}^1)^{an}, j_*\underline{End}(\mathcal{F})) = 2.$$

But on a curve, χ is $h^0 - h^1 + h^2 \leq h^0 + h^2$, so we find

$$h^0((\mathbb{P}^1)^{an}, j_*\underline{Hom}(\mathcal{F}, \mathcal{G})) + h^2((\mathbb{P}^1)^{an}, j_*\underline{Hom}(\mathcal{F}, \mathcal{G})) \geq 2.$$

We rewrite this in terms of ordinary and compact cohomology on U^{an} as

$$h^0(U^{an}, \underline{Hom}(\mathcal{F}, \mathcal{G})) + h_c{}^2(U^{an}, \underline{Hom}(\mathcal{F}, \mathcal{G})) \geq 2.$$

The Poincaré dual of $H_c{}^2(U^{an}, \underline{Hom}(\mathcal{F}, \mathcal{G}))$ is $H^0(U^{an}, \underline{Hom}(\mathcal{G}, \mathcal{F}))$, so we obtain

$$h^0(U^{an}, \underline{Hom}(\mathcal{F}, \mathcal{G})) + h^0(U^{an}, \underline{Hom}(\mathcal{G}, \mathcal{F})) \geq 2.$$

So at least one of the two groups

$$H^0(U^{an}, \underline{Hom}(\mathcal{F}, \mathcal{G})) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$$

or

$$H^0(U^{an}, \underline{Hom}(\mathcal{G}, \mathcal{F})) = \mathrm{Hom}(\mathcal{G}, \mathcal{F})$$

is nonzero. Since $\mathcal{F}$ is irreducible and both $\mathcal{F}$ and $\mathcal{G}$ have the same rank, any nonzero element of either $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ or or $\mathrm{Hom}(\mathcal{G}, \mathcal{F})$ is necessarily an isomorphism.

Now suppose that $\mathcal{F}$ is an irreducible local system of rank $n \geq 1$, which is physically rigid. We will show that

$$\chi((\mathbb{P}^1)^{an}, j_\star\underline{End}(\mathcal{F})) = 2.$$

To do this, it suffices to show that for **any** local system $\mathcal{F}$ of rank $n \geq 1$ which is physically rigid, we have

$$\chi((\mathbb{P}^1)^{an}, j_\star\underline{End}(\mathcal{F})) \geq 2.$$

[If $\mathcal{F}$ is irreducible, both $H^0((\mathbb{P}^1)^{an}, j_\star\underline{End}(\mathcal{F}))$ and its dual $H^2((\mathbb{P}^1)^{an}, j_\star\underline{End}(\mathcal{F}))$ are one-dimensional, so for any nonzero irreducible $\mathcal{F}$, we have $\chi((\mathbb{P}^1)^{an}, j_\star\underline{End}(\mathcal{F})) \leq 2$].

We will resort to a transcendental argument. For a suitable choice of base point u in U^{an}, a suitable numbering $s_1, \ldots, s_k$ of the $k := \mathrm{Card}(S)$ points at ∞, and suitably chosen loops γ_i which run from u to $D^\star(s_i)$, turn once counterclockwise around s_i, and then return to u the same way they came, the fundamental group $\pi_1(U^{an}, u)$ may be described in terms of generators and relations as the abstract group Γ_k with k generators C_i subject to the one relation $\prod_i C_i := C_1C_2\ldots C_k = 1$, via the isomorphism $C_i \mapsto \gamma_i$. Notice that the conjugacy class of C_i in Γ_k is that of local monodromy $\gamma(s_i)$ around s_i.

From this point of view, a rank n local system $\mathcal{F}$ on U^{an} is a collection of k elements A_i in $GL(n, \mathbb{C})$ which satisfy $\prod_i A_i = 1$. Given a second rank n local system $\mathcal{G}$ on U^{an}, corresponding to a collection of k elements D_i in $GL(n, \mathbb{C})$ which satisfy $\prod_i D_i = 1$, $\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy if and only if for each i, A_i and D_i are conjugate in $GL(n, \mathbb{C})$, i.e., if and only if for each i there exists an element B_i in $GL(n,\mathbb{C})$ such that $D_i = B_i A_i B_i^{-1}$. $\mathcal{F}$ and $\mathcal{G}$ are isomorphic if and only if there exists a single element C in $GL(n,\mathbb{C})$, or equivalently in $SL(n,\mathbb{C})$, such that $CA_iC^{-1} = D_i$ for all i.

So suppose that $\mathcal{F}$ is a rank n local system, corresponding to a system of k elements A_i in $GL(n, \mathbb{C})$ which satisfy $\prod_i A_i = 1$. Then $\mathcal{F}$ is physically rigid if and only if given any system of k elements B_i in $GL(n,\mathbb{C})$ such that $\prod_i (B_i A_i B_i^{-1}) = 1$, there exists a single element C in $SL(n,\mathbb{C})$ such that $CA_iC^{-1} = B_i A_i B_i^{-1}$ for all i.

Fix such an $\mathcal{F}$. By the Euler-Poincaré formula, we have

$$\chi((\mathbb{P}^1)^{an}, j_\star \underline{End}(\mathcal{F})) = (2 - k)n^2 + \Sigma_i \dim(\mathfrak{Z}(A_i)),$$

where $\mathfrak{Z}(A_i)$ denotes the commuting algebra of A_i in $M(n, \mathbb{C})$.

Let us denote by $Z(A_i)$ the subgroup of $GL(n, \mathbb{C})$ consisting of all elements which commute with A_i. Then $Z(A_i)$ is a nonempty, and hence dense, open set of the linear space $\mathfrak{Z}(A_i)$, namely it is the open set where the determinant is invertible. Therefore $Z(A_i)$ is irreducible, of $\dim(Z(A_i)) = \dim(\mathfrak{Z}(A_i))$. Thus we may rewrite the above formula as

$$\chi((\mathbb{P}^1)^{an}, j_\star \underline{End}(\mathcal{F})) = (2 - k)n^2 + \Sigma_i \dim(Z(A_i)),$$

Consider the k-fold self product of $GL(n, \mathbb{C})$ with itself,

$$X := (GL(n, \mathbb{C}))^k$$

and the map

$$\pi: X \mapsto SL(n, \mathbb{C})$$

defined by

$$(B_1, \ldots, B_k) \mapsto \prod_i (B_i A_i B_i^{-1}).$$

[Since $\prod_i A_i = 1$, this map π does indeed land in $SL(n, \mathbb{C})$.] Consider the group

$$G := SL(n, \mathbb{C}) \times \prod_i Z(A_i),$$

which acts on X by having an element $(C, Z_1, \ldots, Z_k)$ in G act on X as

$$(B_1, \ldots, B_k) \mapsto (CB_1Z_1^{-1}, \ldots, CB_kZ_k^{-1}).$$

The same group G acts on $SL(n, \mathbb{C})$, by having $(C, Z_1, \ldots, Z_k)$ in G act on $SL(n, \mathbb{C})$ as

$$A \mapsto CAC^{-1}.$$

With respect to these actions of G, the map

$$\pi: X \mapsto SL(n, \mathbb{C})$$

is easily checked to be G-equivariant.

Since the point 1 in $SL(n, \mathbb{C})$ is a fixed point of the G-action, the group G acts on the fibre $\pi^{-1}(1)$. The key tautology is that $\mathcal{F}$ is physically rigid if and only if the group G acts transitively on $\pi^{-1}(1)$. [Indeed, $\mathcal{F}$ is physically rigid if and only if given any point $\{B_i\}_i$ in $\pi^{-1}(1)$, i.e., any system of k elements B_i in $GL(n,\mathbb{C})$ such that $\prod_i (B_iA_iB_i^{-1}) = 1$, there exists an element C in $SL(n,\mathbb{C})$ such that for each i, $CA_iC^{-1} = B_iA_iB_i^{-1}$, i.e., such that $C^{-1}B_i$ is an element $(Z_i)^{-1}$ in $Z(A_i)$, i.e, such that the point $\{B_i\}_i$ is the image of the point $\{1_i\}_i$ under the action of the element $(C, \{Z_i\}_i)$ of G.]

Suppose now that $\mathcal{F}$ is physically rigid. Then G acts transitively on $\pi^{-1}(1)$, so we must have the inequality

$$\dim(G) \geq \dim(\pi^{-1}(1)).$$

Since the point 1 in $SL(n, \mathbb{C})$ is defined in $SL(n, \mathbb{C})$ by $n^2 -1$ equations (an element $(X_{i,j})$ in $SL(n)$ is 1 if and only if $X_{i,j} = \delta_{i,j}$ for each $(i,j) \neq (1,1)$), $\pi^{-1}(1)$ is defined in X by $n^2 -1$ equations. Therefore every irreducible component W of $\pi^{-1}(1)$ has

$$\dim(W) \geq \dim(X) - (n^2 -1).$$

[To see this, recall that $X = GL(n, \mathbb{C})^k$ is equidimensional of dimension $k \times n^2$: at every closed point x of X the local ring $\mathcal{O}_{X,x}$ has dimension $k \times n^2$. If a closed point x of X lies in $\pi^{-1}(1)$, then

$$\mathcal{O}_{\pi^{-1}(1),x} = \mathcal{O}_{X,x}/(\text{an ideal generated by } n^2 -1 \text{ non-units})$$

has dimension

$$\dim(\mathcal{O}_{\pi^{-1}(1),x}) \geq \dim(\mathcal{O}_{X,x}) - (n^2 - 1).]$$

Since the fibre $\pi^{-1}(1)$ is **nonempty** (it contains the point $\{1_i\}_i$), we have

$$\dim(G) \geq \dim(\pi^{-1}(1)) = \sup_{\text{irred compt's } W} \dim(W) \geq$$
$$\geq \dim(X) - (n^2 -1)$$

for $\mathcal{F}$ physically rigid. Recalling the definitions of G and X,

$$G := SL(n, \mathbb{C}) \times \prod_i Z(A_i),$$

$$X := (GL(n, \mathbb{C}))^k,$$

the above inequality

$$\dim(G) \geq \dim(X) - (n^2 - 1)$$

says

$$(n^2 - 1) + \Sigma_i \dim(Z(A_i)) \geq k \times n^2 - (n^2 - 1),$$

i.e.,

$$(2 - k)n^2 + \Sigma_i \dim(Z(A_i)) \geq 2,$$

i.e.,

$$\chi((\mathbb{P}^1)^{an}, j_{\ast}\underline{End}(\mathcal{F})) \geq 2. \qquad \text{QED}$$

Corollary 1.1.3 Notations as in Theorem 1.1.2 above, let $\mathcal{F}$ be an irreducible local system on U^{an} of rank $n \geq 1$, and let $\mathcal{L}$ be a rank one local system on U^{an}. Then the following conditions are equivalent:

1) $\mathcal{F}$ is physically rigid.
2) $\mathcal{F} \otimes \mathcal{L}$ is physically rigid.
3) the dual local system $\mathcal{F}^{\vee} := \underline{Hom}(\mathcal{F}, \mathbb{C}_{U^{an}})$ is physically rigid.

proof Indeed, all three local systems $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}$ and $\mathcal{F}^{\vee}$ are irreducible, and all have the same $\underline{End}$ sheaf on U^{an}, and hence the same $j_{\ast}$End sheaf on $\mathbb{P}^1$. QED

1.2 The case of higher genus

(1.2.1) Using the same technique, we can analyse the situation in higher genus. We return to the situation X a projective smooth connected curve over $\mathbb{C}$, of genus g, S a nonempty finite subset of $X(\mathbb{C})$, $U := X - S$ the open complement, $j: U \to X$ the inclusion. We have already seen that as soon as $g \geq 1$, no nonzero local system on U^{an} can be physically rigid, due to the possibility of tensoring with rank one local systems $\mathcal{L}$ on X. So we introduce two weaker notions. We say that a local system $\mathcal{F}$ on U^{an} is **weakly physically semi-rigid** if there exists a finite collection of local systems $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_d$ on U^{an} with the following property: for any local system $\mathcal{G}$ on U^{an} such that $\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy, there exists a rank one local system $\mathcal{L}$ on X, an index $1 \leq i \leq d$, and an isomorphism

$$\mathcal{G} \approx \mathcal{F}_i \otimes j^{\ast}\mathcal{L}$$

of local systems on U^{an}. We say that a local system $\mathcal{F}$ on U^{an} is **weakly physically rigid** if it is weakly physically semi-rigid and we may take d=1, i.e., if for for any local system $\mathcal{G}$ on U^{an} such that $\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy, there exists a rank one local system $\mathcal{L}$ on X and an isomorphism $\mathcal{G} \approx \mathcal{F}\otimes j^*\mathcal{L}$

Lemma 1.2.2 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g, S a nonempty finite subset of $X(\mathbb{C})$, U := X - S the open complement.

(1) Any local system $\mathcal{F}$ on U^{an} of rank one is weakly physically rigid.

(2) If $\mathcal{F}_1$ is a rank one local system on U^{an}, then for any nonzero local system $\mathcal{G}$ on U^{an}, $\mathcal{G}$ is weakly physically rigid if and only if $\mathcal{G}\otimes\mathcal{F}_1$ is weakly physically rigid.

(3) If $\mathcal{F}$ is a nonzero local system on U^{an}, T any finite subset of $U^{an}(\mathbb{C})$, and $k : U^{an} - T \to U^{an}$ the inclusion, then $\mathcal{F}$ is weakly physically rigid on U^{an} if and only if $k^*\mathcal{F}$ is weakly physically rigid on U^{an} - T.

proof (1) If $\mathcal{F}$ and $\mathcal{G}$ are any two local systems of rank one with isomorphic local monodromy, then $\underline{Hom}(\mathcal{G}, \mathcal{F})$ is a rank one local system on U^{an} with trivial local monodromy, so of the form $j^*\mathcal{L}$ for a unique rank one local system $\mathcal{L}$ (namely $j_*\underline{Hom}(\mathcal{G}, \mathcal{F})$) on X^{an}, whence $\mathcal{F} \approx \mathcal{G}\otimes j^*\mathcal{L}$.

(2) It suffices to show that if $\mathcal{G}$ is weakly physically rigid then $\mathcal{G}\otimes\mathcal{F}_1$ is weakly physically rigid (since $\mathcal{G}$ is $(\mathcal{G}\otimes\mathcal{F}_1)\otimes(\mathcal{F}_1)^{(\otimes-1)}$). If $\mathcal{H}$ and $\mathcal{G}\otimes\mathcal{F}_1$ have isomorphic local monodromy, then $\mathcal{H}\otimes(\mathcal{F}_1)^{(\otimes-1)}$ and $\mathcal{G}$ have isomorphic local monodromy, so by the weak physical rigidity of $\mathcal{G}$, there exists a rank one $\mathcal{L}$ on X^{an} and an isomorphism $\mathcal{G} \approx (j^*\mathcal{L})\otimes\mathcal{H}\otimes(\mathcal{F}_1)^{(\otimes-1)}$ on U^{an}. Tensoring with $\mathcal{F}_1$ gives the required isomorphism $\mathcal{G}\otimes\mathcal{F}_1 \approx (j^*\mathcal{L})\otimes\mathcal{H}$.

(3) Suppose first that $\mathcal{F}$ is weakly physically rigid on U^{an}, and that $\mathcal{G}$ is a local system on U^{an} - T such that $k^*\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy on U^{an} - T. Because $k^*\mathcal{F}$ has trivial local monodromy at each point of T, so also does $\mathcal{G}$, and therefore $\mathcal{F}$

(= $k_*k^*\mathcal{F}$) and $k_*\mathcal{G}$ are two local systems on U^{an} with isomorphic local monodromy. Since $\mathcal{F}$ is weakly physically rigid on U^{an}, there exists a rank one local system $\mathcal{L}$ on X^{an} and an isomorphism $\mathcal{F} \approx k_*\mathcal{G}\otimes j^*\mathcal{L}$ on U^{an}. Restricting this isomorphism to U^{an} - T.gives $k^*\mathcal{F} \approx \mathcal{G}\otimes k^*j^*\mathcal{L}$, as required.

Conversely, suppose $\mathcal{F}$ is a nonzero local system on U^{an}, such that $k^*\mathcal{F}$ is weakly physically rigid on U^{an} - T. Let $\mathcal{G}$ be a local system on U^{an} such that $\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy. Then $k^*\mathcal{F}$ and $k^*\mathcal{G}$ have isomorphic local monodromy on U^{an} - T, so there exists a rank one local system $\mathcal{L}$ on X^{an} and an isomorphism $k^*\mathcal{F} \approx k^*\mathcal{G}\otimes k^*j^*\mathcal{L} = k^*(\mathcal{G}\otimes j^*\mathcal{L})$. Applying k_* gives the required isomorphism $\mathcal{F} = k_*k^*\mathcal{F} \approx k_*(k^*(\mathcal{G}\otimes j^*\mathcal{L})) = \mathcal{G}\otimes j^*\mathcal{L}$. QED

In a more serious vein, we have:

Proposition 1.2.3 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g, S a nonempty finite subset of $X(\mathbb{C})$, $U := X - S$ the open complement, $j: U \to X$ the inclusion. If a nonzero local system $\mathcal{F}$ on U^{an} is weakly physically semi-rigid, then

$$\chi(X^{an}, j_*\underline{\mathrm{End}}(\mathcal{F})) \geq 2 - 2g.$$

proof Once again we resort to a transcendental proof. With suitable base point u in U^{an}, suitable numbering $s_1, ..., s_k$ of the $k := \mathrm{Card}(S)$ points at ∞, and suitable numbering of the g "handles", the fundamental group $\pi_1(U^{an}, u)$ may be described in terms of generators and relations as the abstract group $\Gamma_{g,k}$ with 2g + k generators $E_1, F_1, ..., E_g, F_g, C_1, ..., C_k$ subject to the single relation

$$(\Pi_{j=1,...,g} \{E_j, F_j\})(\Pi_{i=1,...,k} C_i) = 1,$$

where we write {a, b} for the commutator $aba^{-1}b^{-1}$. In this presentation, the elements C_i are the local monodromies around the points s_i at ∞, and $\pi_1(X^{an}, u)$ is the quotient of $\Gamma_{g,k}$ by the normal subgroup generated by the elements $C_1, ..., C_k$.

In terms of this presentation of $\pi_1(U^{an}, u)$, a local system $\mathcal{F}$ on U^{an} of rank $n \geq 1$ is a collection 2g + k elements in $GL(n, \mathbb{C})$,

$$M_1, N_1, \ldots, M_g, N_g, A_1, \ldots, A_k$$

which satisfy $(\prod_{j=1,\ldots,g} \{M_j, N_j\})(\prod_{i=1,\ldots,k} A_i) = 1$.

Fix such an $\mathcal{F}$. By the Euler-Poincaré formula, we have

$$\chi(X^{an}, j_* \underline{End}(\mathcal{F})) = (2 - 2g - k)n^2 + \Sigma_i \dim(Z(A_i)).$$

A rank one local system $\mathcal{L}$ on X^{an} corresponds to an arbitrary system of 2g elements $\mu_1, \nu_1, \ldots, \mu_g, \nu_g$ in $\mathbb{C}^\times$; $\mathcal{F}\otimes j^*\mathcal{L}$ then corresponds to the collection

$$\mu_1 M_1, \nu_1 N_1, \ldots, \mu_g M_g, \nu_g N_g, A_1, \ldots, A_k$$

Consider a second local system $\mathcal{G}$ of the same rank n, corresponding to is a collection 2g + k elements in $GL(n, \mathbb{C})$,

$$P_1, Q_1, \ldots, P_g, Q_g, D_1, \ldots, D_k$$

which satisfy $(\prod_{j=1,\ldots,g} \{P_j, Q_j\})(\prod_{i=1,\ldots,k} D_i) = 1$.

Then $\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy if and only if for each $1 \le i \le k$ there exists an element B_i in $GL(n,\mathbb{C})$ such that

$$D_i = B_i A_i B_i^{-1} \text{ for each } 1 \le i \le k.$$

$\mathcal{F}$ and $\mathcal{G}$ are isomorphic if and only if there exists a single element C in $GL(n,\mathbb{C})$, or equivalently in $SL(n,\mathbb{C})$, such that

$$P_j = CM_jC^{-1} \text{ for } j=1,\ldots,g,$$

$$Q_j = CN_jC^{-1} \text{ for } j=1,\ldots,g,$$

$$D_i = CA_iC^{-1} \text{ for } 1 \le i \le k.$$

Consider the 2g+k-fold self product of $GL(n, \mathbb{C})$ with itself,

$$X := (GL(n, \mathbb{C}))^{2g+k}$$

and the map

$$\pi: X \mapsto SL(n, \mathbb{C})$$

defined by

$$(J_1, K_1, \ldots, J_g, K_g, B_1, \ldots, B_k) \mapsto (\prod_j \{J_j, K_j\})(\prod_i (B_i A_i B_i^{-1})).$$

[Since $(\prod_j \{M_j, N_j\})(\prod_i A_i) = 1$, and commutators lie in $SL(n, \mathbb{C})$, this map π does indeed land in $SL(n, \mathbb{C})$.] Consider the group

$$G := SL(n, \mathbb{C}) \times (\mathbb{C}^\times)^{2g} \times \prod_i Z(A_i).$$

It acts on X by having an element

$$(C, \mu_1, \nu_1, \ldots, \mu_g, \nu_g, Z_1, \ldots, Z_k)$$

in G act on X as

$$(J_1, K_1, \ldots, J_g, K_g, B_1, \ldots, B_k) \mapsto$$

$$C\mu_1 J_1 C^{-1}, C\nu_1 K_1 C^{-1}, \ldots, C\mu_1 J_1 C^{-1}, C\nu_1 K_1 C^{-1}, CB_1 Z_1^{-1}, \ldots, CB_k Z_k^{-1}).$$

The same group G acts on $SL(n, \mathbb{C})$, by having an element

$$(C, \mu_1, \nu_1, ..., \mu_g, \nu_g, Z_1, ..., Z_k)$$
in G act on $SL(n, \mathbb{C})$ as
$$A \mapsto CAC^{-1}.$$
With respect to these actions of G, the map
$$\pi: X \mapsto SL(n, \mathbb{C})$$
is easily checked to be G-equivariant. Since the point 1 in $SL(n, \mathbb{C})$ is a fixed point of the G-action, the group G acts on the fibre $\pi^{-1}(1)$. The key tautology is that $\mathcal{F}$ is weakly physically semi-rigid if and only if under the action of G, $\pi^{-1}(1)$ is a finite union of G-orbits.

Just as above, if $\mathcal{F}$ is weakly physically semi-rigid, we infer that
$$\dim(G) \geq \dim(\pi^{-1}(1)) \geq \dim(X) - (n^2 - 1),$$
which is to say
$$(n^2 - 1) + 2g + \Sigma_i \dim(Z(A_i) \geq (2g + k)n^2 - (n^2 - 1),$$
i.e.,
$$(2 - 2g - k)n^2 + \Sigma_i \dim(Z(A_i) \geq 2 - 2g,$$
i.e.,
$$\chi(X^{an}, j_{*}\underline{End}(\mathcal{F})) \geq 2 - 2g. \qquad \text{QED}$$

Corollary 1.2.4 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g, S a nonempty finite subset of $X(\mathbb{C})$, $k := \mathrm{Card}(S)$, $U := X - S$ the open complement, $j: U \to X$ the inclusion. If $g \geq 2$, and $\mathcal{F}$ is a local system on U^{an} of rank $n \geq 2$, then $\mathcal{F}$ is not weakly physically semi-rigid.

proof For $\mathcal{F}$ of rank $n \geq 1$ on U^{an}, the Euler-Poincaré formula gives
$$\chi(X^{an}, j_{*}\underline{End}(\mathcal{F})) - (2 - 2g)=$$
$$= (2 - 2g - k)n^2 + \Sigma_i \dim(Z(A_i)) - (2 - 2g)$$
$$= (2 - 2g)(n^2 - 1) + \Sigma_i (\dim(Z(A_i) - n^2).$$
Each term $(\dim(Z(A_i) - n^2)$ is ≤ 0, with equality if and only if A_i is scalar. If $g \geq 2$ and $n \geq 2$ then the term $(2 - 2g)(n^2 - 1)$ is < 0, so
$$\chi(X^{an}, j_{*}\underline{End}(\mathcal{F})) < (2 - 2g).$$
Therefore $\mathcal{F}$ is not weakly physically semi-rigid. QED

Corollary 1.2.5 (mise pour memoire) Let S be a nonempty finite subset of $\mathbb{P}^1(\mathbb{C})$, $U := \mathbb{P}^1 - S$ the open complement, $j: U \to \mathbb{P}^1$ the inclusion. Let $\mathcal{F}$ be an irreducible local system on U^{an} of rank $n \geq 1$. The following conditions are equivalent.

(1) $\mathcal{F}$ is physically rigid.
(2) $\mathcal{F}$ is weakly physically rigid.
(3) $\mathcal{F}$ is weakly physically semi-rigid.
(4) $\chi((\mathbb{P}^1)^{an}, j_*\underline{End}(\mathcal{F})) \geq 2$.
(5) $\chi((\mathbb{P}^1)^{an}, j_*\underline{End}(\mathcal{F})) = 2$.

proof The implications (1) ⇒ (2) ⇒ (3) are trivial, and we have proven (3) ⇒ (4) in 1.2.3 above. For $\mathcal{F}$ irreducible, we have already proven (4) ⇒ (5) and (5) ⇒ (1) in the proof of 1.1.2. QED

1.3 The case of genus one

Corollary 1.3.1 Let X a projective smooth connected curve over $\mathbb{C}$, of genus $g=1$, S a nonempty finite subset of $X(\mathbb{C})$, $U := X - S$ the open complement, $j: U \to X$ the inclusion. Let $\mathcal{F}$ be a local system on U^{an} of rank $n \geq 1$. Consider the following conditions:

(1) $\mathcal{F}$ is weakly physically rigid.
(2) $\mathcal{F}$ is weakly physically semi-rigid.
(3) $\chi(X^{an}, j_*\underline{End}(\mathcal{F})) \geq 0 = (2 - 2g)$.
(4) $\chi(X^{an}, j_*\underline{End}(\mathcal{F})) = 0 = (2 - 2g)$.
(5) $\mathcal{F}$ has all its local monodromies scalar.
(6) $j_*\underline{End}(\mathcal{F})$ is a local system on X^{an}.

We have the implications

(1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6).

If in addition $\mathcal{F}$ is irreducible, these conditions are all equivalent.

proof The implication (1) ⇒ (2) is trivial, and (2) ⇒ (3) is the content of Proposition 1.2.3 above. If $g=1$, the Euler Poincaré formula gives

$$\chi(X^{an}, j_*\underline{End}(\mathcal{F})) - (2 - 2g) =$$
$$= (2 - 2g - k)n^2 + \Sigma_i \dim(Z(A_i)) - (2 - 2g)$$
$$= \Sigma_i (\dim(Z(A_i) - n^2).$$

Thus $\chi(X^{an}, j_*\underline{End}(\mathcal{F})) \leq (2 - 2g) = 0$, with strict inequality unless all the local monodromies A_i are scalar. Thus (3) ⇒ (4) and (4) ⇒

(5). If (5) holds, then $\underline{End}(\mathcal{F})$ has trivial local monodromy, so $j_*\underline{End}(\mathcal{F})$ is a local system on the elliptic curve X^{an}. Thus (5) ⇒ (6).

Suppose now that $\mathcal{F}$ is irreducible, and that (6) holds, i.e., $j_*\underline{End}(\mathcal{F})$ is a local system on X^{an}. Let $\mathcal{G}$ be another local system on U^{an} such that $\mathcal{F}$ and $\mathcal{G}$ have ismorphic local monodromy. Then $\underline{Hom}(\mathcal{F}, \mathcal{G})$ and $\underline{End}(\mathcal{F})$ have isomorphic local monodromy. Since. $j_*\underline{End}(\mathcal{F})$ is a local system on X^{an}, $\underline{End}(\mathcal{F})$ has trivial local monodromy. Hence $\underline{Hom}(\mathcal{F}, \mathcal{G})$ has trivial local monodromy, and therefore $j_*\underline{Hom}(\mathcal{F}, \mathcal{G})$ is a local system on X^{an}. Because the π_1 of X^{an} is abelian, any local system on X^{an} is a successive extension of rank one local systems $\mathcal{L}_k$ on X^{an}. So there exists a rank one local system $\mathcal{L}$ on X^{an} and a non-zero map of local systems on X^{an}

$$\mathcal{L} \to j_*\underline{Hom}(\mathcal{F}, \mathcal{G}).$$

Tensoring this map with $\mathcal{L}^{\otimes -1}$ gives a nonzero map of local systems

$$\mathbb{C} \to j_*\underline{Hom}(\mathcal{F}, \mathcal{G})\otimes\mathcal{L}^{\otimes -1} = j_*\underline{Hom}(\mathcal{F}\otimes j^*\mathcal{L}, \mathcal{G}),$$

i.e., a nonzero element in

$$H^0(X^{an}, j_*\underline{Hom}(\mathcal{F}\otimes j^*\mathcal{L}, \mathcal{G})) = H^0(U^{an}, \underline{Hom}(\mathcal{F}\otimes j^*\mathcal{L}, \mathcal{G}))$$
$$= \mathrm{Hom}(\mathcal{F}\otimes j^*\mathcal{L}, \mathcal{G}).$$

Because $\mathcal{F}$ (and hence $\mathcal{F}\otimes j^*\mathcal{L}$) is irreducible and of the same rank as $\mathcal{G}$, any such non-zero map of local systems is an isomorphism. Therefore we have $\mathcal{F}\otimes j^*\mathcal{L} \approx \mathcal{G}$. Thus $\mathcal{F}$ is weakly physically rigid, and so (6) ⇒ (1) for $\mathcal{F}$ irreducible. QED

1.4 The case of genus one: detailed analysis

(1.4.1) We now analyze the case of genus one in greater detail. The first step is to show that up to tensoring with rank one objects, we may reduce to the case when there is only a single point at ∞.

Lemma 1.4.2 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g=1, $S = \{s_1, s_2, ..., s_k\}$ a finite subset of $X(\mathbb{C})$ with $k \geq 2$ points, $U := X - S$ the open complement,

$$j: U \to X,$$

and

$$j_1 : X - S \to X - \{s_1\}$$

the inclusions. Let $\mathcal{F}$ be a local system on U^{an} of rank $n \geq 1$ which is weakly physically rigid. There exists a rank one local system $\mathcal{L}$ on

U^{an}, and a weakly physically rigid local system $\mathcal{F}_1$ on $(X - \{s_1\})^{an}$, such that $\mathcal{F} \approx \mathcal{L}\otimes j_1{}^*\mathcal{F}_1$.

proof If $\mathcal{F}$ on U^{an} of rank $n \geq 1$ is weakly physically rigid, its local monodromy at each point s_i in S is a scalar, say α_i. Think of $\pi_1(U^{an}, u)$ as the abstract group $\Gamma_{1,k}$ with $2 + k$ generators E, F, C_1, ..., C_k subject to the single relation

$$\{E, F\}(\Pi_{i=1,\ldots,k} C_i) = 1.$$

The required $\mathcal{L}$ is any character χ of this group which takes the value α_i at C_i for $i > 1$, e.g., one might take $\chi(E) = \chi(F) = 1$, and $\chi(C_1) := (\Pi_{i=2,\ldots,k}\, \alpha_i)^{-1}$. Then $\mathcal{F}\otimes\mathcal{L}^{\otimes -1}$ has trivial local monodromy at each of s_2, ..., s_k, so it is of the form $j_1{}^*\mathcal{F}_1$.for some local system $\mathcal{F}_1$ on $(X - \{s_1\})^{an}$. By 1.2.2(2), $\mathcal{F}\otimes\mathcal{L}^{\otimes -1}$ on U^{an} is weakly physically rigid, and, by 1.2.2(3), $\mathcal{F}_1$ is weakly physically rigid on $(X - \{s_1\})^{an}$. QED

(1.4.3) We next analyze the irreducible local systems in genus one when there is a single point at infinity.

Lemma 1.4.4 Let X a projective smooth connected curve over $\mathbb{C}$, of genus $g=1$, x_0 in $X(\mathbb{C})$ a point, $U := X - \{x_0\}$ the open complement, $j: U \to X$ the inclusion. Let $\mathcal{F}$ be a local system on U^{an} of rank $n \geq 1$. Then the following conditions are equivalent:

(1) $\mathcal{F}$ is both irreducible and weakly physically rigid.

(2) The local monodromy of $\mathcal{F}$ around x_0 is a scalar ζ which is a primitive n'th root of unity.

proof Suppose first that (1) holds. By the weak physical rigidity of $\mathcal{F}$, and 1.3.1, (1) $\Rightarrow$ (5), we know that the local monodromy of $\mathcal{F}$ around x_0 is a scalar ζ. We first show that ζ is necessarily an n'th root of unity.

Think of $\pi_1(U^{an}, u)$ as the abstract group $\Gamma_{1,1}$ with $2 + 1$ generators E, F, C subject to the single relation

$$\{E, F\}C = 1, \text{ or } C = \{F, E\},$$

C being the local monodromy around x_0. Then $\mathcal{F}$ "is" an n-dimensional $\mathbb{C}$-representation of this group. So its local monodromy around x_0 lies in $SL(n, \mathbb{C})$, because it is the **commutator** of two elements of $GL(n, \mathbb{C})$. Being a scalar, ζ is necessarily an n'th root of unity.

Given an n'th root of unity ζ, we construct an explicit n-dimensional $\mathbb{C}$-representation $\rho_{n,\zeta}$ on $V_{n,\zeta}$ of $\pi_1(U^{an}, u)$, which we now think of as the free group on E and F, as follows:

$V_{n,\zeta}$ is the n-dimensional $\mathbb{C}$-algebra $\mathbb{C}[T]/(T^n - 1)$,

$\rho_{n,\zeta}(E)$ is the automorphism A: $f(T) \mapsto Tf(T)$,

$\rho_{n,\zeta}(F)$ is the automorphism B: $f(T) \mapsto f(\zeta T)$.

Clearly $AB(f)(T) = Tf(\zeta T)$, $BA(f)(T) = \zeta Tf(\zeta T)$, so $BA = \zeta AB$, which is to say $\{B, A\} = \zeta$, or equivalently $\{A, B\}\zeta = 1$.

Interpret the representation $\rho_{n,\zeta}$ as a local system $\mathcal{F}_{n,\zeta}$ on U^{an}. Then its local monodromy around x_0 is the scalar ζ.

We must show that if $\mathcal{F}$ is both irreducible and weakly physically rigid, then ζ is a primitive n'th root of unity. We argue by contradiction. Suppose that for some factorization of $n = dm$, with both m and d integers ≥ 2, ζ were a d'th root of unity. Then $\mathcal{F}$ and $\oplus_{m \text{ copies}} \mathcal{F}_{d,\zeta}$ are two local systems on U^{an} with isomorphic local monodromy (namely ζ).

By the weak physical rigidity of $\mathcal{F}$, there exists a rank one $\mathcal{L}$ on X^{an} and an isomorphism $\mathcal{F} \approx \oplus_{m \text{ copies}} \mathcal{F}_{d,\zeta} \otimes j^*\mathcal{L}$. But this shows that $\mathcal{F}$ is in fact reducible. Thus (1) implies (2).

Suppose now that (2) holds. By 1.3.1, it suffices to prove that $\mathcal{F}$ is irreducible. Interpreting $\mathcal{F}$ as a representation, this results from the following general lemma.

Lemma 1.4.5 Let K be an algebraically closed field, $n \geq 2$ an integer, V an n-dimensional K-vector space, A and B two elements of GL(V), C the commutator $\{B, A\}$. Let $\xi_1, \xi_2, \ldots, \xi_n$ be the n (not necessarily distinct) eigenvalues of C. Suppose that for any proper nonempty subset S of $\{1,2, \ldots, n\}$, the product $\prod_{i \text{ in } S} \xi_i \neq 1$. Then the subgroup of GL(V) generated by A and B acts irreducibly on V. In particular, if C is scalar, equal to a primitive n'th root of unity, then the subgroup of GL(V) generated by A and B acts irreducibly on V.

proof We restate the hypothesis in the following form: for any monic polynomial f(T) of degree $1 \le d < n$ which divides $\det_V(T - C)$, we have $\prod_{\text{all roots } \xi \text{ of } f} \xi \neq 1$.

We argue by contradiction. If W is a nontrivial proper subspace of V which is mapped to itself by both A and B, then W is also mapped to itself by their inverses, and so W is mapped to itself by C = {B, A}. But

$$C \mid W = \{B \mid W, A \mid W\}$$

lies in SL(W) (being a commutator of two elements in GL(W)). Taking $f(T) := \det_W(T - C|W)$, we get a contradiction. QED

Using the local systems $\mathcal{F}_{n,\zeta}$ on $(X - \{x_0\})^{an}$ constructed in the proof of 1.4.4 above, we get a complete description:

Proposition 1.4.6 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g=1, x_0 in $X(\mathbb{C})$ a point, $U := X - \{x_0\}$ the open complement, $j: U \to X$ the inclusion, $n \ge 1$ an integer. The local systems $\mathcal{F}$ on U^{an} of rank n which are both irreducible and weakly physically rigid are precisely those of the form $\mathcal{F}_{n,\zeta} \otimes j^*\mathcal{L}$, with $\mathcal{L}$ of rank one on X^{an}, and with ζ a primitive n'th root of unity.

proof By 1.4.4, we know if ζ is a primitive n'th root of unity, then $\mathcal{F}_{n,\zeta}$, and hence also $\mathcal{F}_{n,\zeta} \otimes j^*\mathcal{L}$, is irreducible. Because $\mathcal{F}_{n,\zeta}$, and hence also $\mathcal{F}_{n,\zeta} \otimes j^*\mathcal{L}$, have scalar local monodromy, [1.3.2, (5) $\Rightarrow$ (1)] shows that $\mathcal{F}_{n,\zeta} \otimes j^*\mathcal{L}$ is weakly physically rigid. Conversely, given a rank n $\mathcal{F}$ which is irreducible and weakly physically rigid, by 1.4.4 its local monodromy at x_0 is a scalar ζ which is a primitive n'th root of unity. So $\mathcal{F}$ and $\mathcal{F}_{n,\zeta}$ have isomorphic local monodromy. By the weak physical rigidity of $\mathcal{F}$, there exists an $\mathcal{L}$ of rank one on X^{an}, and an isomorphism $\mathcal{F} \approx \mathcal{F}_{n,\zeta} \otimes j^*\mathcal{L}$. QED

(1.4.7) We next compute the determinant of $\mathcal{F}_{n,\zeta}$. To state the result, we denote by $\mathcal{L}_{1/2}$ the rank one local system on $(X - \{x_0\})^{an}$ corresponding to the character $E \mapsto -1$, $F \mapsto -1$, and by $\mathcal{L}_0$ the trivial rank one local system on $(X - \{x_0\})^{an}$. [Of course, both of these, like any rank one local system with only one point at ∞, extend uniquely to local systems on the complete curve X^{an}.]

Lemma 1.4.8 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g=1, x_0 in $X(\mathbb{C})$ a point, $U := X - \{x_0\}$ the open complement, $j: U \to X$ the inclusion, $n \geq 1$ an integer, ζ a primitive n'th root of unity. Then

$$\det(\mathcal{F}_{n,\zeta}) = \mathcal{L}_{1/2} \text{ if n is even,}$$
$$\det(\mathcal{F}_{n,\zeta}) = \mathcal{L}_0 \text{ if n is odd.}$$

proof As representation, $\mathcal{F}_{n,\zeta}$ is

$V_{n,\zeta}$ is the n-dimensional $\mathbb{C}$-algebra $\mathbb{C}[T]/(T^n - 1)$,
$\rho_{n,\zeta}(E)$ is the automorphism $A: f(T) \mapsto Tf(T)$,
$\rho_{n,\zeta}(F)$ is the automorphism $B: f(T) \mapsto f(\zeta T)$.

The assertion is that $\det(A) = \det(B) = (\zeta)^{n(n+1)/2}$ ($= (-1)^{n+1}$). To see this for A, notice that the vectors

$$f_i(T) := \Sigma_{j \bmod n}\ \zeta^{-ij}\ T^j,$$

$1 \leq i \leq n$, are an eigenbasis for A, with eigenvalues ζ^i. To see it for B, notice that vectors T^i, $1 \leq i \leq n$, are an eigenbasis for B, with eigenvalues ζ^i. QED

In fact, the local systems $\mathcal{F}_{n,\zeta}$ have a stronger rigidity property.

Proposition 1.4.9 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g=1, x_0 in $X(\mathbb{C})$ a point, $U := X - \{x_0\}$ the open complement, $n \geq 1$ an integer. Let $\mathcal{F}$ be a local system on U^{an} of rank n which is both irreducible and weakly physically rigid. Denote by ζ the primitive n'th root of unity which is the local monodromy of $\mathcal{F}$ around x_0. If $\det(\mathcal{F}) \approx \det(\mathcal{F}_{n,\zeta})$, then $\mathcal{F} \approx \mathcal{F}_{n,\zeta}$.

proof As representation, $\mathcal{F}_{n,\zeta}$ is

$V_{n,\zeta}$ is the n-dimensional $\mathbb{C}$-algebra $\mathbb{C}[T]/(T^n - 1)$,
$\rho_{n,\zeta}(E)$ is the automorphism $A: f(T) \mapsto Tf(T)$,
$\rho_{n,\zeta}(F)$ is the automorphism $B: f(T) \mapsto f(\zeta T)$.

By 1.4.6, there exist nonzero scalars λ, μ in $\mathbb{C}^\times$ such that the representation $\Lambda_{\mathcal{F}}$ corresponding to $\mathcal{F}$ is realized on the same space $V_{n,\zeta}$, but with

$$\Lambda_{\mathcal{F}}(E) := \lambda A,$$
$$\Lambda_{\mathcal{F}}(F) := \mu B.$$

Because $\det(\mathcal{F}) \approx \det(\mathcal{F}_{n,\zeta})$, the scalars λ and μ are n'th roots of unity. Thus we must show that the automorphisms A and B of $V_{n,\zeta}$ have the following property (*)

(*) for any n'th roots of unity λ, μ, there exists an automorphism X of $V_{n,\zeta}$ such that $\lambda A = XAX^{-1}$ and $\mu B = XBX^{-1}$.

To prove (*), recall that $BA = \zeta AB$. Thus

$$BAB^{-1} = \zeta A, \text{ and } ABA^{-1} = \zeta B.$$

So if we write $\lambda = \zeta^i$ and $\mu = \zeta^j$, we may take $X = A^jB^i$. QED

Corollary 1.4.10 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g=1, x_0 in $X(\mathbb{C})$ a point, $U := X - \{x_0\}$ the open complement, $n \geq 1$ an integer. Let $\mathcal{F}$ be a local system on U^{an} of rank n which is both irreducible and weakly physically rigid. Denote by ζ the primitive n'th root of unity which is the local monodromy of $\mathcal{F}$ around x_0. The isomorphism class of $\mathcal{F}$ is determined by the isomorphism class of the data $(n, \zeta, \det(\mathcal{F}))$.

proof The group (under $\otimes$) of isomorphism classes of rank one local systems on U^{an} is divisible (being $\mathbb{C}^\times \times \mathbb{C}^\times$), so tensoring $\mathcal{F}$ with an n'th root of $\det(\mathcal{F}_{n,\zeta})\otimes\det(\mathcal{F})^{-1}$, we reduce to the proposition above. QED

(1.4.11) We now investigate the situation in genus one when there are $k \geq 2$ points at ∞.

Proposition 1.4.12 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g=1, $S = \{s_1, s_2, \ldots, s_k\}$ a finite subset of $X(\mathbb{C})$ with $k \geq 2$ points, $U := X - S$ the open complement,

$$j: U \to X,$$

and $$j_1 : X - S \to X - \{s_1\}$$

the inclusions. Fix an integer $n \geq 1$. The local systems $\mathcal{F}$ on U^{an} of rank n which are both irreducible and weakly physically rigid are precisely those of the form $(j_1)^*(\mathcal{F}_{n,\zeta}$ on $(X - \{s_1\})^{an}) \otimes \mathcal{L}$, with $\mathcal{L}$ of rank one on U^{an}, and with ζ a primitive n'th root of unity.

proof Simply combine 1.4.2 and 1.4.6. QED

Here is an intrinsic characterization.

Lemma 1.4.13 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g=1, $S = \{s_1, ..., s_k\}$ a finite subset of $X(\mathbb{C})$ with $k \geq 1$ points, $U := X - S$ the open complement, $j: U \to X$ the inclusion. Let $\mathcal{F}$ be a local system on U^{an} of rank $n \geq 1$. Then $\mathcal{F}$ is both irreducible and weakly physically rigid if and only if the following two conditions hold:
(1) $\mathcal{F}$ has scalar monodromy ξ_i around each point s_i in S.
(2) $\prod_{i \text{ in } S} \xi_i$ is a primitive n'th root of unity.

proof The assertion is invariant under tensoring with an $\mathcal{L}$ of rank one on U^{an}, so we are reduced, as in the proof of 1.4.2, to the case when k = 1, where it is 1.4.4. QED

Proposition 1.4.14 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g=1, $S = \{s_1, s_2, ..., s_k\}$ a finite subset of $X(\mathbb{C})$ with $k \geq 1$ points, $U := X - S$ the open complement,

$$j: U \to X,$$

and

$$j_1 : X - S \to X - \{s_1\}$$

the inclusions. Let $\mathcal{F}$ be a local system on U^{an} of rank $n \geq 1$ which is both irreducible and weakly physically rigid. Denote by ξ_i the scalar which is the local monodromy of $\mathcal{F}$ around s_i. The isomorphism class of $\mathcal{F}$ is determined by the isomorphism class of the data $(n, \{\xi_i\}_i, \det(\mathcal{F}))$.

proof Given this data, first construct the rank one local system $\mathcal{L}(\{\xi_i\}_i)$ on $(X - \{s_1, ..., s_k\})^{an}$ given as character of π_1 by

$$E \mapsto 1,\ F \mapsto 1,\ C_1 \mapsto \prod_{i \geq 2} \xi_i,\ C_i \mapsto (\xi_i)^{-1} \text{ for } i \geq 2.$$

Tensoring with $\mathcal{L}(\{\xi_i\}_i)$, we reduce to the fact 1.4.10 that

$(j_1)_*(\mathcal{F}\otimes\mathcal{L}(\{\xi_i\}_i))$ on $X - \{s_1\})^{an}$ is determined up to isomorphism by the data $(n, \prod_i \xi_i, \det((j_1)_*(\mathcal{F}\otimes\mathcal{L}(\{\xi_i\}_i)))$. QED

(1.4.15) We now return to the case of a single point at ∞, and give a structure theorem for irreducible, weakly physically rigid local systems of rank $n \geq 1$, which says they are all induced from rank one local systems.

Theorem 1.4.16 Let X a projective smooth connected curve over $\mathbb{C}$, of genus $g=1$, x_0 in $X(\mathbb{C})$ a point, $U := X - \{x_0\}$ the open complement, $j: U \to X$ the inclusion, $n \geq 1$ an integer. Let $\mathcal{F}$ be a local system on U^{an} of rank n which is both irreducible and weakly physically rigid. Denote by ζ the primitive n'th root of unity which is the local monodromy of $\mathcal{F}$ around x_0. Let $\pi : Y \to X$ be **any** connected finite etale covering of degree n (such coverings exist!). Denote by $S \subset Y(\mathbb{C})$ the set $\pi^{-1}(x_0)$, which has $\mathrm{Card}(S) = n$. Then there exists a rank one local system $\mathcal{L}$ on $(Y - S)^{an}$ for which the local monodromy around each of the n points in S is ζ, and for which $\pi_*\mathcal{L} \approx \mathcal{F}$ on U^{an}.

proof Because X has genus one, $\pi_1(X^{an})$ is abelian, $\approx (\mathbb{Z})^2$. So connected finite etale coverings Y of X are in bijective correspondence with the subgroups Γ of $(\mathbb{Z})^2$ of index n, and such Γ clearly exist. By the Hurwitz formula, Y has genus one.

Consider the pullback $\pi^*\mathcal{F}$ on $(Y - S)^{an}$. Because π is finite etale over all of X, the local monodromy of $\pi^*\mathcal{F}$ around each point of S is still the same scalar ζ.

Since S contains n points, and ζ is an n'th root of unity, there exist rank one local systems $\mathcal{L}$ on $(Y - S)^{an}$ for which the local monodromy around each of the n points in S is ζ. Pick any such $\mathcal{L}$. (We will "correct" it later.)

The local system $\underline{\mathrm{Hom}}(\mathcal{L}, \pi^*\mathcal{F})$ on $(Y - S)^{an}$ has trivial local monodromy at each point of S. So denoting by $k: Y - S \to Y$ the inclusion, $k_*\underline{\mathrm{Hom}}(\mathcal{L}, \pi^*\mathcal{F})$ is a local system on Y^{an}, which is (because Y has genus one) necessarily a successive extension of rank one local systems on Y^{an}. So there exists a rank one $\mathcal{L}_0$ on Y^{an}, and a nonzero element of

$$\mathrm{Hom}_{Y^{an}}(\mathcal{L}_0, k_*\underline{\mathrm{Hom}}(\mathcal{L}, \pi^*\mathcal{F})) = \mathrm{Hom}_{(Y-S)^{an}}(\mathcal{L}\otimes k^*\mathcal{L}_0, \pi^*\mathcal{F}).$$

Because $\mathcal{F}$ is irreducible, and π corresponds to a normal subgroup of finite index, $\pi^*\mathcal{F}$ is semisimple. Therefore the group

$$\mathrm{Hom}_{(Y-S)^{an}}(\pi^*\mathcal{F}, \mathcal{L}\otimes k^*\mathcal{L}_0) = \mathrm{Hom}_{U^{an}}(\mathcal{F}, \pi_*(\mathcal{L}\otimes k^*\mathcal{L}_0))$$

is nonzero. Since $\mathcal{F}$ is irreducible, and has the same rank n as $\pi_*(\mathcal{L}\otimes k^*\mathcal{L}_0)$, any nonzero map between them is an isomorphism. Thus $\mathcal{F} \approx \pi_*(\mathcal{L}\otimes k^*\mathcal{L}_0)$. Since $\mathcal{L}_0$ was a rank one local system on all of Y^{an}, $\mathcal{L}$ and $\mathcal{L}\otimes k^*\mathcal{L}_0$ have the same local monodromy at each point of S. So $\mathcal{L}\otimes k^*\mathcal{L}_0$ works as the required "$\mathcal{L}$". QED

Remark 1.4.17 Here is a variant proof. Consider on $(Y - S)^{an}$ any $\mathcal{L}$ for which the local monodromy around each of the n points in S is ζ. Because π is finite etale over all of X, $\pi_*\mathcal{L}$ on U^{an} is a rank n local system which has its local monodromy around x_0 given by the primitive n'th root of unity ζ. By 1.4.4, $\pi_*\mathcal{L}$ is irreducible and weakly physically rigid. So by 1.4.6, $\pi_*\mathcal{L}$ is $\mathcal{F}\otimes j^*(\mathcal{L}_1)^{\otimes -1}$ for some local system $\mathcal{L}_1$ on X^{an}, and so

$$\mathcal{F} \approx (\pi_*\mathcal{L})\otimes(j^*\mathcal{L}_1) = \pi_*(\mathcal{L}\otimes\pi^*j^*\mathcal{L}_1) = \pi_*(\mathcal{L}\otimes k^*\pi^*\mathcal{L}_1),$$

and $\mathcal{L}\otimes k^*\pi^*\mathcal{L}_1$ works as the "$\mathcal{L}$".

(1.4.18) What about **reducible** local systems which are weakly physically rigid (or even semi-rigid)?

Lemma 1.4.19 Let X a projective smooth connected curve over $\mathbb{C}$, of genus g=1, $S = \{s_1, s_2, \ldots, s_k\}$ a finite subset of $X(\mathbb{C})$ with $k \geq 1$ points, $U := X - S$ the open complement,

$$j: U \to X,$$

and

$$j_1 : X - S \to X - \{s_1\}$$

the inclusions. Let $\mathcal{F}$ be a local system on U^{an} of rank $n \geq 1$ which is weakly physically semi-rigid. Then $\mathcal{F}$ is irreducible (and hence weakly physically rigid, by 1.3.1).

proof We first reduce to the case of a single point at ∞. If $k \geq 2$, denote by ξ_i the scalar which is the local monodromy of $\mathcal{F}$ around

s_i, and by $\mathcal{L}(\{\xi_i\}_i)$ the rank one local system on $(X - \{s_1, ..., s_k\})^{an}$ constructed in the proof or 1.4.14 above. Then $\mathcal{F}\otimes\mathcal{L}(\{\xi_i\}_i)$ extends to a local system on $(X - \{s_1\})^{an}$ which is still weakly physically semi-rigid, and it suffices to show that this local system is irreducible on $(X - \{s_1\})^{an}$. [Since the π_1 of $(X - \{s_1, ..., s_k\})^{an}$ maps onto the π_1 of $(X - \{s_1\})^{an}$, a local system $\mathcal{G}$ on $(X - \{s_1\})^{an}$ is irreducible if and only if its restriction $(j_1)^*\mathcal{G}$ to $(X - \{s_1, ..., s_k\})^{an}$ is irreducible.]

We now assume that $k=1$, $S=\{s_1\}$, and show that any weakly physically semi-rigid local system $\mathcal{F}$ on $(X - \{s_1\})^{an}$ is irreducible. Let $\mathcal{F}$ have rank $n \geq 1$, and denote by ζ its local monodromy around the unique point s_1 at ∞. Then ζ is an n'th root of unity, and we know by 1.4.4 that $\mathcal{F}$ is irreducible if and only if ζ is a primitive n'th root of unity. So if ζ has exact order n, we are done.

So suppose that ζ has exact order $d < n$, and define $m:= n/d$, an integer >1. Let $\mathcal{H}$ be any local system on X^{an} of rank m. Then $\mathcal{F}$ and $\mathcal{F}_{d,\zeta}\otimes j^*\mathcal{H}$ are two local systems on $(X - \{s_1\})^{an}$ with isomorphic local monodromy (namely ζ).

By the weak physical semi-rigidity of $\mathcal{F}$, there exists a finite collection of local systems $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_r$ on $(X - \{s_1\})^{an}$ with the following property: for any local system $\mathcal{G}$ on U^{an} such that $\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy, there exists a rank one local system $\mathcal{L}$ on X, an index $1 \leq i \leq r$, and as isomorphism

$$\mathcal{G} \approx \mathcal{F}_i\otimes j^*\mathcal{L}.$$

Applying this to $\mathcal{G} := \mathcal{F}_{d,\zeta}\otimes j^*\mathcal{H}$, as $\mathcal{H}$ runs over all rank m local system on X^{an}, we see that up to tensoring with a rank one local system on X^{an}, we obtain only finitely many isomorphism classes.

We will show this is impossible if $m>1$. Pick a rank one $\mathcal{L}_1$ on X^{an} of infinite order. For each integer $k \geq 1$, define a rank m local system $\mathcal{H}(k)$ on X^{an},

$$\mathcal{H}(k) := \oplus_{1 \leq i \leq m} (\mathcal{L}_1)^{\otimes ki}.$$

Then

$$\mathcal{F}_{d,\zeta}\otimes j^*\mathcal{H}(k) = \oplus_{1 \leq i \leq m} \mathcal{F}_{d,\zeta}\otimes(j^*\mathcal{L}_1)^{\otimes ki}.$$

We claim that even modulo tensoring with a rank one local system on X^{an}, there are still an infinity of isomorphism classes among the local systems $\mathcal{F}_{d,\zeta}\otimes j^*\mathcal{H}(k)$.

Notice that each $\mathcal{F}_{d,\zeta}\otimes j^*\mathcal{H}(k)$ is a direct sum of m pairwise non-isomorphic irreducibles each of the same rank d(namely the $\mathcal{F}_{d,\zeta}\otimes(j^*\mathcal{L}_1)^{\otimes ki}$ with $1\le i\le m$, whose determinants, $\det(\mathcal{F}_{d,\zeta})\otimes(j^*\mathcal{L}_1)^{\otimes dki}$, are already pairwise non-isomorphic, $\mathcal{L}_1$ being of infinite order).

To any direct sum $\mathcal{G}$ of m pairwise non-isomorphic irreducibles $\mathcal{G}_i$, each of the same rank d, we may attach the following invariant: the finite set consisting of the distinct isomorphism classes among the m(m-1) rank one objects $\det(\mathcal{G}_i)/\det(\mathcal{G}_j)$, for all $i \neq j$. This construction visibly attaches the same invariant to $\mathcal{G}$ and to $\mathcal{G}\otimes j^*\mathcal{L}$, for any rank one local system $\mathcal{L}$ on X^{an}.

So it suffices to see that the objects

$$\mathcal{F}_{d,\zeta}\otimes j^*\mathcal{H}(k) = \oplus_{1\le i\le m}\ \mathcal{F}_{d,\zeta}\otimes(j^*\mathcal{L}_1)^{\otimes ki},\ k\ge 1,$$

each give rise to distinct invariants. But this is obvious. The invariant attached to $\mathcal{F}_{d,\zeta}\otimes j^*\mathcal{H}(k)$ consists visibly of the isomorphism classes $(j^*\mathcal{L}_1)^{\otimes kdp}$, $-(m-1)\le p\le m-1$. Because $\mathcal{L}_1$ has infinite order, $(j^*\mathcal{L}_1)^{\otimes kd(m-1)}$, which occurs in the invariant of $\mathcal{F}_{d,\zeta}\otimes j^*\mathcal{H}(k)$, does not occur in the invariant of $\mathcal{F}_{d,\zeta}\otimes j^*\mathcal{H}(N)$ for any $1\le N < k$. QED

2.0 Transition from irreducible local systems on open sets of $\mathbb{P}^1$ to irreducible middle extension sheaves on $\mathbb{A}^1$.

(2.0.1) Let S a nonempty finite subset of $\mathbb{P}^1(\mathbb{C})$, and $U := \mathbb{P}^1 - S$ the open complement, $j : U^{an} \to (\mathbb{P}^1)^{an}$ the inclusion, $\mathcal{F}$ an irreducible $\mathbb{C}$-local system on U^{an} of rank $n \geq 1$. We know by 1.1.2 that $\mathcal{F}$ is physically rigid if and only if $\chi((\mathbb{P}^1)^{an}, j_*\underline{End}(\mathcal{F})) = 2$. Motivated by this fact, we define the **index of rigidity** of $\mathcal{F}$ on U, noted $\mathrm{rig}(\mathcal{F},U)$, to be the integer

$$\mathrm{rig}(\mathcal{F}, U) := \chi((\mathbb{P}^1)^{an}, j_*\underline{End}(\mathcal{F})).$$

Lemma 2.0.2 Let S a nonempty finite subset of $\mathbb{P}^1(\mathbb{C})$, $U := \mathbb{P}^1 - S$ the open complement, $j : U^{an} \to (\mathbb{P}^1)^{an}$ the inclusion, $\mathcal{F}$ an irreducible $\mathbb{C}$-local system on U^{an} of rank $n \geq 1$.

(1) If $\mathcal{L}$ is any rank one local system on U^{an}, we have

$$\mathrm{rig}(\mathcal{F}\otimes\mathcal{L}, U) = \mathrm{rig}(\mathcal{F}, U).$$

(2) If T is any finite subset of $U^{an}(\mathbb{C})$, and $k : U^{an} - T \to U^{an}$ the inclusion, we have

$$\mathrm{rig}(k^*\mathcal{F}, U - T) = \mathrm{rig}(\mathcal{F}, U).$$

proof (1) holds because the local system $\underline{End}(\mathcal{F})$ on U^{an} does not change if we replace $\mathcal{F}$ by $\mathcal{F}\otimes\mathcal{L}$. (2) holds because for any local system $\mathcal{G}$ on U^{an}, we have $\mathcal{G} \approx k_*k^*\mathcal{G}$. Applying this to $\underline{End}(\mathcal{F})$, we get

$$j_*\underline{End}(\mathcal{F}) = j_*k_*k^*\underline{End}(\mathcal{F}) = j_*k_*\underline{End}(k^*\mathcal{F}),$$

and applying $\chi((\mathbb{P}^1)^{an}, \underline{\quad})$ gives the assertion. QED

(2.0.3) Recall [Ka-ESDE, 7.3.1] that on a connected smooth curve $U/\mathbb{C}$, an algebraically constructible sheaf $\mathcal{F}$ of $\mathbb{C}$-vector spaces on U^{an} is called a **middle extension sheaf** if for some (or equivalently for every) nonempty Zariski open set $k: V \to U$ such that $k^*\mathcal{F}$ is a local system on V^{an}, we have $\mathcal{F} \approx k_*k^*\mathcal{F}$. A middle extension sheaf $\mathcal{F}$ on U is called an **irreducible** middle extension if if for some (or equivalently for every) nonempty Zariski open set $k: V \to U$ such that $k^*\mathcal{F}$ is a local system, $k^*\mathcal{F}$ is an irreducible local system.

(2.0.4) We now specialize to the case when U is a Zariski open set of $\mathbb{P}^1$, j: U $\to$ $\mathbb{P}^1$ its inclusion. We will now define the index of rigidity $\mathrm{rig}_U(\mathcal{F})$ of an irreducible middle extension $\mathcal{F}$ on U^{an}. Given such an $\mathcal{F}$, pick a nonempty Zariski open set k: V $\to$ U such that $k^*\mathcal{F}$ is an irreducible local system. The integer

$$\mathrm{rig}(k^*\mathcal{F}, V) := \chi((\mathbb{P}^1)^{an}, j_*k_*\underline{\mathrm{End}}(k^*\mathcal{F}))$$

is independent of the auxiliary choice of V, thanks to lemma 2.0.2 above. We call it $\mathrm{rig}_U(\mathcal{F})$.

(2.0.5) The situation now is this. For any nonempty Zariski open set U in $\mathbb{P}^1$, we have the category IrrME(U) of irreducible middle extensions on U^{an}. Whenever V $\subset$ U, with inclusion k: V $\to$ U, the functors

$$k^* : \mathrm{IrrME}(U) \to \mathrm{IrrME}(V) \text{ and } k_* : \mathrm{IrrME}(V) \to \mathrm{IrrME}(U)$$

are inverse equivalences. Given two nonempty Zariski open sets U_1 and U_2, pick any nonempty Zariski open V in $U_1 \cap U_2$, with inclusions $k_i : V \to U_i$. Then we get an equivalence

$$\varphi_{1,2} := (k_2)_*(k_1)^* : \mathrm{IrrME}(U_1) \to \mathrm{IrrME}(U_2)$$

which is independent of the auxiliary choice of V. Given three nonempty Zariski open sets U_1, U_2, U_3, we have

$$\varphi_{1,3} = \varphi_{2,3} \circ \varphi_{1,2}.$$

So we can canonically identify all of these categories.

(2.0.6) The $\mathbb{Z}$-valued functions $\mathcal{F} \mapsto \mathrm{rig}_U(\mathcal{F})$ on these categories respect these identifications, so we may speak of the single function

$$\mathcal{F} \mapsto \mathrm{rig}(\mathcal{F}).$$

(2.0.7) At first glance, it would seem most natural to work with the single category IrrME($\mathbb{P}^1$). However, it turns out to be better to pick two points in $\mathbb{P}^1(\mathbb{C})$, label them ∞ and 0, and work on the open set $\mathbb{A}^1 := \mathbb{P}^1 - \{\infty\}$. Because we have specified the origin 0, this $\mathbb{A}^1$ has an additive group structure. By embedding the category IrrME($\mathbb{A}^1$) in the slightly larger category IrrPerv($\mathbb{A}^1$) of irreducible perverse sheaves on $\mathbb{A}^1$, we can bring to bear the whole mechanism of additive convolution.

2.1 Transition from irreducible middle extension sheaves on $\mathbb{A}^1$ to irreducible perverse sheaves on $\mathbb{A}^1$

(2.1.1) On any separated $\mathbb{C}$-scheme X of finite type, a sheaf $\mathcal{F}$ of

$\mathbb{C}$-vector spaces on X^{an} is said to be algebraically constructible if there exists a finite partition $X^{red} = \coprod_i Y_i$ as the disjoint union of smooth connected subschemes Y_i, such that on each Y_i, $\mathcal{F} | Y_i^{an}$ is a local system on Y_i^{an}. We denote by $D(X^{an}, \mathbb{C})$ the derived category of the category of all $\mathbb{C}$-sheaves on X^{an}, and by $D^b{}_c(X^{an}, \mathbb{C})$ the full subcategory of $D(X^{an}, \mathbb{C})$ consisting of those objects K for which

1) each cohomology sheaf $\mathcal{H}^i(K)$ is algebraically constructible,
2) only finitely many of the sheaves $\mathcal{H}^i(K)$ are nonzero.

These $D^b{}_c$ support the full Grothendieck formalism of the "six operations". In this formalism, the dualizing complex $K_{X/\mathbb{C}}$ is defined as $f^!\mathbb{C}$, $f: X \to \mathrm{Spec}(\mathbb{C})$ denoting the structural morphism.

(2.1.2) Recall [BBD, Ch. 4] that an object K of $D^b{}_c(X^{an}, \mathbb{C})$ is called **semiperverse** if its cohomology sheaves $\mathcal{H}^i(K)$ satisfy

$$\dim \mathrm{Supp}(\mathcal{H}^i(K)) \le -i, \text{ for every integer } i.$$

An object K is called **perverse** if both K and its Verdier dual $D_{X/\mathbb{C}}K := \mathit{RHom}(K, f^!\mathbb{C})$, $f: X \to \mathrm{Spec}(\mathbb{C})$ denoting the structural morphism, are semiperverse. The main facts about perversity, semiperversity and duality we will use are the following [BBD, Ch. 4]:

(1) if $f:X \to Y$ is an affine morphism, then $K \mapsto Rf_*K$ preserves semiperversity

(2) if $f:X \to Y$ is a quasifinite morphism, then $K \mapsto Rf_!K$ preserves semiperversity

(3) if $f:X \to Y$ is an arbitrary morphism whose geometric fibres all have dimension $\le d$, then $L \mapsto f^*L[d]$ preserves semiperversity

(4) Duality interchanges $Rf_!$ and Rf_*

(5) Duality interchanges $f^!$ and f^*

(6) if $f:X \to Y$ is a smooth morphism everywhere of relative dimension d, then $f^! = f^*[2d](d)$. Consequently $f^*[d](d/2)$ is self-dual, and $K \mapsto f^*K[d]$ preserves perversity

(7) If X is smooth over $\mathbb{C}$, purely of dimension d, then for any local system $\mathcal{F}$ on X, $\mathcal{F}[d]$ is perverse, and $D_{X/k}(\mathcal{F}[d]) = \mathcal{F}^\vee[d](d)$.

(2.1.3) In this discussion, the field $\mathbb{C}$ occurs in two ways, as the ground field over which our variable scheme X is given, and as the coefficient field. Since we speak of X^{an}, the field $\mathbb{C}$ with its classical topology is being used as as the ground field. But the coefficient field $\mathbb{C}$ enters in a purely algebraic way, and could be replaced by any field to which it is isomorphic, for instance by $\overline{\mathbb{Q}}_\ell$ (if we grant the axiom of choice). So we might just as well work with $D^b{}_c(X^{an}, \overline{\mathbb{Q}}_\ell)$ whenever it is convenient.

2.2 Review of $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$

(2.2.1) Let k be a perfect field of characteristic $p \neq \ell$. For variable separated k-schemes of finite type X/k, we can speak of $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$. For morphisms $f: X \to Y$ between separated k-schemes of finite type, one knows (cf. [De-Weil II] for the case when k is either algebraically closed or finite, [Ek], [Ka-Lau], [SGA 4, XVIII, 3]) that these $D^b{}_c$ support the full Grothendieck formalism of the "six operations". In this formalism, the (relative to k) dualizing complex K_X in $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ is defined as $\pi^!\overline{\mathbb{Q}}_\ell$, where π denotes the structural morphism $\pi: X \to \mathrm{Spec}(k)$. In terms of K_X, the Verdier dual $\mathbf{D}(L)$ of an object L of $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ is defined as $\mathit{RHom}(L, K_X)$. One knows that $L \approx \mathbf{DD}(L)$ by the natural map. The duality theorem asserts that for $f : X \to Y$ a morphism of finite type between separated k-schemes of finite type, one has $\mathbf{D}(Rf_!L) \approx Rf_*\mathbf{D}(L)$, $\mathbf{D}(Rf_*L) \approx Rf_!\mathbf{D}(L)$. If X/k is a smooth separated k-scheme of finite type and everywhere of the same relative dimension, noted dimX, then K_X is $\overline{\mathbb{Q}}_\ell[2\dim X](\mathbf{dimX})$, and so $\mathbf{D}(L)$ is $\mathit{RHom}(L, \overline{\mathbb{Q}}_\ell)[2\dim X](\mathbf{dimX})$.

(2.2.2) Given two separated k-schemes X/k and Y/k of finite type, "external tensor product over $\overline{\mathbb{Q}}_\ell$" defines a bi-exact bilinear pairing,

$$D^b{}_c(X, \overline{\mathbb{Q}}_\ell)\times D^b{}_c(Y, \overline{\mathbb{Q}}_\ell) \to D^b{}_c(X\times_k Y, \overline{\mathbb{Q}}_\ell)$$
$$(K, L) \mapsto K\times L := pr_1{}^*K\otimes pr_2{}^*L.$$

One knows that $\mathbf{D}(K\times L) = \mathbf{D}(K)\times\mathbf{D}(L)$.

(2.2.3) If k happens to be $\mathbb{C}$, the comparison theorem gives an exact, fully faithful "passage to the analytic" functor $D^b{}_c(X, \overline{\mathbb{Q}}_\ell) \to D^b{}_c(X^{an}, \overline{\mathbb{Q}}_\ell)$ which is not, however, an equivalence of categories. Everything recalled above about $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ is true also of $D^b{}_c(X^{an}, \overline{\mathbb{Q}}_\ell)$, and all cohomological constructions above commute with the "passage to the analytic" functor. Given any object in $D^b{}_c(X^{an}, \mathbb{C})$, for all $\ell >> 0$

there exists an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ such that the corresponding object in $D^b{}_c(X^{an}, \overline{\mathbb{Q}}_\ell)$ lies in (the essential image of) $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$, cf. [BBD, 6.1]. In 5.9.2, we will give the elementary and down to earth proof of this fact for local systems on open sets of $\mathbb{A}^1$, which will suffice for our purposes.

2.3 Review of perverse sheaves

(2.3.1) We continue to work with X/k as in 2.2.1. An object K of $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ is called **semiperverse** if its cohomology sheaves $\mathcal{H}^i K$ satisfy

$$\dim \mathrm{Supp}(\mathcal{H}^i K) \le -i.$$

An object K of $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ is called **perverse** if both K and its dual $\mathbf{D}(K)$ are semiperverse. If $f : X \to Y$ is an **affine** (respectively a **quasifinite**) morphism, then Rf_* (respectively $f_! = Rf_!$) preserves semiperversity. So if f is both affine and quasifinite (e.g., finite, or an affine immersion), then by duality both $f_! = Rf_!$ and Rf_* preserve perversity. If $f : X \to Y$ is a smooth morphism everywhere of relative dimension d, then $f^*[d]$ preserves perversity. In particular, if K is perverse on X, then its inverse image on $X \otimes_k \overline{k}$ is perverse on $X \otimes_k \overline{k}$. One knows that the full subcategory $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ of $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ consisting of perverse objects is an **abelian** category in which every object is of finite length. If ℓ is fixed, we will often denote $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ simply Perv(X). The objects of Perv(X) are sometimes called "perverse sheaves" on X. However, we will call them "perverse objects" to avoid confusion with "honest" sheaves.

(2.3.1.1) We now recall from [BBD, 1.3] the theory of the perverse truncations ${}^p\tau_{\le i}(K)$ and ${}^p\tau_{\ge i}(K)$ and of the perverse cohomology sheaves ${}^p\mathcal{H}^i(K)$ attached to an object K in $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$. This will be used (only) in section 2.12. Inside $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$, we denote by ${}^pD^{\le 0}$ (respectively ${}^pD^{\ge 0}$) the full subcategory consisting of those objects K which are semiperverse (respectively, those objects K such that DK is semiperverse). For each integer i, we define ${}^pD^{\le i}$ (respectively ${}^pD^{\ge i}$) to be the full subcategory of $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ consisting of those objects K such that K[i] lies in ${}^pD^{\le 0}$ (respectively ${}^pD^{\ge 0}$). Duality interchanges ${}^pD^{\le i}$ and ${}^pD^{\ge -i}$ By [BBD, 1.3.3], the inclusion of ${}^pD^{\le i}$ (respectively of ${}^pD^{\ge i}$) into $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ admits a right (respectively left) adjoint, noted ${}^p\tau_{\leq i}$ (respectively ${}^p\tau_{\geq i}$). It is tautological that

$$({}^p\tau_{\leq i}(K))[i] = {}^p\tau_{\leq 0}(K[i]), \quad ({}^p\tau_{\geq i}(K))[i] = {}^p\tau_{\geq 0}(K[i]).$$

For any i, and any K in $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$, we have a distinguished triangle

$${}^p\tau_{\leq i}(K) \to K \to {}^p\tau_{\geq i+1}(K).$$

For any two integers a, b, there is a canonical isomorphism [BBD, 1.3.5] between the two composites

$${}^p\tau_{\geq a} \circ {}^p\tau_{\leq b} \cong {}^p\tau_{\leq b} \circ {}^p\tau_{\geq a}.$$

Unless $a \leq b$, both of these composites are zero.

Now take the special case a=b=0. In this case, the composite functor above has values in $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$, the intersection of ${}^pD^{\leq 0}$ and of ${}^pD^{\geq 0}$. For K in $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$, we define

$${}^p\mathcal{H}^0(K) := {}^p\tau_{\geq 0}({}^p\tau_{\leq 0}(K)) \cong {}^p\tau_{\leq 0}({}^p\tau_{\geq 0}(K)).$$

For each integer i, we define

$${}^p\mathcal{H}^i(K) := {}^p\mathcal{H}^0(K[i]),$$

or equivalently,

$${}^p\mathcal{H}^i(K)[-i] := {}^p\tau_{\geq i}({}^p\tau_{\leq i}(K)) \cong {}^p\tau_{\leq i}({}^p\tau_{\geq i}(K)).$$

One shows [BBD, 1.3.6] that ${}^p\mathcal{H}^0$ is a cohomological functor from $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ to the abelian category $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$: a distinguished triangle in $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ gives rise to a long exact sequence of perverse cohomology sheaves. The functors ${}^p\tau_{\leq 0}$ and ${}^p\tau_{\geq 0}$ are interchanged by duality, whence

$${}^p\mathcal{H}^0(DK) := D({}^p\mathcal{H}^0(K)),$$

and hence for every i, we have

$${}^p\mathcal{H}^i(DK) := D({}^p\mathcal{H}^{-i}(K)).$$

Given any K in $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$, all but finitely many of its perverse cohomology sheaves ${}^p\mathcal{H}^i(K)$ vanish. [By duality, it suffices to show that ${}^p\mathcal{H}^i(K)$ vanishes for i sufficiently large. But K[i] is (trivially) semiperverse for i sufficiently large, so it suffices to show that for K semiperverse, ${}^p\mathcal{H}^i(K)$ vanishes for $i > 0$. In fact, one knows [BBD, 1.3.7] that K in $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ is semiperverse if and only if ${}^p\mathcal{H}^i(K)$ vanishes for $i > 0$. Therefore ${}^p\mathcal{H}^i(K) = 0$ for i outside [a, b] if and only if K lies in ${}^pD^{[a, b]} := {}^pD^{\geq a} \cap {}^pD^{\leq b}$.] Moreover, any K in $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ is a successive extension of its shifted perverse cohomology sheaves ${}^p\mathcal{H}^i(K)[-i]$, as one sees by induction on the length b-a of the shortest interval [a, b] such that K lies in ${}^pD^{[a, b]}$, using the distinguished triangles ${}^p\tau_{\le i}(K) \to K \to {}^p\tau_{\ge i+1}(K)$.

(2.3.2) If X is smooth over k, everywhere of relative dimension dimX, the simplest example of a perverse object on X is provided by starting with a lisse sheaf $\mathcal{F}$ on X, and taking the object $\mathcal{F}[\dim X]$ of $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ obtained by placing $\mathcal{F}$ in degree $-\dim X$. The object $\mathcal{F}[\dim X]$ is trivially semiperverse, and its dual $\mathbf{D}(\mathcal{F}[\dim X]) = (\mathcal{F}^\vee(\mathbf{dimX}))[\dim X]$, being of the same form, is also. If X is connected, and if $\mathcal{F}$ is irreducible as a lisse sheaf, i.e., as a representation of $\pi_1(X, x)$, then $\mathcal{F}[\dim X]$ is a simple object of Perv(X).

Lemma 2.3.2.1 For X smooth over k, everywhere of relative dimension dimX, consider the following two (rather special) properties of an object K in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$:

a) each of the cohomology sheaves $\mathcal{H}^i(K)$ is lisse,

b) each of the perverse cohomology sheaves ${}^p\mathcal{H}^i(K)$ is of the form (a lisse sheaf on X)[dimX].

These properties are equivalent, and, if they hold, then the perverse and ordinary cohomology sheaves of K are related by

c) $${}^p\mathcal{H}^i(K) = \mathcal{H}^{i-\dim X}(K)[\dim X].$$

proof Suppose first that a) holds. For the usual truncation functors $\tau_{\le n}$, we have distinguished triangles

$$\tau_{\le n-1}K \to \tau_{\le n}K \to \mathcal{H}^n(K)[-n].$$

For n sufficiently negative, we have $\tau_{\le n}K = 0$, and for n sufficiently positive, we have $\tau_{\le n-1}K \cong \tau_{\le n}K \cong K$. Because ${}^p\mathcal{H}^0$ is a cohomological functor, we get a long exact sequence

$$\to {}^p\mathcal{H}^i(\tau_{\le n-1}K) \to {}^p\mathcal{H}^i(\tau_{\le n}K) \to {}^p\mathcal{H}^i(\mathcal{H}^n(K)[-n]) \to {}^p\mathcal{H}^{i+1}(\tau_{\le n-1}K) \to$$

Since $\mathcal{H}^n(K)$ is lisse, $\mathcal{H}^n(K)[\dim X]$ is perverse, and hence

$$\begin{aligned} {}^p\mathcal{H}^a(\mathcal{H}^n(K)[-n]) &= \mathcal{H}^n(K)[\dim X] \text{ for } a=n+\dim X \\ &= 0 \text{ for } a\neq n+\dim X. \end{aligned}$$

Using this fact, and the long exact sequences above, one shows by induction on n that

1)${}^p\mathcal{H}^a(\tau_{\le n}K) = 0$ for $a > n+\dim X$,

2)the map $\tau_{\le n}K \to \mathcal{H}^n(K)[-n]$ induces an isomorphism on ${}^p\mathcal{H}^a$ for $a=n+\dim X$,

3)the map $\tau_{\le n-1}K \to \tau_{\le n}K$ induces an isomorphism on ${}^p\mathcal{H}^a$ for $a<n+\dim X$.

Once we have these facts, then for any a, we get

$${}^p\mathcal{H}^{a+\dim X}(\tau_{\le a}K) \cong {}^p\mathcal{H}^{a+\dim X}(\tau_{\le a+1}K) \cong \ldots \cong {}^p\mathcal{H}^{a+\dim X}(K)$$

by successive application of 3), and then by 2) we get

$${}^p\mathcal{H}^{a+\dim X}(\tau_{\le a}K) \cong {}^p\mathcal{H}^{a+\dim X}(\mathcal{H}^a(K)[-a]) = \mathcal{H}^a(K)[\dim X].$$

Thus we obtain ${}^p\mathcal{H}^{a+\dim X}(K) \cong \mathcal{H}^a(K)[\dim X]$, which proves c), and consequently b).

Now suppose that b) holds. Repeat the above argument, with usual truncation replaced by perverse truncation, with perverse cohomology sheaves replaced by usual cohomology sheaves, and with dimX replaced by -dimX. Again we find c), and consequently a). QED

(2.3.3) Given a locally closed subscheme Y of X such that Y is affine, the inclusion $j: Y \to X$ is both affine and quasifinite (factor it as the open immersion of Y into its closure $\overline{Y}$, followed by the closed immersion of $\overline{Y}$ into X). So for a perverse object K on Y, both $j_!K$ and Rj_*K are perverse on X, and as functors from Perv(Y) to Perv(X) both $j_!$ and Rj_* are exact. There is a natural "forget supports" map from $j_!K$ to Rj_*K, and as Perv(X) is an abelian category it makes sense to form

$$j_{!*}(K) := \mathrm{Image}(j_!K \to Rj_*K) \in \mathrm{Perv}(X),$$

called the "middle extension" from Y to X of the perverse object K. The functor $j_{!*}$ is end-exact (i.e., it preserves both injections and surjections, cf. the appendix to this chapter) from Perv(Y) to Perv(X), it carries simple objects to simple objects, and it commutes with duality. Despite the erroneous assertion in [Ka-ESDE, 8.1.4], the functor $j_{!*}$ is **not** exact in general.

(2.3.3.1) For K and L perverse on Y as in 2.3.3 above, the functors $j_{!*}$ and j^*induce natural maps of Hom groups,

$$\mathrm{Hom}_Y(K, L) \xrightarrow{j_{!*}} \mathrm{Hom}_X(j_{!*}K, j_{!*}L)$$

and

$$\mathrm{Hom}_X(j_{!*}K, j_{!*}L) \xrightarrow{j^*} \mathrm{Hom}_Y(K, L).$$

These maps are inverse isomorphisms. To see this, we argue as follows. The composite functor $K \mapsto j^*j_{!*}K$ is the identity, so the composite map

$$\mathrm{Hom}_Y(K, L) \to \mathrm{Hom}_X(j_{!*}K, j_{!*}L) \to \mathrm{Hom}_Y(K, L)$$

is the identity on $\mathrm{Hom}_Y(K, L)$.

Because $j_{!*}L$ is a subobject of Rj_*K in $\mathrm{Perv}(X)$, the restriction map

$$\mathrm{Hom}_X(j_{!*}K, j_{!*}L) \xrightarrow{j^*} \mathrm{Hom}_Y(K, L).$$

is injective: we have

$$\mathrm{Hom}_X(j_{!*}K, j_{!*}L) \subset \mathrm{Hom}_X(j_{!*}K, Rj_*L) =$$
$$= \mathrm{Hom}_Y(j^*j_{!*}K, L) = \mathrm{Hom}_Y(K, L).$$

Now start with φ in $\mathrm{Hom}_X(j_{!*}K, j_{!*}L)$. Both φ and $j_{!*}j^*(\varphi)$ have the same restriction, namely $j^*(\varphi)$, in $\mathrm{Hom}_Y(K, L)$, hence $\varphi = j_{!*}j^*(\varphi)$ in $\mathrm{Hom}_X(j_{!*}K, j_{!*}L)$.

(2.3.4) One knows that for any simple object S of $\mathrm{Perv}(X)$ there exists an affine locally closed subscheme $j: Y \to X$ such that Y is smooth over k and irreducible, and an irreducible lisse sheaf $\mathcal{F}$ on Y such that S is $j_{!*}(\mathcal{F}[\dim Y])$. Given the simple object S, we construct Y and $\mathcal{F}$ as follows: the closure $\overline{Y}$ of Y is precisely the closure of the support of $\oplus_i \mathcal{H}^i S$, Y is any smooth affine open set of $\overline{Y}$ on which all the $\mathcal{H}^i S$ are lisse, and $\mathcal{F}$ is $\mathcal{H}^{-\dim Y}(S)|Y$.

(2.3.5) An object S of $\mathrm{Perv}(X)$ is called geometrically simple if its inverse image on $X\otimes_k\overline{k}$ is simple. Of course "geometrically simple" $\Rightarrow$ "simple".

(2.3.6) Consider the special case when X/k is a smooth, geometrically connected curve. Then an object K of $D^b{}_c(X, \overline{\mathbb{Q}}_\ell)$ is perverse if and only if

$\mathcal{H}^i K = 0$ for $i \neq -1, 0$,

$\mathcal{H}^{-1} K$ has no nonzero punctual sections,

$\mathcal{H}^0 K$ is punctual.

We call a perverse object K punctual if $K = \mathcal{H}^0(K)$, and we call K "nonpunctual" if $\mathcal{H}^0 K = 0$. If $\mathcal{F}$ is a lisse sheaf on an open nonempty open set $j: U \to X$, then the middle extension $j_{!*}(\mathcal{F}[1])$ is none other than $(j_*\mathcal{F})[1]$. It is for this reason that we adapted the terminology "middle extension" for sheaves of the type $j_*\mathcal{F}$ with $\mathcal{F}$ lisse on U. The dual $\mathbf{D}(j_{!*}(\mathcal{F}[1]))$ of such a middle extension is given by

$$\mathbf{D}(j_{!*}(\mathcal{F}[1])) = \mathbf{D}(j_*\mathcal{F}[1]) = j_{!*}(\mathbf{D}(\mathcal{F}[1])) = j_*(\mathcal{F}^\vee)[1](\mathbf{1}).$$

Any perverse sheaf K on X has a natural two step filtration, whose associated graded pieces are (punctual, middle extension, punctual). To see this we first filter K by its subobject $\mathcal{H}^{-1}(K)[1]$, which sits in the short exact sequence of perverse sheaves

$$0 \to \mathcal{H}^{-1}(K)[1] \to K \to \mathcal{H}^0(K) \to 0.$$

Now denote by $j: U \to X$ the inclusion of a nonempty affine open set U on which $\mathcal{H}^{-1}(K)$ is lisse. Since $\mathcal{H}^{-1}(K)$ has no nonzero punctual sections, we have a short exact sequence of usual sheaves on X

$$0 \to \mathcal{H}^{-1}(K) \to j_*j^*\mathcal{H}^{-1}(K) \to \text{pct'l} \to 0.$$

Shifting by [1] and rotating the triangle, we get a distinguished triangle

$$\text{pct'l} \to \mathcal{H}^{-1}(K)[1] \to (j_*j^*\mathcal{H}^{-1}(K))[1],$$

i.e., a short exact sequence of perverse sheaves

$$0 \to \text{pct'l} \to \mathcal{H}^{-1}(K)[1] \to (j_*j^*\mathcal{H}^{-1}(K))[1] \to 0.$$

The filtration in question is $\text{pct'l} \subset \mathcal{H}^{-1}(K)[1] \subset K$.

There are two types of simple perverse object on X:
(1) the punctual ones, whose Y is a single closed point x of X; the corresponding simple objects are $x_*\mathcal{F}$, where $\mathcal{F}$ is an irreducible representation of $\mathrm{Gal}(\overline{k}/k(x))$ [so if k is algebraically closed, only the delta sheaf $\delta_x := x_*\overline{\mathbb{Q}}_\ell$ supported at x].
(2) the nonpunctual ones, whose Y is a nonempty open set $j: U \to X$ of X; the corresponding simple objects are $(j_*\mathcal{F})[1]$, where $\mathcal{F}$ is an "arithmetically irreducible" lisse sheaf on U, i.e., one whose representation of $\pi_1(U, \overline{u})$ is irreducible [so the nonpunctual simples which are geometrically simple are precisely the $\mathcal{F}[1]$ where $\mathcal{F}$ is an "irreducible middle extension sheaf" in the terminology of 2.0.3.
(2.3.7) If k is $\mathbb{C}$, then for any ℓ the exact, fully faithful "passage to the analytic" functor $D^b_c(X, \overline{\mathbb{Q}}_\ell) \to D^b_c(X^{an}, \overline{\mathbb{Q}}_\ell)$ induces an exact fully

faithful functor $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(X^{an}, \overline{\mathbb{Q}}_\ell)$. Everything said above about $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ holds also for $\mathrm{Perv}(X^{an}, \overline{\mathbb{Q}}_\ell)$, and all cohomological constructions, including middle extension $j_{!*}$, commute with the "passage to the analytic" functor 2.2.3.

2.4 Review of Fourier Transform

(2.4.1) Suppose X/k is $\mathbb{A}^1/k$, k a perfect field with $\mathrm{char}(k) := p > 0$. For each $\ell \neq p$, we fix a nontrivial additive character $\psi: \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$, and its associated lisse, rank one Artin-Schreier sheaf $\mathcal{L}_\psi$ on $\mathbb{A}^1$. The derived category versions of Fourier Transform are defined by

$$FT_{\psi,!}(K) := R(pr_2)_!(pr_1{}^*K \otimes \mathcal{L}_{\psi(xy)})[1],$$

$$FT_{\psi,*}(K) := R(pr_2)_*(pr_1{}^*K \otimes \mathcal{L}_{\psi(xy)})[1].$$

Both are exact functors from $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ to itself, which are essentially interchanged by duality:

$$\mathbf{D}(FT_{\psi,!}K) = FT_{\psi,*}([-1]^* \cdot \mathbf{D}K)(\mathbf{1}).$$

It is easy to prove that $FT_{\psi,!}$ is essentially involutive:

$$FT_{\psi,!} \cdot FT_{\psi,!} \approx [-1]^*(\mathbf{-1});$$

by duality it follows that the same holds for $FT_{\psi,*}$.

(2.4.2) The "miracle" of Fourier Transform is that there is really only one: the natural "forget supports" map $FT_{\psi,!} \to FT_{\psi,*}$ is an isomorphism. We denote it FT_ψ. As FT_ψ (viewed as $FT_{\psi,*}$) preserves semiperversity, it follows from the miracle that FT_ψ preserves perversity, and so defines an exact autoequivalence of $\mathrm{Perv}(\mathbb{A}^1)$. In particular, FT_ψ sends perverse simple objects to perverse simple objects.

2.5 Review of convolution

(2.5.1) Suppose G is a smooth separated k-groupscheme of finite type of relative dimension noted $\dim G$, $\pi: G \times_k G \to G$ the multiplication map, $e: \mathrm{Spec}(k) \to G$ the identity section. Given two objects K and L in $D^b{}_c(G, \overline{\mathbb{Q}}_\ell)$, we define their "compact" or "!" convolution, denoted $K *_! L$, by

$$K *_! L := R\pi_!(K \times L) \in D^b{}_c(G, \overline{\mathbb{Q}}_\ell).$$

We define their "$*$" convolution, denoted $K *_* L$, by

$$K *_{*} L := R\pi_{*}(K \times L) \in D^b{}_c(G, \overline{\mathbb{Q}}_\ell).$$

Duality interchanges the two sorts of convolution:

$$\mathbf{D}(K *_! L) \approx \mathbf{D}(K) *_{*} \mathbf{D}(L),\ \mathbf{D}(K *_{*} L) \approx \mathbf{D}(K) *_! \mathbf{D}(L).$$

By the Leray spectral sequence and the Kunneth formula, we have $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphisms of cohomology algebras

$$H_c{}^*(G\otimes\overline{k}, K *_! L) \approx H_c{}^*((G\times G)\otimes\overline{k}, K\times L) \approx H_c{}^*(G\otimes\overline{k}, K)\otimes H_c{}^*(G\otimes\overline{k}, L),$$

$$H^*(G\otimes\overline{k}, K *_{*} L) \approx H^*((G\times G)\otimes\overline{k}, K\times L) \approx H^*(G\otimes\overline{k}, K)\otimes H^*(G\otimes\overline{k}, L).$$

(2.5.2) If we start with two (usual or perverse) sheaves $\mathcal{F}$ and $\mathcal{G}$ on G, viewed as objects of of $D^b{}_c(G, \overline{\mathbb{Q}}_\ell)$, their convolutions $\mathcal{F} *_! \mathcal{G}$ and $\mathcal{F} *_{*} \mathcal{G}$ are "really" objects of $D^b{}_c(G, \overline{\mathbb{Q}}_\ell)$, and **not** simply single (usual or perverse) sheaves placed in some degree. It is this "instability" of (usual or perverse) sheaves themselves under convolution that makes $D^b{}_c(G, \overline{\mathbb{Q}}_\ell)$ the natural setting for systematically discussing convolution.

(2.5.3) For the convenience of the reader, we collect from [Ka-ESDE, 8.1.9-10] the standard facts about convolution.

(0) If K and L are semiperverse (resp. perverse) objects on G, then $K\times L$ is semiperverse (resp. perverse) on $G\times_k G$. Therefore if G is affine, and if K and L are both semiperverse on G, then $K *_{*} L$ is semiperverse on G. If K and L are both perverse on G and if moreover the natural "forget supports" map is an isomorphism $K *_! L \approx K *_{*} L$, then $K *_! L \approx K *_{*} L$ is perverse (its dual being $\mathbf{D}(K) *_{*} \mathbf{D}(L)$).

(1)Each sort of convolution is associative, and for each the δ-sheaf

$$\delta_e := e_{*}\overline{\mathbb{Q}}_\ell$$

supported at the identity of G is a two-sided identity object. If G is commutative, then each sort of convolution is commutative as well.

(2a) If $\varphi: G\to H$ is a homomorphism of smooth separated k-groupschemes of finite type, then for K and L on G we have

$$R\varphi_{*}(K *_{*} L) \approx (R\varphi_{*}K) *_{*} (R\varphi_{*}L),$$

$$R\varphi_!(K *_! L) \approx (R\varphi_! K) *_! (R\varphi_! L).$$

(2b) If $\varphi : G \to H$ is a homomorphism, then for K on G and L on H we have

$$\varphi^*((R\varphi_! K) *_! L) \approx K *_! (\varphi^* L),$$

$$\varphi^!((R\varphi_{*}K) *_{*} L) \approx K *_{*} (\varphi^! L).$$

(3) For $g \in G(k)$ denote by $T_g : G \to G$ the map $x \mapsto gx$ "left translation by g", and by $\delta_g := (T_g)_*(\delta_e)$ the delta sheaf supported at g. Then for $g \in G(k)$, we have

$$(T_g)_* = R(T_g)_* = (T_g)_! = R(T_g)_!$$
$$(T_g)_*(K*_*L) \approx ((T_g)_*K)*_*L,$$
$$(T_g)_*(K*_!L) \approx ((T_g)_*K)*_!L,$$
$$(T_g)_*(L) \approx (\delta_g)*L.$$

Moreover, if G is commutative, then for g, h in G(k), we have

$$(T_{gh})_*(K*_*L) \approx ((T_g)_*K)*_*((T_h)_*L),$$
$$(T_{gh})_*(K*_!L) \approx ((T_g)_*K)*_!((T_h)_*L).$$

(4) If G is commutative, geometrically connected, and defined over a finite subfield k_0 of k, then for every $\overline{\mathbb{Q}}_\ell$-valued character χ of $G(k_0)$, the associated lisse rank one $\mathcal{L}_\chi$ on G obtained from pushing out the Lang torsor by χ satisfies $\pi^*\mathcal{L}_\chi \approx \mathcal{L}_\chi \times \mathcal{L}_\chi$, whence by the projection formula

$$(K*_!L)\otimes\mathcal{L}_\chi \approx (K\otimes\mathcal{L}_\chi)*_!(L\otimes\mathcal{L}_\chi),$$
$$(K*_*L)\otimes\mathcal{L}_\chi \approx (K\otimes\mathcal{L}_\chi)*_*(L\otimes\mathcal{L}_\chi).$$

(2.5.4) If k is $\mathbb{C}$, then for any ℓ the exact, fully faithful "passage to the analytic" functor $D^b{}_c(X, \overline{\mathbb{Q}}_\ell) \to D^b{}_c(X^{an}, \overline{\mathbb{Q}}_\ell)$ respects both sorts of convolution.

2.6 Convolution operators on the category of perverse sheaves: middle convolution

(2.6.1) Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type of relative dimension noted dimG, $\pi: G\times_k G \to G$ the multiplication map. A perverse sheaf K on G is said to have property $\mathcal{P}$! [respectively property $\mathcal{P}*$] if for any perverse sheaf L on G, the ! convolution $L*_!K$ [respectively the * convolution $L*_*K$] is again perverse. If K has property $\mathcal{P}$! [respectively property $\mathcal{P}*$], the functor $L \mapsto L*_!K$ [respectively $L \mapsto L*_*K$] is an exact functor from Perv(G) to itself. Notice that since duality interchanges the two sorts of convolution,

$$D(L*_!K) \cong D(L)*_*D(K),$$

we have the equivalence.

$$K \text{ has } \mathcal{P}! \Leftrightarrow DK \text{ has } \mathcal{P}*.$$

(2.6.2) A perverse sheaf K on G is said to have property $\mathcal{P}$ if it has both properties $\mathcal{P}!$ and $\mathcal{P}*$. If K has property $\mathcal{P}$, we define, for any perverse L on G, the middle convolution $L*_{mid}K$ to be the perverse sheaf on G defined as the image, in the abelian category of perverse sheaves on G, of the natural "forget supports" map

$$L*_{mid}K := \mathrm{image}(L*_!K \ \to \ L*_*K).$$

For fixed K in $\mathcal{P}$, $L \mapsto L*_{mid}K$ is end-exact (cf. 2.17).

(2.6.3) By definition, the map

$$L*_!K \ \to L*_{mid}K$$

is surjective in Perv, and the map

$$L*_{mid}K \ \to L*_*K$$

is injective in Perv.

(2.6.4) If K and L have $\mathcal{P}!$ [respectively $\mathcal{P}*$], so does $K*_!L$ [respectively $L*_*K$], just by the associativity of convolution. What about $\mathcal{P}$ itself? We will see later that if K and L both have $\mathcal{P}$, and if G is either $\mathbb{G}_m$ or $\mathbb{A}^1$, then $K*_{mid}L$ also has $\mathcal{P}$, but we do not know this for more general G. Returning to the general situation, we have

Lemma 2.6.5 In the situation 2.6.1, if $\mathcal{F}$, K and L are perverse sheaves on G which all have $\mathcal{P}$, then

$$(\mathcal{F}*_{mid}K)*_{mid}L = \mathcal{F}*_{mid}(K*_{mid}L).$$

proof We will show this equality by showing that both are the image of $\mathcal{F}*_!K*_!L$ in $\mathcal{F}*_*K*_*L$. Indeed, we may factor this "forget supports" map as the composition of the two surjective maps

$$(\mathcal{F}*_!K)*_!L \to (\mathcal{F}*_{mid}K)*_!L \text{ (surj. as } *_!L \text{ is exact on Perv)}$$

$$(\mathcal{F}*_{mid}K)*_!L \to (\mathcal{F}*_{mid}K)*_{mid}L$$

and the two injective maps

$$(\mathcal{F}*_{mid}K)*_{mid}L \to (\mathcal{F}*_{mid}K)*_*L$$

$$(\mathcal{F}*_{mid}K)*_*L \to (\mathcal{F}*_*)*_*L \text{ (inj. as} *_*L \text{ is exact on Perv).}$$

Thus $(\mathcal{F}*_{mid}K)*_{mid}L$ is the image of $\mathcal{F}*_!K*_!L$ in $\mathcal{F}*_*K*_*L$. Rearranging the parentheses shows that $\mathcal{F}*_{mid}(K*_{mid}L)$ is also this image. QED

Remark 2.6.6 If we only assume that K and L have $\mathcal{P}$, but not $\mathcal{F}$, then the right hand side $\mathcal{F}*_{mid}(K*_{mid}L)$ isn't **defined**,since we don't know that $K*_{mid}L$ has $\mathcal{P}$. Later, we will know that $K*_{mid}L$ has $\mathcal{P}$if both K and L do, at least on both $\mathbb{A}^1$ and $\mathbb{G}_m$, but it will **not** be true that for any $\mathcal{F}$ in Perv, we have $\mathcal{F}*_{mid}(K*_{mid}L) =$

$(\mathcal{F}*_{mid}K)*_{mid}L$ Indeed, it can be that $K*_{mid}L = \delta_e$, so the left side is $\mathcal{F}$, but that already $\mathcal{F}*_!K = 0$, so a fortiori $\mathcal{F}*_{mid}K = 0$. [example: $\mathcal{F} = \overline{\mathbb{Q}}_\ell[1]$ on $\mathbb{A}^1$, $K = \mathcal{L}_\chi[1]$, $L = \mathcal{L}_{\overline{\chi}}[1]$.]

Lemma 2.6.7 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type. Let K and L be any perverse sheaves on G. Then $K*_!L$ is perverse if and only if it is semiperverse.

proof By definition, $K*_!L$ is perverse if and only both it and its dual are semiperverse. So it suffices to show that for K and L perverse, the dual of $K*_!L$ is semiperverse. But this dual is $D(K*_!L) = D(K)*_*D(L)$, the $*$convolution of the perverse sheaves D(K) and D(L), hence is semiperverse because G is affine (cf. 2.5.3 (0)). QED

Lemma 2.6.8 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type. Let K be perverse on G. Then the following conditions are equivalent:

1) K has property $\mathcal{P}$!.

2) $K*_!M$ is perverse for every perverse irreducible M on G.

proof That 1) implies 2) is trivial. Suppose now that 2) holds, and let L be perverse on G. It is known [BBD, 1.3.6 and 4.3.1 (i)] that the category of perverse sheaves on G is abelian, and every object is of finite length. We proceed by induction on the length of L. If L is irreducible, then $K*_!L$ is perverse. by 2). In general, pick a perverse irreducible M in L, and consider the short exact sequence of perverse sheaves

$$0 \to M \to L \to N \to 0.$$

By induction, both $K*_!M$ and $K*_!N$ are perverse, and hence both are semiperverse. In the derived category $D^b_c(G, \overline{\mathbb{Q}}_\ell)$, the above exact sequence of perverse sheaves is a distinguished triangle. Applying the functor $K*_!(\_)$ to it yields a distinguished triangle

$$0 \to K*_!M \to K*_!L \to K*_!N \to 0$$

whose end terms are semiperverse. The long exact cohomology sequence

$$\to \mathcal{H}^i(K*_!M) \to \mathcal{H}^i(K*_!L) \to \mathcal{H}^i(K*_!N) \to$$

shows that $K*_!L$ is itself semiperverse, and hence (by the previous lemma) perverse. Thus 2) implies 1). QED

The special case of relative dimension one

We now turn to the special case when G in 2.6.1 is further assumed to be of relative dimension one. Although such a G is isomorphic to either $\mathbb{G}_a$ or $\mathbb{G}_m$, we will, to the extent possible, give a unified treatment of the two cases.

Lemma 2.6.9 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. Let K be any perverse irreducible on G whose isomorphism class is not translation-invariant. Then K has property $\mathcal{P}$!.

proof By the above two lemmas, it suffices to show that for any perverse irreducible L on G, $K*_!L$ is semiperverse. Since G is one-dimensional, $K*_!L$ is semiperverse if and only if

$$\mathcal{H}^0(K*_!L) \text{ is punctual, and } \mathcal{H}^i(K*_!L) = 0 \text{ for } i > 0.$$

If K (respectively L) is punctual, then $K*_!L$ is a translate of L (resp. K), and hence $K*_!L$ is certainly perverse. If neither K nor L is punctual, then there exists a dense open set U in G, and irreducible lisse sheaves $\mathcal{F}$ and $\mathcal{G}$ on U, such that, denoting by $j : U \to G$ the inclusion, K and L are $j_*\mathcal{F}[1]$ and $j_*\mathcal{G}[1]$ respectively. The stalk of $\mathcal{H}^i(K*_!L)$ at any geometric point g of G is the cohomology group $H_c{}^{i+2}(G, \mathrm{Trans}_g{}^*(j_*\mathcal{F})\otimes(\mathrm{inv})^*(j_*\mathcal{G}))$. Since G is one-dimensional, this group vanishes for $i > 0$, and hence $\mathcal{H}^i(K*_!L) = 0$ for $i > 0$. It remains to show that $\mathcal{H}^i(K*_!L)$ is punctual, i.e., that for all but at most finitely many g in G(k), we have

$$H_c{}^2(G, \mathrm{Trans}_g{}^*(j_*\mathcal{F})\otimes(\mathrm{inv})^*(j_*\mathcal{G})) = 0.$$

For fixed g in G(k), let us denote by U_g the dense open set

$$U_g := \mathrm{Trans}_g{}^*U \cap (\mathrm{inv})^*U$$

on which both $\mathrm{Trans}_g{}^*(j_*\mathcal{F})$ and $(\mathrm{inv})^*(j_*\mathcal{G})$ are lisse. Then

$$H_c{}^2(G, \mathrm{Trans}_g{}^*(j_*\mathcal{F})\otimes(\mathrm{inv})^*(j_*\mathcal{G}))$$
$$= H_c{}^2(U_g, \mathrm{Trans}_g{}^*(\mathcal{F})\otimes(\mathrm{inv})^*(\mathcal{G})).$$
$$= H^0(U_g, \mathrm{Trans}_{g*}(\mathcal{F}^\vee)\otimes(\mathrm{inv})^*(\mathcal{G}^\vee))^\vee(1)$$
$$= \mathrm{Hom}_{U_g}(\mathrm{Trans}_g{}^*(\mathcal{F}), (\mathrm{inv})^*(\mathcal{G}^\vee))^\vee(1).$$

Since both $\mathcal{F}$ and $\mathcal{G}$ are irreducible, this group vanishes unless

$$\mathrm{Trans}_g{}^*(\mathcal{F}) \cong (\mathrm{inv})^*(\mathcal{G}^\vee) \text{ as lisse sheaves on } U_g,$$

i.e., unless

$$\mathrm{Trans}_g{}^*(j_*\mathcal{F}[1]) \cong (\mathrm{inv})^*(j_*\mathcal{G}^\vee[1]) \text{ as perverse sheaves on } G,$$

i.e., unless

$$\mathrm{Trans}_g{}^*(K) \cong (\mathrm{inv})^*(D(L)) \text{ as perverse sheaves on } G,$$

in which case the group is one-dimensional. By the constructibility of $\mathcal{H}^0(K*_!L)$, either $\mathcal{H}^0(K*_!L)$ is punctual, in which case $K*_!L$ is semiperverse, or there is a dense open set V in G such that $\mathrm{Trans}_g{}^*(K) \cong (\mathrm{inv})^*(D(L))$ for all g in V.

In this last case, we argue as follows. Fix one such g_0; then $\mathrm{Trans}_g{}^*(K) \cong \mathrm{Trans}_{g_0}{}^*(K)$ for all g inV. Then the isomorphism class of $\mathrm{Trans}_{g_0}{}^*(K)$ is invariant by translation by all g in $V_1 := g_0{}^{-1}V$. But the set of g in G(k) such that $\mathrm{Trans}_g{}^*$ fixes any particular isomorphism class (here that of $\mathrm{Trans}_{g_0}{}^*(K)$) is a **subgroup** of G(k). This subgroup, for $\mathrm{Trans}_{g_0}{}^*(K)$, contains the open dense set V_1, and hence contains $V_1V_1 = G$. Thus the isomorphism class of $\mathrm{Trans}_{g_0}{}^*(K)$, and hence of K itself, is translation invariant. QED

Corollary 2.6.10 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. Let K be any perverse irreducible on G whose isomorphism class is not translation-invariant. Then K has property $\mathcal{P}$.

proof By lemma 2.6.9, K has $\mathcal{P}$!. But the isomorphism class of DK is not translation-invariant, so DK also has $\mathcal{P}$!, whence K has $\mathcal{P}*$. QED

Lemma 2.6.11 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. For any perverse objects K and L on G, the not-necessarily perverse object $K*_!L$ on G has $\mathcal{H}^i(K*_!L) = 0$ for $i > 0$.

proof By an obvious devissage, we reduce first to the case when K is perverse irreducible, and then further to the case where L is perverse irreducible as well. If either K or L is punctual, i.e., a delta function, then $K*_!L$ is a translate of either L or K, so is perverse, and hence has $\mathcal{H}^i(K*_!L) = 0$ for $i > 0$. If not, then $K \cong \mathcal{F}[1]$ and $L \cong \mathcal{G}[1]$ for sheaves $\mathcal{F}$ and $\mathcal{G}$ on G. In this case, $\mathcal{H}^i(K*_!L) = R^{i+2}\mathrm{sum}_!(\mathrm{pr}_1{}^*\mathcal{F}\otimes\mathrm{pr}_2{}^*\mathcal{G})$ vanishes for $i \geq 1$ for dimension reasons: G has relative dimension one, hence $R^i\mathrm{sum}_!(\text{any sheaf}) = 0$ for $i \geq 3$. QED

Lemma 2.6.12 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. Let K be perverse on G. Then K has property $\mathcal{P}!$ if and only if $K*_!L$ is perverse for every perverse irreducible L on G whose isomorphism class is translation-invariant.

proof The "only if" is obvious. Suppose now that $K*_!L$ is perverse for every perverse irreducible L on G whose isomorphism class is translation-invariant. We wish to show that K has $\mathcal{P}!$. For this, it suffices (by 2.6.8) to show that $K*_!M$ is perverse for every perverse irreducible M on G. If the isomorphism class of M is translation invariant, we are given the perversity of $K*_!M$ by hypothesis. If the isomorphism class of M is not translation invariant, then by 2.6.10, M itself has $\mathcal{P}!$, and hence $K*_!M = M*_!K$ is perverse. QED

Lemma 2.6.13 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. Let L be a perverse irreducible on G whose isomorphism class is translation-invariant. If G is $\mathbb{G}_m$, then L is the shifted Kummer sheaf $\mathcal{L}_\chi[1]$ for some character χ of $\pi_1{}^{\mathrm{tame}}(\mathbb{G}_m)$. If G is $\mathbb{A}^1$, and char(k) = 0, then L is $\overline{\mathbb{Q}}_\ell[1]$, while if char(k) = p > 0, then L is the shifted Artin-Schreier sheaf $\mathcal{L}_{\psi(\alpha x)}[1]$ for some α in k.

proof Because L is translation invariant, it cannot be punctual, so it must be of the form $\mathcal{L}[1]$ for an irreducible middle extension sheaf $\mathcal{L}$ on G. By translation invariance, $\mathcal{L}$ must be lisse on all of G. Hence $\mathcal{L}$ is a lisse irreducible sheaf on G.

If char(k) = 0, then $\pi_1(\mathbb{G}_m) = \pi_1{}^{\mathrm{tame}}(\mathbb{G}_m)$ is abelian, so $\mathcal{L}$ is an $\mathcal{L}_\chi$, while $\pi_1(\mathbb{A}^1) = 0$, so $\mathcal{L}$ is $\overline{\mathbb{Q}}_\ell$.

If char(k) = p > 0, and G is $\mathbb{G}_m$, the translation invariance of $\mathcal{L}$

forces it to have $\mathrm{swan}_\infty(\mathcal{L}) = 0$ and $\mathrm{swan}_0(\mathcal{L}) = 0$ (cf. [Ver, 1.1], [Ka-GKM, 4.1.6]), hence to be tame. As $\pi_1{}^{\mathrm{tame}}(\mathbb{G}_m)$ is abelian, $\mathcal{L}$ is an $\mathcal{L}_\chi$, as required. Suppose now that G is $\mathbb{A}^1$. Then L is a perverse irreducible whose isomorphism class is translation invariant. Its Fourier transform $\mathrm{FT}_\psi(L)$ is a perverse irreducible, say M, on $\mathbb{A}^1$, whose isomorphism class is invariant under $M \mapsto M\otimes\mathcal{L}_{\psi(\beta x)}$ for all β in k. Look at the $I(\infty)$-representation $M(\infty)$ of such an M. From the fact that the $\mathcal{L}_{\psi(\beta x)}$ give (one for each value of β in k) an infinity of distinct characters of $I(\infty)$, we see (cf. the appendix 2.18 to this chapter) that $M(\infty) = 0$. Therefore M is punctual. Being perverse irreducible, M must be a single delta function δ_α for some α in k, whence L is $\mathcal{L}_{\psi(\alpha x)}[1]$, as required. QED

Lemma 2.6.14 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. Let K be perverse on G. Then K has property $\mathcal{P}!$ if and only if $\mathrm{Hom}_G(K, L) = 0$ for every perverse irreducible L on G whose isomorphism class is translation-invariant.

proof We have shown above that K has $\mathcal{P}!$ if and only if $K*_!L$ is semiperverse for every perverse irreducible L on G whose isomorphism class is translation-invariant. Fix one such L. We know by 2.6.11 that $\mathcal{H}^i(K*_!L) = 0$ for $i > 0$. Thus we must show that $\mathcal{H}^0(K*_!L)$ is punctual if and only if $\mathrm{Hom}_G(K, L) = 0$. Because the isomorphism class of L is translation invariant, at any point g in G(k), the stalk of $\mathcal{H}^0(K*_!L)$ is the same cohomology group $H_c{}^0(G, \mathrm{inv}^*L\otimes K)$. Thus

$\mathcal{H}^0(K*_!L)$ is punctual $\Leftrightarrow$ $\mathcal{H}^0(K*_!L) = 0$ $\Leftrightarrow$ $H_c{}^0(G, \mathrm{inv}^*L\otimes K) = 0$.

By 2.6.13, $L = \mathcal{L}[1]$ with $\mathcal{L}$ lisse of rank one on G, and $\mathcal{L}^\vee \cong \mathrm{inv}^*\mathcal{L}$. Thus, ignoring Tate twists we find

$$H_c{}^0(G, \mathrm{inv}^*L\otimes K) = H_c{}^1(G, \mathrm{inv}^*\mathcal{L}\otimes K) \text{ is dual } (\mathcal{L} \text{ being lisse}) \text{ to}$$
$$H^{-1}(G, \mathrm{inv}^*\mathcal{L}^\vee\otimes D(K)) = H^0(G, \mathrm{inv}^*\mathcal{L}^\vee[-1]\otimes D(K))$$
$$= \mathrm{Hom}(\overline{\mathbb{Q}}_\ell, \mathrm{inv}^*\mathcal{L}^\vee[-1]\otimes D(K))$$
$$= \mathrm{Hom}(\mathrm{inv}^*\mathcal{L}[1], D(K)) \quad (\text{because } \mathcal{L} \text{ is lisse of rank one})$$
$$= \mathrm{Hom}(D(L), D(K)) \quad (\text{because } \mathrm{inv}^*\mathcal{L} \cong \mathcal{L}^\vee)$$
$$\cong \mathrm{Hom}(K, L).$$

Thus we find that $\mathcal{H}^0(K*_!L)$ is punctual if and only if Hom(K, L) = 0. QED

Corollary 2.6.15 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. Let K be perverse on G. Then K has property $\mathcal{P}*$ if and only if $\mathrm{Hom}_G(L, K) = 0$ for every perverse irreducible L on G whose isomorphism class is translation-invariant.
proof Indeed, K has $\mathcal{P}*$ if and only if DK has $\mathcal{P}!$, if and only if Hom(DK, L) = 0 for every perverse irreducible L on G whose isomorphism class is translation-invariant, if and only if Hom(DL, K) = 0 for every such L. But the class of such L is stable by duality. QED

Corollary 2.6.16 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. Let K be perverse on G. If K has $\mathcal{P}!$, then every quotient of K (as perverse sheaf) has $\mathcal{P}!$. If K has $\mathcal{P}*$, then every subobject of K (as perverse sheaf) has $\mathcal{P}*$.
proof Obvious from the Hom(K, L) = 0 and Hom(L, K) = 0 criteria 2.6.14 and 2.6.15 for K to have $\mathcal{P}!$ and $\mathcal{P}*$ respectively. QED

Corollary 2.6.16.1 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. Suppose that

$$0 \to K_1 \to K \to K_2 \to 0$$

is a short exact sequence of perverse sheaves on G. If both K_1 and K_2 have $\mathcal{P}!$ (respectively $\mathcal{P}*$, respectively $\mathcal{P}$), then so does K.
proof Again obvious from the Hom(K, L) = 0 and Hom(L, K) = 0 criteria for K to have $\mathcal{P}!$ and $\mathcal{P}*$ respectively. QED

Corollary 2.6.16.2 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. If K is perverse on G and has $\mathcal{P}!$ (respectively $\mathcal{P}*$) then any perverse irreducible quotient (respectively subobject) M of K as perverse sheaf has $\mathcal{P}$.
proof Indeed by 2.6.16, M has $\mathcal{P}!$ (respectively $\mathcal{P}*$).But M is itself perverse irreducible, hence by (2.6.14) (respectively by (2.6.15), its isomorphism class cannot be translation invariant. So by (2.6.10), M has $\mathcal{P}$. QED

Corollary 2.6.16.3 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. If K is perverse on G and has $\mathcal{P}$, then every perverse irreducible quotient and every perverse irreducible sub-object of K as perverse sheaf has $\mathcal{P}$.
proof Immediate from 2.6.16.2. QED

Remark 2.6.16.4 It is **not** necessarily the case that if K is perverse and has $\mathcal{P}$, then every subobject and every quotient of K as perverse sheaf again has $\mathcal{P}$. To give examples, we work over $\mathbb{C}$ (see 2.10.4 for examples in characteristic p > 0), and use Riemann-Hilbert to work with holonomic RS $\mathcal{D}$-modules instead of perverse sheaves.

On $\mathbb{G}_{a,\mathbb{C}}$:= Spec($\mathbb{C}$[x]), we write $\mathcal{D}$ for $\mathbb{C}[x, \delta]$, where $\delta := d/dx$. Consider the left $\mathcal{D}$-module M := $\mathcal{D}/\mathcal{D}x\delta x$. Because $x\delta x$ is minus its own adjoint, M is self dual. The only irreducible holonomic $\mathcal{D}$-modules whose isomorphism classes are translation invariant are $\mathcal{D}/\mathcal{D}(\delta-\alpha)$, for α in $\mathbb{C}$. Only for $\alpha=0$ is $\mathcal{D}/\mathcal{D}(\delta-\alpha)$ RS. Therefore M has $\mathcal{P}$ if and only if

$$\mathrm{Hom}_{\text{left-}\mathcal{D}\text{-modules}}(M, \mathcal{D}/\mathcal{D}\delta) = 0.$$

We will show that for any α in $\mathbb{C}$, we have

$$\mathrm{Hom}_{\text{left-}\mathcal{D}\text{-modules}}(M, \mathcal{D}/\mathcal{D}(\delta-\alpha)) = 0.$$

This Hom is, via the image of the class of 1 in $\mathcal{D}/\mathcal{D}x\delta x$, just the set of elements in $\mathcal{D}/\mathcal{D}(\delta-\alpha)$ annihilated by $x\delta x$. But $\mathcal{D}/\mathcal{D}(\delta-\alpha) \cong e^{\alpha x}\mathbb{C}[x]$, via $1 \mapsto e^{\alpha x}$, and $e^{\alpha x}\mathbb{C}[x] \subset \mathbb{C}[[x]]$. But $x\delta x$ is obviously (look at lowest degree terms) injective on $\mathbb{C}[[x]]$, and hence M has $\mathcal{P}$. But M admits $\mathcal{D}/\mathcal{D}\delta x$ as quotient, which does not have $\mathcal{P}$, because $\mathcal{D}/\mathcal{D}\delta \subset \mathcal{D}/\mathcal{D}\delta x$ via Right(x) . Similarly, M admits, via Right(x), $\mathcal{D}/\mathcal{D}x\delta$ as a subobject, which again does not have $\mathcal{P}$, because $\mathcal{D}/\mathcal{D}x\delta$ admits $\mathcal{D}/\mathcal{D}\delta$ as a quotient.

On $\mathbb{G}_{m,\mathbb{C}}$:= Spec($\mathbb{C}[x,x^{-1}]$), we write $\mathcal{D}$ for $\mathbb{C}[x, x^{-1}, \delta]$, δ again given by $\delta := d/dx$. Consider the left $\mathcal{D}$-module N := $\mathcal{D}/\mathcal{D}(x-1)\delta(x-1)$. Because $(x-1)\delta(x-1)$ is minus its own adjoint, N is self dual. The irreducible holonomic $\mathcal{D}$-modules whose isomorphism classes are translation invariant are those of the form $\mathcal{D}/\mathcal{D}(x\delta-\alpha)$, for α in $\mathbb{C}$. Thus N has $\mathcal{P}$ if and only if, for every α in $\mathbb{C}$, we have

$$\mathrm{Hom}_{\text{left-}\mathcal{D}\text{-modules}}(N, \mathcal{D}/\mathcal{D}(x\delta-\alpha)) = 0.$$

Just as above, this Hom is the set of elements in $\mathcal{D}/\mathcal{D}(x\delta-\alpha) \cong x^{\alpha}\mathbb{C}[x, x^{-1}]$ which are annihilated by $(x-1)\delta(x-1)$. In terms of the parameter t:= x-1, $x^{\alpha}\mathbb{C}[x, x^{-1}] \subset \mathbb{C}[[t]]$, and $(x-1)\delta(x-1)$ becomes the operator $t(d/dt)t$, which, just as above, is injective on $\mathbb{C}[[t]]$. Therefore N has $\mathcal{P}$. But N admits $\mathcal{D}/\mathcal{D}\delta(x-1)$ as a quotient, which does not have $\mathcal{P}$; indeed, since x is invertible, we have $\mathcal{D}/\mathcal{D}x\delta = \mathcal{D}/\mathcal{D}\delta$, and $\mathcal{D}/\mathcal{D}\delta \subset \mathcal{D}/\mathcal{D}\delta(x-1)$ via Right$(x-1)$. Similarly, N admits, via Right$(x-1)$, $\mathcal{D}/\mathcal{D}(x-1)\delta$ as a subobject, which does not have $\mathcal{P}$, since it in turn admits $\mathcal{D}/\mathcal{D}\delta = \mathcal{D}/\mathcal{D}x\delta$ as a quotient.

(Key) Corollary 2.6.17 Let k be an algebraically closed field. Suppose G is a connected smooth affine k-groupscheme of finite type, of relative dimension one. Suppose that K and L are perverse on G, and each has $\mathcal{P}$. Then their middle convolution $K*_{mid}L$ has $\mathcal{P}$.
proof We have already noted that $K*_!L$ has $\mathcal{P}!$, and that $K*_*L$ has $\mathcal{P}*$, in both cases simply by the associativity of ! and * convolution. But $K*_{mid}L$ is a quotient of $K*_!L$, and hence has $\mathcal{P}!$. Similarly, $K*_{mid}L$ is a subobject of $K*_*L$, and hence has $\mathcal{P}*$. QED

(2.6.18) By the above corollary, when G is either $\mathbb{G}_m$ or $\mathbb{A}^1$, the full subcategory $(\mathcal{P})$ of the category Perv of all perverse sheaves on G consisting of those with property $\mathcal{P}$ is stable by middle convolution $*_{mid}$. This category for $\mathbb{A}^1$, with its middle convolution, will be the essential player in all that follows.

2.7 Interlude: middle direct images (relative dimension one)

(2.7.1) Here is the general setup for "middle direct image".

Proposition 2.7.2 Over an arbitrary base S which is itself separated and of finite type over a field k of characteristic $p \neq \ell$, consider a diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "f"'] & X \arrow[d, "\overline{f}"] & D := X - U \arrow[l, "i"'] \arrow[dl, "\overline{f}|D"] \\ & S & \end{tikzcd}$$

in which $j: U \to X$ is an affine open immersion, $\overline{f}$ is proper, and $f|D : D \to S$ is affine (hence finite, since it is also proper). Suppose K in $D^b{}_c(U, \overline{\mathbb{Q}}_\ell)$ is perverse, and that both Rf_*K and $Rf_!K$ are perverse on S. Then $R\overline{f}_*j_{!*}K$ is perverse on S, and

$$R\overline{f}_*j_{!*}K = \mathrm{Image}(\ Rf_!K \to Rf_*K\ )$$

in Perv(S).

proof On X, we have two short exact sequences of perverse sheaves:

$$0 \to \mathrm{Ker} \to j_!K \to j_{!*}K \to 0,$$
$$0 \to j_{!*}K \to Rj_*K \to \mathrm{Coker} \to 0.$$

Notice that the objects Ker and Coker are supported on D, hence are perverse on D.

In the derived category, these are distinguished triangles, so applying the exact functor $R\overline{f}_*$ gives two distinguished triangles on S,

$$\to R\overline{f}_*\mathrm{Ker} \to Rf_!K \to R\overline{f}_*j_{!*}K \to ,$$
$$\to R\overline{f}_*j_{!*}K \to Rf_*K \to R\overline{f}_*\mathrm{Coker} \to .$$

The objects $R\overline{f}_*\mathrm{Ker}$ and $R\overline{f}_*\mathrm{Coker}$ are perverse on S, because they are both of the form $R(\overline{f}|D)_*$(perverse on D), and $\overline{f}|D$, being finite, preserves perversity.

We first show that $R\overline{f}_*j_{!*}K$ is perverse. The first distinguished triangle above shows that $R\overline{f}_*j_{!*}K$ is semiperverse: its $\mathcal{H}^{-i}$ is caught between the $\mathcal{H}^{-i}$ of one perverse sheaf and the $\mathcal{H}^{-(i-1)}$ of another, so certainly has dimension of support $\le i$. The hypothesis that both Rf_*K and $Rf_!K$ are perverse gives, by duality, that also both Rf_*DK and $Rf_!DK$ are perverse. So repeating the argument we find that $R\overline{f}_*j_{!*}DK = D(R\overline{f}_*j_{!*}K)$ is also semiperverse, whence $R\overline{f}_*j_{!*}K$ is perverse.

Once we know that $R\overline{f}_*j_{!*}K$ is perverse, the two distinguished triangles above are actually short exact sequences in Perv(S)

$$0 \to R\overline{f}_*\mathrm{Ker} \to Rf_!K \to R\overline{f}_*j_{!*}K \to 0,$$
$$0 \to R\overline{f}_*j_{!*}K \to Rf_*K \to R\overline{f}_*\mathrm{Coker} \to 0,$$

which together show that $R\overline{f}_*j_{!*}K = \mathrm{Image}(\ Rf_!K \to Rf_*K\ )$. QED

2.8 Middle additive convolution via middle direct image

(2.8.1) We return to $\mathbb{A}^1$ over an algebraically closed field k of characteristic $\neq \ell$. We will apply the idea of "middle direct image" to computing the middle additive convolution with a perverse object L on $\mathbb{A}^1$ which has property $\mathcal{P}$.

(2.8.2) Given a perverse sheaf K on $\mathbb{A}^1$, we have the perverse sheaf $K_x \otimes L_{t-x}$ on $\mathbb{A}^2$, with coordinates x,t. In these coordinates, the original "sum" map is the projection onto the t-line. By the very definition of additive convolution, we have

$$K*_{!+}L = R(pr_2)_!(K_x\otimes L_{t-x}),$$
$$K*_{*+}L = R(pr_2)_*(K_x\otimes L_{t-x}).$$

Since L lies in ($\mathcal{P}$), both of these objects are perverse.

(2.8.3) We compactify the map pr_2 by compactifying the affine x-line into the projective x-line, all over the $\mathbb{A}^1$ of t's:

$$j: \mathbb{A}^1{}_x\times\mathbb{A}^1{}_t \hookrightarrow \mathbb{P}^1{}_x\times\mathbb{A}^1{}_t.$$

Then we are in the set-up to which the above middle direct image proposition 2.7.2 applies. Thus we find

Proposition 2.8.4 Hypotheses and notations as in 2.8.1, for K perverse on $\mathbb{A}^1$, and L perverse on $\mathbb{A}^1$ with $\mathcal{P}$, the middle additive convolution $K*_{mid+}L$ is

$$R(pr_2)_*(\text{the perverse sheaf } j_{!*}(K_x\otimes L_{t-x}) \text{ on } \mathbb{P}^1\times\mathbb{A}^1).$$

Using this, we get a fibre-by-fibre recipe for $K*_{mid+}L$ over a dense open set.

Corollary 2.8.5 Hypotheses as in proposition 2.8.4 above, suppose in addition that $K = \mathcal{F}[1]$ and $L = \mathcal{G}[1]$ for sheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{A}^1$. Then

1) there exists a dense open set $U \subset \mathbb{A}^1$ over which

a) the formation of $Rj_*(\mathcal{F}_x\otimes\mathcal{G}_{t-x}) \mid \mathbb{P}^1\times U$ commutes with arbitrary change of base on U,

b) $Rj_*(\mathcal{F}_x\otimes\mathcal{G}_{t-x}) \mid \infty\times U$ is lisse on U, of formation compatible with arbitrary change of base on U.

2) Over $\mathbb{P}^1\times U$, we have

a) an isomorphism

$$j_{!*}(K_x\otimes L_{t-x}) \mid \mathbb{P}^1\times U = j_*(\mathcal{F}_x\otimes\mathcal{G}_{t-x})[2] \mid \mathbb{P}^1\times U,$$

b) a short exact sequence of perverse sheaves

$$0 \to j_*(\mathcal{F}_x\otimes\mathcal{G}_{t-x})[1] \mid \infty\times U \to j_!(\mathcal{F}_x\otimes\mathcal{G}_{t-x})[2] \mid \mathbb{P}^1\times U$$
$$\to j_*(\mathcal{F}_x\otimes\mathcal{G}_{t-x})[2] \mid \mathbb{P}^1\times U \to 0.$$

which, as a distinguished triangle, is of formation compatible with arbitrary change of base on U.

3) For E/k any separably closed extension field of k, and any point t in U(E), we have

$$(K*_{mid}L)_t = R\Gamma(\mathbb{P}^1\otimes_k E, (j: \mathbb{A}^1 \to \mathbb{P}^1)_*(\text{the sheaf } x \mapsto \mathcal{F}_x\otimes\mathcal{G}_{t-x}))[2],$$

and a "short exact sequence" (really a distinguished triangle)

$$0 \to (\text{the } I(\infty)\text{-invariants in the sheaf } x \mapsto \mathcal{F}_x \otimes \mathcal{G}_{t-x})[1] \to$$
$$\to (K *_! L)_t \to (K *_{mid} L)_t \to 0$$

which is the fibre at t of a short exact sequence of perverse sheaves on U

$0 \to$ (the lisse sheaf $j_*(\mathcal{F}_x \otimes \mathcal{G}_{t-x}) \mid \infty \times U)[1] \to K *_{!+} L \to K *_{mid+} L \to 0$.

4) If both $\mathcal{F}$ and $\mathcal{G}$ are tame at ∞, then we may take U to be $\mathbb{A}^1$, and the sheaf $j_*(\mathcal{F}_x \otimes \mathcal{G}_{t-x}) \mid \infty \times \mathbb{A}^1$ is lisse, of formation compatible with arbitrary change of base on $\mathbb{A}^1$.

5) If char(k) = 0, then the sheaf $j_*(\mathcal{F}_x \otimes \mathcal{G}_{t-x}) \mid \infty \times \mathbb{A}^1$ is geometrically constant, of formation compatible with arbitrary change of base on $\mathbb{A}^1$.

proof 1) The base-changing statements over a dense open U_1 are a special case of Deligne's generic base change theorem [De-Th.Fin, Cor. 2.9]. The lisseness in 1b) results from the constructibility of $Rj_*(\mathcal{F}_x \otimes \mathcal{G}_{t-x})$, but may require passing to a smaller dense open U in U_1. Once we have 1), 2a) follows from the "successive partial truncation" description [BBD, 2.2.4], and 2b) follows from 1). Once we have 2), 3) results from (proper base change and) the fact that, by 1), the formation of $j_*(\mathcal{F}_x \otimes \mathcal{G}_{t-x}) \mid \mathbb{P}^1 \times U$ commutes with arbitrary change of base on U. To prove 4), notice that the sheaf $(\mathcal{F}_x \otimes \mathcal{G}_{t-x})$ on $\mathbb{P}^1 \times \mathbb{A}^1 - \infty \times \mathbb{A}^1$ is lisse in a Zariski open neighborhood of $\infty \times \mathbb{A}^1$: indeed, if f(x) and g(x) are monic polynomials such that $\mathcal{F}$ (resp. $\mathcal{G}$) is lisse where f (resp. g) is invertible, then $\mathcal{F}_x \otimes \mathcal{G}_{t-x}$ is lisse where the (±)monic-in-x polynomial f(x)g(t-x) is invertible. Moreover, this sheaf is fibre-by-fibre tamely ramified along $\infty \times \mathbb{A}^1$. The result 4) then follows from the relative Abhyankar lemma, cf. [Ka-SE, 4.7.2]. If char(k) = 0, then $\mathcal{F}$ and $\mathcal{G}$ are automatically tame at ∞, so by 4) the sheaf $j_*(\mathcal{F}_x \otimes \mathcal{G}_{t-x}) \mid \infty \times \mathbb{A}^1$ is lisse; because we are in characteristic zero, any lisse sheaf on $\mathbb{A}^1$ is geometrically constant. QED

2.9 Middle additive convolution with Kummer sheaves

(2.9.1) We continue to work over an algebraically closed field of characteristic $\neq \ell$. We now apply 2.8.4 to the special case of computing the middle additive convolution with $j_!\mathcal{L}_\chi[1] = j_*\mathcal{L}_\chi[1]$ on $\mathbb{A}^1$, for $\mathcal{L}_\chi$ a **nontrivial** Kummer sheaf on $\mathbb{G}_m$, and $j : \mathbb{G}_m \to \mathbb{A}^1$ the inclusion. We know that $j_*\mathcal{L}_\chi[1]$ on $\mathbb{A}^1$ is perverse irreducible, and its isomorphism class is not translation invariant (it has a unique point where it is not lisse). So $j_*\mathcal{L}_\chi[1]$ on $\mathbb{A}^1$ has property $\mathcal{P}$.

Exactly as in 2.8.4 we find

Proposition 2.9.2 Hypotheses and notations as in 2.9.1, for K perverse on $\mathbb{A}^1$, and any nontrivial Kummer sheaf $\mathcal{L}_\chi$, the middle additive convolution $K*_{\mathrm{mid}+}j_*\mathcal{L}_\chi[1]$ is

$$R(pr_2)_*(\text{the perverse sheaf } j_{!*}(K_x\otimes\mathcal{L}_{\chi(t-x)}[1]) \text{ on } \mathbb{P}^1\times\mathbb{A}^1).$$

(2.9.3) In order to go further, we need to see what the perverse sheaf $j_{!*}(K_x\otimes\mathcal{L}_{\chi(t-x)}[1])$ on $\mathbb{P}^1\times\mathbb{A}^1$ looks like. To lighten the notational burden in the following lemma, we will sometimes write $\mathcal{L}_\chi$ on $\mathbb{A}^1$ to mean $j_*\mathcal{L}_\chi$ on $\mathbb{A}^1$.

Lemma 2.9.4 Hypotheses and notations as in 2.9.2 above:

1) For any t_0 in $\mathbb{A}^1(k)$, the restriction of $j_{!*}(K_x\otimes\mathcal{L}_{\chi(t-x)}[1])$ to $\mathbb{P}^1 \times \{t_0\}$ is $(j_{t_0}: \mathbb{A}^1{}_x \times \{t_0\} \to \mathbb{P}^1{}_x \times \{t_0\})_{!*}(K_x\otimes\mathcal{L}_{\chi(t_0-x)})[1]$.

2) Denoting by $i: \infty\times\mathbb{A}^1 \to \mathbb{P}^1\times\mathbb{A}^1$ the inclusion, the perverse sheaf $j_{!*}(K_x\otimes\mathcal{L}_{\chi(t-x)}[1])$ on $\mathbb{P}^1\times\mathbb{A}^1$ sits in a short exact sequence of perverse sheaves on $\mathbb{P}^1\times\mathbb{A}^1$

$$0 \to i_*(\text{the constant sheaf } (\mathcal{H}^{-1}(K)\otimes\mathcal{L}_\chi)^{I(\infty)} \text{ on } \infty\times\mathbb{A}^1{}_t)[1] \to$$
$$\to j_!(K_x\otimes\mathcal{L}_{\chi(t-x)}[1]) \to j_{!*}(K_x\otimes\mathcal{L}_{\chi(t-x)}[1]) \to 0.$$

3) The middle and ! additive convolutions of K with $j_*\mathcal{L}_\chi[1]$ sit in an exact sequence of perverse sheaves on $\mathbb{A}^1$

$$0 \to (\text{the constant sheaf } (\mathcal{H}^{-1}(K)\otimes\mathcal{L}_\chi)^{I(\infty)})\,[1] \text{ on } \mathbb{A}^1{}_t \to$$
$$\to K*_{+!}\mathcal{L}_\chi[1]\to K*_{\mathrm{mid}+}\mathcal{L}_\chi[1] \to 0.$$

proof An affine open neighborhood of $\infty\times\mathbb{A}^1$ is the $\mathbb{A}^2$ with coordinates z and t, z=1/x. In these coordinates, $\infty\times\mathbb{A}^1$ is defined by

$z=0$, and so the function $1-zt$ takes the value 1 along $\infty\times\mathbb{A}^1$. Let us denote by U the open in $\mathbb{A}^2$ where $1-tz$ is invertible. On this open set, we have the lisse, rank one sheaf $\mathcal{L}_{\chi(1-tz)}$. Thanks to the identity

$$t - x = (-1/z)(1 - zt), \text{ we have}$$

$$\mathcal{L}_{\chi(t-x)} = \mathcal{L}_{\chi(1-tz)}\otimes\mathcal{L}_{\chi(-1/z)}.$$

In terms of the inclusion

$$j: U - \infty\times\mathbb{A}^1 \to U,$$

we have

$K_x\otimes\mathcal{L}_{\chi(t-x)}[1] \mid U - \infty\times\mathbb{A}^1 =$

$$(j^*\mathcal{L}_{\chi(1-tz)})\otimes(K_{1/z}\otimes\mathcal{L}_{\chi(-1/z)})[1] \text{ on } \mathbb{G}_{m,z}\times\mathbb{A}^1{}_t, \mid U - \infty\times\mathbb{A}^1.$$

The key point here is that $\mathcal{L}_{\chi(1-tz)}$ is lisse on all of U. So taking middle extension across $\infty\times\mathbb{A}^1$, which we may compute Zariski locally along $\infty\times\mathbb{A}^1$, we see that on U we will get

$\mathcal{L}_{\chi(1-tz)}\otimes$(M.E. of $(K_{1/z}\otimes\mathcal{L}_{\chi(-1/z)})[1]$ on $\mathbb{G}_{m,z}\times\mathbb{A}^1{}_t$, across $z=0$).

Since formation of middle extension is compatible with external products (this being true separately for duality, for $j_!$ and (so) for its dual Rj_*), we see that, on U, our middle extension is

$\mathcal{L}_{\chi(1-tz)}\otimes$the restriction to U of the external tensor product sheaf

$$(\text{M.E. of } (K_{1/z}\otimes\mathcal{L}_{\chi(-1/z)})|\mathbb{G}_{m,z} \text{ across } z=0)\otimes(\overline{\mathbb{Q}}_\ell[1] \text{ on } \mathbb{A}^1{}_t).$$

on the $\mathbb{A}^2$ with coordinates z,t. In particular, this shows that 1) holds. If we go back to the original x,t coordinates, we find that the middle extension across $\infty\times\mathbb{A}^1$ is, along $\infty\times\mathbb{A}^1$, the constant perverse sheaf[1] on $\mathbb{A}^1{}_t$ which is

$$(K\otimes\mathcal{L}_{\chi})^{I(\infty)}[1] = (\mathcal{H}^{-1}(K)\otimes\mathcal{L}_{\chi})^{I(\infty)}[2].$$

So on $\mathbb{P}^1{}_x\times\mathbb{A}^1{}_t$, we have, denoting by $i: \infty\times\mathbb{A}^1 \to \mathbb{P}^1\times\mathbb{A}^1$ the inclusion, the short exact sequence of perverse sheaves

$0 \to i_*(\text{the constant sheaf } (\mathcal{H}^{-1}(K)\otimes\mathcal{L}_{\chi})^{I(\infty)} \text{ on } \infty\times\mathbb{A}^1{}_t)[1] \to$

$$\to j_!(K_x\otimes\mathcal{L}_{\chi(t-x)}[1]) \to j_{!*}(K_x\otimes\mathcal{L}_{\chi(t-x)}[1]) \to 0$$

asserted in 2). Taking the total direct image onto the t line, we get the exact sequence

$0 \to(\text{the constant sheaf } (\mathcal{H}^{-1}(K)\otimes\mathcal{L}_{\chi})^{I(\infty)})\ [1] \text{ on } \mathbb{A}^1{}_t \to$

$$\to K*_{+!}\mathcal{L}_{\chi}[1]\to K*_{+\mathrm{mid}}\mathcal{L}_{\chi}[1] \to 0$$

of perverse sheaves on $\mathbb{A}^1$ asserted in 3). QED

Corollary 2.9.5 For K perverse on $\mathbb{A}^1$, and any nontrivial Kummer sheaf $\mathcal{L}_\chi$, the stalk at any point t in $\mathbb{A}^1(k)$ of $K*_{mid+}j_*\mathcal{L}_\chi[1]$ is $R\Gamma(\mathbb{P}^1, j_{!*}(\text{the perverse sheaf } x \mapsto K_x\otimes\mathcal{L}_{\chi(t-x)} \text{ on } \mathbb{A}^1)[1])$.

proof This results from the preceding Proposition 2.9.2, via proper base change and part 1) of the above lemma 2.9.4. QED

Proposition 2.9.6 Let k be an algebraically closed field k. Given any two nontrivial Kummer sheaves $\mathcal{L}_\rho$ and $\mathcal{L}_\chi$ on $\mathbb{G}_{m,k}$, their middle additive convolution $j_*\mathcal{L}_\rho[1]*_{mid+}j_*\mathcal{L}_\chi[1]$ on $\mathbb{A}^1$ is given (geometrically)

$$j_*\mathcal{L}_\rho[1]*_{mid+}j_*\mathcal{L}_\chi[1] \cong j_*\mathcal{L}_{\rho\chi}[1] \text{ if } \rho\chi \neq \mathbb{1},$$
$$\cong \delta_0 \text{ if } \chi\rho = \mathbb{1}.$$

proof Suppose first that $\chi\rho \neq \mathbb{1}$. Then by part 3) of the lemma 2.9.4 above, applied with $K := j_*\mathcal{L}_\rho[1]$, we see that

$$(j_*\mathcal{L}_\rho[1])*_{!+}(j_*\mathcal{L}_\chi[1]) \cong j_*\mathcal{L}_\rho[1]*_{mid+}j_*\mathcal{L}_\chi[1],$$

because $(\mathcal{H}^{-1}(K)\otimes\mathcal{L}_\chi)^{I(\infty)} = (\mathcal{L}_{\rho\chi})^{I(\infty)} = 0$. Thus our middle convolution is given by

$$R(pr_2)_!(\text{the perverse sheaf } \mathcal{L}_{\rho(x)}\otimes\mathcal{L}_{\chi(t-x)}[2]) \text{ on } \mathbb{A}^1\times\mathbb{A}^1).$$

Over t=0, the (geometric) stalk is $R\Gamma_c(\mathbb{G}_m, \mathcal{L}_{\rho\chi}[2]) = 0$. Therefore our middle convolution on $\mathbb{A}^1$ is the extension by zero of its restriction to $\mathbb{G}_m$. So it suffices to show that over $\mathbb{G}_m$, our middle convolution is (geometrically) isomorphic to $\mathcal{L}_{\rho\chi}[1]$.

Over the open set $\mathbb{G}_m$ where t is invertible, we make the change of variable

$$(x, t) \mapsto (tx, t),$$

and we find

$$j_*\mathcal{L}_\rho[1]*_{mid+}j_*\mathcal{L}_\chi[1] \mid \mathbb{G}_m \cong$$
$$\cong R(pr_2)_!(\mathcal{L}_{\rho(tx)}\otimes\mathcal{L}_{\chi(t-tx)}[2]) \text{ on } \mathbb{A}^1\times\mathbb{G}_m).$$
$$= R(pr_2)_!(\mathcal{L}_{\rho(x)}\otimes\mathcal{L}_{\chi(1-x)}[1]\otimes\mathcal{L}_{\rho\chi(t)}[1]) \text{ on } \mathbb{A}^1\times\mathbb{G}_m)$$
$$= \mathcal{L}_{\rho\chi}[1]\otimes R\Gamma_c(\mathbb{A}^1 - \{1,0\}, \mathcal{L}_{\rho(x)}\otimes\mathcal{L}_{\chi(1-x)}[1]).$$

Because $\rho\chi \neq \mathbb{1}$, we have $H^2_c(\mathbb{A}^1 - \{1,0\}, \mathcal{L}_{\rho(x)}\otimes\mathcal{L}_{\chi(1-x)}) = 0$, by considering the local monodromy at ∞ of $\mathcal{L}_{\rho(x)}\otimes\mathcal{L}_{\chi(1-x)}$; because $\mathcal{L}_{\rho(x)}\otimes\mathcal{L}_{\chi(1-x)}$ is tame, and lisse of rank one, we have

$$\chi_c(\mathbb{A}^1 - \{1,0\}, \mathcal{L}_{\rho(x)}\otimes\mathcal{L}_{\chi(1-x)}) = -1,$$

and the only possibly nonvanishing groups are the H^i_c for $i=1,2$. Thus $R\Gamma_c(\mathbb{A}^1 - \{1,0\}, \mathcal{L}_{\rho(x)}\otimes\mathcal{L}_{\chi(1-x)}[1])$ is a one-dimensional $\overline{\mathbb{Q}}_\ell$-vector space, placed in degree zero, as required.

Suppose now that $\chi\rho = \mathbb{1}$. In this case, the corollary 2.9.5 above will give the required assertion. For the stalk of our middle convolution at t in $\mathbb{A}^1(k)$ is given by

$R\Gamma(\mathbb{P}^1, j_{!*}(\text{the perverse sheaf } x \mapsto \mathcal{L}_{\chi(x)}\otimes\mathcal{L}_{\rho(t-x)}[1] \text{ on } \mathbb{A}^1)[1])$.

By definition, the sheaf $\mathcal{L}_{\chi(x)}\otimes\mathcal{L}_{\rho(t-x)}[1]$ on $\mathbb{A}^1$ is lisse outside of $\{t,0\}$, and is extended by zero across $\{t,0\}$. So if we denote by

$$k_{t,0} : \mathbb{A}^1 - \{t,0\} \to \mathbb{A}^1$$

the inclusion, this stalk is

$$R\Gamma(\mathbb{P}^1, j_{!*}(k_{t,0})_!(\mathcal{L}_{\chi(x)}\otimes\mathcal{L}_{\rho(t-x)}[1] \text{ on } \mathbb{A}^1_x - \{t,0\})[1]).$$

Since $\chi\rho = \mathbb{1}$, this stalk is

$$R\Gamma(\mathbb{P}^1, j_{!*}(k_{t,0})_!(\mathcal{L}_{\chi(x/(t-x))}[1] \text{ on } \mathbb{A}^1_x - \{t,0\})[1]).$$

For any t, this coefficient sheaf is lisse of rank one at ∞, so it is lisse of rank one on $\mathbb{P}^1_x - \{t,0\})$ and everywhere tame.

For $t \neq 0$, the sheaf $\mathcal{L}_{\chi(x/(t-x))}$ is not geometrically constant (it has nontrivial monodromy at both 0 and t), and has no nonzero punctual sections, so

$$H^i(\mathbb{P}^1, j_{!*}(k_{t,0})_!(\mathcal{L}_{\chi(x/(t-x))}[1] \text{ on } \mathbb{A}^1_x - \{t,0\})[1]) = 0 \text{ if } i=0, -2.$$

But the Euler characteristic also vanishes, so we find

$$R\Gamma(\mathbb{P}^1, j_{!*}(k_{t,0})_!(\mathcal{L}_{\chi(x/(t-x))}[1] \text{ on } \mathbb{A}^1_x - \{t,0\})[1]) = 0 \text{ for } t \neq 0,$$

and hence

$$j_*\mathcal{L}_\rho[1]*_{\text{mid}+}j_*\mathcal{L}_\chi[1] \text{ is supported at } t=0 \text{ if } \rho\chi = \mathbb{1}.$$

So it remains only to compute the stalk at zero. This stalk is

$$R\Gamma(\mathbb{P}^1, j_{!*}(k_{0,0})_!(\mathcal{L}_{\chi(x/(-x))}[1] \text{ on } \mathbb{A}^1_x - \{0\})[1]).$$

But the perverse sheaf $j_{!*}(k_{0,0})_!(\mathcal{L}_{\chi(x/(-x))}[1])$ is (geometrically) just the constant sheaf $\overline{\mathbb{Q}}_\ell[1]$ on $\mathbb{P}^1 - \{0\}$, extended by zero.

So our stalk at zero is

$$R\Gamma_c(\mathbb{P}^1 - \{0\}, \overline{\mathbb{Q}}_\ell[2]) = R\Gamma_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell[2]),$$

the one-dimensional vector space $H^2{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ placed in degree zero. QED

Theorem 2.9.7 Let k be an algebraically closed field, $\mathcal{L}_\chi$ a nontrivial Kummer sheaf on $\mathbb{G}_m$, j: $\mathbb{G}_m \to \mathbb{A}^1$ the inclusion.
1)On $\mathbb{A}^1$, the operators $K \mapsto K*_{mid+}j_*\mathcal{L}_\chi[1]$ and $K \mapsto K*_{mid+}j_*\mathcal{L}_{\overline{\chi}}[1]$ on $\mathcal{P}$ are automorphisms of $\mathcal{P}$, which are inverses of each other (where we write $\overline{\chi}$ for the character χ^{-1}).
2) On $\mathbb{A}^1$, the operator $K \mapsto K*_{mid+}j_*\mathcal{L}_\chi[1]$ induces an automorphism of the perverse irreducible objects in $\mathcal{P}$, with inverse the operator $K \mapsto K*_{mid+}j_*\mathcal{L}_{\overline{\chi}}[1]$.

proof We know that $j_*\mathcal{L}_\chi[1]$ is perverse irreducible on $\mathbb{A}^1$, and that it lies in $\mathcal{P}$ on $\mathbb{A}^1$ (because it is not translation invariant). In view of the preceding result 2.9.6, 2.9.7 is a special case of the following result.

Theorem 2.9.8 On $\mathbb{A}^1$ over an algebraically closed field k, let K and L be perverse objects in $\mathcal{P}$ with $K*_{mid+}L = \delta_0 = L*_{mid+}K$.
1)On $\mathbb{A}^1$, the operators $X \mapsto X*_{mid+}K$ and $X \mapsto X*_{mid+}L$ on $\mathcal{P}$ are automorphisms of $\mathcal{P}$, which are inverses of each other.
2) The operator $X \mapsto X*_{mid+}K$ induces an automorphism of the perverse irreducible objects in $\mathcal{P}$, with inverse the operator $X \mapsto X*_{mid+}L$.
proof For 1), just use the fact that middle convolution is associative, and that δ_0 is the identity for middle convolution.
For 2), Let X be perverse irreducible in $\mathcal{P}$. We must show that the perverse sheaf $X*_{mid+}K$ is perverse irreducible. By 1), it is nonzero, so it contains a subobject Y which is perverse irreducible. Since $X*_{mid+}K$ lies in $\mathcal{P}$, Y itself lies in $\mathcal{P}$ (by 2.6.16.3). Since $W \mapsto W*_{mid+}L$ is end-exact, it preserves injections, so $Y*_{mid+}L$ is a subobject of $X*_{mid+}K*_{mid+}L = X$. This subobject is nonzero by 1), so by the irreducibility of X, we have $X = Y*_{mid+}L$. Applying $W \mapsto W*_{mid+}K$, we find $X*_{mid+}K = Y*_{mid+}L*_{mid+}K = Y$ is irreducible, as

required. QED

For later use, we record the following corollary.

Corollary 2.9.9 On $\mathbb{A}^1$ over an algebraically closed field k, let K and L be perverse objects in $\mathcal{P}$ with $K*_{mid+}L = \delta_0 = L*_{mid+}K$. Then L and K are perverse irreducible.

proof Indeed, by 2), the functors $X \mapsto X*_{mid+}K$ and $X \mapsto X*_{mid+}L$ carry irreducibles in $\mathcal{P}$ to irreducibles in $\mathcal{P}$. Take X to be δ_0. QED

2.10 Interpretation of middle additive convolution via Fourier Transform

(2.10.1) In this section, we work on $\mathbb{A}^1$ over an algebraically closed field k of characteristic $p > 0$, $p \neq \ell$. Recall [Br, Cor. 9.6] that Fourier Transform interchanges ! convolution and tensor product: more precisely, for K and L in $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$, we have

$$FT_{\psi,!}(K*_!L) = FT_{\psi,!}(K) \otimes FT_{\psi,!}(L)[-1],$$
$$FT_{\psi,!}(K \otimes L)[-1] = FT_{\psi,!}(K)*_!FT_{\psi,!}(L)(-1).$$

Lemma 2.10.2 In the situation of 2.10.1, let K be perverse on $\mathbb{A}^1$. The following conditions are equivalent.

1) K has $\mathcal{P}$!.
2) K has no quotient $\mathcal{L}_{\psi(\alpha x)}[1]$ for any α in k.
3) FT(K) has no nonzero punctual quotient.
4) $\mathcal{H}^0(FT(K)) = 0$,
5) FT(K) is of the form $\mathcal{F}[1]$ for some sheaf $\mathcal{F}$ on $\mathbb{A}^1$ which has no nonzero punctual sections.

proof 1) $\Leftrightarrow$ 2) by 2.6.13 and 2.6.14.
2) $\Leftrightarrow$ 3) since FT carries $\mathcal{L}_{\psi(\alpha x)}[1]$ to $\delta_{-\alpha}$.
3) $\Rightarrow$ 4): For any perverse N on $\mathbb{A}^1$, the short exact sequence of perverse sheaves

$$0 \to \mathcal{H}^{-1}(N)[1] \to N \to \mathcal{H}^0(N) \to 0$$

expresses $\mathcal{H}^0(N)$ as a punctual quotient of N. Applying this to FT(K) gives $\mathcal{H}^0(FT(K)) = 0$.
4) $\Leftrightarrow$ 5) by the concrete description of perverse sheaves in dimension one.
5) $\Rightarrow$ 1): If FT(K) is $\mathcal{F}[1]$ for some sheaf $\mathcal{F}$ on $\mathbb{A}^1$, then

$$L \mapsto FT(K)[-1] \otimes L = \mathcal{F} \otimes L$$

preserves semiperversity. By Fourier inversion, $L \mapsto K*_!L$ preserves semiperversity. By Lemma 2.6.7, K has $\mathcal{P}$!. QED

Lemma 2.10.3 In the situation 2.10.1, for K perverse on $\mathbb{A}^1$, K has $\mathcal{P}$ if and only if FT(K) := N is a middle extension.
proof If K has $\mathcal{P}$, then both K and DK have $\mathcal{P}$!. By the above lemma, both N := $FT_{\psi,!}K$ and and DN = $FT_{\overline{\psi},!}(DK)$ have no nonzero punctual quotient. Thus (by duality) N has no nonzero punctual subobject. By the same lemma, we know N is $\mathcal{F}[1]$ for some sheaf $\mathcal{F}$ on $\mathbb{A}^1$ which has no nonzero punctual sections. This forces N to be a middle extension. For if N is $\mathcal{F}[1]$, with $\mathcal{F} \subset j_\star j^\star\mathcal{F}$ for $j: U \to \mathbb{A}^1$ an open where $\mathcal{F}$ is lisse, then if $\mathcal{F} \neq j_\star j^\star\mathcal{F}$, the short exact sequence of perverse sheaves

$$0 \to (j_\star j^\star\mathcal{F})/\mathcal{F} \to \mathcal{F}[1] = N \to j_\star j^\star\mathcal{F}[1] \to 0$$

exhibits a nonzero punctual subobject of N.

Conversely, suppose that N := $FT_{\psi,!}(K)$ is a middle extension. Then so is DN = $FT_{\overline{\psi},!}(DK)$. By the previous proposition, both K and DK have $\mathcal{P}$!, so K has $\mathcal{P}$. QED

Remarks 2.10.4

1) Here is an slightly variant proof of the above result. We know that a perverse K lies in $\mathcal{P}$ if and only if as a perverse sheaf it has no subobject and no quotient of the form $\mathcal{L}_\psi(\alpha x)[1]$ for any α in k. By Fourier Transform, this becomes the condition that FT(K) as perverse sheaf have no subobject and no quotient which is punctual. But this last condition is equivalent to being a middle extension (compare [Ka-ESDE, 2.9.1]), as one sees using the natural filtration (cf. 2.3.6) of the perverse sheaf FT(K) with associated graded (punctual, middle extension, punctual).

2) Using 2.10.3, we can give a characteristic $p > 0$ example (compare 2.6.16.4) of a perverse K on $\mathbb{A}^1$ which has $\mathcal{P}$, such that K admits a quotient as perverse sheaf which does not have $\mathcal{P}$. Thanks to 2.10.3, it is the same to give an example of a perverse N on $\mathbb{A}^1$ which is a middle extension, but which admits a quotient as perverse sheaf which is not a middle extension. On $\mathbb{G}_m$ over our algebraically closed field k of characteristic $p \neq \ell$, there exists a lisse rank two $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ which is a nontrivial extension of the constant sheaf $\overline{\mathbb{Q}}_\ell$ by itself, because $H^1(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell$. This sheaf is automatically tame, its global monodromy being unipotent, and hence its local monodromy

at zero is a single Jordan block, of dimension two. Denote by $j: \mathbb{G}_m \to \mathbb{A}^1$ the inclusion. Applying j_* to the tautological short exact sequence of sheaves on $\mathbb{G}_m$

$$0 \to \overline{\mathbb{Q}}_\ell \to \mathcal{F} \to \overline{\mathbb{Q}}_\ell \to 0,$$

we get an exact sequence of sheaves on $\mathbb{A}^1$,

$$0 \to \overline{\mathbb{Q}}_{\ell,\mathbb{A}^1} \to j_*\mathcal{F} \to \overline{\mathbb{Q}}_{\ell,\mathbb{A}^1}.$$

The last arrow is not surjective on the stalk at zero, since $j_*\mathcal{F}$ has only a one-dimensional stalk at zero, so we have a short exact sequence of sheaves on $\mathbb{A}^1$

$$0 \to \overline{\mathbb{Q}}_{\ell,\mathbb{A}^1} \to j_*\mathcal{F} \to j_!\overline{\mathbb{Q}}_{\ell,\mathbb{G}_m} \to 0.$$

Shifting by [1], we get a short exact sequence of perverse sheaves on $\mathbb{A}^1$ which exhibits $j_!\overline{\mathbb{Q}}_{\ell,\mathbb{G}_m}[1]$, which is visibly not a middle extension, as a quotient of the middle extension $(j_*\mathcal{F})[1] = j_{!*}(\mathcal{F}[1])$.

Proposition 2.10.5 Let K be perverse on $\mathbb{A}^1$, and suppose K has $\mathcal{P}$. Let $\mathcal{L}_\chi$ be a nontrivial Kummer sheaf on $\mathbb{G}_m$, $j: \mathbb{G}_m \to \mathbb{A}^1$ the inclusion. Write $FT(K) = \mathcal{F}[1]$, with $\mathcal{F}$ a middle extension sheaf on $\mathbb{A}^1$. Then

$$FT(K*_{mid+}j_*\mathcal{L}_\chi[1]) = j_*(j^*FT(K)\otimes\mathcal{L}_{\overline{\chi}}) = j_*(j^*\mathcal{F}\otimes\mathcal{L}_{\overline{\chi}})[1].$$

proof We have already seen in 2.9.4 that we have a short exact sequence of perverse sheaves on $\mathbb{A}^1$

$$0 \to (\text{constant sheaf})[1] \to K*_{!+}\mathcal{L}_\chi[1] \to K*_{mid+}\mathcal{L}_\chi[1] \to 0.$$

Under Fourier Transform, this gives a short exact sequence

$$0\to(\text{pct. sheaf, supp. at }0)\to FT(K*_!j_*\mathcal{L}_\chi[1])\to FT(K*_{mid+}j_*\mathcal{L}_\chi[1]) \to 0.$$

Using the general identity

$$FT(K*_!L) = FT(K)\otimes FT(L)[-1],$$

together with the standard geometric isomorphism

$$FT(j_*\mathcal{L}_\chi[1]) \cong j_*\mathcal{L}_{\overline{\chi}}[1],$$

we rewrite this short exact sequence as

$$0\to(\text{pct. sheaf, supp. at }0) \to FT(K)\otimes j_*\mathcal{L}_{\overline{\chi}} \to FT(K*_{mid+}j_*\mathcal{L}_\chi[1])\to 0.$$

Restricting to $\mathbb{G}_m$, the above exact sequence gives an isomorphism:

$$j^*FT(K)\otimes\mathcal{L}_{\overline{\chi}} \cong j^*FT(K*_{mid+}j_*\mathcal{L}_\chi[1]).$$

Since we know a priori that $FT(K*_{mid+}j_*\mathcal{L}_\chi[1])$ is a middle extension, being FT(an object in ($\mathcal{P}$)), we have

$$FT(K*_{mid+}j_*\mathcal{L}_\chi[1]) \cong j_*j^*FT(K*_{mid+}j_*\mathcal{L}_\chi[1])$$

$$= j_*(j^*FT(K)\otimes\mathcal{L}_{\overline{\chi}}), \text{ as required. QED}$$

Remark 2.10.6 This proposition gives us, in positive characteristic, a second proof of Proposition 2.9.6 (take $K = j_*\mathcal{L}_\rho[1]$).

(2.10.7) In fact, more is true. Under Fourier Transform, the operation $*_{mid+}$ corresponds to the obvious tensor product operation on middle extensions.

Theorem 2.10.8 In the situation 2.10.1, let K and L be perverse on $\mathbb{A}^1$, both K and L having $\mathcal{P}$. Pick a common open set $j\ U \to \mathbb{A}^1$ where both $N := FT(K)$ and $M := FT(L)$ are lisse, and write $N = j_*\mathcal{F}[1]$, $M = j_*\mathcal{G}[1]$, with $\mathcal{F}$ and $\mathcal{G}$ lisse sheaves on U. Then

$$FT(K*_{mid+}L) = j_*(\mathcal{F}\otimes\mathcal{G})[1].$$

proof. The key point is to show that there exists a dense open set $j: U \to \mathbb{A}^1$ such that when we apply FT to the "forget supports" map $K*_{!+}L \to K*_{*+}L$, the map we obtain,

$$FT(K*_{!+}L) \to FT(K*_{*+}L),$$

is an isomorphism on U:

$$j^*FT(K*_{!+}L) \cong j^*FT(K*_{*+}L).$$

If this is so, then from

$$FT(K*_{mid+}L) = \mathrm{Image}(FT(K*_{!+}L) \to FT(K*_{*+}L)),$$

we get

$$j^*FT(K*_{mid+}L) \cong j^*FT(K*_{!+}L) \cong j^*FT(K*_{*+}L)).$$

Since we know a priori that $FT(K*_{mid+}L)$ is a middle extension, we know that

$$FT(K*_{mid+}L) \cong j_*j^*FT(K*_{mid+}L) \cong j_*j^*FT(K*_{!}L).$$

Using $FT(K*_{!+}L) = FT(K)\otimes FT(L)[-1] = \mathcal{F}\otimes\mathcal{G}[1]$ gives the assertion.

It remains to prove that the map

$$FT(K*_{!+}L) \to FT(K*_{*+}L)$$

is an isomorphism on a dense open set of $\mathbb{A}^1$.

Lemma 2.10.9 In the situation 2.10.1, For any two objects K and L in $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$, the natural map

$$FT(\text{"forget supports"}) : FT(K*_{!+}L) \to FT(K*_{*+}L)$$

is an isomorphism on a dense open set.

proof The idea of the proof is to exploit the basic miracle of Fourier Transform, that $FT_{!,\psi} \cong FT_{*,\psi}$. By Deligne's generic base change theorem [De-Th.Fin, Cor. 2.9], there exists a dense open set U in $\mathbb{A}^1$ over which the formation of each of $FT_{*,\psi}(K)$, $FT_{*,\psi}(L)$, and $FT_{*,\psi}(K*_{*+}L)$ commutes with arbitrary change of base. By proper base change, the formation of each $FT_{!,\psi}(K)$, $FT_{!,\psi}(L)$, and $FT_{!,\psi}(K*_{!+}L)$ commutes with arbitrary change of base over all of $\mathbb{A}^1$.

Choose any point α in $\mathbb{A}^1(k)$. The stalk at α of $FT(K*_{!+}L)$, viewed as an $FT_{!,\psi}$, is

$$R\Gamma_c(\mathbb{A}^1,(K*_{!+}L)\otimes\mathcal{L}_{\psi(\alpha x)}[1]).$$

If α lies in U, the stalk at α of $FT(K*_{*+}L)$, viewed as an $FT_{*,\psi}$, is

$$R\Gamma(\mathbb{A}^1,(K*_{*+}L)\otimes\mathcal{L}_{\psi(\alpha x)}[1]),$$

and the map between them is induced by the "forget supports" map $K*_{!+}L \to K*_{*+}L$ on coefficients, followed by the "forget supports" map $R\Gamma_c \to R\Gamma$.

The source is

$$R\Gamma_c(\mathbb{A}^1,(K*_!L)\otimes\mathcal{L}_{\psi(\alpha x)})[1] =$$
$$= R\Gamma_c(\mathbb{A}^1,(R(\mathrm{sum})_!(pr_1{}^*(K)\otimes pr_2{}^*(L))\otimes\mathcal{L}_{\psi(\alpha x)})[1]$$
$$= R\Gamma_c(\mathbb{A}^1, R(\mathrm{sum})_!(pr_1{}^*(K\otimes\mathcal{L}_{\psi(\alpha x)})\otimes pr_2{}^*(L\otimes\mathcal{L}_{\psi(\alpha x)})))[1],$$

(by the projection formula, and the additivity of $\mathcal{L}_\psi$)

$$= R\Gamma_c(\mathbb{A}^2,pr_1{}^*(K\otimes\mathcal{L}_{\psi(\alpha x)})\otimes pr_2{}^*(L\otimes\mathcal{L}_{\psi(\alpha x)}))[1] \quad \text{(by Leray)}$$
$$= R\Gamma_c(\mathbb{A}^1, K\otimes\mathcal{L}_{\psi(\alpha x)})\otimes R\Gamma_c(\mathbb{A}^1, L\otimes\mathcal{L}_{\psi(\alpha x)})[1] \quad \text{(by Kunneth).}$$

In completely analogous fashion, the target, for α in U, is

$$R\Gamma(\mathbb{A}^1, K\otimes\mathcal{L}_{\psi(\alpha x)})\otimes R\Gamma(\mathbb{A}^1, L\otimes\mathcal{L}_{\psi(\alpha x)})[1],$$

and, with these identifications, the map between them is

("forget supports")$\otimes$("forget supports").

Because α lies in U, over which the formation of both $FT_{*,\psi}(K)$ and $FT_{*,\psi}(L)$ commutes with base change, these "forget supports" maps

$$R\Gamma_c(\mathbb{A}^1, K\otimes\mathcal{L}_{\psi(\alpha x)}) \to R\Gamma(\mathbb{A}^1, K\otimes\mathcal{L}_{\psi(\alpha x)}),$$
$$R\Gamma_c(\mathbb{A}^1, L\otimes\mathcal{L}_{\psi(\alpha x)}) \to R\Gamma(\mathbb{A}^1, L\otimes\mathcal{L}_{\psi(\alpha x)}),$$

are just the identity maps of $FT(K)_\alpha[-1]$ and of $FT(L)_\alpha[-1]$ respectively.

Thus the map $FT(K*_{!+}L) \to FT(K*_{*+}L)$ is an isomorphism over U, as required. QED

Corollary 2.10.10 In the situation 2.10.1, for any two objects K and L in $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$, the mapping cone object $[K*_{!+}L \to K*_{*+}L]$ in $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ is a direct sum of shifted $\mathcal{L}_{\psi(\alpha x)}$ sheaves, for various α in k.

Proof This is the Fourier Transform of the statement that the mapping cone $[FT(K*_{!+}L) \to FT(K*_{*+}L)]$ is punctual. QED

Remark 2.10.11 In terms of a relative compactification of pr_2, say

$$\begin{array}{ccccc} \mathbb{A}^1\times\mathbb{A}^1 & \xrightarrow{j} & \mathbb{P}^1\times\mathbb{A}^1 & \xleftarrow{i} & \infty\times\mathbb{A}^1 \\ & & \downarrow pr_2 & & \\ & & \mathbb{A}^1, & & \end{array}$$

the object $[K*_{!+}L \to K*_{*+}L]$ is $i^*Rj_*(K_x\otimes L_{t-x})$. So the above corollary 2.10.10 says that, over an algebraically closed field of characteristic $p > 0$, $i^*Rj_*(K_x\otimes L_{t-x})$ has all of its cohomology sheaves direct sums of $\mathcal{L}_{\psi(\alpha x)}$. This does not seem obvious a priori, and we do not know a "direct" proof of it.

(2.10.12) Here is a slight reformulation of the results of this section. Denote by $GalRep(\mathbb{A}^1/k, \overline{\mathbb{Q}}_\ell)$ the category of those continuous finite-dimensional $\overline{\mathbb{Q}}_\ell$-representations V of $Gal(k(x)^{sep}/k(x))$, the galois group of the function field of $\mathbb{A}^1/k$, with the following two properties:

1) V is definable over a finite extension of $\mathbb{Q}_\ell$,

2) V is unramified outside a finite set of places of k(x).

If we denote by η the generic point of $\mathbb{A}^1$, the functor

$$(\text{perverse middle extension sheaves on } \mathbb{A}^1) \to GalRep(\mathbb{A}^1/k, \overline{\mathbb{Q}}_\ell)$$
$$N = j_*\mathcal{F}[1] \mapsto \mathcal{F}_\eta = (N[-1])_\eta$$

is an equivalence of categories. [Indeed, for each nonempty open U in $\mathbb{A}^1$, this functor induces an equivalence between the full (by 2.3.3.1) subcategory of its source,

(perverse middle extensions on $\mathbb{A}^1$ which are lisse on U),
and the full subcategory of its target
(objects in GalRep($\mathbb{A}^1$/k, $\overline{\mathbb{Q}}_\ell$) which are unramified on U).]
Using this fact, 2.10.3 and 2.10.8 say that the composite functor

$$\text{(perverse sheaves on } \mathbb{A}^1 \text{ with } \mathcal{P}) \to \mathrm{GalRep}(\mathbb{A}^1/k, \overline{\mathbb{Q}}_\ell)$$
$$K \mapsto (\mathrm{FT}(K)[-1])_\eta$$

is an equivalence of categories, under which middle additive convolution goes over to tensor product, and under which the functor $K \mapsto D_-K := [x \mapsto -x]^*DK = D([x \mapsto -x]^*K)$ goes over into the functor $V \mapsto V^\vee :=$ the contragredient representation to V.

(2.10.13) Using this description, we can easily analyze the invertible objects. We say that a perverse sheaf K on $\mathbb{A}^1$ with $\mathcal{P}$ is invertible for the operation of middle convolution if there exists a perverse L with $\mathcal{P}$ such that $K*_{\mathrm{mid}+}L \cong \delta_0$. [If such an L exists, it is unique, by the associativity of middle convolution.]

Theorem 2.10.14 On $\mathbb{A}^1$ over an algebraically closed field of characteristic $p > 0$, a perverse sheaf K on $\mathbb{A}^1$ with $\mathcal{P}$ is invertible for the operation of middle convolution if and only if FT(K)[-1] has generic rank one. In this case, $K*_{\mathrm{mid}+}D_-K = \delta_0$.

proof Obvious by Fourier Transform, where it becomes the question of looking for $\otimes$-invertible objects in GalRep: these are obviously those of dimension one, and for these, the contragredient is the $\otimes$-inverse. QED

Corollary 2.10.15 On $\mathbb{A}^1$ over an algebraically closed field k of characteristic $p > 0$, the perverse sheaves K on $\mathbb{A}^1$ with $\mathcal{P}$ which are both invertible for middle convolution and tame at ∞ are precisely the translated δ-functions δ_α for α in k, and the translated nontrivial Kummer sheaves $j_*\mathcal{L}_{\chi(x-\alpha)}[1]$ for α in k.

proof The objects listed are obviously invertible, and tame at ∞. We must show there are no more. First of all, any invertible object K is irreducible as a perverse sheaf: indeed, from the Fourier Transform description, such a K has no proper subobject in $\mathcal{P}$, while, as already noted in the proof of 2.9.8, any irreducible subobject of a perverse K with $\mathcal{P}$ itself has $\mathcal{P}$.

Among perverse irreducible K's, the punctual ones are already on our list. So we must look for irreducible middle extensions $j_*\mathcal{F}[1]$ with $\mathcal{F}$ tame at ∞, for which $FT(j_*\mathcal{F})$ has generic rank one. For $\mathcal{F}$ tame at ∞, and $\alpha\neq 0$ in k, the stalk $FT(j_*\mathcal{F})_\alpha$ is $H_c{}^1(\mathbb{A}^1, j_*\mathcal{F}\otimes\mathcal{L}_{\psi(\alpha x)})$; both the $H_c{}^2$ and Hc^0 vanish ($j_*\mathcal{F}\otimes\mathcal{L}_{\psi(\alpha x)}$ is totally wild at ∞, and has no nonzero punctual sections). So for $\alpha\neq 0$, the rank of stalk $FT(j_*\mathcal{F})_\alpha$ is

$$-\chi_c(\mathbb{A}^1, j_*\mathcal{F}\otimes\mathcal{L}_{\psi(\alpha x)}) =$$
$$= -\text{ generic rank}(j_*\mathcal{F}\otimes\mathcal{L}_{\psi(\alpha x)}) + \mathrm{Swan}_\infty(j_*\mathcal{F}\otimes\mathcal{L}_{\psi(\alpha x)}) +$$
$$+ \Sigma_{\text{finite sing s}}[\mathrm{Swan}_s(j_*\mathcal{F}\otimes\mathcal{L}_{\psi(\alpha x)}) + \mathrm{drop}_s(j_*\mathcal{F}\otimes\mathcal{L}_{\psi(\alpha x)})].$$

Since $\mathcal{F}$ is tame at ∞, and $\mathcal{L}_{\psi(\alpha x)}$ has slope one at ∞, the first two terms cancel. Since $\mathcal{L}_{\psi(\alpha x)}$ is lisse on $\mathbb{A}^1$, at each finite singularity s of $\mathcal{F}$, the $\mathcal{L}_{\psi(\alpha x)}$ might as well be absent: we find

$$\mathrm{rank}\ FT(j_*\mathcal{F})_\alpha = \Sigma_{\text{finite sing s}}[\mathrm{Swan}_s(j_*\mathcal{F}) + \mathrm{drop}_s(j_*\mathcal{F})].$$

Each of the terms $[\mathrm{Swan}_s(j_*\mathcal{F}) + \mathrm{drop}_s(j_*\mathcal{F})]$ is strictly positive (the drop is nonzero), and each term where $\mathcal{F}$ is not tame is at least two. So if this rank is to be one, there is at precisely one finite singularity, $\mathcal{F}$ is tame there, and has a drop of one. Translating the singularity to the origin, we get an $\mathcal{F}$ which is lisse and tame on $\mathbb{G}_m$, irreducible, and nontrivially ramified at zero. Since it is lisse, tame and irreducible, it must be rank one ($\pi_1{}^{\mathrm{tame}}(\mathbb{G}_m)$ is abelian), so a Kummer sheaf $\mathcal{L}_\chi$ on $\mathbb{G}_m$. Since it is ramified at zero, χ is nontrivial. QED

2.11 Invertible objects on $\mathbb{A}^1$ in characteristic zero

Theorem 2.11.1 On $\mathbb{A}^1$ over an algebraically closed field k of characteristic zero, the perverse sheaves K on $\mathbb{A}^1$ with $\mathcal{P}$ which are invertible for middle convolution are precisely the translated δ-functions δ_α for α in k, and the translated nontrivial Kummer sheaves $j_*\mathcal{L}_{\chi(x-\alpha)}[1]$ for α in k.

proof The listed objects are visibly invertible. We must show there are no more. By 2.9.9, any invertible object K is perverse irreducible. If K is punctual, it is already on our list. So suppose that

K is an irreducible middle extension $j_\star\mathcal{F}[1]$, with $\mathcal{F}$ nonconstant. Such a K lies in $\mathcal{P}$.

step 1) We show that for **any** irreducible middle extension $K = j_\star\mathcal{F}[1]$, with K in $\mathcal{P}$, there exists a surjective map of perverse sheaves $K\star_{mid+}D_-(K) \twoheadrightarrow \delta_0$. [This is obvious in characteristic p, by looking on the Fourier Transform side.] For this, we argue as follows. For any perverse sheaf N on $\mathbb{A}^1$, we have a short exact sequence of perverse sheaves

$$0 \to \mathcal{H}^{-1}(N)[1] \to N \to \mathcal{H}^0(N) \to 0,$$

in which the sheaf $\mathcal{H}^0(N)$ is punctual. So for any α in $\mathbb{A}^1$, we have a surjective map $\mathcal{H}^0(N) \twoheadrightarrow \mathcal{H}^0(N)_\alpha\otimes\delta_\alpha$, so all in all $N \twoheadrightarrow \mathcal{H}^0(N)_\alpha\otimes\delta_\alpha$. Applying this to $N = K\star_{mid+}D_-(K)$, it suffices to show that

$$\mathcal{H}^0(K\star_{mid+}D_-(K))_0$$

is one-dimensional. Because we are in characteristic zero, we can compute **every** fibre of $K\star_{mid+}D_-(K)$ (by 2.8.5, (3) and (4)). Recall that K is $j_\star\mathcal{F}[1]$, $j: U \to \mathbb{A}^1$ a dense open set on which $\mathcal{F}$ is lisse, irreducible, and nonconstant. In terms of $k: \mathbb{A}^1 \to \mathbb{P}^1$ the inclusion, we have

$$(K\star_{mid+}D_-(K))_0 = R\Gamma(\mathbb{P}^1, k_\star(j_\star\mathcal{F}\otimes j_\star(\mathcal{F}^\vee)))[2].$$

Thus

$$\mathcal{H}^0(K\star_{mid+}D_-(K))_0 = H^2(\mathbb{P}^1, k_\star(j_\star\mathcal{F}\otimes j_\star(\mathcal{F}^\vee)))$$
$$= H_c{}^2(U,\mathcal{F}\otimes\mathcal{F}^\vee) = \overline{\mathbb{Q}}_\ell(-1),$$

the last equality because $\mathcal{F}$ is irreducible. [In fact, for such K, one can show that $\mathcal{H}^0(K\star_{mid+}D_-(K))_\alpha = 0$ for $\alpha \neq 0$, provided we are in characteristic zero. In other words, $\mathcal{H}^0(K\star_{mid+}D_-(K))$ is the required δ_0 quotient.]

step 2). We show that if $K\star_{mid+}L = \delta_0 = L\star_{mid+}K$, then L is $D_-(K)$. Indeed, applying the end-exact functor $X \mapsto L\star_{mid+}X$ to the surjection $K\star_{mid+}D_-(K) \twoheadrightarrow \delta_0$. produced in step 1, we get

$$L\star_{mid+}K\star_{mid+}D_-(K) \twoheadrightarrow L\star_{mid+}\delta_0, \text{ i.e., } D_-(K) \twoheadrightarrow L.$$

Since K is irreducible, so is $D_-(K)$, and hence $D_-(K) = L$.

step 3) We show that if $K\star_{mid+}D_-(K) = \delta_0$ with K an irreducible middle extension $j_\star\mathcal{F}[1]$, then $\mathcal{F}$ is a translate of a nontrivial

Kummer sheaf $\mathcal{L}_\chi$. Because we are in characteristic zero, we have, by 2.8.5, (3) and (4), a short exact sequence of perverse sheaves on $\mathbb{A}^1$

$$0 \to (\text{constant sheaf})[1] \to K*_{!+}D_-(K) \to K*_{\mathrm{mid}+}D_-(K) \to 0,$$

which, if $K*_{\mathrm{mid}+}D_-(K) = \delta_0$, reads

$$0 \to (\text{constant sheaf})[1] \to K*_{!+}D_-(K) \to \delta_0 \to 0.$$

We exploit this last exact sequence by computing the difference in the virtual ranks of stalks: for any $t \neq 0$, we have

$$1 = \operatorname{rank} (K*_{!+}D_-(K))_0 - \operatorname{rank} (K*_{!+}D_-(K))_t,$$

i.e.,

$$1 = \chi_c(\mathbb{A}^1, j_*(\mathcal{F})\otimes j_*(\mathcal{F}^\vee)) - \chi_c(\mathbb{A}^1, j_*(\mathcal{F})\otimes[x \mapsto x+t]^*j_*(\mathcal{F}^\vee)).$$

Denote by n the rank of $\mathcal{F}$ (also that of $\mathcal{F}^\vee$), and denote by S the set of finite singularities of $\mathcal{F}$ (also that of $\mathcal{F}^\vee$). At any s in S, both $j_*(\mathcal{F})$ and $j_*(\mathcal{F}^\vee)$ have the same dimensional fibre, say of dimension r_s (the same because r_s is the common number of unipotent Jordan blocks in the local monodromies of $\mathcal{F}$ and of $\mathcal{F}^\vee$ at the point s, cf. 3.1.2). Since we are in characteristic zero, the Euler-Poincaré formula gives

$$\chi_c(\mathbb{A}^1, j_*(\mathcal{F})\otimes j_*(\mathcal{F}^\vee)) = n^2 - \Sigma_{s \text{ in } S} (n^2 - (r_s)^2).$$

For t sufficiently general (not in the finite set $S - S$), the two finite sets S and $S+t$ are disjoint. So for such t,

$$\chi_c(\mathbb{A}^1, j_*(\mathcal{F})\otimes[x \mapsto x+t]^*j_*(\mathcal{F}^\vee)) =$$

$$= n^2 - \Sigma_{s \text{ in } S} (n^2 - (r_s)n) - \Sigma_{s \text{ in } S+t} (n^2 - n(r_s))$$

$$= n^2 - 2\Sigma_{s \text{ in } S} (n^2 - (r_s)n)$$

$$= n^2 - \Sigma_{s \text{ in } S} (2n)(n - r_s)$$

$$= n^2 - \Sigma_{s \text{ in } S} ((n + r_s) + (n - r_s))(n - r_s)$$

$$= n^2 - \Sigma_{s \text{ in } S} (n^2 - (r_s)^2) - \Sigma_{s \text{ in } S} (n - r_s)^2.$$

Subtracting, we find

$$1 = \Sigma_{s \text{ in } S} (n - r_s)^2.$$

Since each term $(n - r_s)^2$ is a strictly positive integer, we conclude that S consists of a single point. Translating that point to the origin, $\mathcal{F}$ becomes a lisse sheaf on $\mathbb{G}_m$ which is irreducible and nonconstant, so necessarily a nontrivial Kummer sheaf $\mathcal{L}_\chi$. QED

2.12 Musings on $*_{\mathrm{mid}\times}$ -invertible objects in $\mathcal{P}$ in the $\mathbb{G}_m$ case

(2.12.1) In this section, we work on $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$.

Lemma 2.12.2 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let K in $D^b_c(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$. The following conditions are equivalent:

1) $R\Gamma_c(\mathbb{G}_m, K\otimes\mathcal{L}_\chi) = 0$ for some Kummer sheaf $\mathcal{L}_\chi$.

2) $R\Gamma(\mathbb{G}_m, K\otimes\mathcal{L}_\chi) = 0$ for some Kummer sheaf $\mathcal{L}_\chi$.

3a) each of the cohomology sheaves $\mathcal{H}^i(K)$ is a successive extension of Kummer sheaves $\mathcal{L}_\chi$.

3b) each of the perverse cohomology sheaves ${}^p\mathcal{H}^i(K)$ is a successive extension of Kummer sheaves $\mathcal{L}_\chi[1]$.

4) $R\Gamma_c(\mathbb{G}_m, K\otimes\mathcal{L}_\chi) = 0$ for all but at most finitely many Kummer sheaves $\mathcal{L}_\chi$.

5) $R\Gamma(\mathbb{G}_m, K\otimes\mathcal{L}_\chi) = 0$ for all but at most finitely many Kummer sheaves $\mathcal{L}_\chi$.

proof By (2.3.2.1), 3a) $\Leftrightarrow$ 3b). The implications 3a) $\Rightarrow$ 4) and 3a) $\Rightarrow$ 5) are obvious, because $R\Gamma_c(\mathbb{G}_m, \mathcal{L}_\chi) = R\Gamma(\mathbb{G}_m, \mathcal{L}_\chi) = 0$ for any nontrivial Kummer sheaf. The implications 4) $\Rightarrow$ 1) and 5) $\Rightarrow$ 2) are trivial. We will prove below that 2) $\Rightarrow$ 3a). Admitting this, we have 2) $\Leftrightarrow$ 3a) $\Leftrightarrow$ 3b) $\Leftrightarrow$ 5). As noted in 2.3.1.1, we have

$${}^p\mathcal{H}^i(DK) := D({}^p\mathcal{H}^{-i}(K)).$$

Therefore 3b) holds for K if and only if 3b) holds for DK. Therefore the equivalences 2) $\Leftrightarrow$ 3b) $\Leftrightarrow$ 5) hold for DK, which by duality means that the equivalences 1) $\Leftrightarrow$ 3b) $\Leftrightarrow$ 4) hold for K.

It remains to show that 2) $\Rightarrow$ 3a). For this, we argue as follows, cf. [Ka-ACT, 2.5.3]. Replacing K by $K\otimes\mathcal{L}_\chi$, we may assume that $R\Gamma(\mathbb{G}_m, K) = 0$. Consider the spectral sequence

$$E_2^{p,q} = H^p(\mathbb{G}_m, \mathcal{H}^q(K)) \Rightarrow H^{p+q}(\mathbb{G}_m, K).$$

It has $E_2^{p,q} = 0$ unless p is 0 or 1 (cohomological dimension of an affine curve), so degenerates at E_2. Therefore for each χ, we have

$$R\Gamma(\mathbb{G}_m, K) = 0 \Leftrightarrow R\Gamma(\mathbb{G}_m, \mathcal{H}^i(K)) = 0 \text{ for all i.}$$

So to prove 2) $\Rightarrow$ 3), it suffices to do so in the case when K is a single sheaf $\mathcal{F}$. In this case, we are reduced to the following sublemma.

Sublemma 2.12.3 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}}_\ell$-sheaf such that $H^*(\mathbb{G}_m, \mathcal{F}) = 0$. Then $\mathcal{F}$ is lisse on $\mathbb{G}_m$ and everywhere tame, hence a successive extension of Kummer sheaves $\mathcal{L}_\chi$.

proof. Let $j : U \to \mathbb{G}_m$ be a dense open set where $\mathcal{F}$ is lisse. Consider the canonical map $\mathcal{F} \to j_*j^*\mathcal{F}$, whose (punctual) kernel we denote $\mathcal{F}_{pct}$. The inclusion $\mathcal{F}_{pct} \subset \mathcal{F}$ induces an inclusion

$$H^0(\mathbb{G}_m, \mathcal{F}_{pct}) \subset H^0(\mathbb{G}_m, \mathcal{F}) = 0,$$

whence $H^0(\mathbb{G}_m, \mathcal{F}_{pct}) = 0$, and consequently $\mathcal{F}_{pct} = 0$. Thus $\mathcal{F}$ has no nonzero punctual sections. For such an $\mathcal{F}$, the Euler Poincaré formula

$$\chi(\mathbb{G}_m, \mathcal{F}) = -\mathrm{Swan}_0(\mathcal{F}) - \mathrm{Swan}_\infty(\mathcal{F})$$
$$-\Sigma_{x \text{ in } \mathbb{G}_m} [\mathrm{drop}_x(\mathcal{F}) + \mathrm{Swan}_x(\mathcal{F})]$$

is a sum of terms which are each nonpositive. Since $\chi(\mathbb{G}_m, \mathcal{F}) = 0$, we see that $\mathcal{F}$ is lisse on $\mathbb{G}_m$, and everywhere tame. Because $\pi_1{}^{tame}(\mathbb{G}_m)$ is abelian, any such $\mathcal{F}$ is a successive extension of Kummer sheaves $\mathcal{L}_\chi$, these being the characters of $\pi_1{}^{tame}(\mathbb{G}_m)$. QED

For ease of later reference, we record here the well-known
Lemma 2.12.3.1 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let K in $D^b{}_c(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ be perverse. Then
1) $\chi(\mathbb{G}_m, K) \geq 0$,
2) $\chi(\mathbb{G}_m, K) = 0$ if and only if K is a successive extension of Kummer sheaves $\mathcal{L}_\chi[1]$.

proof For 1), use the short exact sequence

$$0 \to \mathcal{H}^{-1}(K)[1] \to K \to \mathcal{H}^0(K) \to 0$$

to reduce to the case when K is either punctual, in which case the assertion is obvious, or is of the form $\mathcal{F}[1]$, with $\mathcal{F}$ a sheaf on $\mathbb{G}_m$ with no nonzero punctual sections. In this case the Euler-Poincaré formula shows, as above, that $\chi(\mathbb{G}_m, \mathcal{F}) \leq 0$, i.e., $\chi(\mathbb{G}_m, \mathcal{F}[1]) \geq 0$.

For 2), the vanishing of

$$\chi(\mathbb{G}_m, K) = \chi(\mathbb{G}_m, \mathcal{H}^{-1}(K)[1]) + \chi(\mathbb{G}_m, \mathcal{H}^0(K))$$

together with the non-negativity of each summand shows that $\mathcal{H}^0(K) = 0$, and that $\chi(\mathbb{G}_m, \mathcal{H}^{-1}(K)[1]) = 0$. But $\mathcal{H}^{-1}(K)$ has no nonzero punctual sections, so again as in the proof of 2.12.3 the Euler-Poincaré formula shows that $\mathcal{H}^{-1}(K)$ is a successive extension of Kummer sheaves $\mathcal{L}_\chi$. QED

Lemma 2.12.4 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let K in $D^b{}_c(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$. For all but at most finitely many Kummer sheaves $\mathcal{L}_\chi$, the "forget supports" map

$$R\Gamma_c(\mathbb{G}_m, K\otimes\mathcal{L}_\chi) \to R\Gamma(\mathbb{G}_m, K\otimes\mathcal{L}_\chi)$$

is an isomorphism.

proof By the spectral sequences

$$E_2{}^{p,q} = H_c{}^p(\mathbb{G}_m, \mathcal{H}^q(K)\otimes\mathcal{L}_\chi) \Rightarrow H_c{}^{p+q}(\mathbb{G}_m, K\otimes\mathcal{L}_\chi)$$

and

$$E_2{}^{p,q} = H^p(\mathbb{G}_m, \mathcal{H}^q(K)\otimes\mathcal{L}_\chi) \Rightarrow H^{p+q}(\mathbb{G}_m, K\otimes\mathcal{L}_\chi),$$

we reduce immediately to the case where $\mathcal{F}$ is a single sheaf. In that case, we need only avoid the χ whose inverses occur either in $\mathcal{F}(0)$ or $\mathcal{F}(\infty)$, the $I(0)$ and $I(\infty)$-representations attached $\mathcal{F}$. QED

Lemma 2.12.5 (Gabber-Loeser) On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let K and L be two objects in $D^b{}_c(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$. Then the mapping cone object formed from the "forget supports" map between ! and $*$ multiplicative convolutions,

$$K*_{!\times}L \to K*_{*\times}L,$$

has all of its ordinary and perverse cohomology sheaves successive extensions of Kummer sheaves.

proof (compare the proof of [Ga-Loe, page 28]) Choose a Kummer sheaf $\mathcal{L}_\chi$ such that the "forget supports" map is an isomorphism $R\Gamma_c(\mathbb{G}_m, N\otimes\mathcal{L}_\chi) \cong R\Gamma(\mathbb{G}_m, N\otimes\mathcal{L}_\chi)$ for N any of the three objects K, L, $K*_{!\times}L$. This is possible by the preceding lemma. For this χ, we claim the "forget supports" map

$$R\Gamma(\mathbb{G}_m, (K*_{!\times}L)\otimes\mathcal{L}_\chi) \to R\Gamma(\mathbb{G}_m,(K*_{*\times}L)\otimes\mathcal{L}_\chi)$$

is an isomorphism. Since χ has been chosen so that

$$R\Gamma_c(\mathbb{G}_m, (K*_{!\times}L)\otimes\mathcal{L}_\chi) \cong R\Gamma(\mathbb{G}_m, (K*_{!\times}L)\otimes\mathcal{L}_\chi),$$

it suffices to show that the "forget supports twice" map

$$R\Gamma_c(\mathbb{G}_m, (K*_{!\times}L)\otimes\mathcal{L}_\chi) \to R\Gamma(\mathbb{G}_m,(K*_{*\times}L)\otimes\mathcal{L}_\chi)$$

is an isomorphism. Using the definition of multiplicative convolution, the multiplicativity of $\mathcal{L}_\chi$, the Leray spectral sequence for both ! and $*$ direct image, and the Kunneth formula for both $R\Gamma_c$ and $R\Gamma$, this map becomes the map "forget supports"$\otimes$"forget supports"
$R\Gamma_c(\mathbb{G}_m,K\otimes\mathcal{L}_\chi)\otimes R\Gamma_c(\mathbb{G}_m,L\otimes\mathcal{L}_\chi)\to R\Gamma(\mathbb{G}_m,K\otimes\mathcal{L}_\chi)\otimes R\Gamma(\mathbb{G}_m,L\otimes\mathcal{L})$,
which is an isomorphism by our choice of χ. The lemma now follows from 2.12.2, applied to K := the mapping cone of $K*_{!\times}L \to K*_{*\times}L$.
QED

Corollary 2.12.6 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let K and L be two objects in $D^b{}_c(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ which are perverse. Suppose that K satisfies $\mathcal{P}$ for multiplicative convolution. Then the kernel and cokernel of the "forget supports" map between the perverse sheaves

$$K*_{!\times}L \to K*_{*\times}L$$

are perverse sheaves which are successive extensions of Kummer sheaves $\mathcal{L}_\chi[1]$.

proof The kernel and cokernel are the perverse cohomology sheaves of the mapping cone. QED

Corollary 2.12.7 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let K and L be two objects in $D^b{}_c(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ which are perverse. Suppose that K satisfies $\mathcal{P}$ for multiplicative convolution. Then we have the product formula for middle multiplicative convolution

$$\chi(\mathbb{G}_m, K*_{\mathrm{mid}\times}L) = \chi(\mathbb{G}_m, K)\chi(\mathbb{G}_m, L).$$

proof We have a short exact sequence of perverse sheaves

$$0 \to \ker \to K*_{\mathrm{mid}\times}L \to K*_{!\times}L \to 0$$

on $\mathbb{G}_m$. We know that $\chi(\mathbb{G}_m, \ker) = 0$ because ker is a successive extension of $\mathcal{L}_\chi[1]$'s. Thus we find

$$\chi(\mathbb{G}_m, K*_{\mathrm{mid}\times}L) = \chi(\mathbb{G}_m, K*_{!\times}L).$$

By the ! Kunneth formula we know that

$$\chi_c(\mathbb{G}_m, K*_{!\times}L) = \chi_c(\mathbb{G}_m, K)\chi_c(\mathbb{G}_m, L).$$

Finally, we know that $\chi_c(\mathbb{G}_m, N) = \chi(\mathbb{G}_m, N)$ for any derived category object N. QED

Corollary 2.12.8 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let K and L be two objects in $D^b{}_c(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ which are perverse and which both satisfy $\mathcal{P}$ for multiplicative convolution. Suppose that $K*_{\mathrm{mid}\times}L = \delta_1$, i.e., suppose that K in $\mathcal{P}$ is invertible for multiplicative middle convolution, with inverse L. Then

$$\chi(\mathbb{G}_m, K) = \chi(\mathbb{G}_m, L) = 1.$$

proof Indeed, $\chi(\mathbb{G}_m, \delta_1) = 1$, and $\chi(\mathbb{G}_m, K)$ is a nonnegative integer for any perverse K on $\mathbb{G}_m$, by 2.12.3.1. QED

Theorem 2.12.9 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let K and L in $D^b{}_c(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ be perverse and satisfy $\mathcal{P}$ for multiplicative convolution. Suppose that $K*_{\mathrm{mid}\times}L = \delta_1$.
1)On $\mathbb{G}_m$, the operators $X \mapsto X*_{\mathrm{mid}\times}K$ and $X \mapsto X*_{\mathrm{mid}\times}L$ on $\mathcal{P}$ are automorphisms of $\mathcal{P}$, which are inverses of each other.
2) The operator $X \mapsto X*_{\mathrm{mid}\times}K$ induces an automorphism of the perverse irreducible objects in $\mathcal{P}$, with inverse the operator $X \mapsto X*_{\mathrm{mid}\times}L$.
3) L and K are perverse irreducible on $\mathbb{G}_m$.

proof This is entirely analogous to the additive case, cf. 2.9.7 and 2.9.9. QED

Theorem 2.12.10 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let K in $D^b{}_c(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ be perverse. The following conditions are equivalent.
1) K lies in $\mathcal{P}$ and $\chi(\mathbb{G}_m, K) = 1$.
2) K is perverse irreducible and $\chi(\mathbb{G}_m, K) = 1$.
3) K is an irreducible hypergeometric (cf. [Ka-ESDE, 3.5.4 and 8.5.3]).
4) K lies in $\mathcal{P}$, and there exists a geometric isomorphism

$$K*_{\mathrm{mid}\times}D(\mathrm{inv}^*K) \cong \delta_1.$$

5) K lies in $\mathcal{P}$, and is invertible for multiplicative middle convolution.

proof For 1) $\Rightarrow$ 2), let $L \subset K$ be nonzero perverse irreducible. If L=K, stop. If K/L is nonzero, we get a contradiction as follows. Since K lies in $\mathcal{P}$, L is not any $\mathcal{L}_\chi[1]$, so $\chi(\mathbb{G}_m, L) > 0$, by 2.12.3.1, 2). But by 2.12.3.1, 1),

$$\chi(\mathbb{G}_m, \text{any perverse}) \geq 0,$$

so from the exact sequence

$$0 \to L \to K \to K/L \to 0$$

we get $\chi(\mathbb{G}_m, L) = 1$, and $\chi(\mathbb{G}_m, K/L) = 0$. Thus K/L is a successive extension of $\mathcal{L}_\chi[1]$'s, and so K has an $\mathcal{L}_\chi[1]$ quotient. But this is impossible, because K lies in $\mathcal{P}$.

That 2) $\Rightarrow$ 3) is proven in [Ka-ESDE, 3.5.4 and 8.5.3]. That 4) $\Rightarrow$ 5) is trivial, and 5) $\Rightarrow$ 1) is proven in 2.12.8 above .

It remains only to prove 3) $\Rightarrow$ 4). Suppose first that we are in characteristic $p > 0$. By the structure theorem [Ka-ESDE, 8.5.3] for hypergeometrics, we know that every irreducible hypergeometric is either a δ or is $\mathcal{F}[1]$ for $\mathcal{F}$ a hypergeometric sheaf $\mathcal{H}_\lambda(!, \psi, \chi\text{'s}, \rho\text{'s})$ with disjoint χ's and ρ's. Such a hypergeometric is a successive $*_{\mathrm{mid}\times}$ convolution (because with disjoint χ's and ρ's, each $*_{!\times} \approx *_{*\times}$ in the constructive definition of hypergeometric objects cf. [Ka-ESDE, 8.4.2 (5)]) of sheaves $\mathcal{L}_\chi \otimes \mathcal{L}_\psi[1]$, their multiplicative translates (this only changes the ψ) and their multiplicative inverses. So we are reduced to showing that $K *_{\mathrm{mid}\times} D(\mathrm{inv}^* K) \cong \delta_1$ for $K = \mathcal{L}_\chi \otimes \mathcal{L}_\psi[1]$. Now for all the convolutions on $\mathbb{G}_m$, we have

$$(\mathcal{L}_\chi \otimes A) *_{\mathrm{any},\times} (\mathcal{L}_\chi \otimes B) = \mathcal{L}_\chi \otimes (A *_{\mathrm{any},\times} B).$$

So it remains to show that $K *_{\mathrm{mid}\times} D(\mathrm{inv}^* K) \cong \delta_1$. for $K = \mathcal{L}_\psi[1]$. This will be proven in 2.13.4 below.

If we are in characteristic zero, then by classification every irreducible hypergeometric. is either a δ or is $\mathcal{F}[1]$ for $\mathcal{F}$ a hypergeometric sheaf $\mathcal{H}_\lambda(\chi\text{'s}, \rho\text{'s})$ of type (n,n) with disjoint χ's and ρ's. Such an $\mathcal{F}[1]$ is a successive multiplicative middle convolution (with disjoint χ's and ρ's, each $*_{!\times} \approx *_{*\times}$) of hypergeometrics of type (1,1), $\mathcal{H}_\lambda(\chi, \rho)[1]$ with $\chi \neq \rho$ (compare [Ka-ESDE, 5.3.1] for the $\mathcal{D}$-module analogue). So we are reduced to the case when K is $j_* \mathcal{L}_{\chi(x)(\rho/\chi)(\lambda-x)}[1]$. Just as in characteristic p, we may replace K by $K \otimes \mathcal{L}_{\overline{\chi}}$, and multiplicatively translate λ to 1. This reduces us to treating universally the case when $K = j_* \mathcal{L}_{\chi(1-x)}[1]$, with χ nontrivial. This will be proven in 2.13.3 below.

2.13 Interlude: surprising relations between $*_{mid}$ on $\mathbb{A}^1$ and on $\mathbb{G}_m$

Lemma 2.13.1 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let K in $D^b{}_c(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ be perverse, and let $\mathcal{L}_\chi$ be any nontrivial Kummer sheaf on $\mathbb{G}_m$. Let j: $\mathbb{G}_m \to \mathbb{A}^1$ be the inclusion, and denote $j_*\mathcal{L}_\chi$ on $\mathbb{A}^1$ simply as $\mathcal{L}_\chi$. We have

$$K*_{!\times}\mathcal{L}_{\chi(x-1)}[1] = j^*(\ j_!(K\otimes\mathcal{L}_{\overline{\chi}})*_{!+}\mathcal{L}_\chi[1]\ )$$

$$K*_{*\times}\mathcal{L}_{\chi(x-1)}[1] = j^*(\ Rj_*(K\otimes\mathcal{L}_{\overline{\chi}})*_{*+}\mathcal{L}_\chi[1]\ ).$$

$$K*_{mid\ \times}\mathcal{L}_{\chi(x-1)}[1] = j^*(\ j_{!*}(K\otimes\mathcal{L}_{\overline{\chi}})*_{mid+}\mathcal{L}_\chi[1]\ ).$$

proof Let's start with K on $\mathbb{G}_m$ any derived category object. For χ nontrivial, we have, for $\pi : \mathbb{G}_m\times\mathbb{G}_m \to \mathbb{G}_m$ the map $(x,t) \mapsto t$, and j: $\mathbb{G}_m \to \mathbb{A}^1$ the inclusion,

$$K*_{!\times}\mathcal{L}_{\chi(x-1)} := R\pi_!(K_x\otimes\mathcal{L}_{\chi((t/x)-1)}) =$$

$$= R\pi_!((K\otimes\mathcal{L}_{\overline{\chi}})_x\otimes\mathcal{L}_{\chi(t-x)})$$

$$= j^*(\ j_!(K\otimes\mathcal{L}_{\overline{\chi}})*_{!+}\mathcal{L}_\chi\ ).$$

Dualizing this, we get

$$DK*_{*\times}\mathcal{L}_{\overline{\chi}(x-1)} = j^*(\ Rj_*(DK\otimes\mathcal{L}_\chi)*_{*+}\mathcal{L}_{\overline{\chi}}\ ).$$

Replacing K by DK and χ by $\overline{\chi}$, this gives

$$K*_{*\times}\mathcal{L}_{\chi(x-1)} = j^*(\ Rj_*(K\otimes\mathcal{L}_{\overline{\chi}})*_{*+}\mathcal{L}_\chi\ ).$$

Suppose now that K is perverse on $\mathbb{G}_m$. Then so is $K\otimes\mathcal{L}_{\overline{\chi}}$, and we have equalities on perverse sheaves (perverse because $\mathcal{L}_{\chi(x-1)}[1]$ is $\mathcal{P}$ on $\mathbb{G}_m$, and because $\mathcal{L}_\chi[1]$ is $\mathcal{P}$ on $\mathbb{A}^1$) on $\mathbb{G}_m$:

$$K*_{!\times}\mathcal{L}_{\chi(x-1)}[1] = j^*(\ j_!(K\otimes\mathcal{L}_{\overline{\chi}})*_{!+}\mathcal{L}_\chi[1]\ )$$

$$K*_{*\times}\mathcal{L}_{\chi(x-1)}[1] = j^*(\ Rj_*(K\otimes\mathcal{L}_{\overline{\chi}})*_{*+}\mathcal{L}_\chi[1]\ ).$$

Take the image of the first in the second. On the left side, the map is the "forget supports" map, and we get $K*_{mid\ \times}\mathcal{L}_{\chi(x-1)}[1]$ as image. On the right side, the map is the pullback to $\mathbb{G}_m$ of the map "forget supports twice" of perverse sheaves on $\mathbb{A}^1$

$$j_!(K\otimes\mathcal{L}_{\overline{\chi}})*_{!+}\mathcal{L}_\chi[1] \to Rj_*(K\otimes\mathcal{L}_{\overline{\chi}})*_{*+}\mathcal{L}_\chi[1].$$

This map we may factor into two surjections followed by two injections, as follows:
$j_!(K\otimes\mathcal{L}_{\overline{\chi}})*_{!+}\mathcal{L}_{\chi}[1] \to j_{!*}(K\otimes\mathcal{L}_{\overline{\chi}})*_{!+}\mathcal{L}_{\chi}[1]$ (exactness of $*_{!+}\mathcal{L}_{\chi}[1]$)
$j_{!*}(K\otimes\mathcal{L}_{\overline{\chi}})*_{!+}\mathcal{L}_{\chi}[1] \to j_{!*}(K\otimes\mathcal{L}_{\overline{\chi}})*_{mid+}\mathcal{L}_{\chi}[1]$ (def'n of $*_{mid}$)
$j_{!*}(K\otimes\mathcal{L}_{\overline{\chi}})*_{mid+}\mathcal{L}_{\chi}[1]\to j_{!*}(K\otimes\mathcal{L}_{\overline{\chi}})*_{*+}\mathcal{L}_{\chi}[1]$ (def'n of $*_{mid}$)
$j_{!*}(K\otimes\mathcal{L}_{\overline{\chi}})*_{*+}\mathcal{L}_{\chi}[1]\to Rj_{*}(K\otimes\mathcal{L}_{\overline{\chi}})*_{*+}\mathcal{L}_{\chi}[1]$ (exactness of $*_{*+}\mathcal{L}_{\chi}[1]$)
so the image of this composite map is $j_{!*}(K\otimes\mathcal{L}_{\overline{\chi}})*_{mid+}\mathcal{L}_{\chi}[1]$. QED

In analogous but simpler fashion, we have
Lemma 2.13.2 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $p \neq \ell$, $p > 0$, let K in $D^b{}_c(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ be perverse. Then for $j: \mathbb{G}_m \to \mathbb{A}^1$ the inclusion, we have

$inv^*K*_{!\times}j^*\mathcal{L}_{\psi}[1] = j^*FT(j_!K)$,

$inv^*K*_{*\times}j^*\mathcal{L}_{\psi}[1] = j^*FT(Rj_*K)$,

$inv^*K*_{mid\times}j^*\mathcal{L}_{\psi}[1] = j^*FT(j_{!*}K)$.

proof The first assertion results formally from the definitions, cf [Ka-ESDE, p264, or GKM 8.6.1], and the second is the dual of the first. Since $j^*\mathcal{L}_{\psi}[1]$ on $\mathbb{G}_m$ is in $\mathcal{P}$, being perverse irreducible and not an $\mathcal{L}_{\chi}[1]$, all the objects in the first two assertions are perverse. The third assertion is the image of the first in the second by the "forget supports" map, where on the right we think of the source as $j^*FT_!(j_!K)$ and the target as $j^*FT_*(Rj_*K)$. QED

As a nice application of these last two results, we can now complete the proof of 2.12.10, 3) ⇒ 4).

Corollary 2.13.3 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $\neq \ell$, let $\mathcal{L}_{\chi}$ be any nontrivial Kummer sheaf. Then $K = \mathcal{L}_{\chi}[1]$ satisfies $K*_{mid\times}D(inv^*K) = \delta_1$.
proof For any perverse L on $\mathbb{G}_m$, we have

$L*_{mid\ \times}\mathcal{L}_{\chi(x-1)}[1] = j^*(\ j_{!*}(L\otimes\mathcal{L}_{\overline{\chi}})*_{mid+}\mathcal{L}_{\chi}[1]\ )$.

We take

$L := D(\mathrm{inv}^*\mathcal{L}_{\chi(x-1)}[1]) = \mathcal{L}_{\overline{\chi}((1/x) - 1)}[1] = \mathcal{L}_{\overline{\chi}((1 - x)/x)}[1].$

Then

$$j_{!*}(L \otimes \mathcal{L}_{\overline{\chi}}) = \mathcal{L}_{\overline{\chi}(1 - x)}[1],$$

and the formula becomes

$D(\mathrm{inv}^*\mathcal{L}_{\chi(x-1)}[1]) *_{\mathrm{mid}\ \times} \mathcal{L}_{\chi(x-1)}[1] = j^*(\mathcal{L}_{\overline{\chi}(1 - x)}[1] *_{\mathrm{mid}+} \mathcal{L}_{\chi}[1]).$

This last object $\mathcal{L}_{\overline{\chi}(1 - x)}[1] *_{\mathrm{mid}+} \mathcal{L}_{\chi}[1]$ on $\mathbb{A}^1$ is (geometrically) the additive translation by 1 of $\mathcal{L}_{\overline{\chi}}[1] *_{\mathrm{mid}+} \mathcal{L}_{\chi}[1]$, which we have already seen to be δ_0. So we obtain

$D(\mathrm{inv}^*\mathcal{L}_{\chi(x-1)}[1]) *_{\mathrm{mid}\ \times} \mathcal{L}_{\chi(x-1)}[1] = \delta_1$, as required. QED

Corollary 2.13.4 On $\mathbb{G}_m$ over an algebraically closed field k of characteristic $p \neq \ell$, $p > 0$, $K = \mathcal{L}_{\psi}[1]$ satisfies $K *_{\mathrm{mid}\times} D(\mathrm{inv}^*K) = \delta_1$.

proof For any perverse L on $\mathbb{G}_m$, we have, by 2.13.2 above,

$$\mathrm{inv}^*L *_{\mathrm{mid}\times} j^*\mathcal{L}_{\psi}[1] = j^*FT(j_{!*}L).$$

For $L = j^*\mathcal{L}_{\overline{\psi}}[1]$ on $\mathbb{G}_m$, we have $j_{!*}L = \mathcal{L}_{\overline{\psi}}[1]$ on $\mathbb{A}^1$, and so $FT(j_{!*}L) = \delta_1$. Since $L = DK$, the above formula says precisely

$$\mathrm{inv}^*DK *_{\mathrm{mid}\times} K = \delta_1 \text{ for.} K = j^*\mathcal{L}_{\psi}[1] \text{ on } \mathbb{G}_m.$$

But $\mathrm{inv}^*DK = D(\mathrm{inv}^*K)$, so the commutativity of middle convolution gives

$$K *_{\mathrm{mid}\times} D(\mathrm{inv}^*K) = \delta_1, \text{ as required. QED}$$

2.14 Interpretive remark: Fourier-Bessel Transform We continue to work on $\mathbb{G}_m$ over an algebraically closed field of characteristic $p > 0$, $p \neq \ell$. For any irreducible hypergeometric sheaf $\mathcal{H}$, and in particular for any Kloosterman sheaf $\mathcal{K}$, $\mathcal{H}[1]$ lies in $\mathcal{P}$ and is invertible for middle multiplicative convolution. Thus $*_{\mathrm{mid}}(\mathcal{H}[1])$ is an automorphism of $\mathcal{P}$, with inverse $*_{\mathrm{mid}} D\mathrm{inv}^*(\mathcal{H}[1])$. In particular, if we take $\mathcal{H}$ to be the "classical" rank two Kloosterman sheaf Kl_2, we find the ℓ-adic analogue of the classical Fourier-Bessel Transform.

In characteristic 0, we can only form such a transform with hypergeometrics of type (n,n). But already in the case n=2, where we have the Gauss hypergeometric function, we have a transform which should have received a classical name, and some classical attention.

2.15 Questions about the situation in several variables

Lemma 2.15.1 On $\mathbb{A}^n$ over an algebraically closed field of characteristic $p \neq \ell$, $p > 0$, a perverse K in $D^b{}_c(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell)$ has $\mathcal{P}!$ if and only if FT(K) is of the form $\mathcal{F}[n]$ for some sheaf $\mathcal{F}$.

proof First of all, we already know that for M and K perverse, the dual of $K*_!M$ is semiperverse, so K is in $\mathcal{P}!$ if and only if $K*_!M$ is semiperverse for every perverse M. On the FT side, we are asking when L:= FT(K) has the property that $N \mapsto N\otimes L[-n]$ maps perverses to semiperverses. As soon as L[-n] has some nonzero cohomology sheaf in strictly positive degree, we can take N to be δ_α for some point α where the offending cohomology sheaf lives, and get a non-semiperverse answer. So the condition is necessary, and it is trivially sufficient, since when it holds, $N \mapsto N\otimes L[-n] = N\otimes\mathcal{F}$ only decreases the supports of the $\mathcal{H}^{-i}$. QED

(2.15.2) As an application of this lemma, δ's on $\mathbb{A}^n$ and external products of perverse objects with $\mathcal{P}!$ on each $\mathbb{A}^1$ factor will satisfy $\mathcal{P}!$ on $\mathbb{A}^n$. What about condition $\mathcal{P}$ on $\mathbb{A}^n$? On the Fourier Transform side, we are looking for the sheaves $\mathcal{F}$ on $\mathbb{A}^n$ such that $\mathcal{F}[n]$ is perverse and such that $D(\mathcal{F}[n]) = \mathcal{G}[n]$ for some sheaf $\mathcal{G}$. What is the classification of such sheaves?

(2.15.3) Since K on $\mathbb{A}^n$ has $\mathcal{P}$ iff both K and DK have $\mathcal{P}!$, external products of perverse objects with $\mathcal{P}$ on each $\mathbb{A}^1$ factor will satisfy $\mathcal{P}$ on $\mathbb{A}^n$. What is the intrinsic characterization of $\mathcal{P}$ on $\mathbb{A}^n$? What if any is the analogue of the one-variable criterion

$$\mathrm{Hom}(\mathcal{L}_\psi[1], K) = \mathrm{Hom}(K, \mathcal{L}_\psi[1]) = 0$$

which we had on $\mathbb{A}^1$? There are presumably lots of objects in $\mathcal{P}$ on $\mathbb{A}^n$ other than external products of objects in $\mathcal{P}$ on the factors, and δ's. What are they? How can analyze the situation without FT? What happens in characteristic zero?

(2.15.4) Already on higher dimensional tori $(\mathbb{G}m)^n$ we don't seem to know even that external products of perverse objects with $\mathcal{P}$ on each $\mathbb{G}_m$ factor will satisfy $\mathcal{P}$ on $(\mathbb{G}_m)^n$

2.16 Questions about the situation on elliptic curves

(2.16.1) What about the situation on an elliptic curve E over an algebraically closed field of characteristic $\neq \ell$, where again we may speak of $\mathcal{P}$ (here $\mathcal{P}!$ and $\mathcal{P}*$ are both $\mathcal{P}$)? Any perverse irreducible whose isomorphism class, under translation, has only a finite stabilizer, is automatically in $\mathcal{P}$. We know the stability under $*$ trivially for $\mathcal{P}$ (just as we knew it trivially for $\mathcal{P}!$ in the noncompact case).

(2.16.2) If we are over $\overline{\mathbb{F}}_p$, there is a kind of FT defined using the Lang torsors over finite subfields k_0 over which E is defined,

$$E \to E \text{ by } 1 - \mathrm{Frob}_{k_0}, \text{ with structural group } E(k_0),$$

and pushing out by characters ξ of the group $E(k_0)$ to get rank one lisse sheaves $\mathcal{L}_\xi$ on E against which we can form

$$(K, \xi) \mapsto R\Gamma(E, K\otimes\mathcal{L}_\xi).$$

The Euler characteristic $\chi(E, K\otimes\mathcal{L}_\xi)$ is independent of ξ. In any case, if K is say perverse irreducible, and not itself any $\mathcal{L}_\xi$, then $R\Gamma(E, K\otimes\mathcal{L}_\xi)$ has only its middle cohomology group possibly nonzero.

Just as in the $\mathbb{A}^1$ and $\mathbb{G}_m$ cases, we have

Lemma 2.16.3 On an elliptic curve E over an algebraically closed field of characteristic $\neq \ell$, the necessary and condition that a perverse K on E have $\mathcal{P}$ is that for each lisse rank one $\mathcal{L}$ on E (automatically translation invariant), we have

$$\mathrm{Hom}(\mathcal{L}[1], K) = \mathrm{Hom}(K, \mathcal{L}[1]) = 0.$$

(2.16.4) What are the invertible objects on an elliptic curve E over an algebraically closed field of characteristic $\neq \ell$? Since

$$\chi(E, K*L) = \chi(E, K)\chi(E, L),$$

only those objects in $\mathcal{P}$ whose Euler characteristic $\chi = 1$ can possibly be invertible. Notice that any object in $\mathcal{P}$ with $\chi=1$ is irreducible (since by the Euler Poincaré formula on E, $\chi(E, K) \geq 0$ for any perverse K, with equality if and only if K is $\mathcal{F}[1]$ with $\mathcal{F}$ lisse on E, so a successive extension of lisse $\mathcal{L}$'s of rank one, cf. below). Do we know any non-δ examples of invertible objects? Indeed, do we know any non-δ examples of irreducible perverse sheaves K on E with $\chi(E, K) = 1$?

(2.16.5) On E the Euler Poincaré formula for $j_*\mathcal{F}[1]$ gives

$$\chi(E, j_\star\mathcal{F}[1]) = - \chi(E, j_\star\mathcal{F}) = \Sigma_{x \text{ in } E}(\mathrm{drop}_x(\mathcal{F}) + \mathrm{swan}_x(\mathcal{F})).$$

So if $\chi(E, j_\star\mathcal{F}[1]) = 1$, then there is only one point x with a nonzero contribution. At that point, (which by translation we may as well take to be the origin 0_E) $\mathcal{F}$'s local monodromy is tame (since swan > 0 forces drop > 0) and a pseudoreflection. In fact this pseudoreflection must be unipotent, since $\det(\mathcal{F})$ is lisse of rank one on E - {0}, with tame local monodromy at 0; because E and E-{0} have the same tame $(\pi_1)^{ab}$, $\det(\mathcal{F})$ is also lisse at zero. In particular, $\chi(E, j_\star\mathcal{F}[1]) = 1$ implies $\mathrm{rank}(\mathcal{F}) \geq 2$.

(2.16.6) In characteristic zero, such an $\mathcal{F}$ on E - {0} is a pair A, B in $GL(n, \mathbb{C})$, n:= rank($\mathcal{F}$), with commutator {A,B} a unipotent pseudoreflection. If such a $j_\star\mathcal{F}[1]$ is in $\mathcal{P}$, then, as noted above, $\mathcal{F}$ must be irreducible on E - {0}.

(2.16.7) This brings us to the following question, which we are at present unable to answer, except by explicit calculation, in rank n=2, where the answer is **negative**. Given $n \geq 2$, do there exist elements A, B in $GL(n, \mathbb{C})$ with commutator {A,B} a unipotent pseudoreflection, such that the group <A, B> generated by A and B is an irreducible subgroup of $GL(n, \mathbb{C})$? [As we have seen above, this is equivalent to the question of whether there exist perverse irreducibles K on E with $\chi(E, K) = 1$ of the form $\mathcal{F}[1]$ with $\mathcal{F}$ of generic rank n.] We may scale both A and B, and require that A and B lie in $SL(n, \mathbb{C})$ if we like.

(2.16.8) Let us briefly explain the negative answer to the n=2 case of this question. Here the "trick" is that an element X in $SL(2, \mathbb{C})$ is unipotent if and only if trace(X) = 2. So we first want to find two elements A and B in $SL(2, \mathbb{C})$ whose commutator has trace 2, then check to see if A and B generate an irreducible subgroup of $SL(2, \mathbb{C})$. We may conjugate by $SL(2, \mathbb{C})$ to reduce to the case when A is upper triangular. If A has distinct eigenvalues, then by further conjugation we may assume it is diag(x,y), with $x \neq y$, $xy=1$. If A has repeated eigenvalues, they must be both ±1, so either A = ±1, or A = ±(the standard upper unipotent).

(2.16.9) If A is diagonal and nonscalar, we compute {A,B}, set its trace =2, and find that B is either upper or lower triangular, so <A,B> can't be irreducible in this case. If A is scalar, <A,B> is not irreducible. If A is ±(unipotent), we find that B is upper triangular, and again <A,B> is not irreducible. So it seems that no such irreducibles exist!

(2.16.10) This seems to put a damper on our elliptic hopes, and

makes it likely, or at least plausible, there are no $\chi=1$ irreducibles of any rank > 1. It would be interesting to clarify this point.
(2.16.11) Another point one should clarify is this: the rigid objects $\mathcal{F}_{n,\zeta}$ on $E - \{0_E\}$ constructed in the last chapter give perverse irreducibles $j_*\mathcal{F}_{n,\zeta}[1]$ on E, which have $\chi(E, j_*\mathcal{F}_{n,\zeta}[1]) = n$, hence lie in $\mathcal{P}$. What happens when we convolve them with each other?

2.17 Appendix 1: the basic lemma on end-exact functors

Lemma 2.17.1 We are given two abelian categories $\mathcal{A}$ and $\mathcal{B}$, two exact functors S, T from $\mathcal{A}$ to $\mathcal{B}$, and a morphism of functors $\varphi: S \to T$. Then the functor $\mathrm{Im}(\varphi)$ from $\mathcal{A}$ to $\mathcal{B}$ defined by

$$\mathrm{Im}(\varphi)(A) := \mathrm{Image}\ (\varphi_A: S(A) \to T(A))$$

is end-exact, i.e., it carries injections to injections, and it carries surjections to surjections.

proof Let us begin with a short exact sequence in $\mathcal{A}$,

$$0 \to A \to B \to C \to 0.$$

Applying the functors S and T, we get a commutative diagram in $\mathcal{B}$

$$\begin{array}{ccccccccc} 0 & \to & SA & \to & SB & \to & SC & \to & 0. \\ & & \downarrow\varphi_A & & \downarrow\varphi_B & & \downarrow\varphi_C & & \\ 0 & \to & TA & \to & TB & \to & TC & \to & 0. \end{array}$$

with exact rows. Applying the snake lemma, we get a six term exact sequence

$$0 \to \mathrm{Ker}(\varphi_A) \to \mathrm{Ker}(\varphi_B) \to \mathrm{Ker}(\varphi_C) \xrightarrow{\delta} \mathrm{Coker}(\varphi_A) \to \mathrm{Coker}(\varphi_B) \to$$
$$\to \mathrm{Coker}(\varphi_C) \to 0.$$

We denote by $\mathrm{Ker}(\varphi_C)[\delta] \subset \mathrm{Ker}(\varphi_C)$ the kernel of the coboundary map δ, and extract the short exact sequence

$$0 \to \mathrm{Ker}(\varphi_A) \to \mathrm{Ker}(\varphi_B) \to \mathrm{Ker}(\varphi_C)[\delta] \to 0.$$

Then we have a commutative diagram in $\mathcal{B}$

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ker}(\varphi_A) & \to & \mathrm{Ker}(\varphi_B) & \to & \mathrm{Ker}(\varphi_C)[\delta] & \to & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & SA & \to & SB & \to & SC & \to & 0. \end{array}$$

with exact rows and with all vertical arrows injective. Applying the snake to this diagram gives a short exact sequence

$$0 \to SA/\mathrm{Ker}(\varphi_A) \to SB/\mathrm{Ker}(\varphi_B) \to SC/\mathrm{Ker}(\varphi_C)[\delta] \to 0,$$

which we may rewrite as

$$0 \to \mathrm{Im}(\varphi_A) \quad \to \mathrm{Im}(\varphi_B) \quad \to SC/\mathrm{Ker}(\varphi_C)[\delta] \to 0.$$

Thus the map $\mathrm{Im}(\varphi_A) \to \mathrm{Im}(\varphi_B)$ is injective. The third term $SC/\mathrm{Ker}(\varphi_C)[\delta]$ of this exact sequence surjects onto $SC/\mathrm{Ker}(\varphi_C) \cong \mathrm{Im}(\varphi_C)$, and hence the map $\mathrm{Im}(\varphi_B) \to \mathrm{Im}(\varphi_C)$ is surjective, being the composition of the two surjections

$$\mathrm{Im}(\varphi_B) \twoheadrightarrow SC/\mathrm{Ker}(\varphi_C)[\delta] \twoheadrightarrow SC/\mathrm{Ker}(\varphi_C) \cong \mathrm{Im}(\varphi_C). \quad \text{QED}$$

2.18 Appendix 2: twisting representations by characters

(2.18.1) In analyzing the translation-invariant perverse irreducibles on $\mathbb{A}^1$ over an algebraically closed field k of characteristic $p > 0$, cf. the proof of 2.6.13, we made use of the following lemma, with Γ the group $I(\infty)$, K the field $\overline{\mathbb{Q}}_\ell$, and characters all the $\mathcal{L}_{\psi(\beta x)}$, β in k.

Lemma 2.18.2 Let Γ be a group, K an algebraically closed field of characteristic zero, and M a finite-dimensional K-representation of Γ. Suppose that for an infinity of one-dimensional K-valued characters χ of Γ, there exists an isomorphism $M \cong M \otimes \chi$ (as K-representation of Γ). Then $M = 0$.

We will prove this in the following equivalent form.

Lemma 2.18.2 bis Let Γ be a group, K an algebraically closed field K of characteristic zero, and M a non-zero finite-dimensional K-representation of Γ. Denote by $\mathrm{Stab}(M,\Gamma)$ the subgroup of the character group $\mathrm{Hom}(\Gamma, K^\times)$ consisting of those characters χ of Γ for which there exists an isomorphism $M \cong M \otimes \chi$ (as K-representation of Γ). Then $\mathrm{Stab}(M,\Gamma)$ is finite.

proof

Step 1.We first reduce to the case where M is a faithful representation of Γ. Denote by $\mathrm{Ker}(M) \subset \Gamma$ the kernel of the representation M of Γ. Then for any χ in $\mathrm{Stab}(M,\Gamma)$, $\chi|\mathrm{Ker}(M)$ lies in $\mathrm{Stab}(M|\mathrm{Ker}(M), \mathrm{Ker}(M)) = \mathrm{Stab}(\dim(M) \text{ copies of } \mathbb{1}, \mathrm{Ker}(M))$. Comparing characteristic polynomials, we see that $\chi|\mathrm{Ker}(M)$ is trivial. Thus we find

$$\mathrm{Stab}(M, \Gamma) = \mathrm{Stab}(M, \Gamma/\mathrm{Ker}(M)).$$

Step 2. We henceforth suppose that M is a faithful representation, i.e. that $\Gamma \subset GL(M)$. We denote by G the Zariski closure of Γ in $GL(M)$.

We claim that every character χ in $\mathrm{Stab}(M, \Gamma)$ extends uniquely to an algebro-geometric character $\tilde{\chi}$ in $\mathrm{Hom}(G, \mathbb{G}_m)$.

Indeed, given χ in $\mathrm{Stab}(M, \Gamma)$, there exists by definition an element A in GL(M) such that, for every γ in Γ, we have

$$A\gamma A^{-1} = \gamma\chi(\gamma).$$

We rewrite this as

$$A\gamma A^{-1}\gamma^{-1} = \chi(\gamma),$$

and remember only

$$A\gamma A^{-1}\gamma^{-1} = \text{scalar}.$$

For fixed A in GL(M), the map of GL(M) to itself defined by

$$B \mapsto ABA^{-1}B^{-1}$$

is an algebro-geometric morphism. In the target GL(M), the set of scalar matrices is Zariski closed (equations $X_{i,j} = 0$ for $i \neq j$, $X_{i,i} = X_{1,1}$ for $1 \leq i \leq \dim(M)$). So for fixed A, the set of B in GL(M) for which

$$ABA^{-1}B^{-1} = \text{scalar}$$

is Zariski closed. It is also a group, as one sees in rewriting this equation as

$$ABA^{-1} = (\text{scalar})B.$$

Thus the set of such B's is a Zariski closed subgroup of GL(M), which contains Γ. Therefore it contains G. Thus for every g in G, there exists a scalar, say $\alpha(g)$, such that

$$\alpha(g) := AgA^{-1}g^{-1}.$$

Clearly the function $g \mapsto \alpha(g)$ is an algebro-geometric morphism to $\mathbb{G}_m$ (it is an algebro-geometric morphism to GL(M) which lands in the subgroup consisting of scalars). This function is multiplicative in g, as one sees by rewriting the defining equation as

$$AgA^{-1} = \alpha(g)g.$$

For if g and h are elements of G, we have

$$\alpha(hg)hg = AhgA^{-1} = AhA^{-1}AgA^{-1} = \alpha(h)h\alpha(g)g = \alpha(h)\alpha(g)hg,$$

and multiplying both sides by $(hg)^{-1}$ gives $\alpha(hg) = \alpha(h)\alpha(g)$. Thus $g \mapsto \alpha(g)$ is an algebro-geometric character of G which agrees with χ on Γ. Since Γ is Zariski dense in G, there is at most one such, which we denote $\tilde{\chi}$. Thus

$$\tilde{\chi}(g) := \alpha(g).$$

Step 3. For every χ in $\mathrm{Stab}(M, \Gamma)$, $\tilde{\chi}$ lies in $\mathrm{Stab}(M, G)$, as is clear from the equation

$$AgA^{-1} = \alpha(g)g := \tilde{\chi}(g)g.$$

Thus we have

$$\mathrm{Stab}(M, G) \cong \mathrm{Stab}(M, \Gamma),$$

the isomorphism that of restriction to Γ.

Step 4. For any $\tilde{\chi}$ in Stab(M, G), by taking determinants in the equation

$$AgA^{-1} = \tilde{\chi}(g)g,$$

we see that $\tilde{\chi}(g)^{\dim(M)} = 1$. Thus $\tilde{\chi}$ has finite order, and hence is a character of the finite group G/G^0 of components of G. Therefore Stab(M, G) lies in the finite group $\mathrm{Hom}(G/G^0, K^\times)$, hence is finite. Therefore the isomorphic group $\mathrm{Stab}(M, \Gamma)$ is finite, as required. QED

3.0 Fourier Transform and index of rigidity

(3.0.1) Let ℓ be a prime number. On $\mathbb{A}^1$ over an algebraically closed field of characteristic $\neq \ell$, denote by $k: \mathbb{A}^1 \to \mathbb{P}^1$ the inclusion. Let K in $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ be a perverse sheaf which is a middle extension, i.e., K is $j_*\mathcal{F}[1]$ for a nonempty open set $j: U \to \mathbb{A}^1$, and a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U. We define the **index of rigidity** of K, $\mathrm{rig}(K)$, to be the integer (compare 2.0.3-6)

$$\mathrm{rig}(K) := \chi(\mathbb{P}^1, k_*j_*\underline{\mathrm{End}}(\mathcal{F})).$$

Theorem 3.0.2 Let p and ℓ be prime numbers, $\ell \neq p$. On $\mathbb{A}^1$ over an algebraically closed field of characteristic p, let K in $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ be a perverse irreducible such that neither K nor $FT(K)$ is punctual (i.e., K is neither a δ_α nor an $\mathcal{L}_{\psi_\alpha}[1]$). Then K and $FT(K)$ have the same index of rigidity:

$$\mathrm{rig}(K) = \mathrm{rig}(FT(K)).$$

This is a special case of the slightly more general

Theorem 3.0.3 Let p and ℓ be prime numbers, $\ell \neq p$. On $\mathbb{A}^1$ over an algebraically closed field of characteristic p, let K in $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ be a perverse sheaf on $\mathbb{A}^1$ such that both K and $FT(K)$ are middle extensions from any nonempty open set of $\mathbb{A}^1$. Then

$$\mathrm{rig}(K) = \mathrm{rig}(FT(K)).$$

proof of Theorem 3.0.3 Write $K = j_*\mathcal{F}[1]$ and $FT(K) = j_*\mathcal{G}[1]$, for some dense open $j: U \to \mathbb{A}^1$, and lisse sheaves $\mathcal{F}$ and $\mathcal{G}$ on U. Denote by $k: \mathbb{A}^1 \to \mathbb{P}^1$ the inclusion. By definition,

$$\mathrm{rig}(K) := \chi(\mathbb{P}^1, k_*j_*\underline{\mathrm{End}}(\mathcal{F})) =$$

$$= \dim \underline{\mathrm{End}}(\mathcal{F})^{I(\infty)} + \chi(\mathbb{A}^1, j_*\underline{\mathrm{End}}(\mathcal{F})).$$

The sheaf $j_*\underline{\mathrm{End}}(\mathcal{F})$ on $\mathbb{A}^1$ contains the subsheaf $j_*(\mathcal{F})\otimes j_*(\mathcal{F}^\vee)$; the quotient $j_*\underline{\mathrm{End}}(\mathcal{F})/j_*(\mathcal{F})\otimes j_*(\mathcal{F}^\vee)$ is punctual, supported in $\mathbb{A}^1 - U$:

$$j_*\underline{\mathrm{End}}(\mathcal{F})/j_*(\mathcal{F})\otimes j_*(\mathcal{F}^\vee) = \oplus_{x \text{ in } \mathbb{A}^1} \delta_x \otimes \underline{\mathrm{End}}(\mathcal{F})^{I(x)}/(\mathcal{F}^{I(x)}\otimes(\mathcal{F}^\vee)^{I(x)}).$$

so

$$\mathrm{rig}(K) := \chi(\mathbb{A}^1, j_*(\mathcal{F})\otimes j_*(\mathcal{F}^\vee)) + \dim[\underline{\mathrm{End}}(\mathcal{F})^{I(\infty)}] + $$
$$+ \Sigma_{x \text{ in } \mathbb{A}^1} \dim[\underline{\mathrm{End}}(\mathcal{F})^{I(x)}/(\mathcal{F}^{I(x)}\otimes(\mathcal{F}^\vee)^{I(x)})].$$

Since $K = j_*\mathcal{F}[1]$, $DK = j_*\mathcal{F}^\vee[1]$, and we have

$$j_*(\mathcal{F})\otimes j_*(\mathcal{F}^\vee) = DK\otimes K[-2].$$

Thus

$$\mathrm{rig}(K) := \chi(\mathbb{A}^1, DK\otimes K) + \dim[\underline{\mathrm{End}}(\mathcal{F})^{I(\infty)}] + $$
$$+ \Sigma_{x \text{ in } \mathbb{A}^1} \dim[\underline{\mathrm{End}}(\mathcal{F})^{I(x)}/(\mathcal{F}^{I(x)}\otimes(\mathcal{F}^\vee)^{I(x)})].$$

Now we break up the $I(\infty)$-representation $\mathcal{F}(\infty)$ of $\mathcal{F}$ according to whether slopes are ≤ 1 or >1:

$$\mathcal{F}(\infty) = \mathcal{F}(\infty)(\le 1) \oplus \mathcal{F}(\infty)(>1).$$

Because there are no nonzero $I(\infty)$-equivariant maps between $I(\infty)$-representations with disjoint slopes, we have

$$\underline{\mathrm{End}}(\mathcal{F})^{I(\infty)} := \mathrm{End}_{I(\infty)}(\mathcal{F}(\infty)) =$$
$$= \mathrm{End}_{I(\infty)}(\mathcal{F}(\infty)(\le 1)) + \mathrm{End}_{I(\infty)}(\mathcal{F}(\infty)(>1)).$$

Using this, we rewrite the formula for rig(K) as the sum of four terms

$$\mathrm{rig}(K) = 1(K) + 2(K) + 3(K) + 4(K)$$

$1(K) := \chi(\mathbb{A}^1, DK\otimes K)$,
$2(K) := \dim[\mathrm{End}_{I(\infty)}(\mathcal{F}(\infty)(>1))]$,
$3(K) := \dim[\mathrm{End}_{I(\infty)}(\mathcal{F}(\infty)(\le 1))]$,
$4(K) := \Sigma_{x \text{ in } \mathbb{A}^1} \dim[\underline{\mathrm{End}}(\mathcal{F})^{I(x)}/(\mathcal{F}^{I(x)}\otimes(\mathcal{F}^\vee)^{I(x)})]$.

With this breakup, the equality

$$\mathrm{rig}(K) = \mathrm{rig}(FT(K)).$$

results from the

Theorem 3.0.4 Let p and ℓ be prime numbers, $\ell \neq p$. On $\mathbb{A}^1$ over an algebraically closed field of characteristic p, let K in $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ be a perverse sheaf on $\mathbb{A}^1$ such that both K and FT(K) are middle extensions from any nonempty open set of $\mathbb{A}^1$. Then

$$1(K) = 1(FT(K)),\ 2(K) = 2(FT(K)),\ 3(K) = 4(FT(K)),\ 4(K) = 3(FT(K)).$$

proof We begin with 1), a form of Parseval's formula which is valid for any object K in $D^b{}_c$. We denote by D_K the object $[x \mapsto -x]^*(DK)$. We have

$1(K) := \chi(\mathbb{A}^1, DK\otimes K) = \chi_c(\mathbb{A}^1, DK\otimes K) = \mathrm{rank}_0(D\_K*_{!+}K)$,

the last equality by using proper base change to calculate the stalk at zero of $D\_K*_{!+}K$.

Applying this to FTK gives

$1(FTK) = \mathrm{rank}_0(D\_FTK*_{!+}FTK) = \mathrm{rank}_0(FTDK*_{!+}FTK)$

(using $D\_\circ FT = FT\circ D$)

$= \mathrm{rank}_0(FT(DK\otimes K)[-1])$

(FT interchanges $\otimes$ and $*_{!+}$)

$= \chi(\mathbb{A}^1, DK\otimes K)$

(using the ! form of FT, and proper base change)

$:= 1(K)$, as required.

Assertion 2) follows from taking dimensions in the more precise equality

$$\mathrm{End}_{I(\infty)}(\mathcal{F}(\infty)(>1)) = \mathrm{End}_{I(\infty)}(\mathcal{G}(\infty)(>1)).$$

This holds because, in terms of Laumon's local Fourier Transform $FTloc(\infty,\infty)$, we have

$$\mathcal{G}(\infty)(>1) = FTloc(\infty,\infty)(\mathcal{F}(\infty)(>1)),$$

and one knows (cf. [Lau-TF], [Ka-TL], [Ka-ESDE, 7.4.1]) that $FTloc(\infty,\infty)$ is an autoequivalence of the category of $I(\infty)$-representations with all slopes >1.

The last two assertions,

$$3(K) = 4(FT(K)),\ 4(K) = 3(FT(K)),$$

are in fact equivalent. [They are obtained from each other by replacing K by FTK, and noting that $FT(FTK) \approx [x \mapsto -x]^*K$ up to a Tate twist, while visibly $4(K) = 4([x \mapsto -x]^*K)$.] We will prove that

$$4(K) = 3(FT(K)).$$

For this, we recall the theory of Laumon's local Fourier Transforms $FTloc(x,\infty)$, for x in $\mathbb{A}^1$. Each of these is an equivalence of categories

$I(x)$-representations $\approx$ $I(\infty)$-representations with all slopes <1.

By Laumon's theory of stationary phase [Ka-ESDE, 7.4], we know that

$$\mathcal{G}(\infty)(\leq 1) = \bigoplus_{x \text{ in } \mathbb{A}^1} \mathcal{L}_{\psi_x}\otimes FTloc(x,\infty)(\mathcal{F}(x)/\mathcal{F}(x)^{I(x)}).$$

[In [Ka-ESDE, 7.4.1], the functor $M \mapsto \mathcal{L}_{\psi_x}\otimes FTloc(x,\infty)(M)$ was called $FTloc(x,\infty)$]

There are no nonzero $I(\infty)$-equivariant maps between the distinct summands. [If M and N are any two $I(\infty)$-representations all of whose slopes are <1, then for $x \neq y$, $\mathrm{Hom}_{I(\infty)}(\mathcal{L}_{\psi_x}\otimes M, \mathcal{L}_{\psi_y}\otimes N)$ $= \mathrm{Hom}_{I(\infty)}(M, \mathcal{L}_{\psi_{y-x}}\otimes N) = 0$, since $\mathcal{L}_{\psi_{y-x}}\otimes N$ has all slopes $=1$, while M has all slopes <1.] Therefore

$$\mathrm{End}_{I(\infty)}(\mathcal{G}(\infty)(\leq 1)) = \oplus_{x \text{ in } \mathbb{A}^1} \mathrm{End}_{I(\infty)}(\mathrm{FTloc}(x,\infty)(\mathcal{F}(x)/\mathcal{F}(x)^{I(x)}).$$

Because $\mathrm{FTloc}(x,\infty)$ is an equivalence of categories

$I(x)$-representations $\approx$ $I(\infty)$-representations with all slopes <1,

we have

$$\mathrm{End}_{I(\infty)}(\mathrm{FTloc}(x,\infty)(\mathcal{F}(x)/\mathcal{F}(x)^{I(x)}) = \mathrm{End}_{I(x)}(\mathcal{F}(x)/\mathcal{F}(x)^{I(x)}).$$

Thus we obtain

$$\mathrm{End}_{I(\infty)}(\mathcal{G}(\infty)(\leq 1)) = \oplus_{x \text{ in } \mathbb{A}^1} \mathrm{End}_{I(x)}(\mathcal{F}(x)/\mathcal{F}(x)^{I(x)}).$$

To complete the proof, it remains only to check that

$$\dim[\mathrm{End}_{I(x)}(\mathcal{F}(x)/(\mathcal{F}^{I(x)}\otimes(\mathcal{F}^\vee)^{I(x)}))] = \dim[\mathrm{End}_{I(x)}(\mathcal{F}(x)/\mathcal{F}(x)^{I(x)})].$$

This is proven in the following section, Proposition 3.1.8. QED

3.1 Lemmas on representations of inertia groups

(3.1.1) Throughout this section, we fix a complete discrete valuation ring R with algebraically closed residue field k and with fraction field K. We denote by I the galois group

$$I := \mathrm{Gal}(K^{sep}/K).$$

If $\mathrm{char}(k) = p > 0$, we denote by $P \subset I$ the unique p-Sylow subgroup. If $\mathrm{char}(k) = 0$, we define $P = \{e\}$. The quotient I/P is (noncanonically) the pro-cyclic group $\prod_{\ell\neq p} \mathbb{Z}_\ell$.

(3.1.2) We also fix a prime number $\ell\neq\mathrm{char}(k)$. By an ℓ-adic representation of I, we mean a continuous $\overline{\mathbb{Q}}_\ell$-representation ρ of I on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space V, with the property that there exists a finite extension E_λ of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_\lambda$, a free $\mathcal{O}_\lambda$-module V_0, and a continuous $\mathcal{O}_\lambda$-representation ρ_0 of I on V_0 such that $(\rho_0, V_0)\otimes\overline{\mathbb{Q}}_\ell \approx (\rho, V)$.

(3.1.3) An ℓ-adic representation of I is said to be tame if it is trivial on P. It is said to be unipotent if it is a successive extension of trivial representations of I. Because the upper unipotent subgroup of $GL(n, \mathcal{O}_\lambda)$ is pro-ℓ, any unipotent representation of I is tame. If we fix a topological generator γ of I/P, then isomorphism classes of n-dimensional unipotent representations of I are in bijective correspondence with conjugacy classes of unipotent elements in $GL(n, \overline{\mathbb{Q}}_\ell)$. By the theory of Jordan normal form, there is, up to isomorphism, a unique indecomposable unipotent representation of I of each dimension $n \geq 1$, which we denote Unip(n). We sometimes refer to Unip(n) as the "standard Jordan block of size n".

Lemma 3.1.4 If M and N are inequivalent irreducible ℓ-adic representations of I, then $\mathrm{Ext}_I{}^i(M, N) = 0$ for all i. In particular, $\mathrm{Hom}_I(M, N) = \mathrm{Ext}_I{}^1(M, N) = 0$.

proof $\mathrm{Ext}_I{}^i(M, N) = H^i(I, M^\vee \otimes N) = H^i(I/P, (M^\vee \otimes N)^P)$. In terms of a topological generator γ of I/P, we have

$$H^i(I/P, (M^\vee \otimes N)^P) = \mathrm{Ker}(1-\gamma \mid (M^\vee \otimes N)^P) \text{ if } i = 0,$$
$$= \mathrm{Coker}(1-\gamma \mid (M^\vee \otimes N)^P) \text{ if } i = 1,$$
$$= 0 \text{ for } i \geq 2.$$

This explicit description shows that $\chi(I, M^\vee \otimes N) = 0$. Therefore we have $h^0(I, M^\vee \otimes N) = h^1(I, M^\vee \otimes N)$, i.e.,

$$\dim \mathrm{Ext}_I{}^1(M, N) = \dim \mathrm{Hom}_I(M, N).$$

Since M and N are irreducible and inequivalent, $\mathrm{Hom}_I(M, N) = 0$. QED

Lemma 3.1.5 If N is an irreducible nontrivial ℓ-adic representation of I, then $H^i(I, N) = 0$ for all i.

proof Take M to be the trivial representation in Lemma 3.1.4. QED

Lemma 3.1.6 Any ℓ-adic representation M of I has a canonical direct sum "isotypical" decomposition, indexed by the equivalence classes $\mathcal{A}$ of irreducible representations of I, as $\oplus_{\alpha \text{ in } \mathcal{A}} M_\alpha$, where M_α is a successive extension of irreducible representations all of type α.

proof This is immediate from Lemma 3.1.4. QED

Lemma 3.1.7 Let N be an irreducible ℓ-adic representation of I.
1) Given any two unipotent representations U_1 and U_2 of I, the natural map
"$\otimes N$" defines isomorphisms

$$\mathrm{Hom}_I(U_1, U_2) \approx \mathrm{Hom}_I(N \otimes U_1, N \otimes U_2),$$
$$\mathrm{Ext}_I(U_1, U_2) \approx \mathrm{Ext}_I(N \otimes U_1, N \otimes U_2).$$

2) Given any ℓ-adic representation M of I which is a successive

extension of N by itself, there exists a unipotent representation U of I and an isomorphism $M \approx N \otimes U$ of I-representations. We can recover U intrinsically as the "$\mathbb{1}$-isotypical" component of $N^\vee \otimes M$.
3) Let M be an indecomposable ℓ-adic representation M of $\underline{I}$ which is a successive extension of N by itself. If M has length m as $\overline{\mathbb{Q}}_\ell[I]$-module, then $M \approx N \otimes \mathrm{Unip}(m)$, where Unip(m) is the standard Jordan block of size m.
proof 1) Given U_1 and U_2 unipotent,

$$\mathrm{Ext}^i{}_I(N \otimes U_1, N \otimes U_2) = H^i(I, N^\vee \otimes N \otimes (U_1{}^\vee \otimes U_2)).$$

Because N is irreducible, $N^\vee \otimes N$ is semisimple, and

$$N^\vee \otimes N \approx \mathbb{1} \oplus R, \text{ with } R \approx \oplus \text{ of nontrivial irreducibles.}$$

Because $U_1{}^\vee \otimes U_2$ is unipotent,$R \otimes (U_1{}^\vee \otimes U_2)$ is a successive extension of nontrivial irreducibles, and hence $H^i(I, R \otimes (U_1{}^\vee \otimes U_2)) = 0$ for all i, by Lemma 3.1.5. Therefore the inclusion of $\mathbb{1}$ into $N^\vee \otimes N$ induces isomorphisms

$$H^i(I, U_1{}^\vee \otimes U_2) \approx H^i(I, N^\vee \otimes N \otimes (U_1{}^\vee \otimes U_2)) \text{ for all i.}$$

If we view these groups as Ext groups, then this isomorphism is that induced by "$\otimes N$"

$$\mathrm{Ext}^i{}_I(U_1, U_2) \to \mathrm{Ext}^i{}_I(N \otimes U_1, N \otimes U_2) \text{ for all i.}$$

2) Once we have 1), the first assertion of 2) is proven by induction on the length of M. Once $M \approx N \otimes U$, we have, just as above,

$$N^\vee \otimes M = N^\vee \otimes N \otimes U = (\mathbb{1} \oplus R) \otimes U = U \oplus R \otimes U,$$

with $R \otimes U$ a successive extension of nontrivial irreducibles.
3) If $N \otimes U$ is indecomposable of length m, then U is indecomposable of length m, hence is Unip(m). QED

Proposition 3.1.8 Let M be an ℓ-adic representation of I. Then

$$\dim[\mathrm{End}_I(M)/(M^I \otimes (M^\vee)^I)] = \dim[\mathrm{End}_I(M/M^I)].$$

This will be proven in a series of Lemmas.

Lemma 3.1.9 If Proposition 3.1.8 holds for all unipotent ℓ-adic representations of I, then it holds for all ℓ-adic representations of I.
proof Let M be an ℓ-adic representations of I. In terms of the "isotypical" decomposition of M as $\oplus_{\alpha \text{ in } \mathcal{A}} M_\alpha$, we have

$$\mathrm{End}_I(M) = \oplus_{\alpha \text{ in } \mathcal{A}} \mathrm{End}_I(M_\alpha).$$

Moreover, if we denote by $\mathbb{1}$ the trivial representation of I, and by $M_{\mathbb{1}}$ the corresponding "isotypical" summand (thus $M_{\mathbb{1}}$ is the maximal unipotent subrepresentation of M), then clearly

$$M^I = (M_{\mathbb{1}})^I, \text{ and } (M^\vee)^I = ((M^\vee)_{\mathbb{1}})^I = (M_{\mathbb{1}})^\vee)^I.$$

Thus the "isotypical" decomposition of M/M^I is

$$M/M^I = \oplus_{\alpha \neq \mathbb{1} \text{ in } \mathcal{A}} M_\alpha \oplus M_{\mathbb{1}}/(M_{\mathbb{1}})^I,$$

and hence

$$\mathrm{End}_I(M/M^I) = \oplus_{\alpha \neq \mathbb{1} \text{ in } \mathcal{A}} \mathrm{End}_I(M_\alpha) \oplus \mathrm{End}_I(M_{\mathbb{1}}/(M_{\mathbb{1}})^I).$$

On the other hand,

$$M^I \otimes (M^\vee)^I = (M_{\mathbb{1}})^I \otimes ((M^\vee)_{\mathbb{1}})^I = (M_{\mathbb{1}})^I \otimes ((M_{\mathbb{1}})^\vee)^I,.$$

and hence

$$\mathrm{End}_I(M)/(M^I \otimes (M^\vee)^I) = \oplus_{\alpha \neq \mathbb{1} \text{ in } \mathcal{A}} \mathrm{End}_I(M_\alpha) \oplus$$
$$\oplus \mathrm{End}_I(M_{\mathbb{1}})/((M_{\mathbb{1}})^I \otimes ((M_{\mathbb{1}})^\vee)^I).$$

Thus we see that Proposition 3.1.8 holds for M if and only if it holds for $M_{\mathbb{1}}$. QED

(3.1.10) We now study the case in which M is a **unipotent** ℓ-adic representation U of I, so a direct sum of standard Jordan blocks $\oplus \mathrm{Unip}(n_i)$ of varying sizes $n_i \geq 1$. Given a unipotent representation U of I, we define a sequence of non=negative integers $e_i = e_i(U)$,

$$e_1 \geq e_2 \geq e_3 \geq e_4 \geq \ldots, e_r = 0 \text{ for } r >> 0,$$

by

$$e_i := \text{the number of Jordan blocks of dimension} \geq i.$$

It is obvious from this definition that given two unipotent representations U_1 and U_2 of I, we have

$$e_i(U_1 \oplus U_2) = e_i(U_1) + e_i(U_2).$$

For a single unipotent block Unip(n), we have

$$e_i = 1 \text{ for } i \leq n,$$
$$e_i = 0 \text{ for } i > n.$$

So for $M = \oplus \mathrm{Unip}(n_i)$, the $e_i(M)$ are the partition of dim(M), written in decreasing order, which is dual to the partition of dim(M) given by the block sizes n_i. We call the sequence $(e_1, e_2, e_3,\ldots)$ attached to a unipotent M its "dual partition".

(3.1.11) Clearly if M is unipotent, with dual partition $(e_1, e_2, e_3, e_4,\ldots)$, then M/M^I has dual partition $(e_2, e_3, e_4,\ldots)$ formed from that of M by discarding the leading term. [Indeed, if M = Unip(n), then M/M^I is 0 if n=1, and is Unip(n-1) if n > 1.]

(3.1.12) Notice that if M is a single Jordan block Unip(n), then $M^\vee$ is also a single Jordan block Unip(n). So if $M \approx \oplus\mathrm{Unip}(n_i)$, then also $M^\vee \approx \oplus\mathrm{Unip}(n_i)$. From this we see that

$$\dim(M^I) = \text{the number of Jordan blocks in } M := e_1(M),$$

$$\dim((M^\vee)^I) = e_1(M^\vee) = e_1(M).$$

Remark 3.1.13 One could **define** the dual partition inductively by the properties

$$e_1(M) = \dim(M^I),\ e_{i+1}(M) = e_i(M/M^I), \text{ for } i \geq 1.$$

Lemma 3.1.14 If M and N are unipotent ℓ-adic representations of I, with dual partitions $(e_1, e_2, e_3, e_4,\ldots)$ and $(f_1, f_2, f_3, f_4,\ldots)$, then

$$\dim[\mathrm{Hom}_I(M, N)] = \Sigma_i\, e_i f_i.$$

proof Both sides are additive over direct sums, so it suffices to check in the case when M is Unip(m) and N is Unip(n). In this case the assertion is that

$$\dim[\mathrm{Hom}_I(\mathrm{Unip}(m), \mathrm{Unip}(n))] = \min(m,n).$$

In terms of the action of

$$T := (\text{a topological generator of } I/P) - 1,$$

this is the assertion that over any field E, we have

$$\dim_E \mathrm{Hom}_{E[T]}(E[T]/(T^m), E[T]/(T^n)) = \min(m,n).$$

But by the map $\varphi \mapsto \varphi(1)$,

$\mathrm{Hom}_{E[T]}(E[T]/(T^m), E[T]/(T^n)) \approx$ kernel of T^m in $E[T]/(T^n)$.

$= E[T]/(T^n)$, if $m \geq n$,

$= T^{n-m}E[T]/(T^n) \approx E[T]/(T^m)$ if $n \geq m$. QED

Lemma 3.1.15 If M is a unipotent ℓ-adic representation of I, with dual partition $(e_1, e_2, e_3, e_4,\ldots)$, then

$$\dim[\mathrm{End}_I(M)] = \Sigma_i\, (e_i)^2.$$

proof Take N=M above. QED

Lemma 3.1.16 If M is a unipotent ℓ-adic representation of I,

$$\dim[\mathrm{End}_I(M/M^I)] = \dim[\mathrm{End}_I(M)/(M^I \otimes (M^\vee)^I)].$$

proof If M has dual partition $(e_1, e_2, e_3, e_4, \ldots)$, then M/M^I has dual partition $(e_2, e_3, e_4, \ldots)$, and $\dim(M^I \otimes (M^\vee)^I) = (e_1)^2$, so both sides are equal to $\Sigma_{i\geq 2} (e_i)^2$. QED

In view of Lemma 3.1.9, this completes the proof of Proposition 3.1.8.

3.2 Interlude: the operation $\otimes_{\mathrm{mid}}$

(3.2.1) Let k be a field, X a separated k-scheme of finite type which is irreducible and smooth, everywhere of relative dimension n. Let ℓ be a prime number $\neq$char(k). We look at the full subcategory ME(X) of (ℓ-adic) Perv(X) consisting of those objects which are middle extensions from every nonempty affine open set, i.e., those perverse objects K on X such that for every nonempty affine open set $j:U \to X$, we have $K \cong j_{!*}j^*K$. Given K in ME, we say that K has **generic rank** r if for some (or equivalently, for any) nonempty affine open set $j: U \to X$ on which it is lisse, j^*K is $\mathcal{F}[n]$, with $\mathcal{F}$ a lisse sheaf on U of rank r.

(3.2.2) We define an operation

$$\otimes_{\mathrm{mid}} : \mathrm{ME}(X) \times \mathrm{ME}(X) \to \mathrm{ME}(X)$$

as follows. Given any two objects K and L in ME, pick a common nonempty affine open set $j: U \to X$ on which both K and L are lisse. Thus $j^*K = \mathcal{F}[n]$ and $j^*L = \mathcal{G}[n]$ for lisse sheaves $\mathcal{F}$ and $\mathcal{G}$ on U. We define

$$K\otimes_{\mathrm{mid}}L := j_{!*}((\mathcal{F}\otimes\mathcal{G})[n]) = j_{!*}((j^*K\otimes j^*L)[-n]).$$

This is easily seen to be independent of the auxiliary choice of U, using 2.3.3.1.

Lemma 3.2.3 Let k be a field, X a separated k-scheme of finite type which is irreducible and smooth, everywhere of relative dimension n. Let ℓ be a prime number $\neq$char(k).

1) An object K in ME is perverse irreducible if and only if it is irreducible in ME.

2) Let L in ME have generic rank one. Then

2a)The operation $K \mapsto K\otimes_{\mathrm{mid}}L$ is an autoequivalence of ME with

itself, whose inverse is $K \to K\otimes_{mid}DL$.
2b) The operation $K \mapsto K\otimes_{mid}L$ induces an automorphism of the family of all irreducible perverse sheaves in ME, whose inverse is $K \to K\otimes_{mid}DL$.
proof 1) Since ME is a full subcategory of Perv, any perverse irreducible is an irreducible object of ME. Conversely, suppose that K in ME is not perverse irreducible. Then there exists a perverse irreducible L, and a nonzero element in $Hom_X(L, K)$. If L is in ME, K is not irreducible in ME. If not, then L is supported on a proper closed subvariety Y of X. So there exists an affine open $j :U \to X$ with $j^*L = 0$. But in Perv(X), we have $K = j_{!*}j^*K \subset Rj_*j^*K$, so

$$Hom_X(L, K) \subset Hom_X(L, Rj_*j^*K) = Hom_U(j^*L, j^*K) = 0,$$

contradiction.
2a) is obvious by 2.3.3.1, and by 1) it implies 2b). QED

Lemma 3.2.4 Let ℓ be a prime number. On $\mathbb{A}^1$ over an algebraically closed field of characteristic $\neq\ell$, let L in $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ be a perverse sheaf on $\mathbb{A}^1$ which lies in ME. Suppose that L has generic rank one. Then
1)If K in ME is perverse irreducible, so is $K\otimes_{mid}L$.
2) For any K in ME, $rig(K) = rig(K\otimes_{mid}L)$.
proof Assertion 1) is part 2b) of the preceding lemma. Assertion 2) is obvious from the definitions: if $j: U \to \mathbb{A}^1$ is any nonempty open set on which both K and L are lisse, say $j^*K = \mathcal{F}[1]$ and $j^*L = \mathcal{L}[1]$, with $\mathcal{F}$ lisse of some rank r, and $\mathcal{L}$ lisse of rank one, then on U the sheaves $\underline{End}(\mathcal{F})$ and $\underline{End}(\mathcal{F}\otimes\mathcal{L})$ coincide. QED

3.3 Applications to middle additive convolution

Lemma 3.3.1 Let p and ℓ be prime numbers, $\ell\neq p$. On $\mathbb{A}^1$ over an algebraically closed field of characteristic p, let K and L in $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ be perverse sheaves. Then
1) K is in $\mathcal{P}$ if and only if FTK is in ME.
2) If K and L are both in $\mathcal{P}$, then

$$FT(K*_{mid+}L) = FTK\otimes_{mid}FTL.$$

proof This was proven in 2.10.3 and 2.10.5 above. QED

Lemma 3.3.2 Let p and ℓ be prime numbers, $\ell \neq p$. On $\mathbb{G}_m$ over an algebraically closed field of characteristic $p > 0$, let χ be any continuous nontrivial $\overline{\mathbb{Q}}_\ell^\times$-valued character of $\pi_1(\mathbb{G}_m)^{tame}$, $\mathcal{L}_\chi$ the corresponding Kummer sheaf on $\mathbb{G}_m$, and $j_*\mathcal{L}_\chi$ its extension by direct image to $\mathbb{A}^1$. Then $FT(j_*\mathcal{L}_\chi[1]) = j_*\mathcal{L}_{\overline{\chi}}[1]$ (geometrically).
proof Direct calculation. QED

Theorem 3.3.3 Let p and ℓ be prime numbers, $\ell \neq p$. Over an algebraically closed field of characteristic p, let χ be any continuous nontrivial $\overline{\mathbb{Q}}_\ell^\times$-valued character of $\pi_1(\mathbb{G}_m)^{tame}$. Let K in $D^b_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ be perverse irreducible on $\mathbb{A}^1$.
1) If K is not in $\mathcal{P}$, then
1a) If K is $\overline{\mathbb{Q}}_\ell[1]$, then $K*_{mid+}j_*\mathcal{L}_\chi[1] = 0$.
1b) If K is $\mathcal{L}_{\psi_\alpha}[1]$ with $\alpha \neq 0$, then $K*_{mid+}j_*\mathcal{L}_\chi[1] = \mathcal{L}_{\psi_\alpha}[1]$.
2) If K is in $\mathcal{P}$, then
2a) If K is punctual, say δ_α, $K*_{mid+}j_*\mathcal{L}_\chi[1] = j_*\mathcal{L}_{\chi(x-\alpha)}[1]$.
2b) If K is $j_*\mathcal{L}_{\overline{\chi}(x-\alpha)}[1]$ for some α, then $K*_{mid+}j_*\mathcal{L}_\chi[1] = \delta_\alpha$.
2c) If K is $j_*\mathcal{L}_{\rho(x-\alpha)}[1]$ for some α and some $\rho \neq \overline{\chi}$, then $K*_{mid+}j_*\mathcal{L}_\chi[1] = j_*\mathcal{L}_{\rho\chi(x-\alpha)}[1]$
2d) If K is a perverse irreducible in $\mathcal{P}$ which is not of type 2a, 2b, or 2c, then $K*_{mid+}j_*\mathcal{L}_\chi[1]$ is a perverse irreducible in $\mathcal{P}$ which is not of type 2a, 2b, or 2c.
3) If K is in $\mathcal{P}$ and of type 2d), then
$$\mathrm{rig}(K) = \mathrm{rig}(K*_{mid+}j_*\mathcal{L}_\chi[1]).$$

proof Assertion 1) is proven by direct calculation. Assertion 2a) is proven by direct calculation. We know that middle convolution $*_{mid+}j_*\mathcal{L}_\chi[1]$ with $j_*\mathcal{L}_\chi[1]$ is an automorphism of the irreducibles in $\mathcal{P}$, with inverse $*_{mid+}j_*\mathcal{L}_{\overline{\chi}}[1]$. Using this, 2b) follows from 2a). We also know also that
$$\begin{aligned} j_*\mathcal{L}_\chi[1]*_{mid+}j_*\mathcal{L}_\rho[1] &= j_*\mathcal{L}_{\chi\rho}[1] \text{ if } \chi\rho \neq \mathbb{1}, \\ &= \delta_0 \text{ if } \chi\rho = \mathbb{1}. \end{aligned}$$
Using the associativity of middle convolution, and 2a), we get 2c).

To prove 2d), observe first that for each α, the family of all those irreducibles in $\mathcal{P}$ which are either δ_α or $j_*\mathcal{L}_{\rho(x-\alpha)}[1]$ for some $\rho \neq \mathbb{1}$ is simply the **orbit** of δ_α under the group of operators given by middle convolution with δ_0 and with any $j_*\mathcal{L}_\chi[1]$, for any $\chi \neq \mathbb{1}$. Since the family of all irreducibles in $\mathcal{P}$ is also stable by this group of operators, the complement of a union of orbits is also stable.

To prove 3), we note that by 2d, both sides are middle extensions. Since both sides are also in $\mathcal{P}$, their Fourier Transforms are middle extensions. So by Theorem 3.0.2, we have

$$\mathrm{rig}(K) = \mathrm{rig}(FTK),$$

and

$$\mathrm{rig}(K*_{mid+}j_*\mathcal{L}_\chi[1]) = \mathrm{rig}(FT(K*_{mid+}j_*\mathcal{L}_\chi[1]))$$
$$= \mathrm{rig}(FTK\otimes_{mid}FT(j_*\mathcal{L}_\chi[1])), \text{ by 3.3.1.2}$$
$$= \mathrm{rig}(FTK\otimes_{mid}j_*\mathcal{L}_{\overline{\chi}}[1])), \text{ by 3.3.2}$$
$$= \mathrm{rig}(FTK), \text{ by 3.2.4.2.} \qquad \text{QED}$$

(3.3.4) We now turn to a discussion of the monodromy of middle convolution with a $j_*\mathcal{L}_\chi[1]$. For the convenience of the reader, we recall some of the basic facts about Laumon's local Fourier Transforms.

Theorem 3.3.5 Hypotheses and notations as in the theorem 3.3.3 above, suppose that K is a perverse irreducible in $\mathcal{P}$ of type 2d). Pick a nonempty open set $j: U \to \mathbb{A}^1$ on which both K and $K*_{mid+}j_*\mathcal{L}_\chi[1]$ are lisse, say

$$K = j_{!*}\mathcal{F}[1],\ K*_{mid+}j_*\mathcal{L}_\chi[1] = j_{!*}\mathcal{G}[1]$$

with $\mathcal{F}$ and $\mathcal{G}$ lisse sheaves on U. Then the local monodromies of $\mathcal{F}$ and of $\mathcal{G}$ are related as follows:

1) at s in $\mathbb{A}^1$ - U:

$$FT_\psi loc(s, \infty)(\mathcal{F}(s)/\mathcal{F}(s)^{I(s)})\otimes\mathcal{L}_{\overline{\chi}} = FT_\psi loc(s, \infty)(\mathcal{G}(s)/\mathcal{G}(s)^{I(s)}).$$

1a) In particular, $\mathcal{F}$ is lisse at s if and only if $\mathcal{G}$ is lisse at s.

2) at ∞:

decompose $\mathcal{F}(\infty) = \mathcal{F}(\infty)(\text{slopes} > 1) \oplus (\oplus_{s \text{ in } \mathbb{A}^1} \mathcal{L}_{\psi_s}\otimes\mathcal{F}(\infty,s))$

with $\mathcal{F}(\infty,s)$ an $I(\infty)$-representation with all slopes < 1, and similarly for $\mathcal{G}$. Then

2a) We have the formula

$$FT_\psi loc(\infty, \infty)(\mathcal{F}(\infty)(\text{slopes} > 1))\otimes\mathcal{L}_{\overline{\chi}} = FT_\psi loc(\infty, \infty)(\mathcal{G}(\infty)(\text{slopes} > 1)).$$

2b) For each $s \neq 0$ in $\mathbb{A}^1$,

$$\mathcal{F}(\infty,s) \approx \mathcal{G}(\infty,s).$$

2c) There exists an I(0)- representation M(0) with

$$\mathcal{F}(\infty,0) = FT_{\overline{\psi}}loc(0, \infty)(M(0)/M(0)^{I(0)})$$

$$\mathcal{G}(\infty,0) = FT_{\overline{\psi}}loc(0, \infty)(M(0)\otimes\mathcal{L}_{\overline{\chi}}/(M(0)\otimes\mathcal{L}_{\overline{\chi}})^{I(0)}).$$

2d) In particular, cf. [Ka-ESDE, 7.4.4], $\mathcal{F}$ is tame at ∞ if and only if $\mathcal{G}$ is tame at ∞.

proof This is just the spelling out of identities

$$FT_{\psi}(K*_{mid+}L) = FT_{\psi}K\otimes_{mid}FT_{\psi}L,$$
$$FT_{\overline{\psi}}(FT_{\psi}K\otimes_{mid}FT_{\psi}L) = K*_{mid+}L,$$
$$FT_{\overline{\psi}}(FT_{\psi}K) = K.$$

with $L = j_*\mathcal{L}_{\chi}[1]$, together with the effect of FT upon local monodromies, as expressed via Laumon's FTloc(s, ∞) and FTloc(∞, ∞) functors.

Concretely, let us write $FT_{\psi}K = j_{!*}(\mathcal{A}[1])$, for some sufficiently small nonempty open set $j: V \to \mathbb{A}^1$ and some lisse sheaf $\mathcal{A}$ on V. Then $FT_{\psi}(K*_{mid+}L) = j_{!*}(\mathcal{B}[1])$, with $\mathcal{B} = \mathcal{A}\otimes j^*\mathcal{L}_{\overline{\chi}}$. Their I($\infty$)=representations are related by $\mathcal{B}(\infty) = \mathcal{A}(\infty)\otimes\mathcal{L}_{\overline{\chi}}$, so decomposing them we find

$$\mathcal{B}(\infty)(\text{slope} > 1) = \mathcal{A}(\infty)(\text{slope} > 1)\otimes\mathcal{L}_{\overline{\chi}},$$

and

$$\mathcal{B}(\infty, s) = \mathcal{A}(\infty, s)\otimes\mathcal{L}_{\overline{\chi}}, \text{ for every s in } \mathbb{A}^1.$$

By Laumon (cf. [Ka-ESDE, 7.4.2]) we know that

$$FT_{\psi}loc(s, \infty)(\mathcal{F}(s)/\mathcal{F}(s)^{I(s)}) = \mathcal{A}(\infty, s),$$

$$FT_{\psi}loc(s, \infty)(\mathcal{G}(s)/\mathcal{G}(s)^{I(s)}) = \mathcal{B}(\infty, s) = \mathcal{A}(\infty, s)\otimes\mathcal{L}_{\overline{\chi}},$$

whence

$$FT_{\psi}loc(s, \infty)(\mathcal{F}(s)/\mathcal{F}(s)^{I(s)})\otimes\mathcal{L}_{\overline{\chi}} = FT_{\psi}loc(s, \infty)(\mathcal{G}(s)/\mathcal{G}(s)^{I(s)}).$$

By Laumon, we also know

$$FT_{\psi}loc(\infty, \infty)(\mathcal{F}(\infty)(\text{slope} > 1)) = \mathcal{A}(\infty)(\text{slope} > 1),$$
$$FT_{\psi}loc(\infty, \infty)(\mathcal{G}(\infty)(\text{slope} > 1)) = \mathcal{B}(\infty)(\text{slope} > 1).$$
$$= \mathcal{A}(\infty)(\text{slope} > 1)\otimes\mathcal{L}_{\overline{\chi}},$$

whence

$$FT_{\psi}loc(\infty, \infty)(\mathcal{F}(\infty)(\text{slopes} > 1))\otimes\mathcal{L}_{\overline{\chi}} = FT_{\psi}loc(\infty, \infty)(\mathcal{G}(\infty)(\text{slopes} > 1)).$$

To prove the remaining assertions, we use the inversion formulas

$$FT_{\overline{\psi}}(FT_{\psi}K\otimes_{mid}FT_{\psi}L) = K*_{mid+}L,$$

$$FT_{\overline{\psi}}(FT_{\psi}K) = K.$$

Since our FTL is lisse of rank one on $\mathbb{G}_m$, FTK and $FTK\otimes_{mid}FTL$ give isomorphic representations of I(s) for each $s\neq 0$ in $\mathbb{A}^1$:

$$\mathcal{A}(s) \approx \mathcal{B}(s) \text{ for } s \neq 0 \text{ in } \mathbb{A}^1.$$

By stationary phase,

$$\mathcal{F}(\infty,s) = FT_{\overline{\psi}}loc(s, \infty)(\mathcal{A}(s)/\mathcal{A}(s)^{I(s)}), \text{ and}$$

$$\mathcal{G}(\infty,s) = FT_{\overline{\psi}}loc(s, \infty)(\mathcal{B}(s)/\mathcal{B}(s)^{I(s)}), \text{ for every } s \text{ in } \mathbb{A}^1.$$

But $\mathcal{A}(s) \approx \mathcal{B}(s)$ for $s \neq 0$ in $\mathbb{A}^1$, so $\mathcal{F}(\infty,s) = \mathcal{G}(\infty,s)$ for $s \neq 0$ in $\mathbb{A}^1$.

For $s=0$,

$$\mathcal{A}(0)\otimes\mathcal{L}_{\overline{\chi}} \approx \mathcal{B}(0),$$

so with $M(0) := \mathcal{A}(0)$ we get

$$\mathcal{F}(\infty,0) = FT_{\overline{\psi}}loc(0, \infty)(M(0)/M(0)^{I(0)})$$

$$\mathcal{G}(\infty,0) = FT_{\overline{\psi}}loc(0, \infty)(M(0)\otimes\mathcal{L}_{\overline{\chi}}/(M(0)\otimes\mathcal{L}_{\overline{\chi}})^{I(0)}).$$

QED

Corollary 3.3.6 Hypotheses and notations as in 3.3.3 above, suppose in addition that K is tamely ramified everywhere. Then so is $K*_{mid+}j_*\mathcal{L}_\chi[1]$, and their local monodromies are related as follows:

1) at s in $\mathbb{A}^1$ - U:

$$(\mathcal{F}(s)/\mathcal{F}(s)^{I(s)})\otimes\mathcal{L}_{\chi(x-s)} \approx \mathcal{G}(s)/\mathcal{G}(s)^{I(s)}.$$

2) at ∞: there exists a tame $I(\infty)$-representation $M(\infty)$ with

$$\mathcal{F}(\infty) = M(\infty)/M(\infty)^{I(\infty)},$$

$$\mathcal{G}(\infty) = M(\infty)\otimes\mathcal{L}_\chi/(M(\infty)\otimes\mathcal{L}_\chi)^{I(\infty)}.$$

Moreover,

3) $M(\infty)$ is the unique $I(\infty)$-representation with these properties.

4) We have the formulas

$$\text{rank } M(\infty) = \Sigma_{s \text{ in } \mathbb{A}^1-U} \text{ rank } (\mathcal{F}(s)/\mathcal{F}(s)^{I(s)}),$$

$$\text{rank } M(\infty) = \Sigma_{s \text{ in } \mathbb{A}^1-U} \text{ rank } (\mathcal{G}(s)/\mathcal{G}(s)^{I(s)}).$$

proof At s in $\mathbb{A}^1$ - U, we have

$$FT_{\psi}loc(s, \infty)(\mathcal{F}(s)/\mathcal{F}(s)^{I(s)})\otimes\mathcal{L}_{\overline{\chi}} = FT_{\psi}loc(s, \infty)(\mathcal{G}(s)/\mathcal{G}(s)^{I(s)}).$$

Since $\mathcal{F}(s)$ is tame, and $FT_{\psi}loc(s, \infty)$ and its quasi-inverse both carry tames to tames, we see that $\mathcal{G}(s)/\mathcal{G}(s)^{I(s)}$ is tame, whence $\mathcal{G}(s)$ itself is tame. Moreover, for M a tame I(s)-representation, we have [Ka-ESDE, 7.4.1.3]

$$FT_{\psi}loc(s, \infty)(M)\otimes\mathcal{L}_{\bar{\chi}} = FT_{\psi}loc(s, \infty)(M\otimes\mathcal{L}_{\chi(x-s)}).$$

Indeed, for M tame, we have [Ka-ESDE, 7.4.1.3 and 7.4.4.3]

$$FT_{\psi}loc(s, \infty)(M) = [x-s \mapsto 1/(x-s)]^{*}M \text{ as } I(\infty) \text{ representation.}$$

So from the above isomorphism of $I(\infty)$-representations we infer

$$(\mathcal{F}(s)/\mathcal{F}(s)^{I(s)})\otimes\mathcal{L}_{\chi(x-s)} = \mathcal{G}(s)/\mathcal{G}(s)^{I(s)}.$$

We now turn to the situation at ∞. Since K is tame at ∞, we have $\mathcal{F}(\infty)(\text{slopes} > 1) = 0$. Therefore $FT_{\psi}loc(\infty, \infty)(\mathcal{G}(\infty)(\text{slopes} > 1)) = 0$, and hence $\mathcal{G}(\infty)(\text{slopes} > 1) = 0$. For $s\neq 0$, $\mathcal{F}(\infty,s) = 0$, hence $\mathcal{G}(\infty,s)=0$. For $s=0$, any $I(0)$- representation $M(0)$ with

$$\mathcal{F}(\infty,0) = FT_{\bar{\psi}}loc(0, \infty)(M(0)/M(0)^{I(0)})$$

must be tame [Ka-ESDE, 7.4.4.4]. Moreover, for tame $I(0)$-representations, we have [Ka-ESDE, 7.4.4.3]

$$FT_{\bar{\psi}}loc(0, \infty)(M(0)) = [x \mapsto 1/x]^{*}M(0)$$

as $I(\infty)$-representations. We take our $M(\infty)$ to be $[x \mapsto 1/x]^{*}M(0)$ for $M(0)$ appearing in the previous theorem (i.e., if $FT_{\psi}K = j_{!*}\mathcal{A}[1]$, for a sufficiently small nonempty open set $j: V \to \mathbb{A}^1$ and a lisse sheaf $\mathcal{A}$ on V, then $M(0)$ is $\mathcal{A}(0)$).

To show that $M(\infty)$ is uniquely determined as $I(\infty)$-representation by the two conditions:

$$\mathcal{F}(\infty) = M(\infty)/M(\infty)^{I(\infty)},$$

$$\mathcal{G}(\infty) = M(\infty)\otimes\mathcal{L}_{\chi}/(M(\infty)\otimes\mathcal{L}_{\chi})^{I(\infty)},$$

write the "isotypical decomposition" of $M(\infty)$, (cf. Lemma 3.1.6), say

$$M(\infty) = \bigoplus_{\alpha \text{ in } \mathcal{A}} M(\infty)_{\alpha}.$$

By the first condition,

$$M(\infty)_{\alpha} \approx \mathcal{F}(\infty)_{\alpha} \text{ for } \alpha \neq \mathbb{1}.$$

By the second,

$$(M(\infty)\otimes\mathcal{L}_{\chi})_{\alpha} \approx \mathcal{G}(\infty)_{\alpha} \text{ for } \alpha \neq \mathbb{1},$$

which we rewrite as

$$(M(\infty)\otimes\mathcal{L}_{\chi})_{\alpha\otimes\chi} \approx \mathcal{G}(\infty)_{\alpha\otimes\chi} \text{ for } \alpha\otimes\chi \neq \mathbb{1},$$

i.e.,

$$M(\infty)_{\alpha} \approx (\mathcal{G}(\infty)\otimes\mathcal{L}_{\bar{\chi}})_{\alpha} \text{ for } \alpha \neq \bar{\chi}.$$

In particular,

$$M(\infty)_{\mathbb{1}} \approx (\mathcal{G}(\infty)\otimes\mathcal{L}_{\bar{\chi}})_{\mathbb{1}}.$$

Thus

$$M(\infty) = \oplus_{\alpha\neq \mathbb{1}} \mathcal{F}(\infty)_\alpha \oplus (\mathcal{G}(\infty)\otimes \mathcal{L}_{\overline{\chi}})_{\mathbb{1}}.$$

To compute the rank of $M(\infty)$, recall that if we write $FT_\psi K = j_{!*}(\mathcal{A}[1])$, for a sufficiently small nonempty open set $j: V \to \mathbb{A}^1$ and a lisse sheaf $\mathcal{A}$ on V, then $M(\infty)$ is $[x \mapsto 1/x]^*\mathcal{A}(0)$. By stationary phase,

$$\mathcal{A}(\infty)(\le 1) = \oplus_{x \text{ in } \mathbb{A}^1} \mathcal{L}_{\psi_x}\otimes FTloc(x,\infty)(\mathcal{F}(x)/\mathcal{F}(x)^{I(x)}).$$

$$\mathcal{A}(\infty)(>1) = FTloc(\infty,\infty)(\mathcal{F}(\infty)(> 1)) = 0, \text{ (since } \mathcal{F}(\infty) \text{ is tame).}$$

Since each $\mathcal{F}(x)$ is tame, we have

$$\text{rank } FTloc(x,\infty)(\mathcal{F}(x)/\mathcal{F}(x)^{I(x)}) = \text{rank}(\mathcal{F}(x)/\mathcal{F}(x)^{I(x)}).$$

Thus

$$\text{rank } M(\infty) = \text{rank } \mathcal{A}(0) = \text{rank } \mathcal{A} = \text{rank } \mathcal{A}(\infty) =$$

$$= \Sigma_{x \text{ in } \mathbb{A}^1} \text{rank}(\mathcal{F}(x)/\mathcal{F}(x)^{I(x)}) = \Sigma_{s \text{ in } \mathbb{A}^1-U} \text{rank } (\mathcal{F}(s)/\mathcal{F}(s)^{I(s)}).$$

The second formula for rank $M(\infty)$ is term by term equal to this one: equate ranks in the isomorphism of $I(s)$-representations

$$(\mathcal{F}(s)/\mathcal{F}(s)^{I(s)})\otimes \mathcal{L}_{\chi(x-s)} \approx \mathcal{G}(s)/\mathcal{G}(s)^{I(s)}. \qquad \text{QED}$$

Corollary 3.3.7 (rank formula) Hypotheses and notations as in 3.3.3 above, suppose in addition that K is tamely ramified everywhere. Then so is $K*_{mid+j_*}\mathcal{L}_\chi[1]$, and their ranks and local monodromies are related by the formulas

$$\text{rank } \mathcal{F} = \Sigma_{s \text{ in } \mathbb{A}^1-U} \text{rank } (\mathcal{G}(s)/\mathcal{G}(s)^{I(s)}) - \text{rank}((\mathcal{G}(\infty)\otimes \mathcal{L}_{\overline{\chi}})^{I(\infty)}),$$

$$\text{rank } \mathcal{G} = \Sigma_{s \text{ in } \mathbb{A}^1-U} \text{rank } (\mathcal{F}(s)/\mathcal{F}(s)^{I(s)}) - \text{rank}((\mathcal{F}(\infty)\otimes \mathcal{L}_\chi)^{I(\infty)}).$$

proof These follow immediately from the formulas

$$\mathcal{F}(\infty) = M(\infty)/M(\infty)^{I(\infty)},$$

$$\mathcal{G}(\infty) = M(\infty)\otimes \mathcal{L}_\chi/(M(\infty)\otimes \mathcal{L}_\chi)^{I(\infty)},$$

and

$$\text{rank } M(\infty) = \Sigma_{s \text{ in } \mathbb{A}^1-U} \text{rank } (\mathcal{F}(s)/\mathcal{F}(s)^{I(s)}),$$

$$\text{rank } M(\infty) = \Sigma_{s \text{ in } \mathbb{A}^1-U} \text{rank } (\mathcal{G}(s)/\mathcal{G}(s)^{I(s)}). \qquad \text{QED}$$

3.4 Some open questions about local Fourier Transform

(3.4.1) We know that if M is an irreducible $I(0)$-representation with all slopes a/b, and dimension b, then $FTloc(0,\infty)M$ is an irreducible $I(\infty)$-representation with all slopes $a/(a+b)$, and dimension $a+b$. We also know that $FTloc(0,\infty)$ is an equivalence of categories

$$I(0)\text{-representations} \approx I(\infty)\text{-representations with all slopes} <1.$$

Denote by $(\mathrm{FTloc}(0,\infty))^{-1}$ a quasi-inverse. Then for every nontrivial Kummer sheaf $\mathcal{L}_\chi$, we get an autoequivalence of the category of I(0)-representations, defined by

(3.4.1.1) $\quad M \mapsto (\mathrm{FTloc}(0,\infty))^{-1}(\mathcal{L}_{\overline{\chi}}\otimes\mathrm{FTloc}(0,\infty)(M))$.

This autoequivalence preserves slopes and dimensions, and on tame M is given (cf. [Ka-ESDE, 7.4.4.3]) by $M \mapsto \mathcal{L}_\chi\otimes M$. What is it? Is there a simple formula for it?

(3.4.2) The most naive hope is that this autoequivalence be given by $M \mapsto \mathcal{L}_\chi\otimes M$ on all I(0)-representations, or equivalently that

$$\mathrm{FTloc}(0,\infty)(M\otimes\mathcal{L}_\chi) \approx \mathcal{L}_{\overline{\chi}}\otimes\mathrm{FTloc}(0,\infty)(M) \text{ as } I(\infty)\text{-rep'ns.}$$

This hope is **false**. Here is a simple sequence of counterexamples. For each integer $n \geq 1$, take for M_n the I(0)-representation attached to $\mathrm{inv}^*(\mathrm{Kl}_n(\psi;\ \rho_1, ..., \rho_n))$, where $\mathrm{Kl}_n(\psi;\ \rho_1, ..., \rho_n)$ is any of the rank n Kloosterman sheaves discussed in [Ka-GKM, 7.4.1]. By [Ka-GKM, 8.6.1], we have a geometric isomorphism

$$j^*\mathrm{FT}_\psi(j_!\mathrm{inv}^*(\mathrm{Kl}_n(\psi;\ \rho_1, ..., \rho_n))) \approx \mathrm{Kl}_{n+1}(\psi;\ \mathbb{1}, \rho_1, ..., \rho_n).$$

Becuause $\mathrm{Kl}_{n+1}(\psi;\ \mathbb{1}, \rho_1, ..., \rho_n)$ has all its ∞-slopes < 1 (they are all equal to $1/(n+1)$), stationary phase shows that

$$\mathrm{FTloc}(0,\infty)(M_n) = \mathrm{Kl}_{n+1}(\psi;\ \mathbb{1}, \rho_1, ..., \rho_n)(\infty).$$

If we replace M_n by $M_n\otimes\mathcal{L}_\chi$, we are looking at

$$M_n\otimes\mathcal{L}_\chi := \mathrm{inv}^*(\mathrm{Kl}_n(\psi;\ \overline{\chi}\rho_1, ..., \overline{\chi}\rho_n)) \text{ as } I(0)\text{-representation,}$$

so by the above FT fornmula we have

$$\mathrm{FTloc}(0,\infty)(M_n\otimes\mathcal{L}_\chi) = \mathrm{Kl}_{n+1}(\psi;\ \mathbb{1}, \overline{\chi}\rho_1, ..., \overline{\chi}\rho_n)(\infty).$$

We claim that

$$\mathrm{FTloc}(0,\infty)(M_n\otimes\mathcal{L}_\chi) \neq \mathcal{L}_{\overline{\chi}}\otimes\mathrm{FTloc}(0,\infty)(M_n) \text{ as } I(\infty)\text{-rep'n.}$$

This amounts to the statement that, as $I(\infty)$-representations,

$$\mathrm{Kl}_{n+1}(\psi;\ \mathbb{1}, \overline{\chi}\rho_1, ..., \overline{\chi}\rho_n) \neq \mathcal{L}_{\overline{\chi}}\otimes\mathrm{Kl}_{n+1}(\psi;\ \mathbb{1}, \rho_1, ..., \rho_n).$$

Indeed, the two sides have non-isomorphic determinants. To see this, use the geometric isomorphism ([Ka-7.4.1])

$$\det(\mathrm{Kl}_n(\psi;\ \rho_1, ..., \rho_n) \cong \mathcal{L}_{\rho_1\cdots\rho_{n+1}},$$

valid for $n \geq 2$. The ratio of the two determinants is thus $\mathcal{L}_\chi$, which is nontrivial on $I(\infty)$ so long as χ is nontrivial.

(3.4.3) There is a somewhat unsatisfactory "fomula" for the autoequivalence 3.4.1.1, in terms of middle convolution and the canonical extension. Let M be an irreducible nontame representation of I(0). By the theory of the canonical extension [Ka-LG], there exists a lisse sheaf $\mathcal{F}_M$ on $\mathbb{G}_m$ which is tame at ∞, and whose I(0)-representation is M. This sheaf $\mathcal{F}_M$ is certainly irreducible on $\mathbb{G}_m$, since already its I(0)-representation is irreducible. Denoting by $j: \mathbb{G}_m \to \mathbb{A}^1$ the inclusion, the object $K_M := j_\star\mathcal{F}_M[1]$ is a perverse irreducible on $\mathbb{A}^1$ which (being wild at 0) is visibly of type 2d, i.e., neither punctual nor an $\mathcal{L}_{\psi_\alpha}[1]$ nor a translate of a $j_\star\mathcal{L}_\chi[1]$. Since K_M is tame at ∞ and lisse on $\mathbb{G}_m$, Laumon's stationary phase tells us that FTK_M is lisse on $\mathbb{G}_m$, say $j^*FTK_M = \mathcal{A}[1]$ with $\mathcal{A}$ a lisse sheaf on $\mathbb{G}_m$, and

$$\mathcal{A}(\infty) = FTloc(0, \infty)(M) = FTloc(0, \infty)(\mathcal{F}_M(0)).$$

Now consider the object $L := K_M *_{mid+} j_\star\mathcal{L}_\chi[1]$, whose FT is also lisse on $\mathbb{G}_m$, with $j^*FTL = \mathcal{A}\otimes\mathcal{L}_{\overline{\chi}}[1]$. The object L is lisse on $\mathbb{G}_m$, and tame at ∞ (by 3.3.5, 2d), since K_M is lisse on $\mathbb{G}_m$, and tame at ∞). Say $j^*L = \mathcal{N}[1]$, with $\mathcal{N}$ a lisse sheaf on $\mathbb{G}_m$. Then by stationary phase,

$$(\mathcal{A}\otimes\mathcal{L}_{\overline{\chi}})(\infty) = FTloc(0, \infty)(\mathcal{N}(0)).$$

Thus the autoequivalence

$$M \mapsto (FTloc(0,\infty))^{-1}(\mathcal{L}_{\overline{\chi}}\otimes FTloc(0,\infty)(M)).$$

is given on nontame irreducibles M by

$$M \mapsto \text{ the I(0)-rep'n of } (j_\star\mathcal{F}_M[1]) *_{mid+} (j_\star\mathcal{L}_\chi[1]).$$

(3.4.4) We can also pose a similar question about $FTloc(\infty,\infty)$. If M is an irreducible $I(\infty)$-representation with all slopes $(a+b)/b > 1$ and dimension b, then $FTloc(\infty,\infty)(M)$ is an irreducible $I(\infty)$-representation with all slopes $(a+b)/a > 1$ and dimension a. We know that $FTloc(\infty,\infty)$ is an autoequivalence of

$$\{I(\infty)\text{-representations with all slopes} >1 \},$$

with quasi-inverse $[x \mapsto -x]^*FTloc(\infty,\infty)$. For every nontrivial Kummer sheaf $\mathcal{L}_\chi$, we get another autoequivalence by

(3.4.4.1) $$M \mapsto [x \mapsto -x]^*FTloc(\infty,\infty)(\mathcal{L}_{\overline{\chi}}\otimes FTloc(\infty,\infty)(M)).$$

This autoequivalence preserves slopes and dimensions. What is it? Is there a simple formula for it?

(3.4.5) The most naive hope is that the autoequivalence 3.4.4.1 be given by $M \mapsto \mathcal{L}_{\overline{\chi}} \otimes M$ on all $I(\infty)$-representations with all slopes >1, or equivalently that

$$\mathrm{FTloc}(\infty,\infty)(M \otimes \mathcal{L}_{\chi}) \approx \mathcal{L}_{\chi} \otimes \mathrm{FTloc}(\infty,\infty)(M) \text{ as } I(\infty)\text{-rep'ns.}$$

This hope is **false**. Here is a simple sequence of counterexamples. For each $n \geq 3$ which is prime to p, take for M_n the $I(\infty)$-representation attached to $\mathcal{L}_{\psi(x^n)}$. For any Kummer sheaf $\mathcal{L}_{\chi}$, $\mathrm{FTloc}(\infty,\infty)(M_n \otimes \mathcal{L}_{\chi})$ has rank n-1, and all slopes n/(n-1). We claim that

(3.4.5.1) $\det(\mathrm{FTloc}(\infty,\infty)(M_n \otimes \mathcal{L}_{\chi})) = \mathcal{L}_{\chi}$ as $I(\infty)$ representation.

Admit this for a moment. Then $\mathrm{FTloc}(\infty,\infty)(M_n)$ has rank n-1 and trivial determinant for $n \geq 3$ prime to p, so our naive hope would force $\mathrm{FTloc}(\infty,\infty)(M_n \otimes \mathcal{L}_{\chi})$ to have determinant $\mathcal{L}_{\chi}{}^{n-1}$ rather than $\mathcal{L}_{\chi}$. To show that 3.4.5.1 holds, consider first the case of trivial χ. Then $\mathrm{FT}_{\psi}(\mathcal{L}_{\psi(x^n)})$ is lisse on $\mathbb{A}^1$ of rank n-1, and has all ∞-slopes n/(n-1). As explained in [Ka-GKM, 9.2], its restriction to $\mathbb{G}_m$ descends through the n'th power map, to a Kloosterman sheaf of rank n-1 on $\mathbb{G}_m$. The determinant of this Kloosterman sheaf is tame on $\mathbb{G}_m$, because $n-1 \geq 2$, and hence $\mathrm{FT}_{\psi}(\mathcal{L}_{\psi(x^n)})$ itself has a determinant which is tame on $\mathbb{G}_m$. But as $\mathrm{FT}_{\psi}(\mathcal{L}_{\psi(x^n)})$ is lisse on $\mathbb{A}^1$, so is its determinant, and hence $\det(\mathrm{FT}_{\psi}(\mathcal{L}_{\psi(x^n)})) = \mathbb{1}$ (being both tame on $\mathbb{G}_m$ and trivial at zero). As $\mathrm{FT}_{\psi}(\mathcal{L}_{\psi(x^n)})$ has all its ∞-slopes > 1, $\mathrm{FTloc}(\infty,\infty)(M_n) = \mathrm{FT}_{\psi}(\mathcal{L}_{\psi(x^n)})(\infty)$ as $I(\infty)$-representation, and hence $\mathrm{FTloc}(\infty,\infty)(M_n)$ has trivial determinant, as asserted. Now consider the case when χ is nontrivial. In this case, $\mathrm{FT}_{\psi}(\mathcal{L}_{\psi(x^n)} \otimes \mathcal{L}_{\chi})$ is lisse on $\mathbb{A}^1$ of rank n, and its ∞-slopes are 0 once and n/(n-1) repeated n-1 times. As explained in [Ka-GKM, 9.2.2], the restriction to $\mathbb{G}_m$ of $\mathrm{FT}_{\psi}(\mathcal{L}_{\psi(x^n)} \otimes \mathcal{L}_{\chi})$ descends through the n'th power map to a hypergeometric sheaf of type (n, 1). The determinant of such a hypergeometric sheaf is necessarily tame on $\mathbb{G}_m$ (because $n \geq 3$), and hence $\mathrm{FT}_{\psi}(\mathcal{L}_{\psi(x^n)} \otimes \mathcal{L}_{\chi})$ has trivial determinant (because this determinant is simultaneously lisse on $\mathbb{A}^1$ and tame on $\mathbb{G}_m$). By stationary phase, the $I(\infty)$-representation of $\mathrm{FT}_{\psi}(\mathcal{L}_{\psi(x^n)} \otimes \mathcal{L}_{\chi})$ is a direct sum

$$\mathrm{FT}_{\psi}(\mathcal{L}_{\psi(x^n)} \otimes \mathcal{L}_{\chi})(\infty) = \mathcal{L}_{\overline{\chi}} \oplus \mathrm{FTloc}(\infty,\infty)(M_n \otimes \mathcal{L}_{\chi}).$$

Taking determinants, we find 3.4.5.1, as required.
(3.4.6) Just as in 3.4.3 above, we can give a somewhat unsatisfactory "formula" for the autoequivalence 3.4.4.1 in terms of canonical extensions. In terms of the canonical extension of an $I(\infty)$-representation M with all ∞-slopes > 1 to a lisse sheaf $\mathcal{G}_M$ on $\mathbb{G}_m$ which is tame at zero, we get a description of the autoequivalence 3.4.4.1 as

$$M \mapsto \text{the slope} > 1 \text{ part of the } I(\infty)\text{-rep'n of } (j_*\mathcal{G}_M[1]) *_{\mathrm{mid}+} (j_*\mathcal{L}_\chi[1]).$$

4.0 Good schemes

(4.0.1) Recall that a "good scheme" is one which admits a map f of finite type to a scheme T which is regular of dimension at most one. For variable good schemes X, and ℓ any fixed prime number, we can speak of the triangulated categories $D^b{}_c(X[1/\ell], \overline{\mathbb{Q}}_\ell)$, which admit the full Grothendieck formalism of the "six operations" (cf. [De-Th.Fin], [De-Weil II], [Ek], [Me-SO]).

4.1 The basic setting

(4.1.1) Throughout this section, we fix a prime number ℓ, and work over a ground-ring R which is a normal noetherian integral domain in which ℓ is invertible, and such that Spec(R) is a good scheme. We work on $\mathbb{A}^1{}_R := \mathrm{Spec}(R[x])$. We fix a monic polynomial D(x) in R[x] of some degree $d \geq 1$ whose discriminant Δ is a unit in R, and denote by $D \subset \mathbb{A}^1{}_R$ the divisor defined by the vanishing of D(x). We further assume that the polynomial D(x) factors completely in R[x], say $D(x) = \prod_{1 \leq i \leq d} (x - a_i)$ with $a_i - a_j$ in $R^\times$ for all $i \neq j$. [This "further assumption" always holds after replacing Spec(R) by a connected, finite etale covering of itself.] Then $D = \coprod_i D_i$ is the disjoint union of the sections D_i defined by the vanishing of $x - a_i$.

(4.1.2) We say that an object K in $D^b{}_c(\mathbb{A}^1{}_R, \overline{\mathbb{Q}}_\ell)$ is adapted to the stratification $(\mathbb{A}^1{}_R - D, D = \coprod_i D_i)$ of $\mathbb{A}^1{}_R$, if each of its cohomology sheaves is lisse when restricted either to $\mathbb{A}^1{}_R - D$ or to any D_i.

(4.1.3) We say that an object K in $D^b{}_c(\mathbb{A}^1{}_R, \overline{\mathbb{Q}}_\ell)$ is fibrewise perverse if its restriction to each geometric fibre of $\mathbb{A}^1{}_R$ over Spec(R) is perverse, i.e. if for any algebraically closed field k, and any ring homomorphism $\varphi: R \to k$, the inverse image K_φ of K in $D^b{}_c(\mathbb{A}^1{}_k, \overline{\mathbb{Q}}_\ell)$ is perverse on $\mathbb{A}^1{}_k$.

(4.1.4) We say that an object K in $D^b{}_c(\mathbb{A}^1{}_R, \overline{\mathbb{Q}}_\ell)$ is fibrewise tame if for any algebraically closed field k, and any ring homomorphism $\varphi: R \to k$, the inverse image K_φ of K in $D^b{}_c(\mathbb{A}^1{}_k, \overline{\mathbb{Q}}_\ell)$ is tame on $\mathbb{A}^1{}_k$ in the sense that for any dense open set $U \subset \mathbb{A}^1{}_k$ on which K_φ is lisse, each of the cohomology sheaves $\mathcal{H}^i(K_\varphi) \mid U$ is tamely ramified at each point of $\mathbb{P}^1{}_k - U$.

4.2 Basic results in the basic setting

Proposition 4.2.1 Hypotheses and notations as in 4.1.1 above, let K in $D^b{}_c(\mathbb{A}^1{}_R, \overline{\mathbb{Q}}_\ell)$ be adapted to $(\mathbb{A}^1{}_R - D, D = \coprod_i D_i)$. If the fraction field of R has characteristic zero, then K is fibrewise tame.

proof [Ka-SE, 4.7.1 and Remarque (ii), SGA 1, Expose XIII, 5.5]. QED

Proposition 4.2.2 Hypotheses and notations as in 4.1 above, suppose K in $D^b{}_c(\mathbb{A}^1{}_R, \overline{\mathbb{Q}}_\ell)$ is adapted to $(\mathbb{A}^1{}_R - D, D = \coprod_i D_i)$ and is fibrewise tame. Then the following conditions are equivalent:
1) K is fibrewise perverse.
2) There exists a geometric fibre on which K is perverse.
3) Each of the following three conditions holds.

i) $\mathcal{H}^i(K)$ vanishes for i outside {0, -1},

ii) $\mathcal{H}^0(K) \mid \mathbb{A}^1{}_R - D$ vanishes, and,

iii) denoting by $j: \mathbb{A}^1{}_R - D \to \mathbb{A}^1{}_R$ the inclusion, the natural adjunction map $\mathcal{H}^{-1}(K) \to j_\star j^\star \mathcal{H}^{-1}(K)$ is injective.

proof Fix an algebraically closed field k, and a ring homomorphism $\varphi: R \to k$. The inverse image K_φ of K on $\mathbb{A}^1{}_k$ is perverse if the conditions of 3) above hold **after** the base change $\varphi: R \to k$; let us call these conditions 3φ).

For each integer i, each of the sheaves $\mathcal{H}^i(K) \mid \mathbb{A}^1{}_R - D$ and $\mathcal{H}^i(K) \mid D$ is lisse. So for any $\varphi: R \to k$ as above, 3i) $\Leftrightarrow$ 3φi), and 3ii) $\Leftrightarrow$ 3φii). It is in showing 3iii) $\Leftrightarrow$ 3φiii) that we make essential use of the fibrewise tameness.

The key point [Ka-SE, 4.7.2 and 4.7.3] is that for $\mathcal{F}$ a $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1{}_R$ which is adapted to $(\mathbb{A}^1{}_R - D, D = \coprod_i D_i)$ and fibrewise tame, we have:

a) formation of $j_\star j^\star \mathcal{F}$ commutes with arbitrary change of base on Spec(R), and

b) the sheaf

$$\mathcal{F}_{pct} := \mathrm{Ker}(\mathcal{F} \to j_\star j^\star \mathcal{F})$$

is concentrated on D, lisse on D (i.e., lisse on each D_i), and of formation compatible with arbitrary change of base on Spec(R).

Applying this to $\mathcal{H}^{-1}(K)$, we see that 3iii) ⇔ 3φiii).

Fixing a single φ, we get 2) ⇒ 3). Varying φ, we get 3) ⇒ 1). And 1) ⇒ 2) is trivial. QED

Proposition 4.2.3 Hypotheses and notations as in 4.1.1 above, let K in $D^b_c(\mathbb{A}^1_R, \overline{\mathbb{Q}}_\ell)$ be adapted to $(\mathbb{A}^1_R - D, D = \coprod_i D_i)$, fibrewise perverse and fibrewise tame. Then the following conditions are equivalent:

1) on each geometric fibre, K is the middle extension of its restriction to $\mathbb{A}^1_k - D_k$.

2) there exists a geometric fibre on which K is the middle extension of its restriction to $\mathbb{A}^1_k - D_k$.

3) $\mathcal{H}^0(K) = 0$, and denoting by $j: \mathbb{A}^1_R - D \to \mathbb{A}^1_R$ the inclusion, the adjunction map is an isomorphism $\mathcal{H}^{-1}(K) \cong j_\star j^\star \mathcal{H}^{-1}(K)$.

proof The kernel and cokernel of the adjunction map

$$\mathcal{H}^{-1}(K) \to j_\star j^\star \mathcal{H}^{-1}(K)$$

are concentrated on D, lisse on D, and of formation compatible with arbitrary change of base on Spec(R), cf. [Ka-SE, 4.7.2-3]. Fixing a single φ, we get 2) ⇒ 3). Varying φ, we get 3) ⇒ 1). And 1) ⇒ 2) is trivial. QED

Proposition 4.2.4 Hypotheses and notations as in 4.1.1 above, let K in $D^b_c(\mathbb{A}^1_R, \overline{\mathbb{Q}}_\ell)$ be adapted to $(\mathbb{A}^1_R - D, D = \coprod_i D_i)$, fibrewise perverse and fibrewise tame. Denote by n the rank of the lisse sheaf $\mathcal{H}^{-1}(K) \mid \mathbb{A}^1_R - D$. Fix a subgroup Γ of $GL(n, \overline{\mathbb{Q}}_\ell)$. Then the following conditions are equivalent:

1) on each geometric fibre, K is the middle extension of its

restriction to $\mathbb{A}^1{}_k - D_k$, and the image of $\pi_1(\mathbb{A}^1{}_k - D_k$, base point) in $GL(n, \overline{\mathbb{Q}}_\ell)$ under the monodromy representation of $\mathcal{H}^{-1}(K) \mid \mathbb{A}^1{}_k - D_k$ is conjugate to Γ.
2) On some geometric fibre, K is the middle extension of its restriction to $\mathbb{A}^1{}_k - D_k$, and the image of $\pi_1(\mathbb{A}^1{}_k - D_k$, base point) in $GL(n, \overline{\mathbb{Q}}_\ell)$ under the monodromy representation of $\mathcal{H}^{-1}(K) \mid \mathbb{A}^1{}_k - D_k$ is conjugate to Γ.

proof In view of the preceding result, this results from the fact that for a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $\mathbb{A}^1{}_R - D$ (namely $\mathcal{H}^{-1}(K) \mid \mathbb{A}^1{}_R - D$) of rank denoted n which is fibrewise tame, the function on geometric points of Spec(R) given by

$$(\varphi: R \to k) \mapsto \text{the conjugacy class in } GL(n, \overline{\mathbb{Q}}_\ell) \text{ of the image of } \pi_1(\mathbb{A}^1{}_k - D_k), \text{ any base point})$$

is constant (cf. [Ka-ESDE, 8.17.13]). QED

Corollary 4.2.5 Hypotheses and notations as in 4.1.1 above, let K in $D^b{}_c(\mathbb{A}^1{}_R, \overline{\mathbb{Q}}_\ell)$ be adapted to $(\mathbb{A}^1{}_R - D, D = \coprod_i D_i)$, fibrewise perverse and fibrewise tame. Then the following conditions are equivalent:

1) on each geometric fibre, $K \approx \overline{\mathbb{Q}}_\ell[1]$.
2) there exists a geometric. fibre on which $K \approx \overline{\mathbb{Q}}_\ell[1]$.
proof This is the case $n=1$, $\Gamma = \{1\}$ of the previous result. QED

Corollary 4.2.6 Hypotheses and notations as in 4.1.1 above, let K in $D^b{}_c(\mathbb{A}^1{}_R, \overline{\mathbb{Q}}_\ell)$ be adapted to $(\mathbb{A}^1{}_R - D, D = \coprod_i D_i)$, fibrewise perverse and fibrewise tame. Then the following conditions are equivalent:
1) on each geometric fibre, K is perverse irreducible and is the middle extension of its restriction to $\mathbb{A}^1{}_k - D_k$.
2) there exists a geometric fibre on K is perverse irreducible and is the middle extension of its restriction to $\mathbb{A}^1{}_k - D_k$.
proof This is the case "Γ an irreducible subgroup of $GL(n, \overline{\mathbb{Q}}_\ell)$" of the previous Proposition, for all such Γ. QED

Proposition 4.2.7 Hypotheses and notations as in 4.1.1 above, let K in $D^b_c(\mathbb{A}^1_R, \overline{\mathbb{Q}}_\ell)$ be adapted to $(\mathbb{A}^1_R - D, D = \coprod_i D_i)$, fibrewise perverse and fibrewise tame. Denote by $j: \mathbb{A}^1_R - D \to \mathbb{A}^1_R$ the inclusion. Then the following conditions are equivalent:

1) on each geometric fibre, $K \approx j_*\mathcal{L}_{\chi(x-\alpha)}[1]$ for some nontrivial Kummer sheaf $\mathcal{L}_\chi$ and some α.
2) there exists a geometric fibre on which $K \approx j_*\mathcal{L}_{\chi(x-\alpha)}[1]$ for some nontrivial Kummer sheaf and some α.
3) Each of the following conditions holds:

i) $\mathcal{H}^0(K) = 0$,

ii) $\mathcal{H}^{-1}(K) | \mathbb{A}^1_R - D$ is lisse of rank one,

iii) $\mathcal{H}^{-1}(K) \cong j_*j^*\mathcal{H}^{-1}(K)$,

iiii) $\mathcal{H}^{-1}(K) | D_i$ is lisse of rank one for all but exactly one value i_0 of i, and for this value $\mathcal{H}^{-1}(K) | D_{i_0} = 0$.

proof It is clear that 1) ⇒ 2) and 2) ⇒ 3). To see that 3) ⇒ 1), we may reduce by an additive translation on $\mathbb{A}^1_R$ to the case where the divisor D_{i_0} is the divisor $x=0$. In this case, we must show that for any geometric point $\varphi: R \to k$, the only lisse rank one $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{G}_{m,k}$ which is ramified at 0 and which is tame at both 0 and ∞ is a nontrivial Kummer sheaf $\mathcal{L}_\chi$. But this is a tautology. QED

Proposition 4.2.8 Hypotheses and notations as in 4.1.1 above, let K in $D^b_c(\mathbb{A}^1_R, \overline{\mathbb{Q}}_\ell)$ be adapted to $(\mathbb{A}^1_R - D, D = \coprod_i D_i)$, fibrewise perverse and fibrewise tame. Then the following conditions are equivalent:
1) on each geometric fibre, $K \approx \delta_\alpha$ for some α.
2) there exists a geometric. fibre on which $K \approx \delta_\alpha$ for some α.
3) $\mathcal{H}^{-1}(K)= 0$, and $\mathcal{H}^0(K) | D_i = 0$ for all but exactly one value i_0 of i, and for this value $\mathcal{H}^{-1}(K) | D_{i_0}$ is lisse of rank one.

proof It is clear that 1) ⇒ 2) and 2) ⇒ 3), and 3) ⇒ 1) is obvious as well. QED

4.3 Middle convolution in the basic setting

(4.3.1) We continue to work in the basic setting 4.1.1. Fix a nontrivial character χ of the group

$$\pi_1{}^{tame}(\mathbb{G}_{m,R}) = \prod_{\ell \text{ inv in } R} \mathbb{Z}_\ell(1).$$

(4.3.2) Denote by $\mathcal{C}(R, D)$ the full subcategory of $D^b{}_c(\mathbb{A}^1{}_R, \overline{\mathbb{Q}}_\ell)$ consisting of all objects K which are adapted to $(\mathbb{A}^1{}_R - D, D = \coprod_i D_i)$, fibrewise perverse and fibrewise tame. We will now define a middle convolution functor from $\mathcal{C}(R, D)$ to itself,

$$K \mapsto K*_{mid+}j_*\mathcal{L}_\chi[1],$$

whose formation commutes with arbitrary change of base on R, and which for R a field of characteristic $\neq \ell$ coincides with its namesake 2.6.2. It is defined via the relative compactification of the map pr_2 as in 2.8.3-4,

$$j: \mathbb{A}^1{}_x \times \mathbb{A}^1{}_t \hookrightarrow \mathbb{P}^1{}_x \times \mathbb{A}^1{}_t,$$

by extending $K_x \otimes (j_*\mathcal{L}_\chi[1])_{t-x}$ across $Z := \infty_{\mathbb{A}}1$ by $\tau^Z{}_{\leq -2}Rj_*$, [BBD, 1.4.13] and then taking Rpr_{2*} as in the earlier discussion. The key point is that $Rj_*(K_x \otimes (j_*\mathcal{L}_\chi[1])_{t-x}) \mid Z$ is lisse, and of formation compatible with change of base on R (by the general lemma 4.3.8 below). Therefore the object $\tau^Z{}_{\leq -2}Rj_*(K_x \otimes (j_*\mathcal{L}_\chi[1])_{t-x}) \mid Z$ is lisse, and of formation compatible with change of base on R.

(4.3.3) Since the formation of Rpr_{2*} commute with passage to fibres, by proper base change, we know that $K*_{mid+}j_*\mathcal{L}_\chi[1]$ is fibrewise perverse, of formation compatible with arbitrary change of base on R, and fibre by fibre equal to its namesake.

(4.3.4) We also have an explicit triangle relating this middle convolution to the ! convolution and to a lisse sheaf on $\mathbb{A}^1{}_R$. On $\mathbb{P}^1{}_R \times \mathbb{A}^1{}_R$ we have a distinguished triangle

$$\tau^Z{}_{\leq -3}Rj_*(K_x \otimes (j_*\mathcal{L}_\chi[1])_{t-x}) \to \tau^Z{}_{\leq -2}Rj_*(K_x \otimes (j_*\mathcal{L}_\chi[1])_{t-x}) \to$$
$$\to j_*(\mathcal{H}^{-2}(K_x \otimes (j_*\mathcal{L}_\chi[1])_{t-x})[2] \mid Z.$$

(4.3.5) But $\tau^Z{}_{\leq -3}Rj_*(K_x \otimes (j_*\mathcal{L}_\chi[1])_{t-x})$ is $j_!(K_x \otimes (j_*\mathcal{L}_\chi[1])_{t-x})$, so rotating this triangle gives a distinguished triangle

$$j_*(\mathcal{H}^{-2}(K_x \otimes (j_*\mathcal{L}_\chi[1])_{t-x})[1] \mid Z \;\to\; j_!(K_x \otimes (j_*\mathcal{L}_\chi[1])_{t-x}) \to$$
$$\to \tau^Z{}_{\le -2} Rj_*(K_x \otimes (j_*\mathcal{L}_\chi[1])_{t-x})$$

in which the first term is (a lisse sheaf on Z)[1].

(4.3.6) Applying Rpr_{2*} gives a distinguished triangle on $\mathbb{A}^1{}_R$

$$\text{(a lisse sheaf on } \mathbb{A}^1{}_R)[1] \to K*_{!+}j_*\mathcal{L}_\chi[1] \to K*_{mid+}j_*\mathcal{L}_\chi[1].$$

Using this triangle, we see the adaptedness of $K*_{mid+}j_*\mathcal{L}_\chi[1]$ to the same stratification $(\mathbb{A}^1{}_R - D,\ D = \coprod_i D_i)$ to which K was adapted.

(4.3.7) We know fibre by fibre that tameness is preserved: this is automatic for characteristic zero fibres, and for characteristic $p > 0$ fibres it was proven in 3.3.6.

Lemma 4.3.8 Let S an irreducible noetherian scheme, X/S smooth, and D in X a smooth/S divisor. For $\mathcal{F}$ lisse on X - D and tame along D, $j: X-S \to X$ and $i: D \to X$ the inclusions, we have
1) formation of $j_*\mathcal{F}$ and of $Rj_*\mathcal{F}$ on X commutes with arbitrary change of base on S,
2) the sheaf $i^*j_*\mathcal{F}$ on D is lisse, and formation of $i^*j_*\mathcal{F}$ on D commutes with arbitrary change of base on S.
proof [Ka-SE, 4.7.2 and 4.7.3]. QED

Proposition 4.3.9 Hypotheses and notations as in 4.1.1 above, let K in $D^b{}_c(\mathbb{A}^1{}_R, \overline{\mathbb{Q}}_\ell)$ be adapted to $(\mathbb{A}^1{}_R - D,\ D = \coprod_i D_i)$, fibrewise perverse and fibrewise tame. If K is fibrewise a middle extension, then its fibrewise index of rigidity is constant.
proof Consider the lisse tame sheaf on $\mathbb{P}^1{}_R - \{D \coprod \infty\}$,

$$\mathcal{F} := \underline{\mathrm{End}}(\mathcal{H}^{-1} \mid \mathbb{P}^1{}_R - \{D \coprod \infty\}).$$

Its $j_*\mathcal{F}$ on $\mathbb{P}^1{}_R$ is adapted to $(\mathbb{P}^1{}_R - \{D \coprod \infty\}, \{D \coprod \infty\})$, fibrewise tame, and of formation compatible with arbitrary change of base on Spec(R), by the general lemma above. By the Euler-Poincaré formula, the fibrewise χ of $j_*\mathcal{F}$ is constant. Indeed, if

n := rank of $\mathcal{F}$ on $\mathbb{P}^1{}_R - \{D \coprod \infty\}$,
n_∞ := rank of $j_*\mathcal{F} \mid \infty_R$,
n_i := rank of $j_*\mathcal{F} \mid D_i$, for each of the d sections D_i,

then the fibrewise χ of $j_*\mathcal{F}$, i.e., the fibrewise index of rigidity, is

$$(1 - d)n + n_\infty + \Sigma_i n_i. \qquad \text{QED}$$

Theorem 4.3.10 Hypotheses and notations as in 4.1.1 above, let K in $D^b{}_c(\mathbb{A}^1{}_R, \overline{\mathbb{Q}}_\ell)$ be adapted to $(\mathbb{A}^1{}_R - D, D = \coprod_i D_i)$, fibrewise perverse and fibrewise tame. Suppose K is fibrewise perverse irreducible of type 2d) in the sense of 3.3.3. Then both K and $K*_{\mathrm{mid}+j_*}\mathcal{L}_\chi[1]$ are fibrewise perverse irreducible of type 2d) and have the same index of rigidity.

proof Suppose first that R is not a $\mathbb{Q}$-algebra. Then Spec(R) has points with finite residue characteristic p. Once we know that $K \mapsto K*_{\mathrm{mid}+j_*}\mathcal{L}_\chi[1]$ is stable on fibrewise perverses which are fibrewise tame and adapted to D, we just look fibrewise (legitimate by the earlier fibrewise results) and take a fibre in characteristic $p > 0$, where we can use 3.3.3.3.

How do we get this result if we don't assume R has points in some positive characteristic? By passing to fibres, we first reduce to the case when R is itself an algebraically closed field of characteristic zero. Since π_1 of open curves in characteristic zero doesn't "see" extensions of algebraically closed fields, we may assume that our field R is "just" an algebraic closure of the finitely generated field $\mathbb{Q}$(coef's of D(x)), and try working over the ring $R_0 :=$ $\mathbb{Z}$[coef's of D(x), $1/\Delta$]. This ring R_0 is way too big, but it does have points of characteristic p for all $p >> 0$. Our $\mathcal{H}^{-1} | \mathbb{A}^1{}_R - D$ has a $GL(n, \mathcal{O}_\lambda)$ form, $\mathcal{O}_\lambda$ denoting the ring of integers in a finite extension E_λ of $\mathbb{Q}_\ell$, with finite residue field $\mathbb{F}_\lambda$. The key point is that for $p >> 0$ the group $GL(n, \mathcal{O}_\lambda)$ is prime to p, because it is an extension of the finite group $GL(n, \mathbb{F}_\lambda)$ by the pro-ℓ group $1 + \lambda M_n(\mathcal{O}_\lambda)$. For any $p >>0$, we can, by applying 5.9.3 and then extending the residue field, embed our R_0 inside a complete discrete valuation ring R_1 with algebraically closed residue field of characteristic p.

When we do this, D stays a good D. By the specialization theorem for the prime to p fundamental group of the open curve $\mathbb{A}^1{}_{R_1} - D$, we see that the sheaf $\mathcal{H}^{-1} | \mathbb{A}^1{}_R - D$ extends to a lisse sheaf on $\mathbb{A}^1{}_{R_1} - D$, which will be irreducible if and only if our original $\mathcal{H}^{-1} | \mathbb{A}^1{}_R - D$ was geometrically irreducible. Denoting by

$$j : \mathbb{A}^1{}_{R_1} - D \to \mathbb{A}^1{}_{R_1}$$

the inclusion, $j_*(\mathcal{H}^{-1} \mid \mathbb{A}^1{}_{R_1} - D)$ is adapted, fibrewise perverse, fibrewise tame, and fibrewise a middle extension. By 4.2.6, 4.2.7, and 4.2.5, the fibrewise "type 2d irreducible" condition is still satisfied by our extended K over R_1. Now we apply the already known case of the result over the mixed characteristic R_1 to get the index of rigidity on the general fibre. QED

Theorem 4.3.11 Hypotheses and notations as in 4.1.1 above, let K in $D^b{}_c(\mathbb{A}^1{}_R, \overline{\mathbb{Q}}_\ell)$ be adapted to ($\mathbb{A}^1{}_R$ - D, D = $\amalg_i D_i$), fibrewise perverse and fibrewise tame. Suppose K is fibrewise perverse irreducible of type 2d) in the sense of 3.3.3. For $j : \mathbb{A}^1{}_R - D \to \mathbb{A}^1{}_R$ the inclusion, write

$$j^*K = \mathcal{F}[1],\ j^*(K*_{mid+}j_*\mathcal{L}_\chi[1]) = \mathcal{G}[1],$$

with $\mathcal{F}$ and $\mathcal{G}$ lisse sheaves on $\mathbb{A}^1{}_R$ - D. Then on every geometric fibre, the local monodromies of $\mathcal{F}$ and of $\mathcal{G}$ are related as in 3.3.6 and 3.3.7.

proof Denote by L the set of those prime numbers which are invertible on R. Because $\mathcal{F}$is lisse on $\mathbb{A}^1{}_R - \amalg D_i = \mathbb{P}^1{}_R - \{\infty_R, \amalg D_i\}$ and tame along each of the "missing" sections Z := ∞_R or D_i, its local monodromy along each of these sections "is" a pair

(a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}[Z]$ on Z, an action of $\mathbb{Z}_L(1)$ on $\mathcal{F}[Z]$)

whose formation is compatible with arbitrary change of base on Spec(R) (cf. [De-Weil II, 1.7.8], also [Ka-SE, 4.7.2]). Similarly for $\mathcal{G}$. So in order the verify the asserted relations between the fibrewise local monodromies of $\mathcal{F}$ and $\mathcal{G}$, it suffices to do so on a single geometric fibre.

If Spec(R) has a point of residue characteristic p, we are done by 3.3.6 and 3.3.7. If not, we reduce first to the case when R is an algebraically closed field of characteristic zero, and then we specialize both $\mathcal{F}$ and $\mathcal{L}_\chi$ into characteristic p >> 0 by the argument of the previous theorem, which reduces us to the case when R is a mixed characteristic discrete valuation ring. QED

5.0 Cohomological rigidity

(5.0.1) Let ℓ be a prime number, k an algebraically closed field of characteristic $\neq \ell$. On $\mathbb{A}^1$ over k, denote by k: $\mathbb{A}^1 \to \mathbb{P}^1$ the inclusion. Let K in $D^b{}_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ be a perverse irreducible sheaf which is nonpunctual, i.e., K is $j_*\mathcal{F}[1]$ for a nonempty open set j: U $\to \mathbb{A}^1$, and a lisse irreducible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U. In 3.0.1, we defined the **index of rigidity** of K, rig(K), to be the integer

$$\mathrm{rig}(K) := \chi(\mathbb{P}^1, k_*j_*\underline{\mathrm{End}}(\mathcal{F})).$$

We say that K, or $\mathcal{F}$, is **cohomologically rigid**, if rig(K) = 2.

Theorem 5.0.2 Let ℓ be a prime number, k an algebraically closed field of characteristic $\neq \ell$. In $\mathbb{A}^1$ over k, let j: U $\to \mathbb{A}^1$ be a nonempty open set, $\mathcal{F}$ a lisse irreducible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U. Suppose that $\mathcal{F}$ is cohomologically rigid. Let $\mathcal{G}$ be any lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U which is locally isomorphic to $\mathcal{F}$ in the sense that for each point s of $\mathbb{P}^1$ - U, there exists an isomorphism between the representations $\mathcal{F}(s)$ and $\mathcal{G}(s)$ of the inertia group I(s) afforded by $\mathcal{F}$ and $\mathcal{G}$ respectively. Then there exists an isomorphism $\mathcal{F} \cong \mathcal{G}$ of lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U.

proof. This is a trivial modification of the easy half of 1.1.2. Denote by k: $\mathbb{A}^1 \to \mathbb{P}^1$ the inclusion. Since $\mathcal{F}$ and $\mathcal{G}$ are locally isomorphic, so are the two local systems $\underline{\mathrm{End}}(\mathcal{F})$ and $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$. So from the shape of the Euler-Poincaré formula, which depends only on ranks and on local data at the ramification points and at infinity, we see that

$$2 = \chi(\mathbb{P}^1, k_*j_*\underline{\mathrm{End}}(\mathcal{F})) = \chi(\mathbb{P}^1, k_*j_*\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})).$$

Once we have the inequality $\chi(\mathbb{P}^1, k_*j_*\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})) \geq 2$, we conclude exactly as we did in 1.1.2. QED

5.1 The category $\mathcal{T}_\ell$, and the functors MC_χ and $MT_\mathcal{L}$

(5.1.1) In this section, we work on $\mathbb{A}^1$ over an algebraically closed field k. We fix a prime number $\ell \neq \mathrm{char}(k)$. We are interested in the full subcategory $\mathcal{T}_\ell$ of the category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves $\mathcal{F}$ on $\mathbb{A}^1$ consisting of those which satisfy the following three conditions:

$\mathcal{T}1)$ $\mathcal{F}$ is an irreducible middle extension: there exists a dense open set $j: U \to \mathbb{A}^1$ such that $j^*\mathcal{F}$ is lisse and irreducible on U, and such that $\mathcal{F} \cong j_*j^*\mathcal{F}$.

$\mathcal{T}2)$ $\mathcal{F}$ is tame: $j^*\mathcal{F}$ is tamely ramified at every point of $\mathbb{P}^1 - U$.

$\mathcal{T}3)$ $\mathcal{F}$ has at least two finite singularities: there are at least two distinct points of $\mathbb{A}^1$ at which $\mathcal{F}$ fails to be lisse.

Lemma 5.1.2 In the situation 5.0.1, suppose $\mathcal{F}$ satisfies conditions $\mathcal{T}1)$ and $\mathcal{T}2)$ above. If $\mathcal{F}$ has generic rank ≥ 2, then $\mathcal{F}$ satisfies $\mathcal{T}3)$.
proof Indeed, if $\mathcal{F}$ is tame and irreducible with at most one finite singularity, say α, then $j^*\mathcal{F}$ is an irreducible representation of the abelian group $\pi_1^{\mathrm{tame}}(\mathbb{A}^1 - \{\alpha\})$, so has rank ≤ 1 QED

Lemma 5.1.3 In the situation 5.0.1, the perverse irreducibles K in $D^b_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ which are tame, and of type 2d) in the sense of 3.3.3 are precisely those of the form $\mathcal{F}[1]$, with $\mathcal{F}$ in $\mathcal{T}_\ell$.

proof A nonpunctual perverse irreducible K is precisely an $\mathcal{F}[1]$ for $\mathcal{F}$ satisfying condition $\mathcal{T}1)$ above [and we recover $\mathcal{F}$ as $\mathcal{H}^{-1}(K)$]. Such a perverse irreducible K is tame if and only if $\mathcal{F}$ is tame. Requiring in addition that K have property $\mathcal{P}$ forces $\mathcal{F}$ to be nonconstant (and not to be an $\mathcal{L}_{\psi_\alpha}$, but that case was already eliminated by the tameness requirement). Requiring K to be of type 2d) eliminates the possibility that $\mathcal{F}$ be any $j_*\mathcal{L}_{\chi(x-\alpha)}$. But these last two eliminations, of $\overline{\mathbb{Q}}_\ell$ and of any $j_*\mathcal{L}_{\chi(x-\alpha)}$, amount precisely to condition $\mathcal{T}3)$ above. Indeed, of tame **irreducible** middle extensions, those with at most one finite singularity are either $\overline{\mathbb{Q}}_\ell$ (if there is no finite singularity, because $\pi_1^{\mathrm{tame}}(\mathbb{A}^1)=0$) or they are $j_*\mathcal{L}_{\chi(x-\alpha)}$ for α the unique finite singularity (because $\pi_1^{\mathrm{tame}}(\mathbb{G}_m)$ is abelian). QED

(5.1.4) For any nontrivial continuous character

$$\chi: \pi_1^{\mathrm{tame}}(\mathbb{G}_{m,k}) \to \overline{\mathbb{Q}}_\ell^\times,$$

we denote by $\mathcal{L}_\chi$ the corresponding lisse sheaf of rank one on $\mathbb{G}_m$, and by $j_*\mathcal{L}_\chi$ its direct image to $\mathbb{A}^1$. We may restate 4.3.11 in terms of $\mathcal{T}_\ell$ as follows. Given a finite subset D in $\mathbb{A}^1(k)$, and an $\mathcal{F}$ in $\mathcal{T}_\ell$ which is lisse on $\mathbb{A}^1$ - D, the middle convolution

$$\mathcal{F}[1]*_{\mathrm{mid+}}j_*\mathcal{L}_\chi[1]$$

is of the form $\mathcal{G}[1]$, with $\mathcal{G}$ in $\mathcal{T}_\ell$, and $\mathcal{G}$ lisse on the same $\mathbb{A}^1$ - D. We can recover $\mathcal{F}$ from $\mathcal{G}$ by the inversion formula

$$\mathcal{F}[1] = \mathcal{G}[1]*_{\mathrm{mid+}}j_*\mathcal{L}_{\overline{\chi}}[1].$$

Moreover, the local monodromies of $\mathcal{F}$ and of $\mathcal{G}$ are related by 3.3.6 and 3.3.7. According to 4.3.10, the indices of rigidity of $\mathcal{F}$ and $\mathcal{G}$ (strictly speaking, of the perverse objects $\mathcal{F}[1]$ and of $\mathcal{G}[1]$) are equal:

$$\mathrm{rig}(\mathcal{F}) = \mathrm{rig}(\mathcal{G}).$$

(5.1.5) We define, for χ a nontrivial character as above, the functor

$$MC_\chi: \mathcal{T}_\ell \to \mathcal{T}_\ell$$

by

$$MC_\chi(\mathcal{F}) := \mathcal{G} := (\mathcal{F}[1]*_{\mathrm{mid+}}j_*\mathcal{L}_\chi[1])[-1].$$

These functors have the composition laws

$$MC_\chi \circ MC_\rho = MC_\rho \circ MC_\chi = MC_{\chi\rho} \text{ if } \chi\rho \text{ is nontrivial}$$
$$MC_\chi \circ MC_{\overline{\chi}} = \mathrm{id}.$$

(5.1.6) Here is a relatively concrete description of $MC_\chi(\mathcal{F})$. Let $\mathcal{F}$ in $\mathcal{T}_\ell$. Denote by D the set of finite singularities of $\mathcal{F}$, and by

$$j: \mathbb{A}^1 - D \to \mathbb{A}^1$$

the inclusion. Consider the projection

$$pr_2: \mathbb{A}^2 \to \mathbb{A}^1, (x,t) \mapsto t.$$

On $\mathbb{A}^2$, we have the sheaf $\mathcal{F}_x\otimes\mathcal{L}_{\chi(t-x)}$ (extended by zero across the diagonal t=x). Because $\mathcal{F}$ is lisse on $\mathbb{A}^1$ - D, and in $\mathcal{T}_\ell$, the higher direct images $R^ipr_{2!}(\mathcal{F}_x\otimes\mathcal{L}_{\chi(t-x)})$ and $R^ipr_{2*}(\mathcal{F}_x\otimes\mathcal{L}_{\chi(t-x)})$ are both lisse on $\mathbb{A}^1$ - D, and both vanish for $i\neq1$. Moreover, both are of formation compatible with arbitrary change of base on $\mathbb{A}^1$ - D (by 2.8.5). Thus we may form the lisse sheaf on $\mathbb{A}^1$ - D which is

$$\mathrm{Image}(j^*R^1pr_{2!}(\mathcal{F}_x\otimes\mathcal{L}_{\chi(t-x)}) \to j^*R^1pr_{2*}(\mathcal{F}_x\otimes\mathcal{L}_{\chi(t-x)})).$$

It is tautological that this is none other than $j^*MC_\chi(\mathcal{F})$. But

$MC_\chi(\mathcal{F})$ is a middle extension, so we get the

Explicit Recipes 5.1.7 In the situation 5.1.1, we have:

(1) For $\mathcal{F}$ in $\mathcal{T}_\ell$, and j: $\mathbb{A}^1 - D \to \mathbb{A}^1$ the inclusion of a dense open set on which $j^*\mathcal{F}$ is lisse,

$MC_\chi(\mathcal{F}) = j_* j^* MC_\chi(\mathcal{F}) =$

$$= j_*\mathrm{Image}(j^*R^1pr_{2!}(\mathcal{F}_x\otimes\mathcal{L}_{\chi(t-x)}) \to j^*R^1pr_{2*}(\mathcal{F}_x\otimes\mathcal{L}_{\chi(t-x)})).$$

(2) At any geometric point τ in $\mathbb{A}^1 - D$, the stalk at τ of $MC_\chi(\mathcal{F})$ is given by

$(MC_\chi(\mathcal{F}))_\tau = \mathrm{Image}(H^1{}_c(\mathbb{A}^1,\mathcal{F}\otimes\mathcal{L}_\chi(\tau-x)) \to H^1(\mathbb{A}^1,\mathcal{F}\otimes\mathcal{L}_\chi(\tau-x)))$.

(3) In terms of the inclusion k: $\mathbb{A}^1 \to \mathbb{P}^1$, this image is just the "parabolic" group $H^1(\mathbb{P}^1,k_*(\mathcal{F}\otimes\mathcal{L}_\chi(\tau-x)))$, so

$$(MC_\chi(\mathcal{F}))_\tau = H^1(\mathbb{P}^1,k_*(\mathcal{F}\otimes\mathcal{L}_\chi(\tau-x))),\ \text{for } \tau \text{ in } \mathbb{A}^1 - D.$$

(4) In terms of the inclusion $k_1 : \mathbb{A}^1 - D -\{\tau\} \to \mathbb{P}^1$, we may rewrite this as

$$(MC_\chi(\mathcal{F}))_\tau = H^1(\mathbb{P}^1,k_{1*}((\mathcal{F}\otimes\mathcal{L}_\chi(\tau-x) \mid \mathbb{A}^1 - D -\{\tau\})) =$$

= Image of the "forget supports" map

$$H^1{}_c(\mathbb{A}^1 - D -\{\tau\},\mathcal{F}\otimes\mathcal{L}_\chi(\tau-x)) \to H^1(\mathbb{A}^1 - D -\{\tau\},\mathcal{F}\otimes\mathcal{L}_\chi(\tau-x)).$$

(5.1.8) Let $\mathcal{L}$ be a middle extension sheaf on $\mathbb{A}^1$ which is generically of rank one, and which is tame. We wish to define an operation of "middle tensor product with $\mathcal{L}$" for objects of $\mathcal{T}_\ell$ [compare 3.2]. The naive idea is this: given $\mathcal{F}$ in $\mathcal{T}_\ell$, pick j: $U \to \mathbb{A}^1$ a dense open on which both $\mathcal{F}$ and $\mathcal{L}$ are lisse, and form $j_*((j^*\mathcal{F})\otimes(j^*\mathcal{L}))$. This sheaf, which is independent of the auxiliary choice of U, visibly satisfies both $\mathcal{T}$1) and $\mathcal{T}$2). As it has the same generic rank as $\mathcal{F}$, it will satisfy $\mathcal{T}$3) if $\mathcal{F}$ has generic rank at least 2, thanks to Lemma 5.1.2.

(5.1.9) However, for $\mathcal{F}$ of generic rank one, $j_*((j^*\mathcal{F})\otimes(j^*\mathcal{L}))$ may very well **fail** to satisfy $\mathcal{T}$3): for example, it might be the constant sheaf. In order to deal with this problem, we define

$\mathcal{T}_{\ell,\,\mathrm{rk}\,\geq 2} :=$ the full subcategory of $\mathcal{T}_\ell$ consisting of those objects of generic rank ≥ 2.

On this full subcategory $\mathcal{T}_{\ell,\ \mathrm{rk}\ \geq 2}$, we can define the above "middle tensor product with $\mathcal{L}$", which we denote

$$\mathrm{MT}_{\mathcal{L}}: \mathcal{T}_{\ell,\ \mathrm{rk}\ \geq 2} \to \mathcal{T}_{\ell,\ \mathrm{rk}\ \geq 2},$$
$$\mathrm{MT}_{\mathcal{L}}(\mathcal{F}) := j_{\star}((j^{\star}\mathcal{F})\otimes(j^{\star}\mathcal{L})),$$

for $j: U \to \mathbb{A}^1$ any dense open on which both $\mathcal{F}$ and $\mathcal{L}$ are lisse.
It is clear from the definitions that $\mathrm{MT}_{\mathcal{L}}$ preserves index of rigidity:

$$\mathrm{rig}(\mathrm{MT}_{\mathcal{L}}(\mathcal{F})) = \mathrm{rig}(\mathcal{F}).$$

If we are given a lisse, rank one $\mathcal{L}$ on a dense open $j:U \to \mathbb{A}^1$, we will often write $\mathrm{MT}_{\mathcal{L}}$ instead of the literally correct $\mathrm{MT}_{j_{\star}\mathcal{L}}$.

5.2 The main theorem on the structure of rigid local systems

(5.2.0) Given an everywhere tame lisse $\overline{\mathbb{Q}}_{\ell}$-sheaf on $\mathbb{A}^1 - D$, we denote by $\mathrm{LocMono}(\mathcal{F})$ the subgroup of the group of all continuous characters

$$\chi: I(0)^{\mathrm{tame}} \cong \pi_1{}^{\mathrm{tame}}(\mathbb{G}_{m,k}) \to \overline{\mathbb{Q}}_{\ell}{}^{\times}$$

generated by those which "occur" in the local monodromies of $\mathcal{F}$ at all the points of $\mathbb{P}^1$ [i.e., canonically identify the tame inertia groups $I(\alpha)^{\mathrm{tame}}$ at points α in $\mathbb{P}^1$ to $I(0)^{\mathrm{tame}}$ via automorphisms of $\mathbb{P}^1$ which carry α to 0 (the resulting identifications are independent of the choice)].

Main Theorem 5.2.1 In the situation 5.1.1, suppose $\mathcal{F}$ in $\mathcal{T}_{\ell}$ has generic rank $r(\mathcal{F}) \geq 2$, and is lisse on $\mathbb{A}^1 - D$. Suppose further that $\mathcal{F}$ is cohomologically rigid. Then

1) There exists a lisse, everywhere tame, rank one $\overline{\mathbb{Q}}_{\ell}$-sheaf $\mathcal{L}$ on $\mathbb{A}^1 - D$, and a nontrivial character χ, such that the (necessarily cohomologically rigid) object $\mathcal{G}$ in $\mathcal{T}_{\ell}$ defined by

$$\mathcal{G} := \mathrm{MC}_{\chi}\mathrm{MT}_{\mathcal{L}}(\mathcal{F})$$

has strictly lower generic rank: $r(\mathcal{G}) < r(\mathcal{F})$.

2) In 1) above, we may choose both $\mathcal{L}_{\chi}$ and $\mathcal{L}$ to have all of their local monodromies in the group $\mathrm{LocMono}(\mathcal{F})$, and, if we so choose them, then $\mathcal{G} := \mathrm{MC}_{\chi}\mathrm{MT}_{\mathcal{L}}(\mathcal{F})$has $\mathrm{LocMono}(\mathcal{G}) \subset \mathrm{LocMono}(\mathcal{F})$.

3) If for some integer $N\geq 1$ all the eigenvalues of all the local monodromies of $\mathcal{F}$ are N'th roots of unity, then in 1) above we may choose χ of order dividing N, and we may choose $\mathcal{L}$ to have each of its local monodromies of order dividing N, and if we so choose them,

all the eigenvalues of all the local monodromies of $\mathcal{G}$ will be N'th roots of unity.

(5.2.2) **proof** We will prove the theorem by giving an algorithm for how to choose both $\mathcal{L}$ and χ.
(5.2.2.1) In order to eliminate any ambiguity in the algorithm, for each point α in $\mathbb{P}^1(k)$, we will choose a total ordering of the underlying set of the group of all continuous characters

$$\chi: I(\alpha)^{\mathrm{tame}} \to \overline{\mathbb{Q}}_\ell^\times.$$

One way to do this is as follows. If we pick a topological generator, say γ^{tame}, of the group $I(\alpha)^{\mathrm{tame}}$ (which is canonically the group $\prod_{\ell \neq \mathrm{char}(k)} \mathbb{Z}_\ell(1)$), and a field embedding of $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$, then "evaluation at γ^{tame}" embeds the continuous characters into the group $\mathbb{C}^\times$. So we are reduced to exhibiting a total ordering of $\mathbb{C}^\times$. The simplest way to do this is to use polar coordinates, writing nonzero complex numbers as $re^{i\vartheta}$ with r and ϑ real, $r > 0$ and $0 \leq \vartheta < 2\pi$. Then we say $re^{i\vartheta} < r'e^{i\vartheta'}$ if either $r < r'$ or if $r=r'$ and $\vartheta < \vartheta'$.
(5.2.2.2) **First Step**.We now explain how to choose $\mathcal{L}$. Because $\mathcal{F}$ is lisse on $\mathbb{A}^1$ - D, and everywhere tame, at each point α in $D \amalg \{\infty\}$, the local monodromy of $\mathcal{F}$ defines an $r(\mathcal{F})$-dimensional representation of $I(\alpha)^{\mathrm{tame}}$. We first describe a shorthand to describe this representation of $I(\alpha)^{\mathrm{tame}}$. We write it as a direct sum of (character)$\otimes$(unipotent rep'n.):

$$\mathcal{F} \text{ as } I(\alpha)^{\mathrm{tame}}\text{-rep'n.} = \bigoplus_\chi \mathcal{L}_{\chi(x-\alpha)} \otimes \mathrm{Unip}(\alpha,\chi,\mathcal{F}),\ \alpha \text{ finite},$$

$$\mathcal{F} \text{ as } I(\infty)^{\mathrm{tame}}\text{-rep'n.} = \bigoplus_\chi \mathcal{L}_{\chi} \otimes \mathrm{Unip}(\infty,\chi,\mathcal{F}),\ \text{for } \alpha=\infty.$$

We write $\mathrm{Unip}(\alpha,\chi,\mathcal{F})$ as a direct sum of Jordan blocks, of dimensions $\{n_i(\alpha,\chi,\mathcal{F})\}_i$, and then pass to its "dual partition" [compare 3.1.10], the decreasing sequence of non-negative integers

$$e_1(\alpha,\chi,\mathcal{F}) \geq e_2(\alpha,\chi,\mathcal{F}) \geq \ldots \geq e_k(\alpha,\chi,\mathcal{F}) = 0 \text{ for } k >> 0$$

defined by

$$e_j(\alpha,\chi,\mathcal{F}) := \text{the number of Jordan blocks in } \mathrm{Unip}(\alpha,\chi,\mathcal{F})$$

whose dimension is $\geq j$.

Thus $e_1(\alpha,\chi,\mathcal{F})$ is the multiplicity

$$\dim\mathrm{Hom}_{I(\alpha)}(\{\mathcal{L}_{\chi(x-\alpha)} \text{ if } \alpha \text{ finite},\ \mathcal{L}_\chi \text{ if } \alpha=\infty\},\ \mathcal{F}(\alpha))$$

of χ as simple eigenvalue in $\mathcal{F}(\alpha) := \mathcal{F}$ as $I(\alpha)^{\mathrm{tame}}$-representation (and $e_j(\alpha,\chi,\mathcal{F})=0$ for all j if the character χ does not occur in the local monodromy of $\mathcal{F}$ at α).

Given $\mathcal{F}$ and a point α in D, pick a character χ whose $e_1(\alpha,\chi,\mathcal{F})$ is maximal among all the characters which occur in the local monodromy of $\mathcal{F}$ at α. There may not be a unique such χ, but in case of a tie we choose the one which comes first in the chosen total ordering of the set of all characters of $I(\alpha)^{\mathrm{tame}}$. [The proof works just as well if whenever we are confronted with a tie we make an arbitrary choice, rather than consult some pre-chosen total ordering.] This character we denote $\chi_{\alpha,\mathcal{F}}$.

(5.2.2.3) We now have chosen, at each point α in the finite set $D \subset \mathbb{A}^1$, a continuous character $\chi_{\alpha,\mathcal{F}}$ of $I(\alpha)^{\mathrm{tame}}$. By the known structure of the prime-to-p fundamental group of $\mathbb{A}^1 - D$ (the maximal prime-to-p quotient of the profinite completion of the free group with one generator for each point α in D), there is a unique (up to isomorphism) lisse, everywhere tame, rank one $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}$ on $\mathbb{A}^1 - D$ with the property that, for every α in D,

$$\mathcal{L} \text{ as } I(\alpha)^{\mathrm{tame}}\text{-representation} = (\chi_{\alpha,\mathcal{F}})^{-1}.$$

Concretely, $\mathcal{L}$ is the tensor product, over α in D, of the inverses of the translated Kummer sheaves $\mathcal{L}_{\chi_{\alpha,\mathcal{F}}}(x-\alpha)$.

(5.2.2.4) With this choice of $\mathcal{L}$, the sheaf $MT_{\mathcal{L}}(\mathcal{F})$, which has the same generic rank as $\mathcal{F}$ and the same index of rigidity as $\mathcal{F}$, has the additional property that, at each α in D, the trivial character $\mathbb{1}$ has maximal "simple multiplicity": for each α in D, we have

$$e_1(\alpha,\mathbb{1},MT_{\mathcal{L}}(\mathcal{F})) \geq e_1(\alpha,\chi,MT_{\mathcal{L}}(\mathcal{F})) \text{ for all } \chi.$$

(5.2.2.5) Notice that the local monodromies of $\mathcal{L}$ were all drawn from the group LocMono($\mathcal{F}$), and that, consequently, LocMono($MT_{\mathcal{L}}(\mathcal{F})$) lies in LocMono($\mathcal{F}$).

(5.2.2.6) **Second Step** In order to complete the proof of the theorem, it suffices to apply the following result to $MT_{\mathcal{L}}(\mathcal{F})$.

Main Theorem 5.2.3(= 5.2.1 bis) In the situation 5.1.1, suppose $\mathcal{F}$ in $\mathcal{T}_\ell$ has generic rank $r(\mathcal{F}) \geq 2$, and is lisse on $\mathbb{A}^1 - D$. Suppose that $\mathcal{F}$ satisfies the following condition:

(*) At every point α in D, we have

$$e_1(\alpha,\mathbb{1},\mathcal{F}) \geq e_1(\alpha,\chi,\mathcal{F}) \text{ for all } \chi.$$

If $\mathcal{F}$ is in addition cohomologically rigid, there exists a nontrivial character χ in LocMono($\mathcal{F}$), such that the (necessarily cohomologically rigid) object $\mathcal{G}$ in $\mathcal{T}_\ell$ defined by

$$\mathcal{G} := MC_\chi(\mathcal{F})$$

has strictly lower generic rank: $r(\mathcal{G}) < r(\mathcal{F})$. Moreover, LocMono($\mathcal{G}$) is contained in LocMono($\mathcal{F}$).

(5.2.4) **proof** The "moreover" follows from the explicit recipes of 3.3.6.

(5.2.4.1) We choose the character χ so as to maximize the dimension of the $I(\infty)^{\mathrm{tame}}$ invariants in $\mathcal{L}_\chi \otimes \mathcal{F}$. [If there is more than one χ which works, we choose the first of them in the chosen total ordering, though any choice would work for the proof.]

(5.2.4.2) We claim that **any** χ which maximize the dimension of the $I(\infty)^{\mathrm{tame}}$ invariants in $\mathcal{L}_\chi \otimes \mathcal{F}$ is nontrivial. If not, we would have

$$e_1(\infty, \mathbb{1}, \mathcal{F}) \geq e_1(\infty, \chi, \mathcal{F}) \text{ for all } \chi.$$

Recall that already at all finite singularities α in D, $\mathcal{F}$ has

$$e_1(\alpha, \mathbb{1}, \mathcal{F}) \geq e_1(\alpha, \chi, \mathcal{F}) \text{ for all } \chi.$$

(5.2.4.3) We will show this is impossible if $\mathcal{F}$ in $\mathcal{T}_\ell$ is cohomologically rigid. First, we remark that on $\mathbb{A}^1 - D$, $\mathcal{F}$ is both irreducible and nontrivial (just because it lies in $\mathcal{T}_\ell$). Therefore, denoting by

$$j: \mathbb{A}^1 - D \to \mathbb{P}^1$$

the inclusion, we have

$$H^0(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)) = 0 = H^2(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)),$$

and hence we have the inequality

$$\chi(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)) \leq 0.$$

(5.2.4.4) On the other hand, we are given that $\mathcal{F}$ is cohomologically rigid, i.e.,

$$\chi(\mathbb{P}^1, j_*\underline{\mathrm{End}}(\mathcal{F}|\mathbb{A}^1 - D)) = 2$$

(5.2.4.5) The idea now is to use the Euler-Poincaré formula to make explicit both of these Euler characteristics. Consider any $\mathcal{F}$ in $\mathcal{T}_\ell$ which is lisse on $\mathbb{A}^1 - D$. Because $\mathcal{F}$ is tame,

$$\chi(\mathbb{P}^1, j_*\underline{\mathrm{End}}(\mathcal{F}|\mathbb{A}^1 - D)) =$$

$$= \chi(\mathbb{A}^1 - D, \underline{\mathrm{End}}(\mathcal{F})) + \Sigma_{\alpha \text{ in } D \amalg \{\infty\}} \dim(\mathrm{End}(\mathcal{F})^{I(\alpha)})$$

$$= (1 - \mathrm{Card}(D)) r(\mathcal{F})^2 + \Sigma_{\alpha \text{ in } D \amalg \{\infty\}} \Sigma_\chi \Sigma_i (e_i(\alpha, \chi, \mathcal{F}))^2,$$

the last equality thanks to 3.1.15.

(5.2.4.6) Now at each α in $D\coprod\{\infty\}$, denote by $\chi_{\alpha,\mathcal{F}}$ a choice of character for which $e_1(\alpha,\chi,\mathcal{F})$ is maximal. Since the e_i are a decreasing sequence, we have

$$e_1(\alpha,\chi_{\alpha,\mathcal{F}},\mathcal{F}) \geq e_i(\alpha,\rho,\mathcal{F}) \text{ for all } \rho \text{ and for all } i.$$

Therefore we have the inequality

$$\Sigma_\chi \Sigma_i (e_i(\alpha,\chi,\mathcal{F}))^2 \leq \Sigma_\chi \Sigma_i e_1(\alpha,\chi_{\alpha,\mathcal{F}},\mathcal{F})e_i(\alpha,\chi,\mathcal{F}).$$

But

$$\Sigma_\chi \Sigma_i e_1(\alpha,\chi_{\alpha,\mathcal{F}},\mathcal{F})e_i(\alpha,\chi,\mathcal{F}) = e_1(\alpha,\chi_{\alpha,\mathcal{F}},\mathcal{F})\Sigma_\chi \Sigma_i e_i(\alpha,\chi,\mathcal{F})$$
$$= e_1(\alpha,\chi_{\alpha,\mathcal{F}},\mathcal{F})r(\mathcal{F}),$$

so we get

$$\chi(\mathbb{P}^1, j_*\underline{End}(\mathcal{F}|\mathbb{A}^1 - D)) \leq$$
$$\leq (1 - \mathrm{Card}(D))r(\mathcal{F})^2 + \Sigma_{\alpha \text{ in } D\coprod\{\infty\}} e_1(\alpha,\chi_{\alpha,\mathcal{F}},\mathcal{F})r(\mathcal{F}).$$

This we record as

Basic Inequality 5.2.4.7 For any $\mathcal{F}$ in $\mathcal{T}_\ell$ which is lisse on $\mathbb{A}^1 - D$, we have

$$\chi(\mathbb{P}^1, j_*\underline{End}(\mathcal{F}|\mathbb{A}^1 - D)) \leq$$
$$\leq r(\mathcal{F})[(1 - \mathrm{Card}(D))r(\mathcal{F}) + \Sigma_{\alpha \text{ in } D\coprod\{\infty\}} e_1(\alpha,\chi_{\alpha,\mathcal{F}},\mathcal{F})]$$

(5.2.4.8) Now suppose that

(**) for every α in $D\coprod\{\infty\}$, we have

$$e_1(\alpha,\mathbb{1},\mathcal{F}) \geq e_1(\alpha,\chi,\mathcal{F}) \text{ for all } \chi.$$

Then $e_1(\alpha,\chi_{\alpha,\mathcal{F}},\mathcal{F})$ is just $e_1(\alpha,\mathbb{1},\mathcal{F})$, and we find

$$(1/r(\mathcal{F}))\chi(\mathbb{P}^1, j_*\underline{End}(\mathcal{F}|\mathbb{A}^1 - D)) \leq$$
$$\leq (1 - \mathrm{Card}(D))r(\mathcal{F}) + \Sigma_{\alpha \text{ in } D\coprod\{\infty\}} e_1(\alpha,\mathbb{1},\mathcal{F}).$$

But the Euler Poincaré formula for $j_*\mathcal{F}$ gives

$$\chi(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)) = (1 - \mathrm{Card}(D))r(\mathcal{F}) + \Sigma_{\alpha \text{ in } D\coprod\{\infty\}} e_1(\alpha,\mathbb{1},\mathcal{F}).$$

Thus for $\mathcal{F}$ in $\mathcal{T}_\ell$ satisfying (**), we get the inequality

$$\chi(\mathbb{P}^1, j_*\underline{End}(\mathcal{F}|\mathbb{A}^1 - D)) \leq r(\mathcal{F})\chi(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)).$$

But $\chi(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)) \leq 0$ for any $\mathcal{F}$ in $\mathcal{T}_\ell$, so we get

$$\chi(\mathbb{P}^1, j_*\underline{End}(\mathcal{F}|\mathbb{A}^1 - D)) \leq 0,$$

which is impossible if $\mathcal{F}$ is cohomologically rigid, i.e., if $\chi(\mathbb{P}^1, j_*\underline{End}(\mathcal{F}|\mathbb{A}^1 - D)) = 2$.

(5.2.4.9) This shows that if we choose a character χ so as to maximize the dimension of the $I(\infty)^{\mathrm{tame}}$ invariants in $\mathcal{L}_\chi \otimes \mathcal{F}$, then any such χ is nontrivial, provided $\mathcal{F}$ is cohomologically rigid and satisfies the hypothesis (*) of 5.2.3.

(5.2.4.10) We must now show that for any such choice of χ, the sheaf $\mathcal{G} := MC_\chi(\mathcal{F})$ has strictly lower generic rank: $r(\mathcal{G}) < r(\mathcal{F})$. For this, we use the rank formula 3.3.7, according to which, for any $\mathcal{F}$ in $\mathcal{T}_\ell$,

$$r(\mathcal{G}) = \Sigma_{\alpha \text{ in } D} \operatorname{rank}(\mathcal{F}(\alpha)/\mathcal{F}(\alpha)^{I(\alpha)}) - \operatorname{rank}((\mathcal{F}(\infty)\otimes\mathcal{L}_\chi)^{I(\infty)})$$

$$= \operatorname{card}(D)r(\mathcal{F}) - \Sigma_{\alpha \text{ in } D} \operatorname{rank}(\mathcal{F}(\alpha)^{I(\alpha)}) - \operatorname{rank}((\mathcal{F}(\infty)\otimes\mathcal{L}_\chi)^{I(\infty)}).$$

$$= \operatorname{card}(D)r(\mathcal{F}) - \Sigma_{\alpha \text{ in } D} e_1(\alpha,\mathbb{1},\mathcal{F}) - \operatorname{rank}((\mathcal{F}(\infty)\otimes\mathcal{L}_\chi)^{I(\infty)}).$$

(5.2.4.11) The Euler-Poincaré formula for $\mathcal{F}$ says

$$\chi(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)) = (1 - \operatorname{Card}(D))r(\mathcal{F}) + \Sigma_{\alpha \text{ in } D\amalg\{\infty\}} e_1(\alpha,\mathbb{1},\mathcal{F}),$$

$$= -[\operatorname{card}(D)r(\mathcal{F}) - \Sigma_{\alpha \text{ in } D} e_1(\alpha,\mathbb{1},\mathcal{F})] + r(\mathcal{F}) + e_1(\infty,\mathbb{1},\mathcal{F}).$$

(5.2.4.12) Thus in our formula for $r(\mathcal{G})$ we get, for any $\mathcal{F}$ in $\mathcal{T}_\ell$,

$$r(\mathcal{G}) = r(\mathcal{F}) + e_1(\infty,\mathbb{1},\mathcal{F}) - \chi(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)) - \operatorname{rank}((\mathcal{F}(\infty)\otimes\mathcal{L}_\chi)^{I(\infty)}).$$

(5.2.4.13) We rewrite this as

$r(\mathcal{F}) - r(\mathcal{G}) =$

$$= \chi(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)) + \operatorname{rank}((\mathcal{F}(\infty)\otimes\mathcal{L}_\chi)^{I(\infty)}) - e_1(\infty,\mathbb{1},\mathcal{F}).$$

(5.2.4.14) So we need to show that, for our $\mathcal{F}$, we have

$$\chi(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)) + \operatorname{rank}((\mathcal{F}(\infty)\otimes\mathcal{L}_\chi)^{I(\infty)}) - e_1(\infty,\mathbb{1},\mathcal{F}) > 0.$$

(5.2.4.15) To show this, recall that for any $\mathcal{F}$ in $\mathcal{T}_\ell$ we had the Basic Inequality 5.2.2.7

$$\chi(\mathbb{P}^1, j_*\underline{\mathrm{End}}(\mathcal{F}|\mathbb{A}^1 - D)) \le$$

$$\le r(\mathcal{F})[(1 - \operatorname{Card}(D))r(\mathcal{F}) + \Sigma_{\alpha \text{ in } D\amalg\{\infty\}} e_1(\alpha,\chi_{\alpha,\mathcal{F}},\mathcal{F})].$$

For $\mathcal{F}$ cohomologically rigid and satisfying the hypothesis (*) of 5.2.3, the left hand side is 2 ($\mathcal{F}$ is cohomologically rigid) and on the right each term $e_1(\alpha,\chi_{\alpha,\mathcal{F}},\mathcal{F})$ for α in D is equal to $e_1(\alpha,\mathbb{1},\mathcal{F})$. Thus the Basic Inequality gives

$$2/r(\mathcal{F}) \le (1 - \operatorname{Card}(D))r(\mathcal{F}) + e_1(\infty,\chi_{\infty,\mathcal{F}},\mathcal{F}) + \Sigma_{\alpha \text{ in } D} e_1(\alpha,\mathbb{1},\mathcal{F})$$

$$= \chi(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)) + e_1(\infty,\chi_{\infty,\mathcal{F}},\mathcal{F}) - e_1(\infty,\mathbb{1},\mathcal{F}).$$

Since the left hand side (namely $2/r(\mathcal{F})$) is strictly positive, we get

$$0 < \chi(\mathbb{P}^1, j_*(\mathcal{F}|\mathbb{A}^1 - D)) + e_1(\infty,\chi_{\infty,\mathcal{F}},\mathcal{F}) - e_1(\infty,\mathbb{1},\mathcal{F}).$$

This is precisely the required inequality 5.2.4.14, for by the very choice of the character χ we have

$$e_1(\infty,\chi_{\infty,\mathcal{F}},\mathcal{F}) = \mathrm{rank}((\mathcal{F}(\infty)\otimes\mathcal{L}_\chi)^{I(\infty)}).\quad \text{QED}$$

5.3 Applications and interpretations of the main theorem

(5.3.1) We continue to work in the situation 5.1.1. Fix a dense open set $\mathbb{A}^1 - D$ in $\mathbb{A}^1$, and a subgroup Γ of the group of all continuous $\overline{\mathbb{Q}}_\ell^\times$-valued characters of $I(0)^{\mathrm{tame}}$. Let us denote by $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ the full subcategory of $\mathcal{T}_\ell$ consisting of the objects $\mathcal{F}$ which are lisse on $\mathbb{A}^1 - D$ and for which $\mathrm{LocMono}(\mathcal{F}) \subset \Gamma$.

(5.3.2) We can construct a graph whose vertices are the objects of $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$, and in which there is an edge joining two objects $\mathcal{F}$ and $\mathcal{G}$ if either of the following conditions a) or b) holds:

a) $\mathcal{F}$ and $\mathcal{G}$ both have generic rank >1, and there is a lisse tame rank one $\mathcal{L}$ on $\mathbb{A}^1$ -D, all of whose local monodromies are in Γ, such that $\mathcal{F} \cong \mathrm{MT}_\mathcal{L}(\mathcal{G})$, or equivalently $\mathcal{G} \cong \mathrm{MT}_{\mathcal{L}^{-1}}(\mathcal{F})$.

b) there exists a nontrivial character χ in Γ such that $\mathcal{F} \cong \mathrm{MC}_\chi(\mathcal{G})$, or equivalently $\mathcal{G} \cong \mathrm{MC}_{\chi^{-1}}(\mathcal{F})$.

In terms of this graph, the Main Theorem says precisely

Main Theorem 5.3.3 (= 5.2.1 bis). If $\mathcal{F}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ is cohomologically rigid, then $\mathcal{F}$ (as vertex of the graph 5.3.2) is connected to an object of rank one in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$, and its distance to such an object is at most $2(r(\mathcal{F}) -1)$.

5.4 Some open questions

(5.4.1) In this graph, any two objects which are connected have the same index of rigidity $\chi(\mathbb{P}^1, j_\star \underline{\mathrm{End}(\mathcal{F}|\mathbb{A}^1 - D)})$, by 4.3.10. Let us say that an object $\mathcal{F}$ of $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ is **minimal** if for any object $\mathcal{G}$ to which it is connected, $r(\mathcal{F}) \le r(\mathcal{G})$. Obviously every object is connected to a minimal object which has the same index of rigidity. The Main Theorem 5.3.3 states that the minimal cohomologically rigid objects are exactly those of generic rank one.

(5.4.2) But the index of rigidity, a priori even and ≤ 2, can be **any** even integer ≤ 2. [For instance, the pullback by $x \mapsto x^n$ of the rank two local system on $\mathbb{P}^1 - \{0,1,\infty\}$ attached to the differential equation for the Gauss hypergeometric function $F(1/2,1/2,1;x)$ has index of rigidity $4 - 2n$.] What are the minimal objects $\mathcal{F}$ whose index of rigidity is some given even integer ≤ 0 Already the simplest case of this question, index of rigidity =0, is far from understood. For instance, is it true that if an object is not minimal, then it is at distance at most two from an object of strictly lower rank? Are there minimal objects with index of rigidity =0 of arbitrarily large rank?

(5.4.3) Here is a "candidate" for a construction of minimal objects with index of rigidity =0 of arbitrarily large rank on $\mathbb{A}^1$ - {3 points}. Over $\mathbb{C}$ take an elliptic curve $y^2 = (x-e_1)(x-e_2)(x-e_3)$, an integer $n \geq 1$, and a primitive n'th root of unity ζ. Denote by $\mathcal{F}_{n,\zeta}$ the (extension by zero to E of the) rank n local system on E - {0} with local monodromy around 0 given by the scalar ζ constructed in the proof of 1.4.4. For a rank one local system $\mathcal{L}$ on E for which $\mathcal{L}^{\otimes 2n}$ is nontrivial, there exists no isomorphism (of local systems on E - {0}) $\mathcal{L}\otimes\mathcal{F}_{n,\zeta} \cong [-1]^*(\mathcal{L}\otimes\mathcal{F}_{n,\zeta})$ [compare determinants]. Let $\pi: E \to \mathbb{P}^1$ be the map "x". Then $\mathcal{G} := \pi_*(\mathcal{L}\otimes\mathcal{F}_{n,\zeta})$, for $\mathcal{L}$ as above, is an irreducible middle extension sheaf on $\mathbb{P}^1$ which on $\mathbb{A}^1 - \{e_1, e_2, e_3\}$ is a rank 2n local system with index of rigidity 0. Its local monodromies at the four missing points are semisimple, with eigenvalues

at ∞: each of the two square roots of ζ, repeated n times

at e_i: each of ± 1, repeated n times.

It is easy to show that any object of distance at most two from $\mathcal{G}$ has rank at least that of $\mathcal{G}$. Is $\mathcal{G}$ minimal? If so, $\mathcal{G}$ provides an example of a minimal object of rank 2n, lisse on $\mathbb{A}^1 - \{e_1, e_2, e_3\}$, with index of rigidity =0.

5.5 Existence of universal families of rigids with given local monodromy

(5.5.1) Let k be an algebraically closed field, $n \geq 2$ an integer, $\alpha_1, \alpha_2, \ldots \alpha_n$ a set of n distinct points of $\mathbb{A}^1(k)$, ℓ a prime number invertible in k, $N \geq 1$ an integer which is invertible in k, and ζ a primitive N'th root of unity in k. Let $\mathcal{F}$ be an object of $\mathcal{T}_\ell$ which is lisse on $\mathbb{A}^1 - \{\alpha_1, \alpha_2, \ldots \alpha_n\}$, cohomologically rigid, and such that all eigenvalues of all local monodromies of $\mathcal{F}$ are N'th roots of unity.

(5.5.2) Given the data (n, N, ℓ), we form the ring

$R_{N,\ell} := \mathbb{Z}[\zeta_N, 1/N\ell]$, ζ_N := a primitive N'th root of unity (i.e., $R_{N,\ell}$ is $\mathbb{Z}[1/N\ell][X]/(\Phi_N(X))$, with $\Phi_N(X)$ in $\mathbb{Z}[X]$ the N'th cyclotomic polynomial). We denote by $E = E_N$ the fraction field of $R_{N,\ell}$: thus E is the cyclotomic field $\mathbb{Q}(\zeta_N)$. We fix an embedding

$$R_{N,\ell} \to \overline{\mathbb{Q}}_\ell,$$

i.e., we fix a primitive N'th root of unity in $\overline{\mathbb{Q}}_\ell$. We denote by λ the induced place of the "abstract" field E, and by E_λ the λ-adic completion of E.

(5.5.3) Denote by $S_{N,n,\ell}$ the ring

$$S_{N,n,\ell} := R_{N,\ell}[T_1, \ldots, T_n][1/\Delta],\ \Delta := \prod_{i\neq j}(T_i - T_j).$$

There is a unique ring homomorphism

$$\varphi : S_{N,n,\ell} \to k$$

for which $\varphi(\zeta_N) = \zeta$ and for which $\varphi(T_i) = \alpha_i$ for $1 \le i \le n$.

Over $S_{N,n,\ell}$, we have $\mathbb{A}^1$, with its n disjoint sections $\{T_1, \ldots, T_n\}$. We denote by

$$j: (\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}} \to (\mathbb{A}^1)_{S_{N,n,\ell}}$$

the inclusion.

Theorem 5.5.4 In the situation 5.5.1, we have

1) There exists a lisse E_λ-sheaf

$$\mathfrak{F} \text{ on } (\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$$

which, after the base change $\varphi : S_{N,n,\ell} \to k$ and the extension of scalars $E_\lambda \to \overline{\mathbb{Q}}_\ell$ becomes (the restriction to $\mathbb{A}^1 - \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ of) $\mathcal{F}$.

2) ("mise pour memoire", cf. 4.2.3, 4.3.8, and 4.3.9) The object $j_*\mathfrak{F}$ on $(\mathbb{A}^1)_{S_{N,n,\ell}}$ is of formation compatible with arbitrary change of base on $S_{N,n,\ell}$, and is adapted to the stratification

$$((\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}, \{T_1, \ldots, T_n\}_{S_{N,n,\ell}}) \text{ of } (\mathbb{A}^1)_{S_{N,n,\ell}}.$$

The restriction of $j_*\mathfrak{F}$ to every geometric fibre of $(\mathbb{A}^1)_{S_{N,n,\ell}}$ is (after extension of scalars $E_\lambda \to \overline{\mathbb{Q}}_\ell$) a cohomologically rigid object of $\mathcal{T}_\ell$, all of whose local monodromies have all their eigenvalues N'th roots of unity.

3) The lisse E_λ-sheaf $\mathfrak{F}$ on $(\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$ is pure of some integer weight w, $0 \le w \le \mathrm{rank}(\mathfrak{F}) - 1$, and all its characteristic polynomials of Frobenius at all finite field-valued points have coefficients in the subring $\mathbb{Z}[\zeta_N]$ of E_λ (a subring via the given embedding of $R_{N,\ell}$ into $\overline{\mathbb{Q}}_\ell$).

4) For any prime number ℓ_1, and any embedding

$$\lambda_1: \mathbb{Z}[\zeta_N] \to \overline{\mathbb{Q}}_{\ell_1},$$

there exists on $(\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell_1}}$ a lisse E_{λ_1}-sheaf $\mathfrak{F}_{\lambda_1}$ which satisfies 2) above with ℓ replaced by ℓ_1, which is pure of the same weight w, which at every finite field-valued point has characteristic polynomial of Frobenius with coefficients in the subring $\mathbb{Z}[\zeta_N]$ of E_{λ_1}. Moreover, at any such finite field valued point where ℓ is invertible, the characteristic polynomial of $\mathfrak{F}_{\lambda_1}$ is equal to that of $\mathfrak{F}$ (equality in the common subring $\mathbb{Z}[\zeta_N]$).

5.5.5 construction-proof of 5.5.4

(5.5.5.1) We proceed by induction on the generic rank $r(\mathcal{F})$ of $\mathcal{F}$. Suppose first that $r(\mathcal{F}) = 1$. Then on $\mathbb{A}^1 - \{\alpha_1, \alpha_2, \ldots \alpha_n\}$, $\mathcal{F}$ is a lisse sheaf of rank one, all of whose local monodromies have order dividing N. Denote by χ_i the character of order dividing N of $I(\alpha_i)^{\mathrm{tame}}$ given by $\mathcal{F}$ at α_i. Thus χ_i is a $E_\lambda{}^\times$-valued character of $I(\alpha_i)^{\mathrm{tame}}/NI(\alpha_i)^{\mathrm{tame}} \cong \mu_N(k)$, the isomorphism given by the action of $\mu_N(k)$ as galois group of the finite etale galois connected covering of $\mathbb{A}^1 - \{\alpha_i\}$ of equation $y^N = x - \alpha_i$, on which ζ in $\mu_N(k)$ acts by $(x,y) \mapsto (x,\zeta y)$. Thus the translated Kummer sheaf $\mathcal{L}_{\chi_i(x - \alpha_i)}$ on $\mathbb{A}^1 - \{\alpha_i\}$ is a lisse rank one E_λ-sheaf on $\mathbb{A}^1 - \{\alpha_i\}$, and it has the same local monodromy at α_i that $\mathcal{F}$ does.

(5.5.5.2) Consider the lisse, rank one E_λ-sheaf

$$\otimes_i \mathcal{L}_{\chi_i(x - \alpha_i)} \text{ on } \mathbb{A}^1 - \{\alpha_1, \alpha_2, \ldots \alpha_n\}.$$

It has the same local monodromy as $\mathcal{F}$ at each point α_i, and both it and $\mathcal{F}$ are tame at ∞. Therefore

$$\mathcal{F} \cong \otimes_i \mathcal{L}_{\chi_i(x - \alpha_i)} \text{ on } \mathbb{A}^1 - \{\alpha_1, \alpha_2, \ldots \alpha_n\},$$

because their ratio is lisse and tame on $\mathbb{A}^1$, but $\pi_1{}^{\mathrm{tame}}(\mathbb{A}^1) = 0$.

(5.5.5.3) By means of the fixed choices of primitive N'th roots of unity ζ_N in $R_{N,\ell} \subset S_{N,n,\ell}$ and ζ in k, we may identify the groups $\boldsymbol{\mu}_N(R_{N,\ell}) = \boldsymbol{\mu}_N(S_{N,n,\ell})$ and $\boldsymbol{\mu}_N(k)$. This allows us to view χ_i as a character of the group $\boldsymbol{\mu}_N(S_{N,n,\ell})$ which is the galois group of the finite etale galois connected covering of $(\mathbb{A}^1 - \{T_i\})_{S_{N,n,\ell}}$ of equation $y^N = x - T_i$, on which ζ in $\boldsymbol{\mu}_N(S_{N,n,\ell})$ acts by $(x,y) \mapsto (x,\zeta y)$. Thus we may speak of the translated Kummer sheaf $\mathcal{L}_{\chi_i(x - T_i)}$ as a lisse E_λ-sheaf of rank one on $(\mathbb{A}^1 - \{T_i\})_{S_{N,n,\ell}}$.

(5.5.5.4) We now define $\boldsymbol{\mathcal{F}}$ to be

$$\boldsymbol{\mathcal{F}} := \otimes_i \mathcal{L}_{\chi_i(x - T_i)} \text{ on } (\mathbb{A}^1 - \{T_1, \dots , T_n\})_{S_{N,n,\ell}}.$$

It is obvious that 1) (and hence 2) also) and 3) are satisfied, with the weight w=0. Since the characters χ_i have values in the subgroup $\mu_N(\mathbb{Z}[\zeta_N])$ of $\overline{\mathbb{Q}}_\ell{}^\times$, for any prime number ℓ_1, and any embedding

$$\lambda_1: \mathbb{Z}[\zeta_N] \to \overline{\mathbb{Q}}_{\ell_1},$$

we can view the χ_i as $E_{\lambda_1}{}^\times$-valued characters, and define $\boldsymbol{\mathcal{F}}_{\lambda_1}$ by the same recipe as above, viewing now each $\mathcal{L}_{\chi_i(x - T_i)}$ as a lisse rank one E_{λ_1}-sheaf on $(\mathbb{A}^1 - \{T_i\})_{S_{N,n,\ell_1}}$. It is obvious that 4) is now satisfied. [One should remark that, fibre by fibre, this $\boldsymbol{\mathcal{F}}_{\lambda_1}$ does indeed have at least two finite singularities, since along the section T_i its local monodromy is of exactly the same order as was the local monodromy of the original $\mathcal{F}$ at α_i, namely the order of the character χ_i, and the assumption that $\mathcal{F}$ is in $\mathcal{T}_\ell$ guarantees that χ_i is nontrivial for at least two distinct values of i.] This concludes the construction-proof in the case of generic rank one.

(5.5.5.5) We now explain how to pass to the general case. Thus let $\mathcal{F}$ in $\mathcal{T}_\ell$ be lisse on $\mathbb{A}^1 - \{\alpha_1, \alpha_2, \dots \alpha_n\}$, cohomologically rigid, with all all local monodromy eigenvalues N'th roots of unity, and with generic rank $r(\mathcal{F}) \geq 2$.

(5.5.5.6) According to the main theorem 5.2.1, there exist

1) an object $\mathcal{G}$ in $\mathcal{T}_\ell$, lisse on $\mathbb{A}^1 - \{\alpha_1, \alpha_2, \dots \alpha_n\}$, with all local

monodromy eigenvalues N'th roots of unity, with $r(\mathcal{G}) < r(\mathcal{F})$,
2) a lisse rank one $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}$ on $\mathbb{A}^1 - \{\alpha_1, \alpha_2, \dots \alpha_n\}$, with all local monodromies of order dividing N,
3) a nontrivial $\overline{\mathbb{Q}}_\ell$-valued character χ of $\pi_1(\mathbb{G}_m)^{\mathrm{tame}}$ of order dividing N,
4) an isomorphism

$$\mathcal{F} \cong MT_{\mathcal{L}}(MC_\chi(\mathcal{G})).$$

(5.5.5.7) By applying the argument given in the rank one case to $\mathcal{L}$ on $\mathbb{A}^1 - \{\alpha_1, \alpha_2, \dots \alpha_n\}$, and to $\mathcal{L}_\chi$ on $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$, we construct lisse rank one E_λ-sheaves $\boldsymbol{\mathcal{L}}$ on $(\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}}$ and $\boldsymbol{\mathcal{L}}_\chi$ on $(\mathbb{A}^1 - \{0\})_{S_{N,n,\ell}}$ which after the base change $\varphi : S_{N,n,\ell} \to k$ and the extension of scalars $E_\lambda \to \overline{\mathbb{Q}}_\ell$ become $\mathcal{L}$ and $\mathcal{L}_\chi$ respectively. The conclusions 2), 3) and 4) of the theorem hold for $\boldsymbol{\mathcal{L}}$ on $(\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}}$. These same conclusions hold also for $\boldsymbol{\mathcal{L}}_\chi$ on $(\mathbb{A}^1 - \{0\})_{S_{N,n,\ell}}$ provided we delete the phrase "in $\mathcal{T}_\ell$" from 2).
(5.5.5.8) By induction, we may assume there exists a lisse E_λ-sheaf $\boldsymbol{\mathcal{G}}$ on $(\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}}$ for which after the base change $\varphi : S_{N,n,\ell} \to k$ and the extension of scalars $E_\lambda \to \overline{\mathbb{Q}}_\ell$ becomes $\mathcal{G}$, and for which 2), 3), and 4) hold. We define $\boldsymbol{\mathcal{F}}$ on $(\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}}$ by

$$\boldsymbol{\mathcal{F}}[1] := \boldsymbol{\mathcal{L}} \otimes j^*(j_* \boldsymbol{\mathcal{G}}[1] *_{\mathrm{mid}+} j_{0*} \boldsymbol{\mathcal{L}}_\chi[1]),$$

where

$$j: (\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}} \to (\mathbb{A}^1)_{S_{N,n,\ell}},$$

and $\quad j_0: (\mathbb{A}^1 - \{0\})_{S_{N,n,\ell}} \to (\mathbb{A}^1)_{S_{N,n,\ell}}$

denote the inclusions.
(5.5.5.9) That 1) holds results from 4.3.2, 4.3.3, and 4.3.10. As already remarked, 2) for $\boldsymbol{\mathcal{F}}$ is automatic once we have 1) [Use 4.3.10 to see that (after extension of scalars $E_\lambda \to \overline{\mathbb{Q}}_\ell$) $\boldsymbol{\mathcal{F}} \otimes \boldsymbol{\mathcal{L}}^{-1}$, and hence $\boldsymbol{\mathcal{F}}$, whose generic rank is ≥ 2, is fibre-by-fibre in $\mathcal{T}_\ell$.] To prove 3) for $\boldsymbol{\mathcal{F}}$, it suffices to do so for $(j_* \boldsymbol{\mathcal{G}}[1] *_{\mathrm{mid}+} j_{0*} \boldsymbol{\mathcal{L}}_\chi[1])[-1]$, because we already know that 3) holds for $\boldsymbol{\mathcal{L}}$ itself, with weight $w=0$.
(5.5.5.10) We begin the proof of 3). Let $\mathbb{F}$ be a finite field in which

$N\ell$ is invertible, and $\rho : S_{N,n,\ell} \to \mathbb{F}$ a ring homomorphism. Pick a point τ in $\mathbb{A}^1(\mathbb{F})- \{\rho(T_1), \ldots , \rho(T_n)\}$. Denote by U the open set

$$U := (\mathbb{A}^1- \{\rho(T_1), \ldots , \rho(T_n), \tau\})_{\mathbb{F}}$$

of $\mathbb{A}^1{}_{\mathbb{F}}$. According to 5.1.7, the stalk of $(j_*\mathcal{G}[1]*_{mid+}j_{0*}\mathcal{L}_\chi[1])[-1]$ at the geometric point $(\rho,\tau,\overline{\mathbb{F}})$ is the image of the "forget supports" map

$$H^1{}_c(U\otimes\overline{\mathbb{F}}, \mathcal{G}\otimes\mathcal{L}_\chi(\tau-x)) \to H^1(U\otimes\overline{\mathbb{F}}, \mathcal{G}\otimes\mathcal{L}_\chi(\tau-x)).$$

By induction, $\mathcal{G}$ is pure weight w, $0 \le w \le \mathrm{rank}(\mathcal{G}) - 1$, while $\mathcal{L}_\chi$ is pure of weight zero. By [De-Weil II, 3.2.3 and remark following its statement] the image of the "forget supports" map is precisely the weight = w+1 quotient of of the group $H^1{}_c(U\otimes\overline{\mathbb{F}}, \mathcal{G}\otimes\mathcal{L}_\chi(\tau-x))$, which is a priori of weight $\le$ w+1. Now the cohomology groups

$$H^i{}_c(U\otimes\overline{\mathbb{F}}, \mathcal{G}\otimes\mathcal{L}_\chi(\tau-x)) = 0 \text{ for } i\neq 1$$

(for i=0 because we are lisse on an open curve, for i=2 because in addition we are geometrically irreducible and nontrivial). Therefore the characteristic polynomial of Frobenius $\mathrm{Frob}_{\rho,\tau,\mathbb{F}}$ on $H^1{}_c(U\otimes\overline{\mathbb{F}}, \mathcal{G}\otimes\mathcal{L}_\chi(\tau-x))$ is, by the Lefschetz Trace Formula [Gro-FL], equal to the L-function of the lisse sheaf $\mathcal{G}\otimes\mathcal{L}_\chi(\tau-x)$ on U:

$$L(U,\mathcal{G}\otimes\mathcal{L}_\chi(\tau-x),T)=$$

$$=\det(1 - T\mathrm{Frob}_{\rho,\tau,\mathbb{F}} \mid H^1{}_c(U\otimes\overline{\mathbb{F}}, \mathcal{G}\otimes\mathcal{L}_\chi(\tau-x))).$$

Therefore the characteristic polynomial of Frobenius $\mathrm{Frob}_{\rho,\tau,\mathbb{F}}$ on $(j_*\mathcal{G}[1]*_{mid+}j_{0*}\mathcal{L}_\chi[1])[-1]$ is the "pure of weight w+1 part" of this L-function.

(5.5.5.11) We now consider more closely this L function. Because both $\mathcal{G}$ and $\mathcal{L}_\chi$ have all their characteristic polynomials of Frobenius with coefficients in the subring $\mathbb{Z}[\zeta_N]$ of $\overline{\mathbb{Q}}_\ell$, the Euler product for this L function shows that the L-function itself lies in $1+T(\mathbb{Z}[\zeta_N][[T]])$, hence (being a polynomial) in $1+T(\mathbb{Z}[\zeta_N][T])$, and that it is entirely determined by all the individual characteristic polynomials of Frobenius of both $\mathcal{G}$ and $\mathcal{L}_\chi$. If we factor L(T) as

$$L(T) = \prod_j (1 - \beta_j T),$$

the reciprocal roots β_j are algebraic integers, which (as a set with multiplicity) are stable under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\zeta_N))$. Taking the "part of weight w+1" of L produces the intrinsic divisor of L(T) in which we keep precisely those β_j which, together with all their conjugates an algebraic integers, have complex absolute value $\mathrm{sqrt}((\mathrm{Card}(\mathbb{F}))^{w+1}$. This intrinsic divisor is also stable under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\zeta_N))$, so has coefficients in $\mathbb{Q}(\zeta_N) \cap \{\text{algebraic integers}\} = \mathbb{Z}[\zeta_N]$. Moreover, being intrinsic, it too is entirely determined by all the individual characteristic polynomials of Frobenius of both $\mathcal{G}$ and $\mathcal{L}_\chi$.

(5.5.5.12) It remains to prove 4). Again by induction, all the objects $\mathcal{G}$, $\mathcal{L}$ and $\mathcal{L}_\chi$ have λ_1-adic versions, say $\mathcal{G}_{\lambda_1}$, $\mathcal{L}_{\lambda_1}$ and $\mathcal{L}_{\chi_{\lambda_1}}$. Using these, we define $\mathcal{F}_{\lambda_1}$ out of them in precisely the same way we defined $\mathcal{F}$:

$$\mathcal{F}_{\lambda_1}[1] := \mathcal{L}_{\lambda_1} \otimes j^*(j_*\mathcal{G}_{\lambda_1}[1] *_{\mathrm{mid}+} j_{0*}\mathcal{L}_{\chi_{\lambda_1}}[1]).$$

Now repeat for $j^*(j_*\mathcal{G}_{\lambda_1}[1] *_{\mathrm{mid}+} j_{0*}\mathcal{L}_{\chi_{\lambda_1}}[1])[-1]$ the above discussion of characteristic polynomials of Frobenius. It will establish that its characteristic polynomials of Frobenius have coefficients in $\mathbb{Z}[\zeta_N]$, and are entirely determined by all the individual characteristic polynomials of Frobenius of both $\mathcal{G}_{\lambda_1}$ and $\mathcal{L}_{\chi_{\lambda_1}}$ by exactly the same "part of weight w+1 of an L function" rules as the characteristic polynomials of $\mathcal{F}$ we determined by those of $\mathcal{G}$ and $\mathcal{L}_\chi$. This proves inductively that $j^*(j_*\mathcal{G}_{\lambda_1}[1] *_{\mathrm{mid}+} j_{0*}\mathcal{L}_{\chi_{\lambda_1}}[1])[-1]$ is pure of weight w+1, has characteristic polynomials of Frobenius with coefficients in $\mathbb{Z}[\zeta_N]$, and that for any finite field-valued point of $(\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}}$ where ℓ_1 is invertible, $j^*(j_*\mathcal{G}[1] *_{\mathrm{mid}+} j_{0*}\mathcal{L}_\chi[1])[-1]$ and $j^*(j_*\mathcal{G}_{\lambda_1}[1] *_{\mathrm{mid}+} j_{0*}\mathcal{L}_{\chi_{\lambda_1}}[1])[-1]$ have the "same" (comparison in the common subring $\mathbb{Z}[\zeta_N]$ of $\overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{Q}}_{\ell_1}$) characteristic polynomial of Frobenius. Because we already know that 4) holds for $\mathcal{L}$ and its λ_1-partner $\mathcal{L}_{\lambda_1}$, we may tensor by these to deduce that $\mathcal{F}$ and $\mathcal{F}_{\lambda_1}$ also have the same characteristic polynomials of Frobenius at any finite field-valued point of $(\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}}$ where ℓ_1 is invertible.

(5.5.5.13) It remains only to show that $j_*\mathcal{F}_{\lambda_1}$ satisfies 2). Using 4.3.8, we get most of 2): it remains only to show that $j_*\mathcal{F}_{\lambda_1}$ is fibre-by -fibre in $\mathcal{T}_\ell$. But as rank($\mathfrak{F}_{\lambda_1}$) ≥ 2, it is equivalent to show that $(j_*\mathfrak{G}_{\lambda_1}[1]*_{mid+}j_{0*}\mathfrak{L}_{\chi_{\lambda_1}}[1])[-1]$ is fibre-by -fibre in $\mathcal{T}_\ell$.This results from 4.3.10 and 4.3.11. QED

(5.5.6) We now discuss the weight w of $\mathfrak{F}$ constructed in 5.5.4. If $\mathcal{F}$ has rank one, then w=0. When $\mathcal{F}$ has rank $r(\mathcal{F}) \geq 2$, we can, by 5.3.3, connect $\mathcal{F}$ to a rank one object $\mathcal{L}_0$ through a finite number of steps, each of which is either an $MT_{\mathcal{L}}$ or an MC_χ, say

$$\mathcal{F} = MT_{\mathcal{L}_d}\circ MC_{\chi_d}\circ \dots \circ MT_{\mathcal{L}_1}\circ MC_{\chi_1}(\mathcal{L}_0),$$

in such a way that
1) each application of an MC_χ strictly increases the rank,
2) we have $(\chi_i)^N = \mathbb{1}$ for i=1, ..., d, but each $\chi_i \neq \mathbb{1}$,
3) we have $(\mathcal{L}_i)^{\otimes N} \cong \overline{\mathbb{Q}}_\ell$, for i = 0, ..., d, but $\mathcal{L}_i$ is non-constant for i=0, ..., d-1.

There may be more than one such expression for $\mathcal{F}$, but each choice of such an expression gives rise to an $\mathfrak{F}$. Concretely, we "thicken" each $\mathcal{L}_i$ to $\mathfrak{L}_i$, each $\mathcal{L}_{\chi_i}$ to $\mathfrak{L}_{\chi_i}$, and inductively define $\mathfrak{F}_0 := \mathfrak{L}_0$, $\mathfrak{F}_1$, ..., $\mathfrak{F}_d := \mathfrak{F}$ by (in the notations of 5.5.5.8)

$$\mathfrak{F}_{i+1}[1] := \mathfrak{L}_i\otimes j^*(j_*\mathfrak{F}_{i-1}[1]*_{mid+}j_{0*}\mathfrak{L}_{\chi_i}[1]),$$

for i=0, ..., d-1. The weight of the $\mathfrak{F}$ constructed this way is the number d of steps of type MC_χ.

(5.5.7) We now examine the unicity of an $\mathfrak{F}$ given by 5.5.4. Denote by

$$\pi : (\mathbb{A}^1- \{T_1, \dots , T_n\})_{S_{N,n,\ell}} \to S_{N,n,\ell}.$$

the projection. For any lisse E_λ-sheaf $\mathfrak{L}$ on $S_{N,n,\ell}$, $\mathfrak{F}\otimes\pi^*(\mathfrak{L})$ works just as well as $\mathfrak{F}$ in 5.5.4.1). This is the only ambiguity in $\mathfrak{F}$.

Lemma 5.5.7.1 Hypotheses and notations as in 5.5.4, suppose $\mathfrak{F}$ and $\mathfrak{F}_1$ both satisfy 5.5.4 1). Then there exists a lisse E_λ-sheaf $\mathfrak{L}$ on $S_{N,n,\ell}$, and an isomorphism $\mathfrak{F}_1 \cong \mathfrak{F}\otimes\pi^*(\mathfrak{L})$

proof This follows from the Rigidity Corollary 5.5.7.3 below, applied to S:= Spec($S_{N,n,\ell}$), and U := $(\mathbb{A}^1- \{T_1, \dots , T_n\})_{S_{N,n,\ell}}$.

Lemma 5.5.7.2 (mise pour memoire) Let ℓ be a prime number, S a normal connected noetherian $\mathbb{Z}[1/\ell]$-scheme whose generic point has characteristic zero, X/S a proper smooth curve with geometrically connected fibres, $D \subset X$ a closed subscheme which is finite etale over S of degree $d \geq 0$, $U := X - D$, and $\pi : U \to S$ the structural map. For any finite extension E_λ of $\mathbb{Q}_\ell$, and and lisse E_λ-sheaf $\mathcal{F}$ on U, the sheaves $R^i\pi_!\mathcal{F}$ and $R^i\pi_*\mathcal{F}$ on S are lisse, and of formation compatible with arbitrary change of base on S.

proof Because S has generic characteristic zero, $\mathcal{F}$ is automatically tame on the geometric generic fibre, and the asserted result for $R^i\pi_!\mathcal{F}$ is well-known, cf. [Ka-SE, 4.7.1]. This result, applied to $\mathcal{F}^\vee$, yields the result for the $R^i\pi_*\mathcal{F}$ by duality. QED

Rigidity Corollary 5.5.7.3 In the situation 5.5.7.1, let $\mathcal{F}$ and $\mathcal{G}$ be lisse E_λ-sheaves (respectively $\overline{\mathbb{Q}}_\ell$-sheaves) on U. The following conditions are equivalent.

1) For some geometric point s of S, both $\mathcal{F}|U_s$ and $\mathcal{G}|U_s$ are absolutely irreducible, and there exists an isomorphism $\mathcal{F}|U_s \cong \mathcal{G}|U_s$ of lisse sheaves on U_s.

2) For every geometric point s of S, both $\mathcal{F}|U_s$ and $\mathcal{G}|U_s$ are absolutely irreducible, and there exists an isomorphism $\mathcal{F}|U_s \cong \mathcal{G}|U_s$ of lisse sheaves on U_s.

3) Condition 2) holds, and there exists a lisse, rank one E_λ-sheaf (respectively $\overline{\mathbb{Q}}_\ell$-sheaf) $\mathcal{L}$ on S, and an isomorphism $\mathcal{G} \cong \mathcal{F}\otimes\pi^*(\mathcal{L})$ of lisse sheaves on U.

proof It is trivial that 3) $\Rightarrow$ 2) $\Rightarrow$ 1). We now show that 1 $\Rightarrow$ 3). Consider the lisse sheaf $\mathcal{H} := \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ on U. By 5.5.7.1 applied to $\mathcal{H}$, $\pi_*\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is lisse on S, of formation compatible with arbitrary change of base on S. Taking the fibre at s, we see from 1) that $\mathcal{L} := \pi_*\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is lisse of rank one. We have a canonical map of lisse sheaves on U, $\mathcal{F}\otimes\pi^*(\mathcal{L}) \to \mathcal{G}$, which is is an isomorphism on U_s. But the kernel and cokernel are of this map are lisse sheaves on U, so both must vanish. Therefore $\mathcal{F}\otimes\pi^*(\mathcal{L}) \cong \mathcal{G}$. For every geometric point s of S, we get $\mathcal{F}|U_s \cong \mathcal{G}|U_s$ by passing to fibres. That both $\mathcal{F}|U_s$ and $\mathcal{G}|U_s$ are absolutely irreducible for every geometric point s results from 4.2.6. QED

(5.5.8) We now clarify the sense in which the $\mathfrak{F}$ constructed in 5.5.4 is a <u>universal</u> family of rigids with the same local monodromy as $\mathcal{F}$. We must first specify the precise meaning of "same local monodromy as $\mathcal{F}$". Thus let k_1 be a second algebraically closed field in which ℓN is invertible, ζ_1 in k_1 a primitive N'th root of unity, and $\beta_1, ..., \beta_n$ a set of $n \geq 2$ distinct points of $\mathbb{A}^1(k_1)$. Let $\mathcal{F}_1$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1{}_{k_1} - \{\beta_1, ..., \beta_n\}$ which is everywhere tame and such that all eigenvalues of all local monodromies of $\mathcal{F}_1$ are N'th roots of unity. At any point β in $\{\beta_1, ..., \beta_n, \infty\}$, we can describe the local monodromy of $\mathcal{F}_1$ at β as follows. The tame inertia group $I(\beta)^{\mathrm{tame}}$ is canonically the pro-cyclic group $\prod_{\ell \neq \mathrm{char}(k)} \mathbb{Z}_\ell(1)(k_1)$ $\cong$lim. inv. $\boldsymbol{\mu}_M(k_1)$, the inverse limit taken over M's invertible in k_1. This group maps onto $\boldsymbol{\mu}_N(k_1)$. By hypothesis, $\mathcal{F}_1(\beta)$ as $I(\beta)^{\mathrm{tame}}$-representation is canonically the direct sum of representations of the form

$$(\text{a character of } \boldsymbol{\mu}_N(k_1))\otimes(\text{a unipotent rep'n. of } I(\beta)^{\mathrm{tame}}).$$

A unipotent representation of $I(\beta)^{\mathrm{tame}}$ factors through a unipotent representation of $\mathbb{Z}_\ell(1)(k_1)$, and its isomorphism class is the Jordan normal form of the action of **any** topological generator of $\mathbb{Z}_\ell(1)(k)$.

Therefore $\mathcal{F}_1(\beta)$ as $I(\beta)^{\mathrm{tame}}$-representation factors through the quotient

$$\text{lim. inv.}_\nu\ \mu_{\ell^\nu N}(k_1)$$

of $I(\beta)^{\mathrm{tame}}$. Moreover, for any choice $\gamma(\beta, \zeta_1)^{\mathrm{tame}}$ of a generator of this last group which maps onto ζ_1 in $\boldsymbol{\mu}_N(k_1)$, the Jordan normal form (i.e., conjugacy class in $GL(\mathrm{rank}(\mathcal{F}_1), \overline{\mathbb{Q}}_\ell)$) of its action on $\mathcal{F}_1(\beta)$ is independent of the choice. By "the local monodromy of $\mathcal{F}_1$ at β, relative to ζ_1", we mean the Jordan normal form of the action on $\mathcal{F}_1(\beta)$ of any $\gamma(\beta, \zeta_1)^{\mathrm{tame}}$.

Proposition 5.5.8.1 Consider the situation 5.5.1. Let k_1 be a second algebraically closed field in which ℓN is invertible, ζ_1 in k_1 a primitive N'th root of unity, and $\beta_1, ..., \beta_n$ a set of $n \geq 2$ distinct

points of $\mathbb{A}^1(k_1)$. Let $\mathcal{F}_1$ be an object of $\mathcal{T}_\ell$ on $\mathbb{A}^1{}_{k_1}$ which is lisse on $\mathbb{A}^1{}_{k_1} - \{\beta_1, \ldots, \beta_n\}$, cohomologically rigid, and such that all eigenvalues of all local monodromies of $\mathcal{F}_1$ are N'th roots of unity. Suppose that
a) for i=1, ..., n, the local monodromy of $\mathcal{F}_1$ at β_i, relative to ζ_1, is isomorphic to the local monodromy of $\mathcal{F}$ at α_i, relative to ζ,
b) the local monodromy of $\mathcal{F}_1$ at ∞, relative to ζ_1, is isomorphic to the local monodromy of $\mathcal{F}$ at ∞, relative to ζ.

Denote by $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{F}_1}$ on $(\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$ any choices of the lisse E_λ-sheaves given by 5.5.4, applied to $\mathcal{F}$ and to $\mathcal{F}_1$ respectively. Denote by

$$\pi : (\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}} \to S_{N,n,\ell}.$$

the projection. Then
1) At any geometric point $\varphi: S_{N,n,\ell} \to k_2$ of $S_{N,n,\ell}$, the restriction $\boldsymbol{\mathcal{F}}_\varphi$ of $\boldsymbol{\mathcal{F}}$ to the fibre over φ is a cohomologically rigid object of $\mathcal{T}_\ell$, whose local monodromy at the point $\varphi(T_i)$, i=1, ..., n, (resp. at ∞) relative to $\varphi(\zeta_N)$, is isomorphic to the local monodromy of $\mathcal{F}$ at α_i, i=1, ..., n (resp. at ∞), relative to ζ.
2) Denote by s_1 the geometric point $\varphi_1: S_{N,n,\ell} \to k_1$ with $\varphi_1(\zeta_N) = \zeta_1$ and $\varphi_1(T_i) = \beta_i$ for i=1, ..., n. The restriction $\boldsymbol{\mathcal{F}}_{s_1}$ of $\boldsymbol{\mathcal{F}}$ to the fibre over s_1 is isomorphic to $\mathcal{F}_1$.
3) There exists a lisse E_λ-sheaf $\boldsymbol{\mathcal{L}}$ on $S_{N,n,\ell}$ and an isomorphism $\boldsymbol{\mathcal{F}_1} \cong \boldsymbol{\mathcal{F}} \otimes \pi^*(\boldsymbol{\mathcal{L}})$.

proof Statement 1) results from 5.5.4 2), together with [De-Weil II, 1.7.8] (cf. [Ka-SE, 4.7.2] and the proof of 4.3.11), which tells us that the local monodromy of $\mathcal{F}$ along each section T_i and along the section ∞ is "the same" on all geometric fibres. [Strictly speaking, we should first extend scalars in the universal situation from $\mathbb{Z}[\zeta_N, 1/N\ell]$ to $\mathbb{Z}[\text{all } \zeta_{\ell^\nu N}, 1/N\ell]$, choose a toplogical generator of lim. inv.$_\nu$ $\mu_{\ell^\nu N}(\mathbb{Z}[\text{all } \zeta_{\ell^\nu N}, 1/N\ell])$ which maps to ζ_N, and use the fact that fibre by fibre this generator gives a choice of $\gamma(\beta, \zeta_1)^{\mathrm{tame}}$.] By 1), the restriction $\boldsymbol{\mathcal{F}}_{s_1}$ of $\boldsymbol{\mathcal{F}}$ to the fibre over s_1 has the same local monodromy as $\mathcal{F}_1$, at each of the points $\{\beta_1, ..., \beta_n, \infty\}$. Since $\mathcal{F}_1$ is cohomologically rigid, $\mathcal{F}_1$ is isomorphic to $\mathfrak{F}_{s_1}$. To get 3), apply the rigidity corollary 5.5.7.3 to $\mathfrak{F}$ and to $\mathfrak{F}_1$, and the geometric point s_1. QED

5.6 Remark on braid groups The space $(\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}}$ is the spec of

$$\mathbb{Z}[\zeta_N, 1/N\ell][X, T_1, \dots, T_n][1/[\Pi_{i\neq j}(T_i - T_j))(\Pi_j (X - T_j))].$$

So if name the variables X as T_{n+1}, we can think of this as the ring

$$\mathbb{Z}[\zeta_N, 1/N\ell][T_1, \dots, T_{n+1}][1/(\Pi_{i\neq j}(T_i - T_j))].$$

Thought of this way, we see its fundamental group as an arithmetic-geometrical version of the Artin braid group on n+1 letters. And so we can restate the first assertion of the theorem by the catch-phrase "an irreducible local system on $\mathbb{A}^1$ minus n points which is cohomologically rigid extends to a representation of the braid group on n+1 letters".

5.7 Universal families without quasiunipotence

(5.7.1) We next explore the situation if we drop the hypothesis of quasiunipotence. Thus, we let k be an algebraically closed field, $n \geq 2$ an integer, $\alpha_1, \alpha_2, \dots \alpha_n$ a set of n distinct points of $\mathbb{A}^1(k)$, and ℓ a prime number invertible in k. Let $\mathcal{F}$ be a cohomologically rigid object of $\mathcal{T}_\ell$ which is lisse on $\mathbb{A}^1 - \{\alpha_1, \alpha_2, \dots \alpha_n\}$.

(5.7.2) We denote by p the characteristic of k. If $p > 0$, we denote by $\overline{\mathbb{F}}_p$ the algebraic closure of $\mathbb{F}_p$ in k, and we denote by R the ring $W(\overline{\mathbb{F}}_p)$ of Witt vectors over $\overline{\mathbb{F}}_p$. If char(k) = 0, we denote by R the subfield $\mathbb{Q}$(all roots of unity) of k. There is a canonical ring homomorphism from R to k, which for p=0 is the inclusion, and which for p>0 is the composite $R \to R/pR = \overline{\mathbb{F}}_p \to k$ of reduction mod p and of the inclusion.

(5.7.3) We denote by S_n the ring

$$S_n := R[T_1, \dots, T_n][1/\Delta], \ \Delta := \Pi_{i\neq j}(T_i - T_j).$$

There is a unique ring homomorphism

$$\varphi : S_n \to k$$

which induces the canonical map on R, and for which $\varphi(T_i) = \alpha_i$ for

$1 \leq i \leq n$. Over S_n, we have $\mathbb{A}^1$, with its n disjoint sections $\{T_1, \dots, T_n\}$. We denote by

$$j: (\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_n} \to (\mathbb{A}^1)_{S_n}$$

the inclusion.

A straightforward modification of the proof of the previous theorem 5.5.4 yields:

Theorem 5.7.4 In the situation 5.7.1, we have

1) There exists a lisse $\overline{\mathbb{Q}}_\ell$-sheaf

$$\boldsymbol{\mathcal{F}} \text{ on } (\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_n}$$

which, after the base change $\varphi : S_n \to k$ becomes (the restriction to $\mathbb{A}^1 - \{\alpha_1, \alpha_2, \dots \alpha_n\}$ of $\mathcal{F}$.

2) ("mise pour memoire", cf. 4.2.3, 4.3.8, 4.3.9) The object $j_*\boldsymbol{\mathcal{F}}$ on $(\mathbb{A}^1)_{S_n}$ is of formation compatible with arbitrary change of base on $S_{N,n,\ell}$, and is adapted to the stratification

$$((\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_n}, \{T_1, \dots, T_n\}_{S_n}) \text{ of } (\mathbb{A}^1)_{S_n}.$$

At any geometric point $\varphi: S_n \to k_2$ of S_n, the restriction of $j_*\boldsymbol{\mathcal{F}}$ to the fibre over φ is a cohomologically rigid object of $\mathcal{T}_\ell$, whose local monodromy at each point $\varphi(T_i)$ (resp. at ∞) is isomorphic to the local monodromy of of $\mathcal{F}$ at α_i (resp. at ∞).

5.7.5 Remark The space $(\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_n}$ is the spec of

$$R[T_1, \dots, T_{n+1}][1/(\Pi_{i\neq j}(T_i - T_j))].$$

Its fundamental group is a (less arithmetic) version of the Artin braid group on n+1 letters, and we can again restate the first assertion of the theorem by the catch-phrase "an irreducible local system on $\mathbb{A}^1$ minus n points which is cohomologically rigid extends to a representation of the braid group on n+1 letters".

5.8 The complex analytic situation

Theorem 5.8.1 Over $\mathbb{C}$, suppose given $n \geq 2$ an integer, $\alpha_1, \alpha_2, \dots \alpha_n$ a set of n distinct points of $\mathbb{A}^1(\mathbb{C})$, ℓ a prime number, $N \geq 1$ an integer, ζ_N a primitive N'th root of unity in $\mathbb{C}$, and an embedding of $\mathbb{Q}(\zeta_N)$ into $\overline{\mathbb{Q}}_\ell$. Let $\mathcal{F}$ be a cohomologically rigid object of $\mathcal{T}_\ell$ which is lisse on

$$U := \mathbb{A}^1 - \{\alpha_1, \alpha_2, \dots \alpha_n\},$$

and such that all eigenvalues of all local monodromies of $\mathcal{F}$ are N'th roots of unity. Then there exists on U^{an} a local system $\mathcal{F}_{cycl}$ of finite-dimensional $\mathbb{Q}(\zeta_N)$-vector spaces and an isomorphism of $\overline{\mathbb{Q}}_\ell$-local systems on U^{an}, $\mathcal{F}^{an} \cong \mathcal{F}_{cycl}\otimes_{\mathbb{Q}(\zeta_N)}\overline{\mathbb{Q}}_\ell$.

proof Any lisse, rank one, $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}$ on U whose local monodromies all have order dividing N is a homomorphism from $\pi_1(U)$ to $\mu_N(\overline{\mathbb{Q}}_\ell) = \mu_N(\mathbb{Q}(\zeta_N))$. Now $\mathcal{L}^{an}$ is the same homomorphism, restricted to $\pi_1(U^{an})$, so it too has values in $\mu_N(\mathbb{Q}(\zeta_N))$, so may be viewed as as a rank one $\mathbb{Q}(\zeta_N)$-local system $\mathcal{L}_{cycl}$ on U^{an} which sits in an isomorphism $\mathcal{L}^{an} \cong \mathcal{L}_{cycl}\otimes_{\mathbb{Q}(\zeta_N)}\overline{\mathbb{Q}}_\ell$.

We proceed by induction on the generic rank $r(\mathcal{F})$ of $\mathcal{F}$. If $r(\mathcal{F})=1$, we are done, by the above discussion. Suppose that $r(\mathcal{F}) \geq 2$. By the Main Theorem, we know that there exists on U a rank one lisse $\mathcal{L}$ of order dividing N, a nontrivial character χ of order dividing N, a lisse cohomologically rigid $\mathcal{G}$ with $r(\mathcal{G}) < r(\mathcal{F})$ and with all eigenvalues of all local monodromies N'th roots of unity, and an isomorphism on U

$$\mathcal{F}[1] := \mathcal{L}\otimes j^*(j_*\mathcal{G}[1]*_{mid+}j_{0*}\mathcal{L}_\chi[1]).$$

By induction, the $\mathbb{Q}(\zeta_N)$-local system $\mathcal{G}_{cycl}$ exists on U^{an}, as do $\mathcal{L}_{cycl}$ and $\mathcal{L}_{\chi,cycl}$. We define $\mathcal{F}_{cycl}$ on U^{an} to be

$$\mathcal{F}_{cycl}[1] := \mathcal{L}_{cycl}\otimes j^*(j_*\mathcal{G}_{cycl}[1]*_{mid+}j_{0*}\mathcal{L}_{\chi,cycl}[1]).$$

By the comparison theorem, $\mathcal{F}_{cycl}$ is a local system of finite-dimensional $\mathbb{Q}(\zeta_N)$-vector spaces which sits in an isomorphism of $\overline{\mathbb{Q}}_\ell$-local systems on U^{an}, $\mathcal{F}^{an} \cong \mathcal{F}_{cycl}\otimes_{\mathbb{Q}(\zeta_N)}\overline{\mathbb{Q}}_\ell$. QED

(5.8.2) What happens if we drop the quasiunipotence hypothesis? Over $\mathbb{C}$, $I(\infty)$ and each of the groups $I(\alpha_i)$ are canonically the group $\lim \mathrm{inv}_N\ \boldsymbol{\mu}_{N!}(\mathbb{C})$, a group which has a canonical generator, namely $\{\exp(2\pi i/N!)\}_N$. Thus we may speak of "the eigenvalues of local monodromy", meaning the eigenvalues of the action of this canonical generator, of a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $\mathbb{A}^1 - \{\alpha_1, \alpha_2, \dots \alpha_n\}$ at any of the points ∞ or α_i where it possibly fails to be lisse.

Repeating the above proof yields:

Theorem 5.8.3 Over $\mathbb{C}$, suppose given $n \geq 2$ an integer, $\alpha_1, \alpha_2, \dots \alpha_n$ a set of n distinct points of $\mathbb{A}^1(\mathbb{C})$, ℓ a prime number, Γ a subgroup of $\overline{\mathbb{Q}}_\ell{}^\times$, K the subfield $\mathbb{Q}(\Gamma)$ of $\overline{\mathbb{Q}}_\ell$. Let $\mathcal{F}$ be a cohomologically rigid object of $\mathcal{T}_\ell$ which is lisse on

$$U := \mathbb{A}^1 - \{\alpha_1, \alpha_2, \dots \alpha_n\},$$

and such that all eigenvalues of all local monodromies of $\mathcal{F}$ lie in Γ. Then there exists on U^{an} a local system $\mathcal{F}_\Gamma$ of finite-dimensional K-vector spaces and an isomorphism of $\overline{\mathbb{Q}}_\ell$-local systems on U^{an}, $\mathcal{F}^{an} \cong \mathcal{F}_\Gamma \otimes_K \overline{\mathbb{Q}}_\ell$.

5.9 Return to the original question

(5.9.1) We now return to the question with which we began: what is the structure of irreducible local systems of finite-dimensional $\mathbb{C}$-vector spaces on U^{an} which are physically rigid? We know from 1.1.2 that for such local systems, physical rigidity is equivalent to cohomological rigidity. Thus it "suffices" to understand the structure of cohomologically rigid irreducible local systems of finite-dimensional $\mathbb{C}$-vector spaces on U^{an}. Rather than deal with it directly, we reduce it to the ℓ-adic case on U. This reduction is made possible by the following standard result, which we spell out for the convenience of the reader.

Proposition 5.9.2 Over $\mathbb{C}$, suppose given $n \geq 2$ an integer, $\alpha_1, \alpha_2, \dots \alpha_n$ a set of n distinct points of $\mathbb{A}^1(\mathbb{C})$,

$$U := \mathbb{A}^1 - \{\alpha_1, \alpha_2, \dots \alpha_n\},$$

and $\mathcal{F}_{\mathbb{C},an}$ a local system of finite-dimensional $\mathbb{C}$-vector spaces on U^{an}. There exists an integer $N \geq 1$ such that for all primes ℓ not dividing N, there exists an isomorphism of fields $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$, a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_\ell$ on U, and an isomorphism of $\overline{\mathbb{Q}}_\ell$-local systems on U^{an},

$$(\mathcal{F}_\ell)^{an} \cong \mathcal{F}_{\mathbb{C},an} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell.$$

proof Let $n := \mathrm{rank}(\mathcal{F}_{\mathbb{C},an})$ be the rank of $\mathcal{F}_{\mathbb{C},an}$. Once we pick a base point in $U^{an} := U(\mathbb{C})$, we may interpret $\mathcal{F}_{\mathbb{C},an}$ as a homomorphism of groups

$$\rho: \pi_1(U^{an}) \to GL(n, \mathbb{C}).$$

We know that $\pi_1(U^{an})$ is a finitely generated group, and that its profinite completion is the profinite group $\pi_1(U)$.

Because $\pi_1(U^{an})$ is finitely generated, ρ takes values in $GL(n,R)$ for some subring R of $\mathbb{C}$ which is finitely generated as a $\mathbb{Z}$-algebra. [For instance, if $\{\gamma_i\}_i$ is a finite set of generators, we may take for R the ring $\mathbb{Z}$[entries of all $\rho(\gamma_i)$ and of all $\rho(\gamma_i^{-1})$].] Let us admit temporarily the truth of the following lemma, which is certainly well-known, cf. [BBD, 6.1.2 (A'')], but for which I do not know an explicit reference:

Lemma 5.9.3 Let R be a subring of $\mathbb{C}$ which is finitely generated as a $\mathbb{Z}$-algebra. Then there exists an integer $N \geq 1$ such that for every prime number ℓ which does not divide N, there exists a finite extension E_λ of $\mathbb{Q}_\ell$ with integer ring $\mathcal{O}_\lambda$, and an isomorphism of fields $\iota: \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ under which $\iota(R) \subset \mathcal{O}_\lambda$.

(5.9.4) Granted the lemma, for any ℓ prime to N we get that $\mathcal{F}_{\mathbb{C},an}$ has an $\mathcal{O}_\lambda$-form. As the group $GL(n, \mathcal{O}_\lambda)$ is profinite, the composite homomorphism

$$\rho_\lambda : \pi_1(U^{an}) \to GL(n, R) \subset GL(n, \mathcal{O}_\lambda)$$

extends to a continuous homomorphism of profinite completions

$$\rho_{\lambda,alg} : \pi_1(U) \to GL(n, \mathcal{O}_\lambda).$$

Corresponding to $\rho_{\lambda,alg}$, we have a lisse $\mathcal{O}_\lambda$-sheaf, say $\mathcal{F}_\lambda$, and in terms of $\mathcal{F}_\lambda$ we define $\mathcal{F}_\ell := \mathcal{F}_\lambda \otimes_{\mathcal{O}_\lambda} \overline{\mathbb{Q}}_\ell$. QED

5.9.5 proof of 5.9.3 Denote by $R_\mathbb{Q}$ the $\mathbb{Q}$-subalgebra of $\mathbb{C}$ generated by R. Thus $R_\mathbb{Q}$ is a finitely generated $\mathbb{Q}$-algebra, to which we apply Noether normalization [AK,2.5]: there exists a finite collection of elements $x_1, \ldots ,x_n$ in $R_\mathbb{Q}$ which are algebraically independent over $\mathbb{Q}$, and such that $R_\mathbb{Q}$ is integral over $\mathbb{Q}[x_1, \ldots ,x_n]$. Notice that the ring $\mathbb{Q}[x_1, \ldots ,x_n]$ is unchanged if we replace each x_i by a nonzero integer multiple of itself. Since each x_i is in $R_\mathbb{Q}$, after so replacing it we may assume that each x_i lies in R.

Now every element r in R is integral over $\mathbb{Q}[x_1, \dots ,x_n]$. Writing an equation of integral dependence for r over $\mathbb{Q}[x_1, \dots ,x_n]$, we see that for some integer $N(r) \geq 1$, r is integral over $\mathbb{Z}[1/N(r)][x_1, \dots ,x_n]$. Because R is generated as a $\mathbb{Z}$-algebra by finitely many elements r_i in R, if we define $N := \prod_i N(r_i)$, then each r_i is integral over $\mathbb{Z}[1/N][x_1, \dots ,x_n]$. Since the integral closure of $\mathbb{Z}[1/N][x_1, \dots ,x_n]$ in $R_{\mathbb{Q}}$ is a $\mathbb{Z}[1/N]$-algebra, $R[1/N]$ is integral over $\mathbb{Z}[1/N][x_1, \dots ,x_n]$.

Now pick any prime ℓ not dividing N. Because $\mathbb{Q}_\ell$ is uncountable, it has uncountable transcendence degree over $\mathbb{Q}$. In particular, there exist n elements $y_1, y_2, \dots ,y_n$ in $\mathbb{Q}_\ell$ which are algebraically independent over $\mathbb{Q}$. Multiplying each y_i by a power of ℓ, we may further suppose that each y_i lies in $\mathbb{Z}_\ell$.

Using the axiom of choice, there exists an isomorphism

$$\iota: \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$$

such that $\iota(x_i) = y_i$ for $i=1, \dots ,n$. Under this isomorphism, $\iota(\mathbb{Z}[1/N][x_1, \dots ,x_n]) \subset \mathbb{Z}_\ell$, and hence $\iota(R[1/N])$ is integral over $\mathbb{Z}_\ell$. In particular, every element of $\iota(R[1/N])$ lies in some finite extension of $\mathbb{Q}_\ell$. Since $R[1/N]$ is finitely generated as a $\mathbb{Z}$-algebra, there exists a single finite extension field E_λ of $\mathbb{Q}_\ell$ with $\iota(R[1/N]) \subset E_\lambda$. As $\iota(R[1/N])$ is integral over $\mathbb{Z}_\ell$, we have $\iota(R) \subset \iota(R[1/N]) \subset \mathcal{O}_\lambda$. QED

5.10 The category $\mathcal{T}_{an}(U,\Gamma)$

(5.10.1) Over $\mathbb{C}$, suppose given $n \geq 2$ an integer, $\alpha_1, \alpha_2, \dots \alpha_n$ a set of n distinct points of $\mathbb{A}^1(\mathbb{C})$,

$$U := \mathbb{A}^1 - \{\alpha_1, \alpha_2, \dots \alpha_n\},$$

and Γ a subgroup of $\mathbb{C}^\times$. We denote by $\mathcal{T}_{an}(U,\Gamma)$ the full subcategory of the category of all local systems of finite-dimensional $\mathbb{C}$-vector spaces on U^{an} whose objects are those which are irreducible, have nontrivial local monodromy at two or more of the points α_i, and for which all of eigenvalues of all topological local monodromies lie in Γ.

Theorem 5.10.2 Let χ be any nontrivial $\mathbb{C}^\times$-valued character with values in the subgroup Γ, and $\mathcal{L}_{\chi,an}$ the corresponding Kummer sheaf on $\mathbb{G}_{m,an}$. Then for $\mathcal{F}_{an}$ in $\mathcal{T}_{an}(U,\Gamma)$,the object $\mathcal{G}_{an}$ in $D^b{}_c(U^{an}, \mathbb{C})$ defined by

$$\mathcal{G}_{an}[1] := j^{an*}(j^{an}{}_{*}\mathcal{F}_{an}[1]*_{mid+}j_{0,an*}\mathcal{L}_{\chi,an}[1]),$$

lies in $\mathcal{T}_{an}(U,\Gamma)$, and its local monodromies are related to those of $\mathcal{F}_{an}$ by the rules of 3.3.6 and 3.3.7.

proof For any chosen $\ell >> 0$, there exists a field isomorphism $\iota: \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ under which both $\mathcal{F}_{an}$ and $\mathcal{L}_{\chi,an}$ come from lisse $\overline{\mathbb{Q}}_\ell$-sheaves $\mathcal{F}$ on U and $\mathcal{L}_\chi$ on $\mathbb{G}_m$ respectively. If we now view Γ as a subgroup of $\overline{\mathbb{Q}}_\ell{}^\times$, we may speak of the subgroup Γ_{values} of the group of continuous $\overline{\mathbb{Q}}_\ell{}^\times$-valued characters of I(0) which on the canonical (we are over $\mathbb{C}$) generator take values in Γ. Then $\mathcal{F}$ lies in $\mathcal{T}_\ell(U,\Gamma_{values})$, and χ lies in $\Gamma_{values.}$ By 4.3.11, the object $\mathcal{G}$ of $D^b{}_c(U, \overline{\mathbb{Q}}_\ell)$ defined by

$$\mathcal{G}[1] := j^*(j_*\mathcal{F}[1]*_{mid+}j_{0*}\mathcal{L}_\chi[1]),$$

lies in $\mathcal{T}_\ell(U,\Gamma_{values})$, and its local monodromies are related to those of $\mathcal{F}$ by the rules of 3.3.6 and 3.3.7. Passing to U^{an} and applying the inverse of the isomorphism $\iota: \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$, we recover our assertions about $\mathcal{G}_{an}$. QED

(5.10.3) We write

$$\mathcal{G}_{an} = MC_{\chi,an}(\mathcal{F}_{an}).$$

(5.10.4) We can construct a graph whose vertices are the objects of $\mathcal{T}_{an}(U,\Gamma)$, and in which there is an edge joining two objects $\mathcal{F}_{an}$ and $\mathcal{G}_{an}$ if either of the following conditions)a or b) holds:

a) $\mathcal{F}_{an}$ and $\mathcal{G}_{an}$ both have rank >1, and there is a rank one $\mathbb{C}$-local system $\mathcal{L}_{an}$ on U, all of whose local monodromy eigenvalues are in Γ, such that $\mathcal{F}_{an} \cong \mathcal{L}_{an}\otimes\mathcal{G}_{an}$, or equivalently $\mathcal{G}_{an} \cong \mathcal{L}_{an}{}^{-1}\otimes\mathcal{F}_{an}$.

b) there exists a nontrivial character χ with values in Γ such that $\mathcal{F}_{an} \cong MC_{\chi,an}(\mathcal{G}_{an})$, or equivalently $\mathcal{G}_{an} \cong MC_{\chi_{an}}{}^{-1}(\mathcal{F}_{an})$.

Main Theorem 5.10.5 (complex analytic version of 5.3.3). If $\mathcal{F}_{an}$ is an object of $\mathcal{T}_{an}(U,\Gamma)$ which is rigid, in either of the equivalent senses of being physically rigid or of being cohomologically rigid, then $\mathcal{F}_{an}$ (as vertex of the above graph) is connected to an object of rank one in $\mathcal{T}_{an}(U,\Gamma)$, and its distance to such an object is at most $2(\mathrm{rank}(\mathcal{F}_{an}) -1)$.

proof Once we have the previous result 5.10.2, we can repeat, essentially verbatim, the proof already given in the ℓ-adic case. QED

Corollary 5.10.6 In the situation 5.10.1, let $N \geq 2$ be an integer, and consider the case $\Gamma = \mu_N(\mathbb{C})$. Thus let $\mathcal{F}$ be a local system of finite-dimensional $\mathbb{C}$-vector spaces on U^{an} which is irreducible, has nontrivial local monodromy at two or more of the points α_i, and for which all of eigenvalues of topological local monodromy are N'th roots of unity. Suppose that $\mathcal{F}$ is rigid. Then

1) $\mathcal{F}$ has a $\mathbb{Q}(\zeta_N)$-form, i.e., for any embedding of the abstract field $\mathbb{Q}(\zeta_N)$ into $\mathbb{C}$, there exists a local system $\mathcal{F}_{cycl}$ of $\mathbb{Q}(\zeta_N)$-vector spaces on U^{an} such that $\mathcal{F} \cong \mathcal{F}_{cycl}\otimes_{\mathbb{Q}(\zeta_N)}\mathbb{C}$.

2) Fix a $\mathbb{Q}(\zeta_N)$-form $\mathcal{F}_{cycl}$ of $\mathcal{F}$. For every finite place λ of $E := \mathbb{Q}(\zeta_N)$, with λ-adic completion E_λ and integer ring $\mathcal{O}_\lambda$, there exists a lisse E_λ-sheaf $\mathcal{F}_\lambda$ on U and an isomorphism $(\mathcal{F}_\lambda)^{an} \cong \mathcal{F}_{cycl}\otimes_E E_\lambda$.

3) Fix a $\mathbb{Q}(\zeta_N)$-form $\mathcal{F}_{cycl}$ of $\mathcal{F}$. For every finite place λ of $E := \mathbb{Q}(\zeta_N)$, with λ-adic completion E_λ and integer ring $\mathcal{O}_\lambda$, there exists a local system $\boldsymbol{\mathcal{F}}_\lambda$ of free $\mathcal{O}_\lambda$-modules on U such

$$\mathcal{F}_{cycl}\otimes_E E_\lambda \cong (\boldsymbol{\mathcal{F}}_\lambda\otimes_{\mathcal{O}_\lambda} E_\lambda)^{an}.$$

4) Fix a $\mathbb{Q}(\zeta_N)$-form $\mathcal{F}_{cycl}$ of $\mathcal{F}$. For every finite place λ of $E := \mathbb{Q}(\zeta_N)$, with λ-adic completion E_λ and integer ring $\mathcal{O}_\lambda$, there exists a local system $\boldsymbol{\mathcal{F}}_\lambda{}^{an}$ of free $\mathcal{O}_\lambda$-modules on U^{an} such

$$\mathcal{F}_{cycl}\otimes_E E_\lambda \cong \boldsymbol{\mathcal{F}}_\lambda{}^{an}\otimes_{\mathcal{O}_\lambda} E_\lambda.$$

proof. First let us notice that 4) follows trivially from 3): one takes the $\boldsymbol{\mathcal{F}}_\lambda$ of 3), and defines $\boldsymbol{\mathcal{F}}_\lambda{}^{an}$ to be $(\boldsymbol{\mathcal{F}}_\lambda)^{an}$. Also, 3) follows trivially from 2), since any lisse E_λ-sheaf $\mathcal{F}_\lambda$ on U has an $\mathcal{O}_\lambda$-form.

So it remains to prove 1) and 2). We prove these by induction on the generic rank of $\mathcal{F}$. Both are obvious for rank one lisse sheaves $\mathcal{L}$ on U^{an} all of whose local monodromies are N'th roots of unity, and for $\mathcal{L}_\chi$ with χ nontrivial of order dividing N. Moreover, if 1) and 2) hold for $\mathcal{G}_{an}$, they hold for both $\mathcal{L}_{an} \otimes \mathcal{G}_{an}$ and $MC_{\chi,an}(\mathcal{G}_{an})$ [compare 5.5.5.5-9]. So by the connectedness properties 5.10.5 of the set of rigid points in the graph on $\mathcal{T}_{an}(U,\Gamma)$, with $\Gamma := \mu_N(\mathbb{C})$, 1) and 2) hold for all $\mathcal{F}$ in $\mathcal{T}_{an}(U,\Gamma)$. QED

6.0 Numerical Invariants

(6.0.1) To motivate this section, let us fix an algebraically closed field k, and a prime number $\ell \neq \mathrm{char}(k)$. Let us fix also a finite subset D of $\mathbb{A}^1(k)$ with $\mathrm{Card}(D) \geq 2$. Suppose we are given an object $\mathcal{F}$ in $\mathcal{T}_\ell$ which is lisse on $\mathbb{A}^1 - D$, of rank $r(\mathcal{F})$. Then for any point s in $D \amalg \{\infty\}$, $\mathcal{F}$ gives rise to a representation $\mathcal{F}(s)$ of $I(s)^{\mathrm{tame}}$, which we write as a direct sum, over characters χ of $\pi_1(\mathbb{G}_m)^{\mathrm{tame}}$, of representations of $I(s)^{\mathrm{tame}}$ of the following form:

for s in D: $\mathcal{F}(s) = \oplus_\chi \mathcal{L}_{\chi(x - s)} \otimes \mathrm{Unip}(s, \chi, \mathcal{F})$,
for $s = \infty$: $\mathcal{F}(\infty) = \oplus_\chi \mathcal{L}_{\chi(x)} \otimes \mathrm{Unip}(\infty, \chi, \mathcal{F})$.

[With this naming convention for the characters, if we start with a rank one object $\mathcal{F}$, and denote by χ_s the unique character ρ of $\pi_1(\mathbb{G}_m)^{\mathrm{tame}}$ for which $e_1(s, \rho, \mathcal{F}) = 1$, the characters χ_s are related by the formula $\chi_\infty = \prod_{s \text{ in } D} \chi_s$.]

(6.0.2) For each of the unipotent representations $\mathrm{Unip}(s, \chi, \mathcal{F})$, we denote by

$$e_1(\alpha,\chi,\mathcal{F}) \geq e_2(\alpha,\chi,\mathcal{F}) \geq \ldots \geq e_k(\alpha,\chi,\mathcal{F}) = 0 \text{ for } k \gg 0$$

the sequence of integers defined by

$$e_j(\alpha,\chi,\mathcal{F}) := \text{the number of Jordan blocks in } \mathrm{Unip}(\alpha,\chi,\mathcal{F}) \text{ whose dimension is } \geq j.$$

(6.0.3) The fact that each $\mathcal{F}(s)$ is an $r(\mathcal{F})$-dimensional representation gives the relations:

$$\text{for each s in } D \amalg \{\infty\}, \ \Sigma_{i,\chi}\, e_i(s,\chi,\mathcal{F}) = r(\mathcal{F}).$$

(6.0.4) Recall (3.3.6 and 3.3.7) that there exists an $I(\infty)^{\mathrm{tame}}$-representation $M(\infty,\mathcal{F})$ attached to this situation, with the two properties:

$$M(\infty,\mathcal{F})/M(\infty,\mathcal{F})^{I(\infty)} \cong \mathcal{F}(\infty),$$
$$\mathrm{rank}(M(\infty,\mathcal{F})) = \Sigma_{s \text{ in } D}\, (r(\mathcal{F}) - e_1(s, \mathbb{1}, \mathcal{F})).$$

(6.0.5) These two properties allow us to calculate the invariants attached to $M(\infty,\mathcal{F})$, which we will denote as $E_i(\infty, \chi, \mathcal{F})$. The recipe is

$$E_i(\infty, \chi, \mathcal{F}) = e_i(\infty, \chi, \mathcal{F}) \text{ if } \chi \neq \mathbb{1},$$
$$E_{i+1}(\infty, \mathbb{1}, \mathcal{F}) = e_i(\infty, \mathbb{1}, \mathcal{F}) \text{ for } i \geq 1,$$
$$E_1(\infty, \mathbb{1}, \mathcal{F}) = \mathrm{rank}(M(\infty,\mathcal{F})) - r(\mathcal{F}).$$

(6.0.6) We introduce the notation

$m(\mathcal{F}) := \Sigma_{s \text{ in } D} (r(\mathcal{F}) - e_1(s, \mathbb{1}, \mathcal{F}))$.

Thus $m(\mathcal{F})$ is simply the rank of $M(\infty,\mathcal{F})$, and we have the relations,

$\Sigma_{i,\chi} E_i(\infty,\chi,\mathcal{F}) = m(\mathcal{F})$,

$E_1(\infty, \mathbb{1}, \mathcal{F}) = m(\mathcal{F}) - r(\mathcal{F})$.

(6.0.7) Let us remark in passing that the inequality

$E_1(\infty, \mathbb{1}, \mathcal{F}) \geq E_2(\infty, \mathbb{1}, \mathcal{F})$

may be rewritten as

$m(\mathcal{F}) - r(\mathcal{F}) \geq e_1(\infty, \mathbb{1}, \mathcal{F})$.

(6.0.8) There are a priori linear dependences among these data. If we are given the integer $r(\mathcal{F})$ and all the integers $e_i(s, \chi, \mathcal{F})$ for all s in $D\amalg\{\infty\}$, we may compute $m(\mathcal{F})$ and all the integers $E_i(\infty,\chi,\mathcal{F})$. Conversely, if we are given the integers $E_i(\infty,\chi,\mathcal{F})$ and all the integers $e_i(s, \chi, \mathcal{F})$ for s in D, we may compute both $m(\mathcal{F})$ (namely $\Sigma_{i,\chi} E_i(\infty,\chi,\mathcal{F})$), $r(\mathcal{F})$ (namely $m(\mathcal{F}) - E_1(\infty, \mathbb{1}, \mathcal{F})$), and the $e_i(\infty,\chi,\mathcal{F})$.

(6.0.9) The reason for presenting the data in both (r, all e_i) and (m, E_i at ∞, r, e_i at points of D) formats will become clear when we analyze the effects of the operations $\mathcal{F} \mapsto MT_{\mathcal{L}}(\mathcal{F})$ and $\mathcal{F} \mapsto MC_\chi(\mathcal{F})$ on this numerical data.

(6.0.10) We begin with the more straightforward of the two, $\mathcal{F} \mapsto MT_{\mathcal{L}}(\mathcal{F})$. For each s in $D\amalg\{\infty\}$, we denote by $\chi_{s,\mathcal{L}}$ the unique character ρ with $e_1(s, \rho, \mathcal{L}) = 1$. Then we have

$r(MT_{\mathcal{L}}(\mathcal{F})) = r(\mathcal{F})$,

$e_i(s, \rho\chi_{s,\mathcal{L}}, MT_{\mathcal{L}}(\mathcal{F})) = e_i(s, \rho, \mathcal{F})$ for all s in $D\amalg\{\infty\}$, i, ρ.

(6.0.11) We now turn to the case of $\mathcal{F} \mapsto MC_\chi(\mathcal{F})$. According to 3.3.6 and 3.3.7, we have $M(\infty,MC_\chi(\mathcal{F})) \cong M(\infty,\mathcal{F})\otimes\mathcal{L}_\chi$. This gives the relations

$m(MC_\chi(\mathcal{F})) = m(\mathcal{F})$,

$E_i(\infty, \rho\chi, MC_\chi(\mathcal{F})) = E_i(\infty, \rho, \mathcal{F})$ for all i and ρ.

It allows us to compute all the integers $E_i(\infty, \rho, MC_\chi(\mathcal{F}))$, so in particular to compute

$r(MC_\chi(\mathcal{F})) = m(MC_\chi(\mathcal{F})) - E_1(\infty, \mathbb{1}, MC_\chi(\mathcal{F}))$.

(6.0.12) We know that for s in D, the local monodromies of $\mathcal{F}$ and of $MC_\chi(\mathcal{F})$are related by

$$MC_\chi(\mathcal{F})(s)/MC_\chi(\mathcal{F})(s)^{I(s)} \cong (\mathcal{F}(s)/\mathcal{F}(s)^{I(s)})\otimes\mathcal{L}_{\chi(x - s)}.$$

(6.0.13) Thus we may compute most of the invariants of $MC_\chi(\mathcal{F})(s)$:

$$e_i(s, \rho\chi, MC_\chi(\mathcal{F})) = e_i(s, \rho, \mathcal{F}) \text{ if } \rho \neq \mathbb{1} \text{ and } \rho\chi \neq \mathbb{1},$$

$$e_{i+1}(s, \mathbb{1}, MC_\chi(\mathcal{F})) = e_i(s, \chi^{-1}, \mathcal{F}),$$

$$e_i(s, \chi, MC_\chi(\mathcal{F})) = e_{i+1}(s, \mathbb{1}, \mathcal{F}).$$

(6.0.14) Only $e_1(s, \mathbb{1}, MC_\chi(\mathcal{F}))$ now remains uncomputed, and it is given by equating ranks in the above isomorphism 6.0.12 of $I(s)^{tame}$-representations:

$$r(MC_\chi(\mathcal{F})) - e_1(s, \mathbb{1}, MC_\chi(\mathcal{F})) = r(\mathcal{F}) - e_1(s, \mathbb{1}, \mathcal{F}).$$

(6.0.15) It will also be convenient to give two expressions for the index of rigidity $\chi(\mathbb{P}^1, j_\star\underline{End}(\mathcal{F}|\ \mathbb{A}^1 - D))$ in terms of the numerical data. The first expression is

$$\chi(\mathbb{P}^1, j_\star\underline{End}(\mathcal{F}|\ \mathbb{A}^1 - D)) =$$

$$= \chi(\mathbb{A}^1 - D, \underline{End}(\mathcal{F})) + \Sigma_{s \text{ in } D\amalg\{\infty\}} \dim(End_{I(s)}(\mathcal{F}(s)))$$

$$= \chi(\mathbb{A}^1 - D, \underline{End}(\mathcal{F})) + \Sigma_{s \text{ in } D\amalg\{\infty\}} \Sigma_{i,\chi}\ e_i(s,\chi, \mathcal{F})^2$$

$$= (1 - \mathrm{Card}(D))r(\mathcal{F})^2 + \Sigma_{s \text{ in } D\amalg\{\infty\}} \Sigma_{i,\chi}\ e_i(s,\chi, \mathcal{F})^2.$$

This expression makes it numerically obvious (it is already conceptually obvious !) that $MT_\mathcal{L}(\mathcal{F})$ has the same index of rigidity as does $\mathcal{F}$.

(6.0.16) The second expression for the index of rigidity is more complicated-looking, but it has the merit that is a sum of terms, indexed by the points of $D\amalg\{\infty\}$, each of which is visibly the same for $\mathcal{F}$ and for $MC_\chi(\mathcal{F})$. So this formula has the merit of making numerically obvious the fact that $\mathcal{F}$ and $MC_\chi(\mathcal{F})$ have the same index of rigidity.

Lemma 6.0.17 In the situation 6.0.1, for $\mathcal{F}$ in $\mathcal{T}_\ell$ lisse on $\mathbb{A}^1 - D$, its index of rigidity is given by

$$\chi(\mathbb{P}^1, j_\star\underline{End}(\mathcal{F}|\ \mathbb{A}^1 - D)) =$$

$$\{ -m(\mathcal{F})^2 + \Sigma_{i,\chi}\ E_i(\infty,\chi,\mathcal{F})^2\} +$$

$$+ \Sigma_{s \text{ in } D} \{(r(\mathcal{F}) - e_1(s,\mathbb{1},\mathcal{F}))^2 + \Sigma_{(i,\chi) \neq (1,\mathbb{1})}\ e_i(s,\chi,\mathcal{F})^2\}.$$

proof We begin with the first expression for the index:

$\chi(\mathbb{P}^1, j_*\underline{\mathrm{End}}(\mathcal{F}|\, \mathbb{A}^1 - D)) =$

$= (1 - \mathrm{Card}(D))r(\mathcal{F})^2 + \Sigma_{s \text{ in } D\amalg\{\infty\}} \Sigma_{i,\chi}\ e_i(s,\chi,\ \mathcal{F})^2$

$= (1 - \mathrm{Card}(D))r(\mathcal{F})^2 +$

$+\ \Sigma_{i,\chi}\ e_i(\infty,\chi,\ \mathcal{F})^2 + \Sigma_{s \text{ in } D} \Sigma_{i,\chi}\ e_i(s,\chi,\ \mathcal{F})^2$

$= (1 - \mathrm{Card}(D))r(\mathcal{F})^2 +$

$-E_1(\infty,\mathbb{1},\mathcal{F})^2 + \Sigma_{i,\chi}\ E_i(\infty,\chi,\ \mathcal{F})^2$

$+ \Sigma_{s \text{ in } D} \{e_1(s,\mathbb{1},\mathcal{F})^2\ +\ \Sigma_{(i,\chi) \neq (1,\mathbb{1})}\ e_i(s,\chi,\mathcal{F})^2\}.$

Comparing this with the asserted value, namely

$\{ -m(\mathcal{F})^2 + \Sigma_{i,\chi}\ E_i(\infty,\chi,\mathcal{F})^2\} +$

$+ \Sigma_{s \text{ in } D} \{(r(\mathcal{F}) - e_1(s,\mathbb{1},\mathcal{F}))^2\ +\ \Sigma_{(i,\chi) \neq (1,\mathbb{1})}\ e_i(s,\chi,\mathcal{F})^2\},$

it remains to prove

$(1 - \mathrm{Card}(D))r(\mathcal{F})^2 -E_1(\infty,\mathbb{1},\mathcal{F})^2 + \Sigma_{s \text{ in } D}\ e_1(s,\mathbb{1},\mathcal{F})^2$

$=-m(\mathcal{F})^2 + \Sigma_{s \text{ in } D}\ (r(\mathcal{F}) - e_1(s,\mathbb{1},\mathcal{F}))^2.$

We rewrite this as

$r(\mathcal{F})^2 -E_1(\infty,\mathbb{1},\mathcal{F})^2 + m(\mathcal{F})^2 =$

$= \Sigma_{s \text{ in } D} \{(r(\mathcal{F}) - e_1(s,\mathbb{1},\mathcal{F}))^2 +r(\mathcal{F})^2 - e_1(s,\mathbb{1},\mathcal{F})^2\}.$

Using the identity

$m(\mathcal{F}) - r(\mathcal{F}) = E_1(\infty,\mathbb{1},\mathcal{F}),$

this reduces to

$r(\mathcal{F})^2 -(m(\mathcal{F}) - r(\mathcal{F}))^2 + m(\mathcal{F})^2 =$

$= \Sigma_{s \text{ in } D} \{(r(\mathcal{F}) - e_1(s,\mathbb{1},\mathcal{F}))^2 +r(\mathcal{F})^2 - e_1(s,\mathbb{1},\mathcal{F})^2\},$

which we rewrite

$2r(\mathcal{F})m(\mathcal{F}) = \Sigma_{s \text{ in } D} \{2r(\mathcal{F})^2 - 2r(\mathcal{F})e_1(s,\mathbb{1},\mathcal{F})\}.$

Factoring out $2r(\mathcal{F})$ from both sides, we find the definition of $m(\mathcal{F})$,

$m(\mathcal{F}) = \Sigma_{s \text{ in } D} \{r(\mathcal{F}) - e_1(s,\mathbb{1},\mathcal{F})\}$. QED

6.1 Numerical incarnation: the group NumData

(6.1.1) We now abstract the numerical data attached to an object $\mathcal{F}$ in $\mathcal{T}_\ell$ which is lisse on $\mathbb{A}^1 - D$, and for which all characters occurring in all local monodromies of $\mathcal{F}$ lie in a fixed subgroup Γ of the group of all characters of $\pi_1(\mathbb{G}_m)$.

(6.1.2) Thus we fix:

a finite set D with $\mathrm{Card}(D) \geq 2$,

a set $\{\infty\}$ with one element,
an abelian group Γ, with identity element denoted $\mathbb{1}$.

We define the abelian group NumData(D,Γ) as follows: an element $\mathfrak{N}$ of NumData(D,Γ) is a quadruple (r, m, e, E) consisting of
1) an integer $r(\mathfrak{N})$,
2) an integer $m(\mathfrak{N})$,
3) a $\mathbb{Z}$-valued function e on the product set $(\mathbb{Z}_{\geq 1}) \times (D \coprod \{\infty\}) \times \Gamma$, which we also view as a collection of integers $e_i(s,\chi,\mathfrak{N})$, one for every integer $i \geq 1$, every χ in Γ, and every s in $D \coprod \{\infty\}$,
4) a $\mathbb{Z}$-valued function E on the product set $(\mathbb{Z}_{\geq 1}) \times (\{\infty\}) \times \Gamma$, which we also view as a collection of integers $E_i(\infty,\chi,\mathfrak{N})$ for every $i \geq 1$ and every χ in Γ, and which satisfies the following conditions:

i) the function e has finite support, and for each s in $D \coprod \{\infty\}$, we have $\Sigma_{i,\chi}\ e_i(s,\chi,\mathfrak{N}) = r(\mathfrak{N})$,
ii) $m(\mathfrak{N}) = \Sigma_{s \text{ in } D} \{r(\mathfrak{N}) - e_1(s,\mathbb{1},\mathfrak{N})\}$,
iii) $E_1(\infty,\mathbb{1},\mathfrak{N}) = m(\mathcal{F}) - r(\mathfrak{N})$,
iv) the integers $E_i(\infty,\chi,\mathfrak{N})$ and $e_i(\infty,\chi,\mathfrak{N})$ are related by

$E_i(\infty,\chi,\mathfrak{N}) = e_i(\infty,\chi,\mathfrak{N})$ if $\chi \neq \mathbb{1}$,
$E_{i+1}(\infty,\mathbb{1},\mathfrak{N}) = e_i(\infty,\mathbb{1},\mathfrak{N})$ for all $i \geq 1$.

v) the function E has finite support, and $\Sigma_{i,\chi}\ E_i(\infty,\chi,\mathfrak{N}) = m(\mathfrak{N})$.

(6.1.3) Addition in NumData(D,Γ) is defined componentwise.
(6.1.4) There is a great deal of redundancy in this presentation of an element $\mathfrak{N}$ in NumData(D,Γ). An element $\mathfrak{N}$ is determined by the data $(r(\mathfrak{N}), e)$, which is subject only to condition i). [Then use ii) to find $m(\mathfrak{N})$, then iii) to find $E_1(\infty,\mathbb{1},\mathfrak{N})$, then iv) to define all $E_i(\infty,\chi,\mathfrak{N})$; v) will be automatic, as it is implied by ii), iii) and iv).]
(6.1.5) Alternatively, an element $\mathfrak{N}$ in NumData(D,Γ) is determined by the data $(r(\mathfrak{N}), m(\mathfrak{N}), e$ restricted to $(\mathbb{Z}_{\geq 1}) \times D \times \Gamma$, E), which is subject only to ii), iii), v) and to

i not ∞) the function e on $(\mathbb{Z}_{\geq 1}) \times D \times \Gamma$ has finite support, and for each s in D, we have $\Sigma_{i,\chi}\ e_i(s,\chi,\mathfrak{N}) = r(\mathfrak{N})$.

(6.1.6) We define an even $\mathbb{Z}$-valued quadratic form, called the index of rigidity, denoted "rig", on NumData(D,Γ) by either of the following equivalent (cf. the proof of lemma. 6.0.17) formulas:

$$\mathrm{rig}(\mathcal{N}) := (1 - \mathrm{Card}(D))r(\mathcal{N})^2 + \Sigma_{s \text{ in } D\amalg\{\infty\}} \Sigma_{i,\chi}\ e_i(s,\chi, \mathcal{N})^2,$$

or

$$\mathrm{rig}(\mathcal{N}) := \{ -m(\mathcal{N})^2 + \Sigma_{i,\chi} E_i(\infty,\chi,\mathcal{N})^2\} +$$

$$+ \Sigma_{s \text{ in } D} \{(r(\mathcal{N}) - e_1(s,\mathbb{1},\mathcal{N}))^2 + \Sigma_{(i,\chi) \neq (1,\mathbb{1})} e_i(s,\chi,\mathcal{N})^2\}.$$

(6.1.7) Given an element $\chi \neq \mathbb{1}$ in Γ, we define an endomorphism MC_χ of NumData(D,Γ) as follows: given $\mathcal{N}$ in NumData(D,Γ), $MC_\chi(\mathcal{N})$ is given as follows:

α) $m(MC_\chi(\mathcal{N})) := m(\mathcal{N})$,

β) $E_i(\infty, \rho\chi, MC_\chi(\mathcal{N})) := E_i(\infty, \rho, \mathcal{N})$ for all i and ρ,

γ) $r(MC_\chi(\mathcal{N})) := m(MC_\chi(\mathcal{N})) - E_1(\infty, \mathbb{1}, MC_\chi(\mathcal{N}))$,

δ) for s in D,

δ1) $e_i(s, \rho\chi, MC_\chi(\mathcal{N})) := e_i(s, \rho, \mathcal{N})$ if $\rho \neq \mathbb{1}$ and $\rho\chi \neq \mathbb{1}$,

δ2) $e_i(s, \chi, MC_\chi(\mathcal{N})) := e_{i+1}(s, \mathbb{1}, \mathcal{N})$,

δ3) $e_{i+1}(s, \mathbb{1}, MC_\chi(\mathcal{N})) := e_i(s, \chi^{-1}, \mathcal{N})$,

δ4) $e_1(s, \mathbb{1}, MC_\chi(\mathcal{N})) := r(MC_\chi(\mathcal{N})) - r(\mathcal{N}) + e_1(s, \mathbb{1}, \mathcal{N})$.

It is useful to rewrite δ4) in the more symmetric form

δ4bis) for s in D,

$$r(MC_\chi(\mathcal{N})) - e_1(s, \mathbb{1}, MC_\chi(\mathcal{N})) = r(\mathcal{F}) - e_1(s, \mathbb{1}, \mathcal{N}),$$

and to rewrite γ) as well:

γbis) $r(MC_\chi(\mathcal{N})) + E_1(\infty, \chi^{-1}, \mathcal{N}) := m(\mathcal{N})$.

Lemma 6.1.8 For each $\chi \neq \mathbb{1}$ in Γ, the operator $\mathcal{N} \mapsto MC_\chi(\mathcal{N})$ is an orthogonal automorphism of NumData(D,Γ), with inverse $MC_\chi{-1}$. Moreover, if also $\rho \neq \mathbb{1}$ in Γ, then

$$MC_\rho \circ MC_\chi = MC_{\rho\chi}, \text{ if } \rho\chi \neq \mathbb{1}.$$

proof Given $\mathcal{N}$ in NumData(D,Γ), the above formulas define all of the quantities

(r, m, e restricted to $(\mathbb{Z}_{\geq 1}) \times D \times \Gamma$, E)

needed to define $MC_\chi(\mathcal{N})$, but we must show that $MC_\chi(\mathcal{N})$ satisfies the relations which characterize elements of NumData(D,Γ). That *e* and E have finite support is obvious from the definitions (and the fact that $\mathcal{N}$ was in NumData(D,Γ)). Relation ii) holds, thanks to α) and δ4bis). Relation iii) holds by definition, and relation v) holds by α) and β). To verify the rank formulas in "i) not ∞", rewrite

$$r(MC_\chi(\mathcal{N})) = \Sigma_{i,\chi}\ e_i(s,\chi,MC_\chi(\mathcal{N}))$$

as

$$r(MC_\chi(\mathcal{N})) - e_1(s,\mathbb{1},MC_\chi(\mathcal{N})) = \Sigma_{(i,\rho)\neq(1,\mathbb{1})}\ e_i(s,\rho,MC_\chi(\mathcal{N})).$$

By δ1-3), the right hand side is

$$\Sigma_{(i,\rho)\neq(1,\mathbb{1})}\ e_i(s,\rho,MC_\chi(\mathcal{N}))$$
$$=\Sigma_{i\geq 2}\ e_i(s,\mathbb{1},MC_\chi(\mathcal{N})) + \Sigma_{i\geq 1}\ e_i(s,\chi,MC_\chi(\mathcal{N})) + \Sigma_{i,\rho\neq\mathbb{1},\chi}\ e_i(s,\rho,MC_\chi(\mathcal{N}))$$
$$= \Sigma_{i\geq 1}\ e_i(s,\chi^{-1},\mathcal{N}) + \Sigma_{i\geq 2}\ e_i(s,\mathbb{1},\mathcal{N}) + \Sigma_{i,\rho\neq\mathbb{1},\chi^{-1}}\ e_i(s,\rho,\mathcal{N})$$
$$=\Sigma_{i,\rho}\ e_i(s,\rho,\mathcal{N}) - e_1(s,\mathbb{1},\mathcal{N})$$
$$=r(\mathcal{N}) - e_1(s,\mathbb{1},\mathcal{N})$$
$$=r(MC_\chi(\mathcal{N})) - e_1(s,\mathbb{1},MC_\chi(\mathcal{N})),$$

this last equality by δ4bis).

This shows that MC_χ does in fact map NumData(D,Γ) to itself. That MC_χ is orthogonal is obvious from the second formula for the quadratic form,

$$rig(\mathcal{N}) := \{ -m(\mathcal{N})^2 + \Sigma_{i,\chi}\ E_i(\infty,\chi,\mathcal{N})^2\} +$$
$$+ \Sigma_{s \text{ in } D}\ \{(r(\mathcal{N}) - e_1(s,\mathbb{1},\mathcal{N}))^2 + \Sigma_{(i,\chi)\neq(1,\mathbb{1})}\ e_i(s,\chi,\mathcal{N})^2\}.$$

which writes it as a sum of terms each invariant by $\mathcal{N} \mapsto MC_\chi(\mathcal{N})$.

To show that $(MC_{\chi^{-1}})\circ MC_\chi = id$, we argue as follows. That this is true for m and E is obvious from the definitions. That it is true for r follows from γ),

$$r(MC_\chi(\mathcal{N})) = m(\mathcal{N}) - E_1(\infty,\ \chi^{-1},\ \mathcal{N}),$$

which applied to $MC_{\chi^{-1}}(MC_\chi(\mathcal{N}))$ gives

$$r(MC_{\chi^{-1}}(MC_\chi(\mathcal{N}))) = m(MC_\chi(\mathcal{N})) - E_1(\infty,\ \chi,\ MC_\chi(\mathcal{N}))$$
$$=m(\mathcal{N}) - E_1(\infty,\ \mathbb{1},\ \mathcal{N}),\ \text{using } \alpha) \text{ and } \beta),$$
$$=r(\mathcal{N}),\ \text{by } \gamma).$$

once we know this, then

$$e_1(s,\ \mathbb{1},\ MC_{\chi^{-1}}(MC_\chi(\mathcal{N}))) = e_1(s,\ \mathbb{1},\ \mathcal{N})$$

follows from a double application of δ4bis). Similarly, from δ3) and δ2) we get

$$e_{i+1}(s,\ \mathbb{1},\ MC_{\chi^{-1}}(MC_\chi(\mathcal{N}))) = e_i(s,\ \chi,\ MC_\chi(\mathcal{N})) = e_{i+1}(s,\ \mathbb{1},\ \mathcal{N}),$$
$$e_i(s,\ \chi^{-1},\ MC_{\chi^{-1}}(MC_\chi(\mathcal{N}))) = e_{i+1}(s,\ \mathbb{1},\ MC_\chi(\mathcal{N})) = e_i(s,\ \chi^{-1},\mathcal{N}).$$

For $\rho\neq\mathbb{1}$ and $\rho\chi^{-1}\neq\mathbb{1}$, δ1) gives

$$e_i(s,\ \rho\chi^{-1},\ MC_{\chi^{-1}}(MC_\chi(\mathcal{N}))) = e_i(s,\ \rho,\ MC_\chi(\mathcal{N})) = e_i(s,\ \rho\chi^{-1},\mathcal{N}).$$

This concludes the proof that $MC_{\chi^{-1}}\circ MC_\chi = id$.

That $MC_\rho \circ MC_\chi = MC_{\rho\chi}$ if $\rho\chi \neq \mathbb{1}$ is shown in an altogether similar way: one first checks on m, then on E, then on r, then on $e_1(s, \mathbb{1}, MC_\rho(MC_\chi(\mathcal{N})))$, then on $e_{i+1}(s, \mathbb{1}, MC_\rho(MC_\chi(\mathcal{N})))$, then on $e_i(s, \rho, MC\rho(MC_\chi(\mathcal{N})))$, and finally on $e_i(s, \Lambda, MC\rho(MC_\chi(\mathcal{N})))$ for any Λ other than $\mathbb{1}$ or ρ. QED

(6.1.9) Suppose now we are given an element $\mathcal{L}$ in the group Maps(D, Γ) of all maps of sets from D to Γ (the group structure by pointwise multiplication of values). We write

$$s \mapsto \chi_{s,\mathcal{L}}$$

for the map given by $\mathcal{L}$, and we define

$$\chi_{\infty,\mathcal{L}} := \prod_{s \text{ in } D} \chi_{s,\mathcal{L}}.$$

We write the multiplication in Maps(D, Γ) as $\mathcal{L}_1 \otimes \mathcal{L}_2$.

(6.1.10) Given an element $\mathcal{L}$ in Maps(D, Γ), we define an endomorphism $MT_\mathcal{L}$ of NumData(D, Γ) as follows. We work in the (r, e) presentation of elements of NumData(D, Γ). Given $\mathcal{N}$ in NumData(D, Γ), we define

$$r(MT_\mathcal{L}(\mathcal{N})) := r(\mathcal{N}),$$

$$e_i(s, \rho\chi_{s,\mathcal{L}}, MT_\mathcal{L}(\mathcal{N})) := e_i(s, \rho, \mathcal{N})$$

for any (i, s, ρ) in $(\mathbb{Z}_{\geq 1}) \times (D \amalg \{\infty\}) \times \Gamma$.

Lemma 6.1.11 Given $\mathcal{L}$ in Maps(D, Γ), the operator $\mathcal{N} \mapsto MT_\mathcal{L}(\mathcal{N})$ is an orthogonal automorphism of NumData(D,Γ), with inverse $MT_\mathcal{L}$-1. Moreover, given $\mathcal{L}_1$ and $\mathcal{L}_2$ in Maps(D, Γ), we have

$$MT_{\mathcal{L}_2} \circ MT_{\mathcal{L}_1} = MT_{\mathcal{L}_1 \otimes \mathcal{L}_2}.$$

proof Clear, using the first formula of 6.1.6 for the quadratic form. QED

(6.1.12) We can construct a graph whose vertices are the elements of NumData(D,Γ), and in which there is an edge joining two elements $\mathcal{M}$ and $\mathcal{N}$ if either of the following conditions a) or b) holds:

a) $\mathcal{M}$ and $\mathcal{N}$ each have $r \geq 2$, and there is an $\mathcal{L}$ in Maps(D, Γ) such that $\mathcal{M} = MT_\mathcal{L}(\mathcal{N})$, or equivalently $\mathcal{N} = MT_\mathcal{L}\text{-}1(\mathcal{M})$,

b) for some $\chi \neq \mathbb{1}$ in Γ, $\mathcal{M} = MC_\chi(\mathcal{N})$, or equivalently $\mathcal{N} = MC_\chi\text{-}1(\mathcal{M})$.

6.2 A compatibility theorem

(6.2.1) Let k be an algebraically closed field, $\ell \neq$ char(k) a prime number, D a finite subset of $\mathbb{A}^1(k)$ with Card(D) ≥ 2, and Γ a subgroup of the group of all continuous $\overline{\mathbb{Q}}_\ell$-valued characters of $\pi_1(\mathbb{G}_m)^{\mathrm{tame}}$. Given an object $\mathcal{F}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$, denote by $\mathrm{ND}(\mathcal{F})$ its numerical data, i.e., the element of NumData(D,Γ) given by

$$(r(\mathcal{F}), m(\mathcal{F}), E_i(\infty,\chi,\mathcal{F}) \text{ for all } (i,\chi), e_i(s,\chi,\mathcal{F}) \text{ for all } (i,s,\chi)).$$

To give a lisse, tame, rank one $\mathcal{L}$ on $\mathbb{A}^1 - D$ with all its local monodromies $\chi_{s,\mathcal{L}}$ in Γ, is the same as to give the element in Maps(D, Γ), still denoted $\mathcal{L}$, which is $s \mapsto \chi_{s,\mathcal{L}}$. We may summarize the previous discussion in the following

Compatibility Theorem 6.2.2 In the situation 6.2.1, we have:

1) For $\mathcal{F}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$, $\mathrm{rig}(\mathcal{F}) = \mathrm{rig}(\mathrm{ND}(\mathcal{F}))$.

2) For $\mathcal{F}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ and χ in Γ, $\mathrm{MC}_\chi(\mathrm{ND}(\mathcal{F})) = \mathrm{ND}(\mathrm{MC}_\chi(\mathcal{F}))$.

3) For $\mathcal{F}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ of rank $r \geq 2$, and $\mathcal{L}$ lisse, tame, rank one on $\mathbb{A}^1 - D$ with all local monodromies in Γ, $\mathrm{MT}_\mathcal{L}(\mathrm{ND}(\mathcal{F})) = \mathrm{ND}(\mathrm{MT}_\mathcal{L}(\mathcal{F}))$.

4) Given two objects $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$, $\mathcal{F}$ and $\mathcal{G}$ are connected (as vertices in the graph built on $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$) if and only if their numerical data $\mathrm{ND}(\mathcal{F})$ and $\mathrm{ND}(\mathcal{G})$ are connected (as vertices in the graph built on NumData(D, Γ).

Compatibility Theorem 6.2.3 (Complex analytic variant) If k is $\mathbb{C}$, the above theorem 6.2.2 remains valid with Γ any subgroup of $\mathrm{Hom}(\pi_1((\mathbb{G}_{m,\mathbb{C}})^{\mathrm{an}}), \mathbb{C}^\times) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^\times) = \mathbb{C}^\times$, and with $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ replaced by $\mathcal{T}_{\mathrm{an}}(\mathbb{A}^1 - D, \Gamma)$.

6.3 Realizable and plausible elements

(6.3.1) We continue to work over an algebraically closed field k of characteristic $\neq \ell$. As in 6.0.1, we fix a finite subset D of $\mathbb{A}^1(k)$ with Card(D) ≥ 2.We now consider the question of recognizing those elements $\mathfrak{N}$ in NumData(D, Γ) which are of the form $\mathrm{ND}(\mathcal{F})$ for some $\mathcal{F}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ [or for some $\mathcal{F}$ in $\mathcal{T}_{\mathrm{an}}(\mathbb{A}^1 - D, \Gamma)$, if $k = \mathbb{C}$, and if Γ is a subgroup of $\mathbb{C}^\times$]. We call such such elements of NumData(D, Γ) realizable.

Lemma 6.3.2 In the situation 6.3.1, any realizable element $\mathcal{N}$ satisfies the following ten conditions:

1) $r(\mathcal{N}) \geq 1$,
2) $m(\mathcal{N}) \geq r(\mathcal{N})$,
3) for all χ in Γ and all $i \geq 1$,
$$E_i(\infty,\chi,\mathcal{N}) \geq 0, \text{ and } E_i(\infty,\chi,\mathcal{N}) \geq E_{i+1}(\infty,\chi,\mathcal{N}),$$
4) for all s in $D\amalg\{\infty\}$, all χ in Γ and all $i \geq 1$,
$$e_i(s,\chi,\mathcal{N}) \geq 0, \text{ and } e_i(s,\chi,\mathcal{N}) \geq e_{i+1}(s,\chi,\mathcal{N}),$$
5)there exist at least two distinct s in D for which $r(\mathcal{N}) > e_1(s,\mathbb{1},\mathcal{N})$,
6) $(1 - \mathrm{Card}(D))r(\mathcal{N}) + \Sigma_{s \text{ in } D\amalg\{\infty\}} e_1(s,\mathbb{1},\mathcal{N}) \leq 0$,
7) if $r(\mathcal{N}) \geq 2$, then for any s_0 in $D\amalg\{\infty\}$,
$$(1 - \mathrm{Card}(D))r(\mathcal{N}) + \Sigma_{s\neq s_0 \text{ in } D\amalg\{\infty\}} \mathrm{Max}_\chi\{e_1(s,\chi,\mathcal{N})\} \leq 0,$$

8) $\mathrm{rig}(\mathcal{N}) \leq 2$,
9) $\mathrm{rig}(\mathcal{N})/r(\mathcal{N}) \leq (1 - \mathrm{Card}(D))r(\mathcal{N}) + \Sigma_{s \text{ in } D\amalg\{\infty\}} \mathrm{Max}_\chi\{e_1(s,\chi,\mathcal{N})\}$,
10) in the group Γ written additively, we have
$$\Sigma_\chi(\Sigma_i\, e_i(\infty,\chi,\mathcal{N}))\chi = \Sigma_\chi\,(\Sigma_{s \text{ in } D}\,\Sigma_i\, e_i(s,\chi,\mathcal{N}))\chi.$$

proof If $\mathcal{N}$ is $\mathrm{ND}(\mathcal{F})$ for $\mathcal{F}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ [resp. $\mathcal{T}_{an}(\mathbb{A}^1 - D, \Gamma)$ if $k=\mathbb{C}$], then 5) and 1) hold because $\mathcal{F}$ fails to be lisse at two or more points of D, and so in particular is nonzero. Since $\mathcal{F}(\infty)$ is a quotient of $M(\infty,\mathcal{F})$, we get 2). Since the E_i and the e_i are the numerical invariants of actual representations $M(\infty,\mathcal{F})$ and $\mathcal{F}(s)$ respectively, we get 3) and 4). Since $\mathcal{F}$ is irreducible nontrivial, $\chi(\mathbb{P}^1, j_*\mathcal{F}) \leq 0$, and $\chi(\mathbb{P}^1, j_*\mathrm{End}(\mathcal{F})) \leq 2$, which give 6) and 8). If $\mathcal{F}$ has rank 2 or more, we choose a rank one $\mathcal{L}$ so that $MT_\mathcal{L}(\mathcal{F})$ satisfies
$$\mathrm{Max}_\chi\{e_1(s,\chi,MT_\mathcal{L}(\mathcal{F}))\} = e_1(s,\mathbb{1},MT_\mathcal{L}(\mathcal{F}))$$
for each $s\neq s_0$ in $D\amalg\{\infty\}$. Then we get 7) by writing out
$$\chi(\mathbb{P}^1, j_*MT_\mathcal{L}(\mathcal{F}) \leq 0$$
and dropping the term $e_1(s_0,\mathbb{1},MT_\mathcal{L}(\mathcal{F}))$. The Basic Inequality 5.2.4.7 gives 9). Because $\det(\mathcal{F})$ is lisse on $\mathbb{A}^1 - D$, tame of rank one, we get 10). QED

Definition 6.3.3 An element $\mathcal{N}$ in $\mathrm{NumData}(D, \Gamma)$ is called **plausible** if it satisfies the ten conditions of the above lemma 6.3.2.

(6.3.4) The above lemma 6.3.2 states that every realizable element is plausible. It is **not** true that every plausible element is realizable. We will give an example, due to Deligne, in 6.4 below.

Lemma 6.3.5 In the situation 6.3.1, if $\mathcal{N}$ in NumData(D, Γ) is realizable, then every element $\mathcal{M}$ of NumData(D, Γ) to which $\mathcal{N}$ is connected (as vertex of the graph) is also realizable.
proof By induction on the length of a path which $\mathcal{M}$ to $\mathcal{N}$, it suffices to show that each nearest neighbor of $\mathcal{N}$ is realizable. But such a nearest neighbor is either $MC_\chi(\mathcal{N})$, for some $\chi \neq \mathbb{1}$ in Γ, or, in case $r(\mathcal{N}) \geq 2$, is possibly $MT_\mathcal{L}(\mathcal{N})$, for some $\mathcal{L}$ in Maps(D. Γ). If $\mathcal{N}$ is ND($\mathcal{F}$), these neighbors are $ND(MC_\chi(\mathcal{F}))$ and $ND(MT_\mathcal{L}(\mathcal{F}))$ respectively. QED

Lemma 6.3.6 In the situation 6.3.1, if $\mathcal{N}$ in NumData(D, Γ) is plausible and if $r(\mathcal{N})=1$, then $\mathcal{N}$ is realizable.
proof $\mathcal{N}$ is ND($\mathcal{L}$) for the unique rank one $\mathcal{L}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ [respectively in $\mathcal{T}_{an}(\mathbb{A}^1 - D, \Gamma)$] whose local monodromy at s in $D \amalg \{\infty\}$ is the unique character χ_s for which $e_1(s,\chi,\mathcal{N}) \neq 0$. [This $\mathcal{L}$ exists by condition (10) in the definition of plausibility.] QED

Lemma 6.3.7 In the situation 6.3.1, Suppose that $\mathcal{N}$ in NumData(D, Γ) is plausible, has $r(\mathcal{N}) \geq 2$, and has $rig(\mathcal{N}) = 2$. For each s in D, denote by χ_s any character such that

$$e_1(s,\chi_s,\mathcal{N}) = Max_\chi\{e_1(s,\chi,\mathcal{N})\},$$

and denote by $\mathcal{L}$ the unique rank one $\mathcal{L}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ [respectively in $\mathcal{T}_{an}(\mathbb{A}^1 - D, \Gamma)$] whose local monodromy at s in D is $(\chi_s)^{-1}$. Consider the element $MT_\mathcal{L}(\mathcal{N})$ of NumData(D, Γ). Then either $MT_\mathcal{L}(\mathcal{N})$ is not plausible, or we are in the following situation:
1) $MT_\mathcal{L}(\mathcal{N})$ is plausible,
2) $e_1(\infty,\mathbb{1},MT_\mathcal{L}(\mathcal{N})) < Max_\chi\{e_1(\infty,\chi,MT_\mathcal{L}(\mathcal{N}))\}$,
3) for any character $\chi \neq \mathbb{1}$ such that

$$e_1(\infty,\chi^{-1},MT_\mathcal{L}(\mathcal{N})) = Max_\chi\{e_1(\infty,\chi,MT_\mathcal{L}(\mathcal{N}))\},$$

the element $MC_\chi(MT_\mathcal{L}(\mathcal{N}))$ has $r(MC_\chi(MT_\mathcal{L}(\mathcal{N}))) < r(\mathcal{N})$.
proof If $MT_\mathcal{L}(\mathcal{N})$ is not plausible, we are done. If $MT_\mathcal{L}(\mathcal{N})$ is plausible, we know $rig(MT_\mathcal{L}(\mathcal{N})) = 2$, and by construction we have

$$e_1(s,\mathbb{1},\mathcal{N}) = Max_\chi\{e_1(s,\chi,\mathcal{N})\} \text{ for each s in D,}$$

so 2) results from the plausibility relations 6) and 8). To prove 3), notice that

$$\begin{aligned}
&r(MC_\chi(MT_\mathcal{L}(\mathcal{N}))) = m(MC_\chi(MT_\mathcal{L}(\mathcal{N}))) - E_1(\infty,\mathbb{1},MC_\chi(MT_\mathcal{L}(\mathcal{N})))\\
&= m(MT_\mathcal{L}(\mathcal{N})) - e_1(\infty,\chi^{-1},MT_\mathcal{L}(\mathcal{N}))\\
&= - e_1(\infty,\chi^{-1},MT_\mathcal{L}(\mathcal{N})) \ + \ \Sigma_{s \text{ in } D}\{r(MT_\mathcal{L}(\mathcal{N})) - e_1(s,\mathbb{1},MT_\mathcal{L}(\mathcal{N}))\}\\
&= - e_1(\infty,\chi^{-1},MT_\mathcal{L}(\mathcal{N})) \ + \ \Sigma_{s \text{ in } D}\{r(\mathcal{N}) - e_1(s,\mathbb{1},MT_\mathcal{L}(\mathcal{N}))\}\\
&= - e_1(\infty,\chi^{-1},MT_\mathcal{L}(\mathcal{N})) + \mathrm{Card}(D)r(\mathcal{N}) - \Sigma_{s \text{ in } D}\, e_1(s,\mathbb{1},MT_\mathcal{L}(\mathcal{N}))\\
&= \ \mathrm{Card}(D)r(\mathcal{N}) - \Sigma_{s \text{ in } D\amalg\{\infty\}} \mathrm{Max}_\chi\{e_1(s,\chi,\mathcal{N})\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&r(\mathcal{N}) - r(MT_\chi(MT_\mathcal{L}(\mathcal{N}))) =\\
&= (1 - \mathrm{Card}(D))r(\mathcal{N}) + \Sigma_{s \text{ in } D\amalg\{\infty\}} \mathrm{Max}_\chi\{e_1(s,\chi,\mathcal{N})\}\\
&\geq \mathrm{rig}(\mathcal{N})/r(\mathcal{N}) = 2/r(\mathcal{N}) > 0,
\end{aligned}$$

the penultimate inequality from plausibility condition 8). QED

6.4 Existence algorithm for rigids

(6.4.1) In the situation 6.3.1, suppose that $\mathcal{N}$ in NumData(D, Γ) has rig($\mathcal{N}$) = 2. Here is an algorithm to determine if $\mathcal{N}$ is realizable.

Step I X := $\mathcal{N}$.

Step II Is X plausible? If not, $\mathcal{N}$ is not realizable. Stop.

Step III Is r(X) = 1? If so, $\mathcal{N}$ is realizable. Stop.

Step IV For each s in D, choose a character χ_s such that

$$e_1(s,\chi_s,X) = \mathrm{Max}_\chi\{e_1(s,\chi,X)\}.$$

Denote by $\mathcal{L}$ the unique rank one $\mathcal{L}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ [respectively $\mathcal{T}_{an}(\mathbb{A}^1 - D, \Gamma)$] whose local monodromy at s in D is $(\chi_s)^{-1}$.

X:= $MT_\mathcal{L}$(X).

Step V Is X plausible? If not, $\mathcal{N}$ is not realizable. Stop.

Step VI Choose a character $\chi \neq \mathbb{1}$ such that

$$e_1(\infty,\chi^{-1},X) = \mathrm{Max}_\chi\{e_1(\infty,\chi,X)\}$$

(such χ exist, by 6.3.7 (2) above). X:= MC_χ(X). Go to Step II.

(6.4.2) Since Step VI lowers the rank by at least one, we only need iterate the algorithm at most r($\mathcal{N}$) - 1 times to determine whether or not $\mathcal{N}$ is realizable.

(6.4.3) To see that the algorithm gives the correct answer, assume first that $\mathcal{N}$ is realizable. Then by 5.2.1 the algorithm will correctly tell us that $\mathcal{N}$ is realizable. Suppose that we start with an $\mathcal{N}$ which the algorithm tells us is realizable. Then the algorithm provides us with a path which connects $\mathcal{N}$ to a plausible object of rank one. But any such object is realizable (by 6.3.6), and so our $\mathcal{N}$ is connected to a realizable object, so is itself realizable (by 6.3.5).

6.5 An example

(6.5.1) Here is an example, due to Deligne, of an element $\mathcal{N}$ in NumData(D, Γ) which has rig($\mathcal{N}$)=2, and which is plausible, but which is not realizable, because it has a nearest neighbor which is not plausible. Suppose first that we are not in characteristic 2, and suppose that Γ contains the unique nontrivial character χ_2 of $\pi_1(\mathbb{G}_m)$ of order 2. We take D = {0,1},

$r(\mathcal{N}) = 7$,

$e_1(\infty,\chi_2,\mathcal{N}) = 3$, $e_2(\infty,\chi_2,\mathcal{N}) = 1$, $e_i(\infty,\mathbb{1},\mathcal{N}) = 1$ for $1 \le i \le 3$,

$e_1(0,\mathbb{1},\mathcal{N}) = e_2(0,\mathbb{1},\mathcal{N}) = 3$, $e_3(0,\mathbb{1},\mathcal{N}) = 1$,

$e_1(1,\mathbb{1},\mathcal{N}) = e_2(1,\mathbb{1},\mathcal{N}) = 3$, $e_3(1,\mathbb{1},\mathcal{N}) = 1$,

and all other $e_i(s,\chi,\mathcal{N}) = 0$. One checks easily that rig($\mathcal{N}$) = 2, and that $\mathcal{N}$ is plausible. However, $\mathcal{M} := MC_{\chi_2}(\mathcal{N})$ is readily computed. It turns out to have

$r(\mathcal{M}) = 5$,

$e_1(\infty,\mathbb{1},\mathcal{M}) = 1$, $e_i(\infty,\chi_2,\mathcal{M}) = 1$ for $1 \le i \le 4$,

$e_1(0,\mathbb{1},\mathcal{M}) = 1$, $e_1(0,\chi_2,\mathcal{M}) = 3$, $e_2(0,\chi_2,\mathcal{M}) = 1$,

$e_1(1,\mathbb{1},\mathcal{M}) = 1$, $e_1(1,\chi_2,\mathcal{M}) = 3$, $e_2(1,\chi_2,\mathcal{M}) = 1$,

but this $\mathcal{M}$ is not plausible, since it fails plausibility test 7) with s_0 taken as ∞.

(6.5.2) If we are in characteristic 2, or indeed in any characteristic not 3, suppose that Γ contains a nontrivial element χ_3 of order 3. We take D = {0,1},

$r(\mathcal{N}) = 7$,

$e_1(\infty,\chi_3,\mathcal{N}) = 3$, $e_i(\infty,\mathbb{1},\mathcal{N}) = 1$ for $1 \le i \le 4$,

$e_1(0,\mathbb{1},\mathcal{N}) = e_2(0,\mathbb{1},\mathcal{N}) = 3$, $e_3(0,\mathbb{1},\mathcal{N}) = 1$,

$e_1(1,\mathbb{1},\mathcal{N}) = e_2(1,\mathbb{1},\mathcal{N}) = 3$, $e_3(1,\mathbb{1},\mathcal{N}) = 1$,

and all other $e_i(s,\chi,\mathcal{N}) = 0$. In this case, $\mathcal{M} := MC_{\overline{\chi}_3}(\mathcal{N})$ has

$r(\mathcal{M}) = 5$,

$e_i(\infty,\overline{\chi}_3,\mathcal{M}) = 1$ for $1 \le i \le 5$,

$e_1(0,\mathbb{1},\mathcal{M}) = 1$, $e_1(0,\overline{\chi}_3,\mathcal{M}) = 3$, $e_2(0,\overline{\chi}_3,\mathcal{M}) = 1$,

$$e_1(1,\mathbb{1},\mathcal{M}) = 1,\ e_1(1,\overline{\chi}_3,\mathcal{M}) = 3,\ e_2(1,\overline{\chi}_3,\mathcal{M}) = 1,$$

and again fails to be plausible, for exactly the same reason.

6.6 Open questions How can one determine the realizabilty of elements of $\mathcal{N}$ in NumData(D, Γ) whose index of rigidity is some integer other than 2? A necessary condition for $\mathcal{N}$ to be realizable is that every $\mathcal{M}$ to which $\mathcal{N}$ is connected (in the graph on NumData(D, Γ)) is itself plausible. Can this condition be decided by an algorithm with finite running time? Is this condition sufficient? [It is sufficient for elements $\mathcal{N}$ with rig($\mathcal{N}$) = 2, as is clear from the algorithm.] Much remains to be done.

6.7 Action of automorphisms

(6.7.1) Let $\sigma: \chi \mapsto \chi^{(\sigma)}$ be an automorphism of Γ as abstract group. Given $\mathcal{N}$ in NumData(D, Γ), define $\mathcal{N}^{(\sigma)}$ in NumData(D, Γ) by

$$r(\mathcal{N}^{(\sigma)}) := r(\mathcal{N}),$$
$$m(\mathcal{N}^{(\sigma)}) := m(\mathcal{N}),$$
$$E_i(\infty,\chi^{(\sigma)},\mathcal{N}^{(\sigma)}) := E_i(\infty,\chi,\mathcal{N}),\ \text{for all } i\geq 1,\ \text{all } \chi,$$
$$e_i(s,\chi^{(\sigma)},\mathcal{N}^{(\sigma)}) := e_i(s,\chi,\mathcal{N}),\ \text{for all } i\geq 1,\ \text{all } s \text{ in } D\amalg\{\infty\},\ \text{all } \chi.$$

This defines an orthogonal left action of of the group Aut(Γ) on NumData(D, Γ).

Lemma 6.7.2 In the situation 5.3.1, we have

(1) If $\mathcal{N}$ in NumData(D, Γ) is plausible, then $\mathcal{N}^{(\sigma)}$ is plausible for every σ in Aut(Γ)

(2) If $\mathcal{N}$ and $\mathcal{M}$ in NumData(D, Γ) are adjacent in the graph, then $\mathcal{N}^{(\sigma)}$ and $\mathcal{M}^{(\sigma)}$ are adjacent for every σ in Aut(Γ).

proof (1) is obvious from the definitions. For (2), suppose first that $\mathcal{M} = MC_\chi(\mathcal{N})$. Then $\mathcal{M}^{(\sigma)} = MC_{\chi(\sigma)}(\mathcal{N}^{(\sigma)})$. If $\mathcal{M}$ and $\mathcal{N}$ have common rank ≥ 2 and $\mathcal{M} = MT_{\mathcal{L}}(\mathcal{N})$, then $\mathcal{M}^{(\sigma)} = MT_{\mathcal{L}(\sigma)}(\mathcal{N}^{(\sigma)})$, where $\mathcal{L}^{(\sigma)}$ is the element $\sigma\circ\mathcal{L}$ of Maps(D, Γ). QED

Theorem 6.7.3 (invariance by automorphisms) In the situation 6.3.1, if $\mathcal{N}$ in NumData(D, Γ) is realizable, and has rig($\mathcal{N}$) = 2, then for any σ in Aut(Γ), $\mathcal{N}^{(\sigma)}$ is realizable.

proof If r($\mathcal{N}$)=1, then "realizable" is the same as "plausible", so the result follows from part (1) of the previous lemma. If r($\mathcal{N}$) ≥2, $\mathcal{N}$ is realizable if and only if $\mathcal{N}$ is connected, in the graph, to a plausible element of rank one. Take a connecting sequence and apply σ. By parts (1) and (2) of the previous lemma, this is a sequence which connects $\mathcal{N}^{(\sigma)}$ to a plausible element of rank one. QED

6.8 A remark and a question

(6.8.1) Notice that if an $\mathcal{N}$ in NumData(D, Γ) with rig($\mathcal{N}$) = 2 is realizable, then the object $\mathcal{F}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ with ND($\mathcal{F}$) = $\mathcal{N}$ is already determined up to isomorphism by $\mathcal{N}$; this is precisely the meaning of rigidity. So given $\mathcal{F}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ which is rigid, we may speak of the rigid object $\mathcal{F}^{(\sigma)}$ in $\mathcal{T}_\ell(\mathbb{A}^1 - D, \Gamma)$ for every σ in Aut(Γ). If σ is the automorphism -1 of Γ, then $\mathcal{F}^{(-1)}$ is the dual of $\mathcal{F}$. This description of $\mathcal{F}^{(-1)}$ as the dual of $\mathcal{F}$ makes sense for any $\mathcal{F}$, not just rigid ones, and so -1 preserves the set of realizable elements in NumData(D, Γ).

(6.8.2) In the complex analytic case, with Γ a subgroup of $\mathbb{C}^\times$, if σ in Aut(Γ) is induced by an automorphism $\tilde{\sigma}$ of the field $\mathbb{C}$, then $\mathcal{F}^{(\sigma)}$ has a down to earth interpretation: view $\mathcal{F}$ in $\mathcal{T}_{an}(\mathbb{A}^1 - D, \Gamma)$ as a homomorphism $\Lambda : \pi_1((\mathbb{A}^1 - D)^{an}) \to GL(r, \mathbb{C})$, then $\mathcal{F}^{(\sigma)}$ is the composite homomorphism $\tilde{\sigma}\circ\Lambda$, where $\tilde{\sigma}$ acts on $GL(r, \mathbb{C})$ by conjugating every entry. This description of $\mathcal{F}^{(\sigma)}$ makes sense for any $\mathcal{F}$, not just rigid ones, and shows that Aut($\mathbb{C}$) preserves the set of realizable elements of NumData(D, Γ). [If σ is complex conjugation, and if $\Gamma \subset \mathbb{C}^\times$ lies in the circle S^1, then σ induces -1 on Γ, and $\mathcal{F}^{(\sigma)}$ is the dual of $\mathcal{F}$.]

(6.8.3) However, there are many subgroups Γ of $\mathbb{C}^\times$ on which Aut($\mathbb{C}$) acts trivially, but for which Aut(Γ) is nontrivial. A typical example of such a Γ is the multiplicative subgroup generated by 2 and 3, which as abstract group is free abelian on the elements 2 and 3. Thus Aut(Γ) is GL(2,$\mathbb{Z}$), with the standard upper unipotent element T acting as ($2 \mapsto 6$, $3 \mapsto 3$), the standard involution S acting as ($2 \mapsto 3$, $3 \mapsto 1/2$), and -id acting as ($2 \mapsto 1/2$, $3 \mapsto 1/3$). It is far from clear whether or not, for this Γ, Aut(Γ) preserves the set of realizable elements in NumData(D, Γ). It seems almost miraculous that Aut(Γ) preserves the set of realizable elements which are rigid.

(6.8.4) In the ℓ-adic case, over any algebraically closed field of characteristic zero, once we fix a topological generator γ of

$\pi_1(\mathbb{G}_m)^{\mathrm{tame}}$, we may identify, via evaluation at γ, the groups

$$\mathrm{Hom}_{\mathrm{contin}}(\pi_1(\mathbb{G}_m)^{\mathrm{tame}}, \overline{\mathbb{Q}}_\ell^\times) \cong (\mathcal{O}_{\overline{\mathbb{Q}}_\ell})^\times.$$

We may take for Γ the multiplicative group of $\mathbb{Z}_\ell^\times \subset (\mathcal{O}_{\overline{\mathbb{Q}}_\ell})^\times$ generated by any two distinct primes both different from ℓ, e.g. by 2 and 3 if $\ell \geq 5$, and repeat the same question. [If we are over an algebraically closed field of positive characteristic $p \neq \ell$, we must replace $(\mathcal{O}_{\overline{\mathbb{Q}}_\ell})^\times$ by its subgroup $(\mathcal{O}_{\overline{\mathbb{Q}}_\ell})^\times(\text{not } p)$ consisting of those elements whose image in $\overline{\mathbb{F}}_\ell^\times$ has order prime to p, and we may take for Γ the multiplicative group of $(\mathcal{O}_{\overline{\mathbb{Q}}_\ell})^\times(\text{not } p)$ generated by any two distinct primes q_1 and q_2 which both have order prime to p in $\mathbb{F}_\ell^\times$ (e.g., take both $q_i \equiv 1 \bmod \ell$).

7.0 Diophantine criterion for irreducibility

(7.0.1) Fix a prime number ℓ, and an embedding $\iota: \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. For x in $\overline{\mathbb{Q}}_\ell$ denote by $|x|$ the usual complex absolute value of $\iota(x)$.

Let F be a finite field of characteristic $\neq \ell$, $\overline{F}$ an algebraic closure of F. On $\mathbb{A}^1$ over F, denote by $k: \mathbb{A}^1 \to \mathbb{P}^1$ the inclusion, and by $j: U \to \mathbb{A}^1$ the inclusion of a nonempty open set.

(7.0.2) To motivate this section, recall the well known diophantine criterion for a pure sheaf to be geometrically irreducible (cf. [Ka-MFC, sections II and III]).

Lemma 7.0.3 In the situation 7.0.1, let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U, which is ι-pure of weight zero. Then $\mathcal{F}$ is geometrically irreducible, i.e., irreducible on $U\otimes_F\overline{F}$, if and only if there exists a constant C such that for all finite extensions E of F, of cardinality denoted q_E, we have:

$$(*) \quad \left|\Sigma_{x \text{ in } U(E)} |\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{F})|^2 - q_E\right| \leq C(q_E)^{1/2}.$$

proof Because $\mathcal{F}$ is pure of weight zero, we have

$$|\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{F})|^2 = \mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \underline{\mathrm{End}}(\mathcal{F})),$$

so we may rewrite $(*)$ as

$$\left|\Sigma_{x \text{ in } U(E)} \mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \underline{\mathrm{End}}(\mathcal{F})) - q_E\right| \leq C(q_E)^{1/2}.$$

By the Lefschetz Trace Formula, applied to the lisse sheaf $\underline{\mathrm{End}}(\mathcal{F})$ on U, we know that

$$\Sigma_{x \text{ in } U(E)} \mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \underline{\mathrm{End}}(\mathcal{F})) =$$
$$= \Sigma_{i=1,2} (-1)^i \mathrm{Trace}(\mathrm{Frob}_E \mid H^i_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F})).$$

This we may rewrite $(*)$ as

$$(**) \quad \left|\Sigma_{i=1,2} (-1)^i \mathrm{Trace}(\mathrm{Frob}_E \mid H^i_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F})) - q_E\right| \leq C(q_E)^{1/2}.$$

Because $\underline{\mathrm{End}}(\mathcal{F})$ is ι-pure of weight zero, we know from [De-Weil II, 3.3.1] that $H^i_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F}))$ is mixed of weight $\leq i$, for $i=1,2$, and the H^2_c is pure of weight 2. Again by purity, we also know [De-Weil II, 3.4.1(iii)] that both $\mathcal{F}$ and $\underline{End}(\mathcal{F})$ are geometrically semisimple, and hence

$$H^2_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F})) =$$
$$= (\text{the coinvariants of } \pi_1(U\otimes_F\overline{F}) \text{ in } \underline{End}(\mathcal{F}))(-1)$$
$$= (\text{the invariants of } \pi_1(U\otimes_F\overline{F}) \text{ in } \underline{End}(\mathcal{F}))(-1).$$

Suppose first that $\mathcal{F}$ is geometrically irreducible. Then by Schur's Lemma we have $H^2_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F})) = \overline{\mathbb{Q}}_\ell(-1)$, and hence

$$\mathrm{Trace}(\mathrm{Frob}_E \mid H^2_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F})) = q_E.$$

Since the H^1_c is mixed of weight ≤ 1, taking $C := h^1_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F}))$ gives

$$|\,\mathrm{Trace}(\mathrm{Frob}_E \mid H^1_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F}))\,| \le C(q_E)^{1/2},$$

and (**) is now obvious.

Suppose now that (**) holds. Then at the expense of enlarging C, for every finite extension E of F we have

$$|\mathrm{Trace}(\mathrm{Frob}_E \mid H^2_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F})) - q_E| \le C(q_E)^{1/2}.$$

Since $H^2_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F}))$ is pure of weight 2, the usual compactness argument (cf. [Ka-SE, 2.2.2.1]) shows that $h^2_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F})) = 1$. Because $\mathcal{F}$ is geometrically semisimple (being pure), this one-dimensionality means precisely that $\mathcal{F}$ is geometrically irreducible. QED

7.1 Diophantine criterion for rigidity

Theorem 7.1.1 In the situation 7.0.1, let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U, which is ι-pure of weight zero. The the following conditions are equivalent.

1) $\mathcal{F}$ is geometrically irreducible, i.e., irreducible on $U\otimes_F\overline{F}$, and cohomologically rigid, i.e., $\chi(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F})) = 2$

2) There exists a constant C such that for all finite extensions E of F, of cardinality denoted q_E, we have:

$$|\Sigma_{x \text{ in } U(E)} |\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{F})|^2 - q_E| \le C.$$

proof Because End($\mathcal{F}$) is pure of weight zero, we know by Weil II [De-Weil II, 3.2.3] that for all i, $H^i(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{\mathrm{End}}(\mathcal{F}))$ is pure of weight i. From the long exact sequence of cohomology attached to the short exact (excision) sequence of sheaves on $\mathbb{P}^1$,

$$0 \to k_!j_!\underline{\mathrm{End}}(\mathcal{F}) \to k_*j_*\underline{\mathrm{End}}(\mathcal{F}) \to (\text{punctual, wt.} \le 0) \to 0,$$

we see that

$$H^2{}_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F})) \cong H^2(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{\mathrm{End}}(\mathcal{F})),$$

and that we have a short exact sequence

$$0\to (\text{wt.} \le 0) \to H^1{}_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F})) \to H^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{\mathrm{End}}(\mathcal{F})) \to 0.$$

Suppose first that $\mathcal{F}$ is geometrically irreducible and cohomologically rigid. Then

$$H^2(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{\mathrm{End}}(\mathcal{F})) = \overline{\mathbb{Q}}_\ell(-1),$$
$$H^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{\mathrm{End}}(\mathcal{F})) = 0,$$
$$H^0(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{\mathrm{End}}(\mathcal{F})) = \overline{\mathbb{Q}}_\ell.$$

Therefore we find

$$H^2{}_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F})) = \overline{\mathbb{Q}}_\ell(-1),$$
$$H^1{}_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F})) \text{ is mixed of weight} \le 0.$$

Thus 2) holds, taking $C := h^1{}_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F}))$.

Conversely, suppose that 2) holds. By the previous lemma, $\mathcal{F}$ is geometrically irreducible, and hence

$$H^2{}_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F})) = \overline{\mathbb{Q}}_\ell(-1).$$

Exactly as in the proof the proof of the previous lemma, the Lefschetz Trace Formula allow us to rewrite 2) as

$$|\ \Sigma_{i=1,2}\ (-1)^i\mathrm{Trace}(\mathrm{Frob}_E\ |\ H^i{}_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F}))\ -\ q_E\ |\ \le C.$$

But as $H^2{}_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F})) = \overline{\mathbb{Q}}_\ell(-1)$, this says precisely that for all finite extensions E of F,

$$|\ \mathrm{Trace}(\mathrm{Frob}_E\ |\ H^1{}_c(U\otimes_F\overline{F}, \underline{\mathrm{End}}(\mathcal{F}))\ |\ \le C.$$

From this, the standard "radius of convergence" argument (cf. [Ka-SE, 2.2.1.1]) shows that $H^1{}_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F}))$ is mixed of weight ≤ 0. Therefore $H^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F}))$, being a quotient of this $H^1{}_c$, is itself mixed of weight ≤ 0. But it is also pure of weight one, hence it vanishes: $H^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F})) = 0$. Finally, the geometric irreducibility of $\mathcal{F}$ gives

$$H^0(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F})) = H^0(U\otimes_F\overline{F}, \underline{End}(\mathcal{F})) = \overline{\mathbb{Q}}_\ell.$$

Thus we find $\chi(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F})) = 2$, as required. QED

Variant Theorem 7.1.2 In the situation 7.0.1, let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U, which is ι-pure of weight zero, and let $A \ge 0$ be a non-negative integer. The the following conditions are equivalent.

1) $\mathcal{F}$ is geometrically irreducible, i.e., irreducible on $U\otimes_F\overline{F}$, and $h^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F})) \le A$, i.e., $\chi(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F})) \ge 2 - A$.

2) There exists a constant C such that for all finite extensions E of F, of cardinality denoted q_E, we have:

$$|\Sigma_{x \text{ in } U(E)} |\mathrm{Trace}(\mathrm{Frob}_{x,E} | \mathcal{F})|^2 - q_E | \le A(q_E)^{1/2} + C.$$

3) There exists a constant C such that for all finite extensions E of F, of cardinality denoted q_E, we have:

$$|\Sigma_{x \text{ in } \mathbb{A}^1(E)} |\mathrm{Trace}(\mathrm{Frob}_{x,E} | j_*\mathcal{F})|^2 - q_E | \le A(q_E)^{1/2} + C.$$

proof We first remark that 2) and 3) are trivially equivalent, since at each of the finitely many points of $\mathbb{A}^1 - U$, $j_*\mathcal{F}$ is mixed of weight ≤ 0.

To show that 2) implies 1), we argue as follows. If 2) holds, then by the first lemma 7.0.3, $\mathcal{F}$ is geometrically irreducible, and $H^2{}_c(U\otimes_F\overline{F},\underline{End}(\mathcal{F})) = \overline{\mathbb{Q}}_\ell(-1)$, so 2) says that for all finite extensions E of F,

$$| \mathrm{Trace}(\mathrm{Frob}_E | H^1{}_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F})) | \le A(q_E)^{1/2} + C.$$

From this it follows (cf. [Ka-SE, 2.2.1.1]) that $H^1{}_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F}))$ is mixed of weight ≤ 1, with at most A eigenvalues of weight 1. From the short exact sequence

$$0 \to (\text{wt.} \le 0) \to H^1{}_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F})) \to H^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F})) \to 0.$$

and the fact that $H^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F}))$ is pure of weight 1, we see that $h^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F}))$ is precisely the number of eigenvalues of weight one in $H^1{}_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F}))$. Thus we find

$$h^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F})) \le A, \text{ as required.}$$

Conversely, suppose that 1) holds. By the first lemma, $\mathcal{F}$ is geometrically irreducible, $H^2{}_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F})) = \overline{\mathbb{Q}}_\ell(-1)$, and 2) is the assertion that there exists a constant C such that for all finite extensions E of F,

$$|\,\mathrm{Trace}(\mathrm{Frob}_E\,|\,H^1{}_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F}))\,| \le A(q_E)^{1/2} + C.$$

Since by 1) we have $h^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F})) \le A$, it suffices to show the existence of a constant C such that for all finite extensions E of F,

$$|\,\mathrm{Trace}(\mathrm{Frob}_E\,|\,H^1{}_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F}))\,| \le$$
$$\le h^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F}))(q_E)^{1/2} + C.$$

But this is obvious from the short exact sequence

$$0 \to (\text{wt.} \le 0) \to H^1{}_c(U\otimes_F\overline{F}, \underline{End}(\mathcal{F})) \to H^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F})) \to 0$$

and the fact that $H^1(\mathbb{P}^1\otimes_F\overline{F}, k_*j_*\underline{End}(\mathcal{F}))$ is pure of weight one. QED

(7.1.3) This result allows us to give a quite short proof of the fact that Fourier Transform preserves the index of rigidity in the special case of **pure** objects. It was only after first proving this "special case" that we found the proof of the general case given in 3.0.2.

Theorem 7.1.4 Let F be a finite field of characteristic $\neq \ell$, ψ a nontrivial $\overline{\mathbb{Q}}_\ell$-valued character of the additive group of F. Over F, let K be a perverse sheaf on $\mathbb{A}^1$ which is of the form $j_*\mathcal{F}[1]$ for $\mathcal{F}$ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U, which is ι-pure of weight zero, and geometrically irreducible. Suppose that $\mathcal{F}$ is not geometrically isomorphic to $\mathcal{L}_{\psi(ax)}$ for any a in F. Consider the (Tate-twisted) Fourier Transform $FT_\psi(K)(1/2)$, which is known (cf. [Ka-GKM, 6.2.5 and 8.4.1]) to be of the form $j'_\star\mathcal{G}[1]$ for $j': U' \to \mathbb{A}^1$ the inclusion a nonempty open set, and $\mathcal{G}$ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U', which is ι-pure of weight zero, geometrically irreducible, and not geometrically isomorphic to $\mathcal{L}_{\psi(ax)}$ for any a in F. Then $\mathcal{F}$ and $\mathcal{G}$ have the same index of rigidity: $\chi(\mathbb{P}^1, k_\star j_\star \underline{\mathrm{End}}(\mathcal{F})). = \chi(\mathbb{P}^1, k_\star j'_\star \underline{\mathrm{End}}(\mathcal{G}))$, or equivalently,

$$h^1(\mathbb{P}^1, k_\star j_\star \underline{\mathrm{End}}(\mathcal{F})). = h^1(\mathbb{P}^1, k_\star j'_\star \underline{\mathrm{End}}(\mathcal{G})).$$

proof For every finite extension E of F, the trace functions of $j_\star\mathcal{F}$ and of $j'_\star\mathcal{G}$ as functions on $\mathbb{A}^1(E) = E$ are, up to sign, normalized Fourier Transforms of each other: for y in E, we have

$$\mathrm{Trace}(\mathrm{Frob}_{y,E} \mid j'_\star\mathcal{G}) =$$
$$= (-1/(q_E)^{1/2})\Sigma_{x \text{ in } E}\ \psi_E(yx)\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid j_\star\mathcal{F}),$$

where we have written ψ_E for $\psi\circ\mathrm{Trace}_{E/F}$.

By the Plancherel formula, we have, for every finite extension E of F, the equality

$$\Sigma_{x \text{ in } \mathbb{A}^1(E)} |\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid j_\star\mathcal{F})|^2 =$$
$$= \Sigma_{x \text{ in } \mathbb{A}^1(E)} |\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid j'_\star\mathcal{G})|^2$$

Now apply the equivalence of 1) and 3) in the Variant Theorem 7.1.2 above. By 1) $\Rightarrow$ 3) for $\mathcal{F}$, with A taken as $h^1(\mathbb{P}^1, k_\star j_\star \underline{\mathrm{End}}(\mathcal{F}))$, followed by 3) $\Rightarrow$ 1) for $\mathcal{G}$, with the same A. We see that

$$h^1(\mathbb{P}^1, k_\star j_\star \underline{\mathrm{End}}(\mathcal{F})). \ge h^1(\mathbb{P}^1, k_\star j'_\star \underline{\mathrm{End}}(\mathcal{G})).$$

Reversing the roles of $\mathcal{F}$ and $\mathcal{G}$, we get the opposite inequality. QED

7.2 Appendix: a counterexample

(7.2.1) What happens if, in Theorem 7.1.1, we assume only that the lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U is ι-mixed of weight ≤ 0 (rather than pure of weight zero). By [De-Weil II, 3.4.9], any lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U which is ι-mixed admits a filtration, indexed by real numbers w, by lisse subsheaves

$$\mathcal{F}_{<w} \subset \mathcal{F}_{\le w} \subset \mathcal{F}_{w+\varepsilon} \dots \subset \mathcal{F},$$

such that $\mathcal{F}_{<w}$ (resp. $\mathcal{F}_{\le w}$) is ι-mixed of weight $<w$ (resp. $\le w$), and such that $\mathrm{Gr}_w(\mathcal{F}) := (\mathcal{F}_{\le w})/(\mathcal{F}_{<w})$ is ι-pure of weight w. For $\mathcal{F}$ which is ι-mixed of weight ≤ 0, the pieces $\mathrm{Gr}_w(\mathcal{F}) = 0$ for $w > 0$.

(7.2.2) Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U is ι-mixed of weight ≤ 0, and suppose that there exists a constant C such that for all finite extensions E of F, of cardinality denoted q_E, we have:

$$(*) \qquad \left|\Sigma_{x \text{ in } U(E)} |\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{F})|^2 - q_E\right| \le C.$$

(7.2.3) This estimate **does** imply that $\mathrm{Gr}_0(\mathcal{F})$ is geometrically irreducible. [Indeed the geometric irreducibility of $\mathrm{Gr}_0(\mathcal{F})$ for a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U which is ι-mixed of weight ≤ 0 is equivalent to the existence of a real constant C and of a real $\alpha < 1$, such that for all finite extensions E of F, we have

$$\left|\Sigma_{x \text{ in } U(E)} |\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{F})|^2 - q_E\right| \le C(q_E)^\alpha.]$$

(7.2.4) However, the estimate $(*)$ above does **not** imply that $\mathrm{Gr}_0(\mathcal{F})$ is cohomologically rigid. Here is an example.

(7.2.5) We begin over $\mathbb{Z}$, with a $\mathbb{P}^1$ with homogeneous coordinates (μ,ν), and a $\mathbb{P}^2$ with homogeneous coordinates (X, Y, Z). Over the open set of $\mathbb{P}^1$ where $\nu(\mu^3 - 27\nu^3)$ is invertible, the closed subscheme of $\mathbb{P}^2\times\mathbb{P}^1$ defined by the single equation

$$\nu(X^3 + Y^3 + Z^3) = \mu XYZ$$

is an elliptic curve, say $\pi : \mathcal{E} \to \mathbb{P}^1[1/(\nu(\mu^3 - 27\nu^3))]$. This curve, with say (1,-1,0) as origin, carries an arithmetic level three structure

$$\boldsymbol{\mu}_3 \times \mathbb{Z}/3\mathbb{Z} \cong \mathcal{E}[3]$$

in the sense of [Ka-RA,2.0.4], and in fact this is the universal elliptic curve with arithmetic level three structure.

(7.2.6) If we extend scalars from $\mathbb{Z}$ to $\mathbb{Z}[1/3, \zeta_3]$, we may view this same curve as the universal elliptic curve with usual level 3 structure of determinant ζ_3 over $\mathbb{Z}[1/3, \zeta_3]$-algebras.

(7.2.7) Fix a prime number ℓ. The sheaf

$$\mathcal{G}_\ell := R^1\pi_*\overline{\mathbb{Q}}_\ell \text{ on } \mathbb{P}^1[1/(\ell\nu(\mu^3 - 27\nu^3))]$$

is lisse of rank 2, pure of weight one, and $\det(\mathcal{G}_\ell) \cong \overline{\mathbb{Q}}_\ell(-1)$.

(7.2.8) We next invert the prime 3, i.e., we work over

$$\mathbb{P}^1[1/(3\ell\nu(\mu^3 - 27\nu^3))].$$

The advantage of doing this is that over $\mathbb{Z}[1/3]$, the divisor in $\mathbb{P}^1[1/3]$ of equation $\nu(\mu^3 - 27\nu^3) = 0$, is finite etale of degree 4: indeed over $\mathbb{Z}[1/3, \zeta_3]$ it is the disjoint union of four sections. It is well known (cf. [KM, 14.3.3]) that the local monodromy of $\mathcal{G}_\ell$ along each of these four sections is unipotent and nontrivial.

(7.2.9) In particular, for any prime p other than 3 or ℓ, the restriction

$$\mathcal{G}_{\ell,p} := \mathcal{G}_\ell \mid \mathbb{P}^1[1/(\nu(\mu^3 - 27\nu^3))]\otimes\mathbb{F}_p$$

is lisse of rank 2, pure of weight 1, with nontrivial unipotent local monodromy at each of the four cusps. Since any pure lisse sheaf is geometrically semisimple [De-Weil II, 3.4(iii)], it follows that $\mathcal{G}_{\ell,p}$ is geometrically irreducible (since already under local monodromy at any cusp it is indecomposable). Indeed, the geometric monodromy group of $\mathcal{G}$ must be SL(2), since it is a semisimple subgroup of SL(2) which contains a nontrivial unipotent element.

(7.2.10) Since we know the local monodromy, we can easily compute the index of rigidity of $\mathcal{G}_{\ell,p}$. Denoting by

$$j : \mathbb{P}^1[1/(\nu(\mu^3 - 27\nu^3))]\otimes\overline{\mathbb{F}}_p \to \mathbb{P}^1\otimes\overline{\mathbb{F}}_p$$

the inclusion, we find

$$\chi(\mathbb{P}^1\otimes\overline{\mathbb{F}}_p, j_\ast\mathrm{End}(\mathcal{G}_{\ell,p})) = (2-4)4 + 4\times 2 = 0,$$

or equivalently, $\mathcal{G}_{\ell,p}$ being geometrically irreducible,

$$h^1(\mathbb{P}^1\otimes\overline{\mathbb{F}}_p, j_\ast\mathrm{End}(\mathcal{G}_{\ell,p})) = 2.$$

In particular, $\mathcal{G}_{\ell,p}$ is **not** cohomologically rigid.

Proposition 7.2.11 Hypotheses and notations as in 7.2.5-9 above, fix a prime $p \equiv 2 \bmod 3$, and a prime $\ell\neq p$. Denote by $\mathcal{F}$ the lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $U := \mathbb{P}^1[1/(\nu(\mu^3 - 27\nu^3))]\otimes\mathbb{F}_{p^4}$

$$\mathcal{F} := \mathcal{G}_{\ell,p}(1/2) \oplus \mathcal{G}_{\ell,p}(1),$$

where the half Tate twist is defined using p^2 as $\mathrm{sqrt}(p^4)$. There exists a constant C such that for any finite extension E of $\mathbb{F}_{p^4}$, we have

$$(\ast)\qquad \left|\Sigma_{x \text{ in } U(E)} |\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{F})|^2 - q_E\right| \le C.$$

proof Let us denote $\mathcal{G}_{\ell,p}$ simply as $\mathcal{G}$. The traces of $\mathcal{F}$ and $\mathcal{G}$ are related as follows: for any finite extension E of $\mathbb{F}_{p^4}$, and any x in E,

$\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{F}) = \mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{G})(1/\mathrm{sqrt}(q_E))(1 + 1/\mathrm{sqrt}(q_E))$.
In this formula, q_E is a power of p^4, say $q_E = p^{4n}$, and $\mathrm{sqrt}(q_E)$ is the same power p^{2n} of p^2. Thus

$$|\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{F})|^2 =$$
$$= |\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{G})|^2(1/q_E)(1 + 1/\mathrm{sqrt}(q_E))^2.$$

From its genesis via elliptic curves, we see that the trace function of $\mathcal{G}$ has values in $\mathbb{Z}$. Therefore we may omit the absolute value:

$$|\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{F})|^2 =$$
$$= (\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{G}))^2(1/q_E)(1 + 1/\mathrm{sqrt}(q_E))^2.$$

We must prove the existence of a constant C such that for any any finite extension E of $\mathbb{F}_{p^4}$,

$$(*) \qquad |\Sigma_{x \text{ in } U(E)} |\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{F})|^2 - q_E| \le C.$$

Multiplying through by q_E, and substituting in terms of $\mathcal{G}$, this amounts to

$$(**) |\Sigma_{x \text{ in } U(E)} (\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{G}))^2(1 + 1/\mathrm{sqrt}(q_E))^2 - (q_E)^2| \le Cq_E.$$

Notice that

$$(\mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{G}))^2 = \mathrm{Trace}(\mathrm{Frob}_{x,E} \mid \mathcal{G}\otimes\mathcal{G}).$$

Because $\mathcal{G}\otimes\mathcal{G}$ is pure of weight 2 on U, $j_*(\mathcal{G}\otimes\mathcal{G})$ is punctually mixed of weight ≤ 2 at the cusps. So (**) is equivalent to the existence of a constant C such that for all finite extensions E of $\mathbb{F}_{p^4}$, we have

$$|\Sigma_{x \text{ in } \mathbb{P}^1(E)} (\mathrm{Tr}(\mathrm{Frob}_{x,E} \mid j_*(\mathcal{G}\otimes\mathcal{G}))(1 + 1/\mathrm{sqrt}(q_E))^2 - (q_E)^2| \le Cq_E.$$

We will show that this holds with C=2. Indeed, we will show that

$$(***)\ \Sigma_{x \text{ in } \mathbb{P}^1(E)} (\mathrm{Tr}(\mathrm{Frob}_{x,E} \mid j_*(\mathcal{G}\otimes\mathcal{G}))(1 + 1/\mathrm{sqrt}(q_E))^2 = (q_E - 1)^2,$$

which makes this estimate obvious. Writing

$$(q_E - 1)^2 = (q_E)^2(1 - 1/q_E)^2 = (1 + 1/\mathrm{sqrt}(q_E))^2(1 - 1/\mathrm{sqrt}(q_E))^2,$$

we see that (***) is given by the following

Lemma 7.2.12 If p is congruent to 2 mod 3, then for all finite extensions E of $\mathbb{F}_{p^4}$, we have

$$\Sigma_{x \text{ in } \mathbb{P}^1(E)} (\mathrm{Tr}(\mathrm{Frob}_{x,E} \mid j_*(\mathcal{G}\otimes\mathcal{G})) = (q_E)^2(1 - 1/\mathrm{sqrt}(q_E))^2$$
$$= q_E(q_E + 1) - 2(q_E)^{3/2}.$$

proof Decompose $\mathcal{G}\otimes\mathcal{G}$ as

$$\Lambda^2(\mathcal{G}) \oplus \mathrm{Sym}^2(\mathcal{G}) = \det(\mathcal{G})\oplus \mathrm{Sym}^2(\mathcal{G}) = \overline{\mathbb{Q}}_\ell(-1)\oplus \mathrm{Sym}^2(\mathcal{G}).$$

Thus on $\mathbb{P}^1$ we have

$$j_*(\mathcal{G}\otimes\mathcal{G}) = \overline{\mathbb{Q}}_\ell(-1) \oplus j_*(\mathrm{Sym}^2(\mathcal{G})),$$

and so

$$\Sigma_{x \text{ in } \mathbb{P}^1(E)} (\mathrm{Tr}(\mathrm{Frob}_{x,E} \mid j_*(\mathcal{G}\otimes\mathcal{G})) =$$
$$= q_E(q_E + 1) + \Sigma_{x \text{ in } \mathbb{P}^1(E)} (\mathrm{Tr}(\mathrm{Frob}_{x,E} \mid j_*(\mathrm{Sym}^2(\mathcal{G}))).$$

Thus we are reduced to showing that

$$\Sigma_{x \text{ in } \mathbb{P}^1(E)} (\mathrm{Tr}(\mathrm{Frob}_{x,E} \mid j_*(\mathrm{Sym}^2(\mathcal{G}))) = -2(q_E)^{3/2}.$$

By the Lefschetz Trace Formula, we may rewrite this as

$$\Sigma (-1)^i\mathrm{Tr}(\mathrm{Frob}_E \mid H^i(\mathbb{P}^1\otimes\overline{\mathbb{F}}_p, j_*(\mathrm{Sym}^2(\mathcal{G}))) = -2(q_E)^{3/2}.$$

We claim that the only possibly nonvanishing cohomology group is the H^1. For this, it suffices that $\mathrm{Sym}^2(\mathcal{G})$ be geometrically irreducible and nontrivial. But $\mathrm{Sym}^2(\mathcal{G})$ is geometrically semisimple, because pure, and is indecomposable under each local mondromy, which acts as a single unipotent Jordan block. Therefore $\mathrm{Sym}^2(\mathcal{G})$ is geometrically irreducible and nontrivial. Moreover, at each cusp the local monodromy is tame with a one-dimensional fixed space (because a single unipotent Jordan block).

Thus we are reduced to showing that $H^1(\mathbb{P}^1\otimes\overline{\mathbb{F}}_p, j_*(\mathrm{Sym}^2(\mathcal{G})))$ has dimension 2, and that for all finite extensions E of $\mathbb{F}_p4$, both eigenvalues of Frob_E on this space are $(q_E)^{3/2}$. For this, it suffices to show that the two eigenvalues of $\mathrm{Frob}_{\mathbb{F}_p}$ on this space are $\pm i(p)^{3/2}$.

That $H^1(\mathbb{P}^1\otimes\overline{\mathbb{F}}_p, j_*(\mathrm{Sym}^2(\mathcal{G})))$ has dimension 2 is immediate from the Euler Poincaré formula, which gives

$$\chi(\mathbb{P}^1\otimes\overline{\mathbb{F}}_p, j_*(\mathrm{Sym}^2(\mathcal{G}))) = (2-4)3 + 4*1 = -2,$$

and the vanishing of H^i for $i\neq 1$.

Now $\mathcal{G}(1/2)$ is symplectically self-dual, so $\mathrm{Sym}^2(\mathcal{G})(1)$ is orthogonally self dual, so by Poincaré duality the two eigenvalues of $\mathrm{Frob}_{\mathbb{F}_p}$ on $H^1(\mathbb{P}^1\otimes\overline{\mathbb{F}}_p, j_*(\mathrm{Sym}^2(\mathcal{G})))$ have product equal to p^3. Therefore it suffices to show that

$$\mathrm{Trace}(\mathrm{Frob}_{\mathbb{F}_p} \mid H^1(\mathbb{P}^1 \otimes \overline{\mathbb{F}}_p, j_\ast(\mathrm{Sym}^2(\mathcal{G})))) = 0,$$

for any prime $p \neq \ell$ which is congruent to 2 mod 3.

Implicit in Deligne's proof that "Weil implies Ramanujan" [De-FMRℓ, 4.8-4.9, for n=3] is the following identity of traces: for any prime p not 3 or ℓ, and for any integer $k \geq 0$,

$$\mathrm{trace}(\mathrm{Frob}_{\mathbb{F}_p} \mid H^1(\mathbb{P}^1 \otimes \overline{\mathbb{F}}_p, j_\ast(\mathrm{Sym}^k(\mathcal{G})))) =$$

$$= \mathrm{trace}(T_p \mid \text{cusp forms of weight } k+2 \text{ on } \Gamma(3)).$$

Unfortunately, this identity, which is certainly "well-known" to the specialists, does not seem to appear, at least in so explicit a form, anywhere in the literature. For the sake of completeness, we will give a (somewhat clumsy and convoluted) proof of it in the special case k=2, by making use of results of Scholl [Sch].

The space of weight 4 cusp forms on $\Gamma(3)$ is one-dimensional. Since f:= the cube root of Δ is known [Lang-EF, page 254] to be a weight 4 cusp form on $\Gamma(3)$, f provides a basis of this one dimensional space, and is automatically an eigenfunction of all Hecke operators T_p for all primes $p \neq 3$. The q-expansion of this f at any cusp of $\Gamma(3)$ is

$$q\Pi_{n \geq 1}(1 - q^{3n})^8.$$

If we knew the asserted identity of traces, we would argue as follows. We write this q expansion as $\Sigma a(n)q^n$. By Hecke theory we know that for primes p not 3 we have $T_p(f) = a(p)f$. Therefore for any prime p not 3 or ℓ we have

$$\mathrm{trace}(\mathrm{Frob}_{\mathbb{F}_p} \mid H^1(\mathbb{P}^1 \otimes \overline{\mathbb{F}}_p, j_\ast(\mathrm{Sym}^2(\mathcal{G})))) =$$

$$= \mathrm{trace}(T_p \mid \text{cusp forms of weight 4 on } \Gamma(3)) = a(p).$$

From the q-expansion of f above, it is visible that $a(n) = 0$ unless n is congruent to 1 mod 3.

We now indicate an alternate proof, valid for primes $p \geq 5$, $p \neq \ell$, that

$$\mathrm{trace}(\mathrm{Frob}_{\mathbb{F}_p} \mid H^1(\mathbb{P}^1 \otimes \overline{\mathbb{F}}_p, j_\ast(\mathrm{Sym}^k(\mathcal{G})))) = a(p).$$

Because f has even weight, the element -1 in $SL(2, \mathbb{Z}/3\mathbb{Z})$ fixes f. Therefore f is in fact invariant under $\pm\Gamma(3)$, the group denoted $\Gamma_0(3,3)$ in Atkin-Lehner [AL,page 134] consisting of all elements in $SL(2,\mathbb{Z})$ which reduce mod 3 to diagonal matrices. [In general $\pm\Gamma(N)$ is a proper subgroup of $\Gamma_0(N,N)$: they coincide precisely when ± 1 are the only units in $\mathbb{Z}/N\mathbb{Z}$,i.e., precisely for $N \leq 4$.] Now quite generally, $f(\tau) \mapsto f(N\tau)$ is a bijection between cusp forms of any given weight k on $\Gamma_0(N,N)$ and cusp forms of weight k on $\Gamma_0(N^2)$. Returning to the case at hand, we conclude that the space of weight 4 cusp forms on $\Gamma_0(9)$ is one-dimensional, spanned by a form whose q-expansion at the standard cusp is

$$q\Pi_{n\geq 1}(1 - q^{3n})^8.$$

By the one-dimensionality, this form is automatically an eigenfunction of all Hecke operators T_p for all primes $p\neq 3$. If we write this q expansion as $\Sigma a(n)q^n$, then by Hecke theory we know that for primes p not 3 we have $T_p(f) = a(p)f$. Because we are now on $\Gamma_0(9)$, $T_p(f) = a(p)f$ translates into the following identity on coefficients:

$$a(np) - a(p)a(n) + p^3a(n/p) = 0,$$

with the standard convention that $a(x) = 0$ if x is a non-integer.

For each prime p not 3 or ℓ, let us define

$$A(p) := \mathrm{trace}(\mathrm{Frob}_{\mathbb{F}_p} \mid H^1(\mathbb{P}^1\otimes\overline{\mathbb{F}}_p, j_\ast(\mathrm{Sym}^2(\mathcal{G})))).$$

By the Lefschetz Trace Formula, and the fact that $j_\ast(\mathrm{Sym}^2(\mathcal{G}))$ has a $\mathbb{Z}$-valued trace function (even at the cusps, where its trace is 1) which is independent of the auxiliary choice of ℓ, we see that A(p) lies in $\mathbb{Z}$, and is independent of the auxiliary ℓ. For any prime p not 3 or ℓ we have

$$\det(T - \mathrm{Frob}_{\mathbb{F}_p} \mid H^1(\mathbb{P}^1\otimes\overline{\mathbb{F}}_p, j_\ast(\mathrm{Sym}^2(\mathcal{G})))) = T^2 - A(p)T + p^3.$$

According to Scholl [Sch, Theorem 5.4, applied to k=2 and $\Gamma := \Gamma(3)$: the "M" of Prop. 5.2 is then 3], the q-expansion coefficients a(n) of f satisfy, for each prime p not 3, $p > 3$, the **congruences**

$$a(np) - A(p)a(n) + p^3a(n/p) \equiv 0 \bmod (pn)^3,$$

for every integer n. But we recall that they also satisfy the identities

$$a(np) - a(p)a(n) + p^3a(n/p) = 0.$$

Subtracting, we find

$$(a(p) - A(p))a(n) \equiv 0 \bmod (pn)^3.$$

We wish to infer from this that in fact A(p) = a(p). We take n=1. Because a(1)=1, we get $a(p) - A(p) \equiv 0 \bmod p^3$. But both a(p) and A(p) are usual integers, whose absolute values are at most $2p^{3/2}$. So a(p) - A(p) is an integer of absolute value at most $4p^{3/2}$. But a(p) - A(p) is divisible by p^3, so if nonzero its absolute value is at least p^3. But $p^3 > 4p^{3/2}$ for $p \geq 3$. Therefore $A(p) = a(p)$ for $p > 3$, $p \neq \ell$.

For the sake of completeness, let us show that for $p=2$, $\ell \neq 2$,

$$\mathrm{trace}(\mathrm{Frob}_{\mathbb{F}_2} \mid H^1(\mathbb{P}^1 \otimes \overline{\mathbb{F}}_2, j_*(\mathrm{Sym}^2(\mathcal{G}))) = 0.$$

Because the cohomology groups $H^i(\mathbb{P}^1 \otimes \overline{\mathbb{F}}_2, j_*(\mathrm{Sym}^2(\mathcal{G}))$ vanish for $i \neq 1$,the Lefschetz Trace Formula gives

$$\mathrm{trace}(\mathrm{Frob}_{\mathbb{F}_2} \mid H^1(\mathbb{P}^1 \otimes \overline{\mathbb{F}}_2, j_*(\mathrm{Sym}^2(\mathcal{G}))) =$$
$$- \Sigma_{x \text{ in } \mathbb{P}^1(\mathbb{F}_2)} \mathrm{Trace}(\mathrm{Frob}_{x,\mathbb{F}_2} \mid j_*(\mathrm{Sym}^2(\mathcal{G}))).$$

There are sufficiently few $\mathbb{F}_2$-rational points in $\mathbb{P}^1$ that this computation is quite feasible. The points $x=1$ and $x=\infty$ are both cusps, at each of which the trace of Frobenius is 1 (indeed the restriction to the cusps of $j_*(\mathrm{Sym}^k(\mathcal{G}))$ for any $k \geq 0$ is the constant sheaf). At the point $x=0$, $\mathrm{Sym}^2(\mathcal{G})$ is the Sym^2 of the H^1 of the elliptic curve E of homogeneous equation

$$X^3 + Y^3 + Z^3 = 0$$

over $\mathbb{F}_2$. This curve has 3 $\mathbb{F}_2$ rational points (the three $\mathbb{F}_2$-points where exactly one of X,Y or Z vanishes): as this number of rational points is given by

$$1 + 2 - \mathrm{Trace}(\mathrm{Frob}_{\mathbb{F}_2} \mid H^1(E \otimes \overline{\mathbb{F}}_2, \overline{\mathbb{Q}}_\ell),$$

we see that $\mathrm{Trace}(\mathrm{Frob}_{\mathbb{F}_2} \mid H^1(E \otimes \overline{\mathbb{F}}_2, \overline{\mathbb{Q}}_\ell) = 0$. Therefore the two eigenvalues of $\mathrm{Frob}_{\mathbb{F}_2}$ on $H^1(E \otimes \overline{\mathbb{F}}_2, \overline{\mathbb{Q}}_\ell)$ are $\pm i(2)^{1/2}$. Therefore the three eigenvalues of $\mathrm{Frob}_{\mathbb{F}_2}$ on $\mathrm{Sym}^2(H^1(E \otimes \overline{\mathbb{F}}_2, \overline{\mathbb{Q}}_\ell))$ are -2, -2, 2. Hence for the point $x=0$ we have

$$\mathrm{Trace}(\mathrm{Frob}_{x,\mathbb{F}_2} \mid j_*(\mathrm{Sym}^2(\mathcal{G}))) = (-2) + (-2) + 2 = -2.$$

Thus we find that

$$\Sigma_{x \text{ in } \mathbb{P}^1(\mathbb{F}_2)} \mathrm{Trace}(\mathrm{Frob}_{x,\mathbb{F}_2} \mid j_*(\mathrm{Sym}^2(\mathcal{G}))) = 1 + 1 + (-2) = 0,$$

as required.

8.0 The basic setting

(8.0.1) To motivate this chapter, start with an algebraically closed field k, $n \geq 2$ an integer, $\alpha_1, \alpha_2, \ldots \alpha_n$ a set of n distinct points of $\mathbb{A}^1(k)$, ℓ a prime number invertible in k, $N \geq 1$ an integer which is invertible in k, and ζ a primitive N'th root of unity in k. We are interested in "describing" all objects of $\mathcal{T}_\ell$ which are lisse on $\mathbb{A}^1 - \{\alpha_1, \alpha_2, \ldots \alpha_n\}$, cohomologically rigid, and such that all eigenvalues of all local monodromies of $\mathcal{F}$ are N'th roots of unity. We wish to do this describing in as universal a way as possible.

(8.0.2) Thus we fix an integer $N \geq 1$. As in 5.5.2-3, we have the rings

$$R_{N,\ell} := \mathbb{Z}[\zeta_N, 1/N\ell],\ \zeta_N := \text{a primitive N'th root of unity}$$

and

$$S_{N,n,\ell} := R_{N,\ell}[T_1, \ldots, T_n][1/\Delta],\ \Delta := \prod_{i\neq j}(T_i - T_j).$$

We fix an embedding

$$R_{N,\ell} \rightarrow \overline{\mathbb{Q}}_\ell,$$

i.e., we fix a primitive N'th root of unity in $\overline{\mathbb{Q}}_\ell$. We denote by $E = E_N$ the fraction field of $R_{N,\ell}$. We denote by λ the induced place of the "abstract" field E, and by E_λ the λ-adic completion of E. We denote by φ the unique ring homomorphism

$$\varphi : S_{N,n,\ell} \rightarrow k$$

for which $\varphi(\zeta_N) = \zeta$ and for which $\varphi(T_i) = \alpha_i$ for $1 \leq i \leq n$.

(8.0.3) Over $S_{N,n,\ell}$, we have $\mathbb{A}^1$, with its n disjoint sections $\{T_1, \ldots, T_n\}$, and coordinate X_1. This space

$$(\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$$

is the spec of the ring $S_{N,n+1,\ell}$, in which n+1 variables are denoted $T_1, \ldots, T_n, X_1$. More generally, for each integer $r \geq 0$, we will have occasion to consider the space

$$\mathbb{A}(n, r+1)_{R_{N,\ell}} := \mathrm{Spec}(R_{N,\ell}[T_1, \ldots, T_n, X_1, \ldots, X_{r+1}][1/\Delta_{n,r}],$$

where

$$\Delta_{n,r} := (\prod_{i\neq j}(T_i - T_j))(\prod_{a,j}(X_a - T_j))(\prod_k(X_{k+1} - X_k)).$$

[The indices i and j run in $\{1,2,\ldots,n\}$, the index a in $\{1,\ldots,r+1\}$, the index k in $\{1,\ldots,r\}$; when r=0 the empty product $\prod_k(X_{k+1} - X_k)$ is understood to be 1.]

(8.0.4) A more illuminating way to think of $\mathbb{A}(n, r+1)_{R_{N,\ell}}$ is as lying in the r+1 fold fibre product of $(\mathbb{A}^1 - \{T_1, \dots , T_n\})_{S_{N,n,\ell}}$ with itself over $S_{N,n,\ell}$, as the open set where the function $\prod_k (X_{k+1} - X_k)$ is invertible. Thought of this way, we see r+1 projection maps

$$\mathrm{pr}_i : \mathbb{A}(n, r+1)_{R_{N,\ell}} \to (\mathbb{A}^1 - \{T_1, \dots , T_n\})_{S_{N,n,\ell}}$$
$$(T_1, \dots , T_n, X_1, \dots , X_{r+1}) \mapsto (T_1, \dots , T_n, X_i).$$

(8.0.5) Strictly speaking, $\mathbb{A}(n, r+1)_{R_{N,\ell}}$ depends on the integer n and on the **ordered set** $\{1,2,\dots , r+1\}$ (because $\prod_k (X_{k+1} - X_k)$ depends on the order). For each nonempty subinterval

$$[i,j] := \{i, i+1, \dots , j\} \subset \{1,2,\dots , r+1\},$$

we may form the space $\mathbb{A}(n, [i,j])_{R_{N,\ell}}$, by forming the j+1-i fold fibre product of of $(\mathbb{A}^1 - \{T_1, \dots , T_n\})_{S_{N,n,\ell}}$ with itself over $S_{N,n,\ell}$, and then passing to the open set where $\prod_{k=i,\dots,j-1} (X_{k+1} - X_k)$ is invertible. We have natural projections

$$\mathrm{pr}_{[i,j],[\alpha,\beta]} : \mathbb{A}(n, [i,j])_{R_{N,\ell}} \to \mathbb{A}(n, [\alpha,\beta])_{R_{N,\ell}},$$
$$(T_1, \dots , T_n, X_i, \dots , X_j) \mapsto (T_1, \dots , T_n, X_\alpha, \dots , X_\beta),$$

whenever $[\alpha,\beta] \subset [i,j]$. In this notation, the projection pr_i above is $\mathrm{pr}_{[1,r+1],[i,i]}$.

8.1 Interlude: Kummer sheaves

(8.1.1) We work in the setting 8.0. On $(\mathbb{G}_m)_{R_{N,\ell}}$ with coordinate Z, we have the Kummer covering of degree N, of equation $Y^N = Z$. This is a connected $\mu_N(R_{N,\ell})$-torsor, whose existence defines a surjective homomorphism

$$\pi_1((\mathbb{G}_m)_{R_{N,\ell}}) \twoheadrightarrow \mu_N(R_{N,\ell})$$

The chosen embedding $R_{N,\ell} \to \overline{\mathbb{Q}}_\ell$ gives an embedding $\mu_N(R_{N,\ell}) \to \overline{\mathbb{Q}}_\ell^\times$, which we can think of as a faithful $\overline{\mathbb{Q}}_\ell$-valued character χ_N of the structural group $\mu_N(R_{N,\ell})$. The composite homomorphism

$$\pi_1((\mathbb{G}_m)_{R_{N,\ell}}) \twoheadrightarrow \mu_N(R_{N,\ell}) \to \overline{\mathbb{Q}}_\ell^\times$$

defines the Kummer sheaf $\mathcal{L}_{\chi_N}$ on $(\mathbb{G}_m)_{R_{N,\ell}}$.

(8.1.2) More generally, any character

$$\rho : \mu_N(R_{N,\ell}) \to \overline{\mathbb{Q}}_\ell^\times$$

defines in this way a Kummer sheaf $\mathcal{L}_\rho$ on $(\mathbb{G}_m)_{R_{N,\ell}}$. Any such character ρ is some power of χ_N, say $\rho = (\chi_N)^a$ for some integer a (which is unique mod N).

(8.1.3) For any scheme W and any map
$$f: W \to (\mathbb{G}_m)_{R_{N,\ell}},$$
the Kummer sheaves $f^*\mathcal{L}_\rho := \mathcal{L}_{\rho(f)}$ and $f^*\mathcal{L}_{\chi_N} := \mathcal{L}_{\chi_N(f)}$ on W are related by
$$\mathcal{L}_{\rho(f)} = \mathcal{L}_{\chi_N(f^a)} = (\mathcal{L}_{\chi_N(f)})^{\otimes a}.$$

(8.1.4) An alternative description of the Kummer sheaf $\mathcal{L}_{\rho(f)}$ on W is this. One considers the covering of W of equation $y^N = f$, with structural map
$$\pi: (y^N = f) \to W.$$
The zeroeth direct image sheaf $\pi_*\overline{\mathbb{Q}}_\ell$ has a $\mu_N(R_{N,\ell})$ action, and its ρ-component is the sheaf $\mathcal{L}_{\rho(f)}$ on W.

8.2 Naive convolution on $(\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$.

(8.2.1) We continue in the setting 8.0. Denote by Lisse(N,n,ℓ) the category of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on
$$\mathbb{A}(n, 1)_{R_{N,\ell}} := (\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}.$$
For each nontrivial $\overline{\mathbb{Q}}_\ell^\times$-valued character χ of the group $\mu_N(R_{N,\ell})$, we define an exact functor, "naive convolution with $\mathcal{L}_\chi$", denoted NC_χ, from Lisse(N,n,ℓ) to itself, as follows. Consider the space $\mathbb{A}(n,2)_{R_{N,\ell}}$, with its two projections to $\mathbb{A}(n,1)_{R_{N,\ell}}$. We define
$$NC_\chi(\mathcal{F}) := R^1(pr_2)_!(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)}).$$

Lemma 8.2.2 In the setting 8.2.1, let $\mathcal{F}$ in Lisse(N,n,ℓ), and χ an arbitrary $\overline{\mathbb{Q}}_\ell^\times$-valued character of $\mu_N(R_{N,\ell})$.

1) The sheaves $R^i(pr_2)_!(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)})$ are all in Lisse(N,n,ℓ).

2) If χ is nontrivial, they vanish for $i\neq1$.

3) If χ is nontrivial, the functor NC_χ is exact.

4) If χ is nontrivial, and if $\mathcal{F}$ is mixed of weight $\le w$, then $NC_\chi(\mathcal{F})$ is mixed of weight $\le w+1$.

proof 1) Over the target $\mathbb{A}(n,1)_{R_{N,\ell}}$, the total space is a relative $\mathbb{A}^1$ with coordinate X_1, minus the n+1 disjoint sections $\{T_1, \ldots, T_n, X_2\}$. The sheaf $pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)}$ is lisse on this space. Because the base is normal and connected with generic point of characteristic zero, any lisse sheaf is automatically tamely ramified along each of these missing sections, as well as along ∞. The lisseness of the $R^i(pr_2)_!(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)})$ now results from the standard specialization theorems, cf. [Ka-SE, 4.7.1].
2) Since $\mathcal{F}$ is lisse, and our morphism is affine and lisse of relative dimension one, proper base change shows that the only possibly nonvanishing cohomology sheaves are those with i=1 or i=2. As $\mathcal{F}$ is lisse outside the n disjoint sections $\{T_1, \ldots, T_n\}$, it is lisse along the section X_2. If χ is nontrivial, the sheaf $pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)}$ is totally ramified along the section X_2, and this remains true after passage to any geometric fibre. On a geometric fibre, say ($T_i \mapsto \alpha_i$, $X_2 \mapsto \beta$), the $H^2{}_c$ must vanish: already the inertia group $I(\beta)$ at β acts as a scalar $\zeta\neq1$, so has no nonzero co-invariants, and a fortiori the entire π_1 of the geometric fibre has none either. By proper change, the sheaf $R^2(pr_2)_!(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)})$ vanishes.
3) This results formally from 2), by the long exact cohomology sequence.
4) This is Weil II [De-Weil II, 3.3.1]. QED

Corollary 8.2.3 In the setting 8.2.1, let $r\geq1$ be a positive integer, $\mathcal{F}_1, \ldots, \mathcal{F}_{r+1}$ objects in Lisse(N,n,ℓ), and $\chi_1, \ldots, \chi_r$ nontrivial $\overline{\mathbb{Q}}_\ell{}^\times$-valued characters of $\boldsymbol{\mu}_N(R_{N,\ell})$. Consider the objects $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_r$ in Lisse(N,n,ℓ) defined inductively by

$$\mathcal{G}_0 := \mathcal{F}_1,$$
$$\mathcal{G}_1 := \mathcal{F}_2\otimes NC_{\chi_1}(\mathcal{G}_0),$$
$$.$$
$$.$$
$$\mathcal{G}_r := \mathcal{F}_{r+1}\otimes NC_{\chi_r}(\mathcal{G}_r).$$

On $\mathbb{A}(n, r+1)_{R_{N,\ell}}$, consider the lisse sheaf

$$\mathcal{F}_1(X_1)\otimes(\otimes_{k=1,\ldots,r}(\mathcal{L}_{\chi_k(X_{k+1} - X_k)}\otimes\mathcal{F}_{k+1}(X_{k+1})))$$

and the projection

$$pr_{r+1}: \mathbb{A}(n, r+1)_{R_{N,\ell}} \to (\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}}.$$

We have

$$\mathcal{G}_r = R^r(pr_{r+1})_!(\mathcal{F}_1(X_1)\otimes(\otimes_{k=1,\dots,r}(\mathcal{L}_{\chi_k}(X_{k+1} - X_k)\otimes\mathcal{F}_{k+1}(X_{k+1})))),$$

and for $i\neq r$ we have

$$R^i(pr_{r+1})_!(\mathcal{F}_1(X_1)\otimes(\otimes_{k=1,\dots,r}(\mathcal{L}_{\chi_k}(X_{k+1} - X_k)\otimes\mathcal{F}_{k+1}(X_{k+1})))) = 0.$$

proof This follows from parts 1) and 2) of the previous lemma 8.2.2. Factor the map pr_{r+1} as the composition of successive "one-variable at a time" projections

$$\pi_i := pr_{[i,r+1],[i+1,r+1]} : \mathbb{A}(n, [i,r+1])_{R_{N,\ell}} \to \mathbb{A}(n, [i+1,r+1])_{R_{N,\ell}},$$

$$(T_1, \dots, T_n, X_i, \dots, X_{r+1}) \mapsto (T_1, \dots, T_n, X_{i+1}, \dots, Xr+1).$$

It suffices to show that for each i=1, ..., r, we have

$$R^1(\pi_i)_!(\mathcal{G}_{i-1}(X_i)\otimes(\otimes_{k=i,\dots,r}(\mathcal{L}_{\chi_k}(X_{k+1} - X_k)\otimes\mathcal{F}_{k+1}(X_{k+1}))))$$
$$= \mathcal{G}_i(X_{i+1})\otimes(\otimes_{k=i+1,\dots,r}(\mathcal{L}_{\chi_k}(X_{k+1} - X_k)\otimes\mathcal{F}_{k+1}(X_{k+1}))),$$

and that the other R^j vanish for $j\neq 1$. For this we argue as follows.

For each i, denote by $\mathcal{H}_i$ the lisse sheaf on $\mathbb{A}(n, [i,r+1])_{R_{N,\ell}}$

$$\mathcal{H}_i := \otimes_{k=i,\dots,r}(\mathcal{L}_{\chi_k}(X_{k+1} - X_k)\otimes\mathcal{F}_{k+1}(X_{k+1}))).$$

The map π_i sits in a cartesian diagram

$$\begin{array}{ccc} \mathbb{A}(n, [i,r+1])_{R_{N,\ell}} & \xrightarrow{pr_{[i,r+1],[i,i+1]}} & \mathbb{A}(n, [i,i+1])_{R_{N,\ell}} \\ \pi_i\downarrow & & \downarrow pr_{[i,i+1],[i+1,i+1]} \\ \mathbb{A}(n, [i+1,r+1])_{R_{N,\ell}} & \xrightarrow[pr_{[i+1,r+1],[i+1,i+1]}]{} & \mathbb{A}(n, [i+1,i+1])_{R_{N,\ell}}. \end{array}$$

The sheaf

$$\mathcal{G}_{i-1}(X_i)\otimes(\otimes_{k=i,\dots,r}(\mathcal{L}_{\chi_k}(X_{k+1} - X_k)\otimes\mathcal{F}_{k+1}(X_{k+1})))$$

on the source is the tensor product

$$(\pi_i)^*(\mathcal{F}_{i+1}(X_{i+1})\otimes\mathcal{H}_{i+1})\otimes(pr_{[i,r+1],[i,i+1]})^*(\mathcal{G}_{i-1}(X_i)\otimes\mathcal{L}_{\chi_i}(X_{i+1} - X_i))$$

of a pullback from the base, namely $(\pi_i)^*(\mathcal{F}_{i+1}(X_{i+1})\otimes\mathcal{H}_{i+1})$, which the projection formula takes care of, and of a pullback

$(pr_{[i,r+1],[i,i+1]})^*(\mathcal{G}_{i-1}(X_i)\otimes\mathcal{L}_{\chi_i}(X_{i+1} - X_i))$ from the situation of the previous lemma. QED

8.3 Middle convolution on $(\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$.

(8.3.1) We continue in the setting 8.2.1. For each nontrivial $\overline{\mathbb{Q}}_\ell^\times$-valued character χ of the group $\mu_N(R_{N,\ell})$, we now define a left exact functor, "middle convolution with $\mathcal{L}_\chi$", denoted MC_χ, from Lisse(N,n,ℓ) to itself, as follows. We view the space $\mathbb{A}(n,2)_{R_{N,\ell}}$, with its second projection pr_2 to $\mathbb{A}(n,1)_{R_{N,\ell}}$, as a relative $\mathbb{A}^1$ with coordinate X_1, minus the n+1 disjoint sections $\{T_1, \ldots, T_n, X_2\}$. We then compactify the morphism pr_2 into the relative $\mathbb{P}^1$

$$\overline{pr}_2 : \mathbb{P}^1\times\mathbb{A}(n,1)_{R_{N,\ell}} \to \mathbb{A}(n,1)_{R_{N,\ell}},$$

by "putting back" the n+2 disjoint sections $\{T_1, \ldots, T_n, X_2, \infty\}$:

$$\begin{tikzcd}
\mathbb{A}(n,2)_{R_{N,\ell}} \arrow[r, "j"] \arrow[dr, "pr_2"'] & \mathbb{P}^1\times\mathbb{A}(n,1)_{R_{N,\ell}} \arrow[d, "\overline{pr}_2"] \\
& \mathbb{A}(n,1)_{R_{N,\ell}}.
\end{tikzcd}$$

We then define

$$MC_\chi(\mathcal{F}) := R^1(\overline{pr}_2)_!(j_*(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)})).$$

Lemma 8.3.2 In the situation 8.3.1, let $\mathcal{F}$ in Lisse(N,n,ℓ), and χ an arbitrary $\overline{\mathbb{Q}}_\ell^\times$-valued character of $\mu_N(R_{N,\ell})$.

1) The sheaves $R^i(\overline{pr}_2)_*(j_*(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)}))$ are all in Lisse(N,n,ℓ).

2) If χ is nontrivial, the above sheaves vanish for $i\neq 1$.

3) If χ is nontrivial, and if $\mathcal{F}$ is pure of weight w, then $MC_\chi(\mathcal{F})$ is pure of weight w+1, and we have a short exact sequence of lisse sheaves on $\mathbb{A}(n,1)_{R_{N,\ell}}$,

$0 \to$ (lisse, mixed of weight $\leq$w) $\to NC_\chi(\mathcal{F}) \to MC_\chi(\mathcal{F}) \to 0$.

4) If χ is nontrivial, then $MC_\chi(\mathcal{F})$ is related to the middle convolution of 4.3 as follows. In the notations of chapter 4, take R to

be the ring $S_{N,n,\ell}$, D to be the n disjoint sections $\{T_1, \dots , T_n\}$ of $\mathbb{A}^1$ over $S_{N,n,\ell}$,

$$k: (\mathbb{A}^1 - D)_{S_{N,n,\ell}} \to (\mathbb{A}^1)_{S_{N,n,\ell}}$$

the inclusion, and $K := k_\star\mathcal{F}[1]$. We know that $K\star_{mid+}j_\star\mathcal{L}_\chi[1]$ on $(\mathbb{A}^1)_{S_{N,n,\ell}}$ is adapted to the stratification $(\mathbb{A}^1 - D, D)$, so its restriction to $\mathbb{A}^1 - D$ is of the form (a lisse sheaf)[1]. This lisse sheaf is $MC_\chi(\mathcal{F})$:

$$MC_\chi(\mathcal{F})[1] = k^*(K\star_{mid+}j_\star\mathcal{L}_\chi[1]).$$

proof of 1). Denote by D the divisor $\{T_1, \dots , T_n, X_2, \infty\}$ in $\mathbb{P}^1\times\mathbb{A}(n,1)_{R_{N,\ell}}$, a disjoint union of sections. The key point is that the sheaf

$$j_\star(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)})$$

is adapted to the stratification $(\mathbb{P}^1\times\mathbb{A}(n,1)_{R_{N,\ell}} - D, D)$ (cf. 4.3), and of formation compatible with all change of base. Let us make this explicit. In the exact sequence of sheaves on $\mathbb{P}^1\times\mathbb{A}(n,1)_{R_{N,\ell}}$

$$0 \to j_!(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)}) \to j_\star(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)}) \to$$
$$\to Quot \to 0,$$

the sheaf Quot is the direct sum of sheaves $Quot(T_i)$, concentrated along the section T_i, a sheaf $Quot(X_2)$ concentrated along the section X_2, and of a sheaf $Quot(\infty)$ concentrated along the section ∞. The adaptedness means that each of these sheaves, viewed (via the section on which it is concentrated) as a sheaf on the base, is a lisse sheaf on the base. Therefore $R(\overline{pr}_2)_\star Quot$ is a lisse sheaf on the base, concentrated in degree zero. By the previous lemma 8.2.2, $R(\overline{pr}_2)_\star(j_!(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)}))$ has lisse cohomology sheaves, and thus 1) is obvious from the long exact cohomology sequence.

proof of 2). If χ is nontrivial, use the fact that formation of $j_\star(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)})$ commutes with passage to fibres. Then fibre by fibre we have a sheaf on $\mathbb{P}^1$ with no nonzero punctual sections and whose stalk at some point β vanishes. Such a sheaf has vanishing H^i for $i\neq 1$. By proper base change, we get 2).

proof of 3). If χ is nontrivial, the long exact cohomology sequence appealed to in the proof of 1) above reads

$$0 \to (\overline{pr}_2)_*\mathrm{Quot} \to \to NC_\chi(\mathcal{F}) \to MC_\chi(\mathcal{F}).$$

If $\mathcal{F}$ is pure of weight w, then each of the sheaves $\mathrm{Quot}(T_i)$, $\mathrm{Quot}(X_2)$, and $\mathrm{Quot}(\infty)$ is mixed of weight $\le$ w (cf. [Ka-SE, 4.7.4]). The sheaf $(\overline{pr}_2)_*\mathrm{Quot}$ is just the direct sum of these sheaves, each viewed on the base. Thus $(\overline{pr}_2)_*\mathrm{Quot}$ is lisse, and mixed of weight $\le$w. That $MC_\chi(\mathcal{F})$ is pure of weight w+1 results from the fact that the formation of $j_*(pr_1{}^*(\mathcal{F})\otimes\mathcal{L}_{\chi(X_2 - X_1)})$ on $\mathbb{P}^1\times\mathbb{A}(n,1)_{R_{N,\ell}}$ commutes with arbitrary change of base on $\mathbb{A}(n,1)_{R_{N,\ell}}$, together with Weil II [De-Weil II, 3.2.3] applied fibre by fibre.

proof of 4). This is a tautology, given the definitions of chapter 4.QED

Corollary 8.3.3 In the situation 8.3.1, let $\mathcal{F}$ in Lisse(N,n,ℓ), and χ a nontrivial $\overline{\mathbb{Q}}_\ell{}^\times$-valued character of $\mu_N(R_{N,\ell})$. Suppose that $\mathcal{F}$ is nonzero and geometrically irreducible on each geometric fibre of $(\mathbb{A}^1- \{T_1, \dots , T_n\})_{S_{N,n,\ell}}$

1) If $\mathcal{F}$ is fibrewise in $\mathcal{T}_\ell$, so is $MC_\chi(\mathcal{F})$. In this case, $\mathcal{F}$ and $MC_\chi(\mathcal{F})$ have the same fibrewise index of rigidity.

2) If on a single geometric fibre, $\mathcal{F}$ is constant, then $\mathcal{F}$ is fibrewise constant, and $MC_\chi(\mathcal{F}) = 0$.

3) If on a single geometric fibre, $\mathcal{F}$ is isomorphic to $\mathcal{L}_{\overline{\chi}(X_1 - T_i)}$ for some index i,then $\mathcal{F}$ is fibrewise isomorphic to $\mathcal{L}_{\overline{\chi}(X_1 - T_i)}$, and $MC_\chi(\mathcal{F}) = 0$.

4) If on a single geometric fibre, $\mathcal{F}$ is isomorphic to $\mathcal{L}_{\rho(X_1 - T_i)}$ for some index i, and some $\overline{\mathbb{Q}}_\ell{}^\times$-valued character ρ of $\mu_N(R_{N,\ell})$ with $\rho \ne \overline{\chi}$, then $\mathcal{F}$ is fibrewise isomorphic to $\mathcal{L}_{\rho(X_1 - T_i)}$, and $MC_\chi(\mathcal{F})$ is fibrewise isomorphic to $\mathcal{L}_{\rho\chi(X_1 - T_i)}$.

proof These all result, thanks to part 4) of the above lemma 8.3.2, from what we have already proven about middle convolution with parameters. Part 1) is just a restatement of 4.3.10.

To prove parts 2), 3) and 4), we argue as follows. We first show that if the condition envisioned holds on a single geometric fibre, then it holds on ever fibre. For this, just apply 4.2.5 to $k_*(\mathcal{F})$, $k_*(\mathcal{F}\otimes\mathcal{L}_{\chi(X_1 - T_i)})$, and to $k_*(\mathcal{F}\otimes\mathcal{L}_{\overline{\rho}(X_1 - T_i)})$ respectively, noting that formation of $k_*(\mathcal{F})$ commutes with arbitrary change of base (cf. [Ka-SE, 4.7.2 and 4.7.3]). Once we know that the conditions hold fibrewise, it suffices to check on all geometric fibres in positive characteristic. For parts 2) and 3), this is given in 3.3.3, 1a) and 2b). For part 4), we use 3.3.3, 2b) to show that $MC_\chi(\mathcal{F})$ is isomorphic to $\mathcal{L}_{\rho\chi(X_1 - T_i)}$ on every geometric fibre of positive characteristic, hence in particular on a single geometric fibre. As above, we consider $k_*(MC_\chi(\mathcal{F})\otimes\mathcal{L}_{\overline{\rho}\overline{\chi}(X_1 - T_i)})$ and apply 4.2.5 to it, to conclude that $MC_\chi(\mathcal{F})$ is fibrewise isomorphic to $\mathcal{L}_{\rho\chi(X_1 - T_i)}$. QED

Second Corollary 8.3.4 In the situation 8.3.1, let $r\geq 0$ be a positive integer, $\mathcal{F}_1, ..., \mathcal{F}_{r+1}$ objects in Lisse(N,n,ℓ) which are all pure of weight 0, and $\chi_1, ..., \chi_r$ nontrivial $\overline{\mathbb{Q}}_\ell^\times$-valued characters of $\mu_N(R_{N,\ell})$. Consider the objects $\mathcal{G}_0, \mathcal{G}_1,..., \mathcal{G}_r$ in Lisse(N,n,ℓ) defined inductively by

$$\mathcal{G}_0 := \mathcal{F}_1,$$
$$\mathcal{G}_1 := \mathcal{F}_2\otimes NC_{\chi_1}(\mathcal{G}_0),$$
$$.$$
$$.$$
$$\mathcal{G}_r := \mathcal{F}_{r+1}\otimes NC_{\chi_r}(\mathcal{G}_r).$$

Consider also the objects $\mathcal{H}_0, \mathcal{H}_1,..., \mathcal{H}_r$ in Lisse(N,n,ℓ) defined inductively by

$$\mathcal{H}_0 := \mathcal{F}_1,$$
$$\mathcal{H}_1 := \mathcal{F}_2\otimes MC_{\chi_1}(\mathcal{H}_0),$$
$$.$$
$$.$$
$$\mathcal{H}_r := \mathcal{F}_{r+1}\otimes MC_{\chi_r}(\mathcal{H}_r).$$

For i=0,1,...,r each $\mathcal{H}_i$ is pure of weight i, each $\mathcal{G}_i$ is mixed of weights $\leq$i, and we have a short exact sequence of lisse sheaves on $\mathbb{A}(n,1)_{R_{N,\ell}}$,

$$0 \to (\text{mixed of weight} \le i-1) \to \mathcal{G}_i \to \mathcal{H}_i \to 0.$$

proof For $r=0$, there is nothing to prove. If $r \ge 1$, this results by induction from part 3) of the previous lemma 8.3.2. Indeed, suppose we already have a short exact sequence of lisse sheaves on $\mathbb{A}(n,1)_{R_{N,\ell}}$,

$$0 \to (\text{mixed of weight} \le i-1) \to \mathcal{G}_i \to \mathcal{H}_i \to 0,$$

and $\mathcal{H}_i$ is pure of weight i. Applying NC_{χ_i}, we get (by 8.3.2, parts 3) and 4)) a short exact sequence of lisse sheaves on $\mathbb{A}(n,1)_{R_{N,\ell}}$,

$$0 \to (\text{mixed of weight} \le i) \to NC_{\chi_i}(\mathcal{G}_i) \to NC_{\chi_i}(\mathcal{H}_i) \to 0.$$

From part 3) of the previous lemma 8.3.2, we get a short exact sequence of lisse sheaves on $\mathbb{A}(n,1)_{R_{N,\ell}}$,

$$0 \to (\text{lisse, mixed of weight} \le i) \to NC_{\chi_i}(\mathcal{H}_i) \to MC_{\chi_i}(\mathcal{H}_i) \to 0,$$

and $MC_{\chi_i}(\mathcal{H}_i)$ is pure of weight i+1. The composite map

$$NC_{\chi_i}(\mathcal{G}_i) \to NC_{\chi_i}(\mathcal{H}_i) \to MC_{\chi_i}(\mathcal{H}_i)$$

is surjective, and its kernel is lisse and mixed of weight $\le$i. So we get a short exact sequence of lisse sheaves on $\mathbb{A}(n,1)_{R_{N,\ell}}$,

$$0 \to (\text{lisse, mixed of weight} \le i) \to NC_{\chi_i}(\mathcal{G}_i) \to MC_{\chi_i}(\mathcal{H}_i) \to 0.$$

Tensoring this with $\mathcal{F}_{i+1}$ gives the required short exact sequence

$$0 \to (\text{lisse, mixed of weight} \le i-1) \to \mathcal{G}_{i+1} \to \mathcal{H}i+1 \to 0,$$

and shows that $\mathcal{H}_{i+1}$ is pure of weight i+1. QED

Theorem 8.3.5 In the situation 8.3.1, fix an integer $r \ge 0$. Fix a choice of $(r+1)n$ arbitrary characters

$$\chi_{a,i} : \mu_N(R_{N,\ell}) \to \overline{\mathbb{Q}}_\ell^\times,$$

and a choice of r **nontrivial** characters

$$\rho_k: \mu_N(R_{N,\ell}) \to \overline{\mathbb{Q}}_\ell^\times.$$

On $\mathbb{A}(n, r+1)_{R_{N,\ell}}$, consider the lisse of rank one, pure of weight zero $\overline{\mathbb{Q}}_\ell$-sheaf

$$\mathcal{L} := (\bigotimes_{a,i}\mathcal{L}_{\chi_{a,i}(X_a - T_i)})(\bigotimes_{k=1,\ldots,r}\mathcal{L}_{\rho_k(X_{k+1} - X_k)}).$$

Denote by

$$pr_{r+1} : \mathbb{A}(n, r+1)_{R_{N,\ell}} \to (\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$$

the map

$$(T_1, \ldots , T_n, X_1, \ldots , X_{r+1}) \mapsto (T_1, \ldots , T_n, X_{r+1}).$$

1) The sheaves $R^i(pr_{r+1})_!\mathcal{L}$ on $(\mathbb{A}^1 - \{T_1, \ldots , T_n\})_{S_{N,n,\ell}}$ are lisse and tame, and they vanish for $i \neq r$.

2) The sheaf $\mathcal{K} := R^r(pr_{r+1})_!\mathcal{L}$ is mixed of integral weights in $[0,r]$. It sits in a short exact sequence of lisse sheaves

$$0 \to \mathcal{K}_{\leq r-1} \to \mathcal{K} \to \mathcal{K}_{=r} \to 0$$

where $\mathcal{K}_{\leq r-1}$ is mixed of integral weights in $[0,r-1]$, and where $\mathcal{K}_{=r}$ is punctually pure of weight r.

3) If $\mathcal{K}_{=r}$ is nonzero, then its restriction to every geometric fibre of $(\mathbb{A}^1 - \{T_1, \ldots , T_n\})_{S_{N,n,\ell}}$ over $S_{N,n,\ell}$ is geometrically irreducible and cohomologically rigid, with all eigenvalues of all local monodromies N'th roots of unity.

4) Fix a geometric point of $S_{N,n,\ell}$, i.e., a ring homomorphism

$$\varphi : S_{N,n,\ell} \to k$$

to an algebraically closed field k. Denote $\varphi(T_i) := \alpha_i$ in k. On the corresponding geometic fibre $\mathbb{A}^1 - \{\alpha_1, \ldots , \alpha_n\}$ over k, any tame, geometrically irreducible lisse sheaf which is cohomologically rigid, and such that all eigenvalues of all local monodromies are N'th roots of unity, is isomorphic to (the restriction to that fibre of) a nonzero $\mathcal{K}_{=r}$ for some integer $r \geq 0$, some choice of the characters $\chi_{a,i}$ and some choice of the r nontrivial characters ρ_k.

proof For $a=1, \ldots, r+1$, define $\mathcal{F}_a$ in Lisse(N,n,ℓ) by

$$\mathcal{F}_a(X_a) := \bigotimes_{i=1,\ldots,n}\mathcal{L}_{\chi_{a,i}}(X_a - T_i).$$

Define $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_r$ in Lisse(N,n,ℓ) inductively by

$$\mathcal{G}_0 := \mathcal{F}_1,$$

$$\mathcal{G}_1 := \mathcal{F}_2 \otimes NC_{\rho_1}(\mathcal{G}_0),$$

.

.

$$\mathcal{G}_r := \mathcal{F}_{r+1} \otimes NC_{\rho_r}(\mathcal{G}_r).$$

Define $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_r$ in Lisse(N,n,ℓ) defined inductively by

$$\mathcal{H}_0 := \mathcal{F}_1,$$
$$\mathcal{H}_1 := \mathcal{F}_2 \otimes MC_{\rho_1}(\mathcal{H}_0),$$
$$.$$
$$.$$
$$\mathcal{H}_r := \mathcal{F}_{r+1} \otimes MC_{\rho_r}(\mathcal{H}_r).$$

Assertion 1) results from 8.2.3, which further tells us that $\mathcal{K}$ is the sheaf $\mathcal{G}_r$. Thanks to 8.3.4, we get 2), with the further information that $\mathcal{K}_{=r}$ is the sheaf $\mathcal{H}_r$.

To prove 3), we successively apply 8.3.3 to the sheaves $\mathcal{H}_i$.

To prove 4), we argue as follows. If the $\mathcal{F}$ in question is of rank one, take r=0, and take $\mathcal{F}_1$ to be the unique sheaf of type $\bigotimes_{i=1,\ldots,n} \mathcal{L}_{\chi_{1,i}(X_1 - T_i)}$. whose restriction to the fibre in question is $\mathcal{F}$. If the $\mathcal{F}$ in question is of rank 2 or higher, the main theorem 5.2.1 gives us an explicit algorithm for choosing the $\chi_{a,i}$ and the ρ_k so that with the resulting choice of

$$\mathcal{F}_a(X_a) := \bigotimes_{i=1,\ldots,n} \mathcal{L}_{\chi_{a,i}(X_a - T_i)},$$

our $\mathcal{F}$ is the restriction to its fibre of the object $\mathcal{H}_r$ defined as above. QED

8.4 "Geometric" description of all tame rigids with quasi-unipotent local monodromy

Theorem 8.4.1 In the setting of 8.0, fix an integer $r \geq 0$. Fix a choice of (r+1)n arbitrary integers e(a,i), and a choice of r integers f(k) such that no f(k) is divisible by N. In the product space $\mathbb{G}_m \times \mathbb{A}(n, r+1)_{R_{N,\ell}}$, consider the hypersurface Hyp(e's, f's) of equation

$$Y^N = (\Pi_{a,i}(X_a - T_i)^{e(a,i)})(\Pi_{k=1,\ldots,r}(X_{k+1} - X_k)^{f(k)}).$$

Denote by

$$\pi : \mathrm{Hyp}(e's, f's) \to (\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$$

the map

$$(Y, T_1, \ldots, T_n, X_1, \ldots, X_{r+1}) \mapsto (T_1, \ldots, T_n, X_{r+1}).$$

[Thus we think of Hyp(e's, f's) as a family of hypersurfaces in the r+1 variables $(Y, X_1, \ldots, X_r)$, parameterized by $(T_1, \ldots, T_n, X_{r+1})$.]

Fix a **faithful** $\overline{\mathbb{Q}}_\ell^\times$-valued character χ of the group $\mu_N(R_{N,\ell})$, which acts on Hyp(e's, f's) by moving Y alone.
1) The sheaves $R^i\pi_!\overline{\mathbb{Q}}_\ell$ on $(\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$ are lisse and tame.
2)The χ-component $(R^i\pi_!\overline{\mathbb{Q}}_\ell)^\chi$ vanishes for $i \neq r$.
2) The sheaf $\mathcal{K} := (R^r\pi_!\overline{\mathbb{Q}}_\ell)^\chi$ is mixed of integral weights in [0,r]. It sits in a short exact sequence of lisse sheaves

$$0 \to \mathcal{K}_{\leq r-1} \to \mathcal{K} \to \mathcal{K}_{=r} \to 0$$

where $\mathcal{K}_{\leq r-1}$ is mixed of integral weights in [0,r-1], and where $\mathcal{K}_{=r}$ is punctually pure of weight r.
3) If $\mathcal{K}_{=r}$ is nonzero, then its restriction to every geometric fibre of $(\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$ over $S_{N,n,\ell}$ is geometrically irreducible and cohomologically rigid, with all eigenvalues of all local monodromies N'th roots of unity.
4) Fix a geometric point of $S_{N,n,\ell}$, i.e., a ring homomorphism

$$\varphi : S_{N,n,\ell} \to k.$$

Denote $\varphi(T_i) := \alpha_i$ in k. On the corresponding geometric fibre $\mathbb{A}^1 - \{\alpha_1, \ldots, \alpha_n\}$ over k, any tame, geometrically irreducible lisse sheaf which is cohomologically rigid, and such that all eigenvalues of all local monodromies are N'th roots of unity, is isomorphic to a nonzero $\mathcal{K}_{=r}$ for some integer $r \geq 0$ and some choice of integers e(a,i) and f(k) as in the theorem.

proof Denote by p_2 the projection

$$p_2 : \mathrm{Hyp}(e\text{'s}, f\text{'s}) \to \mathbb{A}(n, r+1)_{R_{N,\ell}}.$$

This map is a μ_N-torsor, and hence we have a direct sum decomposition

$$(p_2)_*\overline{\mathbb{Q}}_\ell = \oplus_\rho ((p_2)_*\overline{\mathbb{Q}}_\ell)^\chi,$$

indexed by all characters $\chi: \mu_N(R_{N,\ell}) \to \overline{\mathbb{Q}}_\ell^\times$. The χ-component is precisely

$$((p_2)_*\overline{\mathbb{Q}}_\ell)^\chi = (\textstyle\bigotimes_{a,i}\mathcal{L}_{\chi_{a,i}(X_a - T_i)})(\bigotimes_{k=1,\ldots,r}\mathcal{L}_{\rho_k(X_{k+1} - X_k)}),$$

where the characters $\chi_{a,i}$ and ρ_k are given by the recipe

$$\chi_{a,i} = \chi^{e(a,i)},\ \rho_k = \chi^{f(k)}.$$

For each integer i and each character χ, we have

$$(R^i\pi_!\bar{\mathbb{Q}}_\ell)^\chi = R^i(pr_{r+1})_!(((p_2)_*\bar{\mathbb{Q}}_\ell)^\chi).$$

Thus we find

$(R^i\pi_!\bar{\mathbb{Q}}_\ell)^\chi =$

$$= R^i(pr_{r+1})_!((\bigotimes_{a,i}\mathcal{L}_{\chi_{a,i}(X_a - T_i)})(\bigotimes_{k=1,\ldots,r}\mathcal{L}_{\rho_k(X_{k+1} - X_k)})).$$

That all of these sheaves are in Lisse(N,n,ℓ) follows from 8.2.2. For χ faithful, the hypothesis that none of f(k) is divisible by N is equivalent to the hypothesis that each ρ_k is nontrivial. Thus this theorem is no more or less than a restatement of the previous one 8.3.5. QED

8.5 A remark and a question

(8.5.1) In the case r=1 of the above theorem, we are looking at the n+1 parameter family of curves in two variables (Y, X_1), with parameters $(T_1, \ldots, T_n, X_2)$, defined by the equation

$$Y^N = (\Pi_i(X_1 - T_i)^{e(1,i)})(\Pi_i(X_2 - T_i)^{e(2,i)})(X_2 - X_1)^{f(1)}.$$

The local systems $\mathcal{K}_{=r}$ in this case are, over $\mathbb{C}$, the (ℓ-adic incarnations of the) Lauricella hypergeometric local systems whose monodromy was studied extensively by Picard, Terada, and, most recently, Deligne-Mostow, cf. the bibliography of [Del-Mo].

We should remark that because the factor $(\Pi_i(X_2 - T_i)^{e(2,i)})$ comes from the base, omitting it simply twists $\mathcal{K}_{=r}$ by the inverse of $\bigotimes_i\mathcal{L}_{\chi_{2,i}(X_2 - T_i)}$). So essentially we are dealing with the n+1 parameter family of curves in two variables (Y, X_1), with parameters $(T_1, \ldots, T_n, X_2)$, defined by the more familiar equation

$$Y^N = (\Pi_i(X_1 - T_i)^{e(1,i)})(X_2 - X_1)^{f(1)}.$$

(8.5.2) In the case of higher r, the local systems $\mathcal{K}_{=r}$ do not seem to have been the object of much systematic study. One might ask whether,over $\mathbb{C}$, the ("Betti realizations" of the) local systems $\mathcal{K}_{=r}$ can have "interesting" monodromy groups also in the case r > 1?

9.0 Introduction

(9.0.1) In this chapter, we will prove (Theorem 9.4.1) Grothendieck's p-curvature conjecture for the regular singular differential equation corresponding, via Riemann-Hilbert, to any irreducible rigid local system on an open set of $\mathbb{P}^1$ over $\mathbb{C}$: such a differential equation has p-curvature zero for almost all p if and only if it has finite monodromy.

9.1 Review of Grothendieck's p-curvature conjecture

(9.1.1) Let S be any smooth connected quasi-projective $\mathbb{C}$-scheme. On S^{an}, we have the category LocSys(S^{an}) of all local systems of finite-dimensional $\mathbb{C}$-vector spaces on S^{an}. On S^{an}, we also have the category DE(S^{an}) of "analytic differential equations on S^{an}", i.e., the category of all pairs (M, ∇), with M a coherent $\mathcal{O}_{S^{an}}$-module M and ∇ an integrable connection $\nabla: M \to M\otimes\Omega^1{}_{S^{an}/\mathbb{C}}$ on M. The functor "sheaf of germs of horizontal sections",

$$M \mapsto M^{\nabla} := \mathrm{Ker}(\nabla: M \to M\otimes\Omega^1{}_{S^{an}/\mathbb{C}}),$$

is an equivalence of categories

$$DE(S^{an}) \cong LocSys(S^{an}).$$

(9.1.2) On $S/\mathbb{C}$ itself, we have the category DE($S/\mathbb{C}$) of all "algebraic differential equations on $S/.\mathbb{C}$", i.e., the category of all pairs (M, ∇), with M a coherent $\mathcal{O}_S$-module M and ∇ an integrable connection $\nabla: M \to M\otimes\Omega^1{}_{S/\mathbb{C}}$ on M. Inside DE($S/\mathbb{C}$), we have the full subcategory RSDE($S/\mathbb{C}$), consisting of those algebraic differential equations with regular singular points "at ∞" in the sense of [De-ED]. Thanks to Deligne's solution of the Riemann-Hilbert problem, cf. [De-ED] and [Ka-ODW23], we know that the functor "passage to the analytic",

$$(M, \nabla) \mapsto (M^{an} := M\otimes_{\mathcal{O}_S}\mathcal{O}_{S^{an}}, \nabla^{an})$$

is an equivalence of categories

$$RSDE(S/\mathbb{C}) \cong DE(S^{an}).$$

Combining this equivalence with the previous one, we see that the functor "sheaf of germs of holomorphic solutions",

$$M \mapsto (M^{an})^{\nabla^{an}},$$

is an equivalence of categories

$$RSDE(S/\mathbb{C}) \cong LocSys(S^{an}).$$

In words, any local system on S^{an} is the monodromy of a unique algebraic differential equation on S with regular singular points.
(9.1.3) Let us next recall from [Ka-NCMT] and [Ka-ASDE, Intro.], what it means for an algebraic differential equation (M, ∇) on $S/\mathbb{C}$, not a priori assumed to have regular singular points, to have "p-curvature zero for almost all p". Given $S/\mathbb{C}$, we can find a subring R of $\mathbb{C}$ which, as a ring, is finitely generated over $\mathbb{Z}$ and smooth over R, and a smooth R-scheme $\mathcal{S}/R$ whose $\mathbb{C}$-fibre (via the given inclusion of R into $\mathbb{C}$ as a subring) is $S/\mathbb{C}$. At the possible expense of enlarging R, but still keeping it finitely generated over $\mathbb{Z}$ and smooth over $\mathbb{Z}$, we can find an affine open $\mathcal{U}$ in $\mathcal{S}$ whose $\mathbb{C}$-fibre $\mathcal{U}_{\mathbb{C}}$ is a Zariski dense affine open U in S, and a pair $(\mathcal{M}, \nabla)$ consisting of a locally free $\mathcal{O}_{\mathcal{U}}$-module of finite rank and an integrable $\mathcal{U}/R$-connection $\nabla : \mathcal{M} \rightarrow \mathcal{M}\otimes\Omega^1{}_{\mathcal{U}/R}$ whose complex fibre is the restriction to $U := \mathcal{U}_{\mathbb{C}}$ of the original (M, ∇) on $S/\mathbb{C}$.
(9.1.4) Having made such choices $(R, \mathcal{U}, \mathcal{M}, \nabla)$, we can ask whether there exists an affine open $\mathcal{V}$ in $\mathcal{U}$, whose $\mathbb{C}$-fibre $\mathcal{V}_{\mathbb{C}}$ is Zariski dense in $\mathcal{U}_{\mathbb{C}}$, such that either of the following two equivalent conditions holds:

1) for every maximal ideal m of R, with (necessarily finite) residue field R/m of positive characteristic p, the restriction to $\mathcal{V}\otimes_R(R/m)$ of the differential equation $(\mathcal{M}/m\mathcal{M}, \nabla)$ on $\mathcal{U}\otimes_R(R/m)$ has p-curvature zero (meaning that $\nabla(D)^p = \nabla(D^p)$ for every R/m -linear derivation D of the affine ring of $\mathcal{V}\otimes_R(R/m)$ to itself).
1') for every prime number p, the restriction to $\mathcal{V}\otimes_R(R/pR)$ of the differential equation $(\mathcal{M}/p\mathcal{M}, \nabla)$ on $\mathcal{U}\otimes_R(R/pR)$ has p-curvature zero (meaning that $\nabla(D)^p = \nabla(D^p)$ for every R/pR -linear derivation D of the affine ring of $\mathcal{V}\otimes_R(R/pR)$ to itself).

(9.1.5) If there exists such a $\mathcal{V}$ (i.e. a $\mathcal{V}$ such that 1) and 1') hold) for one set of choices $(R, \mathcal{U}, \mathcal{M}, \nabla)$, then there exists such a $\mathcal{V}$ for any other set of choices. The existence of such a $\mathcal{V}$ is thus an intrinsic property of the original differential equation (M, ∇) on $S/\mathbb{C}$ (indeed an intrinsic property of the germ of (M, ∇) at the generic point of S), which is called "having p-curvature zero for almost all p".

(9.1.6) It is known [Ka-NCMT, 13.0] that if (M, ∇) on $S/\mathbb{C}$ has p-curvature zero for almost all p, then (M, ∇) has regular singular points, and its local monodromy around any smooth divisor D_i at ∞ in any normal crossing compactification $\overline{S}$ of S (i.e., S open dense in a projective smooth $\overline{S}/\mathbb{C}$ such that $\overline{S} - S$ is a union of smooth divisors D_i with normal crossings) is of finite order.

(9.1.7) Grothendieck's p-curvature conjecture is that if (M, ∇) on $S/\mathbb{C}$ has p-curvature zero for almost all p, then (M, ∇) satisfies the following equivalent (equivalent because (M, ∇) has regular singular points) conditions:

1) $(M, \nabla)^{an}$ has finite monodromy on S^{an}.
2) $(M, \nabla)^{an}$ becomes trivial on a finite etale covering of S^{an}.
3) (M, ∇) becomes trivial on a finite etale covering of S.
4) (M, ∇) has a full set of algebraic solutions.
5) There exists a dense open $U \subset S$ such that $(M, \nabla) | U$ satisfies the preceding conditions 1)-4) on U.

9.2 Interlude: Picard-Fuchs equations and some variants

(9.2.1) Let us denote by K the function field of S. Thus K is a finitely generated extension of $\mathbb{C}$. Given any smooth K-scheme U/K, separated and of finite type, its algebraic de Rham cohomology groups $H^i_{DR}(U/K)$ are finite-dimensional K-spaces endowed with a canonical integrable $\mathbb{C}$-connection ∇, the Gauss-Manin connection. The algebraic differential equations $(H^i_{DR}(U/K), \nabla)$ on $K/\mathbb{C}$ are called the Picard-Fuchs equations (in dimension i, for U/K).

(9.2.2) There are two variations on this theme which will be essential in what follows. The first involves a finite group action. Suppose we are given a finite group G which acts K-linearly on U/K. Then G acts K-linearly and horizontally on each $H^i_{DR}(U/K)$. For each irreducible $\mathbb{C}$-representation ρ of G, we denote by $H^i_{DR}(U/K)(\rho)$ the ρ-isotypical component of $H^i_{DR}(U/K)$. This is a ∇-stable K-subspace of $H^i_{DR}(U/K)$, so corresponds to a subequation $(H^i_{DR}(U/K)(\rho), \nabla)$ of $(H^i_{DR}(U/K), \nabla)$. In fact, this subequation is a direct factor, since $(H^i_{DR}(U/K), \nabla) = \oplus_\rho\, (H^i_{DR}(U/K)(\rho), \nabla)$.

(9.2.3) Hironaka has announced, in the introduction to his paper [Hir-IES] that given U/K and G as above, we can find a G-equivariant normal crossing compactification X/K of U/K, i.e., a proper smooth X/K which contains U as a dense open set, such that X-U is a union of smooth divisors D_i with normal crossings, such that G acts K-linearly on X/K and such that the inclusion of U into X is G-equivariant. Youssin, in the introduction to [You], has announced another proof of this same result. Unfortunately, neither proof of this result has yet appeared (although Hironaka's proof of equivariant resolution for the presumably much harder case of complex analytic spaces has appeared, cf. the bibliography of [Hir-IES]). Although this result was already used freely in [Ka-ASDE], and its use in what follows would slightly simplify the exposition, we will avoid using it here.

(9.2.4) Given U/K as above, by Nagata [Na] there exists a compactification X_0/K of U/K, i.e., a proper K-scheme X_0 which contains U as a dense open set [if U/K is quasiprojective, we may take for X_0 its closure in the ambient projective space]. Applying Hironaka [Hir-RS, Cor 3 of Thm 2] to a compactification X_0/K of U/K, there exists a proper birational K-morphism π: X $\to$ X_0 which is an isomorphism over U, and such that X/K is a normal crossings compactification of U/K, i.e., a proper smooth X/K which contains U as a dense open set, such that X-U is a union of smooth divisors D_i with normal crossings.

(9.2.5) Given a second smooth K-scheme V/K which is separated and of finite type, and a K-morphism f: V $\to$ U, we can find a normal crossings compactification Y/K of V/K, and a K-morphism φ: Y $\to$X which "extends" f in the sense that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{f} & U \\ \cap & & \cap \\ Y & \xrightarrow[\varphi]{} & X. \end{array}$$

[Recall the construction: one takes any compactification Y_0 of Y, and applies Hironaka [Hir-RS, Cor 3 of Thm 2] to the compactification Y_1 of V which is defined to be the closure in $Y_0 \times X$ of the graph of f.]

(9.2.6) Given finitely many K-morphisms f_i : V $\to$ U, i=1, ..., n, we can find a single normal crossings compactification Y/K of V/K and maps φ_i: Y $\to$ X such that φ_i extends f_i for each i; just apply the one map statement to the map $f_1\times\ldots\times f_n$: $V \to U^n$, and the normal crossings compactification X^n of U^n.

(9.2.7) By functoriality, the inclusion of U/K into X/K induces on cohomology a horizontal restriction map

$$(H^i{}_{DR}(X/K), \nabla) \to (H^i{}_{DR}(U/K), \nabla).$$

The image of this map is thus a subequation of $(H^i{}_{DR}(U/K), \nabla)$ which is denoted $(W_iH^i{}_{DR}(U/K), \nabla)$, called the "weight i part" of $(H^i{}_{DR}(U/K), \nabla)$. It was a fundamental insight of Grothendieck's [Gro-Brauer III, 9.1-4], later generalized by Deligne's mixed Hodge theory [De-Hodge II, 3.2.17], and for us an easy consequence of the Weil conjectures and resolution, cf. 9.4.3-5, that the "weight i part" of $(H^i{}_{DR}(U/K), \nabla)$ is independent of the auxiliary choice of the normal crossing compactification X/K used to define it.

(9.2.8) Because $(W_iH^i{}_{DR}(U/K), \nabla)$ is an **intrinsic** subequation of $(H^i{}_{DR}(U/K), \nabla)$, it must, by "pure thought", be stable under the action of G. Here is an explicit geometric way to see this stability. We can find a normal crossings compactification Y/K of U/K such that each of the finitely many maps g: $U \to U$, one for each g in G, extends to a map φ_g: $Y \to X$. Then from the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{g} & U \\ \cap & & \cap \\ Y & \xrightarrow[\varphi_g]{} & X. \end{array}$$

we get a commutative diagram of de Rham cohomology groups

$$\begin{array}{ccc} H^i{}_{DR}(X/K) & \xrightarrow{(\varphi_g)^*} & H^i{}_{DR}(Y/K) \\ \downarrow \text{restr.} & & \downarrow \text{restr.} \\ H^i{}_{DR}(U/K) & \xrightarrow[g^*]{} & H^i{}_{DR}(U/K) \end{array}.$$

But both $H^i{}_{DR}(X/K)$ and $H^i{}_{DR}(Y/K)$ have the **same** image in $H^i{}_{DR}(U/K)$, namely $W_iH^i{}_{DR}(U/K)$. So this diagram shows that $W_iH^i{}_{DR}(U/K)$ is G-stable.

(9.2.9) So for each irreducible representation ρ of G, we may form the ρ-isotypical component $W_iH^i{}_{DR}(U/K)(\rho)$; we call it the weight i part of $H^i{}_{DR}(U/K)(\rho)$. It is a subequation both of $(H^i{}_{DR}(U/K)(\rho), \nabla)$ and of $(W_iH^i{}_{DR}(U/K), \nabla)$. Moreover, we have

$$W_iH^i{}_{DR}(U/K)(\rho) = (W_iH^i{}_{DR}(U/K)) \cap (H^i{}_{DR}(U/K)(\rho)).$$

However, there is another way to describe $W_iH^i{}_{DR}(U/K)(\rho)$ which will be useful later.

(9.2.10) Recall that for an irreducible representation ρ of G, the projector onto the ρ-isotypical component is the central idempotent $P(\rho)$ in the group ring $\mathbb{Z}[1/\mathrm{Card}(G), \zeta_{\mathrm{Card}(G)}][G]$ defined by

$$P(\rho) := (\deg(\rho)/\mathrm{Card}(G))\Sigma_{g \text{ in } G}\, \mathrm{trace}(\rho(g^{-1}))g.$$

We denote by $P(\rho;\varphi)$ the K-linear horizontal map

$$P(\rho;\varphi): H^i{}_{DR}(X/K) \to H^i{}_{DR}(Y/K)$$

defined by

$$P(\rho;\varphi) := (\deg(\rho)/\mathrm{Card}(G))\Sigma_{g \text{ in } G}\, \mathrm{trace}(\rho(g^{-1}))(\varphi_g)^*.$$

We have a commutative diagram

$$\begin{tikzcd} H^i{}_{DR}(X/K) \arrow[r, "P(\rho;\varphi)"] \arrow[d, "\mathrm{restr}_X"] & H^i{}_{DR}(Y/K) \arrow[d, "\mathrm{restr}_Y"] \\ H^i{}_{DR}(U/K) \arrow[r, "P(\rho)."'] & H^i{}_{DR}(U/K) \end{tikzcd}$$

By definition, $W_iH^i{}_{DR}(U/K)(\rho)$ is the result of applying the projector $P(\rho)$ to the subequation $W_iH^i{}_{DR}(U/K)$ of $H^i{}_{DR}(U/K)$. If we think of $W_iH^i{}_{DR}(U/K)$ as the image of $H^i{}_{DR}(X/K)$, we get

$$W_iH^i{}_{DR}(U/K)(\rho) = \mathrm{Image}(\ P(\rho)\circ\mathrm{restr}_X).$$

This corresponds to going around the diagram by the bottom. If instead we go around by the top, we get the alternate description

$$W_iH^i{}_{DR}(U/K)(\rho) = \mathrm{Image}(\ \mathrm{restr}_Y\circ P(\rho;\varphi)).$$

9.3 The main result of [Ka-ASDE] and a generalization

(9.3.1) For the reader's convenience, we recall the statement of the main result of [Ka-ASDE]. Let S be a smooth connected quasi-

projective $\mathbb{C}$-scheme, with function field K Let Let (M, ∇) in $DE(S/\mathbb{C})$.
Consider the following condition (*):
(*) There exists a smooth K-scheme U/K, separated and of finite type, a finite group G acting K-linearly on U/K, an irreducible $\mathbb{C}$-representation χ of G, and an integer $i \geq 0$, such that denoting by $\{\chi^\sigma\}_\sigma$ the distinct $\mathrm{Aut}(\mathbb{C})$-conjugates of χ, the restriction of (M, ∇) to K is isomorphic to $\oplus_\sigma ((H^i{}_{DR}(U/K)(\chi^\sigma), \nabla)$.

Theorem 9.3.2 [Ka-ASDE, 5.7] Let S be a smooth connected quasi-projective $\mathbb{C}$-scheme. Let (M, ∇) in $DE(S/\mathbb{C})$ satisfy the condition (*) above. Then Grothendieck's p-curvature conjecture holds for (M, ∇): if (M, ∇) has p-curvature zero for almost all p, then (M, ∇) has finite monodromy.

(9.3.3) Consider now the following conditions (**) and (***):

(**) There exists a smooth K-scheme U/K, separated and of finite type, a finite group G acting K-linearly on U/K, an irreducible $\mathbb{C}$-representation χ of G, and an integer $i \geq 0$, such that the restriction of (M, ∇) to K is isomorphic to $((H^i{}_{DR}(U/K)(\chi), \nabla)$.

(***) There exists a smooth K-scheme U/K, separated and of finite type, a finite group G acting K-linearly on U/K, an irreducible $\mathbb{C}$-representation χ of G, and an integer $i \geq 0$, such that the restriction of (M, ∇) to K is isomorphic to $((W_i H^i{}_{DR}(U/K)(\chi), \nabla)$.

(9.3.4) The following theorem generalizes and implies the theorem [Ka-ASDE, 5.7] stated above. Its proof is essentially already contained in [Ka-ASDE], but the author only recently understood this fact.

Theorem 9.3.5 Let S be a smooth connected quasi-projective $\mathbb{C}$-scheme. Let (M, ∇) in $DE(S/\mathbb{C})$ satisfy either of the conditions (**) or (***) above. Then Grothendieck's p-curvature conjecture holds for (M, ∇): if (M, ∇) has p-curvature zero for almost all p, then (M, ∇) has finite monodromy.

proof The question is birational on S. So at the expense of shrinking S, we may apply standard "spreading out" techniques to produce

1) a subring R of $\mathbb{C}$, which is finitely generated and smooth over $\mathbb{Z}$,

and which contains the cyclotomic ring $\mathbb{Z}[1/\mathrm{Card}(G), \zeta_{\mathrm{Card}(G)}]$,
2) a smooth affine $\mathcal{S}/R$, whose $\mathbb{C}$-fibre is S,
3) a smooth $\mathcal{U}/\mathcal{S}$, whose restriction to the generic point of $\mathcal{S}_{\mathbb{C}}$ is U/K, and an $\mathcal{S}$-linear action of the finite group G on $\mathcal{U}/\mathcal{S}$ which over the generic point of $\mathcal{S}_{\mathbb{C}}$ is the given action of G on U/K,
4) a normal crossings compactification $\mathcal{X}/\mathcal{S}$ of $\mathcal{U}/\mathcal{S}$, i.e., a proper smooth $\mathcal{X}/\mathcal{S}$ containing $\mathcal{U}$ as a dense open set, such that $(\mathcal{X} - \mathcal{S})^{\mathrm{red}} := \mathcal{D}$ is a union of finitely many smooth over $\mathcal{S}$ divisors $\mathcal{D}_i$ in $\mathcal{X}$ which have normal crossings relative to $\mathcal{S}$,
5) a normal crossings compactification $\mathcal{Y}/\mathcal{S}$ of $\mathcal{U}/\mathcal{S}$, i.e., a proper smooth $\mathcal{Y}/\mathcal{S}$ containing $\mathcal{U}$ as a dense open set, such that $(\mathcal{Y} - \mathcal{S})^{\mathrm{red}} := \mathcal{E}$ is a union of finitely many smooth over $\mathcal{S}$ divisors $\mathcal{E}_i$ in $\mathcal{Y}$ which have normal crossings relative to $\mathcal{S}$,
6) for each g in G, an $\mathcal{S}$-morphism $\varphi_g : \mathcal{Y} \to \mathcal{X}$ which maps $\mathcal{U}$ to $\mathcal{U}$ and induces g on $\mathcal{U}$.

At the expense of further shrinking on $\mathcal{S}$, we may also assume that

1) for each pair of integers (a,b), each of the four Hodge cohomology groups on $\mathcal{S}$

$$H^b(\mathcal{X}, \Omega^a{}_{\mathcal{X}/\mathcal{S}}),\ H^b(\mathcal{X}, \Omega^a{}_{\mathcal{X}/\mathcal{S}}(\log \mathcal{D})),$$
$$H^b(\mathcal{Y}, \Omega^a{}_{\mathcal{Y}/\mathcal{S}}),\ H^b(\mathcal{Y}, \Omega^a{}_{\mathcal{Y}/\mathcal{S}}(\log \mathcal{E})),$$

is a locally free $\mathcal{O}_{\mathcal{S}}$-module of finite rank, whose formation commutes with arbitrary change of base on $\mathcal{S}$.
2) each of the four Hodge-de Rham spectral sequences

(I) $\quad E_1{}^{a,b} = H^b(\mathcal{X}, \Omega^a{}_{\mathcal{X}/\mathcal{S}}) \Rightarrow H^{a+b}(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}})$,

(I log) $\quad E_1{}^{a,b} = H^b(\mathcal{X}, \Omega^a{}_{\mathcal{X}/\mathcal{S}}(\log \mathcal{D})) \Rightarrow H^{a+b}(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}(\log \mathcal{D}))$,

(II) $\quad E_1{}^{a,b} = H^b(\mathcal{Y}, \Omega^a{}_{\mathcal{Y}/\mathcal{S}}) \Rightarrow H^{a+b}(\mathcal{Y}, \Omega^{\bullet}{}_{\mathcal{Y}/\mathcal{S}})$,

(II log) $\quad E_1{}^{a,b} = H^b(\mathcal{Y}, \Omega^a{}_{\mathcal{Y}/\mathcal{S}}(\log \mathcal{E})) \Rightarrow H^{a+b}(\mathcal{Y}, \Omega^{\bullet}{}_{\mathcal{Y}/\mathcal{S}}(\log \mathcal{E}))$,

degenerates at E_1, and is of formation compatible with arbitrary change of base on $\mathcal{S}$.
3) For each g in G, the $\mathcal{S}$-morphism $\varphi_g : \mathcal{Y} \to \mathcal{X}$ induces an isomorphism of spectral sequences (I log) $\cong$ (II log). If we identify (I log) = (II log) via φ_{id}, then $g \mapsto \varphi_g$ defines an action of G on the spectral sequences (I log) and (II log).

4) For each i and j, the restriction maps

$$\mathrm{Fil}^j_{\mathrm{Hodge}} H^i(\mathcal{X}, \Omega^{\bullet}_{\mathcal{X}/\mathcal{S}}) \to \mathrm{Fil}^j_{\mathrm{Hodge}} H^i(\mathcal{X}, \Omega^{\bullet}_{\mathcal{X}/\mathcal{S}}(\log \mathcal{D}))$$

and

$$\mathrm{Fil}^j_{\mathrm{Hodge}} H^i(\mathcal{Y}, \Omega^{\bullet}_{\mathcal{Y}/\mathcal{S}}) \to \mathrm{Fil}^j_{\mathrm{Hodge}} H^i(\mathcal{Y}, \Omega^{\bullet}_{\mathcal{Y}/\mathcal{S}}(\log \mathcal{E})),$$

are maps of locally free $\mathcal{O}_{\mathcal{S}}$-modules of finite rank whose kernels, images, and cokernels are locally free $\mathcal{O}_{\mathcal{S}}$-modules of finite rank whose formation commutes with arbitrary change of base on $\mathcal{S}$. Moreover, via the identification (I log) = (II log) via φ_{id} in 3) above, these image coincide, and are G-stable subspaces of

$$H^i(\mathcal{X}, \Omega^{\bullet}_{\mathcal{X}/\mathcal{S}}(\log \mathcal{D})) = H^i(\mathcal{Y}, \Omega^{\bullet}_{\mathcal{Y}/\mathcal{S}}(\log \mathcal{E})).$$

With all these preliminaries out of the way, we are ready to proceed with the proof of the theorem.

Let us first treat the case (**). Thus we assume that $H^i(\mathcal{X}, \Omega^{\bullet}_{\mathcal{X}/\mathcal{S}}(\log \mathcal{D}))(\chi)$, which "makes sense" thanks to our preliminary assumption 3) above, has p-curvature zero for almost all p. We must show that its complex fibre has finite monodromy on $\mathcal{S}_{\mathbb{C}} = S$. Denote by $\{\chi^{\sigma}\}_{\sigma}$ the distinct Aut($\mathbb{C}$)-conjugates of χ. It suffices, by [Ka-ASDE, 4.2.2.3] to prove that the Hodge filtration on

$$\oplus_{\sigma} H^i(\mathcal{X}, \Omega^{\bullet}_{\mathcal{X}/\mathcal{S}}(\log \mathcal{D}))(\chi^{\sigma})$$

is horizontal, for over $\mathcal{S}_{\mathbb{C}}$ this is (the de Rham "realization" of) a family of mixed Hodge structures whose associated graded family of pure Hodge structures is polarizable. For this, it suffices to prove that each individual term $H^i(\mathcal{X}, \Omega^{\bullet}_{\mathcal{X}/\mathcal{S}}(\log \mathcal{D}))(\chi^{\sigma})$ has its Hodge filtration horizontal (under the assumption that $H^i(\mathcal{X}, \Omega^{\bullet}_{\mathcal{X}/\mathcal{S}}(\log \mathcal{D}))(\chi)$ has p-curvature zero for almost all p).

The key observation here is that every irreducible representation χ of G is defined over the field $\mathbb{Q}(\zeta_{\mathrm{Card}(G)})$. Therefore all the Aut($\mathbb{C}$)-conjugates χ^{σ} of χ are obtained as we let σ vary over the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{\mathrm{Card}(G)})/\mathbb{Q}) = (\mathbb{Z}/\mathrm{Card}(G)\mathbb{Z})^{\times}$, with a in $(\mathbb{Z}/\mathrm{Card}(G)\mathbb{Z})^{\times}$ corresponding to the unique element σ_a of this galois group with $\sigma_a(\zeta) = \zeta^a$ for each Card(G)'th root of unity.

Fix an integer a which is invertible mod Card(G), and consider the automorphism $\sigma := \sigma_a$ in $\mathrm{Gal}(\mathbb{Q}(\zeta_{\mathrm{Card}(G)})/\mathbb{Q})$. In order to show that $H^i(\mathcal{X}, \Omega^\bullet_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi^\sigma)$ has its Hodge filtration horizontal, it suffices to show that for an infinity of primes p,

$H^i(\mathcal{X}, \Omega^\bullet_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi^\sigma)\otimes_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z})$ has its Hodge filtration horizontal.

We will show that $H^i(\mathcal{X}, \Omega^\bullet_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi^\sigma)\otimes_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z})$ has its Hodge filtration horizontal for almost all of the infinitely many (thanks to Dirichlet) primes p which satisfy

$$p \equiv 1/a \bmod \mathrm{Card}(G).$$

If we were to admit the existence of G-equivariant normal crossing compactifications, as we did in [Ka-ASDE], the desired horizontality modulo almost any prime $p \equiv 1/a \bmod \mathrm{Card}(G)$ would be given by [Ka-ASDE, 3.3.2], applied not to χ but rather to χ^σ (for then $(\chi^\sigma)^{(p)}$ in characteristic p is just (the reduction mod p of) χ itself).

Since we wish to avoid assuming the existence of G-equivariant normal crossing compactifications, we must give a slightly more involved, but not essentially different, argument.

To say that $H^i(\mathcal{X}, \Omega^\bullet_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi^\sigma)$ has its Hodge filtration horizontal means that for any derivation D of $\mathcal{S}/R$, acting via the Gauss-Manin connection, the composite map

$$\begin{array}{c} H^i(\mathcal{X}, \Omega^\bullet_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))) \\ \downarrow \nabla(D) \\ H^i(\mathcal{X}, \Omega^\bullet_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))) \\ \downarrow P(\chi^\sigma;\varphi) \\ H^i(\mathcal{Y}, \Omega^\bullet_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E}))) \end{array}$$

maps $\mathrm{Fil}^j{}_{\mathrm{Hodge}}H^i(\mathcal{X}, \Omega^\bullet_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))$ to $\mathrm{Fil}^j{}_{\mathrm{Hodge}}H^i(\mathcal{Y}, \Omega^\bullet_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E}))$ for every integer $j\geq 0$. By Griffiths transversality [Ka-ASDE, 1.4.1.6], $\nabla(D)$, maps $\mathrm{Fil}^j{}_{\mathrm{Hodge}}H^i$ to $\mathrm{Fil}^{j-1}{}_{\mathrm{Hodge}}H^i$. The other map, $P(\chi^\sigma;\varphi)$, respects the Hodge filtration. Thus $H^i(\mathcal{X}, \Omega^\bullet_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi^\sigma)$ has its Hodge filtration horizontal if and only if for each $j\geq 0$ the composite of the associated graded maps

$$\begin{array}{c} gr^{j}{}_{Hodge}H^{i}(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D})) \\ \downarrow \nabla(D) \\ gr^{j-1}{}_{Hodge}H^{i}(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D})) \\ \downarrow P(\chi^{\sigma};\varphi) \\ gr^{j-1}{}_{Hodge}H^{i}(\mathcal{Y}, \Omega^{\bullet}{}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E})) \end{array}$$

vanishes.

Since $\mathcal{S}$ is smooth over R and R is smooth over $\mathbb{Z}$, $\mathcal{S}$ is smooth over $\mathbb{Z}$, so $\mathcal{S}\otimes\mathbb{F}_p$ is smooth over $\mathbb{Z}/p\mathbb{Z}$, and hence $\mathcal{S}\otimes\mathbb{F}_p$ is reduced. Therefore the absolute Frobenius F_{abs} on $\mathcal{S}\otimes\mathbb{F}_p$ is injective. So to show this vanishing on $\mathcal{S}\otimes\mathbb{F}_p$, it suffices to check after base change by the absolute Frobenius F_{abs} of $\mathcal{S}\otimes\mathbb{F}_p$. Thanks to the main technical result [Ka-ASDE, 3.2], and its functoriality for the mappings $\varphi_g : \mathcal{Y} \to \mathcal{X}$, this composite, after reduction mod p and base change on $\mathcal{S}\otimes\mathbb{F}_p$ by absolute Frobenius, becomes (up to sign) the composite of associated graded maps for the conjugate filtration (see [Ka-ASDE, 2.3])

$$\begin{array}{c} gr^{i-j}{}_{con}H^{i}(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))\otimes_{\mathbb{Z}}\mathbb{F}_p \\ \downarrow \psi(D) \\ gr^{1+i-j}{}_{con}H^{i}(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))\otimes_{\mathbb{Z}}\mathbb{F}_p \\ \downarrow P((\chi^{\sigma})^{(p)};\varphi) \\ gr^{1+i-j}{}_{con}H^{i}(\mathcal{Y}, \Omega^{\bullet}{}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E}))\otimes_{\mathbb{Z}}\mathbb{F}_p \end{array}$$

where $\psi(D)$ denotes the p-curvature. Thus we need to see that this composite vanishes, for almost every prime p with

$$p \equiv 1/a \bmod \mathrm{Card}(G).$$

Now the map $P((\chi^{\sigma})^{(p)};\varphi)$ respects the conjugate filtration, while the p-curvature $\psi(D)$ maps $\mathrm{Fil}^{i-j}{}_{con}$ to $\mathrm{Fil}^{1+i-j}{}_{con}$. Thus we are reduced to showing that the composite map

$$\begin{array}{c} H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D})))\otimes_{\mathbb{Z}}\mathbb{F}_p \\ \downarrow \psi(D) \\ H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D})))\otimes_{\mathbb{Z}}\mathbb{F}_p \\ \downarrow P((\chi^\sigma)^{(p)};\varphi) = P(\chi;\varphi) \\ H^i(\mathcal{Y}, \Omega^\bullet{}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E})))\otimes_{\mathbb{Z}}\mathbb{F}_p \end{array}$$

maps

$$\mathrm{Fil}^{i-j}{}_{\mathrm{con}}H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D})))\otimes_{\mathbb{Z}}\mathbb{F}_p$$

to

$$\mathrm{Fil}^{2+i-j}{}_{\mathrm{con}}H^i(\mathcal{Y}, \Omega^\bullet{}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E})))\otimes_{\mathbb{Z}}\mathbb{F}_p.$$

But this composite map is in fact the zero map for almost all primes $p \equiv 1/a \bmod \mathrm{Card}(G)$, precisely by the hypothesis that

$$H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi)$$

has p-curvature zero for almost all p, and hence in particular has p-curvature zero for almost all primes $p \equiv 1/a \bmod \mathrm{Card}(G)$. This concludes the proof of case (**).

We now turn to the proof of the theorem in the case (***). The proof is very similar to that of case (**), but for the sake of completeness we will spell out all the details. Thus we assume that

$$W_iH^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi) :=$$
$$(\mathrm{Image}(H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}) \to H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))))(\chi)$$

has p-curvature zero for almost all p. We must show that its complex fibre has finite monodromy on $\mathcal{S}_{\mathbb{C}} = S$. Denote by $\{\chi^\sigma\}_\sigma$ the distinct $\mathrm{Aut}(\mathbb{C})$-conjugates of χ. It suffices, by [Ka-ASDE, 4.2.1.3] to prove that the Hodge filtration on

$$\oplus_\sigma W_iH^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi^\sigma)$$

is horizontal, for over $\mathcal{S}_{\mathbb{C}}$ this is (the de Rham "realization" of) a family of polarizable pure Hodge structures. For this, it suffices to prove that each individual term $W_iH^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi^\sigma)$ has its Hodge filtration horizontal (under the assumption that $W_iH^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi)$ has p-curvature zero for almost all p).

To say that $W_iH^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi^\sigma)$ has its Hodge filtration horizontal means that for any derivation D of $\mathcal{S}/R$, acting via the Gauss-Manin connection, the composite map

$$\begin{array}{c} H^i(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}) \\ \downarrow \text{restr.} \\ H^i(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))) \\ \downarrow \nabla(D) \\ H^i(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))) \\ \downarrow P(\chi^{\sigma};\varphi) \\ H^i(\mathcal{Y}, \Omega^{\bullet}{}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E}))) \end{array}$$

maps $\mathrm{Fil}^j{}_{\mathrm{Hodge}}H^i(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}})$ to $\mathrm{Fil}^j{}_{\mathrm{Hodge}}H^i(\mathcal{Y}, \Omega^{\bullet}{}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E})))$ for every integer $j\geq 0$. By Griffiths transversality [Ka-ASDE, 1.4.1.6], $\nabla(D)$, maps $\mathrm{Fil}^j{}_{\mathrm{Hodge}}H^i$ to $\mathrm{Fil}^{j-1}{}_{\mathrm{Hodge}}H^i$. The other two maps, restriction and $P(\chi^{\sigma};\varphi)$, respect the Hodge filtration. Thus $W_iH^i(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi^{\sigma})$ has its Hodge filtration horizontal if and only if for each $j\geq 0$ the composite of the associated graded maps

$$\begin{array}{c} \mathrm{gr}^j{}_{\mathrm{Hodge}}H^i(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}) \\ \downarrow \text{restr.} \\ \mathrm{gr}^j{}_{\mathrm{Hodge}}H^i(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))) \\ \downarrow \nabla(D) \\ \mathrm{gr}^{j-1}{}_{\mathrm{Hodge}}H^i(\mathcal{X}, \Omega^{\bullet}{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))) \\ \downarrow P(\chi^{\sigma};\varphi) \\ \mathrm{gr}^{j-1}{}_{\mathrm{Hodge}}H^i(\mathcal{Y}, \Omega^{\bullet}{}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E}))) \end{array}$$

vanishes.

Fix an integer a which is invertible mod Card(G), and suppose that $\sigma := \sigma_a$ in $\mathrm{Gal}(\mathbb{Q}(\zeta_{\mathrm{Card}(G)})/\mathbb{Q})$. In order to show that this last composite vanishes, it suffices to show that it vanishes mod p for an infinity of primes p. We will show that it vanishes mod p for almost all of the infinitely many (thanks to Dirichlet) primes p which satisfy

$$p \equiv 1/a \bmod \mathrm{Card}(G).$$

Since $\mathcal{S}$ is smooth over R and R is smooth over $\mathbb{Z}$, $\mathcal{S}$ is smooth over $\mathbb{Z}$, so $\mathcal{S}\otimes\mathbb{F}_p$ is smooth over $\mathbb{Z}/p\mathbb{Z}$, and hence $\mathcal{S}\otimes\mathbb{F}_p$ is reduced.

Therefore the absolute Frobenius F_{abs} on $\mathcal{S}\otimes\mathbb{F}_p$ is injective. So to show this vanishing on $\mathcal{S}\otimes\mathbb{F}_p$, it suffices to check after base change by the absolute Frobenius F_{abs} of $\mathcal{S}\otimes\mathbb{F}_p$. Thanks to the main technical result [Ka-ASDE, 3.2], and its functoriality for the mappings $\varphi_g : \mathcal{Y} \to \mathcal{X}$, this composite, after reduction mod p and base change on $\mathcal{S}\otimes\mathbb{F}_p$ by absolute Frobenius, becomes (up to sign) the composite of associated graded maps for the conjugate filtration

$$\begin{array}{l}
\mathrm{gr}^{i-j}{}_{\mathrm{con}}H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}})\otimes_{\mathbb{Z}}\mathbb{F}_p \\
\qquad\qquad \downarrow \text{restr.} \\
\mathrm{gr}^{i-j}{}_{\mathrm{con}}H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))\otimes_{\mathbb{Z}}\mathbb{F}_p \\
\qquad\qquad \downarrow \psi(D) \\
\mathrm{gr}^{1+i-j}{}_{\mathrm{con}}H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))\otimes_{\mathbb{Z}}\mathbb{F}_p \\
\qquad\qquad \downarrow P((\chi^\sigma)^{(p)};\varphi) = P(\chi;\varphi) \\
\mathrm{gr}^{1+i-j}{}_{\mathrm{con}}H^i(\mathcal{Y}, \Omega^\bullet{}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E}))\otimes_{\mathbb{Z}}\mathbb{F}_p
\end{array}$$

where $\psi(D)$ denotes the p-curvature. Thus we need to see that this composite vanishes, for every prime p with $p \equiv 1/a \bmod \mathrm{Card}(G)$. Now both the restriction map and the map $P((\chi^\sigma)^{(p)};\varphi)$ respect the conjugate filtration, while the p-curvature $\psi(D^{(p)})$ maps $\mathrm{Fil}^{i-j}{}_{\mathrm{con}}$ to $\mathrm{Fil}^{1+i-j}{}_{\mathrm{con}}$. Thus we are reduced to showing that the composite map

$$\begin{array}{l}
H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}})\otimes_{\mathbb{Z}}\mathbb{F}_p \\
\qquad\qquad \downarrow \text{restr.} \\
H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))\otimes_{\mathbb{Z}}\mathbb{F}_p \\
\qquad\qquad \downarrow \psi(D) \\
H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log(\mathcal{D}))\otimes_{\mathbb{Z}}\mathbb{F}_p \\
\qquad\qquad \downarrow P((\chi^\sigma)^{(p)};\varphi) = P(\chi;\varphi) \\
H^i(\mathcal{Y}, \Omega^\bullet{}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E}))\otimes_{\mathbb{Z}}\mathbb{F}_p
\end{array}$$

maps

$$\mathrm{Fil}^{i-j}{}_{\mathrm{con}}H^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}})\otimes_{\mathbb{Z}}\mathbb{F}_p$$

to

$$\mathrm{Fil}^{2+i-j}{}_{\mathrm{con}}H^i(\mathcal{Y}, \Omega^\bullet{}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{E}))\otimes_{\mathbb{Z}}\mathbb{F}_p.$$

But this composite map is in fact the zero map for almost all primes $p \equiv 1/a \bmod \mathrm{Card}(G)$, precisely by the hypothesis that

$$W_iH^i(\mathcal{X}, \Omega^\bullet{}_{\mathcal{X}/\mathcal{S}}(\log\mathcal{D}))(\chi)$$

has p-curvature zero for almost all p, and hence in particular has p-curvature zero for almost all primes $p \equiv 1/a \bmod \mathrm{Card}(G)$. QED

9.4 Application to rigid local systems

Theorem 9.4.1 The regular singular differential equation corresponding, via Riemann-Hilbert, to any irreducible rigid local system on an open set of $\mathbb{P}^1$ over $\mathbb{C}$ satisfies Grothendieck's p-curvature conjecture: such a differential equation has p-curvature zero for almost all p if and only if it has finite monodromy.

proof Let $\{\alpha_1, \dots, \alpha_n\}$ be $n \geq 2$ distinct complex numbers. Suppose that on $(\mathbb{A}^1(\mathbb{C})- \{\alpha_1, \dots, \alpha_n\})^{an}$ we are given an irreducible rigid local system $\mathcal{F}_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces whose underlying regular singular differential equation, say (M, ∇), has p-curvature zero for almost all p. Thanks to the previous theorem 9.3.5, it suffices to prove that (M, ∇) satisfies the condition (***) of 9.3.3.

To prove that (M, ∇) satisfies (***), we argue as follows. The p-curvature hypothesis implies that $\mathcal{F}_{\mathbb{C}}$ has local monodromy of finite order around ∞ and around each of the points α_i [Ka-NCMT,13.0.2]. So we are reduced to proving the following proposition, which may be of independent interest.

Proposition 9.4.2 Let $\{\alpha_1, \dots, \alpha_n\}$ be $n \geq 2$ distinct complex numbers. Suppose that on $(\mathbb{A}^1(\mathbb{C})- \{\alpha_1, \dots, \alpha_n\})^{an}$ we are given an irreducible rigid local system $\mathcal{F}_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces whose local monodromies are all quasiunipotent. Then the underlying regular singular differential equation satisfies (***) of 9.3.3.

proof Fix an integer N such all the eigenvalues of all the local monodromies of $\mathcal{F}_{\mathbb{C}}$ have order dividing N. According to 5.10.6, $\mathcal{F}$ has a $\mathbb{Q}(\zeta_N)$-form $\mathcal{F}_{\mathrm{cycl}}$ on $(\mathbb{A}^1(\mathbb{C})- \{\alpha_1, \dots, \alpha_n\})^{an}$, and for every finite place λ of $E := \mathbb{Q}(\zeta_N)$, there exists a lisse E_λ-sheaf $\mathcal{F}_\lambda$ on the algebraic variety $(\mathbb{A}^1 - \{\alpha_1, \dots, \alpha_n\})_{\mathbb{C}}$ with $(\mathcal{F}_\lambda)^{an} \cong (\mathcal{F}_{\mathrm{cycl}})\otimes_E E_\lambda$.

Fix one such finite place λ, say of residue characteristic ℓ, and choose (!) an isomorphism of fields $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Denote by $\mathcal{F}_\ell$ the lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_\lambda \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$ on $(\mathbb{A}^1 - \{\alpha_1, \dots, \alpha_n\})_\mathbb{C}$. Then

$$(\mathcal{F}_\ell)^{an} \cong \mathcal{F}_\mathbb{C} \otimes_\mathbb{C} \overline{\mathbb{Q}}_\ell,$$

and $\mathcal{F}_\ell$ is a tame, geometrically irreducible lisse sheaf which is cohomologically rigid, such that all eigenvalues of all local monodromies are N'th roots of unity. For technical reasons, it will be convenient to consider the dual local system $(\mathcal{F}_\ell)^\vee$, which, like $\mathcal{F}_\ell$, is a tame, geometrically irreducible lisse sheaf which is cohomologically rigid, such that all eigenvalues of all local monodromies are N'th roots of unity.

According to 8.4.1, $(\mathcal{F}_\ell)^\vee$ on $(\mathbb{A}^1 - \{\alpha_1, \dots, \alpha_n\})_\mathbb{C}$ arises as follows. For a suitable choice of an integer $r \geq 0$, $(r+1)n$ arbitrary integers $e(a,i)$, r integers $f(k)$ such that no $f(k)$ is divisible by N, and a faithful $\overline{\mathbb{Q}}_\ell^\times$-valued character χ of the group $\boldsymbol{\mu}_N(\mathbb{Z}[1/N\ell, \zeta_N])$, there is a(n explicit) smooth affine hypersurface of relative dimension r, on which the group $\boldsymbol{\mu}_N(\mathbb{Z}[1/N\ell, \zeta_N])$ acts,

$$\pi : \mathrm{Hyp}(e\text{'s}, f\text{'s}) \to (\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}},$$

with the following properties:

1) For all i, the sheaves $R^i\pi_!\overline{\mathbb{Q}}_\ell$ on $(\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}}$ are lisse and tame (and mixed of integral weights in [0,i], by [De-Weil II]).

2)For any **faithful** character ρ of the group $\boldsymbol{\mu}_N(\mathbb{Z}[1/N\ell, \zeta_N])$, the ρ-component $(R^i\pi_!\overline{\mathbb{Q}}_\ell)^\rho$ vanishes for $i \neq r$, and the sheaf $(R^r\pi_!\overline{\mathbb{Q}}_\ell)^\rho$ is lisse and mixed of of integral weights in [0,r].

3) The weight r quotient $((R^r\pi_!\overline{\mathbb{Q}}_\ell)^\chi)_{=r}$ of $(R^r\pi_!\overline{\mathbb{Q}}_\ell)^\chi$, restricted to restricted to $(\mathbb{A}^1 - \{\alpha_1, \dots, \alpha_n\})_\mathbb{C}$, is isomorphic to $(\mathcal{F}_\ell)^\vee$.

We now apply Poincaré duality to this situation. By 1), all of the sheaves $R^i\pi_*\overline{\mathbb{Q}}_\ell$ on $(\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}}$ are lisse and tame, mixed of integer weights $\geq$ i, and of formation compatible with arbitrary change of base on $(\mathbb{A}^1 - \{T_1, \dots, T_n\})_{S_{N,n,\ell}}$. By 2), for any **faithful** character ρ of the group $\boldsymbol{\mu}_N(\mathbb{Z}[1/N\ell, \zeta_N])$, the ρ-component

$(R^i\pi_*\overline{\mathbb{Q}}_\ell)^\rho$ vanishes for $i\neq r$, and the sheaf $(R^r\pi_*\mathbb{Q}_\ell)^\rho$ is lisse and mixed of of integral weights $\geq$ r. By 3), the weight r subsheaf $W_r(R^r\pi_*\mathbb{Q}_\ell)^{\overline{\chi}}$ of $(R^r\pi_*\mathbb{Q}_\ell)^{\overline{\chi}}$, i.e., the $\overline{\chi}$-isotypical component of $W_r(R^r\pi_*\mathbb{Q}_\ell)$, restricted to $(\mathbb{A}^1- \{\alpha_1, ... , \alpha_n\})_\mathbb{C}$, is isomorphic to $\mathcal{F}_\ell$.

Denote by R_0 the subring of $\mathbb{C}$ defined by

$$R_0 := \mathbb{Z}[1/N\ell, \zeta_N, \alpha_1, ... , \alpha_n, 1/\Pi_{i\neq j}(\alpha_i - \alpha_j)].$$

The ring R_0 is finitely generated over $\mathbb{Z}$ as a ring, and there exists a nonzero element δ in R_0 such that $R := R_0[1/\delta]$ is smooth over $\mathbb{Z}$. We have a canonical ring homomorphism

$$S_{N,n,\ell} \to R,\ T_i \mapsto \alpha_i.$$

Via this base change, the open curve $(\mathbb{A}^1- \{T_1, ... , T_n\})_{S_{N,n,\ell}}$ over $S_{N,n,\ell}$ gives rise to an open curve $(\mathbb{A}^1- \{\alpha_1, ... , \alpha_n\})_R$ over R, whose complex fibre (via the given inclusion of R into $\mathbb{C}$) is the complex curve $(\mathbb{A}^1- \{\alpha_1, ... , \alpha_n\})_\mathbb{C}$.

Now consider the **pullback** π_R to $(\mathbb{A}^1- \{\alpha_1, ... , \alpha_n\})_R$ of the smooth hypersurface

$$\pi : \mathrm{Hyp}(e\text{'s}, f\text{'s}) \to (\mathbb{A}^1- \{T_1, ... , T_n\})_{S_{N,n,\ell}},$$

and of the various cohomology sheaves to which it gives rise. Thanks to Hironaka [Hir-RS, Cor. 3 of Thm. 2], we can find a normal crossings compactification of this pulled-back morphism, first over the generic point of $(\mathbb{A}^1- \{\alpha_1, ... , \alpha_n\})_R$, and then, by "spreading out", over a dense open set, say $\mathcal{U}$, of $(\mathbb{A}^1- \{\alpha_1, ... , \alpha_n\})_R$, say

$$\overline{\pi}_R: \mathcal{X} \to (\mathbb{A}^1- \{\alpha_1, ... , \alpha_n\})_R$$

It is standard (a proof is given in 9.4.3 below) that, for every i, we have

$$W_i(R^i\pi_*\mathbb{Q}_\ell) \mid \mathcal{U} = \mathrm{Image}(R^i(\overline{\pi}_R)_*\overline{\mathbb{Q}}_\ell \to R^i(\pi_R)_*\overline{\mathbb{Q}}_\ell).$$

Extending scalars from R to $\mathbb{C}$, we find that over a dense open set U $:= \mathcal{U}_\mathbb{C}$ of $(\mathbb{A}^1- \{\alpha_1, ... , \alpha_n\})_\mathbb{C}$, there exists a normal crossings compactification $\overline{\pi}_\mathbb{C} : X \to U$ of a smooth morphism $\pi_\mathbb{C} : \mathrm{Hyp} \to U$ on which μ_N acts, and a faithful character χ of μ_N such that

$$\mathcal{F}_\ell \mid U = (\mathrm{Image}(R^r(\overline{\pi}_\mathbb{C})_*\overline{\mathbb{Q}}_\ell \to R^r(\pi_\mathbb{C})_*\overline{\mathbb{Q}}_\ell)(\overline{\chi}).$$

Restricting to U^{an}, using the comparison theorem [SGA4, Exp. XVI, 4.1] and the isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we find

$$\mathcal{F}_{\mathbb{C}} \mid U^{an} = (\mathrm{Image}(R^r(\overline{\pi}_{\mathbb{C}})^{an}{}_*\mathbb{C} \to R^r(\pi_{\mathbb{C}})^{an}{}_*\mathbb{C})(\overline{\chi}).$$

Because X/U is a normal crossing compactification of Hyp/U, the relative de Rham cohomology sheaves on U, $H^i{}_{DR}(X/U)$ and $H^i{}_{DR}(\mathrm{Hyp}/U)$, are coherent $\mathcal{O}_U$-modules with integrable connection, of formation compatible with arbitrary change of base on U. They are known [Ka-NCMT] to have regular singular points. Under the Riemann-Hilbert correspondence, they correspond to the local systems $R^i(\overline{\pi}_{\mathbb{C}})^{an}{}_*\mathbb{C}$ and $R^i(\pi_{\mathbb{C}})^{an}{}_*\mathbb{C}$ on U^{an}.

Taking i=r, we see that, on U, the regular singular differential equation corresponding to the local system $\mathcal{F}_{\mathbb{C}}$ is

$$(\mathrm{Image}(H^r{}_{DR}(X/U) \to H^r{}_{DR}(\mathrm{Hyp}/U)))(\overline{\chi}).$$

Restricting to the generic point of U, we see that this differential equation is indeed of type (***). This concludes the proof. QED

(9.4.3) It remains only to recall why

$$W_i(R^i\pi_*\mathbb{Q}_\ell) \mid \mathcal{U} = \mathrm{Image}(R^i(\overline{\pi}_R)_*\overline{\mathbb{Q}}_\ell \to R^i(\pi_R)_*\overline{\mathbb{Q}}_\ell).$$

It suffices to check at all closed points. Since both sides commute with arbitrary change of base, we are reduced to checking that if k is a finite field of characteristic $\neq \ell$, X/k is proper and smooth, and $D = \cup D_i$ is a union of smooth divisors in X with normal crossings, then under the restriction map, we have

$$\mathrm{Image}(H^i(X\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell) \to H^i((X-D)\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell)) = W_iH^i((X-D)\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell).$$

To see this, consider the Leray spectral sequence

$$E_2{}^{p,q} = H^p(X\otimes_k\overline{k}, R^qj_*\overline{\mathbb{Q}}_\ell) \Rightarrow H^{p+q}((X-D)\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell)$$

for the inclusion map j: X-D → X. We have $j_*\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell$, and for each $q \geq 1$,we have

$$R^qj_*\overline{\mathbb{Q}}_\ell = \oplus_{i_1 < i_2 <\ldots< i_q} \overline{\mathbb{Q}}_\ell(-q) \mid D_{i_1}\cap D_{i_2}\cap\ldots\cap D_{i_q}.$$

Thus $E_2{}^{p,q}$ is pure of weight p+2q, and hence $E_r{}^{p,q}$ is pure of weight p+2q for every $r \geq 2$. Since d_r has bidegree (r, 1-r), we have $d_r = 0$ for all $r\geq 3$. Looking at the weights of the $E_\infty = E_3$ terms, we see that $H^i((X-D)\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell)$ is mixed of weight $\geq$ i, and that

$$W_iH^i((X-D)\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell) = E_\infty{}^{i,0} = E_3{}^{i,0}.$$

Since this is a first quadrant spectral sequence, d_2 kills $E_2^{i,0}$, so we have a surjective map $E_2^{i,0} \to E_3^{i,0}$, i.e, a surjective map

$$H^i(X\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell) \to W_iH^i((X-D)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell),$$

as required.

(9.4.4) Here is an alternate proof of a stronger statement. Let X be proper and smooth over a finite field k of characteristic $\neq \ell$, and Z in X any closed subscheme. Then for every i,

$$\mathrm{Image}(H^i(X\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell) \to H^i((X-Z)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell)) = W_iH^i((X-Z)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell).$$

To prove this, we argue as follows. Passing to connected components, we may reduce to the case when X is connected, of some dimension d. Now $H^i((X-Z)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell)$ and $H_c^{2d-i}((X-Z)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell)$ are Poincaré dual, with values in $\bar{\mathbb{Q}}_\ell(-d)$. Looking at weights, we see that under this pairing, the dual of $W_iH^i((X-Z)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell)$ is the weight 2d-i quotient of $H_c^{2d-i}((X-Z)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell)$. So our statement is dual to the statement that

$$\mathrm{Image}(H_c^{2d-i}((X-Z)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell) \to H^{2d-i}(X\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell))$$

is the weight 2d-i quotient of $H_c^{2d-i}((X-Z)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell)$, or what is the same (since $H^{2d-i}(X\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell)$ is pure of weight 2d-i), that

$$\mathrm{Ker}(H_c^{2d-i}((X-D)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell) \to H^{2d-i}(X\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell))$$

is mixed of weight < 2d-i. But this is clear from the excision sequence

$$\ldots H^{2d-i-1}(D\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell)) \to H_c^{2d-i}((X-D)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell) \to H^{2d-i}(X\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell))\ldots$$

and the fact that, Z being proper over k, $H^{2d-i-1}(D\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell))$ is mixed of weight $\leq$ 2d-i-1, by [De-Weil II, 3.3.1].

(9.4.5) As a minor variant on this argument, we could more explicitly exploit the exactness of the functors $gr^W{}_i$ (:= associated graded of weight i for the weight filtration) on the category of $\bar{\mathbb{Q}}_\ell$-finite dimensional $\bar{\mathbb{Q}}_\ell[\mathrm{Gal}(\bar{k}/k)]$-modules. Applying $gr^W{}_{2d-i}$ to the excision sequence gives the injectivity of

$$gr^W{}_{2d-i}(H_c^{2d-i}((X-D)\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell)) \to gr^W{}_{2d-i}(H^{2d-i}(X\otimes_k\bar{k}, \bar{\mathbb{Q}}_\ell))).$$

The Poincaré dual of this injectivity is the surjectivity of

$$\mathrm{gr}^W{}_i(H^i(X\otimes_k \bar{k}, \bar{\mathbb{Q}}_\ell)) \to \mathrm{gr}^W{}_i(H^i((X-D)\otimes_k \bar{k}, \bar{\mathbb{Q}}_\ell)).$$

Since $H^i(X\otimes_k \bar{k}, \bar{\mathbb{Q}}_\ell)$ is pure of weight i, while $H^i((X-D)\otimes_k \bar{k}, \bar{\mathbb{Q}}_\ell)$ is mixed of weight $\geq$ i, this is the required surjectivity of

$$H^i(X\otimes_k \bar{k}, \bar{\mathbb{Q}}_\ell) \to W_iH^i((X-D)\otimes_k \bar{k}, \bar{\mathbb{Q}}_\ell).$$

9.5 Comments and questions

(9.5.1) In addition to the general result [Ka-ASDE, 5.7], Grothendieck's p-curvature conjecture has been proven for several explicit families of differential equations on open sets of $\mathbb{P}^1$. It is striking that in all of these cases, whenever the equation in question is irreducible, it is in fact rigid (in the sense that its local system of germs of holomorphic solutions is a rigid local system). These cases were:

1) the (differential equation satisfied by the) Gauss hypergeomtric function ${}_2F_1$, cf. [Ka-ASDE, 6.2 and 6.9.4],

2) the (differential equation satisfied by the) generalized hypergeometric function ${}_nF_{n-1}$, cf. [B-H,4.8 and 4.9],

3)the ("Pochhammer" differential equation, satisfied by) the hypergeometric functions of Pochhammer type, cf. [Har, Theorems 1.2, 1.3, 2.1].

(9.5.2) In all three of these cases, an essential step is to compute, in terms of the parameters of the equation (i.e., in terms of our "numerical data" of chapter 6, which specifies all the local monodromies) precisely what is the condition to have p-curvature zero for almost all p. In each case, one gets an a posteriori verification that, having begun with a differential equation satisfying (**), one has a direct factor of a differential equation satisfying (*). This was the method of proof in [Ka-ASDE, 6.2]. It was also the method employed in the exposition of [B-H] given in [Ka-ESDE, 5.5].

(9.5.3) However, it should be emphasized that both Beukers-Heckmann and Haraoka prove their results **without** invoking [ASDE, 5.7]. Rather, what they do, at least in the irreducible case with quasiunipotent local monodromy, is to consider the unique (up to scalars) hermitian form carried by the local system $\mathcal{F}_\mathbb{C}$ in question which expresses that the dual local system to $\mathcal{F}_\mathbb{C}$ is just its complex conjugate. They calculate the signature of this hermitian form in terms of the parameters of the equation, and then show that the form is (positive or negative) definite provided the parameters satisfy the conditions of p-curvature zero for almost all p.

(9.5.4) In contrast, while we have proven that Grothendieck's p-curvature conjecture holds for (the regular singular differential equation underlying) any irreducible rigid local system on an open set of $\mathbb{P}^1$, we do **not** know how to tell, in terms of the numerical data of chapter 6, or equivalently in terms of the data of all the local monodromies, whether or not a particular such differential equation has in fact p-curvature zero for almost all p. Presumably there is a simple explicit algorithm for computing, in terms of the numerical data, whether or not we have p-curvature zero for almost all p. What is it? If one is more optimistic, one might ask how to compute, in terms of the numerical data, the dimension of the differential galois group (which for regular singular points is the Zariski closure of the monodromy group), or even the isomorphism class of its Lie algebra. Much remains to be done.

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